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# Lectures on RCFT *

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We review some recent results in two dimensional Rational Conformal Field Theory. We discuss these theories as a generalization of group theory. The relation to a three dimensional topological theory is explained and the particular example of the Chern-Simons-Witten theory is analyzed in detail. This study leads to a natural conjecture regarding the classification of all RCFT's.

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## 1. Introduction – a trip to the Zoo

The fundamental principles of string theory are not yet known. Since conformal field theory [1] plays a crucial role in string theory, many researches believe that a detailed study of conformal field theory will bring us closer to the concepts underlying string theory. It is hoped that a better understanding of the mathematical foundations of conformal field theory will lead to interesting and relevant generalizations of CFT, which might in turn lead to progress in string theory. There are other good reasons to study CFT, on the one hand, the study of CFT might eventually be useful in identifying 2D critical phenomena in nature and on the other it has lead to beautiful results and applications in pure mathematics, and promises to lead to more.

Motivated by the desire to understand better the mathematical structure of conformal field theories one turns to the problem of classifying theories. We are not so much interested in the final list of theories as we are in the techniques used to obtain such a list, and the mathematical structures characteristic of members on that list.

General conformal field theories have not yet been attacked in any meaningful way, but the study of an interesting subclass of theories has been very successful in the past two years. In order to motivate and define these theories let us recall that some theories have the beautiful properties that their correlation functions, partition functions etc. have very simple analyticity properties in the moduli. The prototype of such behavior is the holomorphic factorization of determinants on Riemann surfaces:

$$det\bar{\partial}\partial \sim |F(\tau)|^2$$

which plays a key role in the Belavin-Knizhnik theorem of string theory. Should we focus on this criterion? No: the theories which have this property are too simple – they are basically free theories (on the world sheet!). Holomorphic factorization admits a generalization which leads to a very rich class of conformal field theories, namely, the rational conformal field theories (RCFT). These may be characterized by saying that all correlation functions, partition functions, etc. can be expressed in terms of finite sums of analytic times antianalytic functions:

$$\langle\phi\cdots\phi\rangle \sim \sum_{i=1}^{N<\infty} |\mathcal{F}_i|^2$$

More formally, RCFT's are distinguished amongst the set of all conformal field theories by the existence of a holomorphic (and anti-holomorphic) monodromy-free subalgebra $\mathcal{A}$ (and $\bar{\mathcal{A}}$) of the operator product algebra such that the physical Hilbert space can be decomposed into a finite sum of $\mathcal{A}\times\bar{\mathcal{A}}$ representations:

$$\mathcal{A} = \oplus_{i=1}^{N}\mathcal{H}_i\otimes\bar{\mathcal{H}}_i \qquad .$$

In fact, known theories satisfying this criterion comprise a veritable zoo.

Let us collect some specimens from this zoo. The oldest and most venerable are surely the current algebras – also known as Wess-Zumino-Witten [2] theories. These current algebras have various extended algebras (a notion we explain below). So far, all known extended algebras are related to orbifolds [3] of WZW theories by a subgroup of the center. Another venerable example of rational theories are the minimal models of BPZ [1] and FQS [4] (and their N=1 and N=2 generalizations). These are based on the chiral algebra of the (N=1,or N=2 super-) Virasoro algebra itself, and these have rather nontrivial extensions known as W-algebras and their generalizations, $W_n$-algebras [5]. In addition there are various species of parafermions [6]. Between 1984 and 1986 it was realized [7] [8] [9] that parafermions and the various discrete series could be obtained by the GKO coset construction [10]. Indeed, any coset construction based on two rational chiral algebras will define a rational conformal field theory. Finally whenever the chiral algebra has a discrete symmetry we can form an orbifold [3] theory [11].

Clearly, this zoo should be organized. By trying to formulate all these theories in a unified way, we are led to conjecture: *all* RCFT's are related to certain deformations of groups, this deformation can be described axiomatically or in terms of 3D Chern-Simons-Witten (CSW) gauge theory and is closely related to certain quantum groups. A cynical version of this conjecture would state that nothing new has been found since 1986, so we must be done.

The purpose of these lectures is to make a case that the conjecture is not cynical but based on the insight that RCFT is closely related to group theory, and in fact must be defined by axioms closely related to those defining groups.

These lectures are not meant to be a review of the subject of RCFT. Many groups have contributed to this subject from various points of view. In particular, a completely independent line of development, beginning with the classic papers of Doplicher, Haag, and Roberts, and using the conceptual framework of algebraic quantum field theory has led to similar results [12]. For the most part we will review our own work on the subject [13] [14] [15] [16] [17] and will present it from the point of view developed in these references.

We will assume the reader has some familiarity with conformal field theory, e.g. we will assume familiarity with the material covered in standard review lectures [18]. We have included many exercises, hoping they will help the reader study the subject. It is a good idea to try to work out at least some of them in order to practice the formalism in the text. The answers to most of these exercises can be found in standard CFT reviews or in our papers [13]-[17].

In the next section we give several different definitions of chiral vertex operators. These allow us to have an operator formalism for calculations of conformal blocks and lead to the definition of the duality matrices. In the third section we examine the consistency conditions these matrices have to satisfy. The complete set of independent identities of these matrices is found in section 4. In the fifth section we describe the Tannaka-Krein approach to group theory which is similar to the structure we found in sections 2 - 4. This leads us to the conclusion that RCFT is a generalization of group theory. In section 6 we combine the left moving and right moving conformal blocks into a consistent conformal field theory. Section 7 is devoted to a general discussion about the relation between two dimensional duality (as described in the previous sections) and three dimensional general covariance. This general discussion is made more explicit in sections 8 - 10. In the eighth section we have some comments about quantum groups and the relation of quantum groups to knot invariants and RCFT. In sections 9 - 10 we consider an explicit example of a topological three dimensional field theory. This is the Chern-Simons-Witten (CSW) theory. We first discuss the canonical quantization of the theory (section 9) and explain the connection between the theory and two dimensional conformal field theory.

We then consider different gauge groups in three dimensions (section 10) and show that all known RCFT can be obtained by an appropriate gauge group in three dimensions. Our conclusions are summarized in section 11 where we also present some conjectures about the classification of RCFT.

## 2. Chiral Vertex Operators and Duality Matrices

We need a formalism for manipulating holomorphic parts of vertex operators. Vertex operators will be replaced by objects known as chiral vertex operators ($CVO$'s) [19] [20] [13][14][15] the distinction being that chiral vertex operators are purely holomorphic and keep track of the various internal states and couplings used to form a conformal block. Rather than give the definition immediately, let us build up to it.

Consider the minimal Virasoro models. For every triplet $i, j, k$ of Virasoro representations and $\beta \in \mathcal{H}_j$ we define

$$\Phi^{j,\beta}_{i,k}(z) : \mathcal{H}_k \to \mathcal{H}_i$$

by its matrix elements.

First, consider $\beta$ to be a highest weight vector $\beta = |j>$. For the primaries in $\mathcal{H}_i$ and $\mathcal{H}_k$ we have

$$< i|\Phi^{j,\beta}_{ik}(z)|k> = \| \Phi^{j}_{ik} \| \, z^{-(\Delta_j + \Delta_k - \Delta_i)}$$

where $\Delta$ is the conformal dimension of field. We can compute matrix elements between descendants using the Virasoro algebra and the rule

$$\left[L_n, \Phi^{j,\beta}_{ik}(z)\right] = \left(z^{n+1}\frac{d}{dz} + (n+1)z^n\Delta(\beta)\right)\Phi \ .$$

This only defines $\Phi$ on Verma modules. Demanding that $\Phi$ is defined on the irreducible quotients forces some of the constants $\| \Phi^{j}_{ik} \|$ to vanish.

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• Exercise 2.1 *Null vectors at work.*

a.) Suppose $\phi$ has a nonvanishing weight. Show that if $|0\rangle$ is the $sl(2)$ invariant vacuum then the null vector $L_{-1}|0\rangle$ implies that $||\Phi^{\phi}_{00}|| = 0$.

b.) Consider the Ising model with primary fields $1, \psi, \sigma$ of dimensions $0, 1/2, 1/16$. Use the null vector

$$(L_{-2} - \frac{3}{2}L^2_{-1})|\psi\rangle = 0$$

to show that $||\Phi^{\psi}_{\psi\psi}|| = 0$.

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We initially define the fusion rule $N_{jk}^{i} = 0, 1$ according to whether $\| \Phi \|$ must be zero or not. Having defined $\Phi_{ik}^{j,\beta}$ for highest weight states, we can define it for descendants $\beta = L_{-I}|j>$ (and their linear combinations) by contour integrals:

$$\Phi_{ik}^{j,\beta}(z) \equiv \oint d\xi_1 (\xi_1 - z)^{n_1+1} T(\xi_1) \dots \oint d\xi_\ell (\xi_\ell - z)^{n_\ell+1} T(\xi_\ell) \Phi_{ik}^{j,|j>}(z) \; .$$

For simplicity we will often restrict ourselves to the minimal models. However, we will occasionally point out new elements that arise in more general RCFT's. For example, we define chiral vertex operators for affine Lie algebras $\hat{g}$. Each $\hat{g}$ representation $\mathcal{H}_i$ contains a ground state representation $W_i \subset \mathcal{H}_i$ for the finite dimensional algebra $g$. We first define $\Phi_{ik}^{j,\beta}(z)$ for $\beta \in W_j$. By commutation with the generators of $\hat{g}$ it suffices to define the matrix elements between $\alpha \in W^i$, $\gamma \in W_k$

$$\left\langle \alpha | \Phi_{ik}^{j\beta}(z) | \gamma \right\rangle = t_{\beta\gamma}^{\alpha} z^{-(\Delta_1 + \Delta_j - \Delta_k)}$$

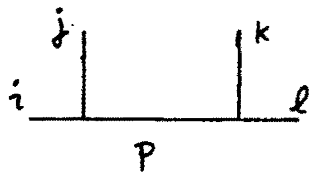
where $t_{\beta\gamma}^{\alpha} \in Inv\left(\bar{W}^i \otimes W^j \otimes W^k\right)$ is an invariant tensor. Other matrix elements and the definition for $\beta$ a descendent can be carried out exactly as before. Again the null vectors will only allow one to define $\Phi$ consistently starting with a *subspace* of $Inv(\bar{W}^i \otimes W^j \otimes W^k)$. This subspace of good couplings

$$V_{jk}^{i} \subset Inv\left(\bar{W}^i \otimes W^j \otimes W^k\right)$$

is called the space of 3-point couplings and $N_{jk}^{i} = dim\ V_{jk}^{i}$ are the fusion rules. Notice that in this case, unlike the discrete series, the integers $N_{ij}^{k}$ are not all zero and one – in some cases there exists more than one invariant coupling. Also, the representations are not all self conjugate. In other words, $N_{i0}^{j} = \delta_i^j$ but $N_{ij}^{0} = \delta_{\bar{i}j}$ where $\bar{i}$ is the conjugate of $i$. In more general theories there are CVO's which vanish for three primary fields but do not vanish for the descendants.

CVO's give an operator formalism for computing conformal blocks. For example, the conformal blocks of the 4-point function for 4 primaries in the minimal models are

$$\mathcal{F}_p^{ijk\ell}(z_2, z_3) = \left\langle i | \Phi_{ip}^{j}(z_2) \Phi_{p\ell}^{k}(z_3) | \ell \right\rangle \sim \qquad (2.1)$$



where the rhs of the above equation illustrates a useful pictorial notation for conformal blocks.

The physical correlation function is given (in the diagonal theory) by

$$\langle \phi^i | \phi^j(z_2) \phi^k(z_3) | \phi^\ell \rangle = \sum_p d_p |\mathcal{F}_p|^2$$

where $d_p$ are constants independent of $z$ and $\bar{z}$. This correlator looks like it depends on many choices. Duality states that many of those choices don't affect the above final result. More precisely, part of duality states that the physical correlators are independent of the choice of basis of conformal blocks. In particular, the order of $\phi^j \phi^k$ on the lhs is irrelevant so one could also have used the blocks

$$\mathcal{F}_p^{ikj\ell}(z_3, z_2) = \left\langle i | \Phi^k_{ip}(z_3) \Phi^j_{p\ell}(z_2) | \ell \right\rangle \ . \tag{2.2}$$

But these blocks must give the same correlation function.

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• Exercise 2.2 *Trivial fact of life*. Show that if $\{f_i\}, \{g_i\}, \{h_i\}, \{k_i\}$ are four sets of linearly independent analytic functions such that

$$\sum_{i=1}^{N} f_i \bar{g}_i = \sum_{i=1}^{M} h_i \bar{k}_i$$

then $N = M$, and $\vec{f} = A\vec{h}$ $\vec{g} = (A^{-1})^\dagger \vec{k}$ for some invertible matrix $A$.

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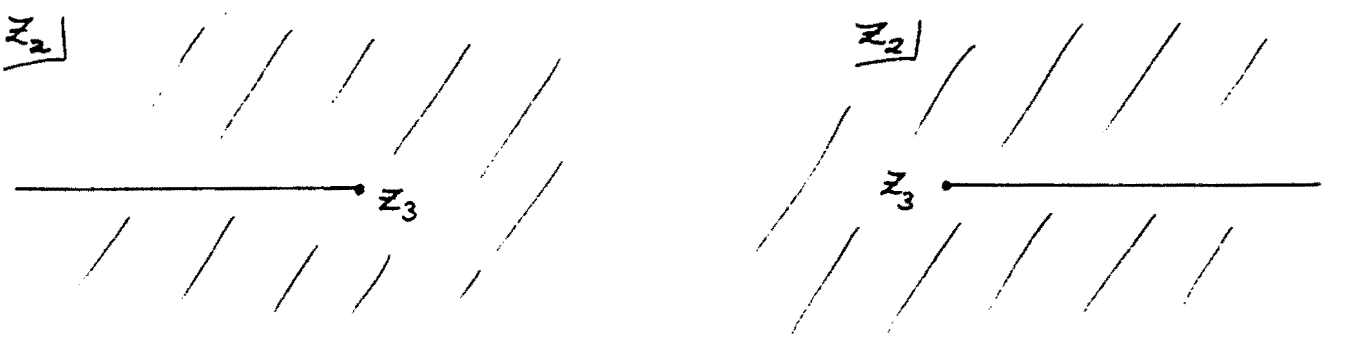
From the above exercise it follows that the two sets of blocks (2.1) and (2.2) are linearly related, and in fact, by considering descendants we have an operator identity:

$$\Phi^j_{ip}(z_1) \Phi^k_{p\ell}(z_2) = \sum_q B_{pq} \begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \Phi^k_{iq}(z_2) \Phi^j_{q\ell}(z_1) \tag{2.3}$$

where that the coefficients $B$ are the same for the primaries and all the descendants.

If one thinks carefully about the above argument he will note that we must choose cuts since the $\mathcal{F}$'s are not globally defined and have monodromy. So we choose the cut: $z_1 - z_2 \in \mathbb{R}^+$. In order to compare (2.1) and (2.2) we must use analytic continuation, and we can only compare these functions on their common domain of definition. In the $z_2$ plane we find that (2.1) and (2.2) are defined on the following regions:




hence the overlap consists of two components and there are in principle two distinct $B$ matrices. We define $B(+)$ by (2.3) for $\mathrm{Im}(z_1 - z_2) > 0$. For $\mathrm{Im}(z_1 - z_2) < 0$ we have, in general a different matrix $B(-)$. If the sign is omitted, we refer to $B(+)$.

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• Exercise 2.3 *Relation to BPZ.* Compare the above discussion with section four of [1] and show that the definition of conformal blocks as matrix elements of $\Phi$ corresponds with that of BPZ.

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All of this has been derived in the simplified notation appropriate for the minimal models, but these considerations apply to arbitrary RCFT's. In the general case, when the space of three-point couplings $V_{jk}^{i}$ is a vector space of dimension larger than one we have linear transformations

$$B_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \; : \; V_{jp}^{i} \otimes V_{k\ell}^{p} \longrightarrow V_{kp}^{i} \otimes V_{j\ell}^{p} \, .$$

The other part of the algebra of the $\Phi$ operators follows from the operator product expansion. We have

$$\Phi_{ip}^{j}(z_1)\Phi_{p\ell}^{k}(z_2) = \underbrace{\sum_{q} \mathcal{F}_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}}_{\substack{\text{Summarises the} \\ \text{representation-theoretic} \\ \text{content of the operator}}} \underbrace{\sum_{Q \in \mathcal{H}_q} \Phi_{i\ell}^{q;Q}(z_2)\langle Q|\Phi_{qk}^{j}(z_{12})|k\rangle}_{\text{sum over descendants}} \tag{2.4}$$

• Exercise 2.4 *Defining the F Matrix.* Prove that the operator product expansion of two $\Phi$ operators has the form

$$\Phi^{i}_{lp}(z_1)\Phi^{j}_{pr}(z_2) = \sum_k F_{pk}\begin{bmatrix} i & j \\ l & r \end{bmatrix} \sum_{K\in\mathcal{H}_k} \Phi^{k,K}_{lr}(z_2)\langle K|\Phi^{i}_{kj}(z_1 - z_2)|j\rangle$$

(Hint: Write out the operator product expansion with arbitrary coefficients. Use translation and scaling invariance to determine some of the structure of the coefficients. Now take the operator product expansion with a third operator $\Phi$ and demand consistency with braiding.)

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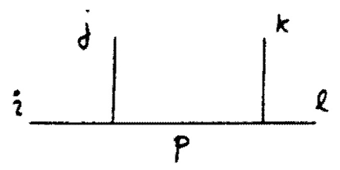
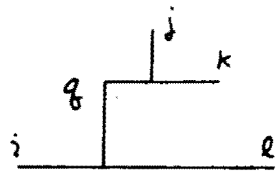
Now going back to our blocks $\mathcal{F}^{ijkl}_p$ we see that we can insert the operator product expansion and define a new basis of conformal blocks, which we may denote pictorially:




$$= \sum_q F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$$




In the general case we have a linear transformation

$$F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} : V^{i}_{jp}\otimes V^{p}_{kl} \to V^{i}_{ql}\otimes V^{q}_{jk}$$

The $F, B$ transformations are the basic duality transformations. The reader may well ask why these objects are of interest. We may answer with two immediate consequences of these considerations.

*First point:* Already the *existence* of $B, F$ have interesting consequences. Since they are defined by a change of basis, the transformation

$$B : \oplus_p V^{i}_{jp}\otimes V^{p}_{k\ell} \longrightarrow \oplus_p V^{i}_{kp}\otimes V^{p}_{j\ell}$$

is an isomorphism. Therefore, matching dimensions, we have

$$\sum_p N^i_{jp} N^p_{k\ell} = \sum_p N^i_{kp} N^p_{j\ell} .$$

This defines Verlinde's fusion rule algebra: [21]

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• Exercise 2.5 *Fusion Rule Algebra.* Using the fact that $B$ and $F$ define isomorphisms show that the matrices

$$(\phi_k)^i_j \equiv N^i_{kj}$$

form a commutative associative algebra. This algebra is known as Verlinde's fusion rule algebra.

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• Exercise 2.6. *Examples of Fusion Rule Algebras.*

a.) Show that the fusion rule algebra for the rational torus (see section 10) is $\mathbb{Z}/N\mathbb{Z}$.

b.) Write out the algebra for the Ising model. Try to determine all physically acceptable fusion rule algebras with three self-conjugate primaries.

c.) Show that the FRA for the WZW model $SU(2)_k$ (the subscript denotes the level) is generated by elements $\phi_\ell$, $\ell = 0, 1/2, \ldots, k/2$ with

$$\phi_{\ell_1}\phi_{\ell_2} = \sum_{|\ell_1-\ell_2|}^{min(\ell_1+\ell_2, k-(\ell_1+\ell_2))} \phi_\ell$$

by considering the null vector $J^{k-2\ell+1}_{-1}|\ell;\ell\rangle$. (See, e.g. [22].)

d.) Consider the WZW model $SU(3)_2$. Show that the fusion rule algebra for the six integrable representations $\mathbf{1}, \mathbf{3}, \mathbf{3}^*, \mathbf{6}, \mathbf{6}^*, \mathbf{8}$ can be determined purely from the known group theoretic decompositions and consistency conditions on the FRA. Note in particular that $N_{\mathbf{888}} = 1$ whereas in group theory it is equal to two.

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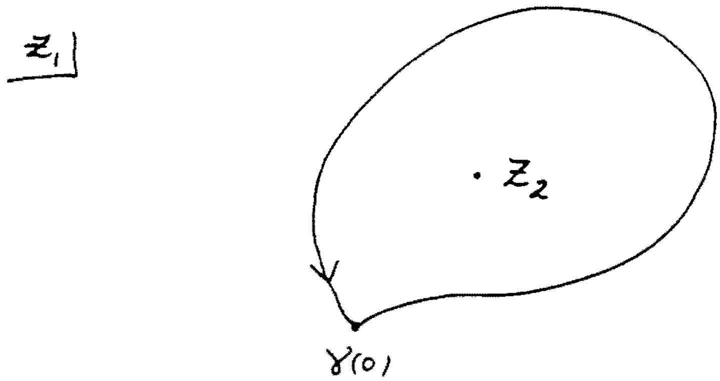
*Second point:* Next, the matrix $B^2$ is not an identity matrix, precisely because of the cuts. In fact, $B^2$ is exactly the monodromy matrix for the analytic continuation of $z_1$ around $z_2$ for the vector of blocks $\mathcal{F}^{ijk\ell}_p(z_1, z_2)$. That is, if $\gamma(s)$ is the following curve:




A curve in $z_1$ plane surrounding $z_2$

Then one can compute the monodromy as in

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• Exercise 2.7 *Monodromy of the blocks.* Show *carefully* that upon analytic continuation we have:

$$\mathcal{F}_p^{ijk\ell}(\gamma(2\pi), z_2) = \sum_q \left( B \begin{bmatrix} k & j \\ i & \ell \end{bmatrix} \; B \begin{bmatrix} j & k \\ i & \ell \end{bmatrix} \right)_{pq} \mathcal{F}_q^{ijk\ell}(\gamma(0), z_2) \; .$$

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Now, the monodromies of conformal field theory are related to the mutual locality factors and therefore to the conformal weights. Thus, the primitive hope is that nontrivial identities on $B, F$ matrices are so restrictive that one can solve them and thus classify RCFT's. This is too naive, but it is on the right track. At any rate, with this hope in mind it is clearly wise to get better acquainted with $B$ as in the following exercise:

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• Exercise 2.8 *B and F with the unit operator.* **By setting various external representations in the four-point function to be the identity we obtain a computable three-point function. Use this observation to evaluate the F and B matrices in the special cases that one of the fields is the identity.** Notice that

$$B \begin{bmatrix} i & j \\ 0 & k \end{bmatrix}$$

**defines a linear map**

$$\Omega^i_{jk} : V^i_{jk} \to V^i_{kj}$$

which may be interpreted as the square-root of a mutual locality factor (compare the previous exercise). Show that

$$(\Omega^i_{jk})^2 = e^{2\pi i(\Delta_j+\Delta_k-\Delta_i)}$$

Therefore

$$\Omega^i_{jk} = \xi^i_{jk} e^{\pi i(\Delta_j+\Delta_k-\Delta_i)}$$

where $\xi = \pm 1$. In simple RCFT's like the discrete series $\xi$ is always +1. In other theories $\xi$ can be -1. For example, in $\hat{SU}(2)$ KM, the sign $\xi$ corresponds to the symmetry or antisymmetry of the tensor coupling the representations. Show that in this example

$$\xi^i_{jk} = (-1)^{2(i+j+k)}$$

where the representations are labeled by their spin (which is integer or half integer). For simplicity, we will limit ourselves in some of the formulae below to the case $\xi = 1$.

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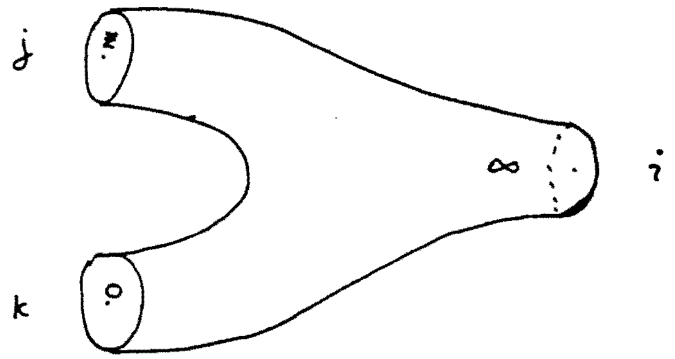
From this discussion it is clear that we need to understand the identities on $B, F$. A number of questions arise: How can we obtain nontrivial identities? What is the full set? What is the minimal set of independent relations? To understand these identities we should understand better what a $CVO$ is. Therefore, let us broaden our viewpoint on chiral vertex operators so that we see more clearly the $S_3$ symmetry of three-point couplings which is fundamental to duality. Instead of choosing the state $\beta$ to define $\Phi$ we should consider a *single* linear operator

$$\begin{pmatrix} i \\ jk \end{pmatrix} : \mathcal{H}_j \otimes \mathcal{H}_k \longrightarrow \mathcal{H}_i \,.$$

that commutes with contour deformation of the chiral algebra. We would like to give this operator a geometrical interpretation. Namely, suppose we have representation spaces on three circles as in the following picture:




three-holed sphere with rep spaces on the three holes

Placing one of the holes about the point at $\infty$ we can define the Virasoro generators acting on the Hilbert space $\mathcal{H}_i$ at $\infty$ by:

$$L_n^{(\infty)} = \oint_{c_\infty} \zeta^{n+1} T(\zeta) d\zeta \ ,$$

But, since $T$ is analytic, these can be deformed to generators around zero and $z$

$$\begin{aligned} L_n^{(\infty)} &= \oint_0 \zeta^{n+1} T(\zeta) + \oint_z \zeta^{n+1} T(\zeta) \\ &= L_n(0) + \sum_{k=0}^{\infty} \binom{n+1}{k} z^{n+1-k} L_{k-1}(z) \ . \end{aligned} \tag{2.5}$$

The chiral vertex operators commute with contour deformation, so we look for operators that satisfy:

$$L_n(\infty) \binom{i}{jk}_z (\beta \otimes \gamma) = \binom{i}{jk}_z \left[ \left( \sum \binom{n+1}{k} z^{n+1-k} L_{k-1}^{(z)} \beta \right) \otimes \gamma + \beta \otimes L_n(0) \gamma \right] \tag{2.6}$$

for any states $\beta, \gamma$. This equation can be interpreted as follows. Think of $L_n(z)$ as a set of Virasoro operators acting on a Hilbert space at $z$, $\mathcal{H}_z$. Then $L_n(z) \otimes L_m(0)$ acts on the Hilbert space $\mathcal{H}_z \otimes \mathcal{H}_0$. The operators $L_n(\infty)$ act on the tensor product $\mathcal{H}_z \otimes \mathcal{H}_0$. They satisfy the Virasoro algebra with the *same* value of the central charge as $L_n(z)$. Therefore, equation (2.5) defines a map $\Delta_z$ from the Virasoro algebra, $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$

$$\Delta_z(L_n) = 1 \otimes L_n + \sum_{k=0}^{\infty} \binom{n+1}{k} z^{n+1-k} L_{k-1} \otimes 1 \ .$$

This "comultiplication" allows us to take tensor products of Virasoro modules with given central charge. Then $CVO$'s are "intertwining operators" for this notion of tensor product. (More on this below.) The above considerations generalize to arbitrary chiral algebras.

We must specify the $z$-dependence of these operators completely and this leads to the condition that

$$\frac{d}{dz}\begin{pmatrix} i \\ jk \end{pmatrix}_z (\beta\otimes\gamma) = \begin{pmatrix} i \\ jk \end{pmatrix}_z (L_{-1}\beta\otimes\gamma) \tag{2.7}$$

In RCFT's there is a finite-dimensional space, $V^i_{jk}$ of operators satisfying (2.6) and (2.7) and we take these equations as our final definition of the $CVO$'s. The connection to our previous description is that

$$\begin{pmatrix} i \\ jk \end{pmatrix}_z (\beta\otimes\cdot) = \Phi^{j,\beta}_{ik}(z)$$

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• Exercise 2.9 Prove the equivalence of these two definitions.

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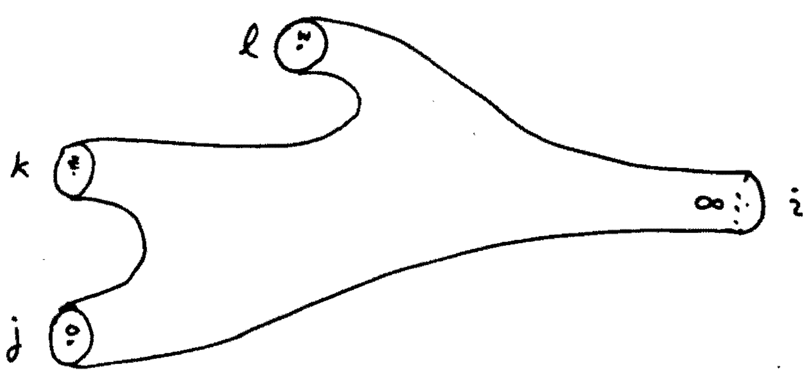
The superiority of our final definition is evident since we can now understand more clearly in terms of the formula (2.6) the statement that the $CVO$ is an operator associated to a 3-holed sphere. Furthermore, it suggests a natural generalization, since we can consider more complicated situations - say, a 4-holed sphere. There will be a finite dimensional vector space $V^i_{jk\ell}$ of operators

$$\mathcal{H}_j\otimes\mathcal{H}_k\otimes\mathcal{H}_\ell \longrightarrow \mathcal{H}_i$$

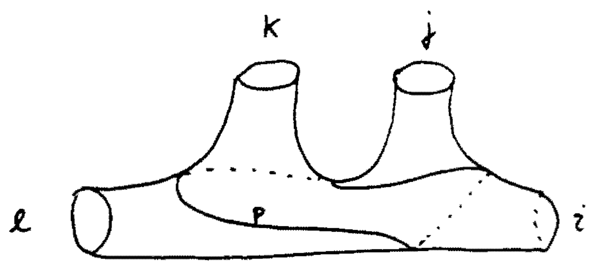
which commute with contour deformation on the surface:

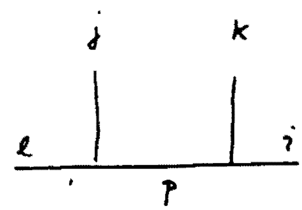



4-holed sphere with representations at each hole

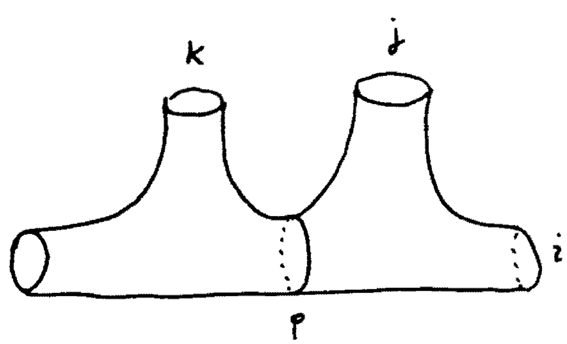
The space of these operators is the same as the space of conformal blocks. This must be true since they are determined by the same equations (which follow from contour deformation arguments) used in more standard descriptions of conformal field theory [1][23]. The new spaces $V^i_{jk\ell}$ can be understood in terms of the simpler spaces of 3-point couplings. Geometrically, we can represent the 4-holed sphere as sewn 3-holed spheres. Analytically, we can use completeness of states to write operators in $V^i_{jk\ell}$ as compositions of CVO's.

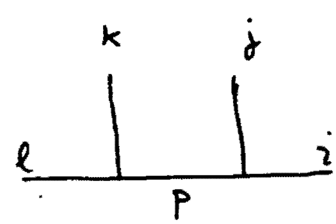
Each sewing has a corresponding composition of CVO's and a corresponding decomposition of $V^i_{jk\ell}$ into simpler spaces:



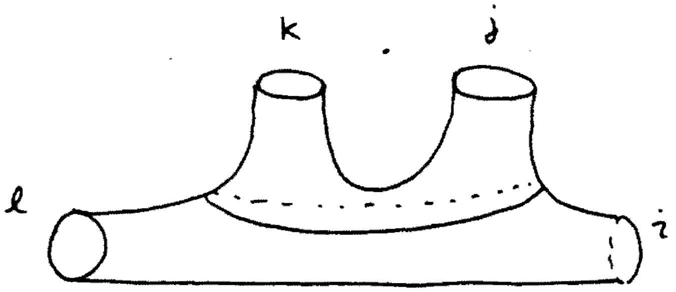



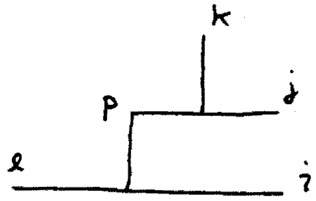

$$V^i_{jkl} \cong \oplus V^l_{jp} \otimes V^p_{ki}$$




$$V^i_{jkl} \cong \oplus V^l_{kp} \otimes V^p_{ji}$$



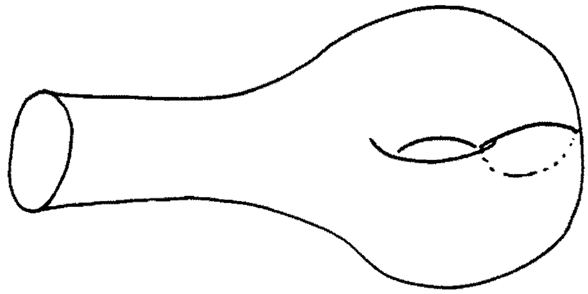




$$V^{i}_{jkl} \cong \oplus V^{l}_{pi} \otimes V^{p}_{kj}$$

Note that each of the sewings corresponds to a different asymptotic region of Teichmüller space. The general construction is the following - any $\phi^3$ diagram can be thickened to give a surface - we can put FN length/twist coordinates on that surface and the region with small lengths corresponds to a region in Teichmuller space. In the asymptotic regions of Teichmuller space where the length coordinate goes to infinity the Riemann surface looks like a $\phi^3$-diagram. In this limit the amplitudes of the conformal field theory and the conformal blocks have poles. The leading singularity corresponds to keeping only one intermediate state in the corresponding channel.

Thus different sewings simply correspond to different bases for $V^{i}_{jk\ell}$. The braiding/fusing isomorphisms express the relationships between these sewings. They are computed from the projectively flat connection on moduli space - according to the picture of the Friedan-Shenker modular geometry [24].

Finally we need the following remark - The compositions described so far only give us $g = 0$ surfaces. For CVO's of type $\underline{i\,|^{j}\,i}$ we can sew to get:



one holed torus obtained by sewing

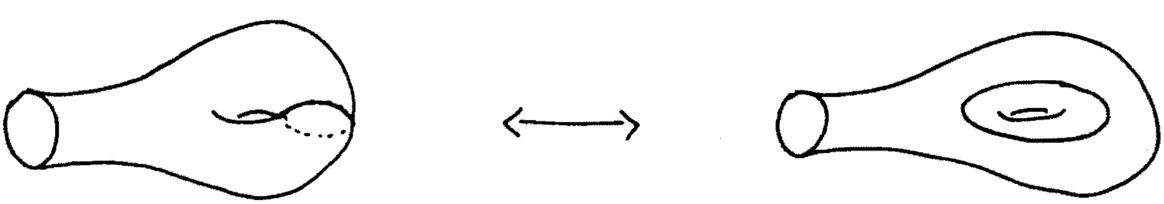
The space of such conformal blocks with channel $i$ will be the space $V_{ji}^{i}$, and the space of all one-point blocks is $\oplus_i V_{ji}^{i}$. In formulas, if we put a state $\beta$ at a puncture on the torus we may form

$$\chi_i^{j,\beta}(z) = Trq^{L_0 - c/24}\left(\Phi_{ii}^{j,\beta}(z)\right)(dz)^{\Delta(\beta)} \tag{2.8}$$

Here $z$ is a point on the complex plane, but the trace essentially identifies $z \sim qz$ so that we actually compute a torus amplitude. If $\beta$ is a Virasoro primary these blocks form a representation of the modular group with the matrix:

$$S(j) : \oplus_i V_{ji}^{i} \longrightarrow \oplus_i V_{ji}^{i} \tag{2.9}$$

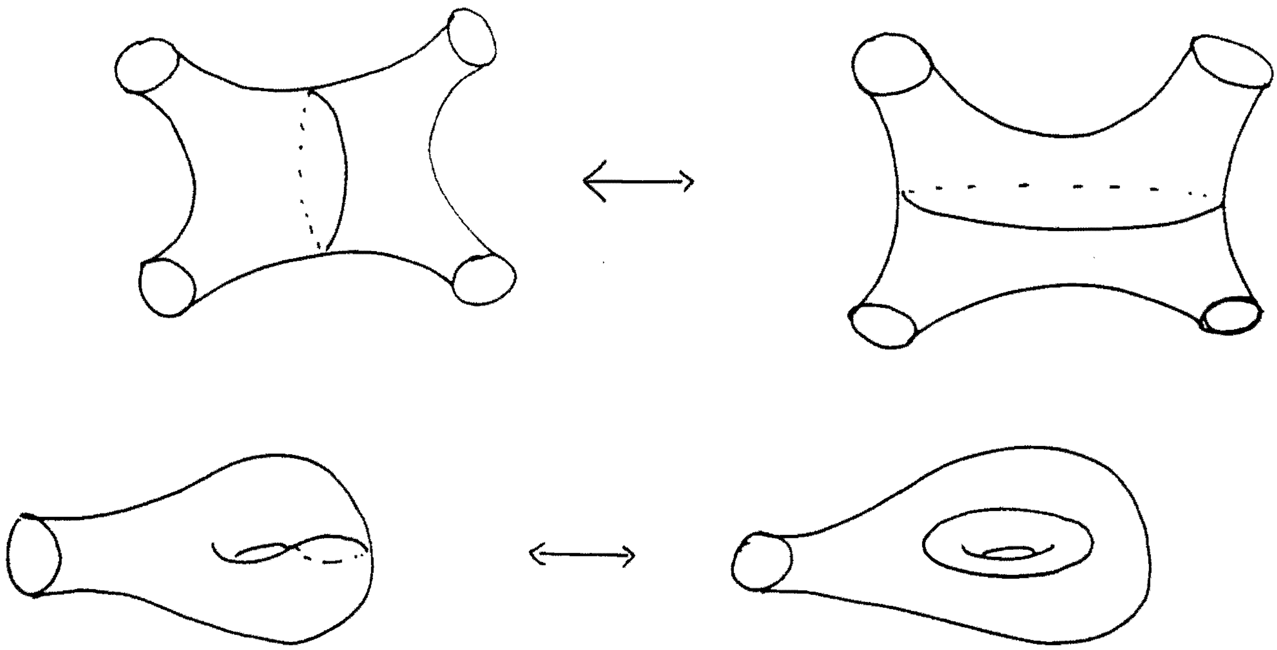
In terms of sewings we are relating the following two diagrams



sewings for S

All these remarks generalize. As first emphasized by Friedan and Shenker [24], to every Riemann surface $\Sigma$ we associate a vector space of conformal blocks $\mathcal{H}(\Sigma)$. This space is intrinsic but can be expressed in terms of the $V^i_{jk}$ in many ways. Each such expression may be associated with a dual diagram. (Which, by its associated pants decomposition is correlated with an asymptotic region of Teichmüller space). Different decompositions of the *same* vector space must be related by isomorphisms. The specific isomorphisms follow from the existence of a projectively flat connection on moduli space. These isomorphisms are known as duality transformations.

An important point is that, in RCFT, all duality transformations can be expressed in terms of a finite number of basic duality transformations. Thus, we need only deal with a finite amount of data. This statement is intuitively obvious. It can be proved [25] that all sewings can be obtained from one another by the two basic moves

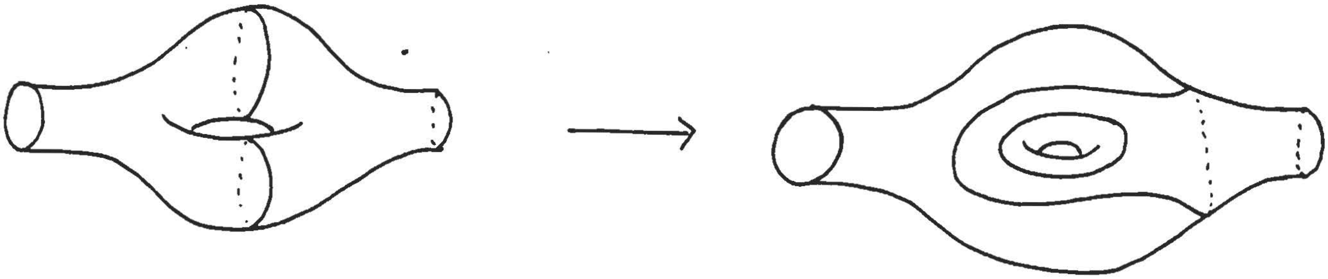


moves on four holed sphere and 1 holed torus

From this, taking into account twists around tubes one sees that all duality transformations can be written in terms of F,B,S and $e^{2\pi i c/24}$.

---

• Exercise 2.10 *Simple Moves*. Decompose the following move ("S for the two-point function") into steps of simple moves:



## 3. Duality Identities

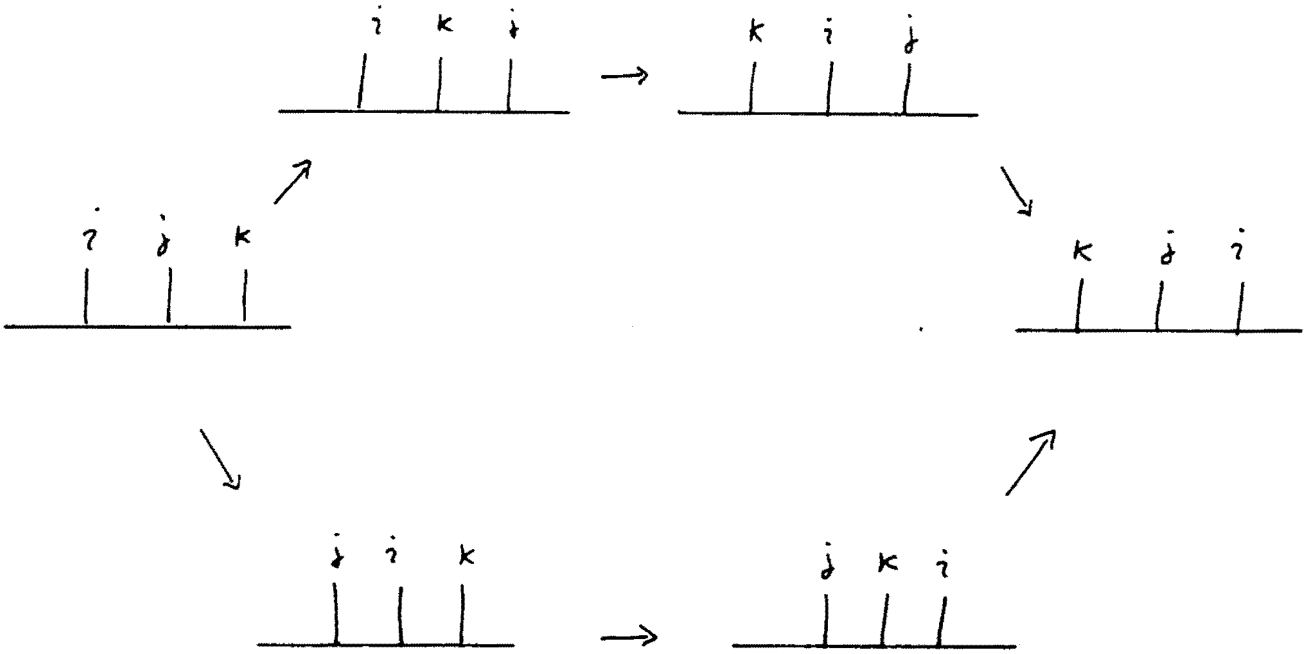
The transformations $F, B, S$ satisfy a large number of nontrivial identities. These identities can be understood in three ways:

a.) The algebra of operators $\Phi$ must be consistent.

b.) The monodromies of conformal blocks form representations of the modular group (duality groupoid).

c.) Different paths of the basic transformations $F, B, S$ relate the same basis of blocks. Thus the identities are intimately connected with the geometry of moduli space.

The simplest example of an identity is the Yang-Baxter relation because it follows immediately from the exchange algebra of the $\Phi$ operators. Consider the following sewings for the 5-point function:




hexagon

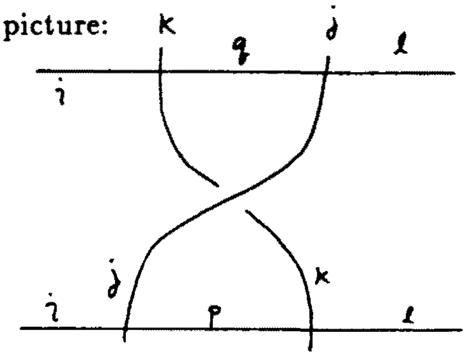
implying an equation of the form $BBB = BBB$ (see below).

---

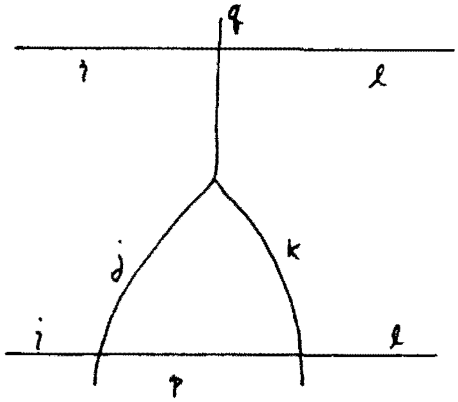
• Exercise 3.1 *Yang-Baxter Equation.* Derive the Yang-Baxter equation for the $B$-matrix by considering the product of three chiral vertex operators and demanding consistency of the braiding algebra.

---

It is very useful to introduce another pictorial formalism for deriving relations between braiding/fusing matrices. We imagine the the braiding and fusing matrices are like amplitudes between conformal blocks with "time" flowing upward as in the following picture:



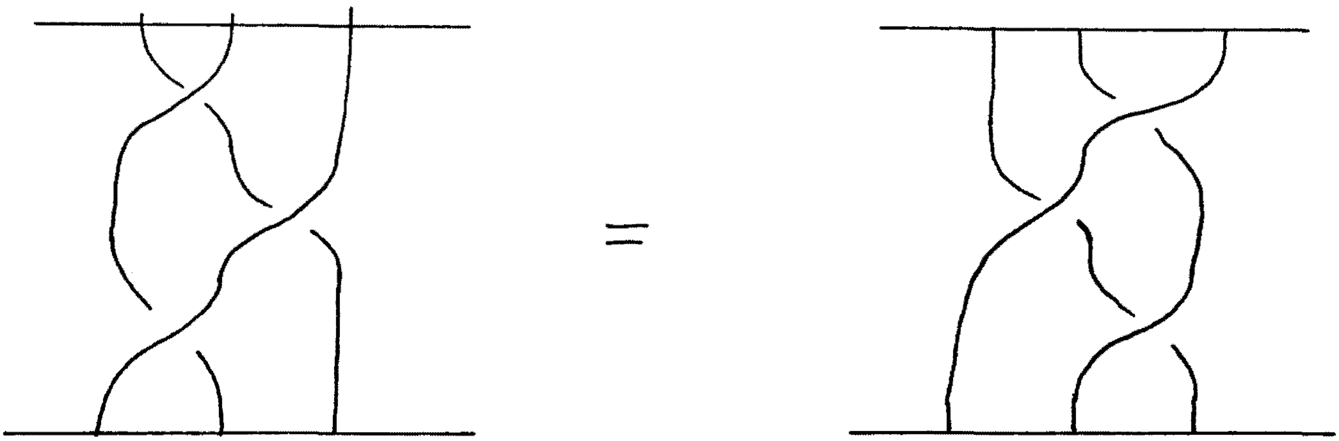

$$\sim \quad B_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$




$$\sim \quad F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$

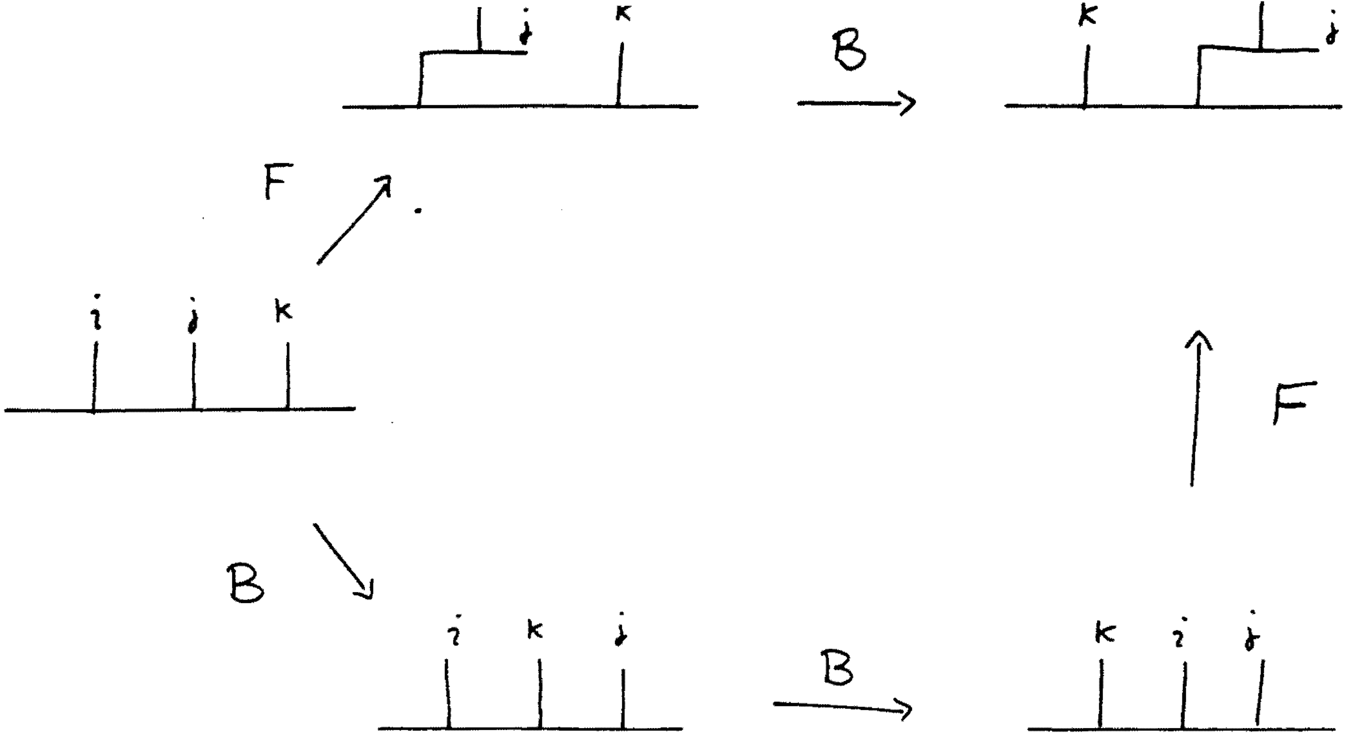
Other pictures for $B$, $F$

(In the 3 dimensional point of view we will see that this interpretation of time can be taken quite literally.) Then we can picture the Yang-Baxter relations as follows:



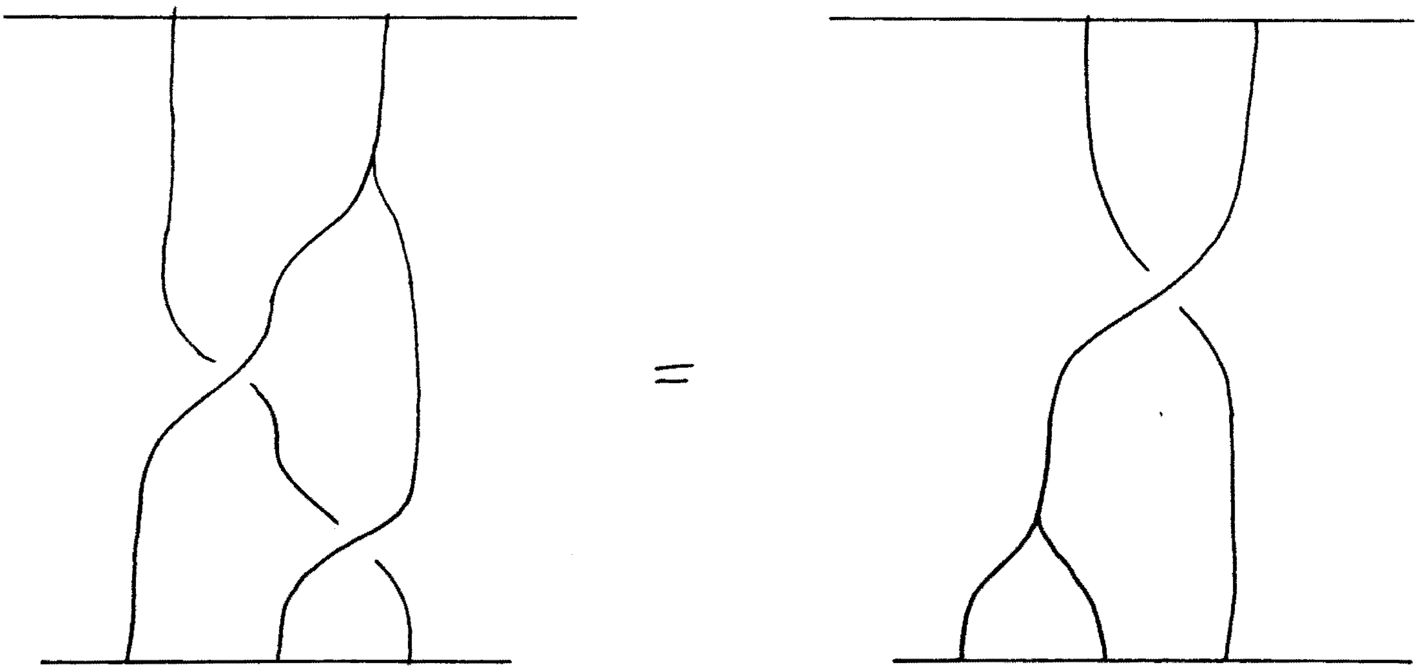
usual picture for Yang-Baxter

Another such identity is the braiding/fusing or pentagon identity which, in terms of duality diagrams may be represented as:




pentagon of dual diagrams

or in the other pictorial notation may be represented as:



**braiding/fusing**

**Clearly, by looking at more and more complicated graphs we will obtain more and more complicated identities. These identities can be neatly characterized as follows. Form a simplicial complex whose different vertices represent different $\phi^3$ decompositions of con-**

formal blocks. Join two vertices, if they are related by a simple move $B$ or $F$. Every loop on the resulting one-complex gives a relation on duality matrices.

---

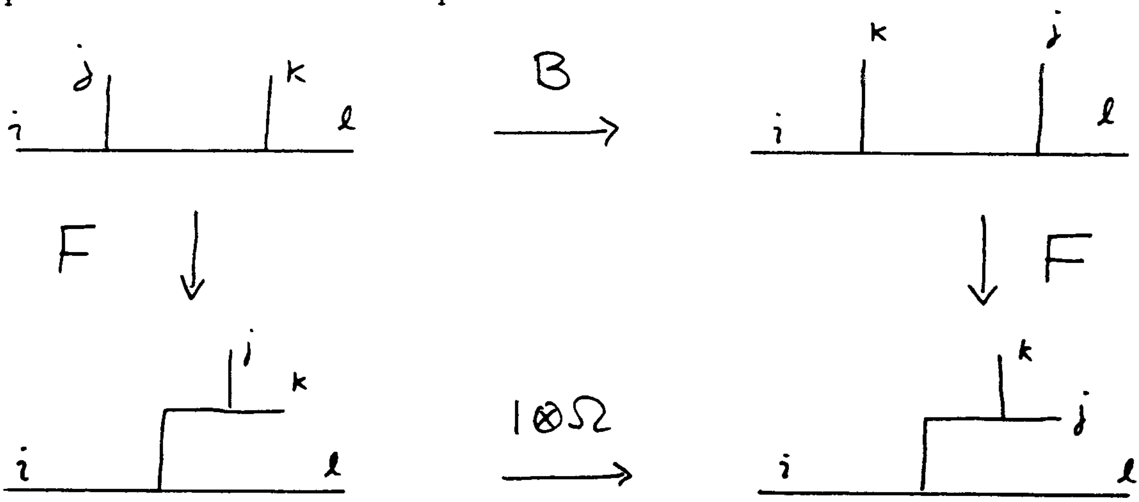
• Exercise 3.2 *The duality complex.* Write out the simplicial complex for the five-point function. Keep initial and final representations fixed in all moves. (Warning: This takes some time.)

---

These identities, and their graphical relations are a great deal of fun to play with - but there are a large number of indices and one can only understand them once he has worked them out for himself. Therefore we urge the reader to work through the following exercise.

---

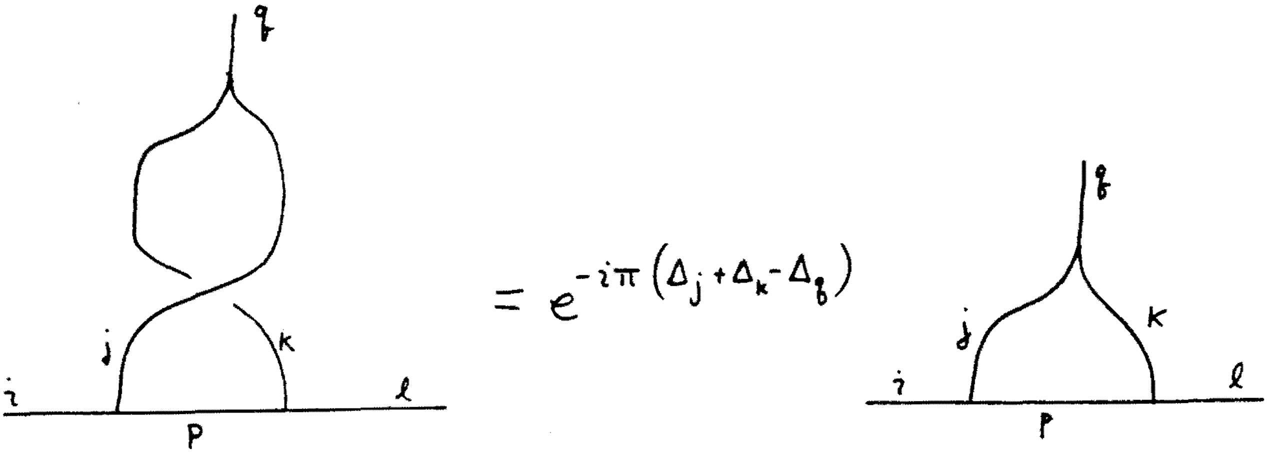
• Exercise 3.3 *Systematic Derivation of Equations.* Consider the 4-point function complex. Show that the closed loop of moves:



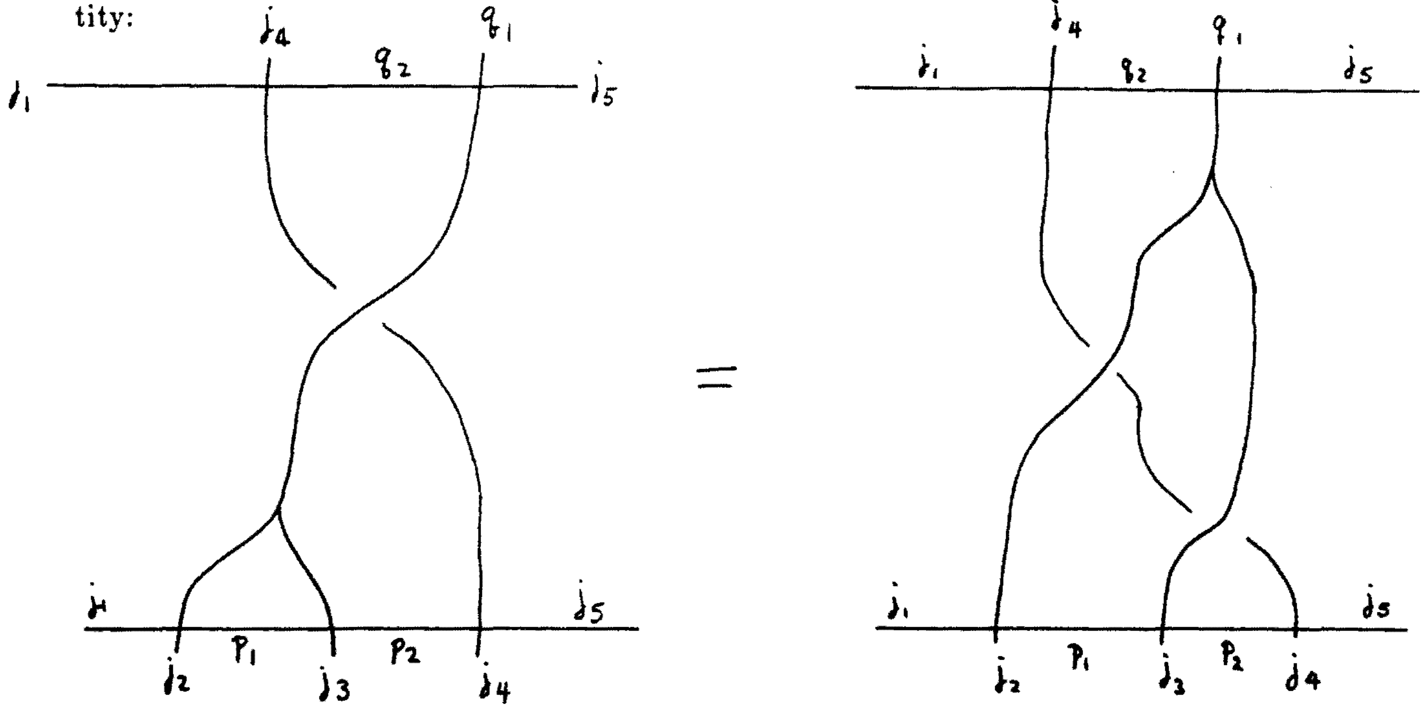

leads to the equation

$$\sum_{p'} B_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\epsilon) F_{p'q}\begin{bmatrix} k & j \\ i & l \end{bmatrix} = F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} e^{-i\pi\epsilon(\Delta_k+\Delta_j-\Delta_q)} \tag{3.1}$$

The $\epsilon$ denotes the sense of the braiding. Note that this identity shows that the eigenvalues of $B$ are the square roots of mutual locality factors. Interpret (3.1) graphically:




Write similar equations involving $F^{-1}, B^{-1}$. Now consider the braiding/fusing identity:




Write the corresponding equation:

$$F_{p_1 q_1}\begin{bmatrix} j_2 & j_3 \\ j_1 & p_2 \end{bmatrix} B_{p_2 q_2}\begin{bmatrix} q_1 & j_4 \\ j_1 & j_5 \end{bmatrix}(\epsilon) = \sum_s B_{p_2 s}\begin{bmatrix} j_3 & j_4 \\ p_1 & j_5 \end{bmatrix}(\epsilon) B_{p_1 q_2}\begin{bmatrix} j_2 & j_4 \\ j_1 & s \end{bmatrix}(\epsilon) F_{s q_1}\begin{bmatrix} j_2 & j_3 \\ q_2 & j_5 \end{bmatrix} \tag{3.2}$$

Now specialize this equation by putting $j_5 = 0$, the identity representation, and derive:

$$F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = e^{-i\pi\epsilon(\Delta_i+\Delta_k-\Delta_p-\Delta_q)} B_{pq}\begin{bmatrix} j & l \\ i & k \end{bmatrix}(\epsilon) \tag{3.3}$$

Now use the relation (draw the picture!):

$$\sum_{p'} B_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\epsilon) B_{p'q}\begin{bmatrix} k & j \\ i & l \end{bmatrix}(-\epsilon) = \delta_{p,q} \tag{3.4}$$

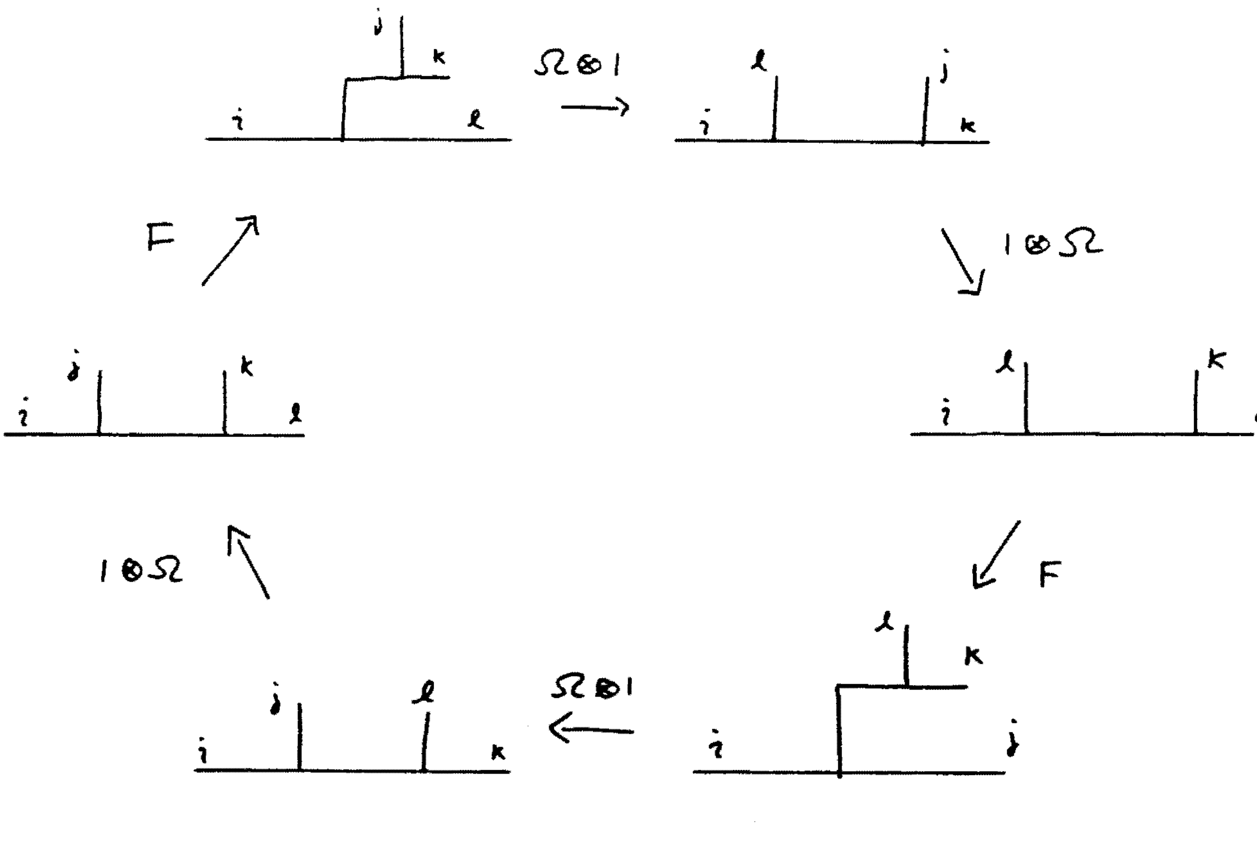
to derive the following two consequences. First

$$\sum_{p'} F_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix} F_{p'q}\begin{bmatrix} l & k \\ i & j \end{bmatrix} = \delta_{p,q} \tag{3.5}$$

and next

$$\sum_{p'} e^{2\pi i\epsilon\Delta_p} B_{pp'}\begin{bmatrix} j & l \\ i & k \end{bmatrix}(\epsilon) e^{2\pi i\epsilon\Delta_{p'}} B_{p'q}\begin{bmatrix} l & j \\ i & k \end{bmatrix}(\epsilon) = \delta_{p,q} e^{2\pi i(\Delta_i+\Delta_k)} \tag{3.6}$$

Interpret (3.5) graphically as a relation following from a closed loop of dual diagrams on the duality complex:



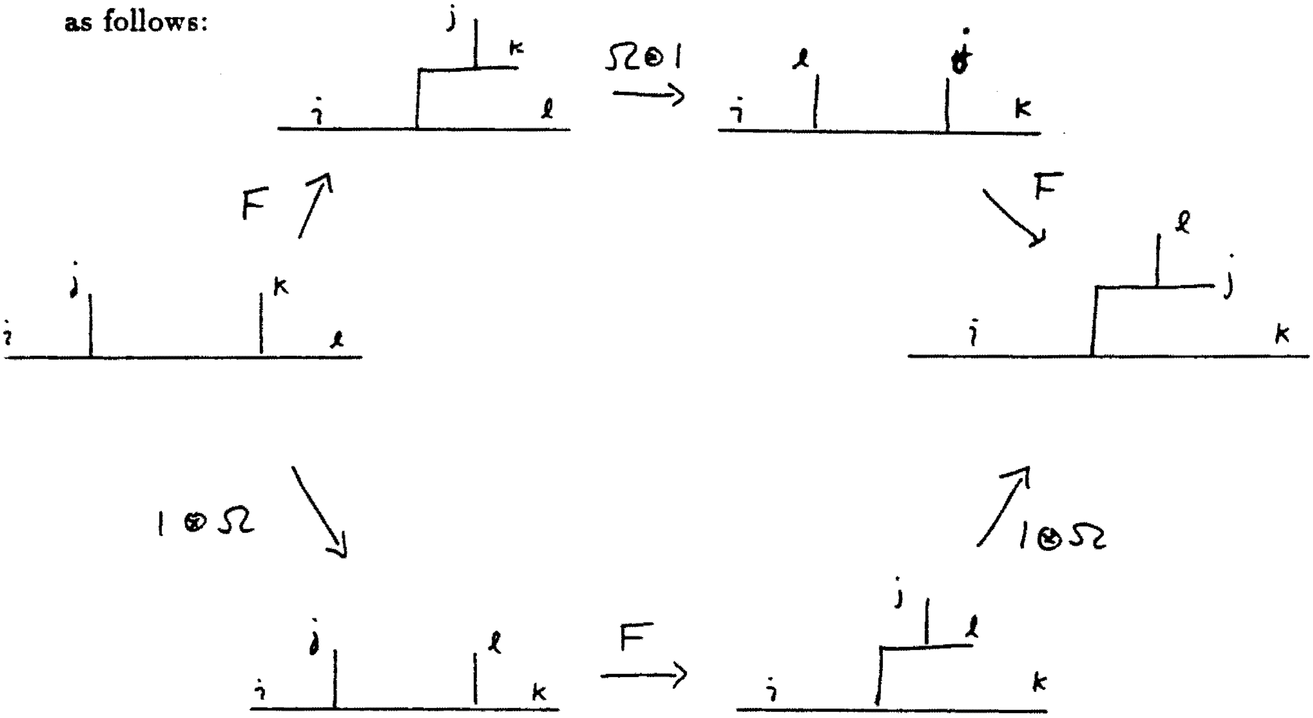

Note that the closed loop is a hexagon.

Note that the determinant of (3.6) gives an interesting constraint on the weights of a rational conformal field theory [26].

Now substitute (3.3) back into (3.1) to get

$$\sum_s B_{ps}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\epsilon) e^{i\pi\epsilon\Delta_s} B_{sq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}(\epsilon) = e^{i\pi\epsilon\Delta_p} B_{pq}\begin{bmatrix} i & k \\ j & l \end{bmatrix}(\epsilon) e^{i\pi\epsilon\Delta_q} e^{-2\pi i\epsilon\Delta_j} \tag{3.7}$$

Write (3.7) in terms of $F$ and interpret in terms of dual diagrams via the hexagon identities as follows:



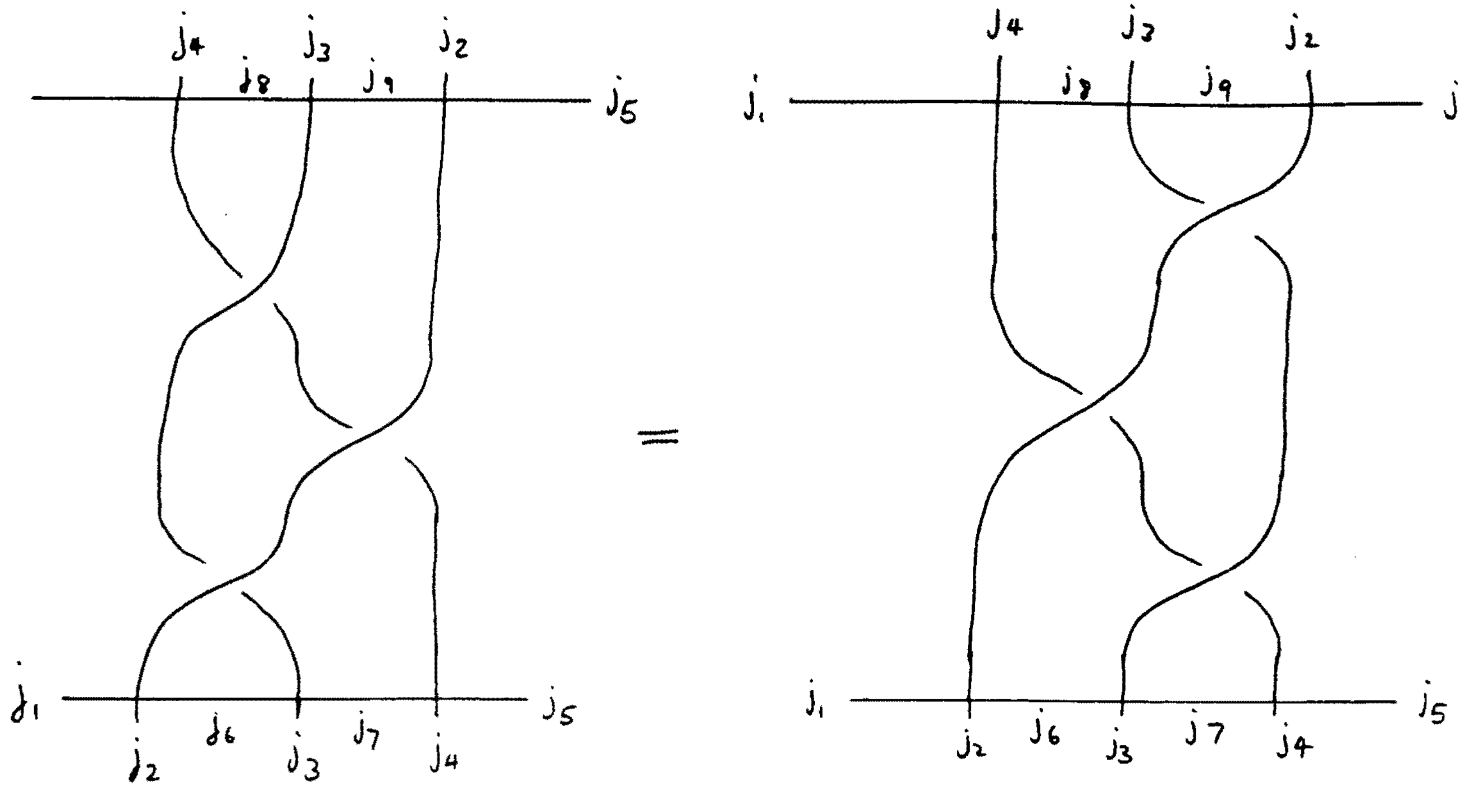
We have thus found three hexagon identities. Show that any one of these hexagons can be deduced from the other two, so there are only two independent hexagons. We will adopt the last two we have just derived.

Now use the equation for $B$ in terms of $F$ to rewrite the braiding/fusing identity:

$$\sum_s F_{p_2 s}\begin{bmatrix} j & k \\ p_1 & b \end{bmatrix} F_{p_1 l}\begin{bmatrix} i & s \\ a & b \end{bmatrix} F_{sr}\begin{bmatrix} i & j \\ l & k \end{bmatrix} = F_{p_1 r}\begin{bmatrix} i & j \\ a & p_2 \end{bmatrix} F_{p_2 l}\begin{bmatrix} r & k \\ a & b \end{bmatrix} \tag{3.8}$$

Interpret this identity as a pentagonal loop of dual diagrams.

Finally, write out the equation corresponding to the figure:




pictorial representation of the Yang-Baxter equation

giving the Yang-Baxter equation:

$$\begin{aligned}\sum_p B_{j_6 p}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_7 \end{bmatrix}(\epsilon) B_{j_7 j_9}\begin{bmatrix} j_2 & j_4 \\ p & j_5 \end{bmatrix}(\epsilon) B_{p j_8}\begin{bmatrix} j_3 & j_4 \\ j_1 & j_9 \end{bmatrix}(\epsilon) = \\ \sum_p B_{j_7 p}\begin{bmatrix} j_3 & j_4 \\ j_6 & j_5 \end{bmatrix}(\epsilon) B_{j_6 j_8}\begin{bmatrix} j_2 & j_4 \\ j_1 & p \end{bmatrix}(\epsilon) B_{p j_9}\begin{bmatrix} j_2 & j_3 \\ j_8 & j_5 \end{bmatrix}(\epsilon)\end{aligned} \tag{3.9}$$

Show that by putting $j_1 = 0$ or $j_5 = 0$ we recover the two hexagon identities. Show moreover that the full Yang-Baxter equation may be deduced from the pentagon and hexagon identities. (Hint: Bring all the $B$ matrices to one side of the equation. Insert $FF^{-1} = 1$ and use the braiding/fusing identity repeatedly.)

---

**The two hexagons and the pentagon are the fundamental genus zero identities.**

---

**• Exercise 3.4** *Gauge Choices.* Note that we did not specify the normalizations $||\Phi^i_{jk}||$ in the definition of the chiral vertex operators. How do the $F, B$ matrices change under a rescaling by $\lambda^i_{jk}$? We refer to such a change as a change of gauge. Show that the polynomial equations of the pervious exercise are gauge invariant.

---

• Exercise 3.5 *Symmetries of the F matrix.* Show, in the case of the discrete series that the matrices satisfy:

$$F_{pp'}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = F_{pp'}\begin{bmatrix} i & l \\ j & k \end{bmatrix}$$
$$= F_{pp'}\begin{bmatrix} l & i \\ k & j \end{bmatrix}$$

Show that these symmetries are gauge invariant. Interpret these symmetries pictorially. In theories other than the minimal models these symmetries typically hold only up to signs. (These signs are described precisely in [15].) When a special choice of gauge is made these matrices sometimes have much more symmetry, similar to the tetrahedral symmetries of Racah coefficients (see below).

---

If we move on to higher genus we get new identities on duality matrices. For example from the one-point block we obtain, as described above, $S_{ii'}(j)$. As is well-known, when the torus is represented as the quotient of the plane by a lattice the square of the transformation $S$ is a 180 degree rotation around the puncture at $z$, so $log z \to -log z$ and we have (in the case where all the representations are self conjugate)

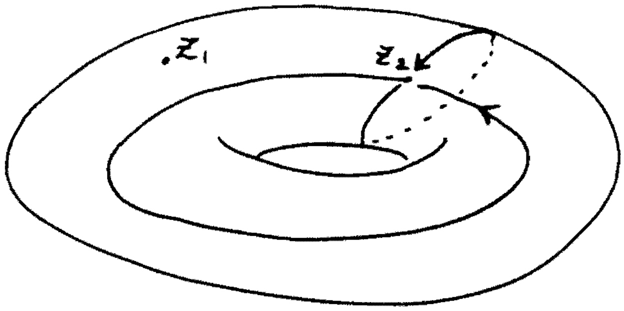
$$S^2(j) = \pm C e^{-i\pi\Delta(\beta)} \tag{3.10}$$

where $C$ is the conjugation matrix on representations and the sign is very similar to the quantity $\xi$ discussed above, and again arises from the symmetry or antisymmetry of a coupling. Similarly we have

$$(ST)^3 = S^2 \tag{3.11}$$

where $T_{jk} = e^{2\pi i(\Delta_j - \frac{c}{24})}\delta_{jk}$. Moving on to several punctures on the torus a new element appears. We may always fix one operator at the standard basepoint, but then there is nontrivial monodromy under the diffeomorphisms which move each of the points around the nontrivial homology cycles of the surface, and around each other.

For a famous example we have for the 2-point function.




two points on a torus with $a, b$ curves

As indicated before, each of these monodromies may be expressed in terms of $F, B, S$. Then the relations of the modular group of the $n$-holed torus imply identities on duality matrices. For example denoting the monodromies of conformal blocks obtained by dragging one operator around the $a, b$ cycles by the same letters $a, b$ we have

$$Sa\ S^{-1} = b\ .$$

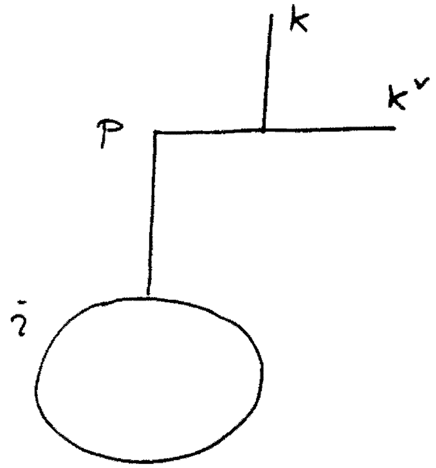
The $a, b$ monodromies can be expressed in terms of $F, B$ matrices. Thus, the above equation implies a new identity relating $F, B, S$. Clearly, these considerations extend to any number of punctures at any genus.

Below we'll begin to bring some order to this chaos of identities. But first let us show that some of these identities can lead to very nice consequences indeed.

For example, the relation $SaS^{-1} = b$ leads to a proof of Verlinde's formula [21]:

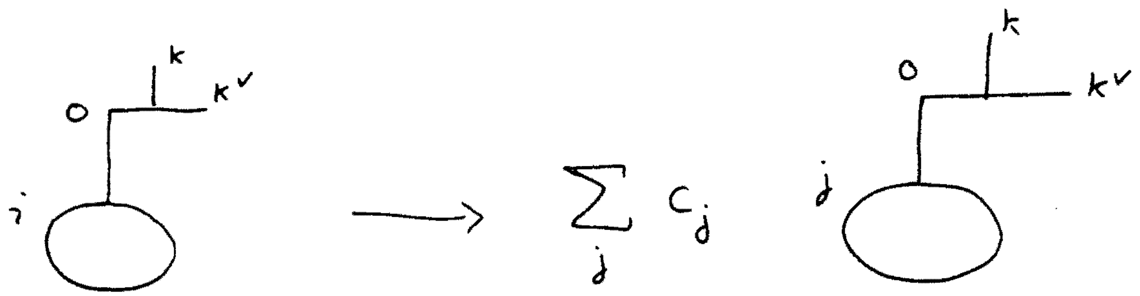
$$N_{ijk} = \sum_{P} \frac{S_{ip} S_{jp} S_{kp}}{S_{0p}}$$

(Here $S = S(0)$, i.e. the transformation matrix on vacuum characters.) To prove this one looks at the blocks:




blocks for two points on the torus in one basis

and computes the $a, b$ monodromies for




transformation of these blocks

Then using the fact that $S$ converts $a$ to $b$ monodromies gives the result. Details are left to the following exercise:

---

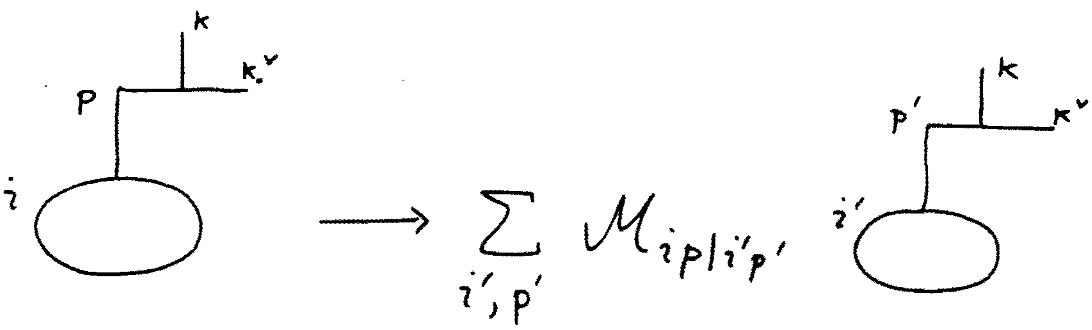
• Exercise 3.6 *Proof of Verlinde's Formula.*

**Verlinde conjectured that the matrix $S = S(0)$ diagonalizes the fusion rule algebra in [21]. There are now, superficially, three different proofs of this statement [13], [27] [28] but all are really equivalent. We will return to a version of Witten's proof later. For now, we proceed with the least elegant, but most straightforward approach.**

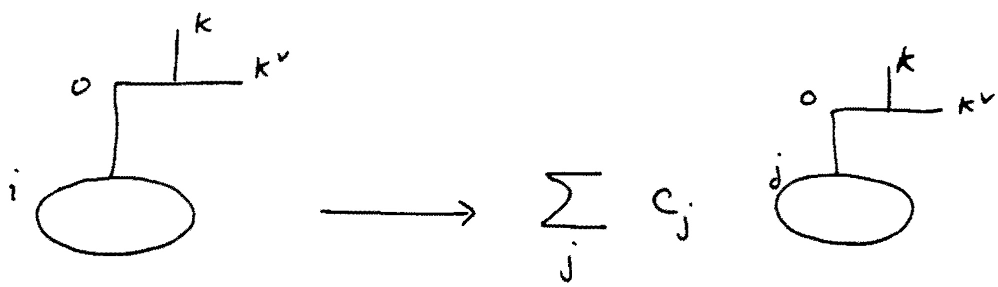
**Consider the discrete series for simplicity. Show that Verlinde's formula**

$$\frac{S_{ij}S_{jk}}{S_{0j}} = \sum_l N_{ikl}S_{lj}$$

**follows from the modular relation $SaS^{-1} = b$ by considering the submatrix element illustrated below:**

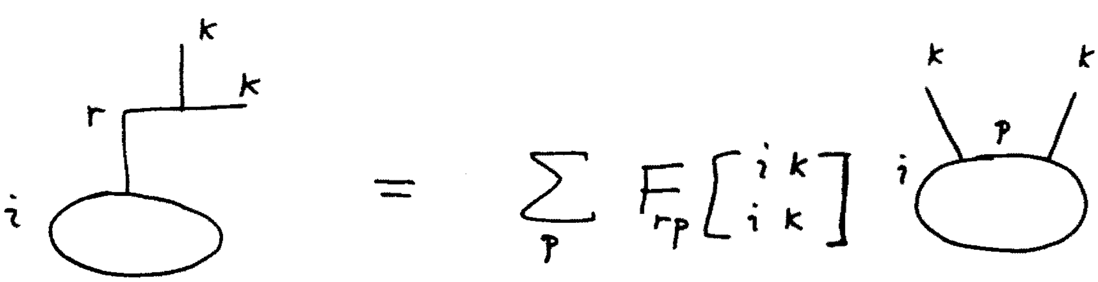



restrict to:




submatrix needed for a proof of Verlinde's formula

Relate the above basis of blocks to the basis




a different basis for the two point function on the torus

Show that the $a$ monodromy in this basis is just $e^{2\pi i(\Delta_i - \Delta_p)}$. Use the identity

$$F_{k0}\begin{bmatrix} i & i \\ j & j \end{bmatrix} F_{0j}\begin{bmatrix} k & i \\ k & i \end{bmatrix} = F_{00}\begin{bmatrix} i & i \\ i & i \end{bmatrix} \equiv F_i$$

to simplify the $b$-monodromy in the original basis and obtain:

$$\sum_j S_{ij} \frac{\left(B\left[\begin{smallmatrix} j & k \\ j & k \end{smallmatrix}\right] B\left[\begin{smallmatrix} k & j \\ j & k \end{smallmatrix}\right]\right)_{00}}{F_k} S_{jl} = N_{ikl}$$

From this derive Verlinde's formula, and show also that

$$\frac{S_{jk}}{S_{00}} = \frac{\left(B\left[\begin{smallmatrix} j & k \\ j & k \end{smallmatrix}\right] B\left[\begin{smallmatrix} k & j \\ j & k \end{smallmatrix}\right]\right)_{00}}{F_k F_j}$$

Note especially the formula for $j = 0$. An argument analogous to the above holds for an arbitrary RCFT.

---

From Verlinde's formula we can deduce many interesting things. As a simple example we can describe the fusion rules for Kac-Moody algebras in a rather elegant way [21]:

---

• Exercise 3.7 *Geometry of the Kac-Moody Fusion Rules.* From Verlinde's formula and the formula for the matrix $S$ of the Weyl-Kac characters show that the one-dimensional representations of the fusion rule algebra:

$$\phi_m \phi_l = \sum_i N^i_{ml} \phi_i$$

in the level $k$ WZW theory are just given by

$$\lambda^{(j)}_m = ch_m\left(2\pi \frac{\mu_j + \rho}{k + h}\right)$$

Here $ch_m$ is the character in the representation $m$, $\mu_j$ is the highest weight of the representation $j$, $\rho$ is the Weyl vector, i.e., half the sum of the positive roots, and $h$ is the dual Coxeter number.

Using this result characterize the fusion rule algebra for the level $k$ WZW theory in terms of reflections in the hyperplane $x \cdot \psi = k + 1$, where $\psi$ is the highest root.

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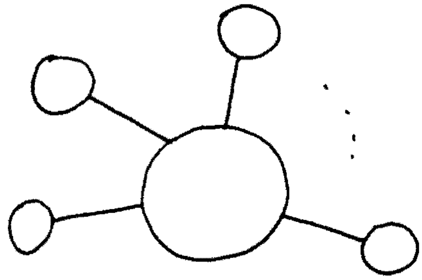
• Exercise 3.8 *Verlinde's Dimension Formula.*

a.) Go to the dual basis for the vacuum characters of the form

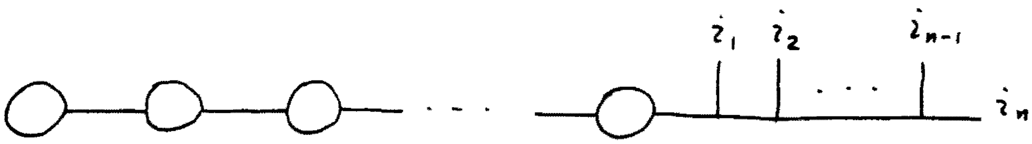


A circle with mirrors emanating from it.

and use Verlinde's formula to show that the dimension of the Friedan-Shenker vector bundle is [21]

$$dim\mathcal{H}(\Sigma_g) = \sum_p \left(\frac{1}{S_{0p}}\right)^{2(g-1)}$$

b.) Go to the dual basis for the n-point functions of the form




to show that the formula for the case with punctures in representations $i_1, \ldots, i_n$ is given by

$$dim\mathcal{H}(\Sigma_g; (P_1, i_1), \ldots (P_n, i_n)) = \sum_p \left(\frac{1}{S_{0p}}\right)^{2(g-1)} \frac{S_{i_1 p}}{S_{0p}} \cdots \frac{S_{i_n p}}{S_{0p}}$$

c.) Verify that $S^2 = C$ guarantees the dimensions behave as expected under sewing.

d.) Substitute the Kac-Peterson formula for $S_{ij}$ into the formula of part (a) to show that for level $k$ WZW theory with simple and simply connected group $G$ we have [29]:

$$dim(\mathcal{H}(\Sigma_g)) = (k+h)^{g-1}(|\Lambda_{rt}^*/\Lambda_{rt}|)^{g-1} \sum_\lambda \frac{1}{\prod_{\alpha\in\Delta}(1 - e^{i\langle\alpha,\theta_\lambda\rangle})^{g-1}}$$

Here $h$ is the dual Coxeter number, $\Lambda_{rt}$ is the root lattice, $\Delta$ is the set of roots, the sum runs over weight vectors $\lambda$ defining level $k$ integrable highest weight representations of the current algebra, and $\theta_\lambda = 2\pi\frac{\lambda+\rho}{k+h}$ is the conjugacy class canonically associated to the Kac-Moody integrable representation $\lambda$. Verlinde has conjectured that this formula can be derived as a fixed point theorem, but such an interpretation has not yet been given.

e.) Write the formula explicitly for $\hat{SU}(2)_k$ and show that, as $k \to \infty$, we have $dim \mathcal{H}(\Sigma) \sim k^{3g-3}$. This behavior is very natural from the Chern-Simons gauge theory viewpoint explained below.

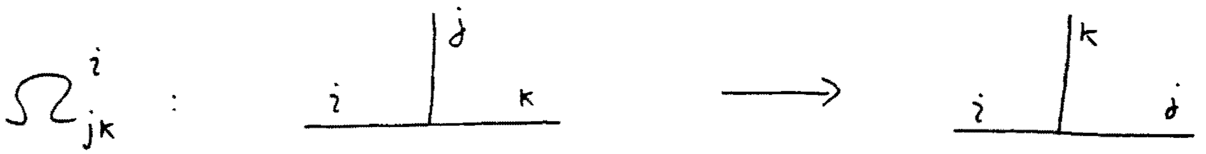
---

## 4. Completeness

In the previous section we said that all duality transformations are expressed in terms of a finite amount of data: $F, B, S$. However, there seemed to be a proliferation of identities. The completeness theorem states that, in fact the number of independent identities is finite.

From the exercises you know that a special case of the B-matrix is

$$\Omega^i_{jk} : V^i_{jk} \to V^i_{kj}$$




a pictorial representation of $\Omega$

Its eigenvalues are just the square roots of mutual locality factors.

The basic genus zero identities are

1) The pentagon

$$\sum_s F_{p_2 s}\begin{bmatrix} j & k \\ p_1 & b \end{bmatrix} F_{p_1 l}\begin{bmatrix} i & s \\ a & b \end{bmatrix} F_{sr}\begin{bmatrix} i & j \\ l & k \end{bmatrix} = F_{p_1 r}\begin{bmatrix} i & j \\ a & p_2 \end{bmatrix} F_{p_2 l}\begin{bmatrix} r & k \\ a & b \end{bmatrix} \tag{4.1}$$

2) The two hexagons

$$\Omega^m_{lk}(\epsilon) F_{mn}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \Omega^l_{jk}(\epsilon) = \sum_r F_{mr}\begin{bmatrix} j & l \\ i & k \end{bmatrix} \Omega^i_{kr}(\epsilon) F_{rn}\begin{bmatrix} k & j \\ i & l \end{bmatrix} \tag{4.2}$$

3) At $g = 1$ there are 3-more identities:

$$S^2(j) = \pm C e^{-i\pi\Delta_j}$$

$$(ST)^3 = S^2$$

$$S a S^{-1} = b$$

Using these identities we can check all the relations on $F, B, S$ following from duality on all surfaces. One would like to present the equations in the most economical possible way. In fact, the last torus equation $SaS^{-1} = b$, which is rather complicated when written out with all its indices, contains a great deal of redundant information. Some of the equations implied by $SaS^{-1} = b$ can be used to solve for $S(p)$ in terms of the braiding and fusing matrices and the normalization term $S_{00}(0)$. In the case of the discrete series, the explicit formula one finds is

$$S_{ij}(p) = S_{00}(0)e^{-i\pi\Delta_p} \frac{F_{i0}\begin{bmatrix} i & i \\ p & p \end{bmatrix}}{F_p F_{p0}\begin{bmatrix} j & j \\ j & j \end{bmatrix} F_{p0}\begin{bmatrix} i & i \\ i & i \end{bmatrix}} \sum_r B_{pr}\begin{bmatrix} i & j \\ i & j \end{bmatrix}(-) B_{r0}\begin{bmatrix} j & i \\ i & j \end{bmatrix}(-) \tag{4.3}$$

and a similar formula holds for an arbitrary RCFT. (The only complication in the general case are some signs measuring the antisymmetry of certain couplings.) This expression is a generalization to arbitrary $p$ of the expression in [13][14]. A nontrivial computation (outlined in section seven below) shows that once this expression is substituted into the remaining equations implied by $SaS^{-1} = b$ one finds no new conditions on $F, B$. Hence, in specifying the fundamental equations, the above three torus equations can be replaced by the definition (4.3) together with the constraint of the first two torus equations, determining that $S$ define a representation of the modular group.

---

• Exercise 4.1 *Example of the Ising model.* Check (4.3) in the Ising model. In this case we have three representations $1, \psi, \sigma$ with the famous fusion rule algebra:

$$\begin{aligned} \psi \times \psi &= 1 \\ \psi \times \sigma &= \sigma \\ \sigma \times \sigma &= 1 + \psi \end{aligned} \tag{4.4}$$

choose a gauge by demanding that:

$$F\begin{bmatrix} \sigma & \psi \\ \sigma & \psi \end{bmatrix} = F\begin{bmatrix} \psi & \sigma \\ \psi & \sigma \end{bmatrix} = F\begin{bmatrix} \psi & \psi \\ \sigma & \sigma \end{bmatrix} = F\begin{bmatrix} \sigma & \sigma \\ \psi & \psi \end{bmatrix} = 1 \tag{4.5}$$

Then show, either by solving the polynomial equations, or by using explicit conformal blocks that we have

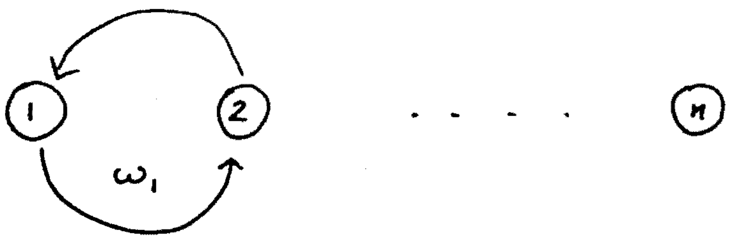
$$F\begin{bmatrix}\psi & \psi\\ \psi & \psi\end{bmatrix} = 1$$
$$F\begin{bmatrix}\sigma & \psi\\ \psi & \sigma\end{bmatrix} = -1$$
$$F\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{4.6}$$
$$B\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}(-) = \frac{e^{-i\pi/8}}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$$

And substitute these into (4.3) . Note, in particular that for the one-point function of $\psi$ the block $\eta(\tau)(dz)^{1/2}$ gives $S(\psi) = e^{-i\pi/4}$, as predicted by (4.3) .

---

Strictly speaking – only the following cases have been carefully checked in all details: ($g = 0$, $n$ holes), ($g = 1$, $n$ holes), ($g$, $n = 0$). We have no doubt that the remaining cases will also work (an argument is given in [15]), but what is needed is a better understanding of why the result should be true which will lead to a more conceptual proof, which should handle all cases simultaneously.

Here we will describe part of the $g = 0$ case in detail. To begin recall the generators and relations for the modular group of the sphere with n holes. The generators are: Firstly, $R_i$ = a Dehn twist around the $i^{th}$ hole. Equivalently, this is a transformation on a local choice of coordinate $dz \to e^{2\pi i}dz$. Secondly, $\omega_i$ = interchange holes $i$ and $i+1$ The action of the generators $\omega_i$ may be pictured as follows:




Illustrations of one of the generating modular transformations.

**The idea of the proof is the following. Recall the simplicial complex from section 3 which is built by declaring that:**

vertices $\longrightarrow$ dual diagrams

edges $\longrightarrow$ simple moves

Define a 2-complex by filling in all faces corresponding to pentagons/hexagons and - in the high genus case - the torus relations. There are no new relations, if the resulting complex is simply connected.

The question can be reduced - in a way which will be indicated below to checking the relations of the modular group. So let's worry about these. The relations we must check are:

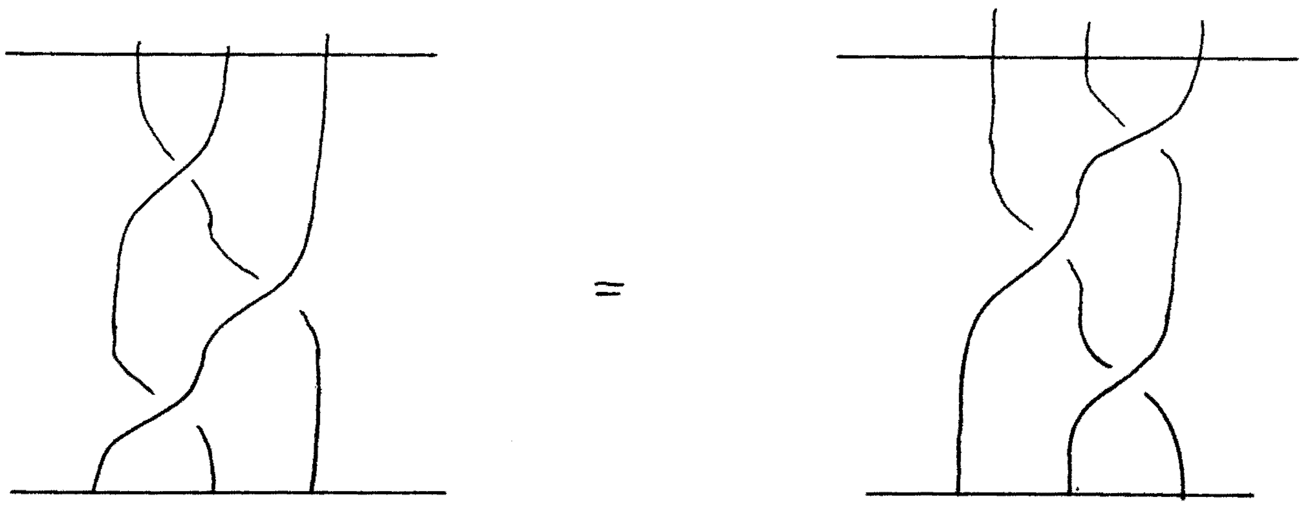
A.

$$\omega_i \omega_j = \omega_j \omega_i \ |i - j| \geq 2$$

$$\omega_i R_j = R_j \omega_i$$

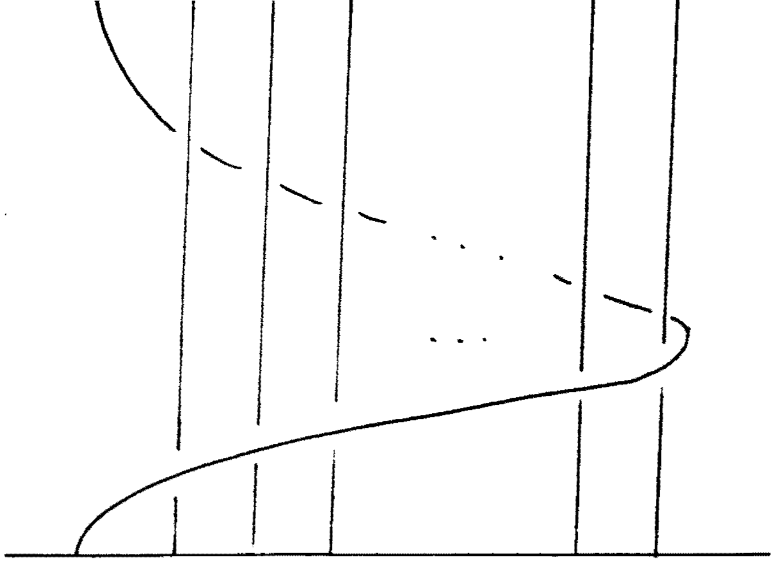
B.

$$\omega_i \omega_{i+1} \omega_i = \omega_{i+1} \omega_1 \omega_{i+1}$$



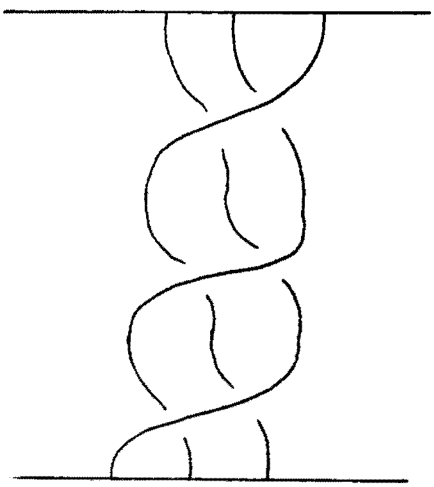
C.

$$\omega_1 \cdots \omega_{n-1}^2 \cdots \omega_1 = R_1^2$$

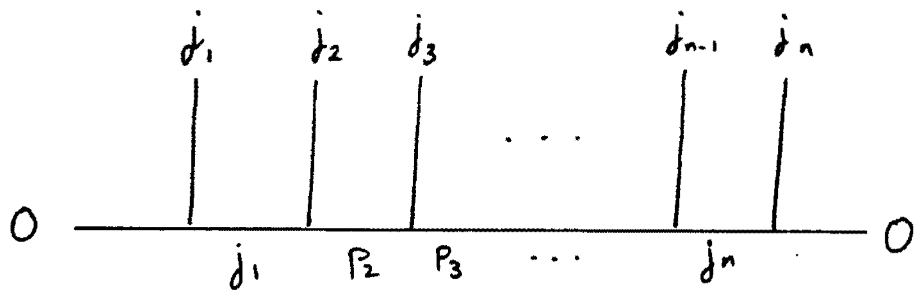


D.

$$(\omega_1 \cdots \omega_{n-1})^n = \Pi R_i$$



Now checking these relations is quite easy. We use the basis of blocks:



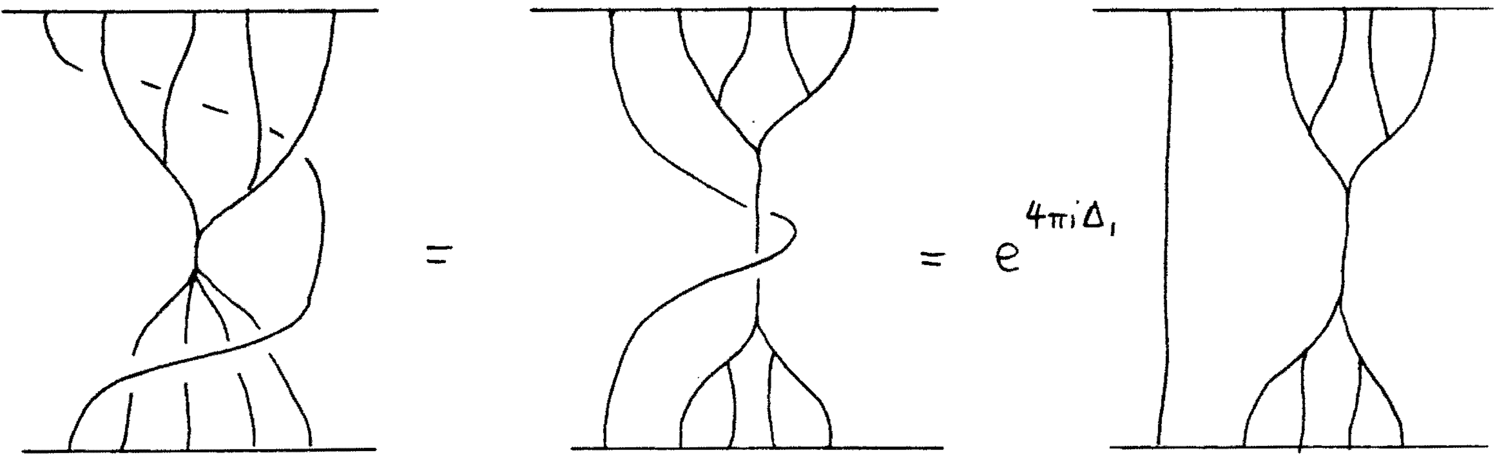

multiperipheral basis

So the representation is just:

$$\rho(R_k) = e^{2\pi i \Delta j_k}$$

$$\rho(\omega_k) = 1 \otimes \cdots \otimes B \begin{bmatrix} j_k & j_{k+1} \\ p_{k-1} & p_{k+2} \end{bmatrix} \otimes \cdots \otimes 1$$

Relation (A) is obviously satisfied. One easily checks that (B) follows from the Yang-Baxter relations. To check (C) we use braiding fusing:




Finally we check (D) similarly.

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• Exercise 4.2 *The barber pole.* Use the braiding/fusing identity and induction to verify the barber pole relation:

$$(\omega_1 \cdots \omega_{n-1})^n = \prod R_i$$

---

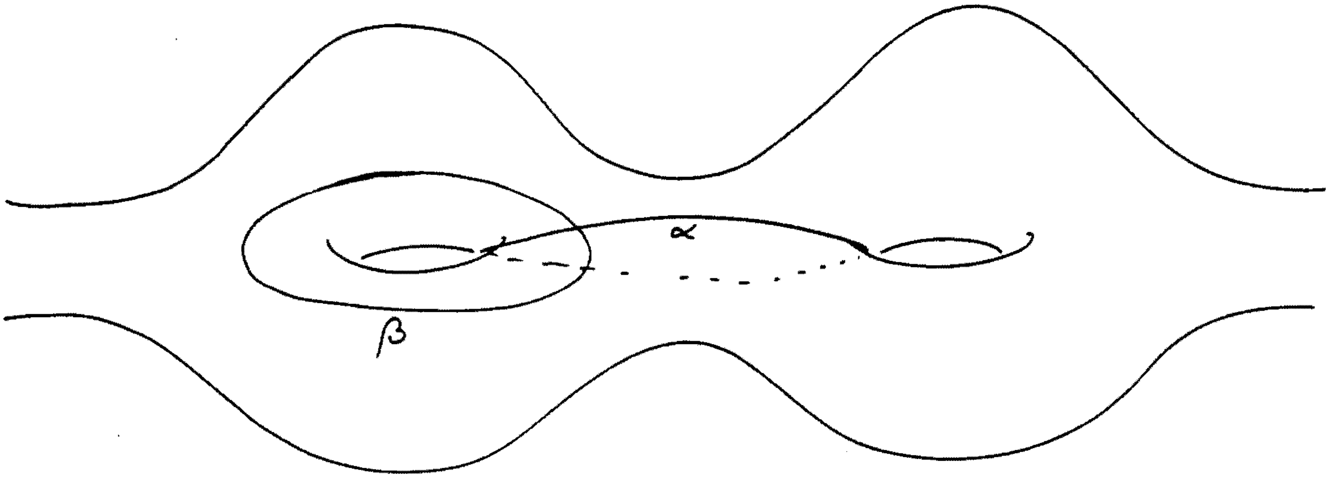
**With considerably more work we can go on to check the modular relations at high genus. An example of a rather tractable one is:**

---

• **Exercise 4.3** ***A Simple High-Genus Relation.***

**a.) Rewrite the equation $(ST)^3 = S^2$ as the equation $\alpha\beta\alpha = \beta\alpha\beta$ where $\alpha, \beta$ are Dehn-twists around the $a, b$ cycles, respectively. (Hint: Show that $\alpha = T^{-1}$ and $\beta = TST$.)**

**b.) Verify geometrically the relation $\alpha\beta\alpha = \beta\alpha\beta$ in the modular group at any genus from the configuration of curves shown below:**




(Hint: Show that the product of Dehn twists $\alpha\beta\alpha^{-1}$ is a single Dehn twist around the image under $\alpha$ of the curve $\beta$.)

c.) Why is this not a new high genus relation on duality matrices?

---

One should still prove that it is enough to check the relations of the modular group. On the sphere, the argument is inductive in the number of external lines. The basic idea in the proof is to use the pentagon to show that there are no new identities from a set of duality transformations starting and ending in the multiperipheral basis. Then, it follows that every closed loop of transformations in the duality complex is homotopically equivalent to a closed loop of transformations in the multiperipheral basis. These transformations form the modular group. Since all the relations in this group are satisfied in this basis, there are no new identities.

The completeness theorem strongly suggests that the equations come close to defining RCFT. Specifically, what it does show is that a solution to the equations allows one to define transition functions for a compatible family of Friedan-Shenker vector bundles on all moduli spaces. This statement can be reformulated in a language currently much in vogue, which we now explain.

In Friedan-Shenker modular geometry the existence of a projectively flat vector bundle means that the data defining the bundle is essentially topological, involving (projective) representations of the Teichmüller modular group. Graeme Segal abstracted the concept, implicitly used from the earliest days of dual model theory and somewhat more precisely described in [1][24][21][13] to the notion of a ***modular functor***. A modular functor may be specified by the following data and axioms:

---

**Axioms for a Modular Functor**

*Data:*

1. Representation labels: A finite set $I$ of labels (i.e. the representations of the chiral algebra) with a distinguished element $0 \in I$ and an involution $i \to i^{\check{}}$ such that $0^{\check{}} = 0$.

2. Conformal blocks: A map

$$(\Sigma, (i_1, v_1, P_1), \dots (i_n, v_n, P_n)) \to \mathcal{H}(\Sigma; (i_1, v_1, P_1), \dots (i_n, v_n, P_n)))$$

from oriented surfaces with punctures, each puncture $P_r$ being equipped with a direction $v_r$ and a label $i_r$, to vector spaces.

3. Duality transformations: A linear transformation $\mathcal{H}(f) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ associated to an automorphism $\Sigma_1 \to \Sigma_2$ (and similarly for punctures).

*Conditions:*

1. Functoriality: $\mathcal{H}(f)$ depends only on the isotopy class of $f$. Thus the mapping class group acts on $\mathcal{H}(\Sigma)$, (and similarly for punctures).

2. Involution: If bar denotes reversal of orientation and application of the involution to the representations then $\mathcal{H}(\bar{\Sigma}) \cong \mathcal{H}(\Sigma)^{*}$.

3. Multiplicativity: $\mathcal{H}(\Sigma_1 \coprod \Sigma_2) \cong \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$.

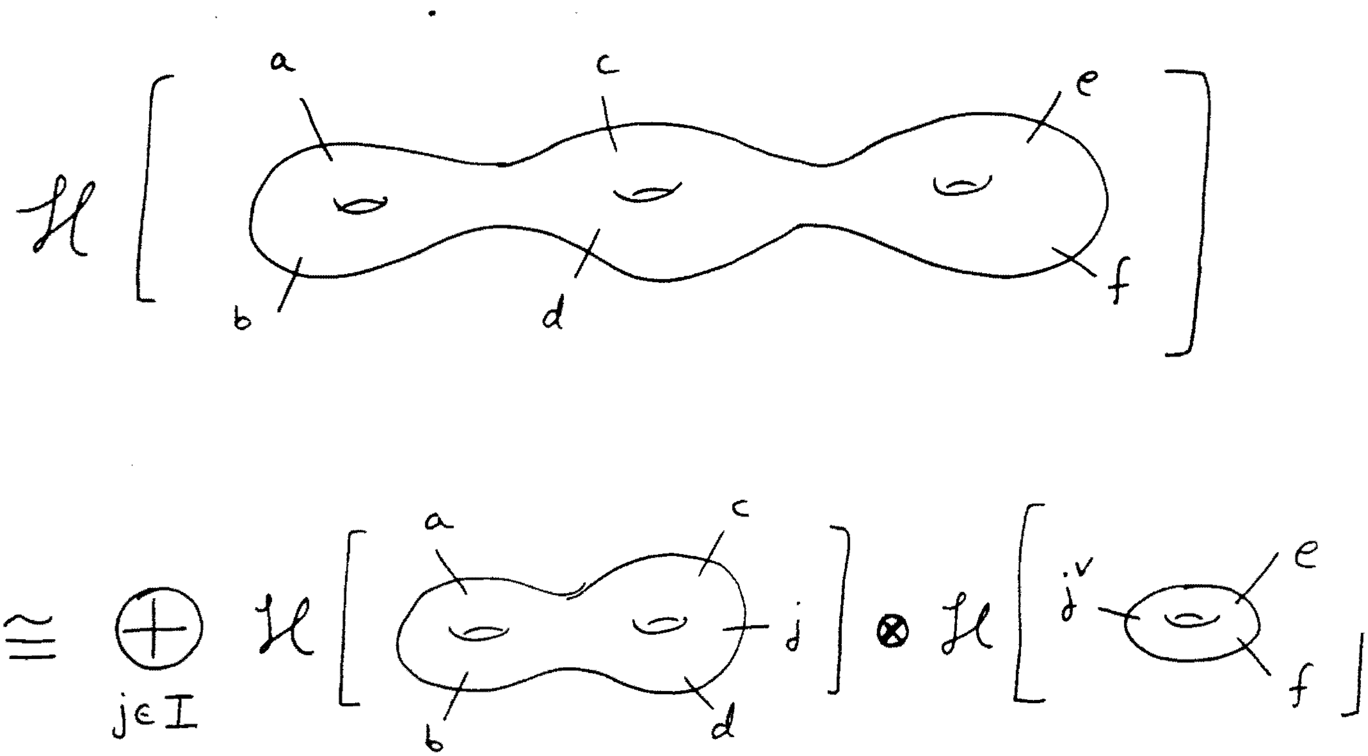
4. Gluing: Pinching $(\Sigma, (i_1, v_1, P_1), \dots (i_n, v_n, P_n))$ along a cycle to obtain a surface (possibly connected or disconnected) $(\tilde{\Sigma}, (i_1, v_1, P_1), \dots (i_n, v_n, P_n), (j, v, P), (j^{\check{}}, v, \tilde{P}))$ with a pair of identified punctures $P, \tilde{P}$ defines vector spaces related by

$$\mathcal{H}(\Sigma; (i_1, v_1, P_1), \dots (i_n, v_n, P_n)) \cong \oplus_{j \in I} \mathcal{H}(\tilde{\Sigma}; \dots (i_n, v_n, P_n), (j, v, P), (j^{\check{}}, v, \tilde{P})).$$

5. Normalization. $\mathcal{H}(S^2; (j, P)) \cong \delta_{j,0} \cdot \mathbf{C}$.

---

The all-important gluing axiom may be illustrated by the figure:




The directions $v_r$ at the punctures are needed to keep track of the nontrivial effects of Dehn twists around the punctures. Geometrically they are needed since conformal blocks should be thought of as differentials on the surface $\Sigma$, i.e., $\mathcal{F} \sim f(z_1, \ldots z_n, \ldots)(dz_1)^{\Delta_1} \cdots (dz_n)^{\Delta_n}$. This subtlety, which shows up in the three-dimensional point of view in the need for framings of links, was first emphasized in [26].

In an obvious way one can change the definitions to define a modular functor which is projective, unitary, and so forth. In this language the completeness theorem states that from a finite amount of data $F, B, S$ satisfying a finite number of conditions one can construct a projective modular functor.

The idea of a modular functor is truly beautiful and allows us to ask many interesting questions in a succinct way. For example we may ask to what extent a modular functor characterizes a rational conformal field theory. Since there are nontrivial theories with trivial modular functors this is a serious question. Or, we may ask if every modular functor arises in some conformal field theory. Simply defining the bundles is not enough for defining physical correlation functions. Whether these bundles have reasonable sections which correspond to blocks in a CFT is another matter which remains undecided. However, there is a closely related problem in mathematics where the answer is known to be in the affirmative in a suitably defined sense, namely the Tannaka-Krein approach to group theory – so we discuss this next.

## 5. Tannaka-Krein theory and Modular Tensor Categories

Let us switch our attention momentarily to an apparently different problem - we want to characterize the sets:

Rep(G) = $\{V | V$ is finite dimensional representation of G$\}$

For example, let us consider $G$ to be a compact Lie group, then there are the most important elements

$$R_i = \text{irreducible representations}$$

Moreover, we can decompose

$$R_i \otimes R_j = \cdots \underbrace{\otimes R_k \otimes \cdots \otimes R_k \otimes}_{n^k_{ij} \text{ times}} \cdots$$

$$= \oplus_k V^k_{ij} \otimes R_k$$

with

$$\dim V^k_{ij} = n^k_{ij}$$

The spaces $V^k_{ij}$ are characterized as the space of a certain kind of intertwiner. Recall that if $W_{1,\rho_1}$ and $W_{2,\rho_2}$ are two representations (that is, $\rho_1 : G \longrightarrow \mathrm{End}(W_1)$ is a homomorphism, etc.) Then an intertwiner $T : W_1 \longrightarrow W_2$ is a group - equivariant map, i.e.

$$\begin{array}{ccc} W_1 & \xrightarrow{T} & W_2 \\ \rho_1(g) \downarrow & & \downarrow \rho_2(g) \\ W_1 & \xrightarrow{T} & W_2 \end{array} \qquad (5.1)$$

commutes for all $g \in G$. In this language $V^k_{ij} = \{intertwiners : R_k \longrightarrow R_i \otimes R_j\}$ e.g. in $SU(2)$ the space of intertwiners is always zero or one-dimensional and is spanned by

$$\begin{pmatrix} J \\ j_1 j_1 \end{pmatrix} = \sum_{M, M_1, M_2} |m_1 j_1 m_2 j_2 >< m_1 j_1 m_2 j_2 | MJ >< M, J|$$

where $< m_1 j_1 m_2 j_2 | MJ >$ are Clebsch-Gordon coefficients.

Now we will examine some nontrivial properties satisfied by these vector spaces, these follow from rather obvious isomorphisms of representations. First, we have the evident isomorphism $\Omega : R_i \otimes R_j \cong R_j \otimes R_i$ since the map $x \otimes y \longrightarrow y \otimes x$ is an intertwiner.

Therefore, if we decompose the above tensor products of representations we learn that:

$$\Omega : V^k_{ij} \cong V^k_{ji}$$

When manipulating these spaces of intertwiners it is good to develop a pictorial notation. Denote

$$V^k_{ij} = \underline{k \;|^{i}\; j}$$

Then

$$\Omega : \underline{k \;|^{i}\; j} \longrightarrow \underline{k \;|^{j}\; i}$$

Note well that an obvious consequence of the fact that the transformation $\Omega$ squares to one is that

$$\Omega^2 = 1$$

as a transformation on $V^k_{ij}$. Thus, when $i = j$ we can diagonalize $\Omega$, the eigenvalues are $\pm 1$ depending on the symmetry of the coupling.

Now consider the second evident isomorphism:

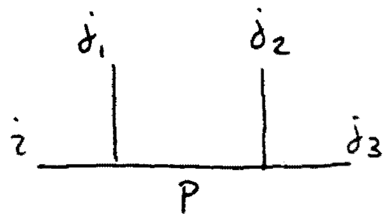
$$F : R_{j1} \otimes (R_{j2} \otimes R_{j3}) \cong (R_{j1} \otimes R_{j2}) \otimes R_{j3}$$

$$x \otimes (y \otimes z) \longrightarrow (x \otimes y) \otimes z$$

When decomposing in terms of irreducible representations we meet compositions of intertwiners, for example we find:

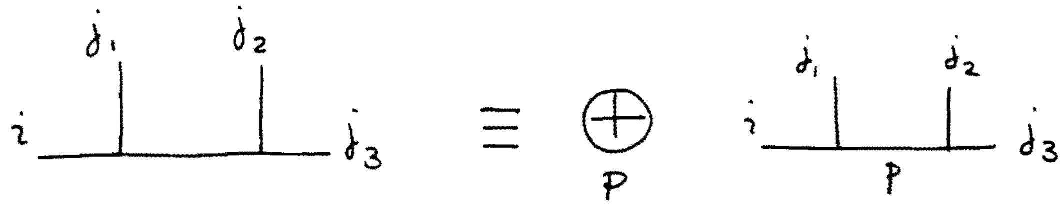
$$\begin{pmatrix} i \\ j_1 p \end{pmatrix} \begin{pmatrix} p \\ j_2 j_3 \end{pmatrix} : R_{j_1} \otimes R_{j_2} \otimes R_{j_3} \to R_i$$

Carrying our pictorial notation further we denote the tensor product of a spaces of intertwiners by

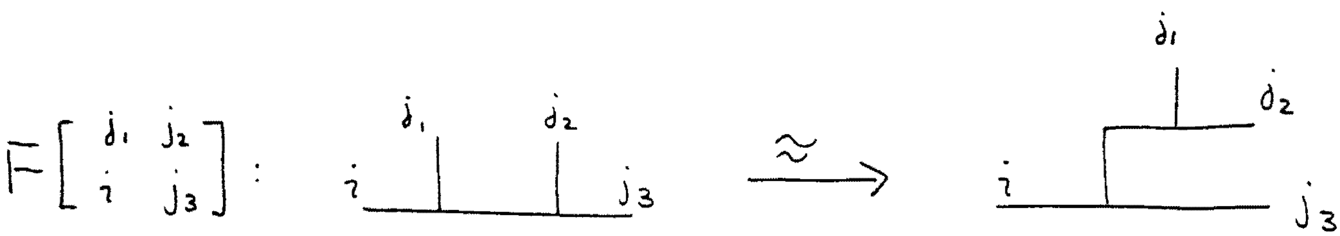



$$\equiv V^{i}_{j_1 p} \otimes V^{p}_{j_2 j_3}$$

If we have a direct sum of these spaces over "intermediate" representations, then we denote the resulting vector space by




Thus, decomposing the second isomorphism in terms of irreducible representations we learn that there must be a transformation:




picture for $F$

Or, in formulas:

$$F\begin{bmatrix} j_1 & j_2 \\ i & j_3 \end{bmatrix} : \oplus_r V^i_{j1,r} \otimes V^r_{j2,j3} \longrightarrow \oplus_s V^i_{s,j_3} \otimes V^s_{j1,j2}$$

In the physics literature the intertwiners are known as Clebsch-Gordan coefficients (3j symbols) and the $F$'s are known as $6j$ or Racah coefficients. Moreover, the fact that $F$ is an isomorphism implies that $n^k_{ij}$ defines a commutative associative algebra which is, in fact, the character ring of the group.

Now the two isomorphisms of representations $\Omega$ and $F$ satisfy simple compatibility conditions. The first is the pentagon relation:

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes (R_3 \otimes R_4)) & \xrightarrow{F} & (R_1 \otimes R_2) \otimes (R_3 \otimes R_4) & \xrightarrow{F} & ((R_1 \otimes R_2) \otimes R_3) \otimes R_4 \\ \downarrow 1 \otimes F & & & & \downarrow F \otimes 1 \\ R_1 \otimes ((R_2 \otimes R_3) \otimes R_4) & & \xrightarrow{F} & & (R_1 \otimes (R_2 \otimes R_3)) \otimes R_4 \end{array} \tag{5.2}$$

for representations $R_1, \ldots, R_4$. The second is the hexagon relation:

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes R_3) & \xrightarrow{F} & (R_1 \otimes R_2) \otimes R_3 & \xrightarrow{\Omega} & R_3 \otimes (R_1 \otimes R_2) \\ \downarrow 1 \otimes \Omega & & & & \downarrow F \\ R_1 \otimes ((R_3 \otimes R_2) & \xrightarrow{F} & (R_1 \otimes R_3) \otimes R_2 & \xrightarrow{\Omega \otimes 1} & (R_3 \otimes R_1) \otimes R_2 \end{array} \tag{5.3}$$

Decomposing these relations in terms of irreducible representations we learn that $F$, $\Omega$ satisfy two corresponding compatibility conditions

$$\sum_s F_{p_2 s}\begin{bmatrix} j & k \\ p_1 & b \end{bmatrix} F_{p_1 l}\begin{bmatrix} i & s \\ a & b \end{bmatrix} F_{sr}\begin{bmatrix} i & j \\ l & k \end{bmatrix} = F_{p_1 r}\begin{bmatrix} i & j \\ a & p_2 \end{bmatrix} F_{p_2 l}\begin{bmatrix} r & k \\ a & b \end{bmatrix} \tag{5.4}$$

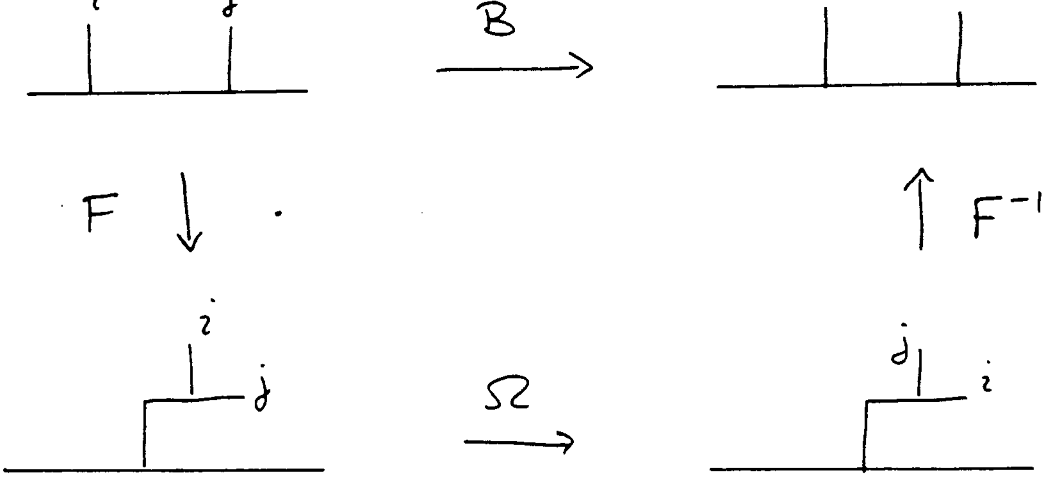
$$\Omega^m_{lk} F_{mn}\begin{bmatrix} j & k \\ i & l \end{bmatrix} \Omega^l_{jk} = \sum_r F_{mr}\begin{bmatrix} j & l \\ i & k \end{bmatrix} \Omega^i_{kr} F_{rn}\begin{bmatrix} k & j \\ i & l \end{bmatrix} \tag{5.5}$$

In the case of $SU(2)$, these relations are known in the physics literature as the Biedenharn sum-rule and Racah's sum-rule.

In category theory there is a theorem, called the MacLane coherence theorem that states that the above two identities are the full set of independent identities on $F, \Omega$.

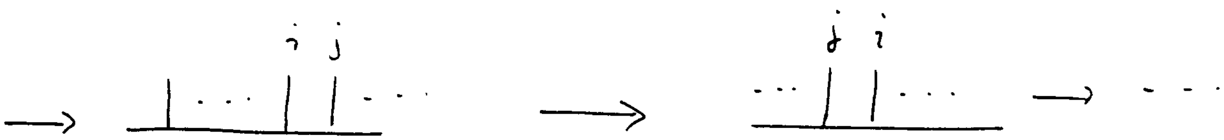
Let us describe the idea of the proof:

Define a simplicial complex where vertices correspond to dual diagrams and edges correspond to simple moves between diagrams. Label these edges by $F, \Omega$ etc. Fill in the pentagons and hexagons to get a two-complex, and show that the resulting two-complex is simply connected. There are two kinds of loops, those involving only the $F$ move and those involving $F, \Omega$. Define the following composite move:




*B* from *F* and Ω

By the pentagon and hexagon we can deform all loops to those involving only multi-peripheral diagrams:



multiperipheral diagrams

Then we need only check that $B$ satisfies the relations of the symmetric group.

There is a clear analogy here with rational conformal field theory, and we have now arrived at the point we were at with RCFT. In the case of group theory it turns out one can go further and state a partial converse to the above results. We would like to know if all solutions to the above axioms in fact come from group theory. It turns out there are solutions to the previous equations that do not come from groups, but we can eliminate these by adding two more axioms.

The first axiom corresponds to the existence of the trivial representation $R_{i=0} = \mathbb{C}$. Note that we have:

$$V^j_{i0} \cong V^j_{0i} \cong \delta^j_i \mathbb{C}$$

Every representation has a conjugate representation:

$$(R_i)^\vee = R_{i^\vee}$$

and $R_i \otimes R_j$ contains the singlet only if $j = i^\vee$, so

$$V_{ij}^0 = \delta_{ij^\vee}\mathbb{C} \ .$$

The second axiom that we must add, which is due to Deligne, [30] involves the special fusion coefficient (for the case $\Omega_{ii}^0 = 1$)

$$F_i = F_{00}\begin{bmatrix} i & i^\vee \\ i & i^\vee \end{bmatrix} :$$

Namely consider the composition

$$R_0 \longrightarrow (R_i)^\vee \otimes R_i \longrightarrow R_0$$

$$1 \longrightarrow \Sigma v_\alpha^\vee \otimes v_\alpha \longrightarrow dim\ R_i$$

We have a map of a one-dimensional vector space to itself, which is, canonically, a complex number. One can compute the value of this number by decomposing in terms of intertwiners, and one finds the answer $\frac{1}{F_i}$.

---

• Exercise 5.1 *Deligne's condition in terms of $F_i$.* By considering the sequence

$$R_i \cong R_0 \otimes R_i \to (R_i^\vee \otimes R_i) \otimes R_i \to (R_i \otimes R_i^\vee) \otimes R_i \to R_i \otimes (R_i^\vee \otimes R_i) \to R_i \otimes R_0 \cong R_i$$

and decomposing the tensor products into irreducible representations, show that

$$\frac{1}{F_i} = \dim\ R_i$$

---

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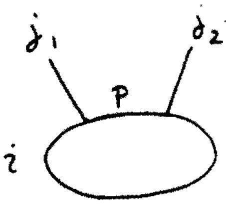
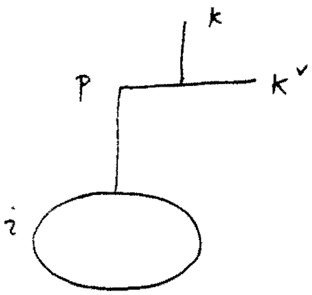
• Exercise 5.2 *Another proof of Deligne's Condition in terms of $F_i$.* Consider the "group theoretic one-loop two-point function":



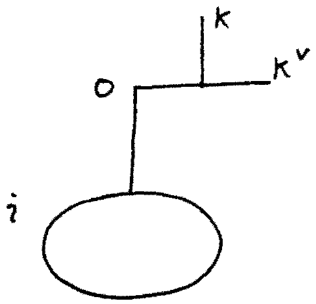

$$= \quad tr_{R_i}\rho_i(g)\binom{i}{j_1 p}(\beta_1 \otimes \cdot)\binom{p}{j_2 i}(\beta_2 \otimes \cdot)$$

Group theoretic two point function

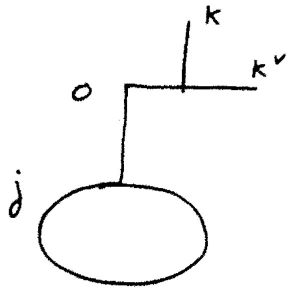
Consider the "monodromy" under $\beta_2 \rightarrow \rho_{j_2}(g)\beta_2$, Where $\binom{i}{jk}$ denote intertwiners. Using the basis of tensors:




show that the monodromy is just:




$$\longrightarrow \quad \sum_j F_k \, n^j_{ik}$$



Take the limit $g \to 1$. Show that the other terms vanish and deduce that

$$\frac{1}{F_k} dim\ R_i = \sum_j n^j_{ik} dim\ R_j$$

and hence

$$\frac{1}{F_k} = dim\ R_k$$

---

The nontrivial result is that these axioms now characterize group theory. This is due to Deligne [30] and, in a slightly different form to Doplicher and Roberts [31] [32]. More precisely, suppose we are given the following:

---

**Axioms for a Tannakian Category**

*Data:*

1. An index set $I$ with a distinguished element 0 and a bijection of $I$ to itself written $i \mapsto i^\vee$.

2. Vector spaces: $V^i_{jk}$ $i, j, k \in I$, with $dim V^i_{jk} = N^i_{jk} < \infty$

3. Isomorphisms:

$$\Omega^i_{jk} : V^i_{jk} \cong V^i_{kj}$$
$$F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix} : \oplus_r V^{i_1}_{j_1 r} \otimes V^r_{j_2 k_2} \cong \oplus_s V^{i_1}_{s k_2} \otimes V^s_{j_1 j_2} \tag{5.6}$$

*Conditions:*

1. $(i^\vee)^\vee = i$ and $0^\vee = 0$.

2. $V^i_{0j} \cong \delta_{ij} C \quad V^0_{ij} \cong \delta_{ij} C \quad V^i_{jk} \cong V^k_{ji} \quad (V^i_{jk})^* \cong V^i_{jk}$

3. $\Omega^i_{jk} \Omega^i_{kj} = 1$ .

4. The identities:

$$F(\Omega \otimes 1)F = (1 \otimes \Omega)F(1 \otimes \Omega)$$

$$F_{23} F_{12} F_{23} = P_{23} F_{13} F_{12}$$

5. The normalization condition:

$$F_i^{-1} \in \mathbb{Z}_+$$

---

From such a set of axioms we can reconstruct a group for which $V^i_{jk}$ are the intertwiners, $F$, the Racah coefficients, etc.

The proof of this result is rather involved, but it would probably be worthwhile to sketch the main ideas of reconstruction which proceeds as follows:

a) Define vector spaces $R_i = \mathbb{C}^{n_i}$, obviously. (In category theory these correspond to simple objects which we must realize with honest vector spaces.)

b) Define the space of intertwiners (morphisms) to be:

$$\begin{aligned} Hom(R_i \longrightarrow R_i) &= \mathbb{C} \\ Hom(R_i \longrightarrow R_j) &= O \; i \neq j \end{aligned}$$

and extend by linearity to $Hom(\oplus R \longrightarrow \oplus R)$.

c) Define tensor products: $R_i \otimes R_j \cong \oplus V^k_{ij} \otimes R_k$. That is, $V^k_{ij}$ is a set of intertwiners.

Now we define the set $\mathbf{Rep} = \{$all sums, products, quotients, duals of the $\mathrm{R_i}\}$.

d) Finally define the set of families of linear transformations:

$$\begin{aligned} \mathcal{G} = \{(\lambda_x)_{x\in\mathbf{Rep}} | \forall x, \lambda_x : x \longrightarrow x \; &is\ an\ invertible\ linear\ transformation. \\ \lambda_{x\otimes y} &= \lambda_x \otimes \lambda_y \\ T : x \rightarrow y \; &\text{ an intertwiner} \Rightarrow T\lambda_x = \lambda_y T\} \end{aligned}$$

$\mathcal{G}$ is a group: This is the group we want! One might naturally wonder whether, had one started with a group $G$, produced the objects $F, \Omega$ etc. and formed the group $Aut$, one would have recovered the same group $G$. This is settled in the following exercise:

---

• **Exercise 5.3** ***On Reconstruction*** **[30][33]. Suppose one begins with a compact group $G$ and constructs the spaces $V^i_{jk}$ as above. We will indicate why the reconstructed group $\mathcal{G}$ defined by the abstract procedure given here is exactly the original group $G$.**

**a.) Note first that $G \subset \mathcal{G}$. Note that every $g \in G$ defines a family $\{\lambda_X\}_{X\in\mathbf{Rep}}$ via $\lambda_X(g) = \rho_X(g)$, where $\rho_X$ is the representation defined by $X$.**

**b.) Show that if $\vec{v} \in X$ is fixed by all of $G$, i.e., if**

$$\forall g \in G : \rho_X(g)\vec{v} = \vec{v}$$

then it is fixed by all of $\mathcal{G}$, i.e.,

$$\lambda_X(\vec{v}) = \vec{v}$$

for any family satisfying the defining axioms of $\mathcal{G}$. (Hint: Show that $\lambda_{R_0} = 1$, and that $z \to z\vec{v}$ is an intertwiner $\mathbb{C} \to X$.) If $G$ is a continuous Lie group we conclude that there are no "broken" generators in $\mathcal{G}/G$ and hence that $\mathcal{G} = G$.

c.) More generally suppose that $G \subset \mathcal{G}$ is a proper subgroup. Then there is some $\lambda_X \in End(X)$ which is not in the set $\{\rho_X(g)|g \in G\} \subset End(X)$. Use the fact that $G$ is compact to show that there must exist a polynomial $P$ on $End(X)$ which vanishes on $\{\rho_X(g)|g \in G\}$, but not at $\lambda_X$. Show that the space $S$ of polynomials of degree $\leq deg(P)$ on $End(X)$ is a representation of $G$. Note that $P \in S$ violates (b), to conclude that $\mathcal{G} = G$.

---

In the above characterization of a Tannakian category we have worked directly with the data $V^i_{jk}$ etc. Alternatively we could have defined the category more directly in terms of objects, with a tensor product of objects satisfying pentagon and hexagon conditions identical to (5.2) and (5.3) , and with some axioms relating to the unit object and dual objects. This is the definition one finds in the literature.

The situation arising in RCFT is more complicated than the one we have described for the Tannakian categories. In RCFT the index set $I$ is finite. Moreover $\Omega^2 \neq 1$. This is crucial: it is the characteristic that leads to interesting monodromy and hence interesting braid representations. The pentagon relation remains but there are two hexagon relations involving $\Omega$ and $\Omega^{-1}$. The category so defined (equivalently, the category defined by axioms on objects and morphisms of objects) is closely related to what is known as a "compact braided monoidal category" which was studied in [34]. Different definitions differ slightly on such details as whether $\check{}$ is involutive, or whether the set $I$ should be finite or not. Thus, roughly speaking, the duality properties of RCFT's on the plane are characterized by "compact braided monoidal categories." Well defined RCFT's have more structure and must be defined on all Riemann surfaces. By the completeness theorem it suffices to define $S(p) : \oplus V^i_{pi} \to \oplus V^i_{pi}$ according to (4.3) and impose the relations of the modular group. We will call the category defined by these axioms a *modular tensor category*. More precisely we have

**Axioms for a Modular Tensor Category**

*Data:*

1. A finite index set $I$ with a distinguished element 0 and a bijection of $I$ to itself written $i \mapsto i\check{}$.

2. Vector spaces: $V^i_{jk}$ $i,j,k \in I$, with $dim V^i_{jk} = N^i_{jk} < \infty$

3. Isomorphisms:

$$\Omega^i_{jk} : V^i_{jk} \cong V^i_{kj}$$
$$F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix} : \oplus_r V^{i_1}_{j_1 r} \otimes V^r_{j_2 k_2} \cong \oplus_s V^{i_1}_{s k_2} \otimes V^s_{j_1 j_2} \tag{5.7}$$

4. A constant $S_{00}(0)$.

*Conditions:*

1. $(i\check{})\check{} = i$, $0\check{} = 0$.

2. $V^i_{0j} \cong \delta_{ij} C \quad V^0_{ij} \cong \delta_{ij} C \quad V^i_{jk} \cong V^k_{ji} \quad (V^i_{jk})\check{} \cong V^i_{jk}$

3. $\Omega^i_{jk}\Omega^i_{kj} \in End(V^i_{jk})$ is multiplication by a phase.

4. The identities:

$$F(\Omega^\epsilon \otimes 1)F = (1 \otimes \Omega^\epsilon)F(1 \otimes \Omega^\epsilon)$$
$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}$$

for $\epsilon = \pm 1$.

5. The identities

$$S^2(p) = \pm e^{-i\pi\Delta_p} C$$
$$(ST)^3 = S^2$$

where $S(p) \in End(\oplus V^i_{pi})$ is defined by

$$S_{ij}(p) = S_{00}(0) e^{-i\pi\Delta_p} \frac{F_{i0}\left[\begin{smallmatrix} i & i \\ p & p \end{smallmatrix}\right]}{F_p F_{p0}\left[\begin{smallmatrix} j & j \\ j & j \end{smallmatrix}\right] F_{p0}\left[\begin{smallmatrix} i & i \\ i & i \end{smallmatrix}\right]} \sum_r B_{pr}\begin{bmatrix} i & j \\ i & j \end{bmatrix}(-) B_{r0}\begin{bmatrix} j & i \\ i & j \end{bmatrix}(-)$$

$C$ represents the action of $\check{}$, the numbers $\pm e^{-i\pi\Delta_p}$ may be deduced from $\Omega$, and $T : V^i_{ji} \to V^i_{ji}$ is scalar multiplication by $e^{2\pi i(\Delta_i - c/24)}$ for a constant $c$. (For more details see [15].)

---

Just as for Tannakian categories we could define modular tensor categories more directly in terms of objects and axioms on the tensor products of objects. In these terms one must define the analog of (4.3) . This may be done in terms of a generating set of simple objects $R_i$ by defining a single morphism $S$ of the object $\oplus_i R_i \otimes R_i^{\vee}$ to itself as follows:

$$\begin{aligned} \oplus_i R_i \otimes R_i^{\vee} &\longrightarrow \oplus_{i,j} R_i \otimes R_j \otimes R_j^{\vee} \otimes R_i^{\vee} \\ &\xrightarrow{\Omega^2 \otimes 1 \otimes 1} \oplus_{i,j} R_i \otimes R_j \otimes R_j^{\vee} \otimes R_i^{\vee} \otimes \\ &\xrightarrow{\Omega^{-1} \otimes \Omega} \oplus_{i,j} R_j \otimes R_i \otimes R_i^{\vee} \otimes R_j^{\vee} \otimes \\ &\longrightarrow \oplus_j R_j \otimes R_j^{\vee} \end{aligned} \tag{5.8}$$

Similarly one may use $\Omega$ to define the data $\pm e^{i\pi\Delta_j}$ as a morphism $R_j \rightarrow R_j$ and from this define $T$ on $\oplus R_i \otimes R_{i^{\vee}}$ and impose a relation on $S^2$ (relating it to $\Omega$) and the relation $(ST)^3 = S^2$.

The name *modular tensor category* was suggested by Igor Frenkel and we will adopt it. We thank him for discussions on this subject and for urging us to express the definition of $S$, (4.3) , in terms of simple objects, along the lines of (5.8).

As we have mentioned, the above axioms are sufficient for establishing the relation $Sa = bS$. Thus we may summarize the main result of [13][15] in the statement that a modular tensor category (henceforth MTC) is equivalent to a modular functor. As in section four we may ask whether all MTC's are associated to some RCFT, and to what extent an MTC characterizes the original RCFT.

From the analogy of Tannakian categories and MTC's one naturally wonders whether there is a reconstruction theorem for MTC's analogous to Deligne's theorem. This is not known at present, but there is some good evidence that such a statement exists. First, there is an analog of the integrality condition in RCFT. From the proof of Verlinde's formula one finds

$$\frac{1}{F_i} = \frac{S_{0i}}{S_{00}}$$

We have already noted that classically the quantity on the LHS is related to the dimension. The quantity on the RHS has been interpreted as the "relative dimension" of the representation spaces. Note that [35]

$$\frac{\text{“}dim H_i\text{”}}{\text{“}dim H_0\text{”}} = \lim_{q \to 1} \frac{tr_{H_i}\, q^{L_0 - c/24}}{tr_{H_0}\, q^{L_0 - c/24}} = \frac{S_{0i}}{S_{00}}$$

All this strongly suggests that some axioms additional to the above polynomial equations in fact characterize RCFT's - and that classifying solutions to these equations is the same as classifying RCFT's.

The relation between the axioms of RCFT as discussed above and the Tannaka-Krein approach to group theory becomes more complete in a certain limit of RCFT. Some RCFT's are labeled by a parameter $k$ such that they simplify considerably in the $k \to \infty$ limit. In this limit the conformal dimensions of all the primary fields approach zero. More generally, there is a subset of the primary fields with a closed fusion rule algebra (namely, if $i$ and $j$ are in the subset then $N_{ij}^{l} \neq 0$ only for $l$ in the set) whose conformal dimensions approach an integer in the $k \to \infty$ limit. We define this limit as the *classical limit of the RCFT.* Examining our axioms at genus zero in this limit we see that they simplify. In particular, since the relevant $\Delta$'s are integers,

$$\Omega^2 = 1 \ . \tag{5.9}$$

Therefore, there are no monodromies in the classical theory and the two hexagons are the same equation. In this limit the axioms of a RCFT are identical to those of group theory in the Tannaka-Krein approach. Since classical RCFT is the same as group theory, it is natural to conjecture that *quantum RCFT is a generalization of group theory.* We'll return to this conjecture below. For the moment we note the following correspondences between group theory and conformal field theory:

| | |
|---|---|
| Group | Chiral algebra |
| Representations | Representations |
| Clebsch-Gordan coefficients/Intertwiners | Chiral vertex operators |
| Invariant tensors | Conformal blocks |
| Symmetry of couplings | $\Omega$ |
| Racah coefficients (6j symbols) | Fusion matrix |

It is also interesting to examine a larger class of CFT's. We refer to them as "quasirational CFT's." In these theories the chiral algebra has an infinite number of irreducible representations . However, the fusion rules are finite, i.e. for given $i, j$, $N_{ij}^k$ is non zer only for a finite number of representations $k$. Because of this condition, the formalisn of the CVO and the duality matrices on the plane is still applicable. Consequently, th polynomial equations on the plane (the pentagon and the two hexagons) are satisfied. On can still define $S(p)$ by (4.3) but since the number of irreducible representations is infinite the torus polynomial equations are not obviously present. The category of representatio spaces of the chiral algebra of a quasirational conformal field theory is also a generalizatio of a tensor category. A well known example of such a theory is the Gaussian model at a irrational value of the square of the radius.

Finally, we must not lose sight of the fact that many interesting irrational (non quasirational) CFT's exist and that the challenge to understand their structure remain unanswered.

## 6. Combining leftmovers with rightmovers.

CFT is not just the study of chiral algebras and their representations. In order to have a consistent conformal field theory, we need to put together left and right-movers to obtain correlation functions with no monodromy.

The left and right chiral algebras $\mathcal{A}$, and $\bar{\mathcal{A}}$ are the algebras of purely holomorphic and anti-holomorphic fields. We can decompose the total Hilbert space of the theory into irreducible representations: $H_r \otimes H_{\bar{r}}$, so t he partition function is:

$$Tr_H\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - c/24} = \sum_{r,\bar{r}} h_{r\bar{r}} \chi_r(q) \chi_{\bar{r}}(\bar{q}).$$

The nonnegative integers $h_{r\bar{r}}$ characterize the field content of the theory.

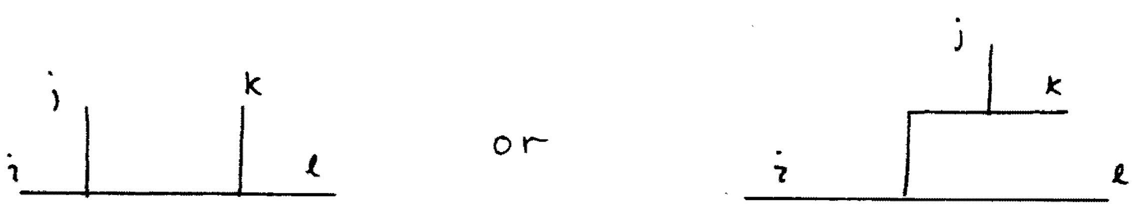
We can write the physical conformal fields in terms of the chiral vertex operators as

$$\phi^{jm,\bar{j}\bar{m}}(z,\bar{z}) = \sum_{j,k} d^{(i\bar{i})}_{(j\bar{j})(k,\bar{k})} \Phi^{j,m}_{ik}(z) \bar{\Phi}^{\bar{j},\bar{m}}_{\bar{i}\bar{k}}(z) \tag{6.1}$$

We assume for simplicity that there is only one field with representation $(i,\bar{i})$ in the theory. Below we'll show that this assumption is always satisfied.

Now the physical correlation function must be independent of the choice of blocks, so there are certain conditions on the $d$-coefficients. For example, from invariance of the partition functions under $T : \tau \to \tau + 1$, we see that $h_{i\bar{i}} = 0$ unless $\Delta_i - \Delta_{\bar{i}} \in \mathbf{Z}$ Proceeding more systematically we could have deduced this from an analysis of 2 and 3 point functions.

Moving on to the four-point function, we must have the same correlator from either basis of blocks:




s and t channel blocks relevant for the four point function this implies

$$\sum_{p,\bar{p}} d^{(i\bar{i})}_{(j\bar{j})(p\bar{p})}\, d^{(p\bar{p})}_{(k\bar{k})(l\bar{l})}\, F_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}\, \bar{F}_{\bar{p}\bar{q}}\begin{bmatrix} \bar{j} & \bar{k} \\ \bar{i} & \bar{l} \end{bmatrix} = d^{(i\bar{i})}_{(q\bar{q})(l\bar{l})}\, d^{(q\bar{q})}_{(j\bar{j})(k\bar{k})} \tag{6.2}$$

---

• Exercise 6.1 *Monodromy invariance.*

a.) Write out the conditions on $d$ following from locality of the three-point function.

b.) Show that the invariance of the physical correlator under $B$ is guaranteed by the condition of part (a) together with the equation for $F$ (6.2).

---

By using the operator product expansion for chiral vertex operators together with (6.2) we may deduce that

$$\phi^{j,m;\bar{j},\bar{m}}(z,\bar{z})\phi^{k,n;\bar{k},\bar{n}}(z,\bar{z}) = \sum_{p',\bar{p}'} d^{(p'\bar{p}')}_{(j\bar{j})(k\bar{k})} \sum_{P\in\mathcal{H}_p, P'\in\mathcal{H}_{p'}} \phi^{p',P';\bar{p}',\bar{P}'}(w,\bar{w}) \langle P'|\Phi^{j,m}_{p'k}(z-w)|n\rangle\overline{\langle P'|\Phi^{j,m}_{p'k}(z-w)|n\rangle} \tag{6.3}$$

Again there is a nice analog of this equation in group theory.

Recall that for a compact group the Hilbert space of $L^2$ functions on the group has an orthonormal basis given by the matrix elements $D^R_{\mu\nu}$ in the irreducible representations $R$. The operator $U(D^R_{\mu\nu})$ on $L^2(G)$ given by multiplication of functions may be represented in terms of intertwiners as

$$U(D^R_{\mu\nu}) = \sum_{R_1,R_2} \sum_{a\in V^{R_1}_{RR_2}} \begin{pmatrix} R_1 \\ RR_2 \end{pmatrix} (\mu\otimes\cdot)_a \begin{pmatrix} R_1^{\check{}} \\ R^{\check{}}R_2^{\check{}} \end{pmatrix} (\nu\otimes\cdot)_{\check{a}} \tag{6.4}$$

Where we sum over a basis of intertwiners and $\check{a}$ is a basis dual to $a$. The algebra of functions on the group manifold is given by

$$D^{R_1}_{\mu_1\nu_1} D^{R_2}_{\mu_2\nu_2} = \sum_{R,\gamma_1,\gamma_2} \sum_{a\in V^R_{R_1R_2}} D^R_{\gamma_1\gamma_2} \langle R,\gamma_1| \begin{pmatrix} R \\ R_1R_2 \end{pmatrix}_a (\mu_1\otimes\mu_2)\rangle \langle R,\gamma_2| \begin{pmatrix} R^{\check{}} \\ R_1^{\check{}}R_2^{\check{}} \end{pmatrix}_{\check{a}} (\nu_1\otimes\nu_2)\rangle$$

Thus we see that in the $k\to\infty$ limit of WZW models the operator product expansion of the fields with $\Delta\to 0$ becomes the algebra of functions on the group $G$, thus providing an explicit example of an old idea of Dan Friedan's for the reconstruction of manifolds from the operator product expansion of CFT. In fact, as described later, in the specific example of current algebra the above ope for finite $k$ is closely related to the algebra of functions on a quantum group. For further discussion of these and related ideas see [36].

We now show how the above equations can be used to deduce some general theorems about the operator content of rational conformal field theories.

---

• Exercise 6.2 *No representation appears more than once.* Consider a RCFT where some representations occur more than once (either $h_{r\bar{r}} > 1$ or both $h_{r\bar{r}}$ and $h_{r\bar{r}'}$ are non zero for $\bar{r} \neq \bar{r}'$).

a. Add indices in equation (6.1) to describe this situation.

b. Rewrite equation (6.2) for this case.

c. Study the four point function of $\langle\phi\phi\phi'\phi'\rangle$ where $\phi$ and $\phi'$ transform the same under $\mathcal{A}$ (the representation $r$) but they are different conformal fields (they might or might not transform the same under $\overline{\mathcal{A}}$) and assume for simplicity that all the representations are self conjugate. Use (3.5) to bring $\overline{F}$ to the other side of the equation and study it for the case where the intermediate representation is 0 on both sides. Simplify the equation by using the fact that the $\mathcal{A}$ ($\overline{\mathcal{A}}$) includes *all* the holomorphic (antiholomorphic) fields i.e. the identity operator is the only primary field under $\mathcal{A} \otimes \overline{\mathcal{A}}$ which is holomorphic. The ope of $\phi\phi$ contains the identity operator and $\phi\phi'$ does not contain the identity operator. Use this fact to show that one side of the equation vanishes. The other side is proportional to $F_r$ and does not vanish. Therefore, we are led to a contradiction and no representation can appear more than once.

---

Notice that in proving this result one uses only the equations on the plane and not the equations on the torus. Hence, this result applies not only in RCFT but also in quasirational theories. On the other hand, this result is not true in theories which are not quasirational [11]. A $\mathbb{Z}_2$ orbifold of the Gaussian model at an irrational value of the square of the radius is not quasirational – the ope of two twist fields includes all the untwisted representations. Since the previous proof does not apply, we are not surprised to see the same representation appearing more than once in the spectrum.

Similarly there is an equation for the $d$'s following from the modular invariance of the $g = 1$, one-point functions.

---

• Exercise 6.3 *Equation for d from genus one.* Write the equation for invariance under $S(p)$ for every $p$. Remember that the characters of the one point function on the torus are defined as differential forms i.e. they have a $z$-dependence $\sim (dz/z)^{\Delta(p)}$ (otherwise they are not invariant). Therefore, there is a phase relating $S$ of the left-movers to $S$ of the right-movers.

---

At this point one may wonder whether there will be further constraints on the $d$ coefficients from duality *invariance* of correlation functions on other Riemann surfaces. The answer is no. Since duality matrices defining an MTC allow us to define duality matrices on all surfaces we know that the conformal blocks are duality *covariant*. To check invariance of left-right combinations of blocks we merely have to check invariance under the generators of duality transformations. Since an MTC defines a modular functor, the generators can be taken to be those duality transformations represented by $F, B, S$. Thus the above duality invariance conditions suffice to guarantee invariance on all surfaces. A similar conclusion was reached independently in [37].

---

• Exercise 6.4 *Every representation of $\mathcal{A}$ occurs in the spectrum.* Show that $S(0)$ is unitary. Use this to show that one of the equations of the previous exercise can be written as

$$\sum_{\bar{j}} h_{i\bar{j}} \overline{S}_{\bar{j}\bar{k}} = \sum_{j} S_{ij} h_{j\bar{k}} \qquad (6.5)$$

Use $h_{0\bar{i}} = h_{i\bar{0}} = \delta_{i0}$, i.e. $\mathcal{A}$ ($\overline{\mathcal{A}}$) includes *all* the holomorphic (antiholomorphic) fields, to show that there is no $r$ such that $h_{r\bar{j}} = 0$ for every $\bar{j}$. Hence, no representation can be omitted.

---

From the last exercises we conclude: If the chiral algebras, $\mathcal{A}$ and $\bar{\mathcal{A}}$ are maximally extended, $h_{r,\bar{r}}$ must be a permutation matrix. We are now ready to tackle

• Exercise 6.5 *The left movers are paired with the right movers by an automorphism of the fusion rule algebra.* Use Verlinde's formula relating the fusion rules to $S$ and (6.5) to prove this.

---

We conclude that $FRA(\mathcal{A}) = FRA(\overline{\mathcal{A}})$ and the pairing of the left movers and the right movers is an automorphism of the fusion rule algebra:

$$h_{r,\bar{r}} = \delta_{r,w(\bar{r})}$$

where

$$N_{\bar{i}\bar{j}\bar{k}} = N_{w(\bar{i})w(\bar{j})w(\bar{k})}$$

The main point here is that the classification of RCFT's is a two-step process. First we classify all chiral algebras and their representation theory, then we look for all automorphisms of the fusion rule algebras.

---

• Exercise 6.6 *No New Conditions on F.* For a unitary diagonal (i.e. $h_{i\bar{i}} = \delta_{i\bar{i}}$) theory, assuming $F$ is real and the fusion rules are zero and one, show that the operator product coefficients may be written

$$d_{ijk}^2 = \frac{F_{0k}\begin{bmatrix} i & j \\ i & j \end{bmatrix}}{F_{k0}\begin{bmatrix} j & j \\ i & i \end{bmatrix}}$$

a.) Use the polynomial equations to show that $d$ is totally symmetric.

b.) Substitute the above equation back into the full set of equations for $d_{ijk}$ on the plane. Show that the resulting identities are guaranteed by the polynomial equations.

---

---

• Exercise 6.7 *Open Problem.* How general is the result of the previous exercise? Do the equations for the torus one-point function follow from the other identities? (Felder and Silvotti [38] have shown that for the discrete series the answer is yes, by direct calculation.) What about non-unitary theories? What about arbitrary fusion rules? Is this true for the non-diagonal theories – when a non-trivial automorphism is used to pair left and right movers?

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• Exercise 6.8 *Modular Invariance of $A_1^{(1)}$ Characters.*

a.) Find the automorphisms of the fusion rule algebra for the level $k$ $SU(2)$ WZW model.

b.) Impose other necessary conditions, e.g. the monodromy invariance of the two-point function.

c.) Using the above point of view interpret the other modular invariants of $A_1^{(1)}$ characters.

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• Exercise 6.9 *Automorphisms of Kac-Moody Fusion Rules.* Using Verlinde's formula for $N_{ijk}$ and Kac's formula for $S_{ij}$, show how automorphisms of the extended Dynkin diagrams can define automorphisms of the fusion rule algebra. An application of this fact can be found in [39].

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• Exercise 6.10 *the d-coefficients and gauge invariance.* How does $d$ transform under the gauge transformations of rescaling the chiral vertex operators? Show that the equations for $d$ are gauge invariant.

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• Exercise 6.11 *Modular invariants for the rational torus.* As we will see in section 10 below, the Gaussian model at radius squared $R^2 = \frac{p}{2q}$ has a chiral algebra which depends only on the quantity $pq$. Compute the automorphisms of the fusion rule algebra of the rational torus and show that they define the different models for which $pq = p'q'$, but $p/q \neq p'/q'$.

---

The analogy between conformal field theory and group theory continues to hold for the combination of left movers with right movers. We can add to the table at the end of section 5 a few more rows:

| | |
|---|---|
| Functions on the group | Physical fields |
| Product of functions on the group | Operator product expansion |
| Average over the group of a product of functions | Physical correlation function |

---

• Exercise 6.12 *Analogy with group theory.* Explain the table. Show that it corresponds to the diagonal theory.

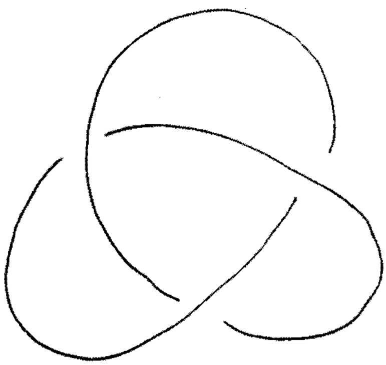
---

The equations for the ope coefficients $d$ can be interpreted as defining a metric [24] on the vector space of the conformal blocks. Therefore, if all the $d$'s are real and positive (and therefore we can pick the gauge $d = 1$), the vector space of the conformal blocks is a Hilbert space. This interpretation will play an important role in the following sections where this Hilbert space will appear in the quantization of a quantum mechanical system.

## 7. 2D Duality vs. 3D General Coordinate Invariance

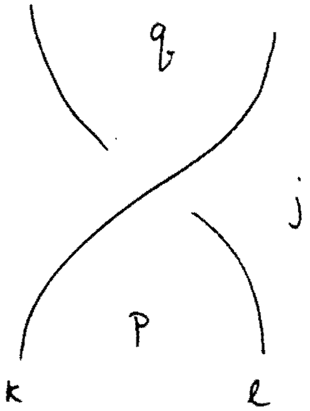
Many people have noticed that RCFT's lead to knot invariants [20][40] [41] [27][42] [43]. One way of producing knot invariants is to view the $B$ matrices as "transition amplitudes" of conformal blocks, then defining an appropriate trace (Markov trace) on these amplitudes the resulting polynomials are, in fact, knot invariants. There is an alternative formalism, used in [40] and elaborated upon in [42][43] which dispenses with the need for a trace at the cost of introducing some new moves. With these new moves the knot invariant becomes the transition amplitude for proceeding from the "null block" to itself with an intervening knot projection. We will present these results from our point of view using the formalism of the previous sections.

Consider the planar projection of a knot from $S^3$, e.g.



A projection of a knot on a plane

We assign a number to this figure by using the graphical formalism described above. For this, we label every line by a representation of a chiral algebra and also label the areas bounded by the lines by such representation. We assign factors of $B$ to

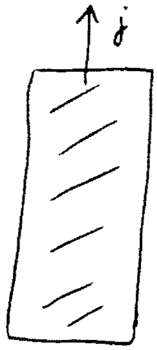



$$\sim \quad B_{pq}\begin{bmatrix} k & \ell \\ i & j \end{bmatrix}$$

Graphical rules for computing a knot invariant

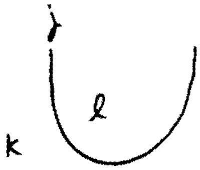
The knot that we consider is a "framed knot." It looks like a ribbon and hence


$$= \quad e^{2\pi i \Delta_j}$$

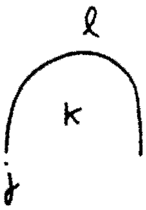



A non-trivial operation on a framed knot

The operation in the figure corresponds to a factor of $e^{2\pi i\Delta_i}$ in the knot invariant. We also need to introduce two new operations on lines for pair creation/annihilation:



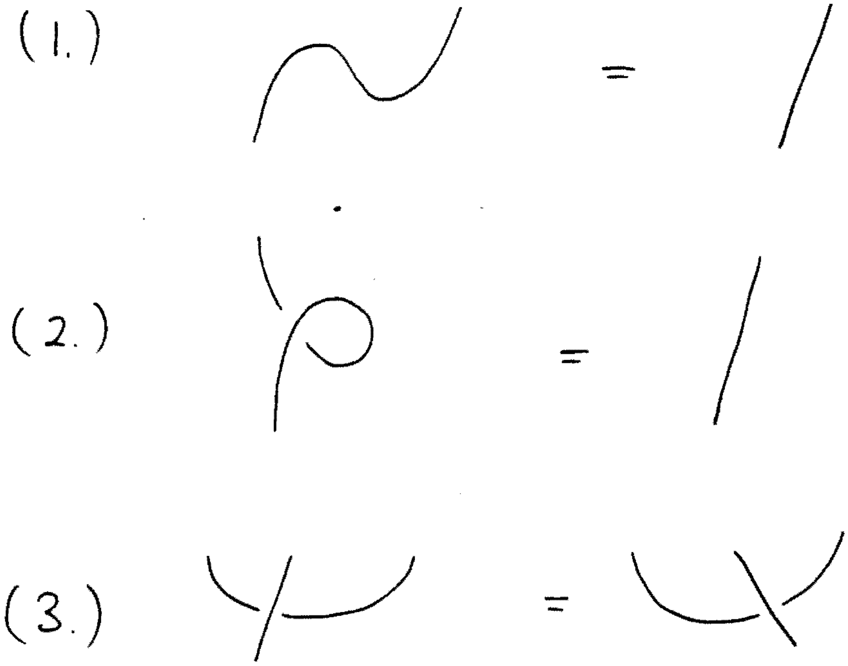

$$\sim \quad C^{j}_{k\ell}$$




$$\sim \quad A^{j}_{k\ell}$$

Pair creation and annihilation moves

The factors for these operations are determined by requiring that:




Consistency conditions on pair creation and annihilation

We make the *ansatz*

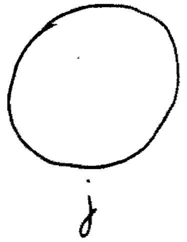
$$\begin{aligned} A^j_{ik} &= \alpha_j F_{k0} \begin{bmatrix} i & i \\ j & j \end{bmatrix} \\ C^j_{ik} &= \beta_j F_{0k} \begin{bmatrix} i & j \\ i & j \end{bmatrix} \end{aligned}$$

and deduce from the first consistency condition that

$$\alpha_i \beta_i = \frac{1}{F_i}$$

Since for a closed graph there is always an equal number of $\alpha_i$ and $\beta_i$, we can set, without loss of generality, $\alpha_i = \beta_i = \frac{1}{\sqrt{F_i}}$.

This result leads to a new interpretation of Deligne's condition discussed earlier. It is simply the requirement that the value of a circle is a trace. Hence it should be an integer in group theory.




$$= n_j = \dim R_j \in \mathbb{Z}_+$$

Deligne's condition

In RCFT it is the relative dimension, as explained in the above. We will see below how this follows from the three-dimensional viewpoint.

---

• Exercise 7.1 *No more consistency conditions.* Show that consistency conditions (2) and (3) are automatically satisfied by using the polynomial equations discussed above and this value of $A^i_{jk}$ and $C^i_{jk}$.

---

The non-trivial problem in knot theory is to prove that this procedure leads to a knot invariant. In other words, different projections of the same knot to two dimensions lead to the same result for the knot invariant. From the discussion in the previous sections and these exercises, it is clear that the polynomial equations guarantee this fact and we indeed find a knot invariant from every RCFT.

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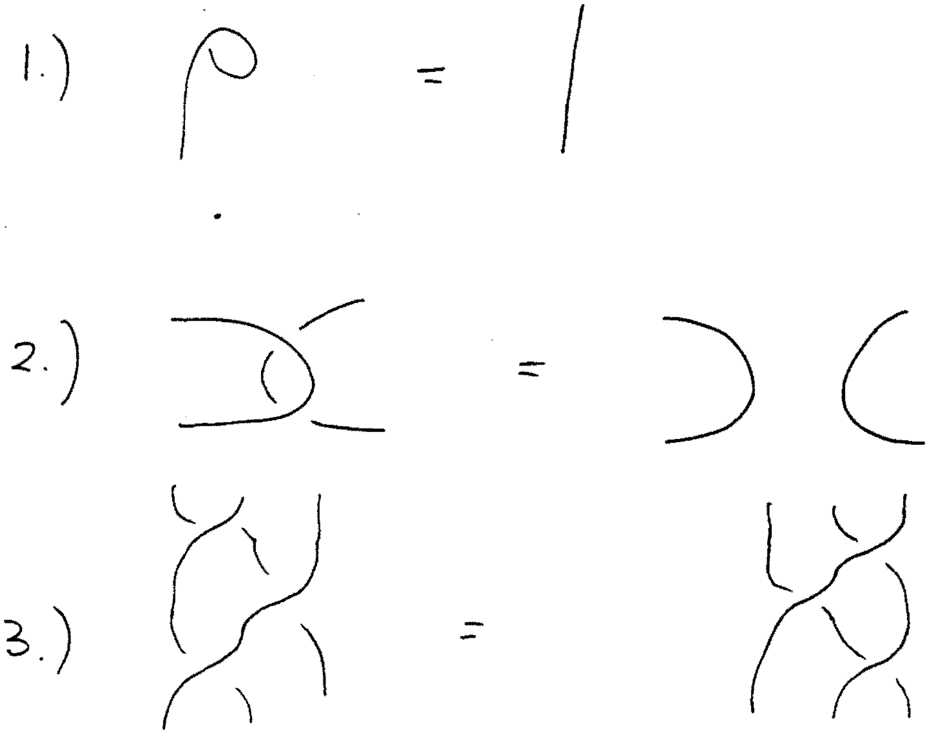
• Exercise 7.2 *Reidemeister Moves.* In the combinatorial approach to knot theory one must check the Reidemeister moves




The three Reidemeister moves.

Check these using the above formalism. Note that the first move is only satisfied up to phase. This may be fixed by discussing framed links or by introducing the writhe, following Kauffmann [44].

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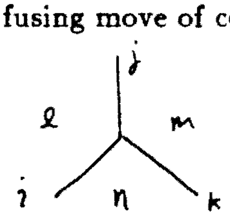
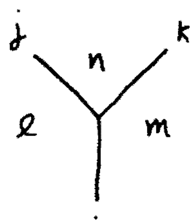
The analysis can easily be generalized to graphs with vertices, which are the analogs of the fusing move of conformal field theory. Define fusing and defusing moves



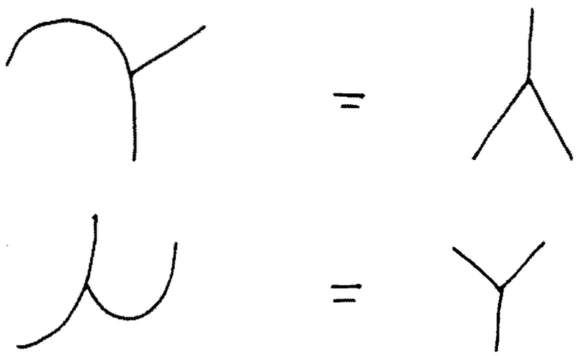

$$= f^{j}_{ik}\, F_{nj}\begin{bmatrix} i & k \\ l & m \end{bmatrix}$$




$$= f^{jk}_{i}\, F^{-1}_{in}\begin{bmatrix} j & k \\ l & m \end{bmatrix} = f^{jk}_{i}\, F_{in}\begin{bmatrix} m & k \\ l & j \end{bmatrix}$$

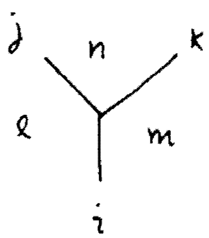
Fusing and defusing

• Exercise 7.3 *Consistency conditions on fusing and defusing.* Impose the relations

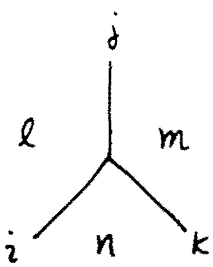


Consistency conditions on fusing and defusing

Derive $\frac{1}{\sqrt{F_i}} f_k^{ij} = \frac{1}{F_{k0}\left[\begin{smallmatrix} j & j\\ i & i\end{smallmatrix}\right]} f_{ik}^{j}$. Normalize the constants $f$ such that if one of the lines corresponds to the identity representation, this line can be dropped from the graph and find the rules

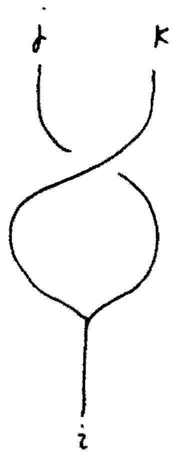



$$= \frac{F_{in}\begin{bmatrix} m & k\\ l & j\end{bmatrix}}{F_{i0}\begin{bmatrix} j & j\\ k & k\end{bmatrix}} \left(\frac{F_jF_k}{F_i}\right)^{1/4} \qquad (7.1)$$

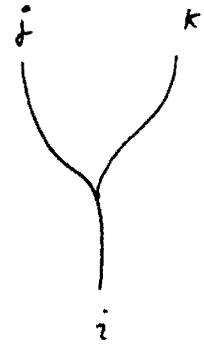



$$= F_{nj}\begin{bmatrix} i & k\\ l & m\end{bmatrix} \left(\frac{F_j}{F_iF_k}\right)^{1/4} \qquad (7.2)$$

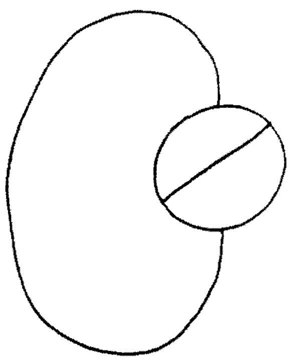
• Exercise 7.4 *Another consistency check.* Use the hexagon to show that

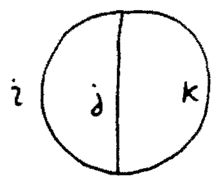



$$= e^{-i\pi(\Delta_j + \Delta_k - \Delta_i)}$$




• Exercise 7.5 *Simple calculations.* Use the rules to compute the invariant of the graphs






Two simple graphs

Use exercise 6.6 to write the second graph as $\frac{d_{ijk}}{\sqrt{F_i F_j F_k}}$ when the conditions of that exercise are fulfilled.

Using these rules one can compute invariants of knotted graphs. As in the case without the vertices, the polynomial equations guarantee the consistency.

• Exercise 7.6 *Gauge invariance.* Show that the invariant of knot without vertices is gauge invariant, i.e. it does not change if we rescale the CVO's and correspondingly the duality matrices. How do knots with vertices transform under such a rescaling? Interpret it.

It is convenient to pick the "good gauge"

$$F_{k0}\begin{bmatrix} i & i \\ j & j \end{bmatrix} = \sqrt{\frac{F_i F_j}{F_k}} \tag{7.3}$$

Write the fusing and the defusing rules in this gauge. Show that when the conditions of exercise 6.6 are fulfilled $d_{ijk} = 1$ in this gauge. Evaluate the two graphs in exercise 7.5 in this gauge. This gauge was used in [40][42].

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- Exercise 7.7 *Symmetries of F*. Use the pentagon to show that

$$F_{n0}\begin{bmatrix} i & i \\ l & l \end{bmatrix} F_{pi}\begin{bmatrix} j & k \\ n & l \end{bmatrix} = F_{p0}\begin{bmatrix} k & k \\ l & l \end{bmatrix} F_{nk}\begin{bmatrix} i & j \\ l & p \end{bmatrix} \tag{7.4}$$

In the good gauge of exercise 7.6 this becomes

$$\sqrt{F_i F_p} F_{pi}\begin{bmatrix} j & k \\ n & l \end{bmatrix} = \sqrt{F_n F_k} F_{nk}\begin{bmatrix} i & j \\ l & p \end{bmatrix}$$

Define $W_{pi}\left[\begin{smallmatrix} j & k \\ n & l \end{smallmatrix}\right] = \sqrt{F_i F_p} F_{pi}\left[\begin{smallmatrix} j & k \\ n & l \end{smallmatrix}\right]$ and use the symmetries of exercise 3.5 to show that

$$\begin{aligned} W_{mn}\begin{bmatrix} i & j \\ k & l \end{bmatrix} &= W_{kj}\begin{bmatrix} l & m \\ n & i \end{bmatrix} \\ &= W_{nm}\begin{bmatrix} j & l \\ i & k \end{bmatrix} \\ &= W_{nm}\begin{bmatrix} l & j \\ k & i \end{bmatrix} \end{aligned} \tag{7.5}$$

These symmetries generate a tetrahedral symmetry generalizing the symmetry satisfied by $SU(2)$ Racah coefficients. Use the results of exercises 7.5 and 7.6 to explain the origin of this symmetry.

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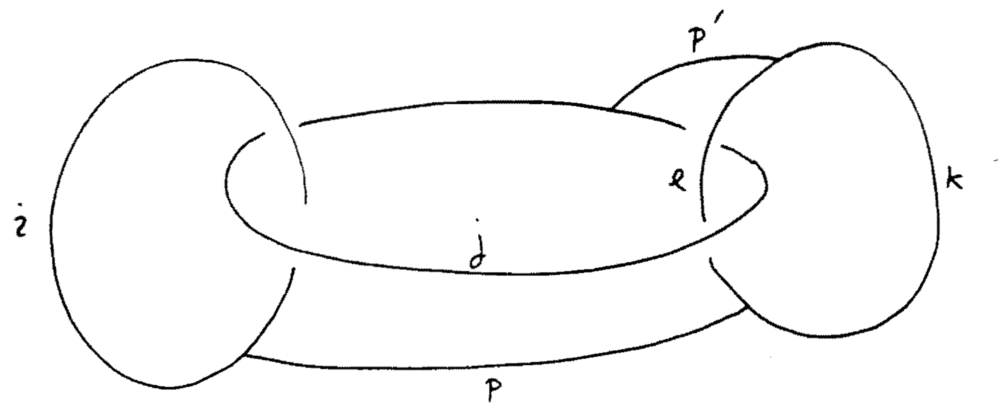
- Exercise 7.8 *Proof of the last equation on the torus*.

The graphical formalism presented here is a very convenient tool in manipulating the duality matrices using the fundamental equations. We'll demonstrate this fact now by showing that the definition (4.3) of $S_{ij}(p)$ in terms of $B$ and $F$ satisfies the last equation on the torus $Sa = bS$. Consider the graph




Graph used to prove $Sa = bS$

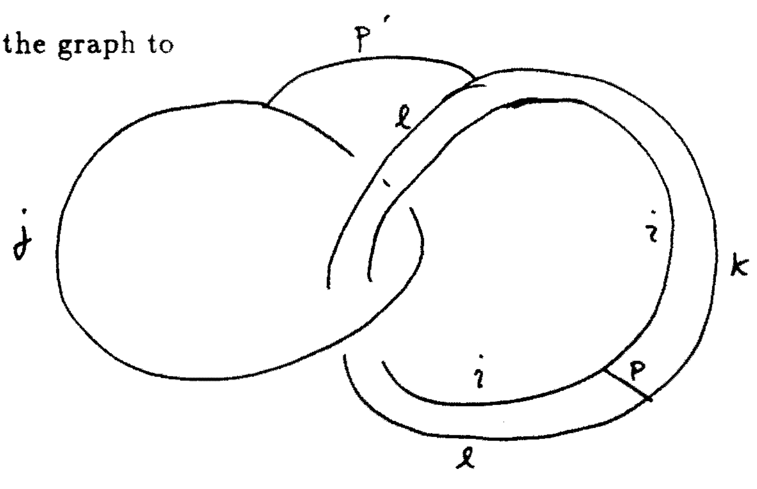
For simplicity, work in the good gauge. Use

$$S_{ij}(p) = S_{00}(0)e^{-i\pi\Delta_p}\frac{\sqrt{F_p}}{F_iF_j}\sum_r B_{pr}\begin{bmatrix} i & j \\ i & j \end{bmatrix}(-)B_{r0}\begin{bmatrix} j & i \\ i & j \end{bmatrix}(-) \tag{7.6}$$

to show that the graph has the value

$$\frac{1}{\sqrt{F_lF_k}S_{00}(0)}e^{i\pi\Delta_p}S_{ij}(p)\sum_r B_{pr}\begin{bmatrix} j & l \\ j & k \end{bmatrix}(+)B_{rp'}\begin{bmatrix} l & j \\ j & k \end{bmatrix}(+) \tag{7.7}$$

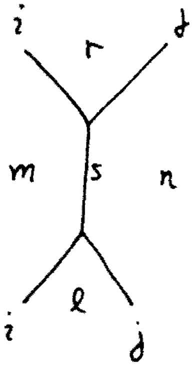
Now, deform the graph to




A deformation of the same graph

which differs from the original graph by a factor of $e^{i\pi(\Delta_k - \Delta_l)}$. Prove the identity

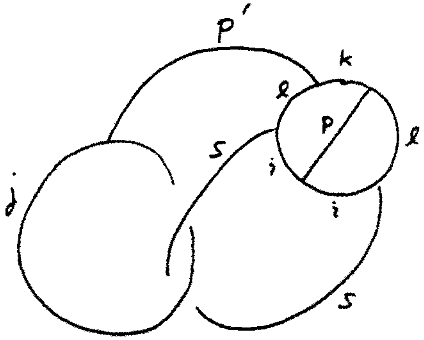
$$\sum_{s}$$




$$\cdot \sqrt{\frac{F_i F_j}{F_s}} = \delta_{l,r}$$

and use it to deform the graph to

$$\sum_{s} \sqrt{\frac{F_l F_i}{F_s}} \; e^{i\pi(\Delta_k - \Delta_l)}$$

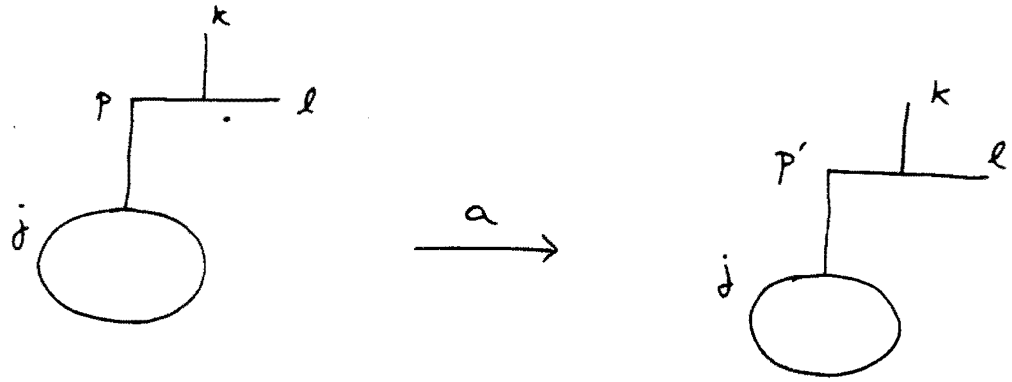



the original graph is equivalent to this graph

Turn this graph upside down and evaluate it. Use the symmetries of $F$ and the expression for $S(p')$ to write it as

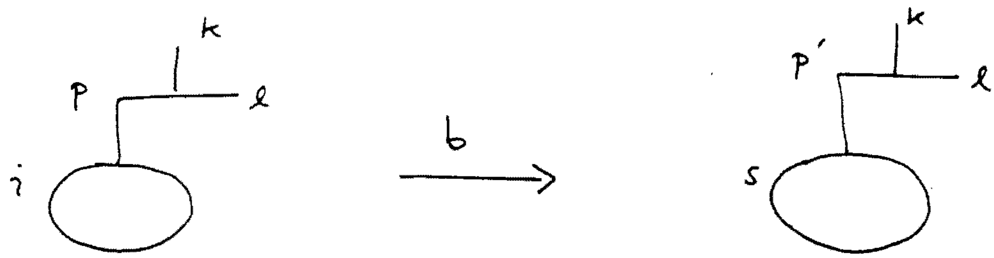
$$\frac{e^{i\pi(\Delta_k - \Delta_l)}}{S_{00}(0)\sqrt{F_k F_l}} \sum_{s} F_{ip'} \begin{bmatrix} s & s \\ k & l \end{bmatrix} F_{ps} \begin{bmatrix} i & l \\ i & k \end{bmatrix} S_{sj}(p') \tag{7.8}$$

Now express the $a$ monodromy




the general $a$ monodromy

and the $b$ monodromy




the general $b$ monodromy

in terms of $F$ and phases. Equate the two different expressions of the same graph (7.7) and (7.8) and use the expressions for these two monodromies to show that

$$Sa = bS$$

**Therefore, this expression for $S$ satisfies the last equation on the torus. Hence, this equation can be dropped from our list of axioms and be replaced by this definition of $S$.**

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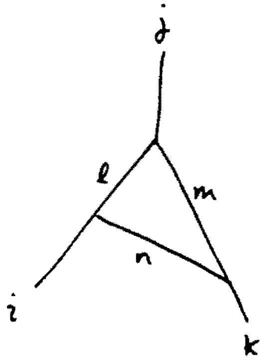
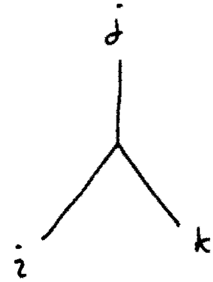
• Exercise 7.9 *more identities for graphs.* Use the pentagon to show that

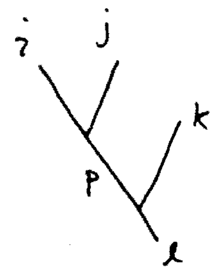
1.



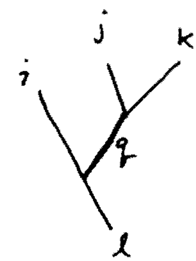

$$= \frac{F_{k\ell}\begin{bmatrix} j & m \\ i & n \end{bmatrix}}{F_{k0}\begin{bmatrix} n & n \\ m & m \end{bmatrix}}$$




2.




$$= \sum_q F_{pq}\begin{bmatrix} j & k \\ i & \ell \end{bmatrix}$$

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In all the manipulations with knots in $S^3$ we use only the polynomial equations on the plane. We do not need the torus equations. Therefore, quasirational as well as rational theories lead to knot invariants in $S^3$.

In the above discussion we have simply *defined* $S_{ij}(p)$ as a combination of certain duality matrices, exactly as in the axioms for a MTC. In order to see directly why, with this definition, $S$ should be related to the modular group of the torus we must pause and discuss Witten's observation [27] that 2-dimensional duality (as axiomatized by the notion of a modular functor) is equivalent to 3 -dimensional general covariance.

One recent application of the knot invariants arising in RCFT has been to the construction of invariants of three manifolds [27][41][43] [45]. These applications are simply one facet of the current interest in studying the geometry and topology of manifolds via quantum field theory, through the general notion of topological QFT's. These were introduced by Witten and recently axiomatized by Atiyah. In 2 + 1 dimensions the Atiyah-Witten axioms, which summarize the formal properties of path integrals for topological field theories, are closely connected to the notion of a modular functor. To see this recall that the Atiyah-Witten axioms are [46] [47],

---

**Axioms for a Topological Field Theory**

*Data:*

1. A map from closed oriented $d$-manifolds to complex finite dimensional vector spaces $\Sigma \to \mathcal{H}(\Sigma)$.

2. A distinguished vector $Z(Y) \in \mathcal{H}(\Sigma)$ associated to $d+1$-manifolds such that $\Sigma = \partial Y$. (In particular if $Y$ is closed $Z(Y)$ is a complex number.)

*Conditions:*

1. Naturality. If $f : \Sigma_1 \to \Sigma_2$ is an automorphism there is an isomorphism $\mathcal{H}(f) : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$ satisfying $\mathcal{H}(f_1 f_2) = \mathcal{H}(f_1)\mathcal{H}(f_2)$. There is a similar naturality condition on the vectors $Z(Y)$.

2. Duality. $\mathcal{H}(\Sigma^*) \cong \mathcal{H}(\Sigma)$.

3. Multiplicativity. $\mathcal{H}(\Sigma_1 \coprod \Sigma_2) \cong \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$. Moreover $\mathcal{H}(\phi) \cong \mathbb{C}$.

4. Gluing. If $Y$ and $Y'$ are glued along a $d$-manifold $\Sigma$ (with opposite orientations for $\Sigma$) to form $\tilde{Y}$ then

$$Z(\tilde{Y}) = \langle Z(Y), Z(Y') \rangle$$

The above makes sense since the opposite orientations of $\Sigma$ allow us to pair a space with its dual.

5. Completeness. The states $Z(Y)$ for all $Y$ with $\partial Y = \Sigma$ span $\mathcal{H}(\Sigma)$.

---

(Note: Atiyah adds a sixth axiom that $Z(Y^*) = Z(Y)^*$, but we will not need this.). Clearly for the case $d = 2$ the above notion is very close to that of a modular functor, in particular in any attempt to pass from one to the other the vector spaces $\mathcal{H}(\Sigma)$ are surely the same. Nevertheless, there are some things to prove. The precise connection was worked out in [48] [49]. To pass from a modular functor to a topological theory the main problem is to construct the vector $Z(Y)$ from the data of the modular functor. This was done in [48][49] by choosing a Morse function, using the data of the modular functor to define "transition amplitudes" between critical points of the Morse function and then checking that the choice of Morse function does not lead to ambiguities. To pass from the topological theory to the modular functor the main problem is to produce the finite set of labels (of "representations") and their fusion rule algebra, etc. An argument that this can be done is presented in [48]. The labels are a basis for the vector space $\mathcal{H}(torus)$.

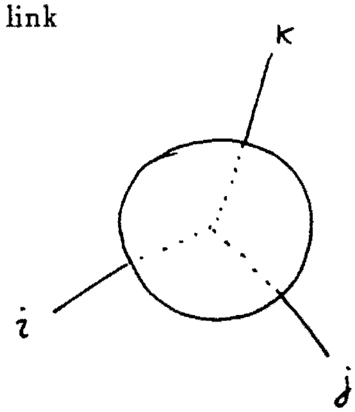
The advantage of the point of view of modular functors and topological field theories is that for any system satisfying the axioms one can compute quantities for nontrivial graphs and nontrivial manifolds via the gluing axiom. In particular, one can compute various quantities using the notion of surgery.

If $\mathcal{H}(\Sigma)$ is an $n$ dimensional vector space, any collection of $n+1$ vectors $Z_i \in \mathcal{H}(\Sigma)$ is linearly dependent; i.e. there are coefficients $a_i$ such that $\sum_i a_i Z_i = 0$. This leads to a linear relation between the invariants of different manifolds. Let $Z_i = Z(Y_i)$ for $n+1$ different $Y_i$. Then,

$$\sum_i a_i Z(\tilde{Y}_i) = \sum_i a_i \langle Z(Y), Z(Y_i)\rangle = 0 \tag{7.9}$$

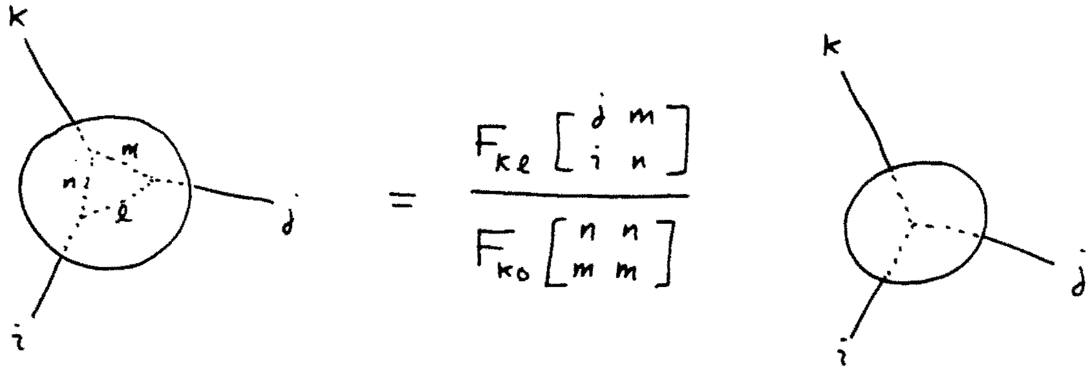
where $\tilde{Y}_i$ is obtained by gluing $Y$ to $Y_i$ along some d-fold $\Sigma$.

Rather than continuing in complete generality, we focus on the particular topological field theory corresponding to a RCFT. As explained above, the labels of the representations label a basis of $\mathcal{H}(T^2)$. The three manifold $Y$ can have links carrying these labels (also links with vertices) and these links may terminate at the boundary of $Y$. For example, for $Y$ a three ball with the link




a link in a three ball

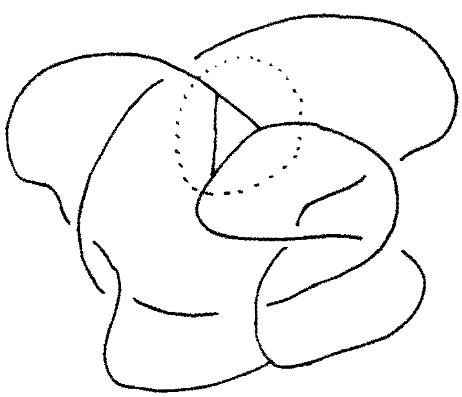
we find a vector $v \in \mathcal{H}(S^2_{ijk})$ where $S^2_{ijk}$ is a sphere with three labeled points $i, j, k$. By the correspondence of a topological field theory and RCFT, $\mathcal{H}(S^2_{ijk}) \cong V_{ijk}$ and its dimension is $N_{ijk}$ (if $N_{ijk} > 1$, we should specify the kind of coupling which is used in the vertex in the link). Continuing to assume for simplicity that $N_{ijk} = 0, 1$, the vector $\tilde{v} \in \mathcal{H}(S^2_{ijk})$ corresponding to




another link in a three ball

is proportional to the original one $\tilde{v} = xv$.

Now, consider a complicated three manifold $Y$ with a link



a complicated link

Remove the three ball which looks like the previous figure (the dashed line) to obtain the three manifold $\tilde{Y}$. By the gluing axiom

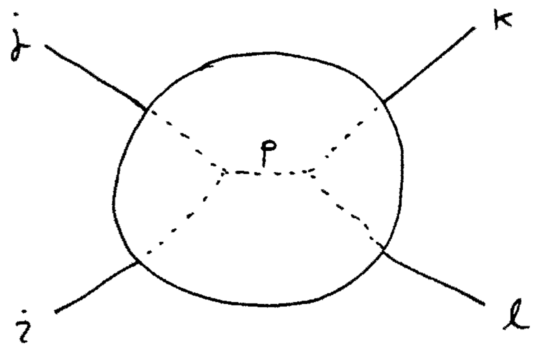
$$Z(Y) = \langle \tilde{v}, Z(\tilde{Y}) \rangle = x^* \langle v, Z(\tilde{Y}) \rangle = x^* Z(Y')$$

where $Y'$ is the same as $Y$ except that the ball is replaced by the simple link. This procedure simplifies the computation of $Z(Y)$ by relating it to a simpler object $Z(Y')$.

• Exercise 7.10 *Interpretation of previous results.*

a.) Use this understanding to interpret the first relation in exercise 7.9. Express $x$ in terms of the duality matrices.

b.) Repeat this analysis for the sphere with four labels $ijkl$. Show that the vectors




a basis for $\mathcal{H}(S^2_{ijkl})$

for all $p$ span $\mathcal{H}(S^2_{ijkl})$. The vector of a given $p$ corresponds in the RCFT to the conformal block with the representation $p$ in the intermediate channel. The second relation in exercise 7.9 expresses duality in RCFT. Interpret it from three dimensions.

c.) Cut the tetrahedron graph (the first figure in exercise 7.5) along the lines $i, j, l, n$ and express the invariant of the graph as an inner product of two vectors in $\mathcal{H}(S^2_{ijln})$. Use part b of this exercise to explain why the tetrahedron graph is proportional to $F$.

d.) Interpret the equations for the ope coefficients $d$ as determining a metric on $\mathcal{H}$ as mentioned in the end of section 6. Use this fact to interpret the second graph in exercise 7.5 as $\frac{d_{ijk}}{\sqrt{F_i F_j F_k}}$.

e.) Interpret the gauge invariance as a freedom in the normalization of the vectors in $\mathcal{H}(\Sigma)$.

---

This interpretation is more powerful when combined with the notion of surgery [27]. First notice that $\mathcal{H}(T^2)$ is spanned by $v_i = Z(M_i)$ where $M_i$ is a solid torus with a line with the label $i$ around the non-contractible cycle. Consider a three manifold $Y_i$ with a closed line with the label $i$. Removing a solid torus $M_i$ surrounding the line from $Y_i$, we find the three manifold $\tilde{Y}$. By the gluing axiom, $Z(Y_i) = \langle Z(\tilde{Y}), v_i \rangle$. Now consider another three manifold $X$ obtained by interchanging the $a$ and $b$ cycles [1] on the boundary of $M_i$ and then gluing it back to $\tilde{Y}$. The relevant inner product is

$$Z(X) = \sum_j S_{ij}\langle Z(\tilde{Y}), v_j\rangle = \sum_j S_{ij} Z(Y_j)$$

As before, we succeeded to express $Z$ of some manifold in terms of $Z$'s of other (simpler) manifolds. Using this procedure it is possible to compute $Z$ for every manifold [27].

---

• Exercise 7.11 *Ambiguity in surgery.* Show that the ambiguity associated with the choice of the b cycle corresponds to the application of $T$ in RCFT. Therefore, it is related to the fact that the lines have to be framed. How does the framing remove the ambiguity?

---

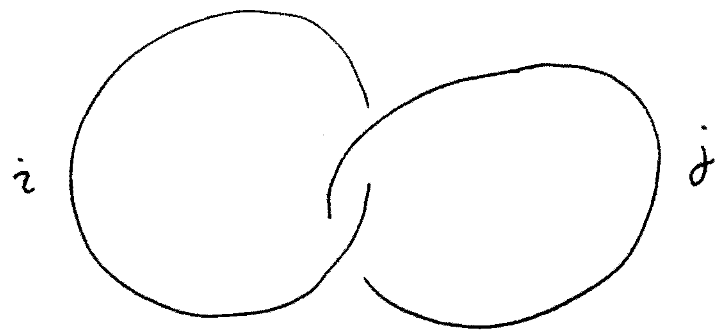
---

• Exercise 7.12 *Some calculations using surgery.*

a.) The invariant for two parallel nonbraiding (="cabled") lines $W_i, W_j$ in $S^2 \times S^1$ is $N_{0ij}$. Why?

b.) Think of $S^2 \times S^1$ as two solid tori whose toroidal boundaries are identified via the identity map $(\sigma^1, \sigma^2) \to (\sigma^1, \sigma^2)$. Change the identification to the transformation: $S : (\sigma^1, \sigma^2) \to (-\sigma^2, \sigma^1)$. Show that the resulting three-manifold is just $S^3$.

c.) Suppose the two solid tori of part (b) contain lines $W_i$ and $W_j$ respectively. Each line wraps along the noncontractible direction. Show that the resulting configuration in $S^3$ is just:




A configuration of lines in $S^3$

[1] The a cycle is the contractible cycle inside $M_i$; however, there is an ambiguity in what we mean by the b cycle. We will return to this ambiguity shortly.

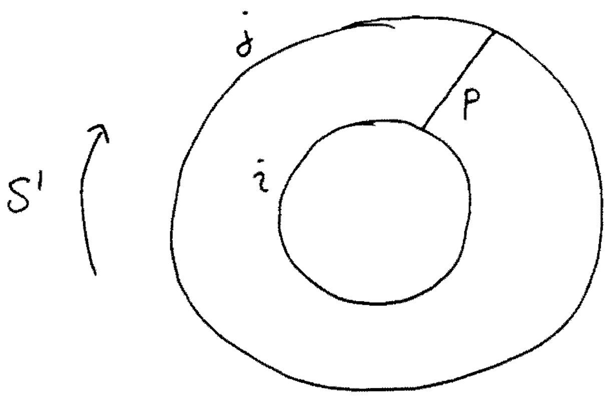
and therefore the invariant of this graph is $S_{ij}$.

d.) Using the graphical formalism described above, compute the figure in part (c) and rederive the formula

$$\frac{S_{ji}}{S_{00}} = \frac{\left(B\left[\begin{smallmatrix} j & i \\ j & i \end{smallmatrix}\right] B\left[\begin{smallmatrix} i & j \\ j & i \end{smallmatrix}\right]\right)_{00}}{F_i F_j}$$

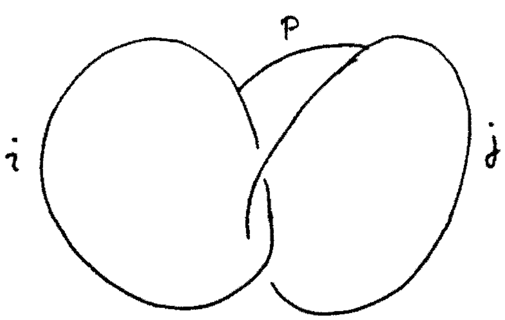
we derived in a previous exercise. Notice that the graphical rules did not include an overall normalization factor of $S_{00}$ for every graph in $S^3$. This factor is natural from the surgery point of view if the invariant in part a of this exercise is normalized to be $N_{0ij}$.

e.) Compute the invariant for two cabled lines $W_i$ and $W_j$ in $S^2 \times S^1$ as before but this time connected by a line with the label $p$:




a configuration in $S^2 \times S^1$

f.) Perform surgery as above using $S(p)$ and turn this into

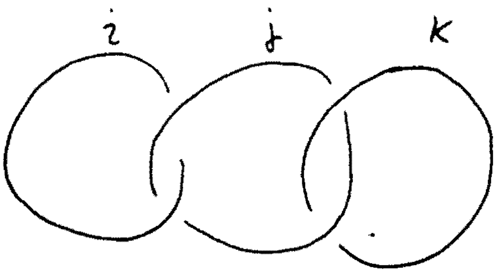



the previous graph after surgery

in $S^3$. Compute this graph using our rules and derive equation (4.3). (Because of the framing, there is a phase ambiguity. The phase $e^{-i\pi\Delta_p}$ is determined by consistency.)

• Exercise 7.13 *Verlinde's formula from Surgery.* We outline a slightly modified proof of E. Witten of Verlinde's formula.

a.) Consider the configuration:

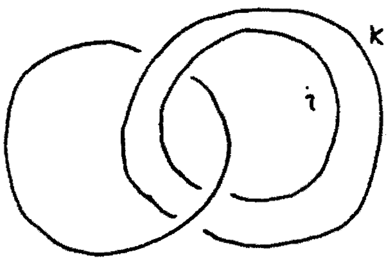



A configuration used in the proof of Verlinde's formula

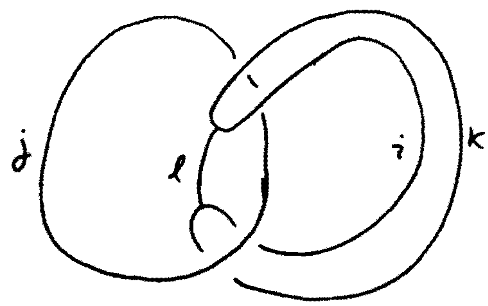
Using the graphical rules and the above formula for $S$ in terms of $B$ show that this has the value:

$$\frac{S_{ij}S_{jk}}{S_{0j}}$$

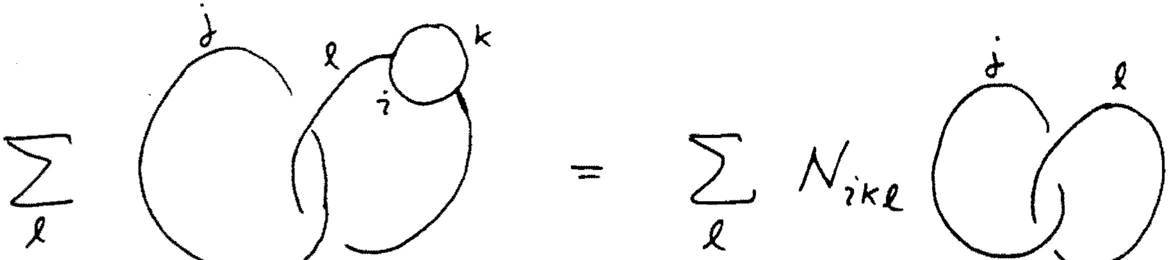
b.) Rewrite the above as




$$= \sum_{\ell}$$




Use the identity $FF^{-1} = 1$ and the braiding/fusing identity to rewrite this as:




From this derive Verlinde's formula.

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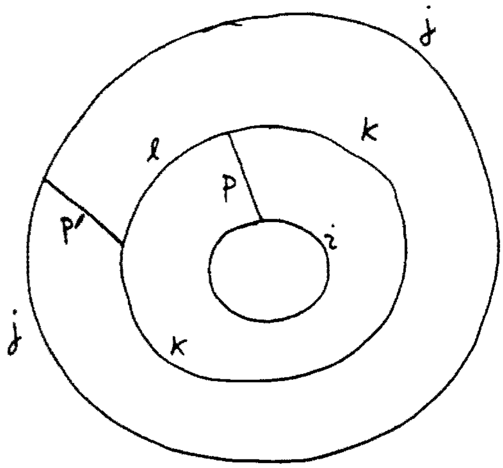
• Exercise 7.14 *a and b monodromies for the two point function on the torus.* Relate the graph




graphical formulas for the b monodromy

in $S^2 \times S^1$ to the b monodromy. Use surgery to relate it to the figure used in exercise 7.8. Find a graph in $S^2 \times S^1$ for the a monodromy and use surgery to relate it to the figure used in exercise 7.8. Thus making the previous proof of $Sa = bS$ somewhat intuitive.

---

We see that the information in surgery is equivalent to the information in the equation $Sa = bS$ which in turn is equivalent to the formula for $S(p)$ in terms of $F$ and $B$.

We have seen that a RCFT defines a modular functor, which has been argued to give rise to a topological 2+1 dimensional theory. Recently L. Crane [45] has shown more directly that the data $F, B, S$ can be used to construct invariants of framed 3-folds through the use of some theorems from combinatorial topology. For example, to identify the invariant associated to a closed 3-fold $Y$ we use a "Heegaard splitting" whereby $Y$ is represented as a glued pair of handlebodies $Y_1, Y_2$ which have as a common boundary the surface $\Sigma$. $Y_1$ is glued to $Y_2$ via a nontrivial diffeomorphism $\phi$ of $\Sigma$. Among the conformal blocks $\mathcal{H}(\Sigma)$ there is a distinguished (normalized) vector $\chi_0$ defined by the condition that the trivial representation be present on all internal lines. Representing $\phi$ by the duality matrix $Z(\phi)$ we have the invariant $Z(Y) = \langle \chi_0, Z(\phi)\chi_0 \rangle$. Since the Heegaard decomposition is not unique it is nontrivial that $Z(Y)$ is an invariant. Using known facts about Heegaard splittings Crane shows that the axioms of an MTC guarantee that $Z(Y)$ is unambiguous up to a factor of $e^{2\pi i c/24}$. Yet another approach, due to Reshetikhin and Turaev [41] will be mentioned in the following section.

So far the discussion was very general and did not depend on a particular three dimensional theory. In [27] Witten considered the Chern-Simons-Witten gauge theory in three dimensions. This is a topological field theory and therefore the general analysis in this section applies there. Moreover, this theory can be solved exactly [27] and explicit expressions for the duality matrices can be obtained. The study of this theory is the subject of sections 9 and 10.

## 8. Quantum group solutions of the polynomial equations

This section contains some remarks intended for those already familiar with basic facts about quantum groups. Thus we assume some familiarity with [50] [40]. A nice review of the subject is [51].

If $A$ is a Hopf algebra then the category of its finite dimensional representations $Rep(A)$ has a tensor product which may be defined by the comultiplication $\Delta$. From the axioms satisfied by a comultiplication there will be an associativity constraint satisfying a pentagon consistency relation. In the previous terminology, the $F$ matrix will exist and will satisfy the pentagon relation. In general there will be no commutativity constraint, i.e., there will be no analog of $\Omega$. If $A$ is a quasitriangular Hopf algebra (see [50], essentially it means that the comultiplication and opposite comultiplication are conjugate by a "universal" $R$ matrix.) then there is a commutativity constraint, but in general $\Omega^2 \neq 1$. In this case there will be two hexagon conditions. These hexagon conditions are equivalent to Drinfeld's formulae $(\Delta \otimes 1)R = R_{13}R_{23}$ and $(1 \otimes \Delta)R = R_{12}R_{23}$. In this case $Rep(A)$ is a braided monoidal category. In [41] a central extension of a quasitriangular Hopf algebra is defined which these authors call a "ribboned Hopf algebra." The extra conditions specified for a ribboned Hopf algebra are such that in this case $Rep(A)$ is a "compact braided monoidal category," which in our terms means that when $F, B$ matrices are suitably identified with quantum group Racah coefficients (in a way precisely analogous to the discussion of group theory above) then the genus zero axioms of a MTC are fulfilled. (Except, perhaps, for the finiteness of the index set $I$.) Correspondingly, in [41] $Rep(A)$ for a ribboned Hopf algebra is used to define invariants of knotted graphs [2] in $\mathbb{R}^3$.

An important special case of ribboned Hopf algebras is provided by the quantized universal enveloping algebras $U_q(\mathcal{G})$ for a Lie algebra $\mathcal{G}$. Applying the machine of [41] one may obtain invariants of knots in $S^3$ for any deformation parameter $q$. However when $q$ is "rational," which means that $q^n = 1$ for some integer $n$, something more remarkable happens. In this case one may truncate the set of representations to a set of 'good' or 'type II' representations [52] [53], characterized as a minimal complete set of representations with

[2] More precisely, invariants of colored directed ribboned tangles.

nonvanishing quantum dimension, such that the truncated space of representations defines a modular tensor category.

The most famous and well-known example of this phenomenon is provided by $U_q(sl(2))$. In this case it has been shown that the braiding and Racah matrices for the case $q = e^{2\pi i/(k+2)}$ are *identical* to those of the conformal field theory $\hat{su}(2)_k$ when we restrict the class of representations and invariant tensors to the "good" ones generated by irreducible representations of dimensions $\leq k+1$ and couplings satisfying the $\hat{su}(2)_k$ fusion rules. The proof of this statement may be obtained as follows. One first computes the braiding matrices for spin 1/2 operators [20] and notices the exact correspondence with the corresponding quantum group objects. In conformal field theory the other braiding matrices may then be obtained by successive use of the braiding/fusing relation. Then one proves that it is valid to truncate the quantum group braiding/fusing relation so that it only includes the good representations. Another argument, using properties of Hecke and TLJ algebras has been advocated by Alvarez-Gaumé, Gomez, and Sierra [51]. With the coincidence of $F, B$ matrices one may define $S$ as in (4.3) and hence the restricted quantum group representation theory defines a MTC. Analogous statements exist for other $U_q(\mathcal{G})$ and full proofs for all cases have been published in [54]. The coincidence of $F, B$ matrices has been widely noted and discussed. Just a few references include [20][54][15][55] [56] [57] [58] [51].

These observations allow one to give very explicit formulae for braiding/fusing matrices (which are more easily obtained by using quantum group technology). For example, very explicit formulae where written down in [40]. As a simple example we quote the well-known result for a braiding matrix of two spin 1/2 fields. The relevant space of conformal blocks is two-dimensional corresponding to intermediate spins $j \pm \frac{1}{2}$ and we have

$$B_{rs}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{bmatrix} = q^{1/4}\delta_{r,s} - q^{-1/4}\frac{\sqrt{S_r S_s}}{S_j}$$

where $S_j = sin\frac{\pi(2j+1)}{k+2}$. Alternatively this may be written

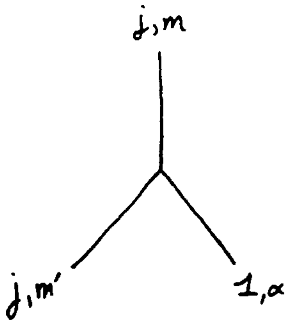
$$B\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{bmatrix} = \frac{1}{[2j+1]}\begin{pmatrix} \frac{-q^{-(j+3/4)}}{\sqrt{q^{-1/2}[2j][2j+2]}} & \sqrt{q^{-1/2}[2j][2j+2]} \\ & q^{j+1/4} \end{pmatrix}$$

where $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$.

In [41] Reshetikhin and Turaev represent 3-manifolds via surgery on links and use the surgery procedures of Witten to reduce the invariants of three-folds to those associated to links (or tangles). Their paper can be viewed as another construction of a three-dimensional topological field theory, starting from the MTC associated to the representation theory of $U_q(sl(2))$ for $q^{k+2} = 1$ (and, in principle, to other $U_q(\mathcal{G})$.) The link or tangle invariants are computed essentially as transition amplitudes of conformal blocks, along the lines described above.

The fact that the type II representation theory of $U_q(\mathcal{G})$ for rational deformation parameters coincides with the MTC of a canonically associated RCFT is still something of a mystery. The statement of this fact has been formulated in a number of conformal field theoretic constructions [51][57][59] [60] but these descriptions make use of the fact rather than explain it. Another connection of CFT to quantum groups has been noted in [61]. In [27] Witten proposed one approach to this problem, which, if successfully brought to conclusion would yield an adequate explanation. More recently Witten has proposed a different explanation in [62]. In the remainder of this section we present an alternative interpretation of Witten's idea.

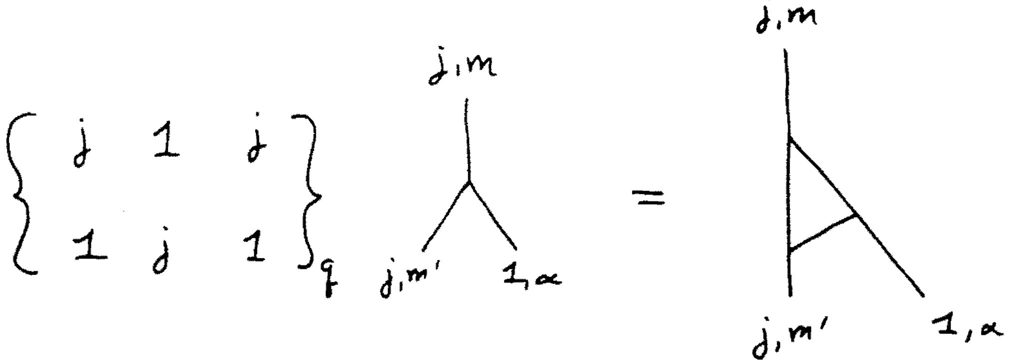
We begin by noting that the quantum $3j$ symbols themselves may be seen to form an algebra. Namely, using the formalism of [40] we have

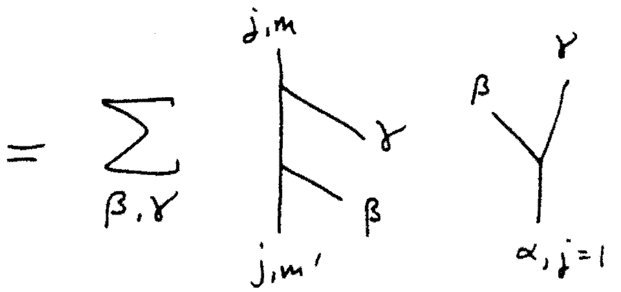



$$= \begin{bmatrix} j & 1 & j \\ m & \alpha & m' \end{bmatrix} \equiv (T_\alpha)_{mm'}$$

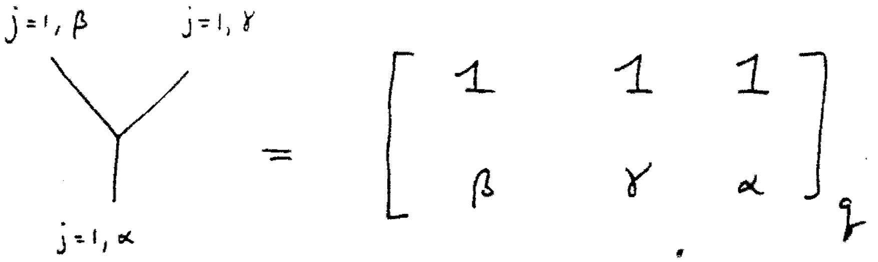
Graphical representation of a $3j$ symbol with one line carrying spin 1.

which we will take to define the matrix elements of three operators $T_{\alpha=-1,0,+1}$. By the very definition of Racah coefficients we may write




where




$3j$ symbol for coupling three spin 1 representations.

and we will denote the Racah coefficient by $A_j$.

Clearly the above formula may be regarded as defining an algebra for the $T_\alpha$ operators, the structure constants being defined by the $3j$ symbols for three spin 1 representations and the Racah coefficient $A_j$. That is, we may write:

$$\sum_{\beta,\gamma} \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{bmatrix} T_\beta T_\gamma = A_j T_\alpha$$

For example, for $U_q(sl(2))$ one may easily compute:

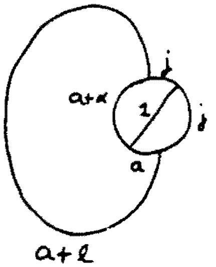
$$q^{-1/2} T_+ T_0 - q^{1/2} T_0 T_+ = A_j T_+$$

$$T_+ T_- - T_- T_+ = (q^{1/2} - q^{-1/2}) T_0^2 + A_j T_0$$

$$q^{-1/2} T_0 T_- - q^{1/2} T_- T_0 = A_j T_-$$

for any value of $q$. This is precisely the algebra derived in [62]. The reason for this is that graphs are computed with quantum Racah or 6j symbols. But, upon analytic continuation away from $|q| = 1$ the $6j$ symbols have large spin limits which are precisely $3j$ symbols. More precisely we have [40]

$\lim_{a \to \infty}$




$$= lim_{a \to \infty} \begin{Bmatrix} a+\alpha & 1 & a \\ j & a+l & j \end{Bmatrix}$$

$$= \frac{1}{[2j+1]^{1/2}} \begin{bmatrix} 1 & j & j \\ -\alpha & l & l-\alpha \end{bmatrix}$$

Thus Witten's lassoing and limiting procedure produces the algebra of $3j$ symbols.

## 9. Chern-Simons-Witten gauge theory – Quantization

The discussion in section 7 was quite general. It can be made much more explicit in a particular field theory – the CSW theory[27]. This is a particular example (we will later mention a conjecture that this is essentially the only example) of a topological field theory. The theory is a gauge theory based on the gauge field $A = A_\mu^a \, T^a \, dx^\mu$ in some Lie algebra $g$ with action

$$S = \frac{k}{4\pi} \int_Y \, Tr(AdA + \frac{2}{3}A^3)$$

for a three manifold $Y$. For simplicity we limit ourselves here to $SU(N)$ gauge theory with a trace in the fundamental representation ($TrT^aT^b = -\delta^{ab}$).

Clearly, the action is independent of the metric on $Y$. To prove that the theory is indeed topological, one needs to show that the measure of the functional integral is also independent of the metric. In what follows, we will assume that this is the case[3].

Perhaps the easiest way to understand the theory is by canonical quantization. Suppose we have a Riemann surface $\Sigma$ and consider the theory on the 3-dimensional manifold $Y = \Sigma \times \mathbb{R}$.

If we canonically quantize the theory we obtain a space of physical states $H(\Sigma)$ associated to the surface $\Sigma$. Witten showed that these states have a natural interpretation in terms of the WZW model for g-current algebra at level $k$. Specifically:

$$\Sigma = \text{closed surface} \Rightarrow \quad H_\Sigma = \begin{cases} \text{vector space of} \\ \text{conformal block for} \\ \text{partition function} \\ \text{on } \Sigma \end{cases}$$

$$\Sigma = \begin{cases} \text{surface pierced by} \\ \text{Wilson line in} \\ \text{Representations } j_1, \cdots j_n \end{cases} \Rightarrow \quad H_\Sigma = \begin{cases} \text{conformal blocks for n-point} \\ \text{function on } H_\Sigma \text{ for n fields} \\ \text{in the representations: } j_1 \cdots j_n \end{cases}$$

Moreover for 3-manifolds interpolating between two surfaces $\Sigma_1$ and $\Sigma_2$ the path integral gives a transformation $H(\Sigma_1) \longrightarrow H(\Sigma_2)$. Witten shows that these transformations are

[3] In [63] Witten showed that the existence of the central charge in two dimensions is related to some dependence on the metric on $Y$ – the theory depends on the "framing on $Y$".

just the duality transformations on the space of blocks. Why is it true? We will explain these matters in a simple physical way.

Choose $A_0 = 0$ gauge: If $\Sigma$ has no boundary then

$$S = \frac{k}{4\pi} \int \epsilon^{ij} Tr A_i \frac{d}{dt} A_j$$

We then have a first order Lagrangian and therefore, the phase space is the space of gauge fields on $\Sigma$. The symplectic structure on this space leads to the commutation relations

$$\left\{A_i^a(x), A_j^b(y)\right\} = \epsilon_{ij}\delta^{(2)}(x-y)\frac{4\pi}{k}.$$

where $\int \delta^{(2)}(z-w)d^2z = 1$. It is convenient to pick a complex structure $\tau$ on $\Sigma$ and to write

$$\left\{A_z^a(z), A_{\bar{z}}^b(w)\right\} = \delta^{(2)}(z-w)\frac{4\pi}{k}.$$

The wave functions in holomorphic quantization are holomorphic functions of $A_z$, $\psi = \psi(A_z)$. The Hilbert space is the space of all these functions. The physical space is the subspace of the Hilbert space which is invariant under the Gauss law.

---

• Exercise 9.1 *Gauss' law.* Show that

$$u(\epsilon) = \frac{ik}{4\pi} \int Tr(\epsilon F)$$

generates an infinitesimal gauge transformation by $\epsilon$:

$$[u(\epsilon), A] = -D\epsilon$$

$$[u(\epsilon_1), u(\epsilon_2)] = u([\epsilon_1, \epsilon_2])$$

---

By integrating $u(\epsilon)$ the operator generating a finite transformation $g = e^\epsilon$ is

$$U(g) = e^{u(\epsilon)}$$

so

$$U(g)AU^{-1}(g) = gAg^{-1} - dgg^{-1}$$

Now how does it act on physical states? We certainly must have:

$$(U(g)\psi)(A_z) = e^{f(A_z;g)}\psi(A_z^g)$$

to find $f$, we impose the group law:

$$U(h)U(g) = U(gh)$$

and find:

$$f(A;gh) = f(A;h) + f(A^h;g) \bmod 2\pi ik$$

The solution is:

$$\begin{aligned} f(A_z;g) = &\frac{ik}{4\pi}\int Trg^{-1}\partial gg^{-1}\bar{\partial}g + k\Gamma_{WZ}(g) \\ &-\frac{ik}{2\pi}\int Tr(A_z g^{-1}\bar{\partial}g) \equiv ikS(g:A_z,0) \end{aligned}$$

So

$$(U(g)\psi)[A_z] = e^{ikS(g;A_z,0)}\psi[A_z^g]$$

This is the key equation. From it we may get the independent physical states as follows.

Physical states are invariant under the Gauss law - so we are looking for linearly independent solutions to the equation

$$\psi(A_z) = e^{ikS(g;A_z,0)}\psi(A_z^g)$$

Now, given *any* functional $\psi_0$ we can generate such a solution by

$$\psi_{phys} = \int DgU(g)\psi_0$$

i.e. we can write:

$$\psi_{phys}(A_z) = \int Dg\ e^{ikS(g;A_z,0)}\psi_0(A_z^g)$$

We will now carry this out for three examples: $\Sigma = T^2$, the torus; $\Sigma = S^2$ pierced by Wilson lines and $\Sigma =$ Disk.

$$\Sigma = T^2$$

From general principles we expect that $H_\Sigma$ will be the space of characters of the affine Lie algebra. The easiest thing to do is choose a complex structure $z = \sigma^1 + \tau\sigma^2$ so we represent the torus by a parallelogram as usual. Define $A_z = \frac{\tau A_1 - A_2}{\tau - \bar{\tau}}$. So

$$[A_z^a(x), A_{\bar{z}}^b(y)] = \frac{-2\pi}{kIm\tau}\,\delta^{ab}\delta^{(2)}(x-y)$$

(In the equations above the factor $Im\tau$ was in the definition of the delta function.)

Now we use a basic fact: we can always gauge $A_z$ to the constant Cartan:

$$A_z = hah^{-1} - \partial h h^{-1}$$

with $h$ in the complexification of the gauge group where $a$ is constant in the Cartan subalgebra. So - by the Gauss law it suffices to know the values $\psi[a_z]$ because $\psi[A_z] = e^{-ikS(h,a,0)}\psi[a]$. Now if we take the family of testfunctions for $\bar{J}_{\bar{z}}$, where $\bar{J}$ is a constant in the Cartan subalgebra,

$$\psi_0^J(A) = e^{\frac{ik}{2\pi}\int Tr A_z J}$$

then the corresponding physical states are

$$\psi_{phys}^J(a) = \int Dg\ e^{ikS(g,a_z,J)}e^{-\frac{ik}{2\pi}\int Tr(aJ)}$$

where $S(g, A_z, A_{\bar{z}})$ is the gauged WZW action:

$$\begin{aligned} S(g, A_z, A_{\bar{z}}) = &\frac{ik}{4\pi}\int Tr g^{-1}\partial g g^{-1}\bar{\partial} g + ik\Gamma \\ &-\frac{ik}{2\pi}\int Tr[Ag^{-1}\bar{\partial} g + \bar{A}\partial g g^{-1} + gAg^{-1}\bar{A} - A\bar{A}] \end{aligned}$$

The value of this path integral is well-known, it is just

$$= \sum_\lambda\ \psi_\lambda(\bar{J})\psi_\lambda(a)$$

where

$$\psi_\lambda(a) = e^{-\frac{kIm\tau}{2\pi}a^2}\ \chi_\lambda(\bar{\tau}, \frac{-iIm\tau}{\hbar}a)$$

where $\chi_\lambda$ are the Weyl-Kac characters. Thus - as we vary $\bar{J}$ we sweep out a space of states spanned by the characters.

---

• Exercise 9.2 *The Weyl Alcove.* Consider quantization of the Chern-Simons-Witten gauge theory on the torus with a real polarization, that is, $\psi = \psi[A_1(x)]$. Take the gauge group to be connected, simply connected and simply laced.

a.) Derive the Gauss law and show that $\psi$ has support on those $A_1$ which are components of a flat connection. Thus the wavefunction is determined by its value for $A_1$ constant and in the Cartan subalgebra.

b.) Show that the Gauss law for the gauge transformations preserving the constant Cartan force $\psi$ to be a periodic delta function whose support is at $\Lambda^{weight}/W \times k\Lambda^{root}$ where $\Lambda^{weight}$ ($\Lambda^{root}$) is the weight (root) lattice and $W$ is the Weyl group. The elements of this coset are in a natural one-to-one correspondence with the integrable highest weight representations of level $k$ of the associated Kac-Moody algebra.

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• Exercise 9.3 *Moduli Space of Flat Connections.*

a.) In his original paper Witten first imposed the constraints and then quantized the resulting phase space. Show that this phase space is just the moduli space of flat connections on $\Sigma$.

b.) A flat connection is characterized by its holonomies, up to conjugation. Show that the real dimension of the resulting phase space is $(2g-2)dimG$ for the gauge group $G$ on a surface of genus $g > 1$.

c.) Use the WKB approximation to show that the number of physical states grows as $k^{(g-1)dimG}$ and compare with exercise 3.8.

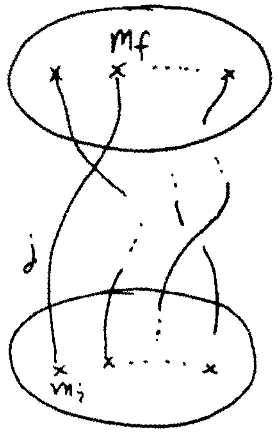
---

$\Sigma = S^2$ punctured by Wilson lines

The Wilson lines for finite transition amplitudes are

$$\langle m_f | P\ exp \int_C A\ | m_k \rangle$$

where the Wilson line carries some representation $j$ and $m_f, m_k$ are states in the representation $j$ as in the following figure



two sphere's with Wilson lines

Since the Hamiltonian of the theory is zero, finite time amplitudes are the same as overlaps of wavefunctions. So we see that the wavefunctions in the case with punctures are simply wavefunctionals valued in the tensor products of representations:

$$\vec{\psi}[A_z] = \sum_{m_i} \psi_{m_1 \cdots m_n}[A_z]\ |m_1\rangle \otimes \cdots \otimes |m_n\rangle$$

We know how Wilson lines transform under gauge transformation, so it is clear that the action of the Gauss law is just:

$$(U(g)\vec{\psi})[A_z] = e^{ikS(g;A_z,0)} \otimes_i \rho_i(g^{-1}(P_i))\vec{\psi}[A_z^g]$$

As before, we may use the basic fact that we can gauge away $A_z$ i.e. $A_z = -\partial_z h h^{-1}$ Thus physical wavefunctions are completely determined by their value at $A_z = 0$:

$$\vec{\psi}^{phys}[A_z = 0] = \int Dg\ e^{ikS(g)} \otimes_i \rho(g^{-1}(P_i))\vec{\psi}_0(-\partial g g^{-1})$$

Now $\vec{\psi}_0$ is an arbitrary functional of the holomorphic current, so, by the holomorphic KM Ward identities we obtain a basis of physical states:

$$\vec{\psi}_{\vec{p}}(A_z) = e^{-ikS(h)} \otimes \rho_i(h(P_i))\vec{\mathcal{F}}_{\vec{p}}(\bar{z}_1, \ldots \bar{z}_n)$$

From this example we see that the transition function given by the path integral for braided Wilson lines is indeed the appropriate duality matrix.

---

• Exercise 9.4 *Knizhnik-Zamolodchikov equations.*

a.) From the discussion of wavefunctions above write the Gauss law for the case of the sphere with sources as:

$$u(\epsilon) = \frac{k}{4\pi}\int Tr\epsilon F + \sum T_i^a \epsilon^a(P_i)$$

We would like to see how the wavefunctions change as the positions $P_i$ of the sources change.

b.) Show that

$$[\mathcal{O}, u(\epsilon)] = \frac{\partial}{\partial \bar{z}_i} u(\epsilon)$$

for $\mathcal{O} = \rho_i(T^a)A_{\bar{z}}^a(P_i)$.

c.) Writing physical states as path integrals show

$$\bar{\partial}_i \vec{\psi}[A; P_i]|_{A=0} = \rho_i(T^a)A_{\bar{z}}^a(P_i)\int Dg e^{ik\left(S - \frac{1}{2\pi}\int TrAg^{-1}\delta g\right)} \otimes_i \rho_i\left(g^{-1}(z_i,\bar{z}_i)\right)\vec{\psi}^0|_{A=0}$$

$$= \int Dg e^{ikS}\frac{1}{k}\bar{J}^a(z_i)\rho_i(T^a)\rho_i(g^{-1}(z_i,\bar{z}_i)) \otimes_{i\neq j} \rho_j(g^{-1}(P_j))\vec{\psi}_0$$

For simplicity (and WLOG) take $\vec{\psi}_0$ to be a constant tensor.

d.) We must define the singular product of operators at $P_i$. We do this by point splitting, then making an appropriate subtraction, which will be uniquely determined from self-consistency. Use the conformal field theory operator product relation (for a proof see [23].):

$$\bar{J}^a(\bar{\zeta})\rho_i(T^a)g^{-1}(z_i,\bar{z}_i) = \frac{C_i}{\bar{\zeta} - \bar{z}_i} + (k+h)\bar{\partial}_i g^{-1}(z_i,\bar{z}_i) + O(\zeta - z_i)$$

where $h$ is the dual Coxeter number and $C_i = C_2(V^{j_i})$ is the Casimir of the representation $V^{j_i}$, to deduce that we must define the singular product of operators by

$$:\rho_i(T^a)\bar{J}^a(\bar{z}_i)\rho_i(g^{-1}(z_i,\bar{z}_i)): \equiv \lim_{\zeta\to z}\left[\rho_i(T^a)\bar{J}^a(\zeta)\rho_i\left(g^{-1}(z_i,\bar{z}_i)\right) - \frac{C_i}{\bar{\zeta} - \bar{z}_i} - h\bar{\partial}_i g^{-1}(z_i,\bar{z}_i)\right]$$

e.) Plugging in this definition and using the Kac-Moody Ward identities for $\bar{J}$ show that physical states satisfy the Knizhnik-Zamalodchikov equations [23]

$$(k+h)\bar{\partial}_i \vec{\psi}[0; P_i] = \sum_{j \neq i} \frac{\rho_i(T^a)\rho_j(T^a)}{\bar{z}_i - \bar{z}_j} \vec{\psi}[0; P_i]$$

---

$\Sigma$=Disk=$D$

Finally, we consider the case of $\Sigma$ with a boundary. In the case where $\Sigma$ is a disk, $H_\Sigma$ is the chiral algebra of the theory[27] .

We consider the path integral on $D \times \mathbb{R}$. Let us try to "evaluate" the path integral

$$\int \frac{DA}{vol\ G} e^{iS}$$

In order to do that we must decide on the appropriate boundary conditions. These are determined by demanding no boundary corrections to the equations of motion:

$$\delta S = \frac{k}{4\pi} \int_{\partial D \times \mathbb{R}} Tr(\delta A A) + \frac{k}{2\pi} \int_{D \times \mathbb{R}} Tr(\delta A F) \quad .$$

So we choose $A_0 = 0$ on the boundary. The gauge group appropriate for these boundary conditions is $\hat{G} = \{g : D \times \mathbb{R} \to G | g|_{\partial D \times \mathbb{R}} = 1\}$

Now let's decompose $A$ into time and space components:

$$A = A_0 + \tilde{A}$$

so

$$d = dt \frac{\partial}{\partial t} + \tilde{d}$$

$$S = \frac{k}{4\pi} \int Tr \left( \tilde{A} \frac{\partial}{\partial t} \tilde{A}\ dt \right) + \frac{k}{2\pi} \int Tr A_0 \left( \tilde{d}\tilde{A} + \tilde{A}^2 \right).$$

Next, do the integral over $A_0$ giving

$$\int \frac{D\tilde{A}}{vol\ G} \delta(\tilde{F}) e^{\frac{ik}{4\pi} \int_{D \times \mathbb{R}} Tr(\tilde{A} \frac{\partial}{\partial t} \tilde{A}\ dt)}.$$

We can solve this to get

$$\tilde{A} = \tilde{d}U\ U^{-1}$$

for $U : D \to G$, since D is simply connected.

Moreover, one can argue that there is no Jacobian

$$D\tilde{A}\delta(\tilde{F}) = DU$$

• Exercise 9.5 *No Jacobian.* Show that in the change of variables

$$\int DA\delta(F)\mathcal{O}(A) = \int DU\mathcal{O}(-U^{-1}dU)$$

for gauge invariant functionals $\mathcal{O}$.

---

Finally, we plug $\tilde{A} = -\tilde{d}UU^{-1}$ back into the Lagrangian to get:

$$S = \frac{k}{4\pi}\int_{\partial D\times \mathbb{R}} Tr\ U^{-1}\partial_\varphi UU^{-1}\partial_t U\ +\ k\Gamma(U)$$

where $\varphi$ is the angular coordinate on the rim of the disk, and $\Gamma$ stands for the Wess-Zumino functional. As is well known, this does not depend on the values of $U$ on the interior - so we can divide out the volume of the gauge group to get the path integral

$$\int DU\ e^{ikS_{wzw}(U)}$$

where

$$U : \partial D \times \mathbb{R} \to G.$$

Quantization of this system is well-known to give the chiral algebra of the $WZW$ model [2].

---

• Exercise 9.6 *A Disk with a source.* Work out the analogous change of variables for the case of a disk with a source in a representation $\lambda$. Represent the source by a quantum mechanics problem with the action [64]

$$\int dtTr\lambda\omega^{-1}(\partial_0 + A_0)\omega(t).$$

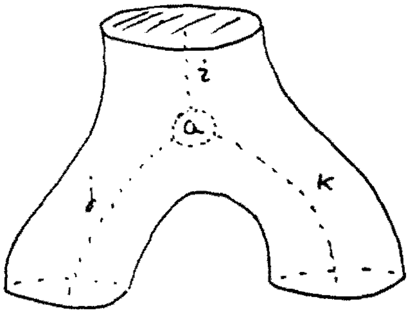
Integrate over $A_0$ to find a constraint on $\tilde{A}$. Show that the holonomy of the flat connection around the source is determined by the representation of the source. Find the effective action on the boundary of $D \times \mathbb{R}$. Its quantization leads to the representation $\lambda$ of Kac-Moody [65]. Use this Lagrangian to find the set of $\lambda$'s which lead to inequivalent effective field theories and hence to the set of integrable representations of Kac-Moody.

---

• Exercise 9.7 *Two sources on $S^2$.* Repeat the analysis of the previous exercise for this case and prove that the Hilbert space is one dimensional if one source is in the conjugate representation to the other source and it is empty otherwise.

---

From these remarks we see that we can also learn about descendents from the $2+1$ dimensional viewpoint. Moreover, note that the quantization on the disk allows us to define a $2+1$ dimensional analog of a chiral vertex operator. Consider the following solid pants diagram threaded by three Wilson lines joined together with an invariant tensor $a$:



Solid pants diagram

The different boundaries are meant to reflect corresponding boundary conditions on the gauge field. From the above exercises we see that the path integral defines an operator from $\mathcal{H}_j \otimes \mathcal{H}_k$ to $\mathcal{H}_i$. Moreover, from general principles of CSW theory this operator has the braiding and fusing properties of a chiral vertex operator. Thus it is natural to suppose that it *is* a chiral vertex operator at some canonical value of $z$, but this has not yet been demonstrated.

Not all aspects of RCFT have been understood from the $2+1$ dimensional viewpoint. We end with the following exercise, part (c) of which is an open problem:

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• Exercise 9.8 *Nontrivial Modular Invariants.*

a.) Show that the natural inner product on quantum wavefunctions for CSGT with connected and simply connected gauge group defines a pairing of representations corresponding to the diagonal modular invariant.

b.) Give the $2+1$ dimensional interpretation of the unitarity of the matrix $S$.

c.) Find a natural interpretation of the nontrivial modular invariants especially exercise 6.5 from the $2+1$ dimensional viewpoint.

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## 10. Chern-Simons-Witten gauge theory – Other RCFT's

In the previous section we saw how KM theories can be reconstructed from connected and simply connected gauge groups in three dimensions. It is therefore natural to ask if other RCFT's can be similarly related to CSW theory for different gauge groups. Here we will show that all known examples of RCFT arise from CSW theory for some gauge group.

Among the other known RCFT's there are three kinds:

1. Extended algebras. Examples include the rational torus, chiral algebras of $D_n$ modular invariants(W-algebras), and other modular invariants obtained by orbifolds of WZW theories.

2. Coset models. Examples include various discrete series

3. Orbifolds of the above.

The holomorphic part of each of these theories can be given a CSGT interpretation:

### 1. Extended KM algebras

Most chiral algebras include high spin fields. Some of them can be obtained by adding extra holomorphic operators to a KM algebra. Theories not finitely decomposable in terms of KM or Virasoro representations might be finitely decomposable with respect to this larger algebra. For example, to form extended algebras one usually uses the "spectral flow" transformation associated to automorphisms of extended Dynkin diagrams. Thus, if we wish to extend level $k$ $\hat{g}$-current algebra we begin with $\theta \in Center(G)$ and write $\theta = e^{2\pi\mu}$ for some weight vector $\mu$. (For simplicity we take $G = SU(n)$, the discussion can be generalized.) The integrable level $k$ representations are given by the points in the Weyl alcove

$$\Lambda_{weight}/W \ltimes k\Lambda_{root}$$

The transformations $\lambda \to \lambda + k\mu$ is equivalent, via the affine Weyl group to a transformation $\lambda \to \mu(\lambda)$ of highest weight representations. For example for $SU(2)$ level $k$ the spin $j$ representation transforms by $j \to k/2 - j$.

Equivalently, we may consider the change in the currents obtained when the boundary conditions are twisted by the multiple-valued "gauge transformation"

$$\Omega(z) = z^{\theta} \tag{10.1}$$

which acts by

$$J(z) \to \Omega(z)J(z)\Omega^{-1}(z) - k\partial\Omega(z)\Omega^{-1}(z) \tag{10.2}$$

In modes we have:

$$\begin{aligned} H_n^i &\to H_n^i + k\theta^i\delta_{n,0} \\ E_n^{\alpha} &\to E_{n+\theta\cdot\alpha}^{\alpha} \\ L_n &\to L_n + \theta^i H_n^i + \tfrac{1}{2}k\theta^2\delta_{n,0} \end{aligned} \tag{10.3}$$

(E,H correspond to simple roots and Cartan elements, respectively) and in the special case of $SU(2)$ this becomes:

$$\begin{aligned} J_n^3 &\to J_n^3 + \frac{k}{2}\delta_{n0} \\ J_n^{\pm} &\to J_{n\pm 1}^{\pm} \\ L_n &\to L_n + \frac{1}{2}J_n^3 + \frac{k}{2}\delta_{n0} \end{aligned} \tag{10.4}$$

In general, for any subgroup $Z \subset Center(G)$ we can "mod out" by this action thus obtaining the extended chiral algebra

$$\mathcal{A} = \oplus_{\mu\in Z}\mathcal{H}_{\mu(0)}$$

A well known example is the rational torus. The toroidal $c = 1$ model with a boson $\phi \sim \phi + 2\pi R$ has a $U(1)$ KM symmetry generated by $J = \partial\phi$ when $R^2 = \frac{p}{2q}$ is rational there are extra holomorphic fields generated by

$$V = e^{\pm i\sqrt{2pq}\,\phi}$$

which generate a large algebra.

It can be shown that this process of extension of the algebra:

$$\{\partial\phi\} \to \left\{\partial\phi, e^{\pm\, i\sqrt{2N}\phi}\right\}$$

corresponds in CSW gauge theory to a change in the gauge group. Namely we can have an Abelian gauge field with action

$$S = \frac{ik}{8\pi}\int AdA$$

but it makes a big difference if the gauge group is $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z} = U(1)$.

If the gauge group is $\mathbb{R}$, the allowed gauge transformations are $A \to A - d\epsilon(x)$ where $\epsilon : Y \to \mathbb{R}$ is a well-defined function. In that case:

1.) We can scale $k$ out of the action

2.) The observables in the theory are the Wilson lines

$$e^{i\alpha \oint A}$$

Recall that the value of $\alpha$ defines a representation - this corresponds to a continuously infinite set of representations in CFT.

3.) No two Wilson lines are equivalent.

On the other hand, if the gauge group is $U(1)$ then around non contractible cycles $\epsilon$ is only well-defined modulo $2\pi$, and this leads to some consequences:

1.) The theory only makes sense for $k = 0 \ mod\ 4$

2.) The observables are

$$W_n(C) = e^{in \oint_C A} \quad n \epsilon Z$$

3.) Two Wilson lines can be equivalent

$$W_n(C) \cong W_{n+k/2}(C)$$

---

• Exercise 10.1 *Level k $U(1)$ Current Algebra.*

a.) Compute explicitly the expectation values of Wilson lines in $S^3$ for the abelian case:

$$\int DA e^{\frac{ik}{8\pi}\int AdA} \prod_i e^{in_i \oint_{C_i} A} = exp\left[\frac{2\pi i}{k}\sum_{i,j} n_i n_j \Phi_{ij}\right]$$

where $\Phi_{ij}$ is the linking number. $\Phi_{ii}$ is ambiguous-but may be regularized and defined up to an integer.

b.) Show that the cross terms are invariant under the change $n \to n + \frac{k}{2}$. Show that the invariance of the self-linking number requires $k = 0 mod\ 4$.

c.) Perform a (singular) gauge transformation $A \to A + d\phi$ where $\phi$ is an angular variable around some Wilson line. Show that this changes $W_n \to W_{n+\frac{1}{2}k}$. This illustrates how changing the gauge group from $\mathbb{R}$ to $U(1) = \mathbb{R}/\mathbb{Z}$ brings about an identification of Wilson lines.

d.) If $k = 4N$ we refer to the corresponding CFT as $U(1)_N$, "level $N$ $U(1)$ current algebra." Show that the conformal field theory is just the holomorphic part of the rational torus $R^2 = p/2q$ where $pq = N$.

e.) The Wilson line $W_{\frac{k}{2}}$ which is a non-trivial operator if the gauge group is $\mathbb{R}$ behaves like the identity operator when the gauge group is $U(1)$. The reason for this is the following. In the $U(1)$ theory one needs to sum over $U(1)$ bundles. The non-trivial bundles can be characterized by an insertion of an 'tHooft operator [66] in the functional integral of the $\mathbb{R}$ theory. Using part c of this exercise, show that the 'tHooft operator is equivalent to $W_{\frac{k}{2}}$. Since we have to sum over the insertions of such operators, the value of the functional integral is not modified if we add another one. Hence, this operator behaves like the identity operator. The two dimensional analog of this is the fact that the representation $\frac{k}{2}$ *extends* the $\mathbb{R}$ KM chiral algebra. This field becomes a descendent of the identity operator (under the larger chiral algebra) and its conformal blocks are the same as those of the identity.

f.) Show that the above considerations extend to any even integral lattice.

g.) Quantize the theory by canonical quantization on $T^2$ as in the previous section. Find the different states as the different representations of $U(1)_N$ and write their wave functions in terms of theta functions of higher level [67].

h.) Quantize the theory on a manifold with boundary. Find the extended chiral algebra by quantization on the disk (hint: because of the boundary conditions, there are non-trivial bundles corresponding to the insertion of $\frac{k}{2}$ in the $\mathbb{R}$ theory) and the different representations by quantization on a disk with a source.

i.) Show that the center of $\mathcal{A}(U(1)_N)$ is simply $\mathbb{Z}/2N\mathbb{Z}$. (Hint: We normally think of the gauge group of a $U(1)$ gauge theory, which is generated by

$$U(\epsilon) = e^{\frac{ik}{4\pi}\int \epsilon(x) F(x)}$$

for smooth functions $\epsilon$ as an abelian group. However we now allow functions like $\epsilon_P \sim \phi$

for $\phi$ an angular coordinate centered at any point $P$. Show that

$$U(\epsilon_P)U(\epsilon)U(\epsilon_P)^{-1} = e^{i\frac{k}{2}\epsilon(P)}U(\epsilon)$$

so that the group becomes nonabelian. Note that the elements of the center are in one-one correspondence with the representations of the rational torus chiral algebra.) Interpret the existence of this center from the two dimensional point of view (hint: the chiral algebra contains charged fields).

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• Exercise 10.2 $G = SO(3) = SU(2)/\mathbb{Z}_2$

a. Show that the only representations which survive have odd dimension. Show moreover that to avoid global anomalies, or to have the extending Wilson line be invisible we must have $k = 0 mod 4$.

b. Show that by the singular gauge transformation we can prove equivalence of the Wilson lines

$$W_j(C) \cong W_{k/2-j}(C)$$

c. Show that the Wilson line $W_{k/4}$ is in fact not the simplest operator in the theory, rather we have $W_{k/4} = \mathcal{O}^+ + \mathcal{O}^-$ where the operators $\mathcal{O}^\pm$ cannot be simply expressed in terms of Wilson lines.

d. Find an expression for $\mathcal{O}^\pm$ in terms of $SU(2)$ theory. (Hint: consider a three point vertex of Wilson lines with one in the representation $k/2$.)

---

Quite generally one can show that all known extended algebras are obtained from 3 dimensional CSW gauge theories by changing the gauge group by

$$G \to G/Z$$

where $Z$ is a subgroup of the center of $G$.

In going from $G$ to $\tilde{G} = G/Z$, three changes in the possible representations take place:

a. Selection rule: of the representations of $G$ current algebra only those which are invariant under $Z$ should be kept.

b. Identification: different irreducible representations of $G$ related by the s pectral fl ow operation are combined into one $\tilde{G}$ irreducible representation.

c. Fixed point: if the spectral flow has a fixed point, there are *different* $\tilde{G}$ representations which are the same as $G$ representations.

These three rules generalize the three parts of the previous exercise.

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• Exercise 10.3 *The three rules from canonical quantization.* Derive these three rules from canonical quantization on the torus. Hints:

1. Rule a follows from gauge transformations which wind around one cycle

2. Rule b from gauge transformations which wind around the other cycle.

3. Rule c is the most subtle. Twisted bundles on the torus are labeled by the subgroup $Z$ used to divide the universal cover to obtain $G$. These bundles may be defined by cutting out a disc and using the transition function $g(\phi) = e^{i\phi\theta}$ where $\phi$ is an angular coordinate and $\theta$ is a weight vector. The flat gauge fields which are sections of the associated $ad(G)$ bundle are characterized [68] by conjugacy classes of solutions of

$$ABA^{-1}B^{-1} = e^{2\pi i\theta} \tag{10.5}$$

where $A, B \in \tilde{G}$ describe holonomies of the flat gauge field.

As a simple example, consider first the nontrivially twisted $SO(3)$ bundle on $\Sigma_1$. Without loss of generality we may rotate $B$ into the maximal torus, taking $B = e^{2\pi i x T^3}$. Then $A$ must be of the form $wA_1$ where $w$ is in the Weyl group and $A_1$ is in the maximal torus. By conjugating with elements of the maximal torus we may set $A_1$ to one. Show that there is exactly one solution, $x = 1/4$ up to conjugacy. Thus the moduli space of twisted flat gauge fields consists of one point, and quantization gives one further state. Recall that in the conformal field theory there are two representations $\mathcal{H}^{\pm}_{k/4}$ of the $\mathcal{A}_k(SO(3))$ chiral algebra. Only one of these was accounted for from the quantization in the untwisted sector, the other comes from the twisted sector. Compare these two different irreducible representations with $\mathcal{O}^{\pm}$ in exercise 10.2. As an example, show that $\mathcal{A}(SO(3)_4) = \mathcal{A}(SU(3)_1)$ and recognize the two different representations of the $SO(3)$ theory as 3 and $\bar{3}$ of $SU(3)$.

These remarks generalize to arbitrary groups. Twisted bundles with transition function $g_\theta$ have one flat connection for the conjugacy class of each (discrete) solution $x, w$ of

$wxw^{-1} + \theta = x mod \Lambda_{rt}$ where $x$ is in the Cartan subalgebra and $w$ is in the Weyl group. Using the conjugacy freedom we can require that $x$ is in the positive Weyl chamber, show that this equation then becomes exactly the condition for a weight $x = \lambda/k$ to be fixed by the spectral flow $\mu_\theta$. Thus, the states arising from quantization on the discrete set of points in the moduli space of twisted flat bundles exactly correspond to the different irreducible representations $\mathcal{H}_i^\omega$ arising from the representations fixed by subgroups of the spectral flow.

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Using these considerations we can easily find new quantization conditions on $k$ in the non-simply connected case (generalizing the $k = 0 mod 4$ in the $U(1)$ theory). The conformal dimension of the extending representation must be an integer. From the three dimensional point of view, this condition is the statement that there is no dependence on the framing of the 'tHooft operator which is used to described the twisted bundles – no global anomalies. The conformal dimension of the representation $\lambda$ is $\Delta_\lambda = \frac{\lambda(\lambda+2\rho)}{2(k+h)}$. If the spectral flow is generated by the representation $\mu$, the extending representation is $k\mu$ and its dimension is $\Delta_{k\mu} = \frac{k\mu(k\mu+2\rho)}{2(k+h)}$. The condition on $k$ is that this number should be an integer. The same result has been obtained by other considerations in [69].

## 2. Coset models $G/H$

They may be obtained as follows: we take gauge fields

$$A^a,\ A^{\bar{a}} \epsilon Lie(G) \qquad \bar{a} \text{ denote directions in } Lie(G)/Lie(H)$$

$$B^a \epsilon Lie(H)$$

and action

$$S = k_1 CS(A) - k_2 CS(B)$$

We must be careful to take the gauge group $(G \times H)/Z$ where $Z$ is the common center of H embedded in G.

To see that this prescription is correct consider the quantization on the disk $D \times \mathbb{R}$, and let us reconsider the boundary conditions. Variation gives

$$\delta S = \frac{k_1}{4\pi} \int_{\partial D \times \mathbb{R}} Tr(\delta A A) - \frac{k_2}{4\pi} \int_{\partial D \times \mathbb{R}} Tr \delta B B + \text{ bulk terms}$$

One possibility is to choose $A_0 = B_0 = 0$ which leads to a $G \times H$ theory. However, when $H \subset G$ and $k_2 = \ell k_1$ ($\ell$ is the index of the embedding) we may choose instead the boundary conditions:

$$\begin{cases} A^a &= B^a \\ A_0^{\bar{a}} &= 0 \end{cases}$$

Performing the change of variables we had before we write (we have chosen $\ell = 1$ for simplicity)

$$A = -dUU^1$$

$$B = -dVV^{-1}$$

and get, as before

$$\int D\lambda DUDV \exp\left[ikS_{wzw}(U) - ikS_{wzw}(V)\right.$$
$$\left. + ik\int Tr\lambda(\partial_\varphi UU^{-1} - \partial_\varphi VV^{-1})\right]$$

where $\lambda$ is a Lagrange multiplier enforcing the boundary condition $A^a = B^a$.

Making the change of variables $U \to gV$, $-\partial_\phi VV^{-1} \to a_\phi$, and $\lambda \to a_t$ we get the path integral

$$\int dadge^{ikS(g,a_\phi,a_t)}$$

which is the gauged WZW model, which is well-known [70] to be the path integral representation of the coset models. Actually, it is quite easy to see why this must be so. The phase spaces are the coadjoint orbits of the pair of $\hat{G}$ and $\hat{H}$ representations $(\Lambda, \lambda)$:

$$(LG/T) \times (LH/T)^*$$

which, upon quantization give the space of states: $\mathcal{H}_\Lambda \otimes \mathcal{H}_\lambda^*$. Now we may impose the *first class* constraints: $\pi_H(\partial_\phi UU^{-1}) - \partial_\phi VV^{-1}$ ($\pi_H$ is a projection from $G$ to $H$) which is an $H$-current algebra with $k = 0$ to obtain the physical states:

$$(\mathcal{H}_\Lambda \otimes \mathcal{H}_\lambda^*)^{LH} \cong Hom_{LH}(\mathcal{H}_\lambda, \mathcal{H}_\Lambda) \cong \mathcal{H}_{\Lambda,\lambda}$$

where the final symbol is the space of states in the coset model, defined by the decomposition $\mathcal{H}_\Lambda = \oplus_\lambda \mathcal{H}_{\Lambda,\lambda} \otimes \mathcal{H}_\lambda$.

• Exercise 10.4 *Example of a Coset.*

a.) Show, using CFT, that the coset model $U(1)_N \times U(1)_M / U(1)_{N+M}$ for the case that $N, M$ have no common factors is equivalent to the rational torus $U(1)_L$ for $L = NM(N+M)$.

b.) Consider the expectation values of Wilson lines in $S^3$ for the action:

$$\frac{N}{2\pi}\int AdA + \frac{M}{2\pi}\int BdB - \frac{N+M}{2\pi}\int CdC$$

where $A, B, C$ are three abelian gauge fields. Show that the expectation value is consistent with the result of part (a).

c.) Show that the quantization of this theory on $T^2$ leads to the correct answer only if the gauge group is $\frac{U(1)\times U(1)\times U(1)}{Z_2}$. In implementing our prescription, we have to view the chiral algebra $U(1)_N$ as non-abelian. See above, exercise 10.1.g.

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• Exercise 10.5 *The $N = 0,1$ discrete series and the role of the center.* Study the coset $\frac{SU(2)_k \times SU(2)_l}{SU(2)_{k+l}}$. For $l = 1$ this is the Virasoro discrete series and for $l = 2$ the super discrete series. The 3d gauge group is $\frac{SU(2)\times SU(2)\times SU(2)}{Z_2}$. The representations are labeled by three spins $j_1, j_2, j_3$ corresponding to the three $SU(2)$. Use rule a above to show that $j_1 + j_2 + j_3$ must be an integer. Use rule b above to show that the representation $(j_1, j_2, j_3)$ is identified with the representation $(\frac{k}{2} - j_1, \frac{l}{2} - j_2, \frac{k+l}{2} - j_3)$. Use rule c to show that if both $k$ and $l$ are even, there are two different representations labeled by $(\frac{k}{2}, \frac{l}{2}, \frac{k+l}{2})$. Rule c applies in the superdiscrete series ($l = 2$) when $k$ is even. What is the difference between the two representations in this case?

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• Exercise 10.6 $c = 7/10$ This conformal field theory can be represented by a coset $\frac{SU(3)_2}{U(2)_2}$. Notice that in the coset we use $U(2)$ rather than $SU(2) \times U(1)$. Why? What are the irreducible representations of $U(2)_2$? What is the three dimensional gauge group? (Don't forget the common center.) What are the irreducible representations of the coset?

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• Exercise 10.7 *Witten's Triple Cosets.* In [63] Witten proposed a generalization of the coset construction. Recall that in the coset construction the fields in the chiral algebra $\mathcal{A}(G/H)$ are all the fields in $\mathcal{A}(G)$ which commute with the fields in $\mathcal{A}(H)$. In particular, $\mathcal{A}(G/H)$ is a subalgebra of $\mathcal{A}(G)$. Thus, if we have a triple of inclusions $K \subset H \subset G$ then we may consider the fields in $\mathcal{A}(G)$ which commute with the fields in $\mathcal{A}(H/K)$. Witten defines this subalgebra of $\mathcal{A}(G)$ to be the triple coset algebra $\mathcal{A}(G/H/K)$. In this exercise we show that the construction of these algebras do not involve any new constructions other than those described above.

a.) As a warmup consider the explicit triple $\hat{SU}(N-1)_1 \subset \hat{SU}(N)_1 \subset \hat{SU}(N+1)_1$. Using the Frenkel-Kac construction of level one current algebra in terms of free scalar fields show that the triple coset is just $\hat{SU}(N-1) \times U(1)_{N(N+1)/2}$.

b.) More generally, show that $\mathcal{A}(G/H/K)$ always contains the subalgebra $\mathcal{A}(G/H) \times \mathcal{A}(K)$. Moreover these have the same central charge and are unitary theories. Thus $\mathcal{A}(G/H/K)$ may be expected to be at most an extended algebra of $\mathcal{A}(G/H) \times \mathcal{A}(K)$. Show that this is indeed the case by decomposing characters:

$$\begin{aligned}\chi_0^G &= \sum_\lambda \chi_{0,\lambda}^{G/H} \chi_\lambda^H \\ &= \sum_{\lambda,\rho} \chi_{\lambda,\rho}^{H/K} \chi_\rho^K \chi_{0,\lambda}^{G/H} \\ &= \sum_{[\lambda,\rho]} \chi_{\lambda,\rho}^{H/K} \sum_{\mu \in C(\mathcal{A}(H)) \cap C(\mathcal{A}(K))} \chi_{\mu(\rho)}^K \chi_{0,\mu(\lambda)}^{G/H}\end{aligned}$$

where $[\lambda, \rho]$ denotes equivalent pairs in the coset module. Thus, in particular, the character of the chiral algebra is just

$$\sum_{\mu \in C(\mathcal{A}(H)) \cap C(\mathcal{A}(K))} \chi_{\mu(0)}^K \chi_{0,\mu(0)}^{G/H}$$

which is a finite extension of $\mathcal{A}(G/H) \times \mathcal{A}(K)$.

c.) Show that this theory may be obtained from 2+1 dimensions using the (schematic) action $CS(G) - CS(H) + CS(K)$ with gauge group $(G \times H \times K)/Z$ and $Z$ is generated by $(\theta, \theta, 1)$ for $\theta \in C(G) \cap C(H)$ and by $(1, \theta, \theta)$ for $\theta \in C(H) \cap C(K)$.

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The MTC of rational orbifolds is fairly complicated in general. In the special case of a rational orbifold obtained from a theory with a trivial MTC, the rational orbifold MTC has a rather beautiful description given in [11][71]. If the finite group $G$ is the orbifold group, the index set $I$ consists of pairs $(\bar{g}, \alpha)$ where $\bar{g}$ is a conjugacy class in $G$ and $\alpha$ is an irreducible representation of the centralizer subgroup of the conjugacy class. The basic data of the MTC can be described in terms of group cohomology [4]. In particular, the fusion rules are elegantly described as a multiplication law in the equivariant $K$-theory of $G$. Fortunately, one can demonstrate by rather general arguments that the holomorphic half of any rational orbifold model can be obtained from a 3D CSW gauge theory based on gauge groups which are not connected [5].

Let $G$ be a connected group with a discrete automorphism group $P$. Then one can construct the semi-direct product group $P \ltimes G$. Quantizing the system on the disk and repeating the steps above, we find that the effective action is the WZW action for a field $U$ on the boundary which takes values in $G$. The phase space is $LG/G$ and leads to $\mathcal{A}(G)$, but because of $P$ gauge invariance, the Hilbert space has to be truncated to the $P$ invariant states (the states are in representations of $P$ because $P$ is an automorphism of $G$). This can be seen by considering the CSGT on $D \times S^1$. The functional integral in this case leads to the trace over the Hilbert space (since the Hamiltonian of the 3D theory vanishes, this trace is infinite). In the functional integral we need to sum over $P$ bundles. This sum projects out the states which are not $P$ invariant. Therefore, $\mathcal{A}(P \ltimes G) = \mathcal{A}(G)/P$. This is the chiral algebra of the orbifold constructed as $G/P$. By quantizing the system on other two surfaces with boundaries we obtain the other representations of the orbifold model.

Orbifolds and cosets are very similar in both two and three dimensions. In 3D we reduced the chiral algebra of the $G$ theory by enlarging the gauge group. In 2D both theories are obtained by considering a $G$ theory and gauging either a continuous subgroup,

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[4] This is also true of theories with "abelian fusion rules" as explained in appendix $E$ of [15].

[5] Initial work with E. Witten first suggested that $O(2)$ would reproduce the rational orbifold. This work motivated the general construction for orbifolds.

$H/Z$ (to obtain $G/H$) or a discrete automorphism group, $P$ (to obtain $G/P$). Finally note that the gauge group $(G \times H)/Z$ of the coset CSGT can also be written as $(H/Z) \ltimes G$ which is the same as the prescription for orbifolds. In the classical limit of these theories the integral weight fields have a closed ope. Therefore, there should be a one to one correspondence between these representations of the chiral algebra and representation spaces of some group. This group is the gauge group of the 3D theory.

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• Exercise 10.8 *The rational orbifold from* $O(2)$. Check that the $O(2)$ CSW gauge theory on $T^2$ leads to the correct number of representations. First use conformal field theory to find that for the rational orbifold of level $N$ there are $N + 7$ representations. When quantizing on $T^2$ the Hilbert space has several sectors. Show that from topologically trivial bundles (those which can be considered to be $SO(2)$ bundles) there are $N+1$ states. Find six twisted $O(2)$ bundles leading to six more states. Hence, the total number of states is $N + 7$.

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• Exercise 10.9 *A more complicated orbifold.* Study the orbifold $SU(2)_k/\mathbf{Z}_2\times\mathbf{Z}_2$, where we take the quotient by 180° degree rotations around orthogonal axes. Unlike the previous exercise here two interesting subtleties arise. First, some of the twisted components of the Hilbert space have more than one state. Second, some of the twisted components in fact contribute no quantum states for some $k$'s, because of a global anomaly in the appropriate sector. Show that the number of quantum states is $(11k+32)/2$ if $k$ is even and $(11k+11)/2$ if $k$ is odd. Derive the same result by the two dimensional considerations of [11].

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The lesson that we learn from this is that all known RCFT's are equivalent to some CSW gauge theory for some compact gauge group. An arbitrary compact group may be disconnected (the quotient of G by its connected component being some finite group) and in turn the connected component may have a finite-sheeted cover consisting of a product of tori and simply connected simple factors. From the previous constructions we see that this full level of generality is needed to order the zoo of known rational conformal field theories.

When working with arbitrary compact groups a further subtlety arises which is analyzed in detail in [69][6]. In order to write the Chern-Simons action in the form

$$S = \frac{k}{4\pi} \int_Y Tr(AdA + \frac{2}{3}A^3)$$

one needs a trivial $G$-bundle over the three-manifold $Y$. By definition, the path integral for theories with $G$ not connected and simply- connected include nontrivial $G$-bundles and one must find another definition of the action. This problem was solved in [69]. The upshot is that the appropriate data needed to specify the action is an element of the cohomology group $\lambda \in H^4(BG; \mathbb{Z})$. For a connected, simply-connected, simple group, $H^4(BG; \mathbb{Z}) = \mathbb{Z}$ and $\lambda$ is simply the integer, usually called $k$, multiplying the Chern-Simons term. For arbitrary connected compact groups the data is equivalent to a nondegenerate symmetric invariant bilinear form on the Lie algebra, needed to define the notion of a trace. In the disconnected case there can be torsion and one must express the data as an element of $H^4(BG; \mathbb{Z})$.

In conclusion, the MTC's of all known RCFT's are organized by simply specifying the pair $(G, \lambda)$ where $G$ is a compact gauge group and $\lambda$ is a cohomology class in $H^4(BG; \mathbb{Z})$.

[6] We thank Dan Freed for very useful discussions on these matters.

## 11. Conclusions and Conjectures

In these lectures we tried to formulate RCFT in an axiomatic way. We were led to define certain axioms which have - rather remarkably - an analog in the TK approach to group theory. Even more remarkably, it is known in the group theory case that a single additional axiom: $F_i^{-1} \in \mathbb{Z}_+$ defines the representation theory of an algebraic group. (To obtain a compact group one has to say a bit more.) Moreover this crucial integrality condition has an analog in RCFT. Thus we conjectured that by adding some axioms to the polynomial equations on $F, B, S$ we will define RCFT purely axiomatically. Making further progress from this point on is difficult: We know that reconstruction will be subtle because there exist nontrivial chiral algebras with one representation and no holonomy (e.g. those obtained from even self-dual lattices of dimension 0 mod 24). This raises a serious question as to how good the notion of a modular functor or a modular tensor category is at identifying a RCFT. Based on the absence of counterexamples we may hope that the only ambiguity comes from tensor products with $c = 24$ purely holomorphic CFT's.

Another difficulty is that it is not exactly obvious what we should say about $F_i^{-1}$. There should be some physical reason based solely on the defining axioms of conformal field theory for why these numbers should take on special values but no one has succeeded in elucidating such a reason [7]. Moreover, it is not obvious that there are not additional axioms with no group theoretic analog (just as there are additional polynomial equations with no group theoretic analog). Nevertheless it ought to be clear from our discussion that RCFT defines some mathematical structure generalizing group theory. Of course, reconstruction is much easier if you know what it is you are trying to reconstruct!

We saw in sections nine and ten that three-dimensional CSW gauge theories can be used to define the MTC of all known RCFT's by taking an appropriate compact gauge group (perhaps neither connected nor simply connected) and action (defined by an appropriate symmetric invariant nondegenerate bilinear form, or, more precisely, by an appropriate class in $H^4(BG; \mathbb{Z})$). Taking account of the general structure of compact groups we

[7] It has been pointed out by many authors that $F_i^{-1}$ is an index for inclusions of finite von Neumann algebras. This is clearly the most fruitful interpretation from which to embark on an investigation of the analog of Deligne's condition.

saw that the full generality is needed to describe CFT's and that the extension from the case of simply-connected simple groups is not entirely trivial. Based on these observations one naturally guesses that the object we are trying to reconstruct is none other than compact CSW theory, and therefore that all RCFT's are equivalent to some compact CSW theory.

The equivalence between compact CSW theories and RCFT's is not one to one. First, there are CSW theories which do not correspond to any RCFT. For instance, if we repeat the $G/H$ construction with $H$ which is not a subgroup of $G$ or with the two coupling constants, $k_G$ for $G$ and $k_H$ for $H$ which are not equal (or not proportional according to the index of the embedding) then the resulting theory does not describe the MTC of the holomorphic half of a RCFT. Also, different CSW theories might lead to the same RCFT. For instance, it is known that the same RCFT can sometimes be described as a coset in two different ways. Other identifications arise for low levels, for example $SU(2)_1$ is the same as $U(1)_1$. According to the philosophy of this paper, these isomorphisms should be viewed as the CFT version of the isomorphisms in the Cartan classification of Lie algebras for algebras of small rank, e.g. $su(2) \cong so(3)$, $su(4) \cong so(6)$, etc.

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• Exercise 11.1 *A Sampling of Isomorphisms.* In the literature on CFT there are often several different realizations of the same theory. Identify the following isomorphisms:

a.) $SU(N)_1 \cong \mathbb{R}^{N-1}/\Lambda_{rt}$ where $\Lambda_{rt}$ is the root lattice.

b.) $U(1)_2 \cong O(2)_1$.

c.) $\frac{SU(2)_N \times U(1)_{-N}}{\mathbb{Z}_2} \cong \frac{SU(N)_1 \times SU(N)_1 \times SU(N)_{-2}}{\mathbb{Z}_N}$

d.) $SO(3)_4 \cong SU(3)_1$

e.) $(SU(3)/\mathbb{Z}_3)_3 \cong SO(8)_1$

f.) $\frac{SU(2)_1 \times SU(2)_1 \times SU(2)_{-2}}{\mathbb{Z}_2} \cong (E_8)_1 \times (E_8)_1 \times (E_8)_{-2}$

g.) $\frac{SU(2)_3 \times SU(2)_1 \times SU(2)_{-4}}{\mathbb{Z}_2} \cong \frac{SU(3)_1 \times SU(3)_1 \times SU(3)_{-2}}{\mathbb{Z}_3}$

There is, at present, no general point of view on how to classify these isomorphisms.

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The relation between the three dimensional and the two dimensional theories arises in two related ways. The Hilbert space of the theory on a manifold without a boundary is the space of conformal blocks. In this case one can study the dimensionality of the vector space and the action of the duality matrices. A more detailed connection between the theories arises upon quantization on a manifold with a boundary. Then all the states in the chiral algebra and in all its representations can be realized.

As we have mentioned, it is sometimes the case that two different theories have the same duality matrices. However, the structure of the representations is different. For instance, if we tensor a theory based on a $c = 24$ self dual lattice with any theory $C$ the duality matrices are those of the theory $C$. The only difference is in the structure of the chiral algebra and its representations.

Correspondingly we may formulate a weak and a strong version of the conjecture alluded to throughout these lectures. The weak version states that the duality properties are reproduced by some CSW theory with compact group. More formally, we may state

**Conjecture 1**: The modular functor of any unitary RCFT is equivalent to the modular functor of some CSW theory defined by the pair $(G, \lambda)$ with $G$ a compact group and $\lambda \in H^4(BG; \mathbf{Z})$.

Let us make some remarks about this conjecture. First, as discussed at the end of section ten, if $G$ is connected then $\lambda$ may be thought of as the data needed to specify the normalizations of the traces in the Chern-Simons action. Alternatively, from the quantum group point of view, $\lambda$ specifies the appropriate roots of unity required for various quantum deformations of relevant simple groups. Second, we expect that the gauge group must be compact for a simple reason. In the WKB approximation one obtains one quantum state for each unit of volume of phase space. The moduli spaces of noncompact groups are noncompact and hence quantization will lead to an infinite number of quantum states, that is, an infinite number of conformal blocks, so the corresponding two-dimensional theory cannot be rational. Recent work of H. Verlinde [72] suggests that this reasoning might be too naive at strong coupling, and that noncompact phase spaces might actually lead to finite dimensional spaces of states. Nevertheless, rational conformal field theories which do seem to be related to noncompact groups also have a description in terms of compact groups. Third, we limit our considerations to unitary theories because CSW theories, which are simply quantum mechanical systems with a finite number of degrees of freedom, are automatically unitary. Every known example of a unitary RCFT fits in with conjecture 1. The situation for nonunitary RCFT's is much less well understood, although there is some preliminary evidence that the correct organizing principle may be found in the theory of compact supergroups [73].

We have taken pains to state conjecture 1 precisely because it is the conjecture we understand best and in which we have the most confidence. Further conjectures in this section will be stated somewhat more loosely. We hope we have convinced the reader that there are substantial reasons for believing conjecture 1 is correct. As we have discussed, one might imagine a proof to proceed along lines very similar to the theorems of Deligne and Doplicher-Roberts. On the other hand, it would be fascinating if there were examples of "sporadic" modular tensor categories arising from conformal field theories. In the introduction we pointed out that an alternative statement of the conjecture says that all RCFT's have already been found. It was probably first stated by Emil Martinec [7] that the nontrivial RCFT's are essentially exhausted by the coset construction, and this was repeated in [9]. It has been reiterated many times in private by Bazhanov, Fröhlich, Gawedzki, Goddard, Reshetikhin, and perhaps others.

Conjecture 1 is a weak conjecture in the sense that it's truth would only classify modular functors of RCFT's. One may hope that a stronger version of the conjecture is true, namely

**Conjecture 2**: The chiral algebra of any unitary RCFT is the physical Hilbert space for canonical quantization of some CSW theory for an appropriate choice of compact gauge group, symmetric bilinear invariant nondegenerate form, and boundary conditions.

Obviously there is no counterexample to this conjecture, but there do exist some examples of chiral algebras which remain to be interpreted along the lines sketched above. Most notably, the chiral algebra of the Monster module remains uninterpreted [8].

[8] We would like to thank W. Nahm for pointing this out to us.

• Exercise 11.2 *Open Problem.* Obtain the chiral algebra of all known $c = 24$ theories with trivial monodromy from quantization of some CSW theory on $D \times \mathbb{R}$.

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• Exercise 11.3 *Dual of a RCFT.* Consider a RCFT with $F, S, \Omega, \Delta, c$. Show that since $F, S, \Omega$ satisfy the polynomial equations so do $F' = F^{-1}, S' = S^{-1}, \Omega' = \Omega^{-1}$. The conformal dimensions of these two solutions are related by $\Delta' = -\Delta mod 1$ and $c' = -c mod 8$. Sometimes there exists a RCFT with $F', S', \Omega'$ (remember, a solution of the polynomial equations does not guarantee that there exists a RCFT with these duality matrices). We define this theory as the dual of the original one.

a.) Show that a theory with one primary field is self dual.

b.) Show that the coset of a self dual theory by the chiral algebra $\mathcal{A}$ is a RCFT which is dual to the RCFT based on $\mathcal{A}$.

c.) Construct a self dual theory by appropriately coupling a theory and its dual.

d.) Use the self dual theory based on $E(8)_1 \times E(8)_1$ and part (b) of this exercise to show that the Ising model is dual to $E(8)_2$. A more sophisticated example of this phenomenon was studied in [74] where it was shown that a certain exceptional modular invariant of $F(4)$ KM is dual to $SU(3)_2$. Using part c a new self dual $c = 24$ theory can be constructed.

e.) Show that the duality matrices of the dual theory can be obtained from three dimensions by reversing the sign of the action – reversing the orientation. Since exercise 10.2 is still an open problem, it is not clear if all the states in the chiral algebra and in its representations for every theory (in particular for the $F(4)$ theory of [74]) can be obtained from three dimensions.

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Another conjecture, related to those above was posed by E. Witten [75]

**Conjecture 3**: All three dimensional topological field theories are CSW theories for some appropriate (super)-group.

As we have seen, any modular functor defines a three dimensional topological field theory so that the truth of conjecture 3 may be expected to imply that of conjecture 1, assuming there is no surprising need to resort to noncompact groups or supergroups.

Finally we should note that there has recently been much progress in abelianizing WZW theories [76] [77] [78] [79] [80] [81] [82] [83] and there has been related progress on abelianizing certain coset models. From this work one is naturally lead to wonder if Kadanoff's old idea that all CFT's are related to the gaussian model might in some sense be correct. More precisely, taking into account some of the recent bosonization results, reference [80] states

**Conjecture 4**: The chiral algebras and representations occuring in RCFT may always be expressed as cohomology spaces for sequences of Fock modules, and all CVO's of RCFT's may be expressed through free field constructions.

The ultimate reduction of RCFT to free field theory would not be in contradiction with the group-theoretic interpretation. Indeed, it is well-known that one can construct representations of groups with harmonic oscillators.

We hope that the truth or falsehood of these conjectures will be established in the near future. Looking beyond the subject of RCFT there are several horizons emerging involving various generalizations, extensions, and applications of the concepts we have used above, but which we have not even mentioned. It is not our intention to discuss these future directions here, should they bear fruit there will be no lack of opportunity for future discussion.

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