IOP Publishing Phys. Scr. 97 (2022) 065203 https://doi.org/10.1088/1402-4896/ac6b60

Physica Scripta

OPEN ACCESS

RECEIVED
27 December 2021

REVISED
12 April 2022

ACCEPTED FOR PUBLICATION
28 April 2022

PUBLISHED
9 May 2022

PAPER

Quantization of nonlocal fields via fractional calculus

Kazım Gökhan Atman[1,2] **and Hüseyin Şirin**[1]

[1] Department of Physics, Science Faculty, Ege University, Izmir 35100, Turkey
[2] School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, United Kingdom

E-mail: k.g.atman@qmul.ac.uk

Keywords: nonlocal phenomena, nonlocal fields, fractional calculus

Abstract

In this study, we investigate the effect of nonlocality in quantum mechanics and propose a fractional approach the theory of quantized fields. For this purpose, we embedded the fractional calculus to broaden theory of quantum fields since the integral and derivative operators are nonlocal in fractional calculus.Additionally, quantum entanglement is discussed to gain comprehension of nonlocality in the foundation of quantum mechanics. Besides, fractional Lagrangian formalism was presented due to fact that the Lagrangian density is the starting point to establish a field theory.Furthermore, to make fractional field operators quantum mechanical, equal-time commutator have been defined for the these operators in terms of Caputo fractional derivative. Thus, a scheme of quantization of fractional fields is introduced and general aspects of the method is illustrated with the theory of massive scalar fields. This approach laid out to a successful generalization of the quantum field theory which is coherent with the standard formalism. Consequently, we developed promising concept for a quantum field theory by introducing nonlocality into standard mathematical formalism.

1. Introduction

As is well known, the nonlocal phenomena which is riddled with the quantum entanglement, has always been the subject of controversy in the theoretical physics. Nevertheless, many important process in physics involves nonlocality in time(memory effects) and space (long-range interactions) [1, 2]. A notable example of nonlocal phenomena is spontaneous fission which occurs due to the quantum tunnelling [3]. Additionally, one problem that has bedevilled the quantum mechanics from the beginning is the question of the nonlocality. Moreover, this fundamental aspect of the theory has been demonstrated by a series of designed experiments [4, 5]. In the light of these facts, it was with a view to obtaining more insight regarding the effects of nonlocality, that the present study was undertaken. For this purpose, we have proposed an approach to interprete quantum entanglement within the framework of fractional time evolution which comprises nonunitary property that describes the nonlocal behavior of quantum systems. It should be also emphasized that our approach relies on the assumption of nonlocal effects occur only in fractional space-time. However, such an opinion might be considered as a violation of basic principle of quantum field theory (QFT), like casuality and unitary. However, in the case of fractional derivative order $\alpha = 1$, space-time becomes continuous hence the our approach satisfies the principles of QFT. Therefore, it seems to be reasonable to assume that standard mathematical QFT is globally local whereas there exist nonlocality in the fractional dimensions.

There is an increasing attention of physicists in nonlocal field theories during recent years. It is also clear that considerable progress have been made in constructing a nonlocal theory of quantized fields. Before the renormalization is well established, first attempts to eliminate ultraviolet divergences in the quantum field theory, have relied on the nonlocal Lagrangians [6–10]. In general, the main idea of these studies is quantized fields by means of the hypothesis of space-time stochasticity [11]. In particular, proposed methods are based on the stochastic space-time R^4 and of averaging in it. According to this, space-time is defined as following [12].

$$\hat{x} = (x_0 + ib_4, x^\mu + g^\mu), \qquad (1)$$

where c represents speed of light, $x_\mu = (x_0 = ct, x^\mu)$, and $g_\mu = (g_4 = c\tau, g^\mu)$ is random vector with a measure $w(g_E^2/l^2)$ which obeys the following conditions,

$$\int dw(g_\mu^2/l^2) = 1, \quad dw(g_\mu^2/l^2) \geqslant 0. \tag{2}$$

Here, l is the fundamental length which represents the domain that the nonlocality occurs. Additionally, canonical quantization of scalar fields which are containing nonlocal terms have been presented by several authors [13, 14]. However, progress achived in nonlocal quantum field theory in during the past decades is connected with probability theory and stochastic processes [15, 16]. Stochastic quantization is first proposed by Parisi and Wu and studies have been made to advance ideas of stochastic quantization since the original paper published [17]. In this manner, stochastic quantization of the Euclidean field is defined in terms of the Langevin type equation with respect to auxiliary time. Beside the stochastic approaches, it is also worth to mention that the constructing a nonlocal theory of fields by making use of fractional calculus is a relatively recent development [18]. Classical fractional fields which are formulated in terms of fractional Lagrangian densities, are proposed by several authors to take into account nonlocality [19, 20]. Moreover, the fractional Dirac equation and its solution were presented [21, 22]. Stochastic quantization of fractional Klein–Gordon field at finite temprature, has also proposed by Lim by generalization of Parisi-Wu stochastic quantization [15, 23]. The underlying idea of this study is to treat Euclidean Klein–Gordon field as a collection of fractional harmonic oscillators.

For this reason, formulation of quantized fields is proposed in terms of Caputo fractional derivative which is nonlocal by its nature. Additionally, it appears that the interest of scientists has increasing attention since it wide application in different fields of science [24–29]. It might be also said that the study of fractional calculus in quantum mechanics is still in infancy. However, fractional Schrödinger equation was proposed via path integral approach for Levy processes by Laskin [30–32]. In addition, fractional quantum calculus which leads the effective way of treating problems that includes the sets of nondifferentiable functions, is proposed in recent [33].

There exist several approaches to quantize field theory. It is also worth to mention that the canonical quantization is mostly used method. Nevertheless, as we emphasized above, stochastic quantization of fractional fields have been also proposed and the basis of this method have been expounded. The main aim of the our study is the present the canonical quantization programme of fractional fields for the first time.

This manuscript is organized as follows: in section 2, the phenomenon of nonlocality is presented in detail and a thought experiment is considered to allow better understanding of implication of nonlocality. In section 3, fractional Lagrangian formalism is introduced by making use of Caputo fractional derivative. Section 4 is devoted to quantization of fractional massive scalar fields. Finally, the conclusions are summarized in section 5.

2. A review of nonlocality

The phenomenon of nonlocality was first illustrated by the famous thought experiment of Einstein, Podolsy and Rosen thus a new window had been opened to our grasp of quantum mechanics by this classic paper [34]. Furthermore, it is clear from the results of series of experiments that the Bell' s inequality is violated [4, 5]. In other words, one may say that it appears that the existence of the nonlocal effect which is called as ' spooky action at a distance' by Einstein. For this reason, in order to allow better understanding of the nonlocal phenomenon in the foundations of the quantum mechanics, it seems to be necessary to develop a sensible language for the theory. The usual formulation of the quantum mechanics is based on the standard mathematical operators which are local. Therefore it seems to us that the utilization of the fractional derivative operators which are nonlocal, gives the deeper understanding of the implications of nonlocality.

Before going forward, let us start with introducing the time fractional evolution operator since we investigate the time evolution of the quantum system. The fractional time evolution operator is given by

$$\mathcal{U}_\alpha(t,\ t_0) = E_\alpha((-i(t-t_0))^\alpha)\mathcal{H}), \tag{3}$$

where $\mathcal{H}$ represents the Hamiltonian of the system [35, 36]. In the above equation $E_\alpha(.)$ is the Mittag-Leffer function which is defined by [37]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n\alpha)}. \tag{4}$$

It is notable that $\mathcal{U}_\alpha(t,\ t_0)$ is not unitary operator which differs the standard time evolution operator.

$$\mathcal{U}_\alpha^\dagger(t,\ t_0)\mathcal{U}_\alpha(t,\ t_0) = |E_\alpha((-i(t-t_0))^\alpha)\mathcal{H})|^2, \tag{5}$$

we shall now consider as a simple example of entanglement with two particles system, namely M and N, which were interacted in the past but the at the present they are situated far from each other. Additionally, it is worth to mention that M and N are entangled. Thus, we assume that the wave function which describes the system is given by,

$$\Phi(mn). \tag{6}$$

Therefore, complete description of the system M is given by the its density matrix [38].

$$\rho_{mm'} = \sum_{n} \Phi^{*}(m'n)\Phi(mn). \tag{7}$$

Now, the question is do measurements at the system N causes the anything to happen at system M, in other words does it cause the change of the density matrix of M. It is well known that according to the postulates of standard quantum mechanics, any evolution of system N must be unitary in keeping with a fundamental principle of quantum mechanics [38]. That is to say evolution of system N is described by the unitary matrix $U_{nn'}$. In this sense, $U_{nn'}$ acts on the wave function hence resulting wave function can be written as following,

$$\Phi_f = \sum_{n'} = U_{nn'}\Phi(mn'). \tag{8}$$

Therefore density matrix of system M which is given by equation (7) becomes

$$\rho_{mm'} = \sum_{n,\, n',\, n''} \Phi^{*}(m'n'')\, U^{\dagger}_{n''n} U_{nn'}\Phi(mn'). \tag{9}$$

It is easy to see that the density matrix of system M does not change since the transformation is unitary. This means that the measurement of N does not cause a particular effect for the system M. However, it would appear from the equation (5) that the a surprising action at a distance occurs due to the non unitary property of fractional time evolution operator. Now, it is appropriate to examine the decay of neutral pi as a simple example to understand this effect. Accordingly, electron and positron are in the singlet state as following

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_M|\downarrow\rangle_N - |\downarrow\rangle_M|\uparrow\rangle_N). \tag{10}$$

Here M represent positron while N represents electron. We shall now suppose that electron and positron are separated in space and electron is subjected by the external magnetic fields **B** which is uniform in the z direction. Therefore, time development of the system is governed by the fractional time evaluation operator:

$$\mathcal{U}_{\alpha} = E_{\alpha}\left(\frac{-iwS_z t^{\alpha}}{\hbar}\right). \tag{11}$$

Hence we obtain state $|EPR\rangle$ at any later time as following

$$|EPR,\, t_0 = 0;\, t\rangle = \frac{1}{\sqrt{2}}\left[E_{\alpha}\left(\frac{iwS_z t^{\alpha}}{\hbar}\right)|\uparrow\rangle_M|\downarrow\rangle_N - E_{\alpha}\left(\frac{-iwS_z t^{\alpha}}{\hbar}\right)|\downarrow\rangle_M|\uparrow\rangle_N\right]. \tag{12}$$

In addition we should mention that the fractional time evolution operator acts on only the second part of state. Thus, according to the the equations (9) and (12) the density matrix of system M can be written as following

$$\begin{aligned}\rho_{mm'} &= \textstyle\sum_{n,\, n',\, n''} |E_{\alpha}\left(-i\frac{w^{\alpha}t^{\alpha}}{2}\right)|^2\langle\uparrow|_M\langle\downarrow|_N - |E_{\alpha}\left(-i\frac{w^{\alpha}t^{\alpha}}{2}\right)|^2|\downarrow\rangle_M|\uparrow\rangle_N\\ &= \begin{pmatrix}\frac{|E_{\alpha}(-i\frac{wt^{\alpha}}{2})|^2}{2} & 0\\ 0 & \frac{|E_{\alpha}(-i\frac{wt^{\alpha}}{2})|^2}{2}\end{pmatrix}.\end{aligned} \tag{13}$$

Hence, it would appear from these results that the detectable effects of nonlocality can be deduced. Additionally, in the case $\alpha = 1$ results of conventional physics are recovered. As a consequence, these results account nonlocal effects in some measure for the quantum nonlocality with superluminal influences.

3. Fractional lagrangian formalism

In this study, partial Caputo derivatives are utilized due to the fact that the initial conditions which are accompanied with the fractional differential equation with Caputo derivatives take the regular form such as for the integer order differantial equations [39]. Hence, partial left and right Caputo derivatives are respectively defined as [40, 41]

$$_{+}\partial_p^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha_p)}\int_{a_p}^{x_p}\frac{\partial_{\tau} f(x_1,\ldots,x_{p-1},\, \tau,\, x_{p+1},\ldots,x_n)}{(x_p-\tau)^{\alpha_p}}d\tau, \tag{14}$$

$$ {}_{-}\partial_p^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha_p)} \int_{a_p}^{x_p} \frac{\partial_\tau f(x_1, \ldots, x_{p-1}, \tau, x_{p+1}, \ldots, x_n)}{(\tau - x_p)^{\alpha_p}} d\tau. \tag{15} $$

It is clear that the Lagrangian density is the starting point to establish a field theory since it guarantees that the theory is Lorentz invariant due to the fact that the Lagrangian density is a scalar quantity. Therefore, it is inevitable to formulate fractional fields in terms of the fractional Lagrangian densities. As in the conventional Lagrange formalism, dynamics of the fractional field is governed by the Lagrangian. Covariant form of the action which contains fractional Lagrangian density is defined as follows,

$$ S[\phi] = \int d^4x \mathcal{L}(\phi(x_\mu), \partial_\mu^\alpha \phi(x_\mu)), \quad \mu = 0, 1, 2, 3, \tag{16} $$

where $\phi(x_\mu)$ is the fractional field and $\partial_\mu^\alpha \phi(x_\mu)$ represents its fractional derivative. In this context, equation of motion is determined by the principle of the least action and using the condition for extremum of a functional [42]. In order to find extremum value of action, variation of function $\phi(x_\mu)$ is considered as,

$$ \phi(x_\mu) = \phi^*(x_\mu) + \epsilon\eta(x_\mu). \tag{17} $$

Hence, we can write the fractional action as,

$$ S[\phi_\mu] = \int dx^4 \mathcal{L}(\phi_\mu, {}_{+}\partial_\mu^\alpha \phi^*(x_\mu) + \epsilon\, {}_{+}\partial_\mu^\alpha \eta, {}_{-}\partial_\mu^\alpha \phi^*(x_\mu) + \epsilon\, {}_{-}\partial_\mu^\alpha \eta). \tag{18} $$

According to the calculus of variations, we obtain

$$ \left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\eta + \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{{}_{+}\partial_\mu^\alpha \phi}({}_{+}\partial_\mu^\alpha \eta) + \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{{}_{-}\partial_\mu^\alpha \phi}({}_{-}\partial_\mu^\alpha \eta)\right] + \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{{}_{+}\partial_\mu^\alpha \phi}({}_{-}\partial_\mu^\alpha \eta)]. \tag{19} $$

By making use of the fractional integration by parts formula we hence obtain fractional Euler–Lagrange equations of motion for fields ϕ_μ.

$$ \frac{\partial \mathcal{L}}{\partial \phi} + \sum_{\mu=0}^{3} {}_{-}\partial_\mu^\alpha \frac{\partial \mathcal{L}}{\partial({}_{+}\partial_\mu^\alpha \phi)} + \sum_{\mu=0}^{3} {}_{+}\partial_\mu^\alpha \frac{\partial \mathcal{L}}{\partial({}_{-}\partial_\mu^\alpha \phi)}. \tag{20} $$

In the case $\alpha = 1$, equation (20) reduces to the standard Euler–Lagrange equation. It is known that the canonical quantization is the method of obtaining the quantum field theory by quantizing the classical theory. Therefore, in order to quantize the fractional classical theory, we need the Hamiltonian formalism of the fractional field theory. For this purpose, we follow formalism which is given by [21] and define the momentum densities π_μ^α and π_μ^β conjugate to field ϕ_μ.

$$ \pi_\mu^\alpha = \frac{\partial \mathcal{L}}{\partial({}_{+}\partial_\mu^\alpha \phi)}, \quad \pi_\mu^\beta = \frac{\partial \mathcal{L}}{\partial({}_{-}\partial_\mu^\beta \phi)}. \tag{21} $$

Therefore, Euler–Lagrange equation can be written as follow,

$$ \frac{\partial \mathcal{L}}{\partial \phi} + \sum_{\mu=0}^{3} {}_{-}\partial_\mu^\beta \pi_\mu^\alpha + \sum_{\mu=0}^{3} {}_{+}\partial_\mu^\alpha \pi_\mu^\beta = 0. \tag{22} $$

According the proposed formalism, Hamiltonian density is given by

$$ \mathcal{H} = \sum_{\mu=0}^{3} (\pi_\mu^\alpha({}_{+}\partial_\mu^\alpha \phi) + \pi_\mu^\beta({}_{-}\partial_\mu^\beta \phi)) - \mathcal{L}. \tag{23} $$

Since we want to quantize classical fields with fractional manner, the given formulation of fields within the Caputo derivatives would serve as a starting point to establish a fractional quantum field theory. In the next section, we expand field operator $\phi(x_\mu)$ in terms of fractional creation / annihilation operators to obtain fractional massive scalar fields and complete our programme of canonical quantization with fractional manner.

4. Fractional massive scalar fields

In this section, we now apply the fractional quantization of the harmonic oscillator to the massive scalar fields in order to obtain fractional extentsion of theory. With regard to this, we write $\phi(x)$ and $\pi_\mu^\alpha(x)$ in terms of linear sum of an infinite number of fractional creation / annihilation operators ${}^\alpha a_p^\dagger$ and ${}^\alpha a_p$,

$$ \phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{(2E_p^\alpha)^{\frac{1}{2}}} ({}^\alpha a_p e^{-ip.x} +^\alpha a_p^\dagger e^{ip.x}), \tag{24} $$

$$\Pi^{\alpha}_{\mu}(x) = \partial^{\alpha}_{\mu}\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E^{\alpha}_{p})^{\frac{1}{2}}}(-ip^{\alpha}_{\mu})({}^{\alpha}a_p e^{-ip.x} + {}^{\alpha}a^{\dagger}_p e^{ip.x}). \tag{25}$$

Here, ${}^{\alpha}a_p$ and ${}^{\alpha}a^{\dagger}_p$ represent the fractional creation/annihilation operators which are interpreted as the increasing or decreasing the energy of state by a fractional unit of energy [43].

Let us introduce the fractional derivative of the exponential function since it is necessary for developing the equal-time commutator for the fractional fields. However,

$${}_{t_0}D^{\alpha}_{t}e^{\rho(t-t_0)} = \rho^k(t-t_0)^{k-\alpha}E_{1,k-\alpha+1}(\rho(t-t_0)). \tag{26}$$

An alternative form of the derivative of exponential function could written as following [44],

$${}_{t_0}D^{\alpha}_{t}e^{\rho(t-t_0)} = \rho^{\alpha}e^{\rho(t-t_0)} + \rho^k F_{\alpha-k}(t-t_0;\rho)). \tag{27}$$

Here, the function $F_{\alpha}(t-t_0;\rho)$ is a monotonically decreasing function which characterizes the deviation from the integer order derivative of the exponential function [44]. Therefore, these deviations vanish for the large factor of exponential and thus,

$${}_{t_0}D^{\alpha}_{t}e^{\rho(t-t_0)} \sim \rho^{\alpha}e^{\rho(t-t_0)}. \tag{28}$$

By making use of equation (28), commutation relation for the fractional fields can be written as following,

$$[\phi(x), \Pi^{\alpha}_{0}(x)] = \int \frac{d^3pd^3q}{(2\pi)^6}\frac{(-i)}{2}\frac{E_q^{\frac{\alpha}{2}}}{E_p^{\frac{\alpha}{2}}}[[{}^{\alpha}a^{\dagger}_q, {}^{\alpha}a_p]e^{i(px-qy)} + [{}^{\alpha}a^{\dagger}_p, {}^{\alpha}a_q]e^{i(qy-px)}]. \tag{29}$$

The commutator for the fractional creation/annihilation operators is given by [43]

$$[{}^{\alpha}a_q, {}^{\alpha}a^{\dagger}_p] = \eta(\epsilon, \alpha), \tag{30}$$

where the ϵ stands for dependency of the particular representation of the Hilbert space. Thus, we obtain the commutation relation between fractional field operators by making use of above equation.

$$\begin{aligned}[\phi(x), \Pi^{\alpha}_{0}(x)] &= \int \frac{d^3}{(2\pi)^3}\frac{(-i)}{2}(-\eta(\epsilon,\alpha)e^{ip(x-y)} - \eta(\epsilon,\alpha)e^{ip(y-x)})\\ &= i\eta(\epsilon,\alpha)\delta^3(x-y).\end{aligned} \tag{31}$$

The main idea of the canonical quantization schema is converting the fields to operator valued function of the space. In this manner, classical fractional field theory have been quantized by defining the commutator for the fractional operators.

To complete our programme of the canonical quantization of the fractional classical fields, we now utilize the expension of the field operator $\phi(x)$ into the Hamiltonian. In this manner, expression for the corresponding energy operator is defined in terms of the fractional creation / annihilation operators.

$$\mathcal{H}^{\alpha} = \int d^3x\frac{1}{2}[[{}_{+}\partial^{\alpha}_{0}\phi(x)]^2 + [\nabla^{\alpha}\phi(x)]^2 + m^{2\alpha}[\phi(x)]^2]. \tag{32}$$

It is reasonable to start by obtaining the momentum density for the calculate Hamiltonian.

$${}_{+}\partial^{\alpha}_{0}\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E^{\alpha}_{p})^{\frac{1}{2}}}(-iE^{\alpha}_{p})({}^{\alpha}a_p e^{-pi.x} - {}^{\alpha}a^{\dagger}_p e^{ip.x}). \tag{33}$$

After the determine momentum density, the corresponding space-like component can be written as following.

$$\nabla^{\alpha}\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}(2E^{\alpha}_{p})^{\frac{1}{2}}}(ip^{\alpha})(({}^{\alpha}a_p e^{-pi.x} - {}^{\alpha}a^{\dagger}_p e^{ip.x})). \tag{34}$$

Thus, by inserting these results into equation (32), calculation of Hamiltonian is completed.

$$\begin{aligned}\mathcal{H}^{\alpha} = \frac{1}{2}\int \frac{d^3xd^3pd^3q}{(2\pi)^3}\Bigg[&-\frac{E_q^{\frac{\alpha}{2}}E_q^{\frac{\alpha}{2}}}{2}({}^{\alpha}a_p e^{-ip.x} - {}^{\alpha}a^{\dagger}_q e^{iq.x})({}^{\alpha}a_q e^{-qi.x} - {}^{\alpha}a^{\dagger}_q e^{iq.x})\\ &+ \frac{1}{2E_q^{\frac{\alpha}{2}}E_q^{\frac{\alpha}{2}}}(ip^{\alpha\alpha}a_p e^{ip.x} - ip^{\alpha\alpha}a^{\dagger}_p e^{-ip.x})(ip^{\alpha\alpha}a_q e^{iq.x} - iq^{\alpha\alpha}a^{\dagger}_q e^{-iq.x})\\ &+ \frac{1}{2E^{\alpha}_{p}E^{\alpha}_{q}}({}^{\alpha}a_p e^{ip.x} + {}^{\alpha}a^{\dagger}_p e^{-ip.x})({}^{\alpha}a_q e^{iq.x} + {}^{\alpha}a^{\dagger}_q e^{-iq.x})\Bigg].\end{aligned} \tag{35}$$

By making use of $\int d^3x e^{ip.x} = (2\pi)^3\delta^{(3)}(x)$, we obtain,

$$\begin{aligned}\mathcal{H}^\alpha &= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}E_p^\alpha[{}^\alpha a_p^{\dagger\alpha}a_p + [{}^\alpha a_p, {}^\alpha a_p^\dagger]],\\ &= \int\frac{d^3p}{(2\pi)^3}E_p^\alpha({}^\alpha a_p^{\dagger\alpha}a_p),\\ &= \int\frac{d^3p}{(2\pi)^3}E_p^\alpha\hat{n}^\alpha,\end{aligned} \tag{36}$$

where $\hat{n}^\alpha = {}^\alpha a_p^{\dagger\alpha}a_p$ is the number operator. Additionally, in order to deal with the infinity, fractional creation and annihilation operators were written as normal ordered.In this manner, fractional theory of quantized fields is completed. The programme of the concomitant formalism which is aiming to the fractional quantization of the massive scalar field, can be briefly summarized as following:

(i) The proposed model is defined starting with a Lagrangian density for scalar massive field theory.

$$\mathcal{L}^\alpha = \frac{1}{2}[{}_+\partial\phi(x)]^2 - \frac{1}{2}m^2[\phi(x)]^2. \tag{37}$$

(ii) After the writing down Lagrangian density, momentum density and Hamiltonian are calculated in terms of the fractional fields.

$$\Pi_\mu^\alpha = \frac{{}_+\partial_\mu^\alpha\mathcal{L}^\alpha}{\partial({}_+\partial_\mu^\alpha\phi(x))}. \tag{38}$$

hence, Hamiltonian can be written in terms of the momentum density as following,

$$\mathcal{H}^\alpha = \Pi_0^\alpha\,{}_+\partial_0^\alpha\phi(x) - \mathcal{L}^\alpha. \tag{39}$$

(iii) To quantize the given fractional field, we impose the canonical commutation relation,

$$[\phi(x), \Pi_0^\alpha(x)] = i\eta(\epsilon, \alpha)\delta^3(x - y). \tag{40}$$

(iv) Last step is the expanding the fields in terms of fractional creation and annihilation operators.

$$\phi(x) = \int\frac{d^3p}{(2\pi)^{\frac{3}{2}}}\frac{1}{(2E_p^\alpha)^{\frac{1}{2}}}({}^\alpha a_p e^{-ip.x} + {}^\alpha a_p^\dagger e^{ip.x}). \tag{41}$$

Thus, we have demonstrated that the by making use of the fractional calculus leads to nonlocal quantum field theory.

5. Conclusion

In this study, we have presented the fractional quantum field theory which has been enriched by new powerful method that relies on the ideas of nonlocal fractional derivative operators, to take into account nonlocal effects. For this reason, in order to realize the quantization of fractional classical fields within framework of canonical quantization, fractional equal-time commutation relation has been derived. In this way, we have completed the canonical quantization schema of massive scalar fields for the first time.

We have also shown that the viewpoint of fractional calculus leads to new insights and surprising interpretation of the nonlocal phenomena in the foundations of quantum mechanics. We have provided with a simple thought experiment to gain a deeper understanding of nonlocality in quantum mechanics. As a result, there exist measurable consequences of nonlocal effects for the maximally entangled states. However, in the case of $\alpha = 1$ where the space-time becomes continuous, the standard mathematical expressions are recovered which means that the nonlocal effects are vanished. In other words, it must clearly be assumed that nonlocality occurs only in the course of fractional time. Thus we believe that, this idea gives proper foundation for the concept of nonlocality in quantum mechanics.

Consequently, we have developed promising concept for a quantum field theory by introducing nonlocality into standard mathematical formalism. Accordingly, it has been shown that the proposed formalism leads to surprising interrelations of quantized fields and nonlocal effects that remain unconnected until now. Nevertheless, effects that predicted by the concomitant formalism are coherent with the outcomes obtained by the standard mathematical quantum mechanics. The advantage of the quantization scheme developed, compared to existing studies is that nonlocality is not only introduced for interacting fields but also for free fields as well. As a possible direction for future research, renormalization problem and arising divergences will be investigated in a separate work since in our quantization scheme the ultraviolet divergences are absent.

Acknowledgments

This research has received no external funding.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

References

[1] Ertik H, Demirhan D, Şirin H and Büyükkılıç F 2009 A fractional mathematical approach to the distribution functions of quantum gases: Cosmic microwave background radiation problem is revisited *Physica A* **388** 4573–85
[2] Ertik H, Şirin H, Demirhan D and Büyükkılıç F 2012 Fractional mathematical investigation of bose-einstein condensation in dilute 87rb, 23na and 7li atomic gases *Int. J. Mod. Phys.* B **26** 1250096
[3] Martin B R and Shaw G 2019 *Nuclear and Particle Physics: An Introduction* (Hoboken,NJ, USA.: John Wiley Sons, Inc.)
[4] Aspect A, Dalibard J and Roger G 1982 Experimental test of bell's inequalities using time-varying analyzers *Phys. Rev. Lett.* **49** 1804–7
[5] Aspect A, Grangier P and Roger G 1982 Experimental realization of einstein-podolsky-rosen-bohm gedankenexperiment: a new violation of bell's inequalities *Phys. Rev. Lett.* **49** 91–4
[6] Efimov G V 1967 Non-local quantum theory of the scalar field *Comm. Math. Phys.* **5** 42–56
[7] Wataghin G 1934 Bemerkung über die selbstenergie der elektronen *Zeitschrift für Physik* **88** 92–8
[8] Yukawa H 1950 Quantum theory of non-local fields. i. free fields *Phys. Rev.* **77** 219–26
[9] Yukawa H 1950 Quantum theory of non-local fields. ii. irreducible fields and their interaction *Phys. Rev.* **80** 1047–52
[10] Efimov G V 1977 Strong coupling in the quantum field theory with nonlocal nonpolynomial interaction *Comm. Math. Phys.* **57** 235–58
[11] Brooke J, Guz E and Prugovecki W 1982 Reciprocity principle in stochastic quantum mechanics *Hadronic J.* **5** 1717–33
[12] Namsrai K 1986 Nonlocal quantum field theory and stochastic quantum mechanics *Ser. Fundamental Theories of Physics* (Dordrecht, Netherlands: Reidel) 13
[13] do Amaral R L P G and Marino E C 1992 Canonical quantization of theories containing fractional powers of the d'Alembertian operator *J. Phys.* A **25** 5183–200
[14] Barci D, Oxman L and Rocca M 1996 Canonical quantization of nonlocal field equations *Int. J. Mod. Phys.* A **11** 2111–26
[15] Lim S and Muniandy S 2004 Stochastic quantization of nonlocal fields *Phys. Lett.* A **324** 396–405
[16] Kleinert H 2012 Fractional quantum field theory, path integral, and stochastic differential equation for strongly interacting many-particle systems *EPL (Europhysics Letters)* **100** 10001
[17] Parisi G and Wu Y 1981 Perturbation Theory without Gauge Fixing *Sci. Sinica* **24** 483
[18] Tarasov V E 2021 Lattice fractional quantum field theory: exact differences approach *Mod. Phys. Lett.* A **36** 2140001
[19] Baleanu D and Muslih S I 2005 Lagrangian formulation of classical fields within riemann-liouville fractional derivatives *Phys. Scr.* **72** 119–21
[20] Goldfain E 2006 Complexity in quantum field theory and physics beyond the standard model *Chaos, Solitons Fractals* **28** 913–22
[21] Muslih S I, Agrawal O P and Baleanu D 2010 A fractional dirac equation and its solution *J. Phys. A: Math. Theor.* **43** 055203
[22] Raspini A 2001 Simple solutions of the fractional dirac equation of order 2/3 *Phys. Scr.* **64** 20–2
[23] Lim S 2006 Fractional derivative quantum fields at positive temperature *Physica A* **363** 269–81
[24] Advances in fractional calculus: theoretical developments and applications in physics and engineering Agarwal P *et al* (ed) 2019 *ICFDA 2018 Amman Jordan, July 16–18*
[25] Ali M A, Abbas M, Budak H, Agarwal P, Murtaza G and Chu Y 2021 New quantum boundaries for quantum simpson's and quantum newton's type inequalities for preinvex functions *Advances in Difference Equations* **2021** 1–21
[26] Gençoğlu M T and Agarwal P 2021 Use of quantum differential equations in sonic processes *Applied Mathematics and Nonlinear Sciences* **6** 21–8
[27] Khan Y 2021 Fractal higher-order dispersions model and its fractal variational principle arising in the field of physcial process *Fluctuation and Noise Letters* **20** 2150034
[28] Khan Y 2021 Novel solitary wave solution of the nonlinear fractal schrödinger equation and its fractal variational principle *Multidiscipline Modeling in Materials and Structures* **17** 630–5
[29] Khan Y and Faraz N 2021 Hausdorff fractal new coupled nonlinear schrÖdinger model and its novel solitary wave solution *Fractals* **29** 2150252
[30] Laskin N 2000 Fractals and quantum mechanics *Chaos* **10** 780–90
[31] Laskin N 2000 Fractional quantum mechanics *Phys. Rev.* E **62** 3135–45
[32] Laskin N 2002 Fractional schrödinger equation *Phys. Rev.* E **66** 056108
[33] Tariboon J, Ntouyas S K and Agarwal P 2015 New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations *Advances in Difference Equations* **2015** 1–19
[34] Einstein A, Podolsky B and Rosen N 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47** 777–80
[35] Ertik H, Demirhan D, Şirin H and Büyükkłłłç F 2010 Time fractional development of quantum systems *J. Math. Phys.* **51** 082102
[36] Dong J and Xu M 2008 Space-time fractional schrödinger equation with time-independent potentials *J. Math. Anal. Appl.* **344** 1005–17

[37] Hilfer R 2000 *Applications of Fractional Calculus in Physics* (Singapore: World Scientific)
[38] Sakurai J J and Napolitano J 2017 *Modern Quantum Mechanics* II edn (Cambridge: Cambridge University Press)
[39] Podlubny I 1998 *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications* (New York: Academic Press)
[40] Samko S, Kilbas A and Marichev O I 1993 *Fractional Integrals and Derivatives: Theory and Applications* (Amsterdam: Gordon and Breach)
[41] Spanier S K B O J 1974 *The Fractional Calculus : Theory and Applications of Differentiation and Integration to Arbitrary Order* (New York: Academic)
[42] Agrawal O 2002 Formulation of euler-lagrange equations for fractional variational problems *J. Math. Anal. Appl.* **272** 368–79
[43] Atman K G and Şirin H 2020 Nonlocal phenomena in quantum mechanics with fractional calculus *Rep. Math. Phys.* **86** 263–70
[44] Garrappa R, Kaslik E and Popolizio M 2019 Evaluation of fractional integrals and derivatives of elementary functions: Overview and tutorial *Mathematics* **7** 407 https://mdpi.com/2227-7390/7/5/407