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Article

On the Anomalous Dimension in QCD

Koichi Yamawaki

Kobayashi-Maskawa Institute for the Origin of Particles and the Universe (KMI), Nagoya University, Nagoya 464-8602, Japan; yamawaki@kmi.nagoya-u.ac.jp

Abstract: The anomalous dimension $\gamma_m = 1$ in the infrared region near the conformal edge in the broken phase of the large N_f QCD has been shown by the ladder Schwinger–Dyson equation and also by the lattice simulation for $N_f = 8$ and for $N_c = 3$. Recently, Zwicky made another independent argument (without referring to explicit dynamics) for the same result, $\gamma_m = 1$, by comparing the pion matrix element of the trace of the energy-momentum tensor $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = \left\langle \pi(p_2)|\theta_\mu^\mu|\pi(p_1)\right\rangle = 2M_\pi^2$ (up to trace anomaly) with the estimate of $\left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = 2M_\pi^2$ through the Feynman–Hellmann theorem combined with an assumption $M_\pi^2 \sim m_f$ characteristic of the broken phase. We show that this is not justified by the explicit evaluation of each matrix element based on the dilaton chiral perturbation theory (dChPT) : $\left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = 2M_\pi^2 + [(1-\gamma_m)M_\pi^2\cdot 2/(1+\gamma_m)] = 2M_\pi^2\cdot 2/(1+\gamma_m) \neq 2M_\pi^2$ in contradiction with his estimate, which is compared with $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = (1+\gamma_m)M_\pi^2 + [(1-\gamma_m)M_\pi^2] = 2M_\pi^2$ (both up to trace anomaly), where the terms in [ ] are from the σ (pseudo-dilaton) pole contribution. Thus, there is no constraint on γ_m when the σ pole contribution is treated consistently for both. We further show that the Feynman–Hellmann theorem is applied to the inside of the conformal window where dChPT is invalid and the σ pole contribution is absent, and with $M_\pi^2 \sim m_f^{2/(1+\gamma_m)}$ instead of $M_\pi^2 \sim m_f$, we have the same result as ours in the broken phase. A further comment related to dChPT is made on the decay width of $f_0(500)$ to $\pi\pi$ for $N_f = 2$. It is shown to be consistent with the reality, when $f_0(500)$ is regarded as a pseudo-NG boson with the non-perturbative trace anomaly dominance.

Keywords: non-perturbative anomalous dimension; QCD; σ pole; pseudo-dilaton; dilaton chiral perturbation theory; trace of energy-momentum tensor; non-perturbative trace anomaly; infrared fixed point; Feynman–Hellmann theorem; hyperscaling

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1. Introduction

The anomalous dimension $\gamma_m = 1$ together with the pseudo-dilaton σ as a candidate for the composite Higgs in the gauge theory with the spontaneously broken chiral/scale symmetry is an essence of the walking technicolor [1,2] (walking technicolor was also advocated in [3–5] without the notion of the anomalous dimension and the scale symmetry/dilaton). Such a theory is realized by the asymptotically free QCD-like $SU(N_c)$ gauge theories with a large number of flavors $N_f(\gg N_c)$ ("large N_f QCD") in the spontaneous chiral symmetry breaking phase [6]. It is inspired by the near scale-invariant/conformal structure around the Caswell–Banks–Zaks (CBZ) infrared (IR) fixed point α_* in the two-loop beta function at large N_f ($8 \le N_f \le 16$ for $N_c = 3$) [7,8], where the coupling $\alpha(\mu) \simeq \alpha_*$ in the infrared region $\mu < \Lambda_{\rm QCD}$ stays almost constant ("walking"), which must be larger than the critical coupling $\alpha_{\rm cr}$ for the spontaneous symmetry breaking to occur. In fact, $\gamma_m \simeq 1$ in the broken phase for the walking coupling has been shown by explicit dynamical calculations, the ladder Schwinger–Dyson equation [9] (and references therein) and also by the lattice simulation for QCD with $N_f = 8$ through an (approximate) hyperscaling fit [10] (and references therein) [11,12].

The ladder SD equation in such a theory, with almost constant coupling $\alpha(\mu) \simeq \alpha \gtrsim \alpha_{\rm cr}$ for $\mu < \Lambda_{\rm QCD}$ (with the intrinsic scale $\Lambda_{\rm QCD}$ acting as the UV cutoff Λ), which gives in the chiral limit the scale-dependence of the chiral condensate characterized by the dynamical mass of the fermion $m_D \ll \Lambda_{\rm QCD}$: $\langle\bar\psi\psi\rangle_{m_D} \sim m_D^3$ and $\langle\bar\psi\psi\rangle_{\Lambda_{\rm QCD}} = Z_m^{-1}\cdot\langle\bar\psi\psi\rangle_{m_D} \sim \Lambda_{\rm QCD} m_D^2$ with $Z_m^{-1} = (\Lambda_{\rm QCD}/\mu)^{\gamma_m(\alpha(\mu))}$ [13]; thus, the anomalous dimension $\gamma_m(\alpha(\mu)) \simeq 1$ for $\alpha(\mu) \simeq \alpha$ in the infrared region $m_D < \mu < \Lambda_{\rm QCD}$ (the perturbative IR fixed point α_* is washed out due to $m_D \neq 0$, since the fermions with m_D get decoupled from the loop at $\mu < m_D$). The chiral condensate also breaks the scale symmetry not only spontaneously but also explicitly due to the new scale $m_D \neq 0$, giving rise to the non-perturbative trace anomaly, which produces a pseudo-NG boson and a pseudo-dilaton σ, with the mass $m_\sigma = \mathcal{O}(m_D) \neq 0$ even in the chiral limit [9].This is contrasted to the UV region $\mu > \Lambda_{\rm QCD}$ of the same theory where the perturbative coupling runs asymptotically free in units of the intrinsic scale $\Lambda_{\rm QCD}$, which originated from the scale symmetry violation due to regularization (perturbative trace anomaly), with the perturbative anomalous dimension vanishing logarithmically: $\gamma_m(\mu) \sim \alpha(\mu) \sim 1/\ln(\mu^2/\Lambda_{\rm QCD}^2) \to 0$ for $\mu \gg \Lambda_{\rm QCD}$.

As for the lattice studies of $N_f = 8$ QCD, with the data in the region away from the chiral limit, the measured mass of all hadrons (except for π and σ) is dominated by the explicit chiral symmetry breaking, $M_H/2 \sim m_f^R(\mu = m_R)$, which obeys the hyperscaling $M_H/2 \sim m_f^R = Z_m^{-1}\cdot m_f^0 \sim (m_f^0)^{1/(1+\gamma_m)}$ [14,15], with $\gamma_m \simeq 1$, where $m_f^R = m_f^R(\mu = m_R)$ is the renormalized mass of the fermion at $\mu = m_R$ and m_f^0 is the bare mass (input mass) in the lattice action, and both are related as $m_f^R = Z_m^{-1} m_f^0$, with $Z_m^{-1} = (\Lambda/m_f^R)^{\gamma_m}$ and Λ regarded as the inverse lattice spacing.

Recently, Roman Zwicky [16] presented another independent argument (without referring to explicit dynamics) for the same result, $\gamma_m = 1$, near the conformal edge in the broken phase of the large N_f QCD. He evaluated

$$< \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1) >= 2M_\pi^2 \tag{1}$$

through the Feynman–Hellmann theorem combined with an additional assumption, $M_\pi^2 \sim m_f$. He further showed that the result coincides with the double use of the soft pion theorem. This was then compared with the standard generic evaluation of the matrix element $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = \left\langle \pi(p_2)|\beta(\alpha)/(4\alpha)\cdot G_{\mu\nu}^2 + (1+\gamma_m)\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = \left\langle \pi(p_2)|\theta_\mu^\mu|\pi(p_1)\right\rangle = 2M_\pi^2$, with an additional assumption of the IR fixed point (even in the broken phase and $M_\pi^2 \neq 0$) of ignoring the trace anomaly contribution. He then concluded that $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle = \left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1)\right\rangle$, i.e., $\gamma_m = 1$ (=γ_* under his assumption of an IR fixed point with vanishing trace anomaly).

In this paper, we show that the explicit evaluation of each matrix element based on the nonlinear realization Lagrangian of scale and chiral symmetries, namely the dilaton chiral perturbation theory (dChPT) [17,18], gives:

$$< \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1) >= 2M_\pi^2 + \left[\frac{2}{1+\gamma_m}\cdot(1-\gamma_m)M_\pi^2\right] = \frac{2}{1+\gamma_m}\cdot 2M_\pi^2, \tag{2}$$

$$< \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar\psi_i\psi_i|\pi(p_1) >= (1+\gamma_m)M_\pi^2 + \left[(1-\gamma_m)M_\pi^2\right] = 2M_\pi^2, \tag{3}$$

(both up to trace anomaly), where the terms in [ ] are from the σ pole contribution. Note that Equation (3) is consistent with the well-known generic result $< \pi(p_2)|\theta_\mu^\mu|\pi(p_1) >= 2M_\pi^2$ based on the form-factor argument only when including the σ pole contribution. Thus, including (or ignoring) the σ pole contribution for both matrix elements consistently, there is no constraint on γ_m in contrast to Zwicky's result [16]. Even including the trace anomaly, we will show that the result keeps the relation $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle = (1+\gamma_m)/2\cdot\left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle$, which is consistent with Equations (2) and (3).

The same result is also obtained based on the Feynman–Hellmann theorem, within the conformal window where dChPT is invalid and no σ pole contribution exists, with π now as a non-pseudo-NG boson having the mass to obey the hyperscaling, $M_\pi^2 \sim m_f^{2/(1+\gamma_m)}$, instead of the pseudo-NG boson case $M_\pi^2 \sim m_f$ in the broken phase:

$$\begin{aligned} < \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)> &= 2\frac{\partial}{\partial \ln m_f} < \pi(p_2)|\mathcal{H}|\pi(p_1)> \\ &= \frac{\partial}{\partial \ln m_f} 2E_\pi^2 = \frac{2}{1+\gamma_m}2M_\pi^2 \neq 2M_\pi^2, \end{aligned} \quad (4)$$

$$\begin{aligned} < \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)> &= (1+\gamma_m)\frac{\partial}{\partial \ln m_f} < \pi(p_2)|\mathcal{H}|\pi(p_1)> \\ &= \frac{\partial}{\partial \ln m_f}(1+\gamma_m)E_\pi^2 = 2M_\pi^2, \end{aligned} \quad (5)$$

which is the same as Equations (2) and (3) in the broken phase (up to the trace anomaly term). Thus, the result is independent of the phases, broken or conformal, as it should be. Actually, the Feynman–Hellmann theorem is insensitive to the spontaneous symmetry breaking, giving the same kinetic term form in M_π^2 independently of the phase, and the combined use of $M_\pi^2 \sim m_f$ characteristic to the broken phase is not justified, as it would result in $< \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)> = 2M_\pi^2$ and $< \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)> = (1+\gamma_m)M_\pi^2$, which is the same as the wrong results neglecting the σ pole contribution in Equations (2) and (3), still keeping the inequality between the two anyway. If the theorem were to be used in the broken phase, then all of the hadron masses including M_π should be regarded as a simple Coulombic bound state $M_H \sim 2m_f^{(R)} \sim m_f^{1/(1+\gamma_m)}$ as in the conformal phase, in which case the result would coincide with the correct one.

Incidentally, Zwicky unjustifiably identifies $M_\pi^2 \sim m_f$ in the broken phase as the hyperscaling with $\gamma_m = 1$ in the generic broken phase (including the deeply broken phase like $N_f = 2$) [16]. In fact, it was shown on the lattice [10] that for $N_f = 4$, generic hadron spectra (including F_π) other than M_π (and M_σ) do not obey the hyperscaling at all, and hence $M_\pi^2 \sim m_f$ cannot be understood as hyperscaling. For $N_f = 8$ near the conformal window, on the other hand, spectra other than M_π (and M_σ) do obey the hyperscaling with $\gamma_m \simeq 1$, while M_π (M_σ as well) does only non-universally with $\gamma_m \sim 0.6$ due to m_f-dependence away from the chiral limit as a pseudo-NG boson, which is different from the others obeying hyperscaling.

As to the double soft pion theorem for $< \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)>$, which he claims [16] gives an equivalent result to that from the Feynman–Hellmann theorem combined with his assumption $M_\pi^2 \sim m_f$, it ignores the σ pole contribution, i.e., the term in [ ] of Equation (2). Actually, the same double soft pion theorem applied consistently for both matrix elements would give $\left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle = 2M_\pi^2$ and $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i| \pi(p_1)\right\rangle = (1+\gamma_m)M_\pi^2$; thus, again there is no constraint on the value of γ_m (or γ_*). Inclusion of the σ pole contribution for both gives the correct results (up to the trace anomaly): $\left\langle \pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle = 2M_\pi^2 + [(1-\gamma_m)M_\pi^2\cdot 2/(1+\gamma_m)] = 2M_\pi^2\cdot 2/(1+\gamma_m) \neq 2M_\pi^2$, while $\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle = (1+\gamma_m)M_\pi^2 + [(1-\gamma_m)M_\pi^2] = 2M_\pi^2$, to be consistent with the generic form-factor argument, where the term in [ ] of each result is from the σ (pseudo-dilaton) pole contribution. Thus, there is no constraint on γ_m when the σ pole contribution is consistently included/ignored for both.

Also, we shall make a comment related to dChPT on the decay width of $f_0(500)$ to $\pi\pi$ for $N_f = 2$, where the spontaneously broken scale symmetry is also broken explicitly by the non-perturbative trace anomaly and the quark mass. It is shown to be consistent with the reality, when both π and $f_0(500)$ are regarded as pseudo-NG bosons, based on this dChPT, with the non-perturbative trace anomaly dominance. This is contrasted with the decay width evaluated by the low-energy theorem for the scale symmetry, which regards $f_0(500)$ as a pseudo-NG boson but π as a matter field and not as a pseudo-NG boson and is 50 times smaller than the real data, which is a long-standing problem and has long been a puzzle when $f_0(500)$ is regarded as a pseudo-dilaton σ.

2. Nonlinear Realization of the Chiral and Scale Symmetries

Let us start with the basic formula based on the Ward–Takahashi (WT) identity for N_f QCD (with the same mass m_f for N_f flavors) for θ^μ_μ [9]:

$$\theta^\mu_\mu = \partial_\mu D^\mu = \frac{\beta^{(\mathrm{NP})}(\alpha)}{4\alpha} G^2_{\mu\nu} + (1+\gamma_m) \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i, \tag{6}$$

with ψ_i for a single flavor within the degenerate N_f flavors, and where $\frac{\beta^{(\mathrm{NP})}}{4\alpha} G^2_{\mu\nu}$ is the non-perturbative trace anomaly, $\left\langle 0 | \frac{\beta^{(\mathrm{NP})}(\alpha)}{4\alpha} G^2_{\mu\nu} | 0 \right\rangle = -\mathcal{O}(\Lambda^4_{\mathrm{IR}})$ (up to factor $N_f N_c$), due to the dynamically generated IR mass scale Λ_{IR} (or dynamical quark mass $m_D \sim M_\rho/2 \sim M_N/3$) in the chirally broken phase with $\langle 0|(\bar{\psi}\psi)_R|0)\rangle = -\mathcal{O}(\Lambda^3_{\mathrm{IR}})$. Here, the perturbative trace anomaly $< \frac{\beta^{(\mathrm{perturbative})}}{4\alpha} G^2_{\mu\nu} >= -\mathcal{O}(\Lambda^4_{\mathrm{QCD}})$ due to the regularization, with the UV scale Λ_{QCD} characterizing the asymptotically-free running of the perturbative coupling, is irrelevant to the IR physics and is thus subtracted out from Equation (6).

A concrete picture of this is given in the ladder SD equation, in the broken phase near the conformal window with $\alpha_* \gtrsim \alpha_{\mathrm{cr}}$, where α_{cr} is the critical coupling for the condensate to be generated. The dynamically generated fermion mass m_D takes the form of the essential singularity [13] of BKT (Berezinsky–Kosterlitz–Thouless) type: $m_D \sim \Lambda \cdot \exp[-a/(\alpha - \alpha_{\mathrm{cr}})^r] \to 0 \quad (a, r > 0)$, for $\alpha (\lesssim \alpha_*) \searrow \alpha_{\mathrm{cr}}$, where Λ is the UV scale to be identified with the intrinsic scale Λ_{QCD}. Due to $m_D \neq 0$, the perturbative IR fixed point α_* is washed out in contradiction to Zwicky's assumption. The coupling for $\alpha > \alpha_{\mathrm{cr}}$ runs non-perturbatively due to this mass generation, with $\beta^{(\mathrm{NP})}(\alpha)$ now having a UV fixed point at α_{cr} instead of an IR fixed point: the ladder SD equation gives $a = \pi, r = 1/2, \alpha_{\mathrm{cr}} = \pi/4$ for $N_c = 3$ near the conformal window $\alpha_* \gtrsim \alpha_{\mathrm{cr}}$ and $\beta^{(\mathrm{NP})}(\alpha) = \frac{\partial \alpha(\Lambda)}{\partial \ln \Lambda} = -\frac{2\alpha_{\mathrm{cr}}}{\pi} \left(\frac{\alpha}{\alpha_{\mathrm{cr}}} - 1 \right)^{3/2}$, which vanishes at $\alpha \searrow \alpha_{\mathrm{cr}}$, while $\left\langle 0 | G^2_{\mu\nu} | 0 \right\rangle \sim \left(\frac{\alpha}{\alpha_{\mathrm{cr}}} - 1 \right)^{-3/2} m^4_D$ blows up, to precisely cancel the vanishing $\beta^{(\mathrm{NP})}(\alpha)$, resulting in $< 0| \frac{\beta^{(\mathrm{NP})}(\alpha)}{4\alpha} G^2_{\mu\nu} |0 >= -\mathcal{O}(m^4_D)$. See, e.g., Ref. [9].

From the pole-dominated WT identity for Equation (6) we have:

$$\begin{aligned} M^2_\sigma F^2_\sigma &= i\mathcal{F.T.} \langle 0|T(\partial_\mu D^\mu(x) \cdot \partial_\mu D^\mu(0))|0\rangle\big|_{q_\mu \to 0} = \langle 0|[-iQ_D, \partial_\mu D^\mu(0)]|0\rangle \\ &= \langle 0| - \delta(\partial_\mu D^\mu(0))]|0\rangle \\ &= 4 \cdot \langle 0| - \frac{\beta^{(\mathrm{NP})}(\alpha)}{4\alpha} G^2_{\mu\nu} |0\rangle + (3-\gamma_m) \cdot \langle 0| - (1+\gamma_m) \sum_{i=1}^{N_f} m_f \bar{\psi}^i \psi^i |0\rangle, \end{aligned} \tag{7}$$

with the scale dimension $d_{G_{\mu\nu}^2} = 4$, $d_{\bar{\psi}\psi} = 3 - \gamma_m$. Similarly, the pole-dominated WT identity for the non-singlet axial-vector current $A_\mu^\alpha (\alpha = 1, 2, 3)$ for each doublet $\psi^i (i = 1, 2)$ gives the Gell–Mann–Oakes–Renner (GMOR) relation:

$$F_\pi^2 M_\pi^2 \cdot \delta^{\alpha\beta} = \langle 0|[-iQ_5^\alpha, \partial^\mu A_\mu^\beta(0)]|0\rangle = \langle 0| - \sum_{i=1}^{2} m_f \bar{\psi}^i \psi^i |0\rangle \cdot \delta^{\alpha\beta}, \quad (8)$$

$$\text{i.e.,} \quad \langle 0| - \sum_{i=1}^{N_f} m_f \bar{\psi}^i \psi^i |0\rangle = \frac{N_f}{2} F_\pi^2 M_\pi^2. \quad (9)$$

Equations (7) and (8) (usually derived using the soft pion theorem in the broken phase) are simply based on the pole dominance, and hence valid in both the broken and the conformal phases. Then, we have [17]:

$$M_\sigma^2 = m_\sigma^2 + (3 - \gamma_m)(1 + \gamma_m) \frac{\frac{N_f}{2} F_\pi^2 M_\pi^2}{F_\sigma^2}, \quad m_\sigma^2 \equiv \frac{1}{F_\sigma^2} \langle 0| - \frac{\beta^{(\mathrm{NP})}(\alpha)}{\alpha} G_{\mu\nu}^2 |0\rangle, \quad (10)$$

independently of the phases. Any effective theory should reproduce Equation (10) for the σ mass M_σ^2.

As such, in the broken phase we use the dilaton ChPT (dChPT) Lagrangian [17] corresponding to Equation (6):

$$\mathcal{L} = \mathcal{L}_{\mathrm{inv}} + \mathcal{L}_{\mathrm{hard}} + \mathcal{L}_{\mathrm{soft}}, \quad (11)$$

$$\mathcal{L}_{\mathrm{inv}} = \frac{F_\sigma^2}{2} (\partial_\mu \chi)^2 + \frac{F_\pi^2}{4} \chi^2 \mathrm{tr}[\partial_\mu U^\dagger \partial^\mu U], \quad (12)$$

$$\mathcal{L}_{\mathrm{hard}} = -\frac{F_\sigma^2}{4} m_\sigma^2 \chi^4 \left(\ln \frac{\chi}{S} - \frac{1}{4} \right), \quad (13)$$

$$\begin{aligned} \mathcal{L}_{\mathrm{soft}} &= \mathcal{L}_{\mathrm{soft}}^{(1)} + \mathcal{L}_{\mathrm{soft}}^{(2)}, \\ \mathcal{L}_{\mathrm{soft}}^{(1)} &= \frac{F_\pi^2}{4} \left(\frac{\chi}{S} \right)^{3-\gamma_m} \cdot S^4 \mathrm{tr}[\mathcal{M}^\dagger U + U^\dagger \mathcal{M}], \\ \mathcal{L}_{\mathrm{soft}}^{(2)} &= -\frac{(3-\gamma_m) F_\pi^2}{8} \chi^4 \cdot \mathrm{tr}\mathcal{M}, \end{aligned} \quad (14)$$

where $U = e^{2i\pi/F_\pi}$, $\chi = e^{\sigma/F_\sigma}$, and $\mathcal{M}$ and S are spurion fields introduced so as to incorporate the explicit breaking effects of the chiral and scale symmetry, respectively. Under the chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry, these building blocks transform as $U \to g_L \cdot U \cdot g_R^\dagger$, $\mathcal{M} \to g_L \cdot \mathcal{M} \cdot g_R^\dagger$, $\chi \to \chi$ and $S \to S$ with $g_{L,R} \in SU(N_f)_{L,R}$, while under the scale symmetry, they are infinitesimally transformed as $\delta U(x) = x_\nu \partial^\nu U(x)$, $\delta \mathcal{M}(x) = x_\nu \partial^\nu \mathcal{M}(x)$, $\delta\chi(x) = (1 + x_\nu \partial^\nu)\chi(x)$, and $\delta S = (1 + x_\nu \partial^\nu) S(x)$, with the vacuum expectation values of the spurion fields $\mathcal{M}$ and S, $\langle \mathcal{M} \rangle = M_\pi^2 \times \mathbf{1}_{N_f \times N_f}$ and $\langle S \rangle = 1$.

This effective Lagrangian is the same as that of Ref. [18] except for Equation (13),

$$\mathcal{L}_{\mathrm{hard}} = -\frac{1}{16} F_\sigma^2 m_\sigma^2 - \frac{1}{2} m_\sigma^2 \sigma^2 + \cdots, \quad (15)$$

which is absent in Ref. [18] and gives the σ mass in the chiral limit due to the trace anomaly:

$$\begin{aligned} F_\sigma m_\sigma^2 &= <0|\theta_\mu^\mu|_{m_f=0}|\sigma> = <0|\frac{\beta^{(\mathrm{NP})}(\alpha)}{4\alpha} G_{\mu\nu}^2 |\sigma> = <0| - \delta\mathcal{L}_{\mathrm{hard}} |\sigma> = <0| F_\sigma^2 m_\sigma^2 \chi^4 \ln\chi |\sigma> \\ &= \frac{4}{F_\sigma} <0| - \theta_\mu^\mu|_{m_f=0}|0> = \frac{1}{F_\sigma} <0| - \frac{\beta^{(\mathrm{NP})}(\alpha)}{\alpha} G_{\mu\nu}^2 |0> = \frac{16}{F_\sigma} <0|\mathcal{L}_{\mathrm{hard}}|0>, \end{aligned} \quad (16)$$

to be compared with Equation (10). On the other hand, $\mathcal{L}_{\rm soft}$ has two terms: $\mathcal{L}^{(1)}_{\rm soft}$ corresponds to the fermion mass term [18]:

$$\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i = -\mathcal{L}^{(1)}_{\rm soft} = -\frac{F_\pi^2}{4}(\chi)^{3-\gamma_m}\cdot {\rm tr}[\mathcal{M}^\dagger U + U^\dagger \mathcal{M}]$$
$$= \left[1+(3-\gamma_m)\frac{\sigma}{F_\sigma}+\frac{1}{2}(3-\gamma_m)^2\frac{\sigma^2}{F_\sigma^2}\right]\left(-\frac{N_f}{2}F_\pi^2M_\pi^2+\frac{M_\pi^2}{2}\pi^a\pi^a\right)+\cdots, \quad (17)$$

which correctly reproduces the π mass term as in the standard ChPT, $\frac{M_\pi^2}{2}\pi^a\pi^a$, and the GMOR relation Equation (8), but would imply that σ is a tachyon that can destabilize the vacuum (in the case of $m_\sigma^2=0$): $M_\sigma^2 = -(3-\gamma_m)^2\frac{N_f}{2}F_\pi^2M_\pi^2/F_\sigma^2 < 0$. Then, $\mathcal{L}^{(2)}_{\rm soft}$ was introduced in Ref. [18] to prevent σ from being a tachyon:

$$-\mathcal{L}^{(2)}_{\rm soft} = \frac{(3-\gamma_m)F_\pi^2}{8}\chi^4\cdot{\rm tr}\mathcal{M} = \frac{3-\gamma_m}{4}\left(1+4\frac{\sigma}{F_\sigma}+8\frac{\sigma^2}{F_\sigma^2}\right)\frac{N_f}{2}F_\pi^2M_\pi^2+\cdots, \quad (18)$$

which is essential for the correct σ mass term $-M_\sigma^2\sigma^2/2$ (in addition to $-m_\sigma^2\sigma^2/2$ in Equation (15)) given as a combination of the two terms of $\mathcal{L}_{\rm soft}$:

$$M_\sigma^2 = m_\sigma^2+\left[-(3-\gamma_m)^2+4(3-\gamma_m)\right]\frac{\frac{N_f}{2}F_\pi^2M_\pi^2}{F_\sigma^2} = m_\sigma^2+(3-\gamma_m)(1+\gamma_m)\frac{\frac{N_f}{2}F_\pi^2M_\pi^2}{F_\sigma^2}, \quad (19)$$

thus correctly reproducing the σ mass formula derived using the WT identity Equation (10).

The same mass formula is also obtained through the trace of the energy-momentum tensor $<0|\theta_\mu^\mu|\sigma>= M_\sigma^2F_\sigma$, with $\theta_\mu^\mu=-\delta\mathcal{L}$:

$$(M_\sigma^2-m_\sigma^2)F_\sigma = <0|(1+\gamma_m)\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i|\sigma>=(1+\gamma_m)<0|-\delta(\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i)|0>$$
$$= <0|-\delta\mathcal{L}_{\rm soft}|\sigma>=(1+\gamma_m)\cdot(3-\gamma_m)\frac{\frac{N_f}{2}F_\pi M_\pi^2}{F_\sigma^2}, \quad (20)$$

where $<0|-\delta(\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i)|0>=(3-\gamma_m)<0|-\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i)|0>$ with Equation (9), while:

$$-\delta\mathcal{L}_{\rm soft} = -(\delta\mathcal{L}^{(1)}_{\rm soft}+\delta\mathcal{L}^{(2)}_{\rm soft})$$
$$= (3-\gamma_m)\chi^{3-\gamma_m}\left[-\frac{N_f}{2}F_\pi^2M_\pi^2+\frac{M_\pi^2}{2}\pi^a\pi^a\right]+(3-\gamma_m)\chi^4\frac{N_f}{2}F_\pi^2M_\pi^2+\cdots$$
$$= (3-\gamma_m)[-(3-\gamma_m)+4]\frac{N_f}{2}F_\pi^2M_\pi^2\frac{\sigma}{F_\sigma}+(3-\gamma_m)\frac{M_\pi^2}{2}\pi^a\pi^a+\cdots, \quad (21)$$

both lines in Equation (20) thus giving the same result. This is compared with $\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i$ in Equation (17) having no contribution of $\mathcal{L}^{(2)}_{\rm soft}$:

$$<0|2\cdot\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i|\sigma> = 2\cdot<0|-\delta(\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i)|0>\frac{1}{F_\sigma}=2\cdot<0|\delta\mathcal{L}^{(1)}_{\rm soft}|0>\frac{1}{F_\sigma}$$
$$= 2\cdot(3-\gamma_m)\frac{\frac{N_f}{2}F_\pi^2M_\pi^2}{F_\sigma}=\frac{2}{1+\gamma_m}\cdot(M_\sigma^2-m_\sigma^2)F_\sigma. \quad (22)$$

Equations (20) and (22) are crucial to later compare the σ pole contribution to the $<\pi|(1+\gamma_m)\cdot\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i|\pi>$ and $<\pi|2\cdot\sum_{i=1}^{N_f}m_f\bar{\psi}_i\psi_i|\pi>$, respectively.

The result of Equations (19) and (20) coincides with that of Ref. [18] for $m_\sigma^2 = 0$ (thus, Ref. [18] implicitly assumes that $\frac{\beta(\alpha)}{4\alpha} G_{\mu\nu}^2 = 0$, or $m_\sigma^2 = 0$, in the broken phase, which is in contradiction to their own calculation by the ladder SD equation, which shows no massless dilaton in the chiral limit; see also Ref. [9]). Zwicky also assumes $m_\sigma^2 = 0$. He evaluated $\left\langle \pi(p_2) | (1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i | \pi(p_1) \right\rangle$ through the form-factor argument on $\left\langle \pi(p_2) | \theta_\mu^\mu | \pi(p_1) \right\rangle = \left\langle \pi(p_2) | \frac{\beta(\alpha)}{4\alpha} G_{\mu\nu}^2 + (1+\gamma_m) \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i | \pi(p_1) \right\rangle$, which is known to give $2M_\pi^2$ at $q^2 = (p_1 - p_2)^2 \to 0$. Then, he needed the assumption of the existence of the IR fixed point (in the broken phase with $M_\pi^2 \neq 0$) in order to drop out the contribution of $\left\langle \pi(p_2) | \frac{\beta(\alpha)}{4\alpha} G_{\mu\nu}^2 | \pi(p_1) \right\rangle (\propto m_\sigma^2)$ to conclude $\left\langle \pi(p_2) | (1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i | \pi(p_1) \right\rangle = 2M_\pi^2$.

However, this term is actually irrelevant to the discussion here to directly compute $\left\langle \pi(p_2) | (1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i | \pi(p_1) \right\rangle$ using dChPT without referring to $\left\langle \pi(p_2) | \theta_\mu^\mu | \pi(p_1) \right\rangle$, and then compare it with $\left\langle \pi(p_2) | 2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i | \pi(p_1) \right\rangle$ computed on the same footing based on the dChPT. At any rate, our result with Equations (20) and (22) obviously shows the same conclusion even including the trace anomaly, $m_\sigma^2 \neq 0$. Hence, the following discussion is irrelevant to Zwicky's assumption that there exists the IR fixed point, $\frac{\beta(\alpha)}{4\alpha} G_{\mu\nu}^2 = 0$, even in the broken phase with the condensate $\langle \bar{\psi}\psi \rangle|_{m_f=0} \neq 0$ (its mass scale is the explicit as well as spontaneous breaking of the scale symmetry) and $M_\pi^2 \neq 0$ (the explicit breaking of the scale symmetry as well as chiral symmetry). At any rate, such an assumption itself has been shown to be in contradiction with the explicit calculation in the ladder SD equation (see the second paragraph in Section 2).

3. Evaluation of Matrix Element between Pion States on the Mass Shell

Before evaluation by the dChPT Lagrangian, we first see the generic argument for $\langle \pi(p_2) | \theta^{\mu\nu}(x^\mu = 0) | \pi(p_1) \rangle$ based on the form factor:

$$\begin{aligned} \langle \pi(p_2) | \theta^{\mu\nu} | \pi(p_1) \rangle &= 2P^\mu P^\nu F(q^2) + (g^{\mu\nu} q^2 - q^\mu q^\nu) G(q^2), \\ P^\mu = (p_1^\mu + p_2^\mu)/2,\ q^\mu &= p_2^\mu - p_1^\mu,\ F(0) = 1,\ G(q^2)|_{M_\sigma^2 \neq 0} \text{ regular at } q^2 \to 0 \end{aligned} \tag{23}$$

$$\begin{aligned} \left\langle \pi(p_2) | \theta_\mu^\mu | \pi(p_1) \right\rangle &= 2M_\pi^2 F(q^2) + q^2 \left[3G(q^2) - F(q^2)/2 \right] \\ &\to 2M_\pi^2 \quad \text{at} \quad q^2 \to 0. \end{aligned} \tag{24}$$

It should be noted that in this formula, the σ pole contribution is invisible at $q^2 \to 0$ and the result is valid independently of the phases, either the broken phase or the conformal phase.

Now, we evaluate the same quantity through the dChPT Lagrangian Equation (11) for the broken phase [18]:

$$\begin{aligned} \left\langle \pi(p_2) | \theta_\mu^\mu | \pi(p_1) \right\rangle &= 4M_\pi^2 - 2p_1 \cdot p_2 + \left\langle 0 | \theta_\mu^\mu | \sigma(q) \right\rangle \tfrac{1}{M_\sigma^2 - q^2} G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2) \\ &= 2M_\pi^2 + q^2 + \tfrac{q^2}{M_\sigma^2 - q^2} \left[(1-\gamma_m) M_\pi^2 + q^2 \right], \end{aligned} \tag{25}$$

where $\left\langle 0 | \theta_\mu^\mu | \sigma(q) \right\rangle = F_\sigma q^2$ and

$$F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2) = (3-\gamma_m) M_\pi^2 - 2p_1 \cdot p_2 = (1-\gamma_m) M_\pi^2 + q^2, \tag{26}$$

with the $\sigma - \pi - \pi$ vertex $G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2)$ given by $F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2) = (1-\gamma_m) M_\pi^2 + q^2$ as a sum of $(3-\gamma_m) M_\pi^2$ from the explicit breaking term in Equation (14) and $-2p_1 \cdot p_2 = q^2 - 2M_\pi^2$ from the pion kinetic term in Equation (12) (Equation (26) was also obtained in Ref. [19] in a different context). Equation (25) is consistent with the form-factor argument Equation (24):

$$\left\langle \pi(p_2)|\theta^\mu_\mu|\pi(p_1)\right\rangle \to 2M_\pi^2 \quad \text{at } q^2 \to 0, \tag{27}$$

again with the σ pole contribution being invisible at $q^2 \to 0$. Note that this implies that the trace anomaly term giving $m_\sigma^2 \neq 0$ does not contribute to $\left\langle \pi(p_2)|\theta^\mu_\mu|\pi(p_1)\right\rangle$ at $q^2 \to 0$, even without the assumption of the IR fixed point.

On the other hand, we have:

$$\begin{aligned}
&\left\langle \pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle = \langle\pi(p_2)|-\delta\mathcal{L}_{\rm inv}-\delta\mathcal{L}_{\rm soft}|\pi(p_1)\rangle \\
&= [4-(3-\gamma_m)]M_\pi^2 + <0|-\delta\mathcal{L}_{\rm soft}|\sigma> \frac{1}{M_\sigma^2-q^2}G_{\sigma\pi\pi} \\
&= (1+\gamma_m)M_\pi^2 + \frac{M_\sigma^2-m_\sigma^2}{M_\sigma^2-q^2}\left[(1-\gamma_m)M_\pi^2+q^2\right] \\
&= \left[2M_\pi^2+q^2+\frac{q^2}{M_\sigma^2-q^2}\left[(1-\gamma_m)M_\pi^2+q^2\right]\right] - \frac{m_\sigma^2}{M_\sigma^2-q^2}\left[(1-\gamma_m)M_\pi^2+q^2\right],
\end{aligned} \tag{28}$$

where use has been made of Equations (20), (21) and (26). Note that the σ pole term of Equation (28) is from the pole of σ in the scalar density $\bar{\psi}\psi$ coupled to two π's, with the $\sigma-\pi-\pi$ coupling $F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2)$ in Equation (26). Equation (28) is identical to Equation (25), with the last term being precisely the same as the σ pole contribution to the trace anomaly, Equation (16):

$$\begin{aligned}
<\pi(p_2)|\frac{\beta^{(\rm NP)}(\alpha)}{4\alpha}G_{\mu\nu}^2|\pi(p_1)> &= \frac{<0|\frac{\beta^{(\rm NP)}(\alpha)}{4\alpha}G_{\mu\nu}^2|\sigma>}{M_\sigma^2-q^2}G_{\sigma\pi\pi}(q^2,M_\pi^2,M_\pi^2) \\
&= \frac{m_\sigma^2}{M_\sigma^2-q^2}\left[(1-\gamma_m)M_\pi^2+q^2\right],
\end{aligned} \tag{29}$$

to be cancelled by each other for $<\pi(p_2)|\theta^\mu_\mu|\pi(p_1)>$ in Equation (25). At $q^2 \to 0$ we have:

$$<\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)>= 2M_\pi^2 - \frac{m_\sigma^2}{M_\sigma^2}(1-\gamma_m)M_\pi^2 \quad \text{at } q^2 \to 0\,. \tag{30}$$

Now to the matrix element $<\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)>$. From Equation (22), we have:

$$\begin{aligned}
<\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)> &= 2\cdot<\pi(p_2)|-\mathcal{L}_{\rm soft}^{(1)}|\pi(p_1)> \\
&= 2M_\pi^2 + 2<0|\delta\mathcal{L}_{\rm soft}^{(1)}|0> \frac{1}{M_\sigma^2-q^2}G_{\sigma\pi\pi} \\
&= 2M_\pi^2 + \frac{2}{1+\gamma_m}\frac{M_\sigma^2-m_\sigma^2}{M_\sigma^2-q^2}\left[(1-\gamma_m)M_\pi^2+q^2\right] \\
&\longrightarrow_{q^2\to 0} \frac{2}{1+\gamma_m}\cdot 2M_\pi^2 - \frac{2}{1+\gamma_m}\frac{m_\sigma^2}{M_\sigma^2}(1-\gamma_m)M_\pi^2,
\end{aligned} \tag{31}$$

where Equation (22) was used. By contrast, if we used the conventional non-scale-invariant ChPT Lagrangian with Equation (17) replaced by one ignoring the σ terms, then the RHS would be just $2M_\pi^2$, which is also obtained by the double soft pion theorem as claimed by Zwicky. See also the later discussion.

From Equations (30) and (31), we conclude:

$$
\begin{aligned}
< \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1) > \quad &= \quad \frac{2}{1+\gamma_m} < \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1) >, \\
&\neq \quad < \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1) >, \qquad (32)
\end{aligned}
$$

and thus there is no constraint on γ_m, in contradiction to Zwicky's claim [16], even including the trace anomaly $m_\sigma^2 \neq 0$, and hence it is independent of the IR fixed point argument.

The crucial point of the results is the contribution of the pole of σ, $(1-\gamma_m)M_\pi^2$ and $2\frac{1-\gamma_m}{1+\gamma_m}M_\pi^2$, without which we would erroneously conclude that $\left\langle \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i| \pi(p_1)\right\rangle = (1+\gamma_m)M_\pi^2$ (as emphasized in Ref. [18]) and $\left\langle \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle = 2M_\pi^2$, compared with the correct ones, $2M_\pi^2$ and $\frac{2}{1+\gamma_m} \cdot 2M_\pi^2$, respectively (up to the trace anomaly m_σ^2 term). This implies that Zwicky's argument corresponds to the inclusion of the σ pole for the former, while neglecting the latter.

In fact, Zwicky's arguments (assuming $m_\sigma^2 = 0$) are equivalent to the *neglect of the σ pole* contribution in Equation (31) to arrive at $\left\langle \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle = 2M_\pi^2$, which he in fact showed to be equivalent to the double use of the soft pion theorem (unjustifiably *removing the σ pole contribution*). Actually, if we use the soft pion theorem $\left\langle \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle|_{p_1^2=M_\pi^2, p_2\to 0} = \left\langle 0|\left[iQ_5^a, 2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i\right]|\pi(p_1)\right\rangle|_{p_1^2=M_\pi^2, p_2\to 0}/F_\pi$, the resultant expression removes the σ pole contribution, since σ is a chiral singlet, $[iQ_5^a, \sigma] = 0$ (the dilaton σ is different from the "sigma" ($\hat{\sigma}$) in the linear sigma model, which is a chiral partner of $\hat{\pi}^a$, with the correspondence to σ as [20]: $\hat{\sigma}^2 + (\hat{\pi}^a)^2 = (F_\pi \cdot \chi)^2 = F_\pi^2 \cdot e^{2\sigma/F_\pi}$). On the other hand, for $\left\langle \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle$, he equated it with the generic result Equation (25) (where σ pole is invisible at $q^2 \to 0$), although the same result $\left\langle \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i \; |\pi(p_1)\right\rangle = 2M_\pi^2$ through the direct computation is obtained *only when including the σ pole*, as shown in Equation (30). Equating the two results dealing with the σ pole differently, he concluded that $2 = 1+\gamma_m$, i.e., $\gamma_m = 1$.

Putting it differently, we may consistently use the same double soft pion theorem on both $\left\langle \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle$ and $\left\langle \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1)\right\rangle$ (though both π's are not on the mass shell in contrast to the main stream of the present discussion), which implies *neglecting the σ pole for both*. By this we would obtain:

$$
\begin{aligned}
< \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1) > |_{p_1,p_2\to 0} \quad &= \quad (1+\gamma_m) \cdot < 0| - \sum_{i=1}^{2} m_f \bar{\psi}_i \psi_i|0 > /F_\pi^2 \\
&= \quad (1+\gamma_m)M_\pi^2,, \\
< \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar{\psi}_i \psi_i|\pi(p_1) > |_{p_1,p_2\to 0} \quad &= \quad 2 \cdot < 0| - \sum_{i=1}^{2} m_f \bar{\psi}_i \psi_i|0 > /F_\pi^2 \\
&= \quad 2M_\pi^2 \qquad (33)
\end{aligned}
$$

which coincides with the result neglecting the σ pole contributions in Equations (28) and (31), where the GMOR relation $M_\pi^2 = -\left\langle 0|\sum_{i=1}^{2} m_f \bar{\psi}_i \psi_i|0\right\rangle/F_\pi^2$, Equation (8), was used. It should be noted that the GMOR relation is based on the single use of the soft pion theorem for the *axial-vector current* which has no pole of σ, while the *flavor-singlet scalar density* $\bar{\psi}\psi$ has the same quantum number as σ and both $(1+\gamma_m)\bar{\psi}\psi$ and $2\bar{\psi}\psi$ equally have a σ pole with the coupling to two π's given in Equation (26). Differently from GMOR, the double use of the soft pion theorem for $\bar{\psi}\psi$ ignoring the σ pole contribution is not justified).

Thus, again, $<\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)>\neq<\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)>$, namely, there is no constraint on the value of γ_m (or γ_*) in contradiction to Zwicky's argument claiming both sides equally to be $2M_\pi^2$.

Of course, the inequality is trivially true, with the same matrix element $\left\langle\pi(p_2)|\sum_{i=1}^{N_f} m_f \bar{\psi}_i\psi_i|\pi(p_1)\right\rangle|_{p_1,p_2\to 0}$ evaluated by the same method being simply multiplied by the different numerical factor $1+\gamma_m$ vs. 2. However, the message of this trivial game is as follows: the double use of the soft pion theorem for the scalar density (coupled to σ) simply misses the (massive) σ pole contribution $(1-\gamma_m)M_\pi^2$ for $\left\langle\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle$, the inclusion of which gives the correct results, $2M_\pi^2$, which are consistent with the form-factor argument as shown in Equation (30), while including the σ pole also in $\left\langle\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle$ would no longer keep $2M_\pi^2$, but rather $2/(1+\gamma_m)\cdot 2M_\pi^2$, thus again arriving at inequality, when the σ pole is included in both consistently, i.e., the equality $1+\gamma_m=2$ is lost anyway.

More strikingly, the double use of the soft pion theorem also implies that $\left\langle\pi(p_2)|\theta_\mu^\mu|\pi(p_1)\right\rangle|_{p_1,p_2\to 0}=\left\langle\pi(p_2)|(1+\gamma_m)\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle|_{p_1,p_2\to 0}$, since the trace anomaly term $\beta(\alpha)/(4\alpha)G_{\mu\nu}^2$ is a chiral singlet, $\left[iQ_5^a,\beta(\alpha)/(4\alpha)G_{\mu\nu}^2\right]=0$, and the soft pion theorem makes its contribution zero, *independently of Zwicky's assumption of the IR fixed point*. Hence, we would obtain $\left\langle\pi(p_2)|\theta_\mu^\mu|\pi(p_1)\right\rangle|_{p_1,p_2\to 0}=(1+\gamma_m)M_\pi^2\neq 2M_\pi^2$, in contradiction to the form-factor argument that Zwicky's arguments are crucially based on.

So far, we presented the dChPT result. We now comment on Zwicky's argument based on the Feynman–Hellmann theorem, Equation (2.20) in Ref. [16]:

$$\begin{aligned}\left\langle\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle &= 2\frac{\partial}{\partial\ln m_f}\langle\pi(p_2)|\mathcal{H}|\pi(p_1)\rangle=\frac{\partial}{\partial\ln m_f}(2E_\pi\cdot E_\pi)\\ &=\frac{\partial}{\partial\ln m_f}2M_\pi^2=2M_\pi^2,\end{aligned}\tag{34}$$

up to the order of m_f^2. The last equation depends crucially on *his assumption of the combined use of* $M_\pi^2\sim m_f$, which is characteristic of the pion as a pseudo-NG boson in the broken phase. However, if we used the same theorem for $\left\langle\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle=(1+\gamma_m)\frac{\partial}{\partial\ln m_f}\langle\pi(p_2)|\mathcal{H}|\pi(p_1)\rangle$ with the same assumption $M_\pi^2\sim m_f$, then we would obtain $\left\langle\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle=(1+\gamma_m)M_\pi^2\neq 2M_\pi^2$ in contradiction with the generic result in Equation (24). The theorem is insensitive to the spontaneous symmetry breaking, giving the same form in M_π^2 before taking derivative $\frac{\partial}{\partial\ln m_f}$ for both the broken phase and conformal phase.

Actually, if we apply the same theorem to the conformal phase where dChPT is invalid and without the σ pole contribution, we may use the hyperscaling $M_\pi\sim m_f^{1/(1+\gamma_m)}$ to obtain:

$$\begin{aligned}\left\langle\pi(p_2)|2\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle &= 2\frac{\partial}{\partial\ln m_f}\langle\pi(p_2)|\mathcal{H}|\pi(p_1)\rangle\\ &=\frac{\partial}{\partial\ln m_f}2E_\pi^2=\frac{2}{1+\gamma_m}\cdot 2M_\pi^2,\\ \left\langle\pi(p_2)|(1+\gamma_m)\cdot\sum_{i=1}^{N_f} m_f\bar{\psi}_i\psi_i|\pi(p_1)\right\rangle &= (1+\gamma_m)\frac{\partial}{\partial\ln m_f}\langle\pi(p_2)|\mathcal{H}|\pi(p_1)\rangle\\ &=\frac{\partial}{\partial\ln m_f}(1+\gamma_m)E_\pi^2=2M_\pi^2,\end{aligned}\tag{35}$$

with the latter now being consistent with the generic phase-independent result in Equation (24), as it should be. Equation (35) is the same result as in the broken phase through the dChPT, up to the trace anomaly term $m_\sigma^2 \neq 0$ (which is the pole term). It is curious that the combined use of the Feynman–Hellmann theorem and $M_\pi^2 \sim m_f$ coincides with the wrong result of the double-soft pion theorem Equation (33) ignoring the σ pole contribution, while the combined use of the hyperscaling (followed by the simple Coulombic bound state) even for the pion in the broken phase gives the correct result phase-independently.

4. Additional Comments

(1) $\sigma - \pi - \pi$ vertex in Equation (26)

One might be concerned about the $\sigma - \pi - \pi$ vertex in Equation (26). It is different from the well-known low-energy theorem of the scale symmetry [20] (and references therein),

$$F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2) = 2M_\pi^2, \quad q^2 \to 0, \tag{36}$$

which is also obtained by the dispersion representation, $\left\langle \pi(p_2)|\theta_\mu^\mu|\pi(p_1)\right\rangle = M_\sigma^2/(M_\sigma^2 - q^2) \cdot F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2)$, compared with the form-factor argument Equation (24). Both are valid for σ as a pseudo-dilaton but not for π *as a non-NG boson* (massive matter field) like the ρ meson.

On the other hand, Equation (26) is the result for the case of *both σ and π being pseudo-NG bosons*, since it is a sum of $(3-\gamma_m)M_\pi^2$ from the explicit breaking term Equation (14) and $-2p_1 \cdot p_2 = q^2 - 2M_\pi^2$ from the pion kinetic term Equation (12), which are both characteristic of the spontaneously broken scale and chiral symmetries for σ and π. At $q^2 = 0$, it reads $F_\sigma G_{\sigma\pi\pi}(0, M_\pi^2, M_\pi^2) = (1-\gamma_m)M_\pi^2$, which is obviously different from the low-energy theorem of the scale symmetry.

(2) $f_0(500)$ meson for $N_f = 2$ as a massive dilaton

John Ellis [19] obtained the same result as Equation (26): $F_\sigma G_{\sigma\pi\pi}(q^2, M_\pi^2, M_\pi^2) = -\lambda M_\pi^2 - 2p_1 \cdot p_2 = (1-\gamma_m)M_\pi^2 + q^2$, with $\lambda = -(3-\gamma_m)$. He instead focused on the on-shell σ coupling $F_\sigma G_{\sigma\pi\pi}(M_\sigma^2, M_\pi^2, M_\pi^2) = (1-\gamma_m)M_\pi^2 + M_\sigma^2$, but with the M_σ^2 free parameter.

This is compared with our case, where M_σ is not a free parameter but is constrained as [17]:

$$M_\sigma^2 = m_\sigma^2 + (3-\gamma_m)(1+\gamma_m)(N_f/2)(F_\pi^2/F_\sigma^2) \cdot M_\pi^2, \tag{37}$$

which is derived not only through the dChPT Lagrangian Equation (11) valid in the broken phase, Equation (19), but also more generally through the WT identity, Equation (10), and hence is valid both for the broken phase and the conformal phases. Were it not for $m_\sigma^2 = -\left\langle 0|\beta(\alpha)/(\alpha)G_{\mu\nu}^2|0\right\rangle/F_\sigma^2$ as in Zwicky's case, we would have $M_\sigma^2 = \mathcal{O}(M_\pi^2)$ for $(3-\gamma_m)(1+\gamma_m) \cdot (N_f/2)(F_\pi^2/F_\sigma^2) = \mathcal{O}(1)$, and hence $F_\sigma G_{\sigma\pi\pi}(M_\sigma^2, M_\pi^2, M_\pi^2) = \mathcal{O}(M_\pi^2)$, which is roughly the same as the low-energy theorem for σ: $F_\sigma G_{\sigma\pi\pi}(0, M_\pi^2, M_\pi^2) = 2M_\pi^2$. In fact, $N_f = 8$ LatKMI data [10] read $M_\sigma^2 \simeq M_\pi^2 \gg m_\sigma^2$ and hence $F_\sigma G_{\sigma\pi\pi}(M_\sigma^2, M_\pi^2, M_\pi^2) \simeq M_\sigma^2 \simeq M_\pi^2$.

On the other hand, for the real $N_f = 2$ QCD in the deep broken phase near the chiral limit, the σ mass should be mainly due to the trace anomaly $m_\sigma^2 = -\left\langle 0|\beta(\alpha)/(\alpha)G_{\mu\nu}^2|0\right\rangle/F_\sigma^2 \gg M_\pi^2$, such that $M_\sigma^2 \simeq m_\sigma^2 \gg M_\pi^2$, suggesting the identification of σ as $f_0(500)$. Then, thanks to the trace anomaly dominance in the mass formula above, the formula Equation (26) definitely predicts $\sigma - \pi - \pi$ coupling for the σ on the mass shell $q^2 = M_\sigma^2$ ($\gg (1-\gamma_m)M_\pi^2$):

$$G_{\sigma\pi\pi}(M_\sigma^2, M_\pi^2, M_\pi^2) \simeq M_\sigma^2/F_\sigma \simeq M_\sigma^2/F_\pi \gg 2M_\pi^2/F_\pi, \tag{38}$$

with $F_\sigma \simeq F_\pi$. If it is the case, the width of $f_0(500)$ will be enhanced by $[M_\sigma^2/(2M_\pi^2)]^2 tobe \sim$ 50 times larger than the low theorem value in Equation (36), in rough agreement with the reality, and $f_0(500)$ may be regarded as a pseudo-NG boson and pseudo-dilaton (though very massive and far from the invariant limit). The crucial point is that in addition to the *dominance of the non-perturbative trace anomaly for M_σ^2*, the formula Equation (26) for the $G_{\sigma\pi\pi}(M_\sigma^2, M_\pi^2, M_\pi^2)$ is valid only when *both* σ *and* π *are treated as pseudo-NG bosons* in contrast to the low-energy theorem Equation (36) treating π as a matter field and not as a pseudo-NG boson (or if we use the low-energy theorem, we should regard π as a matter field, not a pseudo-NG boson, i.e, put $M_\pi^2 \sim M_\rho^2$ as a typical matter field, in which case the width would also give a result roughly consistent with the reality, although M_π^2 is far from the reality).

5. Summary and Discussion

The anomalous dimension in the infrared region $\gamma_m(\mu = \Lambda_{\rm IR} < \Lambda_{\rm QCD})$ in QCD is determined by the non-perturbative dynamics, both in the broken and conformal phases, in contrast to the perturbative one at the asymptotically-free UV region $\gamma_m(\mu > \Lambda_{\rm UV} = \Lambda_{\rm QCD}) \sim 1/\ln(\mu/\Lambda_{\rm QCD}) \simeq 0$ determined by the perturbative dynamics. The present paper shows that the value of $\gamma_m(\mu = \Lambda_{\rm IR})$ is *not determined without explicit dynamical calculations* such as the ladder SD equation and lattice calculations. This was contrasted to the recent claim of Ref. [16] without explicit dynamical calculations, which is based on the inconsistent treatment of the σ (pseudo-dilaton) pole contribution to the pion matrix elements $\left\langle \pi(p_2)|2 \cdot \sum_{i=1}^{N_f} m_f \bar\psi_i\psi_i|\pi(p_1)\right\rangle$ and $\left\langle \pi(p_2)|(1+\gamma_m) \cdot \sum_{i=1}^{N_f} m_f \bar\psi_i\psi_i|\pi(p_1)\right\rangle$ in the broken phase, where both π and σ are pseudo-NG bosons.

In fact , as mentioned in the Introduction, the ladder SD equation and lattice calculations for $N_f = 8$ QCD near the conformal window both give $\gamma_m(\mu < \Lambda_{\rm QCD}) \simeq 1$ (as well as a light scalar σ pseudo-dilaton), which is crucial to the walking technicolor model that is expected to have such a large anomalous dimension in the infrared region $\mu < \Lambda_{\rm TC} \simeq \Lambda_{\rm ETC}$ [1] to suppress the flavor-changing neutral currents (FCNC), where $\Lambda_{\rm TC}$ is the intrinsic UV scale of the walking technicolor, and is an analogue of the $\Lambda_{\rm QCD}$, which is relevant to the asymptotically-free perturbative running coupling in the UV region, while $\Lambda_{\rm ETC}$ is the scale where the technicolor group is unified into a large gauge group, the Extended Technicolor.

Incidentally, both the explicit dynamical calculations in the SD equation [21] and the lattice calculations [22] for $N_f = 12$ to be inside the conformal window show $\gamma_m \simeq 0.4$–0.5 near the IR fixed point $\alpha(\mu) \simeq \alpha_* \gg 0$, which is also large compared to the perturbative one. While the ladder SD equation in the conformal phase is at a leading order, with $\gamma_m = 1 - \sqrt{1 - \alpha/\alpha_{\rm cr}} \sim 0.8$ at $\alpha = \alpha_*$, including non-leading terms for the large mass and finite size, it gives $\gamma_m \sim 0.5 - 0.6$ when fitted by a simple hyperscaling form as in the lattice analyses. Moreover, it was found on the lattice for $N_f = 12$ [23] that M_σ is even smaller than the M_π for a non-zero quark mass, indicating that the scale symmetry is spontaneously (as well as explicitly) broken due to the gluon condensate generated by the non-zero quark mass [14]; thus, σ is a pseudo-NG boson in contrast to π, which is not a pseudo-NG boson since the chiral symmetry is broken only explicitly.

Further explicit dynamical calculations of the non-perturbative anomalous dimension in the IR region as well as a light pseudo-dilaton σ in the QCD are highly desired in the future, for many choices of N_c and N_f, including not only the broken phase but also the conformal phase.

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