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A method for constructing quaternary Hermitian self-dual codes and an application to quantum codes

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Abstract

We introduce quaternary modified four μ-circulant codes as a modification of four circulant codes. We give basic properties of quaternary modified four μ-circulant Hermitian self-dual codes. We also construct quaternary modified four μ-circulant Hermitian self-dual codes having large minimum weights. Two quaternary Hermitian self-dual [56, 28, 16] codes are constructed for the first time. These codes improve the previously known lower bound on the largest minimum weight among all quaternary (linear) [56, 28] codes. In addition, these codes imply the existence of a quantum [[56, 0, 16]] code.

Keywords Self-dual code · Quaternary code · Hermitian self-dual code

Mathematics Subject Classification 94B05

1 Introduction

Self-dual codes are one of the most interesting classes of (linear) codes. This interest is justified by many combinatorial objects and algebraic objects related to self-dual codes (see e.g., [6], [20] and [26]).

Let $\mathbb{F}_{q^2}$ denote the finite field of order q^2, where q is a prime or a prime power. A code C over $\mathbb{F}_{q^2}$ of length n is said to be Hermitian self-dual if $C = C^{\perp_H}$, where the Hermitian dual code $C^{\perp_H}$ of C is defined as $C^{\perp_H} = \{x \in \mathbb{F}_{q^2}^n \mid \langle x, y\rangle_H = 0 \text{ for all } y \in C\}$ under the Hermitian inner product $\langle x, y\rangle_H$. By the Gleason–Pierce theorem, there are nontrivial divisible Hermitian self-dual codes over $\mathbb{F}_{q^2}$ for $q = 2$ only. This is one of the reasons why much work has been done concerning Hermitian self-dual codes over $\mathbb{F}_4$ (see e.g., [1, 4, 5, 9–11, 16–19, 21–25] and [27]). In this paper, we study Hermitian self-dual codes over $\mathbb{F}_4$.

It is a fundamental and challenging problem in self-dual codes to classify self-dual codes and determine the largest minimum weight among all self-dual codes for a fixed length. A

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code over $\mathbb{F}_4$ is called quaternary. All quaternary Hermitian self-dual codes were classified in [5], [16], [17] and [25] for lengths $n \le 20$. Also, the largest minimum weight $d(n)$ among all Hermitian self-dual codes is determined for lengths $n \le 30$ (see [9, Table 5] for the current information on $d(n)$).

For small fields $\mathbb{F}$, many four circulant (negacirculant) self-dual codes over $\mathbb{F}$ having large minimum weights are known (see e.g., [7], [13], [14], [15] and the references given therein). In this paper, by modifying four circulant self-dual codes, we give a method for constructing quaternary Hermitian self-dual codes based on μ-circulant matrices, which are called modified four μ-circulant codes. Some basic properties of modified four μ-circulant quaternary Hermitian self-dual codes are given. We also give numerical results of quaternary modified four μ-circulant Hermitian self-dual codes together with an application to quantum codes.

This paper is organized as follows. In Sect. 2, we give some definitions, notations and basic results used in this paper. In Sect. 3, we define quaternary modified four μ-circulant codes as a certain modification of four circulant codes. We also give basic properties of quaternary modified four μ-circulant Hermitian self-dual codes. In particular, we give a condition for quaternary modified four μ-circulant codes to be Hermitian self-dual. In addition, we observe equivalences of quaternary modified four μ-circulant Hermitian self-dual codes. In Sect. 4, we present numerical results of quaternary modified four μ-circulant Hermitian self-dual codes. By computer search based on basic properties presented in Sect. 3, we give a classification of quaternary modified four μ-circulant Hermitian self-dual codes having the currently known largest minimum weights for lengths 24, 28, 32 and 36 (Proposition 7). For larger lengths, we also construct quaternary modified four μ-circulant Hermitian self-dual codes having large minimum weights. We emphasize that quaternary Hermitian self-dual [56, 28, 16] codes are constructed for the first time (Proposition 10). These codes $C_{56,1}$ and $C_{56,\omega}$ improve the previously known lower bounds on the largest minimum weight among all quaternary (linear) [56, 28] codes (Corollary 11). In Sect. 5, we give an application of $C_{56,1}$ and $C_{56,\omega}$ to quantum codes. More precisely, $C_{56,1}$ and $C_{56,\omega}$ imply the existence of a quantum [[56, 0, 16]] code.

2 Preliminaries

In this section, we give some definitions, notations and basic results used in this paper.

2.1 Quaternary codes

We denote the finite field of order 4 by $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$, where $\overline{\omega} = \omega^2 = \omega + 1$. A *quaternary* linear $[n, k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_4^n$. All codes in this paper are quaternary and linear unless otherwise noted, so we omit linear and we often omit quaternary. The parameter n is called the *length* of C. A *generator matrix* of C is a $k \times n$ matrix such that the rows of the matrix generate C. The *weight* $\mathrm{wt}(x)$ of a vector $x \in \mathbb{F}_4^n$ is the number of non-zero components of x. The *weight enumerator* of C is given by $\sum_{c \in C} y^{\mathrm{wt}(c)}$. A vector of C is called a *codeword* of C. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C. A *quaternary* $[n, k, d]$ *code* is a quaternary $[n, k]$ code with minimum weight d.

2.2 Quaternary Hermitian self-dual codes

The *Hermitian dual* code $C^{\perp_H}$ of a quaternary code C of length n is defined as

$$C^{\perp_H} = \{x \in \mathbb{F}_4^n \mid \langle x, y\rangle_H = 0 \text{ for all } y \in C\},$$

under the following Hermitian inner product

$$\langle x, y\rangle_H = \sum_{i=1}^{n} x_i y_i^2$$

for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{F}_4^n$. A quaternary code C is said to be *Hermitian self-dual* if $C = C^{\perp_H}$. All codewords of a quaternary Hermitian self-dual code have even weights [25, Theorem 1].

All matrices in this paper are matrices over $\mathbb{F}_4$, so we write simply matrices. Throughout this paper, let I_n denote the identity matrix of order n, and let A^T denote the transpose of a matrix A. Moreover, let $\overline{A}$ denote the matrix (a_{ij}^2) for a matrix $A = (a_{ij})$. The following lemma is a criterion for Hermitian self-duality.

Lemma 1 [25, Theorems 1 and 4] *Let C be a quaternary $[2n, n]$ code with generator matrix $\begin{pmatrix} I_n & M \end{pmatrix}$. If $M\overline{M}^T = I_n$, then C is Hermitian self-dual.*

It was shown in [25] that the minimum weight d of a quaternary Hermitian self-dual code of length n is bounded by:

$$d \le 2\left\lfloor \frac{n}{6} \right\rfloor + 2. \tag{1}$$

A quaternary Hermitian self-dual code of length n and minimum weight $2\lfloor n/6\rfloor + 2$ is called *extremal*.

Two quaternary Hermitian self-dual codes C and C' are *equivalent* if there is a monomial matrix P over $\mathbb{F}_4$ with $C' = C \cdot P$, where $C \cdot P = \{xP \mid x \in C\}$ (see [25]). Throughout this paper, two equivalent quaternary Hermitian self-dual codes C and C' are denoted by $C \cong C'$. All quaternary Hermitian self-dual codes were classified in [5], [16], [17] and [25] for lengths up to 20. All extremal quaternary Hermitian self-dual codes of length 22 were also classified in [17].

3 Definition and basic properties of modified four μ-circulant codes

In this section, we define quaternary modified four μ-circulant codes and we give their basic properties.

An $n \times n$ matrix of the following form

$$\begin{pmatrix}
r_0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\
\mu r_{n-1} & r_0 & r_1 & \cdots & r_{n-3} & r_{n-2} \\
\mu r_{n-2} & \mu r_{n-1} & r_0 & \cdots & r_{n-4} & r_{n-3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\mu r_1 & \mu r_2 & \mu r_3 & \cdots & \mu r_{n-1} & r_0
\end{pmatrix}$$

is called μ-*circulant*, where $\mu \in \{1, \omega, \overline{\omega}\}$. In particular, if $\mu = 1$, then this is well-known as a *circulant* matrix. It is trivial that a μ-circulant matrix with first row $(r_0, r_1, \dots, r_{n-1})$ is written as $\sum_{i=0}^{n-1} r_i E_n(\mu)^i$, where

$$E_n(\mu) = \begin{pmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \mu & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma 2 *Suppose that* $\mu \in \{1, \omega, \overline{\omega}\}$.

(i) *If* A *and* B *are* $n \times n$ μ*-circulant matrices, then* $AB = BA$.
(ii) *If* A *is an* $n \times n$ μ*-circulant matrix with first row* $(r_0, r_1, \dots, r_{n-1})$*, then* $\overline{A}^T$ *is a* μ*-circulant matrix with first row* $(r_0^2, (\mu r_{n-1})^2, \dots, (\mu r_1)^2)$.

Proof The assertion (i) follows from the fact that a μ-circulant matrix with first row $(r_0, r_1, \dots, r_{n-1})$ is written as $\sum_{i=0}^{n-1} r_i E_n(\mu)^i$. The assertion (ii) follows from the fact that $\overline{A}^T$ is written as $r_0^2 I_n + \sum_{i=1}^{n-1} (\mu r_{n-i})^2 E_n(\mu)^i$. □

By modifying four circulant self-dual codes (see e.g., [15] for the definition), we introduce the following method for constructing quaternary Hermitian self-dual codes. Suppose that $\mu \in \{1, \omega, \overline{\omega}\}$. Let A and B be $n \times n$ μ-circulant matrices. We say that a quaternary $[4n, 2n]$ code with generator matrix of the following form

$$\begin{pmatrix} I_{2n} & \begin{matrix} A & B \\ \overline{B}^T & \overline{A}^T \end{matrix} \end{pmatrix} \tag{2}$$

is *modified four* μ*-circulant*. A modified four 1-circulant code is also called *modified four circulant*. We denote the code with generator matrix (2) by $C_\mu(A, B)$.

Remark 3 As a different modification of four circulant codes, codes with generator matrices of the following form

$$\begin{pmatrix} I_{2n} & \begin{matrix} A^T C J & \overline{B} \\ B^T C J & \overline{A} \end{matrix} \end{pmatrix}$$

are given in [27], where A, B and C are circulant matrices and J is the exchange matrix.

Now we give some basic properties of modified four μ-circulant Hermitian self-dual codes. Although the following lemmas are somewhat trivial, we give proofs for the sake of completeness.

Lemma 4 *Suppose that* $\mu \in \{1, \omega, \overline{\omega}\}$*. A quaternary modified four* μ*-circulant code* $C_\mu(A, B)$ *is Hermitian self-dual if* $A\overline{A}^T + B\overline{B}^T = I_n$.

Proof By Lemma 2 (i), $AB + BA = O_n$, where O_n denotes the $n \times n$ zero matrix. By Lemma 2 (ii), $\overline{A}^T$ and $\overline{B}^T$ are μ-circulant. Again by Lemma 2 (i), $\overline{A}^T\overline{B}^T = \overline{B}^T\overline{A}^T$ and $A\overline{A}^T = \overline{A}^T A$. Thus, we have

$$\overline{A}^T\overline{B}^T + \overline{B}^T\overline{A}^T = O_n \text{ and } \overline{A}^T A + \overline{B}^T B = I_n.$$

Let $M(A, B)$ denote the $2n \times 2n$ matrix $\begin{pmatrix} A & B \\ \overline{B}^T & \overline{A}^T \end{pmatrix}$. Then we have

$$M(A, B)\overline{M(A, B)}^T = \begin{pmatrix} A\overline{A}^T + B\overline{B}^T & AB + BA \\ \overline{A}^T\overline{B}^T + \overline{B}^T\overline{A}^T & \overline{A}^T A + \overline{B}^T B \end{pmatrix} = I_{2n}.$$

The result follows from Lemma 1. □

Lemma 5 *Suppose that $C_\mu(A, B)$ is a quaternary modified four μ-circulant Hermitian self-dual code, where $\mu \in \{1, \omega, \overline{\omega}\}$. Then the following statements hold.*

(i) $C_\mu(A, B) \cong C_\mu(\omega A, \omega B) \cong C_\mu(\overline{\omega} A, \overline{\omega} B)$.
(ii) $C_\mu(A, B) \cong C_\mu(B, A)$.
(iii) $C_\mu(A, B) \cong C_\mu(\overline{A}^T, \overline{B}^T)$.
(iv) $C_\mu(A, B) \cong C_\mu(A, \overline{B}^T)$.

Proof The assertions (i), (ii) and (iii) are trivial. The Hermitian dual code $C_\mu(A, B)^{\perp_H}$ of $C_\mu(A, B)$ has the following generator matrix

$$\begin{pmatrix} \overline{A}^T & B & \\ \overline{B}^T & A & I_{2n} \end{pmatrix}.$$

Since $C_\mu(A, B) = C_\mu(A, B)^{\perp_H}$, the above matrix is also a generator matrix of $C_\mu(A, B)$. It follows from (iii) that $C_\mu(A, B) \cong C_\mu(A, \overline{B}^T)$. □

Lemma 6 *Let C be a quaternary modified four μ-circulant Hermitian self-dual code, where $\mu \in \{1, \omega, \overline{\omega}\}$. Then there is a quaternary modified four μ-circulant Hermitian self-dual code $C_\mu(A, B)$ such that $C \cong C_\mu(A, B)$ and the first nonzero coordinate of the first row of A is* 1.

Proof Suppose that $C = C_\mu(A', B')$ and the first nonzero coordinate of the first row of A' is ω (resp. $\overline{\omega}$). Then $C_\mu(\overline{\omega} A', \overline{\omega} B')$ (resp. $C_\mu(\omega A', \omega B')$) is a modified four μ-circulant code such that nonzero coordinate of the first row of $\overline{\omega} A'$ (resp. $\omega A'$) is 1. By Lemma 5 (i), we have that $C \cong C_\mu(\overline{\omega} A', \overline{\omega} B')$ (resp. $C \cong C_\mu(\omega A', \omega B')$). The result follows. □

The above lemma substantially reduces the number of codes which need be checked when a classification of modified four μ-circulant Hermitian self-dual codes is completed and the largest minimum weight among all modified four μ-circulant Hermitian self-dual codes is determined in the next section.

4 Numerical results of modified four μ-circulant Hermitian self-dual codes

In this section, we present numerical results of quaternary modified four μ-circulant Hermitian self-dual codes. We emphasize that Hermitian self-dual [56, 28, 16] codes are constructed. These codes are the first examples of not only Hermitian self-dual [56, 28, 16] codes but also (linear) [56, 28, 16] codes. All computer calculations in this section were done using programs in MAGMA [2] unless otherwise specified.

Table 1 Values $d^K(n)$

n	24	28, 32	36, 40, 44	48, 52, 56	60, 64	68, 72, 76	80
$d^K(n)$	8	10	12	14	16	18	20

4.1 Classification of modified four μ-circulant Hermitian self-dual codes

As described in Sect. 2, all quaternary Hermitian self-dual codes of lengths up to 20 were classified in [5], [16], [17] and [25]. From now on, we consider Hermitian self-dual codes for only lengths $n \geq 24$.

Let $d(n)$ denote the largest minimum weight among all Hermitian self-dual codes of length n. Let $d^K(n)$ denote the largest minimum weight among previously known Hermitian self-dual codes of length n. For $n \in \{24, 28, \dots, 80\}$, the values $d^K(n)$ are listed in Table 1, noting that $d(24) = 8$ and $d(28) = 10$ (see [9, Table 5]).

Here we give a classification of modified four μ-circulant Hermitian self-dual codes having minimum weight $d^K(n)$ for length $n \in \{24, 28, 32, 36\}$. We describe how to complete our classification briefly. Our exhaustive computer search based on Lemmas 4 and 6 found all distinct generator matrices (2) of modified four μ-circulant Hermitian self-dual $[n, n/2, d^K(n)]$ codes $C_\mu(A, B)$, which must be checked further for equivalences. To test equivalence of two modified four μ-circulant Hermitian self-dual $[n, n/2, d^K(n)]$ codes, we used Magma function `IsIsomorphic`. Moreover, in the process of finding these codes, we verified that there is no modified four μ-circulant Hermitian self-dual code of length n and minimum weight $d > d^K(n)$ for lengths $n = 32$ and 36. Then we have the following proposition.

Proposition 7 (i) *Up to equivalence, there are* 7 *quaternary modified four circulant Hermitian self-dual* $[24, 12, 8]$ *codes. Up to equivalence, there are* 9 *quaternary modified four* μ*-circulant Hermitian self-dual* $[24, 12, 8]$ *codes for* $\mu \in \{\omega, \overline{\omega}\}$.

(ii) *Up to equivalence, there are* 3 *quaternary modified four* μ*-circulant extremal Hermitian self-dual* $[28, 14, 10]$ *codes for* $\mu \in \{1, \omega, \overline{\omega}\}$.

(iii) *Up to equivalence, there are* 59 *quaternary modified four* μ*-circulant Hermitian self-dual* $[32, 16, 10]$ *codes for* $\mu \in \{1, \omega, \overline{\omega}\}$*. If* $d \geq 12$*, then there is no quaternary modified four* μ*-circulant Hermitian self-dual* $[32, 16, d]$ *code for* $\mu \in \{1, \omega, \overline{\omega}\}$.

(iv) *Up to equivalence, there is a unique quaternary modified four* μ*-circulant Hermitian self-dual* $[36, 18, 12]$ *code for* $\mu \in \{1, \omega, \overline{\omega}\}$*. If* $d \geq 14$*, then there is no quaternary modified four* μ*-circulant Hermitian self-dual* $[36, 18, d]$ *code for* $\mu \in \{1, \omega, \overline{\omega}\}$.

For $n \in \{24, 28, 32, 36\}$, by $C_{n,\mu,i}$ $(i \in \{1, 2, \dots, N_\mu(n)\})$, we denote the modified four μ-circulant Hermitian self-dual $[n, n/2, d^K(n)]$ codes described in the above proposition, where

$$N_1(24) = 7,\ N_\mu(24) = 9\ (\mu \in \{\omega, \overline{\omega}\}),\ N_\mu(28) = 3\ (\mu \in \{1, \omega, \overline{\omega}\}),$$
$$N_\mu(32) = 59\ (\mu \in \{1, \omega, \overline{\omega}\}),\quad N_\mu(36) = 1\ (\mu \in \{1, \omega, \overline{\omega}\}).$$

For these codes $C_{n,\mu,i} = C_\mu(A, B)$ $(\mu = 1, \omega, \overline{\omega})$, the first rows r_A (resp. r_B) of A (resp. B) are listed in Tables 2, 3, 4, 9, 10 and 11.

Table 2 Modified four μ-circulant Hermitian self-dual [24, 12, 8] codes

Code	r_A	r_B	A_8
$C_{24,1,1}$	$(1, 0, 1, 0, 1, 1)$	$(\omega, \overline{\omega}, \omega, 1, 0, 1)$	513
$C_{24,1,2}$	$(1, \omega, 1, 1, \omega, 1)$	$(\omega, 1, \omega, 1, 0, 1)$	594
$C_{24,1,3}$	$(1, \overline{\omega}, \overline{\omega}, 0, \omega, 1)$	$(\overline{\omega}, 0, 0, \overline{\omega}, 0, 0)$	594
$C_{24,1,4}$	$(0, 1, 1, 0, \omega, \omega)$	$(\omega, \overline{\omega}, \omega, 1, 0, 1)$	837
$C_{24,1,5}$	$(0, 1, 1, 0, \omega, \omega)$	$(1, 1, 1, 0, 0, 0)$	837
$C_{24,1,6}$	$(1, 1, 1, 0, \omega, \overline{\omega})$	$(1, \omega, 1, 1, 0, 0)$	837
$C_{24,1,7}$	$(1, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1, \overline{\omega})$	$(\overline{\omega}, \overline{\omega}, \overline{\omega}, 0, 0, 0)$	837
$C_{24,\omega,1}$	$(1, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, \omega)$	$(\overline{\omega}, 0, 0, \overline{\omega}, 1, 0)$	513
$C_{24,\omega,2}$	$(1, 1, 1, \omega, \overline{\omega}, 0)$	$(\overline{\omega}, 0, 1, \overline{\omega}, \omega, 0)$	513
$C_{24,\omega,3}$	$(1, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, 0)$	$(\omega, 0, \overline{\omega}, \overline{\omega}, 1, 0)$	513
$C_{24,\omega,4}$	$(1, \overline{\omega}, \overline{\omega}, 1, \omega, 0)$	$(0, 0, 0, \overline{\omega}, 1, 0)$	513
$C_{24,\omega,5}$	$(1, 1, 1, \omega, 0, \overline{\omega})$	$(1, \omega, \overline{\omega}, 1, \omega, \omega)$	513
$C_{24,\omega,6}$	$(1, \overline{\omega}, \overline{\omega}, 1, 1, \overline{\omega})$	$(\omega, \omega, \overline{\omega}, \omega, 1, 0)$	513
$C_{24,\omega,7}$	$(1, \overline{\omega}, \overline{\omega}, 1, \omega, 1)$	$(0, 0, \omega, \omega, 1, 0)$	513
$C_{24,\omega,8}$	$(0, 1, 1, \omega, \overline{\omega}, \overline{\omega})$	$(\omega, 0, 1, \overline{\omega}, \omega, 0)$	513
$C_{24,\omega,9}$	$(0, 1, \omega, \overline{\omega}, 0, 0)$	$(1, \overline{\omega}, 0, 1, \overline{\omega}, 0)$	513
$C_{24,\overline{\omega},1}$	$(0, 0, 1, 1, 0, \omega)$	$(\overline{\omega}, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},2}$	$(1, 0, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1)$	$(0, \omega, 0, \overline{\omega}, \omega, 1)$	513
$C_{24,\overline{\omega},3}$	$(1, 0, \overline{\omega}, \overline{\omega}, 1, 0)$	$(0, \omega, 1, \omega, \omega, 1)$	513
$C_{24,\overline{\omega},4}$	$(0, 0, 1, 1, 0, 0)$	$(\omega, \omega, \omega, 0, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},5}$	$(0, 0, 1, 1, 0, 0)$	$(\omega, 0, 1, \omega, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},6}$	$(0, 0, 1, 1, 1, 1)$	$(\overline{\omega}, 0, \overline{\omega}, 1, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},7}$	$(0, 0, 1, 1, \overline{\omega}, 0)$	$(\omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},8}$	$(0, 0, 1, 1, 1, 0)$	$(\overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, \omega)$	513
$C_{24,\overline{\omega},9}$	$(0, 0, 1, 1, 1, 0)$	$(0, \omega, 1, 0, \overline{\omega}, \omega)$	513

Remark 8 By MAGMA function `IsIsomorphic`, we have the following

$$\begin{array}{ll} C_{24,\omega,i} \cong C_{24,\overline{\omega},i} & (i \in \{1, 2, \dots, 9\}), \\ C_{28,1,i} \cong C_{28,\omega,i} \cong C_{28,\overline{\omega},i} & (i \in \{1, 2, 3\}), \\ C_{32,1,i} \cong C_{32,\omega,i} \cong C_{32,\overline{\omega},i} & (i \in \{1, 2, \dots, 59\}), \\ C_{36,\omega,1} \cong C_{36,\overline{\omega},1}, & \end{array}$$

and there is no other pair of equivalent codes among the codes described in Proposition 7.

For $n = 24, 32$ and 36, the possible weight enumerators of quaternary Hermitian self-dual $[n, n/2, d^K(n)]$ codes can be written using $A_{d^K(n)}$ (see [1] and [22, Sect. III]). Note that the possible weight enumerator of an extremal Hermitian self-dual code of a fixed length is uniquely determined. For the above codes $C_{n,\mu,i}$ ($n = 24, 32$ and 36), the numbers $A_{d^K(n)}$ of codewords of minimum weight $d^K(n)$ are also listed in Tables 2, 4, 9, 10 and 11. This was calculated by the MAGMA function `NumberOfWords`.

Table 3 Modified four μ-circulant Hermitian self-dual [28, 14, 10] codes

Code	r_A	r_B
$C_{28,1,1}$	$(0, 1, \overline{\omega}, 0, 1, 0, \overline{\omega})$	$(\overline{\omega}, \omega, \overline{\omega}, \omega, 0, 0, \overline{\omega})$
$C_{28,1,2}$	$(1, \omega, 1, 0, \omega, 0, 0)$	$(\overline{\omega}, \overline{\omega}, 1, \omega, \omega, 1, 1)$
$C_{28,1,3}$	$(0, 1, \omega, 0, \overline{\omega}, 0, 0)$	$(\omega, 1, \omega, 1, 1, 0, \omega)$
$C_{28,\omega,1}$	$(1, \omega, \omega, 0, \omega, 0, 1)$	$(\omega, 0, 0, \overline{\omega}, \omega, 0, \omega)$
$C_{28,\omega,2}$	$(1, \omega, \omega, 0, \omega, 0, 0)$	$(1, \overline{\omega}, \overline{\omega}, \omega, \omega, 1, \omega)$
$C_{28,\omega,3}$	$(0, 1, \omega, 0, \omega, 0, 0)$	$(\overline{\omega}, \omega, 0, \omega, 1, 1, \omega)$
$C_{28,\overline{\omega},1}$	$(1, 0, \overline{\omega}, \omega, \overline{\omega}, 0, 1)$	$(\omega, 0, 0, 0, 1, 1, \overline{\omega})$
$C_{28,\overline{\omega},2}$	$(1, \omega, 1, \overline{\omega}, 1, 1, \omega)$	$(\overline{\omega}, 0, \overline{\omega}, 0, 0, \omega, 1)$
$C_{28,\overline{\omega},3}$	$(1, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 0, 1)$	$(0, \overline{\omega}, \overline{\omega}, 0, 0, 0, \overline{\omega})$

Table 4 Modified four μ-circulant Hermitian self-dual [36, 18, 12] codes

Code	r_A	r_B	A_{12}
$C_{36,1,1}$	$(1, 1, 1, 1, 1, 1, \omega, \overline{\omega}, \omega)$	$(1, 1, 0, 0, 1, 0, 1, 0, 0)$	20844
$C_{36,\omega,1}$	$(1, \omega, 1, 1, 1, 1, \overline{\omega}, 1, 1)$	$(1, \omega, \overline{\omega}, \overline{\omega}, 0, \overline{\omega}, 1, 0, 0)$	19548
$C_{36,\overline{\omega},1}$	$(1, \overline{\omega}, 1, 1, 1, 1, \omega, 1, 1)$	$(1, \overline{\omega}, \omega, \omega, 0, \omega, 1, 0, 0)$	19548

4.2 Largest minimum weights of modified four μ-circulant Hermitian self-dual codes

We give some observations on the largest minimum weight $d(n)$ among all Hermitian self-dual codes of length n and the largest minimum weight $d_\mu(n)$ ($\mu = 1, \omega, \overline{\omega}$) among all modified four μ-circulant Hermitian self-dual codes of length n. For lengths $n = 40$ and 44, by a method similar to the above, our exhaustive computer search based on Lemmas 4 and 6 verified that there is no modified four μ-circulant Hermitian self-dual $[n, n/2, d]$ code with $d > d^K(n)$ for $\mu \in \{1, \omega, \overline{\omega}\}$ (see Table 1 for the minimum weights $d^K(n)$). In addition, we found a modified four μ-circulant Hermitian self-dual $[n, n/2, d^K(n)]$ code $C_{n,\mu}$ for $\mu \in \{1, \omega, \overline{\omega}\}$. This implies the following proposition.

Proposition 9 *For* $\mu \in \{1, \omega, \overline{\omega}\}$, $d_\mu(40) = 10$ *and* $d_\mu(44) = 12$.

For the above codes $C_{40,\mu} = C_\mu(A, B)$ and $C_{44,\mu} = C_\mu(A, B)$, the first rows r_A (resp. r_B) of A (resp. B) are listed in Table 5. The numbers $A_{d^K(n)}$ of codewords of minimum weight $d^K(n)$ are also listed in the table. This was calculated by the Magma function `NumberOfWords`. The numbers show that these codes are inequivalent.

For lengths 48, 52, ..., 76 and 80, by a non-exhaustive search based on Lemmas 4 and 6, we continued finding modified four μ-circulant Hermitian self-dual codes having large minimum weights. Then we found a modified four μ-circulant Hermitian self-dual code $C_{n,\mu}$ of length n and minimum weight d for

$$
\begin{aligned}
(n, \mu, d) =&(52, 1, 14), (52, \omega, 14), (52, \overline{\omega}, 14), (56, 1, 16), (56, \omega, 16), (56, \overline{\omega}, 14),\\
&(60, 1, 16), (60, \omega, 16), (60, \overline{\omega}, 16), (64, 1, 16), (64, \omega, 16), (64, \overline{\omega}, 16),\\
&(68, 1, 18), (68, \omega, 18), (68, \overline{\omega}, 18), (72, 1, 18), (72, \omega, 18), (72, \overline{\omega}, 18),\\
&(76, 1, 18), (76, \omega, 18), (76, \overline{\omega}, 18), (80, 1, 20), (80, \omega, 20), (80, \overline{\omega}, 20).
\end{aligned}
$$

Table 5 Modified four μ-circulant Hermitian self-dual codes $C_{40,\mu}$ and $C_{44,\mu}$

Code	r_A	r_B	$A_{d^K(n)}$
$C_{40,1}$	$(1, 0, 0, 1, \overline{\omega}, 1, 0, 0, 1, 0)$	$(\omega, \omega, 1, 1, \omega, \omega, 0, \omega, \omega, 0)$	5220
$C_{40,\omega}$	$(1, \overline{\omega}, \overline{\omega}, 1, 1, 1, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega})$	$(\omega, 1, 0, \omega, \overline{\omega}, \omega, \omega, \overline{\omega}, 0, 0)$	5130
$C_{40,\overline{\omega}}$	$(1, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 0, \omega)$	$(1, \omega, 0, 1, \overline{\omega}, 0, \overline{\omega}, 0, \omega, 0)$	5040
$C_{44,1}$	$(1, \omega, \overline{\omega}, 0, 0, \omega, 0, \overline{\omega}, \omega, 1, \omega)$	$(\overline{\omega}, \omega, \overline{\omega}, \omega, \omega, \overline{\omega}, 0, \omega, 0, 0, 0)$	1188
$C_{44,\omega}$	$(1, 0, 0, \overline{\omega}, \omega, 1, \overline{\omega}, 0, \omega, 1, 0)$	$(\overline{\omega}, 0, \overline{\omega}, 0, \omega, 0, 0, \overline{\omega}, 1, \omega, 0)$	1551
$C_{44,\overline{\omega}}$	$(1, 0, \overline{\omega}, 0, 1, 1, 1, \omega, 0, \overline{\omega}, \omega)$	$(\omega, 1, \overline{\omega}, \overline{\omega}, 0, \overline{\omega}, \overline{\omega}, \omega, \omega, \overline{\omega}, 0)$	1749

Table 6 Largest minimum weights $d_1(n)$, $d_\omega(n)$ and $d_{\overline{\omega}}(n)$

n	$d_1(n)$	$d_\omega(n)$	$d_{\overline{\omega}}(n)$	n	$d_1(n)$	$d_\omega(n)$	$d_{\overline{\omega}}(n)$
24	8	8	8	56	16–20	16–20	14–20
28	10	10	10	60	16–22	16–22	16–22
32	10	10	10	64	16–22	16–22	16–22
36	12	12	12	68	18–24	18–24	18–24
40	12	12	12	72	18–26	18–26	18–26
44	12	12	12	76	18–26	18–26	18–26
48	14–18	14–18	14–18	80	20–28	20–28	20–28
52	14–18	14–18	14–18				

For the above codes $C_{n,\mu} = C_\mu(A, B)$, the first rows r_A (resp. r_B) of A (resp. B) are listed in Table 12. We have the following proposition.

Proposition 10 *There are quaternary Hermitian self-dual* [56, 28, 16] *codes.*

We emphasize that $C_{56,1}$ and $C_{56,\omega}$ are the first examples of not only Hermitian self-dual [56, 28, 16] codes but also (linear) [56, 28, 16] codes [8]. We give the weight enumerators of these codes in the next subsection.

In Table 6, we summarize the current information on $d_1(n)$, $d_\omega(n)$ and $d_{\overline{\omega}}(n)$. The upper bounds on $d_1(n)$, $d_\omega(n)$ and $d_{\overline{\omega}}(n)$ follow from (1). The lower bounds on $d_1(n)$, $d_\omega(n)$ and $d_{\overline{\omega}}(n)$ follow from Table 12.

4.3 $C_{56,1}$ and $C_{56,\omega}$

It is a main problem in coding theory to determine the largest minimum weight $d_q(n, k)$ among all $[n, k]$ codes over a finite field of order q for a given (q, n, k). The current information on $d_4(n, k)$ can be found in [8]. For example, it was previously known that $15 \le d_4(56, 28) \le 21$. As a consequence of Proposition 10, we have the following corollary.

Corollary 11 $16 \le d_4(56, 28) \le 21$.

Now we determine the weight enumerators of $C_{56,1}$ and $C_{56,\omega}$. It is well known that the possible weight enumerators of quaternary Hermitian self-dual codes can be determined by the Gleason type theorem [24, p. 804] (see also [25, Theorem 13]). The weight enumerator

Table 7 Possible weight enumerator $W_{56,16}$

i	A_i
0	1
16	α
18	β
20	$113963850 - 78\alpha - 15\beta$
22	$1616214600 + 520\alpha + 99\beta$
24	$35022262275 - 1495\alpha - 357\beta$
26	$467452738368 + 1344\alpha + 612\beta$
28	$4854958425240 + 5560\alpha + 612\beta$
30	$37999586848608 - 28576\alpha - 7140\beta$
32	$223928221341825 + 79170\alpha + 23868\beta$
34	$991894905892800 - 170560\alpha - 51714\beta$
36	$3272633909885340 + 309452\alpha + 82654\beta$
38	$7961209635178800 - 471120\alpha - 102102\beta$
40	$14053893738878070 + 586586\alpha + 99450\beta$
42	$17629097730552000 - 584000\alpha - 76908\beta$
44	$15262097167863000 + 457080\alpha + 47124\beta$
46	$8759255147042400 - 276640\alpha - 22644\beta$
48	$3144896807802750 + 126685\alpha + 8364\beta$
50	$646962821144640 - 42432\alpha - 2295\beta$
52	$65864956983210 + 9810\alpha + 441\beta$
54	$2485731965640 - 1400\alpha - 53\beta$
56	$14512944519 + 93\alpha + 3\beta$

W of a quaternary Hermitian self-dual code of length n is written as:

$$W = \sum_{j=0}^{\lfloor \frac{n}{6} \rfloor} a_j (1+3y^2)^{\frac{n}{2}-3j} (y^2(1-y^2)^2)^j, \tag{3}$$

using some integers a_j. The possible weight enumerator $W_{56,16} = \sum_{i=0}^{56} A_i y^i$ of a quaternary Hermitian self-dual [56, 28, 16] code is determined by (3), where A_i are listed in Table 7 together with $\alpha = A_{16}$ and $\beta = A_{18}$. Only this calculation was done by MATHEMATICA [28]. By the MAGMA function `NumberOfWords`, we calculated that

$$(A_{16}, A_{18}) = (48825, 2275560) \text{ and } (47544, 2282700),$$

for $C_{56,1}$ and $C_{56,\omega}$, respectively. This determines the weight enumerators of $C_{56,1}$ and $C_{56,\omega}$.

4.4 Largest minimum weights *d*(*n*)

In Table 8, we summarize the current information on the largest minimum weights $d(n)$ for $n \in \{24, 28, \ldots, 80\}$. The upper bounds on $d(n)$ follow from (1). The references about the lower bounds on $d(n)$ are also listed in the table.

In [9, Table 5], the largest minimum weights $d(n)$ were considered for $n \le 80$. Here we investigate the largest minimum weights $d(n)$ for $n \in \{84, 88, 92, 96, 100\}$. A Hermitian

Table 8 Largest minimum weights $d(n)$

n	$d(n)$	References	n	$d(n)$	References
24	8	(see [4, p. 140])	64	16–22	[9, Table 5]
28	10	(see [18, Theorem 9])	68	18–24	[9, Table 5]
32	10–12	[10, Table I]	72	18–26	[9, Table 5]
36	12–14	[10, Table I]	76	18–26	[9, Table 5]
40	12–14	[10, Table I]	80	20–28	[9, Table 5]
44	12–16	[9, Table 5]	84	**20**–30	$C_{84,\omega}$ in Table 12
48	14–18	[9, Table 5]	88	**20**–30	$C_{88,\omega}$ in Table 12
52	14–18	[9, Table 5]	92	22–32	G_{92} (see [8])
56	**16**–20	$C_{56,1}, C_{56,\omega}$ in Table 12	96	**22**–34	$C_{96,\omega}$ in Table 12
60	16–22	[9, Table 5]	100	22–34	G_{100} (see [8])

self-dual code of length n and minimum weight 22 is given in [8] for $n = 92$ and 100. We denote the two codes by G_{92} and G_{100}, respectively. As information, we briefly give the construction of G_{92} and G_{100}. Let $G_{91,1}$ and $G_{91,2}$ denote the cyclic codes of length 91 with generator polynomials g_1 and g_2, respectively, where

$$\begin{aligned} g_1 =& x^{46} + \omega x^{44} + x^{43} + \overline{\omega}x^{42} + \overline{\omega}x^{41} + \omega x^{40} + \overline{\omega}x^{39} + \omega x^{38} + x^{37} + x^{35} \\ &+ x^{34} + \omega x^{33} + x^{31} + \omega x^{30} + \overline{\omega}x^{28} + x^{27} + x^{26} + \omega x^{25} + \omega x^{24} + x^{21} \\ &+ x^{20} + x^{19} + x^{18} + \overline{\omega}x^{16} + \omega x^{15} + \overline{\omega}x^{14} + \overline{\omega}x^{13} + \overline{\omega}x^{12} + \overline{\omega}x^{10} \\ &+ \overline{\omega}x^{8} + \overline{\omega}x^{7} + x^{6} + \omega x^{5} + x^{4} + \omega x^{3} + \overline{\omega}x^{2} + x + 1, \\ g_2 =& x^{45} + x^{44} + \overline{\omega}x^{43} + \omega x^{42} + x^{41} + \omega x^{40} + \overline{\omega}x^{38} + x^{37} + x^{34} + \omega x^{32} \\ &+ \omega x^{31} + \overline{\omega}x^{30} + x^{29} + x^{28} + \omega x^{27} + \overline{\omega}x^{26} + \omega x^{25} + \omega x^{23} + \omega x^{22} \\ &+ \omega x^{21} + \overline{\omega}x^{20} + \omega x^{19} + \overline{\omega}x^{18} + \omega x^{17} + \omega x^{16} + x^{15} + \overline{\omega}x^{14} \\ &+ \overline{\omega}x^{12} + \overline{\omega}x^{9} + \overline{\omega}x^{8} + \overline{\omega}x^{6} + \omega x^{5} + x^{3} + \overline{\omega}x^{2} + 1. \end{aligned}$$

The code G_{92} is constructed from $G_{91,1}$, $G_{91,2}$ and the [1, 1] code by Construction X. The code G_{100} is equivalent to the double circulant code with generator matrix $\begin{pmatrix} I_{50} & R \end{pmatrix}$, where R is the circulant matrix with the first row

$$(\omega, \omega, 1, \overline{\omega}, \omega, 1, 1, \omega, \overline{\omega}, 0, 1, 0, \overline{\omega}, 1, \omega, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, 0, 1, \overline{\omega}, \overline{\omega}, 0, \\ \omega, 0, 0, 1, 1, \omega, \omega, \omega, 1, \omega, \overline{\omega}, 1, 1, 1, 1, \omega, 1, \overline{\omega}, 1, \omega, \omega, \omega, 0, \overline{\omega}, 1).$$

For $(n, d) = (84, 20), (88, 20)$ and $(96, 22)$, by a non-exhaustive search based on Lemmas 4 and 6, we found a modified four ω-circulant Hermitian self-dual code $C_{n,\omega} = C_\omega(A, B)$ of length n and minimum weight d. For the above codes, the first rows r_A (resp. r_B) of A (resp. B) are listed in Table 12. In Table 6, we give lower and upper bounds on the largest minimum weights $d(n)$ for $n \in \{84, 88, 92, 96, 100\}$. The upper bounds on $d(n)$ follow from (1). The references about the lower bounds on $d(n)$ are also listed in the table. For

$$(n, d) = (56, 16), (84, 20), (88, 20) \text{ and } (92, 22),$$

a Hermitian self-dual code of length n and minimum weight d is constructed for the first time. In Table 8, the minimum weights of these codes are given in bold.

Table 9 Modified four circulant Hermitian self-dual [32, 16, 10] codes $C_{32,1,i}$

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
1	$(0,1,0,0,0,\overline{\omega},\overline{\omega},\omega)$	$(\omega,\overline{\omega},1,\omega,1,0,1,\omega)$	1200	31	$(0,1,1,0,0,1,\overline{\omega},\omega)$	$(\omega,\omega,1,0,\overline{\omega},0,1,\omega)$	1776
2	$(0,1,1,0,0,\overline{\omega},\omega,\omega)$	$(1,\overline{\omega},\omega,\overline{\omega},0,0,1,\omega)$	1200	32	$(0,1,0,0,0,\overline{\omega},\omega,0)$	$(\overline{\omega},\omega,\omega,\omega,0,0,1,\omega)$	1776
3	$(0,1,1,0,0,\overline{\omega},\omega,\omega)$	$(\omega,\overline{\omega},0,\overline{\omega},1,0,1,\omega)$	1200	33	$(0,1,0,0,0,1,\overline{\omega},\overline{\omega})$	$(\overline{\omega},1,1,\overline{\omega},1,0,1,\omega)$	1776
4	$(1,\omega,1,0,0,1,\omega,1)$	$(0,\omega,1,0,1,0,1,\omega)$	1344	34	$(0,1,0,0,0,\overline{\omega},\overline{\omega},\omega)$	$(\omega,\omega,\overline{\omega},\omega,\omega,0,1,\omega)$	1776
5	$(1,\omega,1,0,0,\overline{\omega},\omega,\overline{\omega})$	$(\overline{\omega},\overline{\omega},\omega,\omega,0,1,1,\omega)$	1344	35	$(1,1,0,0,0,\overline{\omega},\omega,1)$	$(\overline{\omega},1,\overline{\omega},\omega,1,\omega,1,\omega)$	1776
6	$(0,1,0,1,0,\overline{\omega},\omega,1)$	$(\omega,\overline{\omega},\omega,\omega,1,1,1,\omega)$	1392	36	$(1,1,1,0,0,\omega,\omega,1)$	$(\overline{\omega},1,\omega,\overline{\omega},\overline{\omega},0,1,\omega)$	1776
7	$(1,\overline{\omega},0,0,0,1,\omega,1)$	$(1,\omega,0,\overline{\omega},1,0,\overline{\omega},1)$	1392	37	$(1,\overline{\omega},\overline{\omega},0,0,1,\omega,0)$	$(1,\omega,\omega,1,1,\overline{\omega},\overline{\omega},1)$	1776
8	$(0,1,0,1,0,\omega,\omega,\omega)$	$(\overline{\omega},\overline{\omega},\omega,1,1,1,1,\omega)$	1392	38	$(1,\overline{\omega},\overline{\omega},0,0,1,\omega,0)$	$(\omega,\overline{\omega},0,1,\omega,0,\overline{\omega},1)$	1776
9	$(0,1,1,0,0,\omega,\omega,0)$	$(\omega,1,\omega,\omega,1,0,1,\omega)$	1392	39	$(1,1,1,0,0,1,1,1)$	$(\omega,0,\overline{\omega},1,0,0,1,\omega)$	1776
10	$(1,\overline{\omega},\omega,0,0,1,\omega,\omega)$	$(1,0,\omega,0,\omega,0,\omega,\overline{\omega})$	1488	40	$(1,0,0,0,0,\overline{\omega},\overline{\omega},1)$	$(\overline{\omega},0,\omega,\overline{\omega},0,0,1,\omega)$	1776
11	$(1,\overline{\omega},\overline{\omega},0,0,\overline{\omega},\omega,\omega)$	$(\omega,\omega,1,\omega,1,0,\overline{\omega},1)$	1488	41	$(1,0,0,0,0,\omega,\omega,1)$	$(\overline{\omega},1,\omega,\overline{\omega},\overline{\omega},0,1,\omega)$	1776
12	$(1,0,1,0,0,1,\overline{\omega},\omega)$	$(\overline{\omega},0,\overline{\omega},1,1,0,1,\omega)$	1488	42	$(1,1,0,0,0,1,1,\omega)$	$(\omega,1,\omega,1,\omega,1,1,\omega)$	1776
13	$(0,1,0,0,0,\overline{\omega},\overline{\omega},\omega)$	$(\omega,\overline{\omega},\omega,\omega,\omega,0,1,\omega)$	1488	43	$(1,\overline{\omega},0,0,0,\overline{\omega},1,\overline{\omega})$	$(\omega,\omega,1,\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},1)$	1776
14	$(0,1,0,0,0,\overline{\omega},\omega,0)$	$(\overline{\omega},\omega,\overline{\omega},1,\overline{\omega},1,1,\omega)$	1536	44	$(1,1,1,0,0,1,1,1)$	$(\overline{\omega},1,\overline{\omega},\omega,1,0,1,\omega)$	1776
15	$(0,1,0,0,0,\omega,1,\overline{\omega})$	$(\overline{\omega},\omega,\omega,1,1,0,1,\omega)$	1632	45	$(0,1,0,\overline{\omega},0,\omega,\overline{\omega},\overline{\omega})$	$(\overline{\omega},\omega,1,1,1,1,\overline{\omega},1)$	1824
16	$(1,\overline{\omega},\overline{\omega},0,0,1,\overline{\omega},\omega)$	$(\overline{\omega},0,\overline{\omega},0,\omega,0,\overline{\omega},1)$	1632	46	$(1,1,1,0,0,1,1,1)$	$(\omega,1,\overline{\omega},\overline{\omega},\omega,0,1,\omega)$	1824

Table 9 continued

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
17	$(1, \overline{\omega}, 0, \overline{\omega}, 0, \overline{\omega}, 1, 1)$	$(\omega, \overline{\omega}, 1, \overline{\omega}, 1, 0, \overline{\omega}, 1)$	1632	47	$(1, 0, \overline{\omega}, 0, 0, \omega, \overline{\omega}, \omega)$	$(\overline{\omega}, \omega, 1, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, 1)$	1824
18	$(0, 1, 0, \overline{\omega}, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\omega, 1, \omega, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, 1)$	1632	48	$(1, \overline{\omega}, 0, 0, 0, 1, \omega, 1)$	$(\omega, 0, \omega, \overline{\omega}, \overline{\omega}, 0, \overline{\omega}, 1)$	1824
19	$(1, 0, \overline{\omega}, 0, 0, \overline{\omega}, 0, 1)$	$(\overline{\omega}, 0, 1, 0, 1, 0, \overline{\omega}, 1)$	1632	49	$(1, 1, 0, 0, 0, 1, 1, \omega)$	$(\omega, 0, 1, \overline{\omega}, \overline{\omega}, 0, 1, \omega)$	1824
20	$(1, 1, 0, 0, 0, 1, 1, \omega)$	$(\overline{\omega}, 1, 1, \overline{\omega}, \omega, 1, 1, \omega)$	1632	50	$(0, 0, 0, 0, 0, 1, 1, 0)$	$(\omega, 1, \overline{\omega}, \overline{\omega}, \omega, 0, 1, \omega)$	1824
21	$(0, 1, 0, 0, 0, 1, \omega, \omega)$	$(\omega, 1, \omega, \overline{\omega}, \omega, 0, 1, \omega)$	1680	51	$(1, 0, \overline{\omega}, 0, 0, \overline{\omega}, \overline{\omega}, \overline{\omega})$	$(\omega, \omega, 1, 0, \overline{\omega}, 0, \overline{\omega}, 1)$	1920
22	$(0, 1, 0, 0, 0, \overline{\omega}, 1, \omega)$	$(\omega, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0, 1, \omega)$	1680	52	$(1, 1, 0, 0, 0, \omega, 1, 0)$	$(\overline{\omega}, 1, 0, 1, 0, 0, 1, \omega)$	1920
23	$(1, 1, 0, 0, 0, \overline{\omega}, 1, 0)$	$(\overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \omega, 0, 1, \omega)$	1680	53	$(1, 0, 0, 0, 0, \omega, 1, 0)$	$(\omega, 0, 1, 1, 1, 0, 1, \omega)$	1920
24	$(0, 0, 1, 0, 0, \omega, \omega, \omega)$	$(\overline{\omega}, 1, 1, \omega, \omega, 0, 1, \omega)$	1680	54	$(1, 0, 0, 0, 0, \overline{\omega}, 1, 0)$	$(\overline{\omega}, \overline{\omega}, 1, \omega, 1, 1, 1, \omega)$	1920
25	$(1, 0, 1, 0, 0, \overline{\omega}, \omega, 1)$	$(1, \omega, 1, 1, \overline{\omega}, 1, 1, \omega)$	1680	55	$(1, \overline{\omega}, \overline{\omega}, 0, 0, \omega, 1, \omega)$	$(1, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 0, \overline{\omega}, 1)$	1920
26	$(1, 1, 1, 0, 0, \overline{\omega}, 1, \omega)$	$(\omega, 0, \overline{\omega}, 0, \omega, 0, 1, \omega)$	1680	56	$(1, \overline{\omega}, \overline{\omega}, 0, 0, 1, \omega, 0)$	$(1, 1, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1)$	1920
27	$(1, \overline{\omega}, \omega, 1, 0, 1, \omega, \overline{\omega})$	$(\omega, 1, 1, 1, 1, 1, 1, \omega)$	1680	57	$(1, \overline{\omega}, \omega, 0, 0, \overline{\omega}, \omega, 1)$	$(\overline{\omega}, 1, \omega, \omega, 1, 0, \omega, \overline{\omega})$	1920
28	$(0, 1, 1, 1, 0, \omega, \overline{\omega}, \omega)$	$(\omega, 1, 1, \omega, \overline{\omega}, 0, 1, \omega)$	1776	58	$(0, 1, 1, 0, 0, \omega, \omega, 0)$	$(\overline{\omega}, \overline{\omega}, 1, 1, \omega, 0, 1, \omega)$	1920
29	$(1, \overline{\omega}, 1, 0, 0, \overline{\omega}, 1, 1)$	$(\omega, 0, \omega, 0, 1, 0, 1, \omega)$	1776	59	$(1, 0, \overline{\omega}, 0, 0, \overline{\omega}, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, \omega, 1, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, 1)$	1968
30	$(1, \overline{\omega}, 0, \overline{\omega}, 0, \overline{\omega}, 1, 1)$	$(1, \omega, 1, \omega, \omega, 0, \overline{\omega}, 1)$	1776				

Table 10 Modified four ω-circulant Hermitian self-dual [32, 16, 10] codes $C_{32,\omega,i}$

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
1	$(0, 1, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(\overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, 0, 1, 1, \omega)$	1200	31	$(1, 1, 1, 0, 0, 1, \omega, 0)$	$(1, \overline{\omega}, 0, \overline{\omega}, 0, 1, 1, \omega)$	1776
2	$(0, 1, 1, 0, 0, \overline{\omega}, \overline{\omega}, \overline{\omega})$	$(0, 1, 1, \overline{\omega}, \overline{\omega}, 0, 1, \omega)$	1200	32	$(0, 1, 0, 0, 0, 1, 1, 0)$	$(1, 1, \overline{\omega}, \omega, 0, 0, 1, \omega)$	1776
3	$(1, \omega, \omega, 0, 0, 1, \omega, 0)$	$(\overline{\omega}, 0, \omega, 1, 0, \overline{\omega}, \omega, \overline{\omega})$	1200	33	$(0, 1, 0, 0, 0, \omega, 1, \omega)$	$(\omega, \omega, 0, 1, 1, 1, 1, \omega)$	1776
4	$(1, 1, 0, 0, 0, \omega, 1, 1)$	$(1, \omega, 0, \overline{\omega}, 1, 0, 1, \omega)$	1344	34	$(0, 1, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(\omega, \omega, \overline{\omega}, 1, 1, 0, 1, \omega)$	1776
5	$(1, \overline{\omega}, \overline{\omega}, 0, 0, \omega, \omega, 1)$	$(1, \overline{\omega}, 1, 1, \omega, 0, \overline{\omega}, 1)$	1344	35	$(1, 1, 0, 1, 0, \overline{\omega}, \omega, 0)$	$(1, \overline{\omega}, \omega, \omega, 1, 1, 1, \omega)$	1776
6	$(0, 1, 0, 1, 0, \omega, \omega, 1)$	$(\omega, \omega, \overline{\omega}, \omega, 1, 1, 1, \omega)$	1392	36	$(1, \overline{\omega}, 1, 0, 0, \overline{\omega}, 1, \omega)$	$(\overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0, 1, \omega)$	1776
7	$(1, 1, 0, 0, 0, 1, \overline{\omega}, 1)$	$(\overline{\omega}, 0, 1, 1, 0, 1, 1, \omega)$	1392	37	$(0, 1, 1, 0, 0, 1, 1, \omega)$	$(\omega, 1, \overline{\omega}, 1, 1, 1, 1, \omega)$	1776
8	$(1, 1, 0, 1, 0, \overline{\omega}, 0, 1)$	$(\omega, 1, 1, \omega, 1, 1, 1, \omega)$	1392	38	$(0, 1, 1, 0, 0, 1, 1, \omega)$	$(\omega, 1, 0, 0, \overline{\omega}, \omega, 1, \omega)$	1776
9	$(0, 1, 1, 0, 0, 1, \overline{\omega}, 0)$	$(\omega, \overline{\omega}, 1, 1, \omega, 0, 1, \omega)$	1392	39	$(0, 1, 1, 0, 0, 1, 1, \omega)$	$(1, \omega, \overline{\omega}, 1, 0, 0, 1, \omega)$	1776
10	$(0, 1, 0, 1, 0, \omega, \omega, 1)$	$(\overline{\omega}, 1, 0, \omega, 1, 0, 1, \omega)$	1488	40	$(1, 0, 1, 0, 0, \overline{\omega}, 0, \overline{\omega})$	$(0, 1, \omega, \omega, 0, 0, 1, \omega)$	1776
11	$(1, \omega, \omega, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, \omega, 0, \overline{\omega}, \overline{\omega}, \omega, \omega, \overline{\omega})$	1488	41	$(1, 0, 0, 0, 0, 1, \omega, \omega)$	$(\omega, 1, \overline{\omega}, \omega, 1, 0, 1, \omega)$	1776
12	$(1, 0, 1, 0, 0, \omega, 1, \omega)$	$(\overline{\omega}, 0, \omega, \omega, \omega, 0, 1, \omega)$	1488	42	$(1, \omega, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(\overline{\omega}, 1, 1, \overline{\omega}, \overline{\omega}, \omega, \omega, \overline{\omega})$	1776
13	$(0, 1, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(1, \overline{\omega}, \overline{\omega}, 1, 1, 0, 1, \omega)$	1488	43	$(1, 1, 0, 0, 0, \omega, 1, 1)$	$(1, \omega, 1, \overline{\omega}, 1, 1, 1, \omega)$	1776
14	$(0, 1, 0, 0, 0, \overline{\omega}, \omega, 0)$	$(\overline{\omega}, \omega, \omega, \omega, 1, 1, 1, \omega)$	1536	44	$(1, 1, 1, 1, 0, \overline{\omega}, \omega, 1)$	$(1, \omega, \overline{\omega}, 1, 0, 0, 1, \omega)$	1776

Table 10 continued

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
15	$(0, 1, 0, 0, 0, 1, \overline{\omega}, \omega)$	$(\omega, \overline{\omega}, \omega, \omega, \omega, 0, 1, \omega)$	1632	45	$(0, 1, 0, 1, 0, 1, 1, \omega)$	$(1, \omega, 1, \overline{\omega}, \omega, \omega, 1, \omega)$	1824
16	$(0, 1, 0, 1, 0, \omega, \omega, 1)$	$(\omega, 1, \overline{\omega}, \omega, 0, 0, 1, \omega)$	1632	46	$(1, \omega, \overline{\omega}, 0, 0, \overline{\omega}, 1, \omega)$	$(1, \overline{\omega}, 1, \overline{\omega}, 0, \overline{\omega}, \overline{\omega}, 1)$	1824
17	$(1, 1, 0, 1, 0, \omega, 1, 1)$	$(1, \overline{\omega}, 1, \omega, 1, 0, 1, \omega)$	1632	47	$(1, 0, 1, 0, 0, \omega, \omega, 1)$	$(\overline{\omega}, \omega, 1, \overline{\omega}, \omega, 1, 1, \omega)$	1824
18	$(0, 1, 0, 1, 0, 1, 1, \omega)$	$(\overline{\omega}, \omega, \omega, \omega, 1, 1, 1, \omega)$	1632	48	$(1, 1, 0, 0, 0, 1, \overline{\omega}, 1)$	$(\overline{\omega}, 0, \omega, \omega, \omega, 0, 1, \omega)$	1824
19	$(1, 0, 0, 0, 0, \overline{\omega}, \overline{\omega}, 1)$	$(1, 0, \omega, 0, 1, 0, 1, \omega)$	1632	49	$(1, \omega, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(1, 0, \omega, \overline{\omega}, \omega, 0, \omega, \overline{\omega})$	1824
20	$(1, \omega, 0, 0, 0, \overline{\omega}, 1, \overline{\omega})$	$(\overline{\omega}, \omega, 1, \omega, \omega, \omega, \omega, \overline{\omega})$	1632	50	$(0, 0, 0, 0, 0, 1, \omega, 0)$	$(\overline{\omega}, \omega, \overline{\omega}, \omega, 0, \omega, \omega, \overline{\omega})$	1824
21	$(0, 1, 0, 0, 0, \omega, \omega, \overline{\omega})$	$(\omega, \omega, \omega, 1, 0, 1, 1, \omega)$	1680	51	$(1, 0, 1, 0, 0, 1, \omega, \overline{\omega})$	$(\overline{\omega}, \overline{\omega}, \omega, 0, \overline{\omega}, 0, 1, \omega)$	1920
22	$(0, 1, 0, 0, 0, 1, \overline{\omega}, \omega)$	$(1, \overline{\omega}, 1, \omega, 1, 0, 1, \omega)$	1680	52	$(1, 1, 0, 0, 0, \overline{\omega}, 1, 0)$	$(1, \overline{\omega}, 0, 0, 1, 0, 1, \omega)$	1920
23	$(1, 1, 0, 0, 0, \overline{\omega}, 1, 0)$	$(\overline{\omega}, \omega, 1, \omega, \overline{\omega}, 0, 1, \omega)$	1680	53	$(1, 0, 0, 0, 0, \omega, 1, 0)$	$(\overline{\omega}, 0, \omega, \overline{\omega}, 1, 0, 1, \omega)$	1920
24	$(0, 0, 1, 0, 0, \omega, \overline{\omega}, 1)$	$(\overline{\omega}, 1, 1, 1, 1, 0, 1, \omega)$	1680	54	$(1, 0, 0, 0, 0, \omega, 1, 0)$	$(\overline{\omega}, \omega, 1, 1, 1, 1, 1, \omega)$	1920
25	$(1, 0, 1, 0, 0, \overline{\omega}, 1, 1)$	$(1, \omega, \overline{\omega}, \omega, 1, 1, 1, \omega)$	1680	55	$(1, \overline{\omega}, \overline{\omega}, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\omega, \overline{\omega}, \omega, 0, 1, \overline{\omega}, \overline{\omega}, 1)$	1920
26	$(1, 1, 1, 0, 0, \omega, \overline{\omega}, 1)$	$(\overline{\omega}, 0, \overline{\omega}, 0, \overline{\omega}, 0, 1, \omega)$	1680	56	$(0, 1, 1, 0, 0, 1, 1, \omega)$	$(\omega, \omega, 1, 1, \overline{\omega}, 1, 1, \omega)$	1920
27	$(1, 1, 1, 1, 0, \overline{\omega}, \omega, 1)$	$(1, \omega, \overline{\omega}, 1, \overline{\omega}, 1, 1, \omega)$	1680	57	$(1, 1, 1, 0, 0, \omega, \omega, \omega)$	$(\omega, 1, \omega, 1, 0, 1, 1, \omega)$	1920
28	$(1, \omega, \omega, \omega, 0, \overline{\omega}, \overline{\omega}, 1)$	$(\omega, 0, \overline{\omega}, \overline{\omega}, \omega, 0, \omega, \overline{\omega})$	1776	58	$(0, 1, 1, 0, 0, 1, \overline{\omega}, 0)$	$(\omega, 1, \omega, 1, 0, 1, 1, \omega)$	1920
29	$(1, \overline{\omega}, \overline{\omega}, 0, 0, 1, 1, \omega)$	$(\omega, 0, 1, 0, 1, 0, \overline{\omega}, 1)$	1776	59	$(1, 0, 1, 0, 0, 1, \omega, \overline{\omega})$	$(\overline{\omega}, \omega, 1, \overline{\omega}, \omega, 1, 1, \omega)$	1968
30	$(1, 1, 0, 1, 0, \omega, 1, 1)$	$(\omega, \overline{\omega}, \omega, \omega, \omega, 0, 1, \omega)$	1776				

Table 11 Modified four $\overline{\omega}$-circulant Hermitian self-dual [32, 16, 10] codes $C_{32,\overline{\omega},i}$

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
1	$(0, 1, 0, 0, 0, \omega, 1, \omega)$	$(1, 1, 1, \overline{\omega}, \overline{\omega}, 0, 1, \omega)$	1200	31	$(0, 1, 1, 0, 0, \overline{\omega}, \omega, \overline{\omega})$	$(\omega, \omega, \omega, 0, \omega, 0, 1, \omega)$	1776
2	$(0, 1, 1, 0, 0, \omega, \omega, \omega)$	$(1, 0, \overline{\omega}, \omega, 0, 1, 1, \omega)$	1200	32	$(0, 1, 0, 0, 0, 1, 1, 0)$	$(\omega, \omega, 1, \overline{\omega}, 0, 0, 1, \omega)$	1776
3	$(0, 1, 1, 0, 0, \omega, \omega, \omega)$	$(\omega, \omega, \overline{\omega}, 1, 0, 0, 1, \omega)$	1200	33	$(0, 1, 0, 0, 0, \overline{\omega}, \overline{\omega}, \omega)$	$(1, \overline{\omega}, 1, \omega, \overline{\omega}, 0, 1, \omega)$	1776
4	$(1, \omega, \omega, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(0, \overline{\omega}, \overline{\omega}, 0, 1, 0, \omega, \overline{\omega})$	1344	34	$(0, 1, 0, 0, 0, 1, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, \omega, 0, 1, \omega)$	1776
5	$(1, \omega, \omega, 0, 0, \overline{\omega}, \overline{\omega}, 1)$	$(\overline{\omega}, \omega, \omega, 1, 0, \overline{\omega}, \omega, \overline{\omega})$	1344	35	$(1, 1, 0, 0, 0, 1, \overline{\omega}, \omega)$	$(1, \omega, \overline{\omega}, 1, 1, 1, 1, \omega)$	1776
6	$(0, 1, 0, 1, 0, \overline{\omega}, \overline{\omega}, 1)$	$(\omega, 1, 1, \omega, 1, 1, 1, \omega)$	1392	36	$(1, 1, \overline{\omega}, 0, 0, \omega, 1, \overline{\omega})$	$(1, \omega, \overline{\omega}, 1, \overline{\omega}, 0, \overline{\omega}, 1)$	1776
7	$(1, 1, 0, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, 1, 0, \overline{\omega}, 1, 0, 1, \omega)$	1392	37	$(0, 1, 1, 0, 0, \omega, 1, \omega)$	$(\overline{\omega}, \overline{\omega}, \omega, \omega, 1, \omega, 1, \omega)$	1776
8	$(1, 1, 0, 1, 0, \omega, 0, 1)$	$(\overline{\omega}, \overline{\omega}, 1, \omega, 1, 1, 1, \omega)$	1392	38	$(0, 1, 1, 0, 0, 1, 1, \overline{\omega})$	$(\omega, 1, \omega, \overline{\omega}, 0, 0, 1, \omega)$	1776
9	$(0, 1, 1, 0, 0, 1, \omega, 0)$	$(\overline{\omega}, \omega, \omega, 1, 0, \omega, 1, \omega)$	1392	39	$(1, \overline{\omega}, \omega, 0, 0, \omega, 1, \overline{\omega})$	$(1, 0, 1, \omega, 0, 0, \omega, \overline{\omega})$	1776
10	$(1, 1, 1, 0, 0, \omega, \overline{\omega}, \omega)$	$(\omega, 0, \overline{\omega}, 0, 1, 0, 1, \omega)$	1488	40	$(1, 0, 0, 0, 0, \overline{\omega}, \omega, \overline{\omega})$	$(\omega, 0, 1, \omega, 0, 0, 1, \omega)$	1776
11	$(1, \overline{\omega}, \overline{\omega}, 0, 0, \overline{\omega}, \omega, \omega)$	$(1, \overline{\omega}, \overline{\omega}, 1, 1, 0, \overline{\omega}, 1)$	1488	41	$(1, 0, 0, 0, 0, 1, \overline{\omega}, \overline{\omega})$	$(\omega, \overline{\omega}, 1, \omega, 1, 0, 1, \omega)$	1776
12	$(1, 0, \overline{\omega}, 0, 0, 1, 1, \overline{\omega})$	$(1, 0, \omega, \overline{\omega}, \omega, 0, \overline{\omega}, 1)$	1488	42	$(1, \overline{\omega}, 0, 0, 0, \omega, 1, 1)$	$(\overline{\omega}, 1, 1, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, 1)$	1776
13	$(0, 1, 0, 0, 0, 1, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, 1, 1, \overline{\omega}, \omega, 0, 1, \omega)$	1488	43	$(1, 1, 0, 0, 0, \overline{\omega}, 1, 1)$	$(\omega, 1, \overline{\omega}, \omega, \omega, \omega, 1, \omega)$	1776
14	$(0, 1, 0, 0, 0, \omega, \overline{\omega}, 0)$	$(1, 1, \omega, \omega, 1, 1, 1, \omega)$	1536	44	$(1, \overline{\omega}, \omega, 0, 0, \omega, 1, \overline{\omega})$	$(\overline{\omega}, 1, 1, 1, 1, 0, \omega, \overline{\omega})$	1776
15	$(0, 1, 0, 0, 0, \omega, \omega, \omega)$	$(\omega, \omega, 1, 1, \overline{\omega}, 0, 1, \omega)$	1632	45	$(0, 1, 0, 1, 0, \overline{\omega}, \omega, \omega)$	$(\omega, \overline{\omega}, \omega, 1, \overline{\omega}, 1, 1, \omega)$	1824
16	$(1, 1, 1, 0, 0, \overline{\omega}, \omega, \omega)$	$(1, 0, \omega, 0, 1, 0, 1, \omega)$	1632	46	$(1, \overline{\omega}, \omega, 0, 0, \omega, 1, \overline{\omega})$	$(1, \omega, \overline{\omega}, \omega, \omega, 0, \omega, \overline{\omega})$	1824
17	$(1, 1, 0, 1, 0, \overline{\omega}, 1, 1)$	$(\overline{\omega}, \omega, \overline{\omega}, \omega, 1, 0, 1, \omega)$	1632	47	$(1, 0, 1, 0, 0, \overline{\omega}, \overline{\omega}, 1)$	$(\omega, \overline{\omega}, 1, \omega, \omega, 1, 1, \omega)$	1824
18	$(0, 1, 0, 1, 0, \overline{\omega}, \omega, \omega)$	$(1, 1, \omega, \omega, 1, 1, 1, \omega)$	1632	48	$(1, 1, 0, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\omega, 0, \overline{\omega}, 1, \overline{\omega}, 0, 1, \omega)$	1824

Table 11 continued

i	r_A	r_B	A_{10}	i	r_A	r_B	A_{10}
19	$(1, 0, 1, 0, 0, 1, 0, \overline{\omega})$	$(1, 0, 1, 0, \omega, 0, 1, \omega)$	1632	49	$(1, \overline{\omega}, 0, 0, 0, \omega, 1, 1)$	$(\omega, 0, 1, 1, \overline{\omega}, 0, \overline{\omega}, 1)$	1824
20	$(1, \overline{\omega}, 0, 0, 0, \omega, 1, 1)$	$(1, \omega, 1, 1, 1, 1, \overline{\omega}, 1)$	1632	50	$(0, 0, 0, 0, 0, 1, \overline{\omega}, 0)$	$(\omega, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, 0, \overline{\omega}, 1)$	1824
21	$(0, 1, 0, 0, 0, \overline{\omega}, \overline{\omega}, \omega)$	$(1, 1, \omega, 1, 1, 0, 1, \omega)$	1680	51	$(1, 0, 1, 0, 0, 1, \overline{\omega}, \omega)$	$(\overline{\omega}, \overline{\omega}, \overline{\omega}, 0, 1, 0, 1, \omega)$	1920
22	$(0, 1, 0, 0, 0, 1, \omega, \overline{\omega})$	$(\overline{\omega}, \omega, \overline{\omega}, \omega, 1, 0, 1, \omega)$	1680	52	$(1, 1, 0, 0, 0, \omega, 1, 0)$	$(\omega, \overline{\omega}, 0, 1, 0, 0, 1, \omega)$	1920
23	$(1, 1, 0, 0, 0, \omega, 1, 0)$	$(1, \omega, 1, 1, 1, 0, 1, \omega)$	1680	53	$(1, 0, 0, 0, 0, \overline{\omega}, 1, 0)$	$(\overline{\omega}, 0, 1, 1, \omega, 0, 1, \omega)$	1920
24	$(0, 0, 1, 0, 0, 1, 1, \omega)$	$(\omega, 1, 1, \overline{\omega}, 1, 0, 1, \omega)$	1680	54	$(1, 0, 0, 0, 0, \overline{\omega}, 1, 0)$	$(\omega, \omega, 1, 1, 1, 1, 1, \omega)$	1920
25	$(1, 0, 1, 0, 0, \omega, 1, 1)$	$(\omega, 1, \overline{\omega}, \omega, 1, 1, 1, \omega)$	1680	55	$(1, \omega, \omega, 0, 0, \overline{\omega}, \omega, \omega)$	$(1, 1, \overline{\omega}, \overline{\omega}, 1, 0, \omega, \overline{\omega})$	1920
26	$(1, \overline{\omega}, \omega, 0, 0, \overline{\omega}, 1, \omega)$	$(1, 0, 1, 0, \overline{\omega}, 0, \omega, \overline{\omega})$	1680	56	$(0, 1, 1, 0, 0, 1, 1, \overline{\omega})$	$(1, \omega, 1, 1, \overline{\omega}, 1, 1, \omega)$	1920
27	$(1, 1, 1, 1, 0, \omega, \overline{\omega}, 1)$	$(\overline{\omega}, \omega, 1, \overline{\omega}, \omega, \omega, 1, \omega)$	1680	57	$(1, 1, 1, 0, 0, \overline{\omega}, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, 1, \omega, 1, 1, 0, 1, \omega)$	1920
28	$(1, 1, 1, 1, 0, 1, \overline{\omega}, \overline{\omega})$	$(1, 0, \overline{\omega}, 1, 1, 0, 1, \omega)$	1776	58	$(0, 1, 1, 0, 0, 1, \omega, 0)$	$(\overline{\omega}, 1, \omega, 1, 1, 0, 1, \omega)$	1920
29	$(1, \overline{\omega}, 1, 0, 0, \omega, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, 0, 1, 0, \overline{\omega}, 0, 1, \omega)$	1776	59	$(1, 0, 1, 0, 0, 1, \overline{\omega}, \omega)$	$(\omega, \overline{\omega}, 1, \omega, \omega, 1, 1, \omega)$	1968
30	$(1, 1, 0, 1, 0, \overline{\omega}, 1, 1)$	$(\omega, \omega, 1, 1, \overline{\omega}, 0, 1, \omega)$	1776				

Table 12 Modified four μ-circulant Hermitian self-dual codes with large minimum weights

Code	d	r_A	r_B
$C_{48,1}$	14	$(1, 1, 0, 1, \omega, \omega, 0, \overline{\omega}, \overline{\omega}, 1, 1, 0)$	$(\overline{\omega}, 1, \overline{\omega}, 1, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1, \overline{\omega})$
$C_{48,\omega}$	14	$(0, 1, \overline{\omega}, 1, \overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0)$	$(\omega, 0, \overline{\omega}, 0, 1, \omega, \overline{\omega}, 0, 1, \overline{\omega}, \overline{\omega}, \omega)$
$C_{48,\overline{\omega}}$	14	$(0, 0, 1, 1, \overline{\omega}, \omega, \omega, 0, 0, \overline{\omega}, \overline{\omega}, 1)$	$(0, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0, \omega, 0, 1, \overline{\omega})$
$C_{52,1}$	14	$(1, 0, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, 1, 1, \omega, 0, \omega, \omega, 1)$	$(\omega, \overline{\omega}, 0, 1, \overline{\omega}, 1, \omega, 0, 1, \omega, 1, 0, 1)$
$C_{52,\omega}$	14	$(1, 0, 1, \overline{\omega}, 0, 1, 1, 0, 0, \omega, 1, 0, 1)$	$(\overline{\omega}, \overline{\omega}, 0, 1, 0, \overline{\omega}, 1, \omega, 1, 1, \overline{\omega}, 1, \overline{\omega})$
$C_{52,\overline{\omega}}$	14	$(1, 0, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, 1, 1, \omega, 0, \omega, \omega, 1)$	$(\overline{\omega}, 0, 0, \omega, 0, 1, 1, 0, 1, \omega, 1, 0, 1)$
$C_{56,1}$	16	$(0, 1, 0, 0, \omega, \omega, \omega, \overline{\omega}, 1, 0, \overline{\omega}, \omega, \omega, \overline{\omega})$	$(1, 1, 0, 0, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \omega, 0, \omega, 1, 1, 1)$
$C_{56,\omega}$	16	$(1, \omega, \overline{\omega}, 0, 1, \overline{\omega}, \omega, 1, \omega, 0, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega})$	$(0, 0, \overline{\omega}, \omega, 1, 0, \overline{\omega}, 0, 0, \overline{\omega}, \omega, \overline{\omega}, \omega, \overline{\omega})$
$C_{56,\overline{\omega}}$	14	$(1, \omega, \omega, 0, 0, \omega, 1, 0, 1, 0, 1, 0, 0, \omega)$	$(\omega, 0, 1, \omega, \omega, 0, 0, 0, 0, \omega, \overline{\omega}, \omega, 1, \omega)$
$C_{60,1}$	16	$(1, 0, \omega, 1, \omega, \overline{\omega}, 0, 1, 1, 1, \overline{\omega}, 0, 1, 1, \overline{\omega})$	$(\overline{\omega}, 0, \omega, 1, 0, \omega, \overline{\omega}, \omega, \overline{\omega}, 0, 0, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega})$
$C_{60,\omega}$	16	$(1, \overline{\omega}, 1, 0, 0, 0, \overline{\omega}, \omega, 0, \omega, \overline{\omega}, 1, \overline{\omega}, \omega, \omega)$	$(\overline{\omega}, 0, \omega, 0, 1, \omega, \omega, 0, \overline{\omega}, 1, 1, \omega, \overline{\omega}, 1, 1)$
$C_{60,\overline{\omega}}$	16	$(1, \overline{\omega}, 1, 0, 0, 0, \overline{\omega}, \omega, 0, \omega, \overline{\omega}, 1, \overline{\omega}, \omega, \omega)$	$(\overline{\omega}, \omega, 1, 0, 0, \omega, 0, 0, \omega, 0, 1, \omega, \overline{\omega}, 1, 1)$
$C_{64,1}$	16	$(1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0, 1, 1, 1, 1, \omega, \overline{\omega}, \omega, \omega, \omega, \omega)$	$(1, \omega, 0, \overline{\omega}, \omega, 1, 1, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, 0, 1, \omega, \omega)$
$C_{64,\omega}$	16	$(1, \omega, 0, 1, 1, 1, \omega, \overline{\omega}, \omega, \overline{\omega}, 0, 1, 0, 1, \overline{\omega}, 0)$	$(0, 1, 1, 1, 1, 0, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1, 1, \overline{\omega}, 0, 1, \overline{\omega}, 1)$
$C_{64,\overline{\omega}}$	16	$(1, \omega, 0, 1, 1, 1, \omega, \overline{\omega}, \omega, \overline{\omega}, 0, 1, 0, 1, \overline{\omega}, 0)$	$(0, 1, 1, 1, 1, 0, \overline{\omega}, \overline{\omega}, 1, 0, \overline{\omega}, 1, \omega, \omega, \omega, \overline{\omega})$
$C_{68,1}$	18	$(1, 0, \overline{\omega}, \omega, \omega, \overline{\omega}, 1, \overline{\omega}, 0, \overline{\omega}, 0, 1, 1, 1, 0, \omega, \overline{\omega})$	$(\omega, 1, 0, 1, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, \omega, 0, \overline{\omega}, \omega, \overline{\omega}, 0, \overline{\omega})$
$C_{68,\omega}$	18	$(1, \overline{\omega}, \omega, \omega, \overline{\omega}, 0, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, 0, \overline{\omega}, \omega, \overline{\omega}, 0, 0)$	$(\overline{\omega}, 1, 0, 1, \omega, \overline{\omega}, \overline{\omega}, \omega, 1, 0, \omega, 0, 1, 0, \omega, \omega, 0)$
$C_{68,\overline{\omega}}$	18	$(0, 1, \omega, 0, \omega, 0, \omega, \omega, 1, \overline{\omega}, \omega, \omega, \omega, \omega, \overline{\omega}, 1, \overline{\omega})$	$(0, 1, \omega, \overline{\omega}, \omega, 0, 0, 1, 1, \overline{\omega}, 1, \overline{\omega}, 1, \overline{\omega}, \omega, 0, \omega)$
$C_{72,1}$	18	$(1, \overline{\omega}, 0, \omega, 0, \omega, \overline{\omega}, 0, 0, 0, \omega, \overline{\omega}, 0, \overline{\omega}, 0, \omega, \overline{\omega}, \omega)$	$(\overline{\omega}, \omega, 0, 1, 0, \omega, 1, 0, 1, 0, \omega, 1, 1, 1, 0, 0, 1, 1)$
$C_{72,\omega}$	18	$(1, 0, \omega, \overline{\omega}, \omega, 0, 0, \omega, \omega, 0, \overline{\omega}, 1, 1, 1, 1, \omega, 1, \overline{\omega})$	$(\overline{\omega}, \overline{\omega}, 1, \overline{\omega}, 1, 1, \overline{\omega}, \omega, \overline{\omega}, 0, \omega, 0, 1, \omega, 1, 1, 1, 0)$
$C_{72,\overline{\omega}}$	18	$(0, 1, 1, \omega, 0, 1, \omega, \omega, \omega, 1, \overline{\omega}, 1, \overline{\omega}, 1, \omega, 0, \omega, 0)$	$(1, \overline{\omega}, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0, \omega, 0, 0, 0, \omega, 0, 1, \overline{\omega}, \overline{\omega}, 0, 1, 0)$
$C_{76,1}$	18	$(0, 1, \overline{\omega}, 1, \overline{\omega}, 0, 0, \omega, 1, 1, \overline{\omega}, \omega, \omega, \overline{\omega}, 1, \omega, 0, \overline{\omega}, \overline{\omega})$	$(0, \omega, 1, 0, 0, \overline{\omega}, 1, \overline{\omega}, 1, 0, 0, 1, \omega, 0, \overline{\omega}, 0, \overline{\omega}, 0, 0)$

Table 12 continued

Code	d	r_A	r_B
$C_{76,\omega}$	18	$(0, 1, \overline{\omega}, \omega, 0, 1, \omega, 1, \overline{\omega}, \overline{\omega}, 0, 0, 1, \overline{\omega}, 0, \omega, 1, \overline{\omega}, \overline{\omega})$	$(\overline{\omega}, 1, 1, 1, 0, \omega, 1, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, 0, 1, 1, 1, 0, \overline{\omega}, \omega, 0)$
$C_{76,\overline{\omega}}$	18	$(0, 1, \omega, \overline{\omega}, \overline{\omega}, 1, 0, 0, 1, 1, 0, 1, 0, \omega, 1, 1, \overline{\omega}, \omega, 1)$	$(1, \overline{\omega}, 1, 1, 1, 0, 0, 1, \omega, \omega, 0, \omega, 1, \overline{\omega}, \overline{\omega}, 0, 1, 0, 0)$
$C_{80,1}$	20	$(1, \overline{\omega}, 0, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, 1, \omega, 0, \omega, 1, 1, 0, \overline{\omega}, 1, 1, \omega, \omega, 0)$	$(1, 1, 0, 0, \overline{\omega}, 1, 1, \omega, \overline{\omega}, 0, 0, 0, \overline{\omega}, 1, \omega, \overline{\omega}, \overline{\omega}, 1, 1, \omega)$
$C_{80,\omega}$	20	$(1, 0, \overline{\omega}, 0, \omega, 0, 1, \overline{\omega}, \omega, 1, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, \omega, 0, 0)$	$(0, 0, \omega, 0, 1, 0, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, \omega, 0, \overline{\omega}, 0, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, 1)$
$C_{80,\overline{\omega}}$	20	$(0, 1, \overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, 0, 1, \omega, 0, \omega, 0, 1, 1, \omega, \overline{\omega}, 1, \omega)$	$(1, 1, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 0, 0, 1, 1, \overline{\omega}, \omega, 0, \overline{\omega}, 1, 0, 1, 0, \omega, \omega)$
$C_{84,\omega}$	20	$(1, \omega, \overline{\omega}, \overline{\omega}, 0, 0, 1, \overline{\omega}, 1, 1, \omega, \omega, \overline{\omega}, 0, 1, 0, \omega, 1, 1, \omega, 1)$	$(\omega, 0, 0, 0, 0, \omega, \overline{\omega}, 0, 0, 1, 1, 1, \omega, 0, 0, \overline{\omega}, 0, 0, 0, 0, 0)$
$C_{88,\omega}$	20	$(0, 0, 1, 0, 0, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, 0, \omega, 1, 1, \overline{\omega}, \omega, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \overline{\omega}, 0)$	$(0, 0, 0, \overline{\omega}, \omega, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 0, 1, \overline{\omega}, \omega, \omega, 0, \overline{\omega}, 0, 0, 0, 0, 0, 0)$
$C_{96,\omega}$	22	$(1, 1, 1, \overline{\omega}, 1, \overline{\omega}, 1, 0, 0, 0, 0, 0, 1, 0, \omega, \overline{\omega}, 1, 0, \overline{\omega}, \omega, 0, \overline{\omega}, \omega, 1)$	$(1, 0, \overline{\omega}, \omega, \overline{\omega}, 0, 1, \overline{\omega}, 0, 1, 0, \omega, \overline{\omega}, 1, \omega, 1, \omega, 0, 1, \overline{\omega}, 1, 1, 1, \omega)$

5 Application to quantum codes

In this section, we consider an application of the quaternary Hermitian self-dual [56, 28, 16] codes $C_{56,1}$ and $C_{56,\omega}$ found in the previous section to quantum codes.

A quaternary *additive* $(n, 2^k)$ code $\mathcal{C}$ is an additive subgroup of $\mathbb{F}_4^n$ with $|\mathcal{C}| = 2^k$. The *dual* code $\mathcal{C}^*$ of a quaternary additive $(n, 2^k)$ code $\mathcal{C}$ is defined as

$$\mathcal{C}^* = \{x \in \mathbb{F}_4^n \mid x * y = 0 \text{ for all } y \in \mathcal{C}\},$$

under the following trace inner product

$$x * y = \sum_{i=1}^{n}(x_i y_i^2 + x_i^2 y_i)$$

for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{F}_4^n$. A quaternary additive code $\mathcal{C}$ is called *self-orthogonal* and *self-dual* if $\mathcal{C} \subset \mathcal{C}^*$ and $\mathcal{C} = \mathcal{C}^*$, respectively. Note that a quaternary Hermitian self-dual $[n, n/2, d]$ code is a quaternary additive self-dual $(n, 2^n)$ code with minimum weight d (see e.g., [12]).

A useful method for constructing quantum codes from quaternary additive self-orthogonal codes was given by Calderbank, Rains, Shor and Sloane [3] (see [3] for undefined terms concerning quantum codes). A quaternary additive self-orthogonal $(n, 2^{n-k})$ code $\mathcal{C}$ such that there is no vector of weight less than d in $\mathcal{C}^* \setminus \mathcal{C}$, gives a quantum $[[n, k, d]]$ code, where $k \neq 0$. A quaternary additive self-dual $(n, 2^n)$ code with minimum weight d gives a quantum $[[n, 0, d]]$ code. Let $d_{\max}(n, k)$ denote the largest minimum weight d among quantum $[[n, k, d]]$ codes. Similar to the classical coding theory, it is a fundamental problem to determine $d_{\max}(n, k)$. A table on $d_{\max}(n, k)$ is given in [3, Table III] for $n \leq 30$. An extended table is obtained electronically from [8]. For example, it was previously known that $15 \leq d_{\max}(56, 0) \leq 20$ [8].

In Sect. 4, quaternary Hermitian self-dual [56, 28, 16] codes $C_{56,1}$ and $C_{56,\omega}$ were constructed for the first time. By the above method, a quantum [[56, 0, 16]] code is obtained. Hence, we have the following proposition.

Proposition 12 (i) *There is a quantum* [[56, 0, 16]] *code.*
(ii) $16 \leq d_{\max}(56, 0) \leq 20$.

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Data availibility Available from the corresponding author on reasonable request.

Code availability Not applicable.

Declarations

Conflict of interest The author declares there are no conflicts of interest.

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