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The Validity of the Ehrenfest Theorem in Quantum Gravity Theory

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Abstract

The Ehrenfest theorem is a well-known theoretical result of quantum mechanics. It relates the dynamical evolution of the expectation value of a quantum operator to the expectation value of its corresponding commutator with the Hermitian Hamiltonian operator. However, the proof of validity of the Ehrenfest theorem for quantum gravity field theory has remained elusive, while its validation poses challenging conceptual questions. In fact, this presupposes a number of minimum requirements, which include the prescription of quantum Hamiltonian operator, the definition of scalar product, and the identification of dynamical evolution parameter. In this paper, it is proven that the target can be established in the framework of the manifestly covariant quantum gravity theory (CQG theory). This follows as a consequence of its peculiar canonical Hamiltonian structure and the commutator-bracket algebra that characterizes its representation and probabilistic interpretation. The theoretical proof of the theorem for CQG theory permits to elucidate the connection existing between quantum operator variables of gravitational field and the corresponding expectation values to be interpreted as dynamical physical observables set in the background metric space-time.

Keywords: covariant quantum gravity; Ehrenfest theorem; Hamiltonian theory

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1. Introduction

The original formulation of the Ehrenfest theorem was established in the framework of non-relativistic quantum mechanics [1]. The theorem relates the dynamical evolution of the expectation value of a quantum operator Q to the expectation value of its corresponding commutator with the Hamiltonian operator $H^{(q)}$ of the system as

$$\frac{d}{dt}\langle Q\rangle = \frac{1}{i\hbar}\left\langle \left[Q, H^{(q)}\right]\right\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle, \tag{1}$$

where $[\dots]$ denotes the commutator bracket and t is the absolute coordinate time of the non-relativistic reference frame. In addition, according to customary notation and for a generic operator Q, the bracket $\langle Q \rangle$ denotes the quantum expectation value defined over the configuration space Ω with the differential measure d^3x as

$$\langle Q \rangle \equiv \int_\Omega d^3x \psi^* Q \psi, \tag{2}$$

where ψ is the complex quantum wave function entering the Schrödinger equation and ψ^* its complex conjugate. The Ehrenfest theorem is proved as a theoretical result implicit in the foundational formalism of non-relativistic quantum mechanics, namely the validity of the Schrödinger equation together with the unitary evolution and probabilistic interpretation of the quantum wave function. The theorem has a notable importance, as it establishes a unique mathematical relationship between quantum variables and corresponding observable expectation values. This helps in understanding the conceptual meaning of the so-called correspondence principle, namely the connection rules existing between classical and quantum theories, like the semiclassical limit.

However, besides this outcome, it has to be noted that a satisfactory extension of the Ehrenfest theorem to other branches of quantum physics has remained incomplete, despite the progresses made in theoretical physics in the last century. This falls within the more general question inherent in the consistency of the conceptual framework of canonical quantum mechanics with the principle of covariance as the basis of special and general relativity theories [2]. As a result, the realization of the Ehrenfest theorem for relativistic quantum mechanics or quantum field theory, including quantum gravity, still represents an open problem today [3].

In fact, a number of difficulties arise when trying to formulate the Ehrenfest theory implementing the covariant formalism of the relativistic framework. Two main problems are met in this respect. The first one is the formal achievement of a dynamical equation that can be consistently interpreted as the relativistic analogous of the Ehrenfest equation. The second problem is a conceptual one, namely the possibility of assigning this equation the same correct physical interpretation of the non-relativistic Ehrenfest theory, consistent with the foundations of quantum mechanics itself. In more detail, from the physical point of view, the critical issues for relativistic quantum mechanics are related to the following:

(1) Lack of an absolute (and possibly invariant) time evolution parameter measuring the dynamics at both classical and corresponding quantum levels [4–6]. The same time parameter should be identified unambiguously and should play the role of the absolute coordinate time t of the non-relativistic picture. The absence of such an absolute time variable has been customarily ascribed to the property of invariance of the theory under time parametrization in the relativistic framework where time and space variables are treated on equal footing. This has motivated the search of multiple but questionable alternative suggestions for the identification of temporal-like variables, namely physical properties or quantities of the system whose intrinsic dynamical change can be put in correspondence with an analogue time variable. As a result, in such a setting the identification of a plausible Ehrenfest equation is found to be formally involved, while in the end its meaning would remain trivial or even empty of physical content due to absence of temporal dynamics.

(2) Lack of a notion of unitary operator, with particular reference to unitary dynamical evolution generated by the Hamiltonian operator, which follows from the missing concept of time evolution parameter.

(3) Lack of a probabilistic interpretation of quantum wave function, namely the notion that positive-defined probability and probability density is associated with the norm of the quantum wave function. The deficiency is related in turn to the mathematical structure of the relativistic quantum-wave equation, which usually differs from a Schrödinger-like differential equation. This does not warrant validity of customary probability-density

conservation theorem, as it happens in non-relativistic regime. A well-known example of this occurrence is provided by the Klein–Gordon wave equation, which predicts the conservation of a 4-current density j^{μ} rather than a probability density. However, the 4-vector character of j^{μ} is manifestly inadequate to cast a probabilistic quantum theory as it can lead to states with either positive, null or negative squared norm.

Analogous issues are met when attempting to a formulation of the Ehrenfest theorem in quantum field theory [3]. From a conceptual point of view, also in this case the target should be to establish a relationship of the type (1) that applies to continuous fields and currents, with the Hamiltonian operator being replaced by a Hamiltonian density. Then, the quantum-field Ehrenfest theorem should relate classical field equations of motion to quantum dynamics through suitably defined quantum expectation values. However, it is known that the resulting equations for expectation values recover classical field equations only under particular conditions, e.g., linear theories, non-interacting fields or for appropriate semiclassical limits. The reason lies in the commutation properties among differential operators and quantum expectation values, which do not commute when (non-linear) mutual field interactions are included through the Hamiltonian potential term. The discrepancy with the standard representation of the Ehrenfest theorem in quantum mechanics is explained as being due to quantum fluctuation corrections showing up at characteristic scale lengths. A convenient framework that permits a perturbative treatment of these quantum fluctuation contributions is provided by so-called Effective Field Theories, with potential impact also on quantum gravity [7–9]. However, apart from considerations of general grounds, a comprehensive non-perturbative and non-asymptotic theoretical treatment of the Ehrenfest theorem in quantum field theory established in terms of an unambiguous mathematical setup appears hard to be found.

The extension of the Ehrenfest theorem to quantum gravity has been much less straightforward than relativistic quantum mechanics, and the proof of its validity in such a framework has remained elusive. In fact, the establishment of the Ehrenfest theorem for quantum gravity raises challenging targets that pertain to both theoretical physics and mathematics, as well as formidable conceptual and philosophical questions that are rooted in the understanding of the very nature of space and time and quantum aspects of emerging gravitational field. The literature on the subject is very limited and only a few works have tackled the problem from a mathematical physics perspective (see Refs. [10,11]). The setup usually considered, among different candidate theories, is the so-called loop quantum gravity (LQG) yielding the Wheeler–DeWitt equation as quantum wave equation [12,13]. This choice is motivated by analogies of such a theory with the customary approach to relativistic quantum field theory canonical representation, which pertain both the formal representation of quantum equations as well as their physical content.

Trying to establish the Ehrenfest theorem in the framework of LQG inherits however the same problems outlined above for relativistic quantum mechanics concerning the definition of time and the probabilistic interpretation of the wave function [14]. Besides these aspects, the same procedure meets also additional obstacles, as it presupposes a number of minimum requirements to be satisfied altogether, which include the constraints placed by covariance principle, the prescription of quantum Hamiltonian operator and again the identification of Hilbert space on which scalar products can be defined. Overall, this must be performed in a conceptual framework where space-time should arise as an emerging phenomenon and where the quantum gravity theory itself is demanded to be formally independent of such a concept.

These issues are addressed below according to the following scheme. In Section 2, a summary of the Arnowitt–Deser–Misner (ADM) theoretical approach to the Hamiltonian formulation of Einstein equations and the quantization rule yielding the Wheeler–DeWitt

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equation is given. In Section 3, the manifestly covariant DeDonder–Weyl formalism is introduced for the variational treatment of general relativity (GR) and some relevant qualitative properties are discussed. Sections 4 and 5 deal with the Lagrangian and Hamiltonian formulations of the manifestly covariant classical gravity theory obtained in terms of novel synchronous variational principle, respectively. In Section 6, the classical Hamiltonian theory of GR is cast in evolution form, which is appropriate for its canonical quantization. The corresponding manifestly covariant quantum gravity theory (CQG theory) and its fundamental principles are then established in Section 7. This provides the theoretical basis for the proof of the Ehrenfest theorem which is given in Section 8 and is shown to hold for CQG theory as a consequence of its peculiar quantum Hamiltonian structure. Finally, in Section 9, a discussion is proposed on the physical meaning of the dynamical evolution parameter of the Ehrenfest theory, while concluding remarks are reported in Section 10.

2. ADM Formalism and Wheeler–DeWitt Equation

In this section we provide a short review of ADM formalism and the consequent derivation of the Wheeler–DeWitt quantum equation. The content of this section will be useful below to understand the formulation of the Ehrenfest theorem within the manifestly covariant framework of CQG theory. In view of the scope of this paper, it is therefore worth recalling the main assumptions of ADM approach.

In detail, the quantum theory of LQG is rooted on the so-called ADM classical variational theory of GR, commonly referred to as the Hamiltonian formulation of GR [15,16]. This approach was originally introduced in field theory by Dirac and subsequently extended to GR, where the natural prescription concerned the identification of canonical momenta as coordinate-time partial derivatives of the metric tensor [17,18] in the framework of so-called Dirac theory of constrained dynamics [19]. In this respect, we notice that in the literature different names were adopted to refer to the constrained Dirac dynamics, including in particular the Dirac–Bergmann and the Rosenfeld–Dirac–Bergmann formalisms [20–22]. According to this approach, any variational tensor field $g(r) \equiv \{g_{\mu\nu}\}$ is assumed to identify also a metric tensor of the space-time $\{\mathbf{Q}^4, g\}$, with $Q^4 \equiv \mathbb{R}^4$ being a Lorentzian differential manifold. As a consequence, any variational tensor $g(r)$ can raise and lower tensor indices, and therefore it belongs to the constrained functional setting $\{g\}_C$ defined as

$$\{g\}_C \equiv \left\{ g(r) \in C^2(\mathbf{Q}^4) \middle| g_{\mu\nu} g^{\mu k} = \delta^k_\nu \right\}, \tag{3}$$

where the orthogonality constraint $g_{\mu\nu} g^{\mu k} = \delta^k_\nu$ implies the "normalization" condition $g_{\mu\nu} g^{\mu\nu} = 4$. Then, the variational metric tensor $g_{\mu\nu}(r) \in \{g\}_C$ is represented in terms of the set of non-4-tensor variational variables $(h_{ab}(r), N(r), N_a(r))$, to be denoted as ADM Lagrangian variables, as follows:

$$\begin{aligned} g_{00}(r) &= \frac{1}{c^2}(-N^2 + N_a N^a), \\ g_{0a}(r) &= g_{a0} = \frac{1}{c} N_a, \\ g_{ab}(r) &= h_{ab}. \end{aligned} \tag{4}$$

Here, h_{ab} is the 3×3 variational matrix (the Latin indices a, b run from 1 to 3, while the Greek ones range as usual from 0 to 3), while N and N^a are, respectively, the "lapse" function and the "shift" 3-vector. In this sense, ADM variables express a foliation of the four-dimensional space-time as being composed by 3-space surface sections $\Sigma_t \equiv h_{ab}(r)$ that are separately characterized by constant coordinate-time t.

As an illustrative example of the theory we treat here the vacuum case (i.e., no external fields), but retaining a cosmological constant term, so that $\Lambda \neq 0$. In terms of these definitions, the foliation of space-time implies the representation of the four-dimensional differential integration element d^4r as $d^4r = dtd\Sigma$ with $d\Sigma$ being the surface element on Σ_t. Then, the ADM variational functional is expressed by the multiple integral

$$S_{ADM}(G) = \int dt \int_{\Sigma_t} d\Sigma L_{ADM}(G). \tag{5}$$

Here, $G(r) \equiv \left\{ h_{ab}, p^{ab}, N, N_a \right\}$ is the set of ADM canonical variables, where $\{h_{ab}, N, N_a\}$ are the Lagrangian variables and p^{ab} are suitably defined "canonical momenta" conjugate to h_{ab}. The ADM variational Lagrangian density $L_{ADM}(G)$ is found to be

$$L_{ADM}(G) \equiv p^{ab}\dot{h}_{ab} - NH_{\perp}(G) - N_a H^a(G), \tag{6}$$

where $\dot{h}_{ab}$ represents the Lie derivative of h_{ab} with respect to coordinate-time, which in ADM variables reduces to the partial time derivative $\dot{h}_{ab} \equiv \frac{\partial}{\partial t} h_{ab}$, while $H_{\perp}$ and H^a are the so-called "Hamiltonians" given by

$$H_{\perp} = \sqrt{|h|}\left[-{}^{(3)}R - 2\Lambda + |h|^{-1} p^{ab} p_{ab} - \frac{1}{2}|h|^{-1} p^2\right], \tag{7}$$

$$H^a = -2\sqrt{|h|} D_b\left(|h|^{-1/2} p^{ab}\right), \tag{8}$$

where $|h|$ is the determinant of h_{ab}, $p = p^a_a$ and D_c identifies the covariant derivative prescribed on the section Σ_t, which acts on p^a_b as $D_c p^a_b = h^a_d h^e_b h^f_c \nabla_f p^d_e$. Finally, ${}^{(3)}R = h^{lm(3)} R_{lm}$ represents the three-dimensional spatial curvature scalar defined by index saturation of the three-dimensional Ricci tensor ${}^{(3)}R_{lm}$.

Implementing the Frechet derivative on the ADM variational functional yields for the extremal ADM fields $G = (G)_{extr}$ the set of Euler–Lagrange equations, to be referred to as ADM initial-value equations, given by

$$\begin{cases} \dot{h}_{ab} - N\frac{\partial}{\partial p^{ab}} H_{\perp} - N_c \frac{\partial}{\partial p^{ab}} H^c = 0, \\ \dot{p}^{ab} + N\frac{\partial}{\partial h_{ab}} H_{\perp} + N_c \frac{\partial}{\partial h_{ab}} H^c = 0, \\ H_{\perp} = 0, \\ H^a = 0, \end{cases} \tag{9}$$

subject to initial conditions of the type $(h_{ab}(t_o), p^{ab}(t_o))$, to be assigned at the initial coordinate time t_o. Here, the first two Hamilton evolution equations apply for the couple of conjugate variables (h_{ab}, p^{ab}), while the last two equations are instead a 3-scalar and a 3-vector constraint equations expressed in terms of the "Hamiltonian" functions $H_{\perp}$ and H^a, respectively, which are required to vanish identically for the extremal values of the ADM variables (h_{ab}, p^{ab}, N, N_a).

These results are sufficient to comment on the critical aspects of the ADM formalism and the LQG representation that follows from it. The first point is represented by the violation of the principle of manifest covariance [23], which is a foundational concept of GR that should remain satisfied at all levels of representation of its variational theory, variable identification and extremal field equations, therefore including also the Hamiltonian theory. In agreement with the notion of invariance that arises for relativistic theories, the principle of manifest covariance requires all the physical laws to be manifestly covariant, namely that it should always be possible to cast them in 4-tensor form. It is immediately verifiable that, while the notion of general covariance applies to the whole set of ADM Lagrangian variables, the slicing of space-time that leads to the definition of lapse and shift functions necessarily violates manifest covariance. The choice of lapse and shift functions is comparable to fixing the gauge, so that the tensor notation warranting manifest covariance is consequently lost. In fact, the 3 + 1 decomposition of GR in ADM formalism remains coordinate invariant but is foliation dependent, since it holds for a preferred choice of observers for which space-time is split correspondingly into time and a 3-space, both to be identified uniquely with ADM variables [24]. Nevertheless, manifest covariance is lost because of the adoption of non-4−tensor Lagrangian and Hamiltonian variables inherent in the ADM approach. In fact, none of the ADM variables identifies a 4-tensor per se, since the whole ensemble of variables $G(r)$ is equivalent to the ten independent components of the metric tensor.

The immediate consequence is that the realization of the ADM formalism, and therefore of LQG itself, only applies in particular reference frames where the 3 + 1 slice is admitted in that particular sought form. The ADM structure remains instead forbidden if the slice is not applicable or it is destroyed when expressed in another reference frame obtained under the action of a general coordinate transformation that mixes space and time coordinates.

More serious issues, however, affect ADM and LQG theories. In fact, the ADM approach does not realize a canonical Hamiltonian formulation of GR. For example, the relationships between canonical momenta and generalized field velocities are not linear nor invertible as required in order for the Legendre transform to be meaningful. This is primarily a consequence of the non-linear functional dependence on the metric tensor (and therefore also on the sub-matrix h_{ab}) and its second-order partial derivatives carried by the Ricci tensor in the variational Lagrangian density. Additional critical features arise also in the variational treatment of the Ricci scalar ${}^{(3)}R$ expressed in ADM variables, as is discussed in detail in Ref. [25]. This implies that a canonical Hamiltonian quantization established in terms of a regular invertible classical Hamiltonian structure cannot be applied to the whole set of ADM Equations corresponding to the set of ADM variables. As a consequence, LQG cannot be claimed to represent a Hamiltonian quantum theory of the full set of GR equations. In this sense, also the connection between classical and quantum variables in LQG, as well as expectation values, remains inhibited, with obvious unfortunate consequences on the possibility of establishing a meaningful version of the Ehrenfest theorem.

Finally, it must be noted that the quantum wave equation in such a framework is postulated according to the Wheeler–DeWitt prescription [26], which is based only on the constraint scalar equation $H_\perp = 0$. This can be obtained by requiring the Hamiltonian $H_\perp$ to become an operator, i.e., letting $H_\perp \to \widehat{H}_\perp$ and by applying $\widehat{H}_\perp$ to the quantum gravitational-field wave function Ψ, yielding the Wheeler–DeWitt equation:

$$\widehat{H}_\perp \Psi = 0. \tag{10}$$

Because of the underlying ADM foliation of space-time, the last equation represents a scalar relation which is however not set in 4-scalar invariant form. Furthermore, since the operator $\widehat{H}_\perp$ carries second-order functional derivatives, the explicit realization of the previous equation requires the additional implementation of an appropriate (and, hopefully, unique) regularization ordering scheme. Finally, as is evident from (10), the constraint Wheeler–DeWitt equation realizes a purely stationary relationship, since it lacks any dynamical content for the evolution of the wave function Ψ to be expressed in terms of a given temporal parameter. In this sense it is distinguished from a Schrödinger-like wave equation, a prerequisite demanded in order to prove the validity of Ehrenfest theorem.

These features together mine the meaning of the formalism attempted in Refs. [10,11] to reach such a result.

It is clear from the previous considerations that, in order to set the Ehrenfest theorem for quantum gravity, the non manifestly covariant frameworks provided by Hamiltonian classical ADM and quantum WdW theories are conceptually inadequate and too formally intricate to be dealt with. In this respect, the objection raised by Hawking on the ADM setting appears well-motivated, who stated that "the split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of relativity" [27,28]. A novel and independent point of view must be therefore invoked to gain any progress in this respect. We suggest here that the latter must be sought within the formalism provided by the manifestly covariant DeDonder–Weyl Lagrangian and Hamiltonian treatments of continuous fields [29–35], which is detailed in the next sections.

3. DeDonder–Weyl Formalism for Gravitational Field: Qualitative Properties

In this section we summarize the qualitative properties that characterize the DeDonder–Weyl manifestly covariant approach to continuous field variational theory [36–38], to be applied to the gravitational field and the Einstein field equations of GR [39]. The choice of this framework is motivated by the more general compelling requirement about the possibility of determining coordinate-independent characterizations of classical GR geometries, which could permit corresponding invariant classifications of space-time solutions [40,41]. These kinds of approaches must rely on the identification of the properties of suitable sets of space-time invariants, for example those referred to as curvature and Killing invariants [42–45].

More precisely, by implementing the DeDonder–Weyl formalism it is possible to establish a manifestly covariant theory of classical gravity (CCG) and its corresponding manifestly covariant theory of quantum gravity (CQG) related by canonical quantization of the Hamiltonian state. The Hamiltonian theory obtained in this way is consistent with both classical mechanics and the principle of manifest covariance, and it is constraint-free. The Hamiltonian structure of the DeDonder–Weyl theory provides also the appropriate formalism for the establishment of a corresponding manifestly covariant quantum field theory through canonical quantization method. The quantization of the gravitational field obtained with this Lagrangian-Hamiltonian formalism offers the possibility of a comparison with semiclassical gravity models of quantum gravity theory in the literature [46–49]. Semiclassical gravity attempts a unification of quantum mechanics and gravitational physics by treating gravity as a classical field [48,49]. In the semiclassical approach, this is accomplished by solving the so-called semiclassical non-vacuum Einstein field equations in which the stress-energy tensor is replaced with its quantum counterpart expressed by a quantum expectation value of a stress-energy operator on the quantum state of source matter [49]. Thus, a point in common between semiclassical gravity with CCG and CQG theories is represented by the attempt in both frameworks to make GR and quantum mechanics compatible to each other by warranting the validity of the postulates and the structure of GR at the classical level. However, the primary goal of CCG and CQG theories is the quantization of space-time itself, and not only the source term of non-vacuum field equations. The quantum gravitational field of CQG theory is found to obey a well-determined quantum-gravity wave equation which yields the Einstein field equations under an appropriate semiclassical continuous limit. This marks a substantial difference with semiclassical gravity approaches. Nevertheless, semiclassical models can provide reference background GR solutions or insights on how to treat non-vacuum configurations in the presence of quantum source fields.

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Let us then introduce the foundational concepts of DeDonder–Weyl formalism underlying the classical theory of gravitational field and see in what it distinguishes itself from other previous literature approaches to the issue, with particular reference to ADM formalism. From the point of view of the classical setup, CCG-theory realizes a Lagrangian and Hamiltonian representation of classical GR and the Einstein field equations in terms of a novel variational approach referred to as the synchronous variational principle [50]. The latter implements a manifestly covariant and superabundant variable approach in which the 4-scalar Lagrangian function depends on the couple of variables $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$. Accordingly, in the variational action integral, the variational tensor $g \equiv \{g_{\mu\nu}\}$ must be distinguished from the background metric tensor $\widehat{g} \equiv \{\widehat{g}_{\mu\nu}\}$, which instead is considered as non-variational. The inclusion of the tensor $\widehat{g}$ is necessary for the validity of the properties of covariance and background independence for the quantum theory. This means that $\widehat{g}_{\mu\nu}(r)$ has a geometric character, it satisfies the condition of orthogonality $\widehat{g}_{\mu\nu}\widehat{g}^{\mu k} = \delta^k_\nu$ and it prescribes the tensor transformation laws of any other tensor field. Hence, $\widehat{g}$ can raise or lower tensor indices, it has a vanishing covariant derivative $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu} = 0$ (metric compatibility condition), and it permits the definition of the standard Christoffel connections and the curvature tensors of space-time. On the contrary, in the unconstrained framework the variational tensor g is such that $g_{\mu\nu} g^{\mu k} \neq \delta^k_\nu$. Similarly, for variational (i.e., virtual) curves we have that $\widehat{\nabla}_\alpha g_{\mu\nu} \neq 0$, which permits the introduction of "generalized field velocities" in the Lagrangian function. This is permissible since the variational domain must be intended here as the preliminary setting for moving away from the pure classical regime to the quantum regime to be obtained through canonical quantization of Hamiltonian structure. In the same setting the momenta will then arise due to the covariant derivative of the metric tensor being non-zero.

It must be clarified that the apparent "doubling of the metric" realized by the simultaneous introduction of field variables $g_{\mu\nu}$ and $\widehat{g}_{\mu\nu}$ is instrumental for the effective realization of a manifestly covariant DeDonder–Weyl Lagrangian and Hamiltonian formulations of GR, as proven in Sections 4 and 5 below. However, in the variational principle the variational Lagrangian variable is identified only with $g_{\mu\nu}$, while $\widehat{g}_{\mu\nu}$ is effectively treated as non-variational. Then, in the classical theory, $\widehat{g}$ is determined "a posteriori" by the extremal (Einstein) field equations, where the identity $g = \widehat{g}$ applies, thus restoring the property of having a single metric tensor.

This idea was pursued in some attempts in past literature, where similar albeit incomplete conceptual assumptions were made. For example, one can follow the reasoning behind the distinction between Hamiltonian and super-Hamiltonian functions proposed by Misner, Thorne, and Wheeler in Ref. [39] for relativistic particle dynamics. An analogous approach for the construction of an unconstrained Hamiltonian theory to be implemented for classical and quantum gravity is represented by the formalism originally developed for relativistic particle dynamics by Fock and Stueckelberg [51], where an unconstrained Schrödinger-like dynamical evolution is generated by relaxing the Hamiltonian constraints. The procedure generates a wider class of equations and solutions that ultimately require again the imposition of the constraints "a posteriori". However, in CCG-theory the distinction between g and $\widehat{g}$ applies only in the variational principle, since the identity $g = \widehat{g}$ is recovered in the extremal equations. A similar reasoning concerns the definition of the invariant 4-volume element of the action integral, which takes the "synchronous" form $d\widehat{\Omega} = d^4 r \sqrt{-|\widehat{g}|}$, implying vanishing variation $\delta d\widehat{\Omega} = 0$. Here, $d^4 r$ is the corresponding canonical measure of $d\widehat{\Omega}$ and $|\widehat{g}|$ denotes as usual the determinant of the metric tensor $\widehat{g}(r)$. This feature provides a point of connection between the synchronous setting and literature approaches known as non-metric volume forms, or modified measures, defined for example in Refs. [52,53], or the so-called non-Riemannian space-time volume elements [54,55].

These works propose variational approaches to the GR equations in which the volume elements of integration in the action principles are non-variational and metric-independent entities, to be determined dynamically by the inclusion of new scalar fields. In contrast with this, however, the synchronous setting relies only the use of the field variables $g_{\mu\nu}$ and does not require any additional field.

In the context of synchronous variational principle to GR, both the variational fields $x_R = \{g, \pi\}$ and the Lagrangian coordinates $g(r) \equiv \{g_{\mu\nu}(r)\}$ are left unconstrained, while the background space-time $\left(\mathbf{Q}^4, \widehat{g}(r)\right)$ is considered defined "a priori" in terms of $\widehat{g}_{\mu\nu}(r)$ (background space-time picture). It follows that CCG-theory is characterized by the property of emergent gravity [56,57], so that the geometrical properties of space-time expressed by $\widehat{g}_{\mu\nu}(r)$ should emerge as a mean field description of microscopic stochastic or quantum degrees of freedom associated with the variational fields $g_{\mu\nu}(r)$ underlying the classical solution. The representation of the action functional in terms of the superabundant variables $g_{\mu\nu}(r)$ and $\widehat{g}_{\mu\nu}(r)$ is therefore fundamental for the identification of the covariant Hamiltonian structure associated with the classical gravitational field and provides a fertile physical scenario for the development of a corresponding manifestly covariant quantum treatment of GR.

A concluding remark must be detailed here concerning a notation convention adopted through the paper in order to avoid notation misunderstandings. In fact, according to DeDonder–Weyl formalism, the theories of manifestly covariant classical and quantum gravity proposed in the manuscript below—which are the basis for the proof of validity of Ehrenfest theorem in the same context—have the characteristic feature of being expressed in a manifestly covariant form. This means that all the relevant physical and dynamical quantities are expressed in tensorial form. In particular, the classical Hamiltonian function, the corresponding Hamiltonian operator, the quantum-wave equation and the quantum wave function introduced in the following sections are all represented by 4-scalars, which are invariant entities and identify physical observables. The tensorial representation in terms of 4-tensors pertains instead to the resulting extremal or Euler–Lagrange equations, namely the Einstein field equations of general relativity, or equivalently their representation in terms of a set of Hamilton equations (see Sections 4–6 below). Nevertheless, only for the purpose of symbolic notation, use of explicit metric components is also adopted for the specification of the functional dependence of physical observables (i.e., 4-scalars or gauge-invariant tensors) on the variational or quantum tensor $g_{\mu\nu}$ and the background metric tensor $\widehat{g}_{\mu\nu}$. In the case of 4-scalars this means that these tensors can contribute to the functional dependence through index saturation and/or scalar products that preserve the 4-scalar character of the same invariant observables.

4. CCG-Theory: Lagrangian Formulation

Let us now consider the problem of determining a manifestly covariant Lagrangian representation for CCG-theory from a mathematical point of view, based on the DeDonder–Weyl formalism [58]. We then assume to have a 4-scalar Lagrangian function with the following functional form

$$L\left(Z, \widehat{\nabla} Z, \widehat{Z}\right) \equiv L\left(Z_{\mu\nu}, \widehat{\nabla}_\alpha Z_{\mu\nu}, \widehat{Z}_{\mu\nu}\right), \tag{11}$$

which is taken for completeness to depend on the tensorial variational field $Z \equiv Z_{\mu\nu}$, its covariant derivative $\widehat{\nabla} Z \equiv \widehat{\nabla}_\alpha Z_{\mu\nu}$ through at most a quadratic term, and a set of extremal tensor fields $\widehat{Z} \equiv \widehat{Z}_{\mu\nu}$. Consistent with the qualitative analysis pointed out in Section 3, the set of Lagrangian variables $\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)\equiv\left(Z_{\mu\nu},\widehat{\nabla}_{\alpha}Z_{\mu\nu},\widehat{Z}_{\mu\nu}\right)$ is then identified with the fields

$$\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)\equiv\left(g_{\mu\nu},\widehat{\nabla}_{\alpha}g_{\mu\nu},\widehat{g}_{\mu\nu}\right), \tag{12}$$

where $\widehat{\nabla}_{\alpha}$ denotes the covariant derivative operator and hereafter all hatted quantities are defined with respect to the background metric tensor $\widehat{g}_{\mu\nu}$. It is instructive to compare the previous set of tensorial Lagrangian variables with the non-tensorial variable realization provided by ADM variables (4).

Again, to illustrate the issue, we consider without loss of generality the case of vacuum field equations with non-vanishing cosmological constant. The starting point is then provided by the synchronous Lagrangian variational principle originally developed in Ref. [50], which in terms of the Lagrangian action functional $S_L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)$ requires the identity

$$\delta S_L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)=0 \tag{13}$$

to apply for arbitrary variations in Z. Here, δ denotes the synchronous variational operator defined in terms of the Frechet derivative. If $\widehat{Z}$ represents a background extremal tensor field, then the operator δ acts such that $\delta\widehat{Z}\equiv 0$. The action functional S_L is defined as

$$S_L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)=\int d\widehat{\Omega}L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right), \tag{14}$$

where the 4-scalar synchronous Lagrangian function $L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)$ takes the representation

$$L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)\equiv-\kappa\left[\left(g^{\mu\nu}\widehat{R}_{\mu\nu}-2\Lambda\right)h-\frac{1}{2}\widehat{\nabla}^{k}g_{\mu\nu}\widehat{\nabla}_{k}g^{\mu\nu}\right]. \tag{15}$$

Here, κ is the dimensional constant $\kappa=\frac{c^3}{16\pi G}$, while $h\equiv h\left(Z,\widehat{Z}\right)$ is the 4-scalar factor

$$h\left(Z,\widehat{Z}\right)\equiv\left(2-\frac{1}{4}g^{\alpha\beta}g_{\alpha\beta}\right). \tag{16}$$

Furthermore, $\widehat{R}_{\mu\nu}$ is the Ricci 4-tensor and $\widehat{\nabla}_k$ the covariant derivative operator, both evaluated for the background metric tensor $\widehat{g}_{\mu\nu}(r)$. It must be noted that here the variational tensor $g_{\mu\nu}(r)$ does not raise/lower indices. In addition, the covariant and contravariant representations of varied functions $g_{\alpha\beta}(r)$ and $g^{\alpha\beta}(r)$ are mutually transformed into each other through $\widehat{g}_{\alpha\beta}$ as $g_{\alpha\beta}=\widehat{g}_{\alpha\mu}\widehat{g}_{\beta\nu}g^{\mu\nu}$.

Then, the symbolic Euler-Lagrangian equations for the variational Lagrangian $L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)$ can be shown to be expressed in manifestly covariant form as

$$\widehat{\nabla}_i\frac{\partial L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)}{\partial\left(\widehat{\nabla}_i g^{\mu\nu}\right)}-\frac{\partial L\left(Z,\widehat{\nabla}Z,\widehat{Z}\right)}{\partial g^{\mu\nu}}=0, \tag{17}$$

namely explicitly

$$-\widehat{\nabla}_i\widehat{\nabla}^i g_{\mu\nu}-\frac{\partial}{\partial g^{\mu\nu}}\left[\left(g^{\alpha\beta}\widehat{R}_{\alpha\beta}-2\Lambda\right)h\right]+\frac{1}{2}\widehat{\nabla}^k g_{\alpha\beta}\widehat{\nabla}_k g^{\alpha\beta}\frac{\partial}{\partial g^{\mu\nu}}h=0, \tag{18}$$

which yield, as a particular realization, the Einstein field equations. The latter are recovered when the variational field $g_{\mu\nu}$ is identified with the background tensor $\widehat{g}_{\mu\nu}$ in the extremal equation, since then $h(\widehat{g}^{\mu\nu}(r)) = 1$ and $\widehat{\nabla}_\alpha \widehat{g}^{\mu\nu}(r) = 0$, so that the previous equation reduces to

$$\widehat{R}_{\mu\nu} - \frac{1}{2}\widehat{R}\widehat{g}_{\mu\nu} + \Lambda\widehat{g}_{\mu\nu} = 0. \tag{19}$$

The following aspects on the choice of the Lagrangian function (15) must be emphasized. In fact, it is true that from the symmetries of the extremal metric tensor, namely for $g_{\mu\nu} = \widehat{g}_{\mu\nu}$, any contribution proportional to the covariant derivative $\widehat{\nabla}_\alpha \widehat{g}^{\mu\nu}(r)$ necessarily vanishes, which might raise concerns in particular concerning the prescription of the "kinetic" term $\frac{1}{2}\widehat{\nabla}^k g_{\mu\nu}\widehat{\nabla}_k g^{\mu\nu}$ and its uniqueness. However, fundamental physical and mathematical reasons support the solution (15), namely:

(1) The locality of the Lagrangian function and its covariance under diffeomorphisms.

(2) The need of having a Lagrangian that is quadratic in first background-covariant derivatives in order to reach a representation of the functional that is formally analogous to customary variational principles holding in classical field theory.

(3) The possibility of obtaining a regular Legendre mapping yielding a corresponding manifestly covariant Hamiltonian theory, see Section 5 below.

(4) The on-shell GR limit, namely the possibility of recovering both the standard Hilbert-Einstein Lagrangian from $L\left(Z, \widehat{\nabla}Z, \widehat{Z}\right)$ as well as the Einstein field equations from Euler–Lagrange equations when the theory is evaluated on the sector where $g_{\mu\nu} = \widehat{g}_{\mu\nu}$.

5. CCG-Theory: Hamiltonian Formulation

Based on the Lagrangian formulation outlined above, we introduce the formulation of the manifestly covariant canonical Hamiltonian theory of GR that is associated with the Lagrangian function $L\left(Z, \widehat{\nabla}Z, \widehat{Z}\right)$ in Equation (15) [56]. Implementing the standard method according to the DeDonder–Weyl formalism, the canonical momenta are expressed by the tensor $\Pi^\alpha_{\mu\nu}$ as follows:

$$\Pi^\alpha_{\mu\nu} = \frac{\partial L\left(Z, \widehat{\nabla}Z, \widehat{Z}\right)}{\partial\left(\widehat{\nabla}_\alpha g^{\mu\nu}\right)} = \kappa\widehat{\nabla}^\alpha g_{\mu\nu}. \tag{20}$$

We can therefore represent the canonical state as $\{x\} = \left\{Z \equiv g^{\mu\nu}, \Pi^\alpha_{\mu\nu}\right\}$, where the symmetry property $\Pi^\alpha_{\mu\nu} = \Pi^\alpha_{\nu\mu}$ applies by construction, while the same tensor $\Pi^\alpha_{\mu\nu}$ is non-vanishing when $g_{\mu\nu} \neq \widehat{g}_{\mu\nu}$. We notice that Equation (20) is invertible, which yields a form analogous to the customary relationship between momenta and generalized velocities occurring in relativistic particle dynamics. The Hamiltonian $H = H(x, \widehat{x})$ associated with the Lagrangian $L\left(Z, \widehat{\nabla}Z, \widehat{Z}\right)$ is then provided by the Legendre transform

$$L\left(Z, \widehat{\nabla}Z, \widehat{Z}\right) \equiv \Pi^\alpha_{\mu\nu}\widehat{\nabla}_\alpha g^{\mu\nu} - H(x, \widehat{x}). \tag{21}$$

The same Lagrangian can equivalently be written in canonical variables, namely letting $L\left(Z, \widehat{\nabla}_\mu Z, \widehat{Z}\right) = L(x, \widehat{x})$, where

$$L(x, \widehat{x}) = -\kappa h\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right] + \frac{1}{2\kappa}\Pi^\alpha_{\mu\nu}\Pi^{\mu\nu}_\alpha, \tag{22}$$

so that the Hamiltonian $H(x, \widehat{x})$ is found to be

$$H(x, \widehat{x}) = \frac{1}{2}\frac{1}{\kappa}\Pi^\alpha_{\mu\nu}\Pi^{\mu\nu}_\alpha + \kappa h\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right]. \tag{23}$$

We can now define the Hamiltonian action functional associated with the Lagrangian $L(x,\widehat{x})$ given by Equation (22) as

$$\begin{aligned} S_H(x,\widehat{x}) &= \int d\widehat{\Omega} L(x,\widehat{x}) \\ &= \int d\widehat{\Omega}\Big[\Pi^{\alpha}_{\mu\nu}\widehat{\nabla}_{\alpha}g^{\mu\nu} - H(x,\widehat{x})\Big]. \end{aligned} \tag{24}$$

Then, the synchronous Hamiltonian variational principle

$$\delta S_H(x,\widehat{x}) = 0 \tag{25}$$

is required to hold for arbitrary independent variations $\delta g^{\mu\nu}(r)$ and $\delta\Pi^{\alpha}_{\mu\nu}(r)$ in the respective functional classes. The corresponding variational derivatives yield the continuous Hamilton equations

$$\frac{\delta S_H(x,\widehat{x})}{\delta g^{\mu\nu}(r)} \equiv -\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}} - \widehat{\nabla}_{\alpha}\Pi^{\alpha}_{\mu\nu} = 0, \tag{26}$$

$$\frac{\delta S_H(x,\widehat{x})}{\delta \Pi^{\alpha}_{\mu\nu}(r)} \equiv \widehat{\nabla}_{\alpha}g^{\mu\nu} - \frac{\partial H(x,\widehat{x})}{\partial \Pi^{\alpha}_{\mu\nu}(r)} = 0, \tag{27}$$

which explicitly become

$$\widehat{\nabla}_{\alpha}\Pi^{\alpha}_{\mu\nu} = -\frac{\partial H(x,\widehat{x})}{\partial g^{\mu\nu}}, \tag{28}$$

$$\widehat{\nabla}_{\alpha}g^{\mu\nu} = \frac{1}{\kappa}\Pi^{\mu\nu}_{\alpha}. \tag{29}$$

These equations are equivalent to the Euler–Lagrange Equation (18), whereby the second equation recovers the definition of the canonical momentum. For extremal curves, when the identification $g_{\alpha\beta} = \widehat{g}_{\alpha\beta}$ is made, the previous equations reduce to the Einstein Equation (19) and the condition $\widehat{\Pi}^{\alpha}_{\mu\nu} = 0$, respectively.

We stress that in the DeDonder–Weyl framework realized above for the gravitational field, the Lagrangian is regular, so that the Legendre transform is a non-singular smooth bijection. This Hamiltonian structure emerges only in the framework of the synchronous principle by the adoption of tensorial superabundant variables. The feature in fact permits the display of the contribution of the canonical momenta in the phase-space where they are non-vanishing. This property can only be met when $g_{\mu\nu} \neq \widehat{g}_{\mu\nu}$, implying that $\widehat{\nabla}^{\alpha}g_{\mu\nu} \neq 0$ and therefore also that $\Pi^{\alpha}_{\mu\nu} \neq 0$, according to Equation (20).

6. Evolution Form of Hamiltonian Theory

We now pose the issue of representing the manifestly covariant Hamilton equations in evolution form [59]. The goal is reached in terms of a reduced continuous Hamiltonian theory for GR, to be represented by the classical Hamiltonian system $\{x_R, H_R\}$, where x_R is an appropriate 4-tensor canonical state and $H_R(x_R,\widehat{x}_R(r),r,s)$ a suitable 4-scalar Hamiltonian. In particular, it is required that fields and reduced momenta form a couple of second-rank conjugate 4-tensors in the corresponding reduced canonical state. The requisite to be imposed is that such a Hamiltonian theory must recover the Einstein field equations, to be generated equivalently by the corresponding reduced continuous Hamilton equations cast in evolution form. These are referred to as GR-Hamilton equations of CCG-theory and are expressed as

$$\begin{cases} \frac{Dg_{\mu\nu}(s)}{Ds} = \frac{\partial H_R(x_R,\widehat{x}_R(r),r,s)}{\partial \pi^{\mu\nu}(s)}, \\ \frac{D\pi_{\mu\nu}(s)}{Ds} = -\frac{\partial H_R(x_R,\widehat{x}_R(r),r,s)}{\partial g^{\mu\nu}(s)}, \end{cases} \tag{30}$$

to be supplemented by initial conditions of the type

$$\begin{cases} g_{\mu\nu}(s_o) \equiv g^{(o)}_{\mu\nu}(s_o), \\ \pi_{\mu\nu}(s_o) \equiv \pi^{(o)}_{\mu\nu}(s_o). \end{cases} \tag{31}$$

Here, s is the proper-time that parameterizes an arbitrary geodesic curve $r(s) \equiv \{r^\mu(s)\}$ of the background space-time, while

$$x_R(s) \equiv \{g_{\mu\nu}(r(s)), \pi_{\mu\nu}(r(s))\} \tag{32}$$

is the s—parameterized reduced-dimensional variational canonical state, with $g_{\mu\nu}(r)$ and $\pi_{\mu\nu}(r)$ being the corresponding continuous Lagrangian coordinates and the conjugate momenta, so that the background state becomes $\widehat{x}_R(s) \equiv \{\widehat{g}_{\mu\nu}(r(s)), \widehat{\pi}_{\mu\nu}(r(s)) \equiv 0\}$. In addition, $\frac{D}{Ds} = \frac{\partial}{\partial s} + t^\alpha(s)\widehat{\nabla}_\alpha$ is the covariant s—derivative, while $t^\alpha(s)$ and $\widehat{\nabla}_\alpha$ are, respectively, the tangent 4-vector to the geodesics $r(s) \equiv \{r^\mu(s)\}$ and the covariant derivative expressed in terms of the background metric tensor $\widehat{g}_{\mu\nu}(r)$.

In order to proceed, we must first prescribe the reduced-dimensional Hamiltonian $H_R(x, \widehat{x}, s) \equiv H_R(x_R, \widehat{x}_R(r), r, s)$ which follows from Equation (23) and is represented as

$$H_R(x, \widehat{x}, s) \equiv \frac{1}{2}\frac{1}{\kappa}\pi_{\mu\nu}\pi^{\mu\nu} + \kappa h\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right], \tag{33}$$

where the canonical momenta $\Pi^\alpha_{\mu\nu}$ have been replaced by the reduced-dimensional ones $\pi_{\mu\nu}$ in the "kinetic" term of the Hamiltonian. The connection between the "extended" Hamiltonian system $\{x, H\}$ expressed by the "extended" canonical state $x \equiv \left\{g^{\mu\nu}, \Pi^\alpha_{\mu\nu}\right\}$ and the "reduced" Hamiltonian system $\{x_R, H_R\}$ is obtained upon identifying $\Pi^\alpha_{\mu\nu} = t^\alpha \pi_{\mu\nu}$, from which it follows that

$$\pi_{\mu\nu} \equiv t_\alpha \Pi^\alpha_{\mu\nu}, \tag{34}$$

so that $\pi_{\mu\nu}(r)$ arises from the projection of $\Pi^\alpha_{\mu\nu}(r)$ along $t_\alpha(s)$. Then, based on the notion of the Lagrangian path (LP) [60,61], a trajectory-based parametrization of the Hamiltonian state can be given. This follows upon identifying with $t^\gamma(\widehat{g}(r), r)$ the tangent 4-vector to a geodesic at an arbitrary 4-position $r \equiv \{r^\mu\}$ of the space-time $\left(\mathbf{Q}^4, \widehat{g}(r)\right)$ [62] and by defining the LP with the geodetic curve $\{r^\mu(s)\} \equiv \left\{r^\mu(s) | \, \forall s \in \mathbb{R}, \, r^\mu(s_o) = r^\mu_o\right\}$. The proper-time parametrization is then obtained by replacing $r \equiv \{r^\mu\}$ with $r(s) \equiv \{r^\mu(s)\}$, namely by setting

$$\begin{cases} g^{\mu\nu}(s) \equiv g^{\mu\nu}(r(s)), \\ \pi^{\mu\nu}(s) \equiv \pi^{\mu\nu}(r(s)), \\ \widehat{x}(r) \equiv \widehat{x}(r(s)). \end{cases} \tag{35}$$

This yields for the Hamiltonian H_R the so-called LP-parametrization that must be implemented in the reduced Hamilton Equation (30), where for greater generality the representation $H_R(s) \equiv H_R(x_R(s), \widehat{x}_R(r), r(s), s)$ is adopted, i.e., H_R also includes a possible explicit dependence in terms of the proper-time s.

For better understanding of the theoretical procedure, the following remarks must be specified concerning the reduced contraction presented here:

(1) The reduction given by Equation (34) preserves the Hamiltonian structure of the original DeDonder–Weyl formulation given in Section 5. This is true also concerning the physical dimensions of the Hamiltonian and the covariance property of the theory, since by definition t^α is a dimensionless 4-vector.

(2) The projection is performed with the intention of obtaining a representation of the GR Hamiltonian theory in evolution form, yielding a formalism analogous to the

one holding in particle dynamics. However, this does not mean that GR field theory is effectively reduced to a particle dynamics theory. In fact, the reduction through the introduction of t^{α} and the trajectory-based representation concerns the LP parametrization of the GR field theory, an occurrence analogous, for example, to the possibility of casting hydrodynamics equations in equivalent Eulerian or Lagrangian forms. Nevertheless, GR field theory preserves its continuous structure.

(3) The request of having an evolution form for the Hamiltonian theory of GR is a necessary prerequisite to further proceed with canonical quantization of the same theory implementing standard Poisson-brackets formalism, as shown in detail below.

(4) The reduction proposed here introduces the invariant evolution parameter identified with the proper-time s, which is a 4-scalar and therefore exhibiting the correct tensorial property in agreement with the postulate of manifest covariance set as the basis of the present Hamiltonian theory of GR. As a consequence, the same invariant parameter must be generally distinguished from alternative non-invariant coordinate-time parametrizations available in the literature. These include, for example, the coordinate-time t generated by the $3+1$ slicing of space-time arising in the ADM formalism or the so-called "cosmic time" associated with the temporal variation in the scale factor of the particular GR Friedmann–Lemaître–Robertson–Walker solution of Einstein field equations modeling a homogeneous and expanding cosmological universe. However, these kinds of parametrizations must be excluded from consideration in favour of proper-time, as they represent specific components of coordinate systems for particular reference frames, i.e., space-time metric tensor representations. As a consequence, they do not exhibit the correct invariant property demanded by manifestly covariant CQG theory, namely they do not preserve their character and meaning under general coordinate transformations among GR reference frames.

(5) The invariant proper-time introduced here arises in association with the notion of LP, but it can be also equivalently interpreted a posteriori as an intrinsic field-theory evolution parameter associated with the non-stationary character of quantum cosmological constant predicted by the same CQG theory, see Ref. [63] and Section 9 below. This proves that indeed the reduced continuous Hamiltonian field theory of GR preserves the character of field theory, while the particle-dynamics formalism pertains only the parametrization of field dynamics as a convenient representation of Hamilton equations.

In terms of the reduced Hamiltonian state it is possible to introduce a reduced Hamiltonian variational principle that yields Equation (30). Thus, the variational functional is identified with the real 4-scalar

$$S_R(x,\widehat{x}) \equiv \int d\widehat{\Omega} L_R(x,\widehat{x},r,s), \tag{36}$$

in which the variational Lagrangian $L_R(x,\widehat{x},r,s)$ is the Legendre transform of the variational Hamiltonian $H_R(x,\widehat{x},r,s)$, namely

$$L_R(x,\widehat{x},r,s) \equiv \pi_{\mu\nu}\frac{D}{Ds}g^{\mu\nu} - H_R(x,\widehat{x},r,s). \tag{37}$$

Then, the synchronous variational principle associated with the functional $S_R(x,\widehat{x})$ is written as

$$\delta S_R(x,\widehat{x}) = 0, \tag{38}$$

in which both the state $\widehat{x}$ and the 4-scalar volume element $d\widehat{\Omega}$ are treated as non-variational by definition. The symbolic form of the 4-tensor Euler–Lagrange equations equivalent to the Hamilton Equation (30) then follows:

$$\begin{cases} \frac{\delta S_R(x,\widehat{x})}{\delta g^{\mu\nu}} = 0, \\ \frac{\delta S_R(x,\widehat{x})}{\delta \pi_{\mu\nu}} = 0. \end{cases} \tag{39}$$

These equations can be written also in the Poisson-bracket representation as

$$\frac{D}{Ds} x_R(s) = [x_R, H_R(s)]_{(x_R)}, \tag{40}$$

where explicitly the Poisson bracket $[,]_{(x_R)}$ becomes

$$[x_R, H_R(s)]_{(x_R)} = \frac{\partial x_R}{\partial g^{\mu\nu}} \frac{\partial H_R(s)}{\partial \pi_{\mu\nu}} - \frac{\partial x_R}{\partial \pi_{\mu\nu}} \frac{\partial H_R(s)}{\partial g^{\mu\nu}}. \tag{41}$$

It is immediate to verify that Equation (40) yield the GR-Hamilton equations in evolution form given above by Equation (30). Invoking the explicit representation of $H_R(s)$ given by (33) yields

$$\frac{\partial H_R(s)}{\partial g^{\mu\nu}(s)} = \kappa h(s) \widehat{R}_{\mu\nu} - \kappa g_{\mu\nu}(s) \frac{1}{2} \left(g^{\alpha\beta}(s) \widehat{R}_{\alpha\beta} \right). \tag{42}$$

Hence, the canonical Equation (30) reduce to the LP-parameterized Lagrangian equation for $g_{\mu\nu}(s)$:

$$\frac{D}{Ds} \left[\frac{D}{Ds} g_{\mu\nu}(s) \right] + h(s) \widehat{R}_{\mu\nu} - g_{\mu\nu}(s) \frac{1}{2} \left[g^{\alpha\beta}(s) \widehat{R}_{\alpha\beta} - 2\Lambda \right] = 0. \tag{43}$$

The final step in this derivation is about the proof of the relationship between the canonical Equation (30) and the Einstein theory of GR. This can be obtained assuming that the Hamiltonian is of the form $H_R = H_R(x_R, \widehat{x}_R(r), r)$, namely it does not depend explicitly on proper-time s. Then, since $\widehat{g}_{\mu\nu}(s) \widehat{g}^{\mu\nu}(s) = \delta^\mu_\mu$ and $\frac{D}{Ds} \widehat{g}_{\mu\nu}(s) \equiv 0$, the identity $\widehat{\pi}_{\mu\nu}(s) \equiv 0$ also holds, so that the canonical equation for $\widehat{\pi}_{\mu\nu}(s)$ (or equivalently Equation (43)) yields

$$\widehat{R}_{\mu\nu} - \widehat{g}_{\mu\nu}(s) \frac{1}{2} \widehat{g}^{\alpha\beta}(s) \widehat{R}_{\alpha\beta} + \Lambda \widehat{g}_{\mu\nu}(s) = 0, \tag{44}$$

which coincides with the Einstein field Equation (19). Therefore, the latter arise as a stationary solution of the GR-Hamilton Equation (30) with respect to proper-time, i.e., imposing the initial conditions

$$\begin{cases} g_{\mu\nu}(s_o) \equiv \widehat{g}_{\mu\nu}(s_o), \\ \pi_{\mu\nu}(s_o) \equiv \widehat{\pi}_{\mu\nu}(s_o) = 0, \end{cases} \tag{45}$$

together with the identity $\widehat{\pi}_{\mu\nu}(s) = 0$ for all $s \in I$.

7. Manifestly Covariant Quantum Gravity Theory

Given the results of the previous sections, we now proceed with the formulation of a theory of quantum gravitational fields based on the manifestly covariant Hamiltonian theory determined above, to be referred to as the manifestly covariant quantum gravity theory (CQG theory) [64,65]. To this aim we will adopt a canonical axiomatic derivation starting from the classical reduced Hamiltonian theory and the Hamiltonian structure $\{x_R, H_R\}$. We notice however the following remarkable feature of CQG theory, namely the fact that it can be equivalently obtained by implementing a Hamilton-Jacobi quantization equivalent to the Hamiltonian theory, see the proof in Ref. [66].

Contrary to LQG or WdW theories based on ADM representation, the CQG state is expressed by a complex 4-scalar function $\psi(s)$ of the form

$$\psi(s) \equiv \psi(g, \widehat{g}(r), r(s), s), \tag{46}$$

to be referred to as CQG wave-function. This can be interpreted as being associated with a spin-2 quantum particle with invariant rest mass $m_o > 0$, thanks to the validity of the scalar product relationship $\psi(s) = \widehat{g}_{\mu\nu}\psi^{\mu\nu}$. It is assumed that $\psi(s)$ can depend explicitly on the proper-time s or implicitly through the tensor field $g \equiv \{g_{\mu\nu}\} \in U_g$, the background field $\widehat{g}(r(s)) \equiv \{\widehat{g}_{\mu\nu}(r(s))\}$, and the LP-parametrization in terms of the geodesics $r(s) \equiv \{r^\mu(s)\}$. The function ψ defined by Equation (46) spans a Hilbert space Γ_ψ, i.e., a finite-dimensional linear vector space in which the following scalar product between arbitrary elements $\psi_{a,b}(s) \equiv \psi_{a,b}(g, \widehat{g}(r), r(s), s)$ is defined:

$$\langle \psi_a | \psi_b \rangle \equiv \int_{U_g} d(g) \psi_a^*(g, \widehat{g}(r), r(s), s) \psi_b(g, \widehat{g}(r), r(s), s), \tag{47}$$

with $d(g) \equiv \prod_{\mu,\nu=1\text{to}4} dg_{\mu\nu}$ being the canonical measure on U_g, while ψ_a^* is the complex conjugate of ψ_a. It follows that the real function $\rho(s) \equiv \rho(g, \widehat{g}(r), r(s), s)$ defined as

$$\rho(s) \equiv |\psi(s)|^2 \tag{48}$$

identifies the quantum probability density function (CQG-PDF) of $g \equiv \{g_{\mu\nu}\}$ associated with the CQG state in the configuration space U_g. We notice a remarkable difference with respect to non-manifestly covariant approaches to quantum gravity, which consists in the fact that in the present framework the PDF retains the same physical meaning of non-relativistic quantum mechanics, and it has a tensorial character; namely, it is represented by an invariant 4-scalar defined on the four-dimensional space-time with respect to the background field $\widehat{g}_{\mu\nu}$. For consistency, the normalization condition for $\rho(s)$ is then introduced:

$$\langle \psi | \psi \rangle \equiv \int_{U_g} d(g) \rho(s) = 1. \tag{49}$$

The next step concerns the prescription of the expectation values of tensor functions and CQG observables. Thus, if $X(s) \equiv X(g, \widehat{g}(r), r(s), s)$ denotes an arbitrary tensor function or a local tensor operator acting on an arbitrary wave-function $\psi(s) \in \Gamma_\psi$, the weighted integral

$$\langle \psi | X\psi \rangle \equiv \int_{U_g} d(g) \psi^*(s) X(s) \psi(s) \tag{50}$$

identifies the CQG expectation value of X. By denoting the complex conjugate of $X(s)$ as $X^*(s)$ if

$$\langle \psi | X\psi \rangle = \langle X^* \psi | \psi \rangle \equiv \int_{U_g} d(g) \psi(s) X^*(s) \psi^*(s), \tag{51}$$

then $\langle \psi | X\psi \rangle$ is real and X identifies a CQG observable. The latter ones include the following, for example:

(A) The 4-tensor function $X \equiv g_{\mu\nu}$, for which the integral

$$\langle \psi | g_{\mu\nu}\psi \rangle = \int_{U_g} d(g) g_{\mu\nu} \rho(g, \widehat{g}(r), r(s), s) = \widetilde{g}_{\mu\nu}(\widehat{g}(r), r(s), s) \tag{52}$$

identifies the corresponding CQG expectation value at $r \equiv r(s) \in (\mathbf{Q}^4, \widehat{g}(r))$.

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(B) The CQG expectation value of the momentum operator $\pi^{(q)\mu\nu} \equiv -i\hbar\frac{\partial}{\partial g_{\mu\nu}}$. This is given by the integral

$$\left\langle \psi | \pi^{(q)\mu\nu} \psi \right\rangle = \int\limits_{U_g} d(g)\psi(s)\left(-i\hbar\frac{\partial}{\partial g_{\mu\nu}}\right)\psi^*(s) \equiv \widetilde{\pi}^{(q)\mu\nu}(\widehat{g}(r), r(s), s), \tag{53}$$

which is assumed to exists, with $\widetilde{\pi}^{(q)\mu\nu}$ being a real tensor field.

The quantum Hamiltonian operator follows from the prescription of the classical Hamiltonian H_R. A dimensional normalization of the Hamiltonian structure is required in order for the canonical momenta to have the dimension of an action: $\{x_R, H_R\} \to \{\overline{x}_R, \overline{H}_R\}$. This yields the classical dimensionally normalized Hamiltonian structure $\{\overline{x}_R, \overline{H}_R\}$ defined in terms of the canonical state $\overline{x}_R \equiv \left\{\overline{g}_{\mu\nu}, \overline{\pi}_{\mu\nu}\right\}$ and the Hamiltonian $\overline{H}_R$. Here, $\overline{g}_{\mu\nu} \equiv g_{\mu\nu}$ and $\overline{\pi}_{\mu\nu} = \frac{\alpha L}{k}\pi_{\mu\nu}$ is the normalized conjugate momentum, where $\kappa = \frac{c^3}{16\pi G}$, L is a 4-scalar scale length and α is a suitable dimensional 4-scalar, both defined in Ref. [59]. Instead, $\overline{H}_R$ is defined as the real 4-scalar field

$$\overline{H}_R(\overline{x}_R, \widehat{g}, r, s) = \overline{T}_R(\overline{g}, \widehat{g}, r, s) + \overline{V}(\overline{g}, \widehat{g}, r, s), \tag{54}$$

with $\overline{T}_R(\overline{g}, \widehat{g}, r, s) \equiv \frac{\overline{\pi}^{\mu\nu}\overline{\pi}_{\mu\nu}}{2\alpha L}$ and $\overline{V}(\overline{g}, \widehat{g}, r, s) \equiv h\alpha L\left[g^{\mu\nu}\widehat{R}_{\mu\nu} - 2\Lambda\right]$ being the normalized effective kinetic and potential densities. Then, the canonical quantization rules for CQG theory are based on the CQG correspondence principle realized by the map

$$\begin{cases} \overline{g}_{\mu\nu} \equiv g_{\mu\nu} \to g^{(q)}_{\mu\nu} \equiv g_{\mu\nu}, \\ \overline{\pi}_{\mu\nu} \to \pi^{(q)}_{\mu\nu} \equiv -i\hbar\frac{\partial}{\partial g^{\mu\nu}}, \\ \overline{H}_R \to \overline{H}^{(q)}_R = \overline{T}^{(q)}_R(\overline{\pi}) + \overline{V}, \end{cases} \tag{55}$$

where $\overline{H}^{(q)}_R$ is the Hermitian CQG Hamiltonian operator, $x^{(q)} \equiv \left\{g^{(q)}_{\mu\nu}, \pi^{(q)}_{\mu\nu}\right\}$ is the quantum canonical state and $\pi^{(q)}_{\mu\nu}$ is the quantum momentum operator prescribed so that the commutator $\left[g^{(q)}_{\mu\nu}, \pi^{(q)\alpha\beta}\right] = i\hbar\delta^\alpha_\mu\delta^\beta_\nu$ exactly. In addition, $\overline{T}^{(q)}_R(\overline{\pi})$ is the kinetic quantum operator

$$\overline{T}^{(q)}_R(\overline{\pi}) = \frac{\pi^{(q)\mu\nu}\pi^{(q)}_{\mu\nu}}{2\alpha L}. \tag{56}$$

The quantum-gravity wave equation of CQG theory that advances in proper-time the quantum state ψ is reached by promoting the Poisson brackets to quantum commutator. One obtains the CQG quantum wave equation

$$i\hbar\frac{\partial}{\partial s}\psi(s) + \left[\psi(s), \overline{H}^{(q)}_R\right] = 0, \tag{57}$$

where $\left[\psi(s), \overline{H}^{(q)}_R\right] \equiv -\overline{H}^{(q)}_R\psi(s)$. The CQG wave Equation (57) describes the dynamical evolution of the quantum state $\psi(s)$ along the geodetics of the background metric tensor $\widehat{g}_{\mu\nu}(r)$. Remarkably, the same Equation (57) is realized by a first-order partial differential equation, which must be supplemented by initial conditions prescribing for all $r(s_o) = r_o \in (\mathbf{Q}^4, \widehat{g}(r))$ the function $\psi(s_o) = \psi_o(g, \widehat{g}(r_o), r_o)$, as well as the limiting boundary conditions $\lim_{g\to\infty}\psi(g, \widehat{g}(r), r(s), s) = 0$ on the improper boundary of configuration space U_g.

8. The Ehrenfest Theorem for CQG Theory

The theoretical developments proposed in the previous sections build the mathematical framework for the establishment of the Ehrenfest theorem holding for CQG theory.

This represents an unprecedented result in comparison with alternative non manifestly covariant literature theories of quantum gravity. In fact its validity appears as a unique feature permitted by the manifestly covariant Lagrangian and Hamiltonian structures and the corresponding quantum gravity theory determined above. In particular, it is proved here that the target can be established in the framework of CQG theory as a consequence of its peculiar canonical Hamiltonian structure and the commutator-bracket algebra that characterizes its representation and probabilistic interpretation. The theoretical proof of the theorem for CQG theory permits to elucidate the connection existing between quantum operator variables of gravitational field and the corresponding expectation values, to be interpreted as dynamical physical observables referred to the background metric space-time. It must be stressed that, as a characteristic feature of the present theory and consistent with the manifestly covariant classical and quantum theoretical frameworks developed above, the Ehrenfest theorem proved here also holds for gauge-invariant observables. This means that the Ehrenfest theorem and corresponding expectation values apply to gauge-invariant fields representing physical observables.

Let us now proceed with the establishment of the theorem. The starting point is the CQG quantum wave Equation (57)

$$i\hbar\frac{\partial}{\partial s}\psi(s) = \overline{H}_R^{(q)}\psi(s), \tag{58}$$

which takes the form of a hyperbolic evolution equation with respect to the invariant parameter s. The same equation must be supplemented by an initial condition of the type

$$\psi(s_0) = \psi_0(g, \widehat{g}(r(s_0), s_0), r(s_0)), \tag{59}$$

with $\psi_0(g, \widehat{g}(r(s_0), s_o), r(s_0))$ denoting an appropriate initial wave function. Then, within the theoretical framework established by CQG theory, the following proposition holds.

Theorem 1 (The Ehrenfest theorem)**.** *Let us assume validity of the following set of assumptions:*

(1) The notions of manifest covariance and background space-time $\widehat{g}_{\mu\nu}(r)$ underlying the Lagrangian and Hamiltonian theories of classical and quantum gravity, respectively, CCG and CQG theories.

(2) The functional form of the 4-scalar quantum wave function $\psi(s) \equiv \psi(g, \widehat{g}(r), r(s), s)$ which depends on the quantum tensor field $g_{\mu\nu}$, on the LP-parametrized background metric tensor $\widehat{g}_{\mu\nu}(r(s))$ and possibly also explicitly on the proper-time s.

(3) The manifestly covariant CQG wave equation expressed by Equation (58) evolving the 4-scalar quantum wave function $\psi(s) \equiv \psi(g, \widehat{g}(r), r(s), s)$ of the Hilbert space Γ_ψ in terms of the invariant evolution parameter s. According to the Lagrangian-path representation of CQG theory, the latter parameter is identified with the proper-time associated with massive gravitons following geodetic curves in the background space-time [63].

(4) The quantum 4-scalar Hermitian Hamiltonian operator $\overline{H}_R^{(q)}$ defined by Equation (55).

(5) The notion of local scalar product defined in Equation (47) on the Hilbert space Γ_ψ in terms of which quantum expectation values are defined.

Then, for a generic quantum operator Q of CQG theory with a tensorial character defined with respect to the background space-time and expectation value $\langle Q\rangle$, the following relationship holds

$$\frac{d}{ds}\langle Q\rangle = \frac{1}{i\hbar}\left\langle \left[Q, \overline{H}_R^{(q)}\right]\right\rangle + \left\langle \frac{\partial Q}{\partial s}\right\rangle, \tag{60}$$

to be referred to as the CQG–Ehrenfest theorem.

Proof. To prove the thesis we start by direct evaluation of the following quantity

$$\frac{d}{ds}\langle Q\rangle = \frac{d}{ds}\int_{U_g} d(g)\psi^*(s)Q\psi(s), \tag{61}$$

where we have used the definition of expectation value for the operator Q. In the previous expression, the differential operator $\frac{d}{ds}$ takes the Eulerian representation

$$\frac{d}{ds} = \left.\frac{d}{ds}\right|_s + \left.\frac{d}{ds}\right|_r. \tag{62}$$

Here, first $\left.\frac{d}{ds}\right|_s \equiv t^\alpha \nabla_\alpha$ identifies the directional covariant derivative, with

$$t^\alpha = \frac{dr^\alpha(s)}{ds} \equiv \left.\frac{d}{ds}\right|_s r^\alpha(s) \tag{63}$$

being the tangent to the geodetic curve $r(s) \equiv \{r^\alpha(s)\}$. Second, $\left.\frac{d}{ds}\right|_r$ denotes the covariant s—partial derivative. When it operates on a 4-scalar this coincides with the ordinary partial derivative, so that $\left.\frac{d}{ds}\right|_r = \frac{\partial}{\partial s}$, and consequently in this case

$$\frac{d}{ds} = \left.\frac{d}{ds}\right|_s + \frac{\partial}{\partial s} \equiv D_s. \tag{64}$$

Let us therefore elaborate the right-hand side of Equation (61), noting that only the partial derivative $\frac{\partial}{\partial s}$ commutes under the integral, since the dependence on $r^\alpha(s)$ is only through the background metric tensor. This yields

$$\begin{aligned}\frac{d}{ds}\langle Q\rangle &= \int_{U_g} d(g)\left(\frac{\partial\psi^*(s)}{\partial s}\right)Q\psi(s) \\ &\quad+\int_{U_g} d(g)\psi^*(s)\left(\frac{\partial Q}{\partial s}\right)\psi(s) \\ &\quad+\int_{U_g} d(g)\psi^*(s)Q\left(\frac{\partial\psi(s)}{\partial s}\right),\end{aligned} \tag{65}$$

Now we notice that, by definition, the second term on the right-hand side of the previous equation is the expected value of $\frac{\partial Q}{\partial s}$, namely

$$\int_{U_g} d(g)\psi^*(s)\left(\frac{\partial Q}{\partial s}\right)\psi(s) \equiv \left\langle\frac{\partial Q}{\partial s}\right\rangle. \tag{66}$$

Hence, we can write

$$\begin{aligned}\frac{d}{ds}\langle Q\rangle &= \int_{U_g} d(g)\left(\frac{\partial\psi^*(s)}{\partial s}\right)Q\psi(s) + \left\langle\frac{\partial Q}{\partial s}\right\rangle \\ &\quad+\int_{U_g} d(g)\psi^*(s)Q\left(\frac{\partial\psi(s)}{\partial s}\right).\end{aligned} \tag{67}$$

On the other hand, from the validity of the CQG wave Equation (58) we have that necessarily

$$\frac{\partial \psi(s)}{\partial s} = \frac{1}{i\hbar} \overline{H}_R^{(q)} \psi(s), \tag{68}$$

so that its complex conjugate gives

$$\frac{\partial \psi^*(s)}{\partial s} = -\frac{1}{i\hbar} \psi^*(s) \overline{H}_R^{(q)*} = -\frac{1}{i\hbar} \psi^*(s) \overline{H}_R^{(q)}, \tag{69}$$

since the quantum Hamiltonian operator $\overline{H}_R^{(q)}$ is Hermitian. Hence, replacing this last result in Equation (65) yields

$$\frac{d}{ds}\langle Q \rangle = \frac{1}{i\hbar} \int_{U_g} d(g) \psi^*(s) \left(Q\overline{H}_R^{(q)} - \overline{H}_R^{(q)} Q \right) \psi(s) + \left\langle \frac{\partial Q}{\partial s} \right\rangle. \tag{70}$$

Upon defining the commutator

$$\left[Q, \overline{H}_R^{(q)} \right] \equiv Q\overline{H}_R^{(q)} - \overline{H}_R^{(q)} Q, \tag{71}$$

and invoking again the definition of expectation value for $\left[Q, \overline{H}_R^{(q)} \right]$ finally permits to reach the conclusion, namely to express Equation (70) in the form

$$\frac{d}{ds}\langle Q \rangle = \frac{1}{i\hbar} \left\langle \left[Q, \overline{H}_R^{(q)} \right] \right\rangle + \left\langle \frac{\partial Q}{\partial s} \right\rangle, \tag{72}$$

which proves the thesis.

Q.E.D. □

The following comments are pointed out concerning Theorem 1:

(1) It must be stressed that Equation (72) represents the covariant generalization of the quantum-mechanics Equation (1), holding for the case of quantum gravitational field in the framework of CQG theory. In particular, besides the formal analogy between the two equations, it is important to notice the following points of difference. The first one is that the absolute coordinate time t of Equation (1) is replaced here by the invariant proper-time s which is peculiar to CQG theory. The second issue is that the expectation-value operator $\langle \ldots \rangle$ of Equation (1) acts on the three-dimensional coordinate space d^3x, while in Equation (72) the same mean-value operator is defined on the configuration space of quantum tensor fields $g_{\mu\nu}$ with measure $d(g)$. Finally, the quantum Hamiltonian operator $\overline{H}_R^{(q)}$ is peculiar to CQG theory, demanding validity of underlying manifestly covariant Lagrangian and Hamiltonian theories of classical gravity [65].

(2) It is important to mention that the formal analogy between Equations (1) and (72) is due to the fact that both in quantum mechanics and CQG theory a particle interpretation of the quantum field applies. More precisely, in the case of CQG theory this refers to the quanta of the gravitational field, namely the massive quantum gravitons which follow sub-luminal geodesic curves with respect to the background space-time. This property gives also a meaning to the interpretation of proper-time s. From the validity of the CQG–Ehrenfest Equation (72), it follows that the latter proper-time parameter is the one governing the dynamical evolution of both quantum fields and corresponding classical expectation values.

(3) Incidentally, we remark that the proof of the Ehrenfest theorem proposed in Theorem 1 is obtained within the framework of CQG theory and the canonical covariant quantization performed on the underlying classical Lagrangian and Hamiltonian structures

of GR. As such, the proof of validity of the theorem is obtained without performing any kind of perturbation schemes, asymptotic expansions, approximation techniques or representation restrictions on the configuration domain, symmetry properties and space-time variable identifications.

(4) A crucial aspect follows from inspection of Equation (72). This concerns the fact that, generally, the differential operators with respect to proper-time and the integral operator of quantum average, namely the quantum expectation value $\langle \dots \rangle$, do not commute. This is true for a generic quantum operator Q as expressed by the theorem, but evidently it applies also to functional dependences of either explicit or implicit types. In particular, if $Q = Q(g_{\mu\nu})$, then necessarily one has that $Q(\langle g_{\mu\nu} \rangle) \neq \langle Q(g_{\mu\nu}) \rangle$. This property on operator ordering reflects the non-linearity of the theory. In this way, the quantum average of functional dependences expressed by non-linear scalar products and index saturations of the quantum field variables generate non-trivial contributions of non-linear type. Terms of this type can be physically associated with higher-order moments or statistically meaningful quantum fluctuations of the quantum gravitational field that do not average to zero but can be determinant in distinguishing the relative importance of the expectation-value terms in the Ehrenfest theory. Explicit non-linear contributions to the Ehrenfest theory arise in particular by the quantum-average term $\left\langle \left[Q, \overline{H}_R^{(q)} \right] \right\rangle$ in Equation (72), due to the combined action of non-linear functional dependences contained in the Hamiltonian $\overline{H}_R^{(q)}$ (e.g., through the quadratic function h in the potential term, see Equation (16)), the commutator with the operator Q and, finally, the integral operator $\langle \dots \rangle$. Examples of analytical treatments of non-linear dependences of this kind in CQG theory can be found: (a) in Refs. [59,67] in reference to the construction of discrete spectrum of stationary solutions to the CQG wave equation with non-vanishing Hamiltonian potential; (b) in Ref. [68] for cosmological scenarios in connection with the peculiar CQG screening effect of the cosmological constant. Non-linear quantum fluctuations of this kind that are inherited at quantum level by the underlying Lagrangian-Hamiltonian theory are expected to have an impact on the application of the Ehrenfest theorem to the treatment of non-linear phenomena in CQG theory, to be further ascertained in future studies.

(5) Equation (72) is an integro-differential equation supporting the notion of non-locality. This arises both in configuration space, because of the action of the quantum expectation value operator, and in the parametrization of proper-time and proper-time dependences, because of the integral definition of the same parameter s, see Equation (74) below. As a result, the non-locality of CQG theory showing up in Equation (72) reconciles with the concept of emergent space-time background and possible effect of quantum entanglement (see also discussion in Section 9).

(6) A basic difference characterizes the Ehrenfest theorem for quantum field theory as a background dependent theory, and the extension of the Ehrenfest theorem to CQG theory. In fact, quantum field theory is usually carried out in a given background space-time, whose solution for the corresponding metric tensor is assumed to be known and prescribed. Then, the quantum dynamical evolution of the field occurs in such a space-time, and typically it is assumed that the same field can be treated as a test field, namely it is such that its energy content does not modify significantly the background space-time. On the other hand, the framework of CQG theory is intrinsically different. The Ehrenfest theorem (72) depends on the background space-time metric tensor $\hat{g}_{\mu\nu}$, which however is a dynamical variable by itself. This is related to the quantum gravitational tensors $g_{\mu\nu}$, e.g., through the emergent gravity phenomenon (see Section 9 below) and therefore its solution is not assigned "a priori". The same reasoning applies to the proper-time used for parametrization of functional dependences. This feature reflects the conceptual

and mathematical complexities behind Equation (72) which characterize the non-linear character of classical GR and quantum gravity theories.

(7) A final concluding remark must be made on the validity of the Ehrenfest theory proposed here. This concerns the fact that the same Ehrenfest theorem is established on the basis of CQG theory presented in the above sections, where the fundamental quantum-gravity wave equation is provided by Equation (57). The whole CQG theory implemented here is cast in unitary representation, which means in turn that necessarily the quantum evolution of the wave-function $\psi(s)$ is unitary too. Therefore, the possible occurrence of non-unitary effects are excluded from the present treatment. The latter ones could be associated for example with the presence of classical horizons and quantum graviton sinks localized on their surrounding [69], or with black-hole entropy production phenomena [70] leading to dissipative terms in the semi-classical limit of gravitational evolution. Applicability of the present formalism to non-unitary settings is therefore excluded, while an appropriate generalization of Ehrenfest theory to such domains would require a preliminary non-unitary generalization of CQG theory as outlined in Ref. [69]. On similar grounds, foundational issues related to measurements effects and wave-function collapse are not treated in the present work [71,72]. This includes in particular possible backreaction phenomena of quantum-gravity system with measuring apparatus [73] that could affect the definition of proper-time evolution parameter in the Ehrenfest theorem. The issue is reflected once more by the implementation of unitary CQG theory.

9. Proper-Time as a Dynamical Variable

Additional remarks must be stressed concerning the meaning of proper-time in CQG theory. First of all, the adoption of the proper-time parametrization permits recovering, formally, also for CQG theory the concepts of standard quantum mechanics. These are associated with the Hamiltonian structure of the theory and the possibility of casting the Hamilton equations in evolution form as well as with the physical meaning and probabilistic interpretation to be assigned to the quantum wave function. From one side the choice of a 4-scalar proper-time (instead of the coordinate time) is consistent with the manifest covariance principle, while from the other side it plays the role of "time" dynamical variable. The invariant proper-time s is the dynamical parameter of CQG theory, and its notion is necessary for the representation and physical interpretation of the same theory. For this reason, it cannot be confused with a coordinate-time parameter (for a discussion of the issue see Refs. [5,6]).

In detail, in the framework of background CQG theory the metric tensor $\widehat{g}_{\mu\nu}$ determines the differential Riemann distance ds on the space-time $\{\mathbf{Q}^4, \widehat{g}\}$ by means of the 4-scalar equation

$$ds^2 = \widehat{g}_{\mu\nu} dr^\mu dr^\nu, \tag{73}$$

where the 4-tensor displacement dr^μ belongs to the subset of $\{\mathbf{Q}^4, \widehat{g}\}$ where $\widehat{g}_{\mu\nu} dr^\mu dr^\nu \geq 0$. By integration, it follows that

$$s - s_1 = \int_{r_1}^{r} \sqrt{\widehat{g}_{\mu\nu} dr^\mu dr^\nu}, \tag{74}$$

where $r \equiv r(s)$ and $r_1 \equiv r(s_1)$ denote two 4-positions on an arbitrary curve $r(s)$ of a massive graviton. This provides a geometric interpretation of proper-time. In particular, in accordance to Ref. [59], for an arbitrary GR-frame, the worldline on which the Riemann distance is evaluated can be identified with one of the (infinite possible) non-null field geodetics $r(s) \equiv \{r^\mu(s)\}$ prescribed so that at proper-time $s > 0$ it coincides with the observer's position according to the initial (crossing) condition $r^\mu = r^\mu(s)$.

However, since geodetics are intrinsically non-unique, the precise value of s still depends both on the choice of the space-time curve on which it is measured and that of the reference 4-position $r_1 = r(s_1)$ on the same curve. However, the notion of proper-time makes sense only if s is an observable. As such, it can be prescribed in either of the following two ways. In the first case, proper-time is a local observable having different realizations for each observer, to be identified with GR-frames that are mutually connected through local-point transformations. In the second case, proper-time is a global observable which coincides for all observers that are properly "synchronized" with each other.

In the derivation of Ehrenfest theory proposed in this paper, the role of proper-time appears essential for the representation of the equations of CQG theory as well as the underlying Hamiltonian theory cast in evolution form. A relevant issue that arises in this respect concerns the consistency of this dependence on the choice of the nature of time with the property of background independence, to be regarded as a fundamental property that should be inherent to any comprehensive quantum theory of gravity. According to the notion of background independence, the structure of the continuous space-time described by classical GR should possess an emergent character; namely, it should arise "a posteriori" from the underlying quantum-dynamical gravitational field. As proven in Ref. [74], a statistical formulation of background independence can be consistently obtained in CQG theory, showing that the same notion of background independence is an intrinsic feature of CQG theory. Accordingly, the classical background metric tensor $\widehat{g}_{\mu\nu}$ can be expressed in terms of a statistical average of the stochastic quantum gravitational field tensor. The relationship demonstrated in Ref. [74] can be expressed mathematically by the identity

$$\widehat{g}_{\mu\nu} \equiv \langle \Delta g_{\mu\nu} \rangle_{stoch}, \tag{75}$$

where $\Delta g_{\mu\nu}$ identifies a quantum-gravity stochastic tensor associated with quantum gravitational fluctuations, while $\langle \rangle_{stoch}$ is the stochastic average operator of CQG theory defined in Ref. [74]. As a result, the background space-time metric tensor is shown to arise consistently from the quantum nature of the gravitational field, which is characterized by an intrinsic stochastic behavior. Given these premises, it follows that the current notion of proper-time invoked for the parametrization of CQG theory equations is consistent with the property of background independence, thanks to the combined validity of definitions (73) and (74) with (75). Since the proper-time is a derived invariant quantity of the emergent tensor $\widehat{g}_{\mu\nu}$ according to Equation (74), then it is found to be consistent with background independence of CQG theory and not to be in conflict with this notion.

Finally, concerning the physical meaning of s, it has been proved that an alternative point of view can be given in terms of a cosmological interpretation that applies to both CCG and CQG theories [63]. In fact, a correspondence can be established between the evolution parameter and the CQG quantum representation of the cosmological constant generated by non-linear Bohm quantum-vacuum interaction [66]. Accordingly, the proper-time s can be associated with the intrinsic non-stationary character of the cosmological constant Λ_{CQG} predicted by CQG theory. As discussed in Ref. [63], from one side this conclusion yields a plausible solution to the so-called "problem of time" affecting alternative approaches to quantum gravity available in the literature and which predict stationary quantum-gravity states in which the notion of time evolution seems to appear a trivial one [75,76]. From the other side, the same result realizes a consistent conceptual framework related to the notions of "relational evolution" and "evolving constants of motion", according to which physical observables in quantum gravity would not evolve dynamically in terms of a temporal parameter, but rather their mutual relational evolution should provide an independent way for the identification of the dynamics itself [77,78]. In this sense in fact, the Hamiltonian dynamics of CCG and CQG theories can be measured either by the dynamical parameter s, or equivalently by the intrinsic change of the non-stationary quantum field Λ_{CQG}. Such a feature, in turn, makes also a point of contact with alternative formulations of GR known as "unimodular gravity" [51], in which the cosmological constant Λ is promoted to be a dynamical field [79,80] generating an effective cosmological time [81], in analogy to the non-constant relativistic particle rest mass of the Fock–Stueckelberg theory. It must be stressed however that despite conceptual connections with the approaches mentioned here, the scheme of CCG and CQG theories remains distinguished and independent, in primis because of the consistency with the principle of manifest covariance and the existence of a canonical Hamiltonian structure.

10. Conclusions

An established theoretical result of non-relativistic quantum mechanics is represented by the Ehrenfest theorem. This relates the dynamical evolution of the expectation value of a quantum operator to the expectation value of its corresponding commutator with the Hamiltonian operator of the system, to be measured in terms of the absolute time-coordinate of the non-relativistic reference frame. The Ehrenfest theorem arises as a relationship implied by the mathematical setting of non-relativistic quantum mechanics, in particular the validity of the quantum wave equation (in either so-called Schrödinger or Heisenberg pictures) together with the unitary evolution and probabilistic interpretation of the quantum wave function. In addition to the theoretical meaning of the theorem per se, its relevance also involves foundational questions of quantum mechanics and can have impact on its logical and philosophical interpretation, in particular regarding the validity and significance of the correspondence principle between quantum and classical realms. However, the extension of the Ehrenfest theorem to relativistic quantum mechanics and field theory has been partly limited, and despite several long-lasting attempts to the problem, no equivalent unambiguous result exists in these frameworks. In fact, conceptual difficulties arise in relativistic settings due to the lack of Schrödinger-like quantum wave equation and absolute dynamical evolution time-parameter, as well as concerning the notion of unitary definition of the quantum theory and the probabilistic interpretation to be attached to the quantum wave function. Similar issues arise in the context of quantum field theory, especially for mutually non-linear interacting fields. In this case, the customary formulation of the Ehrenfest theorem is expected to be affected by additional quantum-fluctuation contributions appearing at characteristic scale-lengths, which can be treated in asymptotic or perturbative ways, e.g., in the framework of effective field theories.

The situation appears even more serious when considering specifically the case of quantum gravitational fields, where the principles of quantum mechanics clash with foundational issues pertinent both to GR (like the principle of covariance and manifest covariance) and to deep questions on the meaning of quantum gravity itself. Debates in this reference concern for example the very nature of space and time and the meaning of gravitational wave function. Several alternative approaches have been proposed in the literature as candidates to represent plausible theories of quantum gravity. However, for none of them has it been possible to firmly establish, so far, an extended version of the Ehrenfest theorem. Attempts were proposed in the case of Wheeler–DeWitt quantum equation arising in the framework of Loop Quantum Gravity and the underlying ADM formulation of GR, which realize so-called canonical theories of classical and quantum gravity. However, besides recovering the same problems affecting relativistic quantum mechanics, the additional violation of manifest covariance and the lack of a true canonical and regular Hamiltonian theory and of a true dynamical quantum wave equation in the Wheeler–DeWitt theory have inhibited any further progress in this respect.

On the other hand, the possibility of establishing a quantum-gravity version of the Ehrenfest theorem could represent by itself a proof of validity and correctness of a quantum gravity theory and related conceptual framework among other attempts. Following this reasoning, in this paper we have addressed the issue in the setting provided by the manifestly covariant quantum gravity theory (CQG theory). This is established from canonical quantization of the corresponding classical manifestly covariant Lagrangian and Hamiltonian formulations (CCG-theory) of GR, consistent with the prescription of the DeDonder–Weyl tensorial formalism for Hamiltonian theory of continuous field dynamics. Characteristic features of CQG theory are

(1) The consistency with manifest covariance principle, so that the theory maintains 4-covariance and any 3 + 1 decomposition of space-time is avoided;

(2) The existence of a regular and invertible Hamiltonian structure as well as commutator algebra formalism;

(3) The prescription of a Schrödinger-like quantum-gravity wave equation expressing unitary evolution of quantum-wave function;

(4) The probabilistic interpretation attributed to the quantum wave function and probability density;

(5) The definition of Hilbert space as the configuration space for quantum gravitational field on which scalar products and expectation values are defined.

In particular, thanks to the joint validity of these principles, it has been proved that an analogue representation of the Ehrenfest theorem of quantum mechanics can be established for CQG theory, to be referred to as the CQG–Ehrenfest theorem. This represents a peculiar and unique feature of CQG theory and its theoretical framework, which warrants consistency with the principle of manifest covariance. The theorem reproduces, for the quantum gravitational field, the same result that applies in non-relativistic quantum mechanics, both at the level of the formal representation of the Ehrenfest equation, which retains the same mathematical structure, as well as its physical interpretation. Thus, the manifestly covariant formalism permits to recover for GR and quantum gravity settings the validity of the Ehrenfest theorem of quantum mechanics, providing a theoretical background for a critical analysis of quantum-classical correspondence of gravity. This outcome distinguishes the theoretical principles at the root of CCG and CQG theories as plausible candidates in the understanding of quantum phenomena of gravitational field and the advance of their future investigations.

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