Rexigel et al. *EPJ Quantum Technology* (2025) 12:22
https://doi.org/10.1140/epjqt/s40507-025-00327-4
EPJ.org
EPJ Quantum Technology
a SpringerOpen Journal
RESEARCH Open Access

Investigating the use of multiple representations in university courses on quantum technologies

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Abstract

The field of Quantum Information Science and Technology (QIST) education presents unique challenges for both students and educators, such as the necessity of understanding abstract properties of quantum systems. To provide a more intuitive understanding of quantum systems, a multitude of qubit representations have been developed in recent years. Given the diversity of the field, a specific representation may be more suitable in one content area of than in another. Consequently, the choice of representation may vary considerably depending on the course orientation. However, no exhaustive analysis has been conducted into the differences between the representation of single- and multi-qubit systems in higher education QIST courses. Furthermore, the factors which influence the selection of a suitable representation remain open. To close this gap, we conducted an online survey with 25 educators at different German and Austrian universities on their use of representations in QIST-related courses. The results confirm the pivotal role of mathematical formalism in QIST education regardless of the specific course characteristics but also reveal an untapped potential for enhancing student learning through the intentional and comprehensive use of multiple external representations (MERs), especially in the case of multi-qubit systems. The findings are discussed within the context of the field of QIST and current insights into learning with MERs.

Keywords: Quantum technologies; Higher education; Multiple representations

1 Introduction

The simplest non-trivial quantum system employed in Quantum Information Science & Technology (QIST) is the two-level system, also called a qubit. Depending on the context, a qubit can be realised in different ways, such as the two long-lasting energetic levels of a trapped ion, the possible spin orientations of an electron, or the polarisation of a single photon. The non-classical characteristics of qubits form the basis of quantum computation and simulation, quantum sensing, or quantum communication and are therefore also of special interest in respective educational courses. Furthermore, the important role of mathematics in the representation and communication of quantum physics introduces an additional layer of complexity for prospective stakeholders. Even the foundational description of a single qubit's state as a superposition of two basis states necessitates a fundamental comprehension of linear algebra and complex numbers. Given the interdisciplinary nature of QIST, it is not reasonable to assume that all learners possess the requisite substantial mathematical understanding to engage in the respective scientific discourse readily. Consequently, the effective acquisition of QIST content constitutes a considerable challenge for both learners and educators.

Given the extensive scope of the field, the various content aspects of qubits imply the existence of a wide variety of representations of qubit states. While certain representations, such as the Dira/Bra-ket notation (e.g. [1]), the density matrix formalism (e.g. [2]) and the Bloch sphere (e.g. [3]), have been in use in the field of quantum physics for several decades, a variety of other representations have been developed to facilitate the effective communication of quantum properties and to provide students with the necessary support in the field of QIST. These include applications of established representations, such as geometric vector representations [4], as well as new representations, such as the Q-Sphere [5] or circle notation [6]. However, graphical representations of qubits in particular can be limited in their applicability, as they often rely on classical analogies and focus on the presentation of specific qubit properties. The teaching of QIST content therefore requires a flexible handling of various representations to select the appropriate representation(s) of the property considered.

Current insights in educational research demonstrate the relevance of appropriate combinations of representations for students success in science learning (e.g. [7]), and especially quantum education [8]. However, currently there is no clear guidance for educators on the use of relevant representations across the young field of QIST education. Moreover, while there may be a general consensus regarding the graphical representation of single-qubit states, such as the Bloch sphere [9], there seems to be a lack of agreement, particularly with regard to the graphical representation of multi-qubit systems. The initial step in developing transparent pathways is to identify the specific representations employed in each context. This provides a comprehensive overview and a foundational starting point. In order to fulfill this objective, we conducted a survey of QIST educators on the use of representations in their university courses. We then analysed the responses in the context of the QIST field and current findings from educational research.

2 Research background

In the following, we outline the evolving field of QIST education in Sect. 2.1 and provide an overview of current research insights into learning with Multiple External Representations (MERs) in Sect. 2.2. Both sections serve as a basis for the formulation of the research questions presented in Sect. 2.3.

2.1 The field of QIST education

In recent decades, the field of quantum physics has attracted significant interest from industry due to its potential applications in developing new technologies such as quantum sensing and quantum computing, thereby sparking the second quantum revolution [10]. In addition to this, the role of higher education in the second quantum revolution has been explored [11–15] and multiple university programmes and courses have been developed or are currently being developed [16–20]. In recent years, research has focused not only on the content coverage of QIST courses [21, 22], but also on the needs of the industry with respect to the content of the respective courses [23]. Although this constitutes a multitude of endeavours to define and evolve the field of QIST education, it is still in its infancy.

An overarching categorisation of the field of QIST is given by the European Competence Framework (CF) [23]. The framework divides the field of QIST into eight domains, including Concepts and Foundations, Physical Foundations, Enabling Technologies and Techniques, Quantum Hardware, Quantum Computing and Simulation, Quantum Sensors and Imaging Systems, Quantum Communication and Networks, and Valorisation. Each of these domains consists of multiple subdomains that include topics and subtopics. In this way, the content curricula can be mapped to the CF in detail. The framework continues with the definition of proficiency levels similar to language proficiency levels according to the Common European Framework of Reference for languages scale [24] and the development of qualification profiles based on interviews with industry representatives [23]. In doing so, the CF provides a structure to define exactly what to teach and with what goals in mind.

Another framework in the field of QIST education is given by [25]. Here, the authors identify three categories of relevant skills in the field of QIST, which are Theory & Analytics, Computation & Simulation and Experiment & Real World Application. Separately from the specific content, these categories allow an additional structuring of relevant course characteristics. In addition, the authors propose to define explicit questions using Bloom's pyramid taxonomy of remembering, understanding, applying, analysing, evaluating, and creating [26] and relate teaching approaches to the appropriate level of inquiry [27], sufficient scaffolding [28], and teaching formats such as cooperative learning [29]. Furthermore, due to the high level of abstraction in the field, appropriate representations should be chosen and created with careful consideration following the Design, Functions, and Tasks (DeFT) framework [7] (see also Sect. 2.2).

To optimise the learning of QIST content, various educational aspects should be considered. As a key aspect, in recent decades, many research efforts have proven the relevance of the appropriate use of multiple external representations (MERs) in education [30–32]. In the following, we focus on the respective research background.

2.2 Learning with multiple representations

The term "representation" is a collective term that can be interpreted in different ways depending on the context in which it is used. In this study, we focus on external representations, which provide information to a learner in an external format. During the learning process, a mental representation is constructed in working memory, which may not necessarily align with the external representation [33, 34]. Educational and psychological research over the past several decades has shown that learning in the context of Science, Technology, Engineering, & Mathematics (STEM) can be improved by not relying solely on a single external representation, but rather on MERs (for an overview, see, e.g., [35–37]). We use the term MERs to summarise all combinations of external representations that have different visual encoding. It is important to note that external representations may not necessarily be received visually but can also be provided in different modalities (e.g. [38]). For example, we can also perceive spoken text through the ear. However, in this study, we focus on representations that are perceived visually through the eyes. Following the definition of Schnotz, independent of a representation modality, a distinction can be made between symbolic (descriptive) representations and graphical (depictive) representations [34, 39]. Symbolic external representations, such as text, formulas, or equations, are based on symbols that do not bear a surface similarity to the information they represent. The association between the representation and the information presented, its referent, is based on convention [39]. In contrast, graphical external representations, such as tables, illustrations, and diagrams, are based on icons and share structural characteristics, such as similarity, with the referent [39]. The choice of external representation in a given context may vary depending on the specific content characteristics, that is, the referent. For example, when representing qubit states, some external representations may prove particularly valuable for representing superposition as a specific qubit characteristic, while others may be more appropriate in order to represent entanglement. Typically, symbolic representations are more abstract than graphical representations and are therefore often useful for representing more complex information structures [39, 40]. However, there are also contexts in which a graphical external representation has proven to be more effective than a symbolic one [34]. As a prominent example, the advantages of learning with text and pictures, also known as multimedia learning, compared to text alone have been extensively studied under the term "multimedia effect" (for an overview, see [30]). The existence of the multimedia effect can be explained in terms of information processing and the dual channel structure of the working memory (e.g. [34], [33]). Theories such as the Cognitive Theory of Multimedia Learning (CTML) [33] and the Integrated Theory of Text and Picture Comprehension (ITPC) [34] provide explanatory approaches to the multimedia effect. Although the two theories diverge in their explanations of the cognitive processes involved in multimedia learning, they nevertheless share a number of fundamental similarities. These include the limited capacity of working memory, which aligns with Sweller's cognitive load theory [41], the dual-channel structure of working memory, which is in line with Paivio's dual-channel theory [42], and the active processing of information when learning, characterised by the selection, organisation and integration of relevant information presented. In doing so, the use of symbolic and graphical external representations is argued to enable the distribution of cognitive load throughout the learning process across both mental channels more efficiently, thus reducing the risk of cognitive overload [33, 34]. However, the use of text and pictures does not unconditionally facilitate enhanced learning. Over the past few decades, a multitude of learning principles have been elucidated to promote a learning-enhancing use of MERs [33]. For instance, research has demonstrated that presenting MERs with high spatial and temporal contiguity can result in enhanced learning effectiveness [43].

The benefits of learning with text and picture over learning with text alone have been extensively researched and documented (e.g. [30]). However, learning environments may exceed the mere implementation of text and pictures, encompassing a multitude of diverse symbolic and graphical representations, such as formulas, equations, or schematic diagrams. In order to assess the efficacy of incorporating more complex combinations of multiple symbolic or graphical representations into students' learning, multimedia learning theories such as ITPC and CTML are limited in their application. Here, Ainsworth's Design, Functions and Tasks (DeFT) framework [7] provides a comprehensive theoretical basis. Within the DeFT framework, Ainsworth initially discusses the relevant design aspects of MERs, among others the number and type of combined representations [7]. The framework posits that MERs can fulfil three different functions to support students in their learning. First, they can be complementary in either the processes they support or in the information they contain. Second, MERs can constrain each other in interpretation. For instance, one representation may be more familiar, and thereby constraining the interpretation of another less familiar one. An alternative approach is to use individual representational characteristics. To illustrate, a more precise representation can constrain the interpretation of a less detailed one. Third, MERs can facilitate the construction of deeper understanding by integrating information from different representations. However, in order to benefit from the presentation of MERs, learners must engage with the cognitive tasks associated with them. This requires the possession of representational competence [44]. In order to engage successfully with MERs, it is necessary to have conceptual and perceptual competencies. These include the ability to understand the visual and connectional aspects of the representations, as well as the ability to perceive them fluently [32]. Consequently, to effectively integrate representations into students' learning, educators must consider the individual prerequisites of the learners.

2.3 Research questions

Compared to other contexts of classical science, representations in QIST education must bear particular peculiarities. Learning quantum physics content based on classical knowledge is limited [45–47]. As graphical representations often exploit classical analogies, many common representations are only applicable to a limited extent. However, previous research suggests that limitations based on the abstract and counter-intuitive nature of quantum physics are apparent not only for graphical external representations but also for symbolic ones. The limitations of language for communicating non-classical concepts serve as a major reason for conceptual difficulties [48]. In line with this, using multiple external representations fluently depending on the specific context has been shown a pivotal factor in understanding quantum properties [49]. Educators are currently facing a scattered field of representations to communicate QIST content to students. In this context, it can be helpful for instructors to get an overview of the most commonly used representations. This may be particularly important in the context of multi-qubit systems, due to the much higher complexity of these systems. Given the emerging nature of quantum technologies and the diverse range of specialisms within this field, a plethora of learning resources have been developed over the past few decades (e.g. [6, 9, 50–52]). Although new learning resources and qubit representations are constantly being introduced, we currently have no overview of which representations are actually used in university courses and under what circumstances. In order to close this research gap, our first research question is the following.

RQ 1: Which representations are used in higher education courses on QIST to represent single- and multi-qubit states?

Due to the interdisciplinarity of the field, we expected that the representations used would depend on the specific characteristics of the course. This includes, on the one hand, the orientation of the courses, as indicated by the related CF domains [23] and learner skills [25] and, on the other hand, the content characteristics focused on in the respective courses, such as superposition or entanglement. Therefore, our second research question is:

RQ 2: How do the incorporated representations differ depending on the course orientation and content characteristics?

3 Methods

To answer our research questions, we conducted an online survey and invited educators of QIST university courses from different locations in Germany and Austria. The online survey was created with SoSci Survey [53] and was conducted between October 2023 and January 2024. Educators were invited to participate in the survey via an online link.

3.1 Participants

In total, 25 complete data sets were recorded. Participants can be assigned to TU Munich (k = 5), RPTU Kaiserslautern-Landau (k = 3), TH Mittelhessen (k = 1), Leibniz University Hannover, TU Dresden, FH Aachen, FH Dortmund, FH Oberösterreich, TH Ingolstadt, HS Kaiserslautern (k = 1 each). In seven cases, no institution was stated.

3.2 Questionaire design

In the first step, participants were asked to provide information about their course orientation relevant to RQ 2. In doing so, participating educators were asked to give the approximate percentage of students for the disciplines physics, mathematics, computer science, and engineering and analogously the percentage of students for the students' intended academic degree, involving Bachelor's (or comparable), Master's (or comparable) and Ph.D. (or comparable). More information on other disciplines or intended academic degrees could be added manually. Subsequently, the participants were asked to select the relevant domains of the CF [23] and the learner skills primarily targeted in the respective course according to [25]. Due to the distance from the scientific content, we decided to neglect the CF domain of Valorisation and focus on the other seven categories, including 1. Concepts and Foundations, 2. Physical Foundations, 3. Enabling Technologies and Techniques, 4. Quantum Hardware, 5. Quantum Computing and Simulation, 6. Quantum Sensors and Imaging Systems, and 7. Quantum Communication and Networks. Furthermore, we adapted the learner skill Computation & Simulation as proposed by [25] to Numerics & Simulation to distinguish it more clearly from the other skills Theory & Analytics and Experiment & Real World.

In order to gain further insight into the content characteristics of each course, also with respect to RQ 2, participants were then asked to indicate the most important single-qubit characteristics of their course as free text. In the next step, with respect to both research questions, they were asked to provide the representations used to present a single-qubit state in the respective course. A selection of commonly used known representations was given to choose from, and further representations could be added manually if necessary. The representations suggested in the questionnaire can be classified as symbolic representations, such as Dirac/Bra-ket notation (e.g. [1]), column/row vector and density matrix [2] or as graphical representations like the 2D and 3D Bloch sphere (e.g. [3]), geometric vector representation [4], circle notation [6], phase disk [54], Q-sphere [5], pie chart (e.g. [55]), knot structures [56], spindrops [57, 58], real-world experiment and (interactive) simulation [59]. It should be noted that, in keeping with the focus on visually perceived external representations, possible effects of interaction with the real-world experiment and the (interactive) simulation were not considered in this context. Figure 1 illustrates examples of the graphical representations provided to participants for consideration.

Similarly to the questions on single-qubit states, participants had to give information on relevant characteristics and used representations of multi-qubit states in line with the

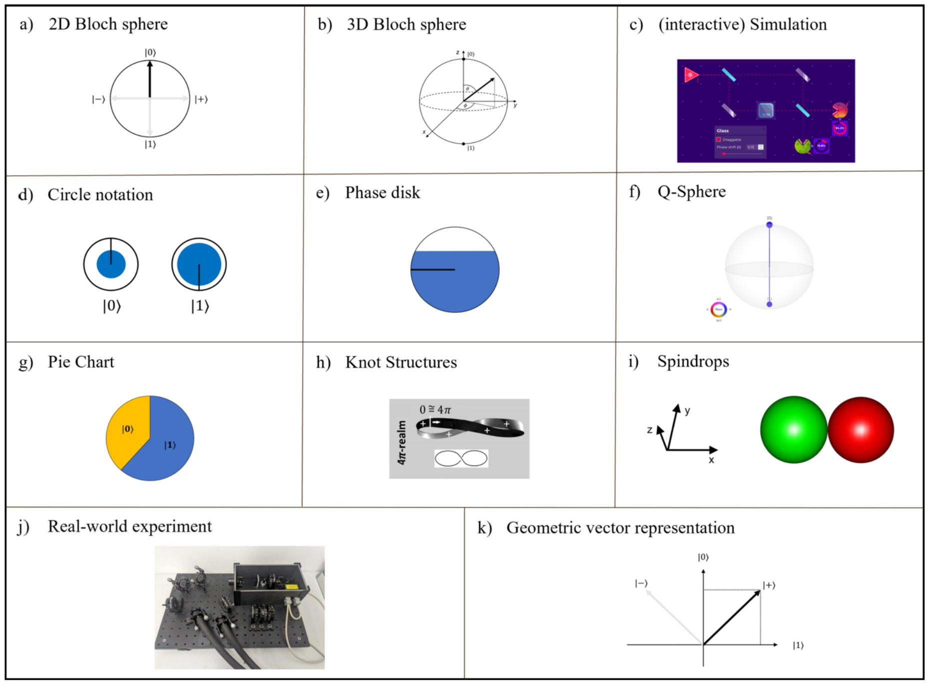


Figure 1 Selection of known graphical representations of single-qubit states provided for consideration with examples. c) from "Visualizing quantum mechanics in an interactive simulation – Virtual Lab by Quantum Flytrap" by Migdał, P., Jankiewicz, K., Grabarz, P., Decaroli, C., and Cochin, P., 2022, Optical Engineering, 61(08), 81808 (https://doi.org/10.1117/1.OE.61.8.081808). Copyright 2022 by the authors. d) redrawn from "Visualizing entanglement in multiqubit systems" by Bley, J., Rexigel, E., Arias, A., Longen, N., Krupp, L., Kiefer-Emmanouilidis, M., Lukowicz, P., Donhauser, A., Küchemann, S., Kuhn, J., and Widera, A., 2024, Physical Review Research, 6(2), 23077 (https://doi.org/10.1103/PhysRevResearch.6.023077). Copyright 2024 by the American Physical Society. e) redrawn from IBM Quantum (https://quantum-computing.ibm.com/composer/) f) created using IBM Quantum (https://quantum-computing.ibm.com/composer/) h) from "The Topological Origin of Quantum Randomness" by Heusler, S. ,Schlummer, P., and Ubben, M., 2021, Symmetry, 13(4), 581 (https://doi.org/10.3390/sym13040581). Copyright 2021 by the authors. i) redrawn from "SpinDrops" by Glaser, S., Tesch, M., Glaser, N. (https://spindrops.org/). Copyright 2018 by the authors

focus on single- and multi-qubit states in RQ 1. The most common representations for multi-qubit states were included in this section of the questionnaire like Dirac/Bra-ket notation (e.g. [1]), column/row vector and density matrix [2], the Bloch hypersphere [60], multiple 2D and 3D Bloch spheres, multiple points in the Bloch sphere [61], the geometric vector representation [4], geometric figures [52, 62, 63], circle notation [6], multiple phase disks [54], Q-sphere [5], pie chart (e.g. [55]), knot structures [56], Spindrops [57], real-world experiment and (interactive) simulation [59, 64]. As with the external representations of single-qubit states, possible effects of interaction with the real-world experiment or the (interactive) simulation were not addressed in keeping with the focus on visually perceived representations. Examples of the proposed graphical representations of multi-qubit state are shown in Fig. 2. Participants also had the possibility to include an additional representation which is not listed via a free text field. An overview of the questionnaire used is provided as supplementary material.

3.3 Analysis

The objective of this study is to analyse the use of representations of qubit states in QIST university courses. To this end, we focus on the number and type of representations re-

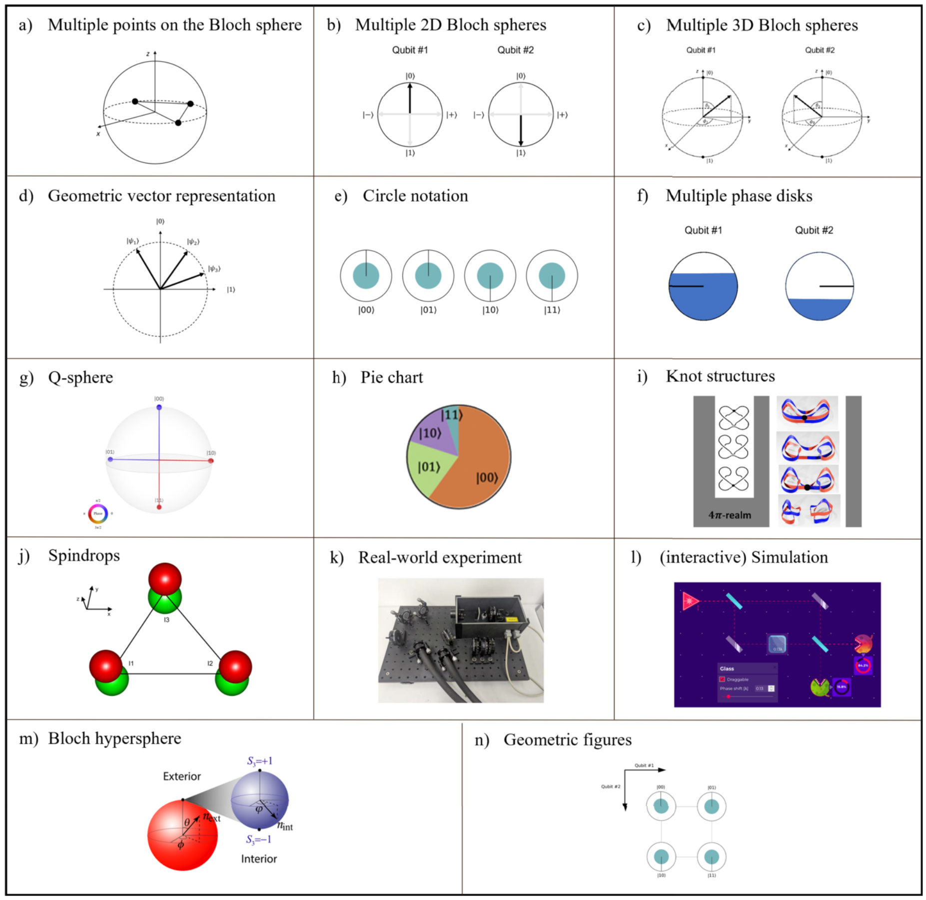


Figure 2 Selection of known graphical representations of multi-qubit states provided for consideration with examples. a) redrawn from "N -qubit states as points on the Bloch sphere" by Mäkelä, H., and Messina, A., 2010, Physica Scripta, T140, 14054 (https://doi.org/10.1088/0031-8949/2010/T140/014054). e) redrawn from "Visualizing entanglement in multiqubit systems" by Bley, J., Rexigel, E., Arias, A., Longen, N., Krupp, L., Kiefer-Emmanouilidis, M., Lukowicz, P., Donhauser, A., Küchemann, S., Kuhn, J., and Widera, A., 2024, Physical Review Research, 6(2), 23077 (https://doi.org/10.1103/PhysRevResearch.6.023077). Copyright 2024 by the American Physical Society. f) redrawn from IBM Quantum (https://quantum-computing.ibm.com/composer/) g) created using IBM Quantum (https://quantum-computing.ibm.com/composer/) i) from "The Topological Origin of Quantum Randomness" by Heusler, S., Schlummer, P., and Ubben, M., 2021, Symmetry, 13(4), 581 (https://doi.org/10.3390/sym13040581). Copyright 2021 by the authors. j) redrawn from "SpinDrops" by Glaser, S., Tesch, M., Glaser, N. (https://spindrops.org/). Copyright 2018 by the authors. l) from "Visualizing quantum mechanics in an interactive simulation – Virtual Lab by Quantum Flytrap" by Migdał, P., Jankiewicz, K., Grabarz, P., Decaroli, C., and Cochin, P., 2022, Optical Engineering, 61(08), 81808 (https://doi.org/10.1117/1.OE.61.8.081808). Copyright 2022 by the authors. m) from "Nested-sphere description of the N -level Chern number and the generalized Bloch hypersphere" by Kemp, C. J. D., Cooper, N. R., and Ünal, F. N., 2022, Physical Review Research, 4(2) (https://doi.org/10.1103/PhysRevResearch.4.023120) n) created using DCN-Webtool (https://dcn.physik.rptu.de/)

ported by the participants. First, we compare the number of representations for single- and multi-qubit systems using a paired samples t-test. The assumptions necessary for the implementation have been tested and are fulfilled. We then analyse the types of representations reported for single- and multi-qubit systems by classifying the responses into symbolic and graphical representations.

The potential impact of course orientation and content characteristics on the representations employed for both single- and multi-qubit states was examined. In order to analyse the potential effects of course orientation on the representations employed, we classified the reported representations into heterogeneous combinations of symbolic and graphical representations and homogeneous combinations of solely symbolic representations. Subsequently, the frequency of the respective representational combination was calculated for each CF domain [23] and learner skills as described in [25]. Furthermore, we conducted an investigation to determine the potential influence of content characteristics on the utilisation of representations. Consequently, free-text responses related to course-relevant qubit characteristics were categorised into overarching themes. For example, the theme "Superposition and Coherence" encompasses all responses that can be assigned to both superposition and coherence, including superposition, coherence, coherent control, phase, coherence time, and interference. Since all the characteristics stated by the participants as free-text responses were of a low inferential nature, it was not necessary to ensure reliability in the coding process. As in the investigation of the potential impact of course orientation, we proceeded to calculate the prevalence of heterogeneous and homogeneous combinations of representations across each content theme.

4 Results

4.1 Course characteristics

To account for individual course characteristics, participating educators were first asked to provide information on the percentage composition of students' disciplines within their course. The results indicate that the underlying courses are directed at students of various disciplines, including physics ($M = 47.36\%,\ SD = 42.40\%$), computer science ($M = 34.56\%,\ SD = 42.37\%$), engineering ($M = 7.20\%,\ SD = 13.81\%$), and mathematics ($M = 6.08\%,\ SD = 19.82\%$). Individual courses also focus on the disciplines business and chemistry.

Similarly, participants were asked to state the percentage composition of the level of education of the students within the underlying courses. The results range from bachelor students ($M = 38.52\%, SD = 37.82\%$), over master students ($M = 41.40\%,\ SD = 37.00\%$), to Ph.D. students ($M = 10.52\%,\ SD = 26.11\%$). In four cases, teacher training was explicitly stated as the relevant education level ($M = 11.36\%,\ SD = 29.21\%$).

Furthermore, the courses investigated cover all the domains of CF [23], as well as the three given skills Theory & Analytics, Numerics & Simulation, and Experiment & Real World taken from [25] (see Fig. 3). Only for the domain of Quantum Communication and Networks, none of the investigated courses related to Numerics & Simulation. The mean number of CF domains and learner skills indicated by each participant was 4.48 ($SD = 1.77$) and 2.12 ($SD = 0.82$), respectively.

4.2 Representations of single- and multi-qubit states

Participants were asked to state the representations used in their courses. The indicated representations to present single-qubit states are listed according to the frequency with which they are mentioned in Fig. 4. Relevant representations of multi-qubit states are listed in Fig. 5 accordingly.

Number of representations Participants identified a mean number of 5.20 ($SD = 1.98$) representations of single-qubit states to be used in their course(s). In particular, each participant stated at least two representations. The mean number of representations of multi-qubit states was 3.92 ($SD = 1.61$). Here, 6 participants indicated to use only one single

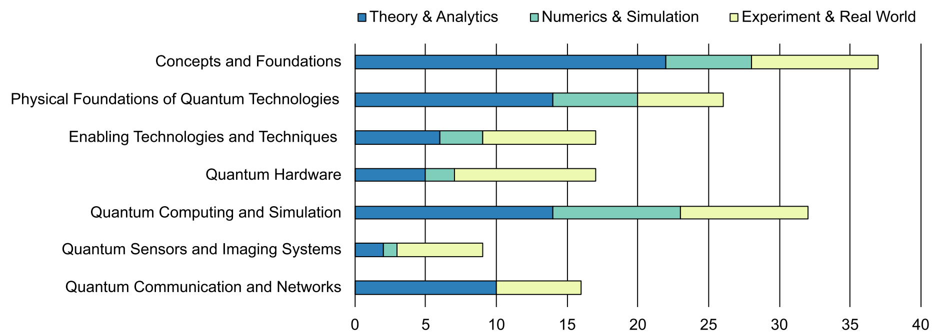


Figure 3 Number of selected CF domains and skills for the $N = 25$ datasets

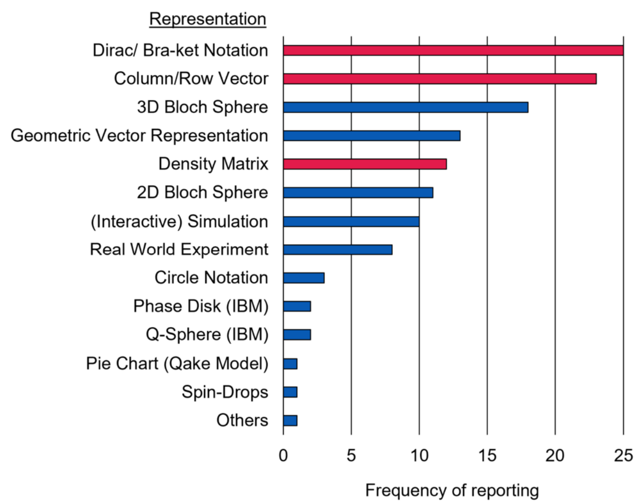


Figure 4 Relevant representations of single-qubit states. Incorporated symbolic (red) and graphical (blue) representations of single-qubit states are listed according to the frequency with which they are mentioned. Others were specified with the cube model [52]

representation in their course(s). A t-test of paired samples indicates a significant effect ($t(24) = 3.892, p < .001^{***}, d = .766$). According to [66], the effect can be interpreted as medium.

Types of representations For representations of single qubit states, all the indicated combinations are heterogeneous combinations of at least one symbolic and graphical representation each. In the case of multi-qubit states, $k = 6$ of the 25 participants specified only symbolic representations to be used in their course(s), neglecting graphical representations.

4.3 Effect of course characteristics on representations used

To investigate possible factors that influence the use of MERs, we analysed the type of representational combinations incorporated, as stated by the participants, according to the respective orientation of the course and the characteristics of the content.

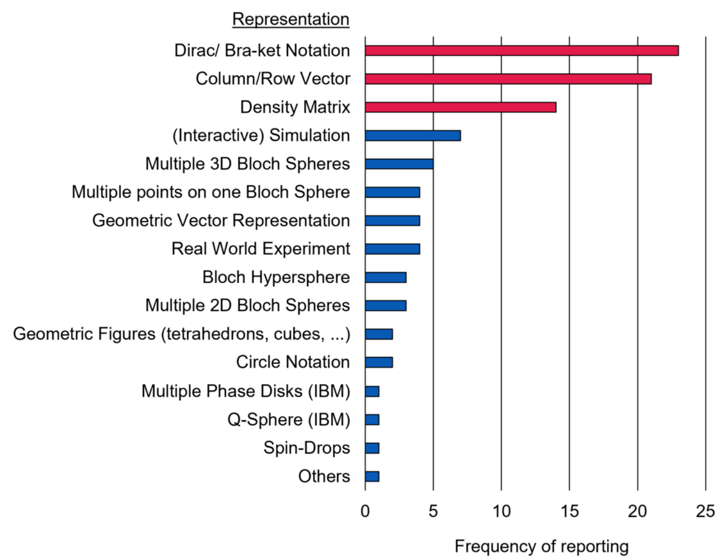


Figure 5 Relevant representations of multi-qubit states. The incorporated symbolic (red) and graphical (blue) representations of multi-qubit states are listed in accordance with the frequency of their mention. Others was specified with the macroscopic Bloch sphere, e.g. [65]

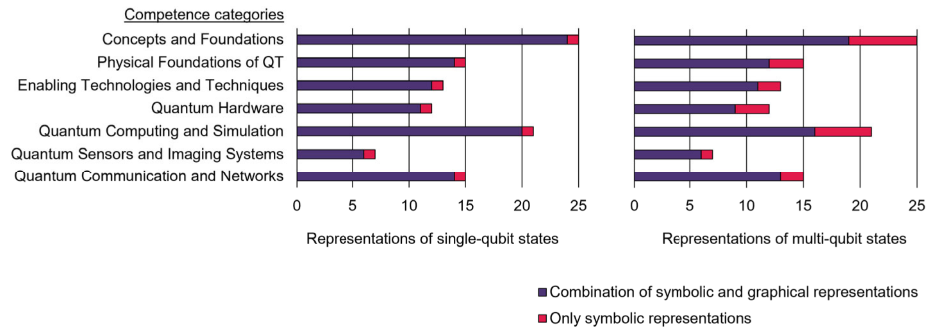
Effect of course orientation For each domain of the CF [23] and the learner skills taken from [25], we identified the type of representational combination stated by the participants. As an underlying course may cover more than one domain or learning skill at the same time, there may be some overlap in the responses of the participants. It should be acknowledged that, in such a case, possible differences in the representations used depending on the CF domains or learning skills cannot be resolved. In this paper, we differentiated between answers that included combinations of symbolic and graphical representations and those that included solely symbolic representations. The results of the reported combination depending on the respective CF domain are presented in Fig. 6a. The corresponding results, depending on the chosen learning skill, are summarised in Fig. 6b.

Effect of content characteristics Similar to the previous analyses, we also identified the representational combinations incorporated depending on the content characteristics of the courses. Each participating educator was allowed to enter a maximum of three qubit-state characteristics deemed most relevant within the context of the corresponding course. In this study, we distinguished the most relevant characteristics of single- and multi-qubit states. The answers provided are summarised in Table 1.

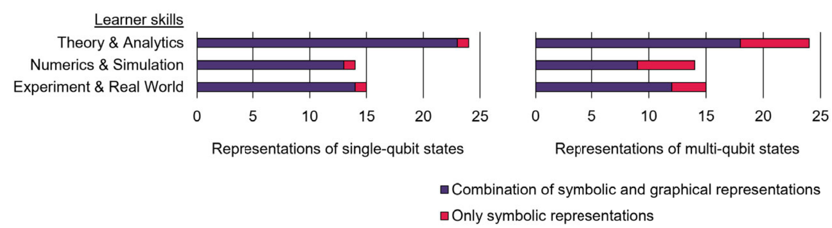
To determine the potential impact of the content characteristic on the representations employed in the course, we calculated the frequency of heterogeneous combinations of symbolic and graphical representations in comparison to solely symbolic combinations for each content theme. The respective results are presented in Fig. 7.

5 Discussion

With regard to our first research question (Which representations are used to represent single- and multi-qubit states?), the results indicate that a multitude of representations are currently used in QIST university courses to teach characteristics of single- as well



(a) Effect of CF domain on incorporated representations.



(b) Effect of targeted learner skills on incorporated representations.

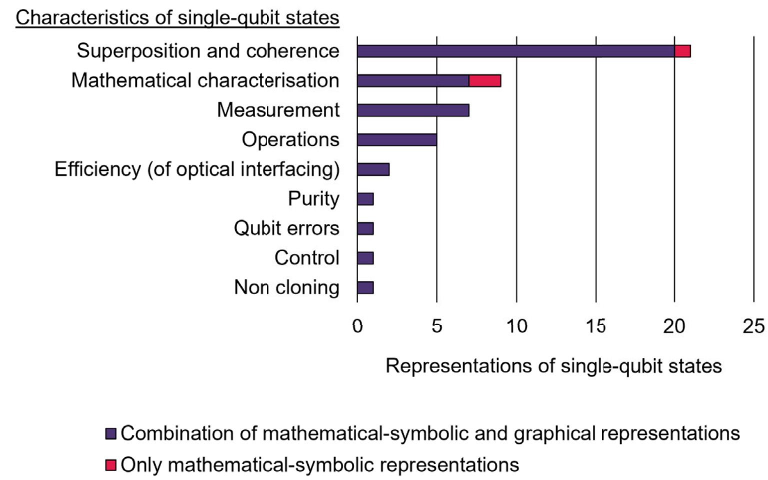
Figure 6 Frequency of reported representational combinations for single- and multi-qubit states depending on the chosen (a) CF domain [23] and (b) learner skills [25]. If a course covers more than one CF domain or learner skill, possible differences in the representations used depending on the domains or skills cannot be resolved

Table 1 Relevant characteristics of single- and multi-qubit states stated within the survey

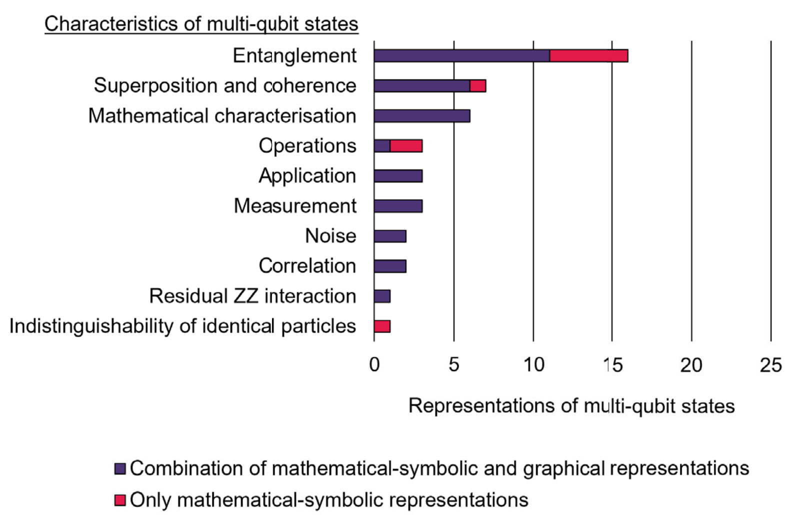
Single-qubit state		Multi-qubit state	
Characteristic	*k*	Characteristic	*k*
Superposition and coherence	21	Entanglement	16
Mathematical characterisation	9	Superposition and coherence	7
Measurement	7	Mathematical characterisation	6
Operations	5	Operations	3
Efficiency (of optical interfacing)	2	Application	3
Purity	1	Measurement	3
Qubit errors	1	Noise	2
Control	1	Correlation	2
Non cloning	1	Residual ZZ interaction	1
		Indistinguishability of identical particles	1

Note: *k* represents the total number of entries of the respective characteristic.

as multi-qubit states. A closer look at the implemented representations reveals a dominance of only a few symbolic representations (see Fig. 4). First, the Dirac/Bra-ket notation is the main mathematical description used in all courses, followed by the column/row vector representation, which is neglected in only two of the 25 courses. Third, in half of the courses the density matrix formalism is also implemented to represent single-qubit states. Although the Dirac/Bra-ket formalism is implemented in all courses, in 23 of the 25 courses, at least one additional symbolic description is also included to teach characteristics of single-qubit states. In contrast to the focus on a few less symbolic representations, the results demonstrate that a substantial number of graphical representations are used



(a) Effect of the qubit characteristics most relevant in the course on the incorporated representations of single-qubit states.



(b) Effect of the qubit characteristics most relevant in the course on the incorporated representations of multi-qubit states.

Figure 7 Frequency of reported representational combinations depending on the courses' most relevant qubit characteristics stated by the participating educators for (a) single-qubit states and (b) multi-qubit states

in QIST university courses. In more than half of the courses, single-qubit states are presented either by the 3D Bloch sphere or the geometric vector representations (see Fig. 4). A possible explanation for the extensive use of the geometric vector representation is its intrinsic simplicity. This is achieved through the omission of complex numbers from the representation and an emphasis on only real coefficients. There is a notable similarity to photon polarisation and it can be employed to illustrate fundamental quantum concepts such as measurements. However, the Bloch sphere may be less accessible initially due to the doubling of the polar angle and the three-dimensional representation. Nevertheless, it is a bijective representation of single-qubit states and is particularly effective for presenting time evolution, including single-qubit quantum gates as rotations around axes. Besides these two graphical representations, nine more graphical representations have been mentioned to be used in the courses, most of them only in individual courses of three or less.

The three symbolic representations, Dirac/Bra-ket notation, column/row vector, and density matrix formalism, not only dominate the representation of single-qubit states but are also the predominant ones used to represent multi-qubit states (see Fig. 5). The incorporation of graphical representations decreases compared to single-qubit states, but at the same time, the variety of relevant representations increases. One reason for this might be the limited applicability of some common representations. A striking example is the Bloch sphere. When presenting single-qubit states, in 18 of the 25 courses the 3D Bloch sphere is used, and the 2D Bloch sphere in nearly half of the courses. In the case of multi-qubit states the Bloch sphere, regardless of whether 3D or 2D, reaches its limits. In order to keep the representational basis, a multitude of extensions and modifications are in use. Besides multiple 3D or 2D Bloch spheres or multiple points on one Bloch sphere, related graphical representations include the Bloch hypersphere and the macroscopic Bloch sphere. In doing so, students have to further develop available representation competence in order to understand multi-qubit states content.

Comparing the representations used to educate characteristics of single- and multi-qubit states shows not only differences in the individual representations, but also in the representational combinations. In the context of single-qubit states, all educators stated that they incorporate MERs in their courses. In addition, all specified combinations are heterogeneous combinations of at least one symbolic and graphical representation. However, for multi-qubit systems, the number of incorporated representations decreases significantly with a medium effect ($t(24) = 3.892, p < .001^{***}, d = .766$). The results indicate that with progress in content from single- to multi-qubit states, the incorporation of graphical representations recedes, and educators focus on the use of symbolic descriptions. This fact is also supported by the answers of 6 of the 25 participants that stated to solely rely on symbolic representations, neglecting graphical representations of multi-qubit states completely. One reason for the more prevalent use of only symbolic external representations for multi-qubit states could lie in the limited applicability of many graphical external representations. Although the use of symbolic external representations, such as Dirac/Bra-ket notation, requires a significant investment of effort for a competent understanding of the representation [67], developing an appropriate representational competence with regard to these generalisable representations can facilitate a deeper understanding of quantum physics and technologies [49]. The extension to more complex topics, including multi-qubit systems, can be considered a relatively straightforward endeavour, given that these representations have already been introduced in the context of single-qubit systems. Although graphical external representations are used extensively in the context of single-qubit representations, they are limited in their applicability and their adequate presentation of quantum physics content [45–47]. This is frequently due to the fact that they are founded upon classical analogies, which are unable to adequately represent more advanced quantum properties. This may serve to elucidate the prevalence of symbolic external representations, such as Dirac/Bra-ket notation, in the context of multi-qubit systems. A second factor contributing for the subordinate role of graphical external representations in the context of multi-qubit systems may be the amount of time spent with multi-qubit states in these courses. If multi-qubit states are only discussed briefly, educators might not find the time to introduce various representations. In contrast, the density matrix formalism is introduced by some educators only in the context of multi-qubit states. This may be attributed to its efficacy in describing even complex multi-qubit states and pertinent characteristics, such as entanglement. Nevertheless, given its prevalence in the context of multi- rather than single-qubit states, the density matrix formalism appears to present a more substantial barrier to entry than the other symbolic representations.

In line with our second research question (How do the incorporated representations differ depending on the course characteristics?) we investigated possible influencing factors for the representational use in QIST university courses. In doing so, we first focus on the CF domains of the courses [23], and the associated learner skills, taken from [25]. Due to the interdisciplinarity of QIST, it was assumed that QIST university courses also differ in their use of representations. Although the underlying data can be assigned to a diverse set of all CF domains and learner skills, the types of representational combination used are independent of both course characteristics. For all of the seven CF domains, heterogeneous combinations of symbolic and graphical representations clearly predominate solely symbolic combinations (see Figs. 6a and 6b).

Furthermore, we took into account the most relevant qubit characteristics within the courses as indicated by the educators. Although there were no indications of an influencing effect of the general course orientation on the representations used in QIST university courses, taking into account the specific relevant content aspects allows for a more precise analysis of possible effects. Figure 7b indicates that the focus on only symbolic representations is predominantly present with respect to the characteristics of entanglement and qubit operations. When courses focused on entanglement as one relevant characteristic of multi-qubit systems, nearly one third of the educators stated to solely rely on symbolic descriptions. In those courses that focused on qubit operations, even two of three educators stated to use only symbolic representations to describe multi-qubit states. Nevertheless, given the limited number of only three participants who have included qubit operations in their courses, this aspect necessitates further research. Although university courses in QIST vary in their competence and skill orientation, they are based on a similar use of representational combinations. However, differences can be found on a more precise content level, taking into account the relevant qubit characteristics in the courses. The results of our survey indicate the existence of untapped possibilities in the use of heterogeneous combinations, especially for courses in the context of entanglement and qubit operations.

In interpreting the results, it is important to consider the potential limitations that can affect the generalisability of the findings. First, the study does not elucidate the manner in which the representations stated were provided and combined. For example, in accordance with the principles of temporal and spatial contiguity, external representations should be presented in close temporal and spatial relation. However, at this stage, it is not possible to determine to what extent external representations have been used within the courses in accordance with these principles. For instance, it is possible that an educator may introduce the Bloch sphere in the initial lecture of a semester and then focus solely on the Dirac/Bra-ket notation in further lectures on distinct content without making connections to the Bloch sphere. This approach may not be conducive to learning with MERs. Consequently, further research is required to gain insight into the question of whether the stated external representations have been used effectively.

Second, the data does not allow any conclusions to be drawn regarding the extent to which specific representations were used in the individual categories of competences or learner skills. Participants were asked to provide information on course characteristics and the representations used in a pooled form. Consequently, it is not possible to resolve any possible differences in the representations used between the specified course characteristics. Furthermore, we do not have any information on the frequency of active use of each representation. It is possible that the focus will still be on a single representation, although further representations are addressed occasionally in the course. It would be beneficial for future research to concentrate on the extent to which the individual relevant representations are utilised.

Third, the study aimed to investigate the integration of MERs in QIST courses. More research is needed to investigate the effectiveness or ineffectiveness of the relevant combinations of representations. Especially the focus on symbolic representations of multi-qubit provides a starting point for future research investigating the efficacy of merely homogeneous combinations of symbolic representations on students' learning outcomes. To gain insight into the learning effectiveness of common combinations of MERs, future studies should investigate the effect of the respective combinations on relevant learner variables, such as performance and cognitive load.

Fourth, the participating educators were solely from Germany and Austria. Local characteristics may influence the use of representations, which may limit the extent to which the results can be transferred to courses in other countries. However, when focussing on Germany and Austria, participants covered a wide range of different locations, disciplines, and achieved academic levels of the students, supporting the comprehensiveness of the underlying data. To enhance the generalisability of the findings, it would be advantageous to extend the data set to encompass courses from a range of international academic institutions.

Fifth, this study concentrated on higher education courses as a significant element in training qualified workers in the field of QIST. When considering different educational levels, it is important to recognise that courses may vary not only in terms of their depth of content, but also in their use of representations, due to the differing characteristics of the learners. More research is needed to assess the extent to which the results can be applied to lower levels of education. In light of the aforementioned limitations and potential points of connection, the study can serve as a foundation for future research in order to support QIST educators in the effective use of relevant MERs and facilitate the learning of QIST content.

Sixth, especially in the case of the real-world experiment and the (interactive) simulation the external representations may not only be understood as visually perceived external representations, but may also influence students' learning through interaction effects. In the questionnaire, we did not impose any restrictions on the participants as to whether the representations were used in an interactive way or only for demonstration purposes. To uncover possible differences in the use of interactive representations in QIST courses, further research is needed.

6 Conclusion

To our knowledge, the present study is the first to provide comprehensive information on the use of MERs and the factors influencing their use in higher education courses in

QIST. The study indicates that, despite the considerable heterogeneity of the QIST field, the corresponding courses follow similar strategies in the presentation of relevant content. Regardless of the specific focus, combinations of symbolic and graphical representations are used in all courses to depict single-qubit states. However, the use of graphical representations for the presentation of multi-qubit states is lower than for single qubits. The exact reasons for this remain to be investigated. However, in line with current educational research, we propose a deliberate but more extensive inclusion of MERs not only in the context of single-qubit states, but especially in teaching the characteristics of multi-qubit states.

Our study indicates that content complexity is a significant factor in the design of educational programmes in the field of QIST and reveals an untapped potential for the integration of MERs, particularly in the context of multi-qubit systems. To facilitate the creation and implementation of beneficial courses in QIST, future research should focus on investigating the learning effectiveness of relevant representational combinations on students.

Abbreviations

QIST, Quantum Information Science & Technology; STEM, Science, Technology, Engineering, & Mathematics; DeFT, Design, Functions, and Tasks; MERs, Multiple External Representations; CF, European Competence Framework; ITPC, Integrated Theory of Text and Picture Comprehension; CTML, Cognitive Theory of Multimedia Learning.

Supplementary information

Supplementary information accompanies this paper at https://doi.org/10.1140/epjqt/s40507-025-00327-4.

Additional file 1. (PDF 1001 kB)

Author contributions

ER, JB, AA and AW planned the study and designed the survey materials used. ER collected the data. ER, JB, AA, LQ and AW analysed and discussed the data. ER, JB and AA wrote the first draft of the manuscript. LQ, SK, JK and AW reviewed the manuscript and provided feedback. SK, JK and AW supervised the study. All authors read and approved the final manuscript.

Funding

Open Access funding enabled and organized by Projekt DEAL. This work was supported by the German Federal Ministry of Education and Research (BMBF) as part of the QuanTUK project (FKZ13N15995).

Data Availability

The survey material and collected data can be accessed online at https://osf.io/98hjm/?view_only=7e247e9b4e064a21914d06159fa563d9.

Declarations

Competing interests

The authors declare no competing interests.

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Received: 1 October 2024 Accepted: 7 February 2025 Published online: 13 February 2025

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