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Article

# Boson Models with Interactions of Arbitrary Order

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## Abstract

The paper considers quantal many-boson systems that are described by a rotationally invariant and boson-number conserving Hamiltonian. The properties of a generic model are studied, which treats  $N$  bosons of  $p$  different kinds with non-zero angular momenta  $\ell_1, \ell_2, \dots, \ell_p$ , possibly augmented with a (number of) scalar  $s$  boson(s). The order  $k$  of the interaction between the bosons is arbitrary, and closed formulas are given for matrix elements between  $N$ -boson states for any  $k$  if  $p = 1$  and  $p = 2$ . A recursive procedure is defined for arbitrary  $k$  and  $p$ . With the expressions derived in the paper, it is possible to express symbolically a Hamiltonian matrix element between  $N$ -boson states as a linear combination of  $k$ -body interaction matrix elements. More generally, the formulas allow the evaluation of matrix elements of tensor operators that are not necessarily scalar nor boson-number conserving. The numerical implementation of the formalism is discussed and illustrated with a few examples.

**Keywords:** interacting boson models;  $k$ -body interactions; symbolic calculation

## 1. Introduction

Many quantum systems can be modeled in terms of bosons. Although the system's basic constituents are fermions (e.g., neutrons and protons in an atomic nucleus), clusters of an even number of them can be treated approximately as bosons, leading to a significant simplification of the original many-fermion problem. However, since the bosons are not elementary and violate the Pauli principle that acts among the fermions, the price to pay is an interaction between the bosons that is complicated and usually of higher order.

The purpose of this paper is to present a formalism that deals with quantal systems consisting of bosons with complex higher-order interactions. It is assumed that the system is described by a Hamiltonian that is rotationally invariant so that its eigenstates are characterized by an angular momentum  $J$  and its projection on the  $z$  axis  $M$ . Furthermore, the Hamiltonian is also assumed to conserve the total number of bosons  $N$ . The bosons can be of  $p$  different kinds that have the respective angular momenta  $\ell_i, i = 1, \dots, p$ . The number  $n_i$  of  $\ell_i$  bosons is *not* conserved in general, and interactions may transform a boson with angular momentum  $\ell_i$  into another one with  $\ell_{i'}$ . With these assumptions the expressions given in this paper enable the construction of matrix elements of a Hamiltonian with  $k$ -body interactions, or any higher-order non-scalar operator, between  $N$ -boson states for arbitrary order  $k$  and arbitrary number  $p$  of different kinds of bosons.

A system of  $N$  bosons of one kind with angular momentum  $\ell$  can be considered as the most 'elementary' one. Models of this type shall be referred to as the  $\ell$ -interacting boson model or  $\ell$ -IBM. In Section 3, a generic algorithm is presented, which enables the solution of the eigenvalue problem in  $\ell$ -IBM for interactions between the bosons that are, in principle, of arbitrary order  $k$ .



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The elementary  $\ell$ -IBM can be generalized to  $\ell_1 \dots \ell_p$ -IBM. For  $p = 2$ , explicit formulas are given in Section 4 for the  $N$ -boson matrix elements in  $\ell_1 \ell_2$ -IBM, valid for arbitrary angular momenta  $\ell_1$  and  $\ell_2$  and arbitrary order  $k$  of the interaction. For  $p > 2$ , explicit formulas become cumbersome because of the complicated angular momentum recoupling. In that case a recursive procedure is defined in Section 5, reducing a matrix element in  $\ell_1 \dots \ell_p$ -IBM to a linear combination of products of  $p$  matrix elements in  $\ell_i$ -IBM,  $i = 1, \dots, p$ .

The set of bosons  $\{b_{\ell_i}, i = 1, \dots, p\}$  may contain a scalar  $s$  boson with zero intrinsic angular momentum,  $\ell = 0$ . This possibility covers two important models, namely the vibron model of molecules [1] with  $s$  and  $p$  ( $\ell = 1$ ) bosons and the IBM of nuclei [2] with  $s$  and  $d$  ( $\ell = 2$ ) bosons. As will be shown below, the  $s$ -boson dependence of matrix elements can be factored out if operators are written in normal-ordered form with creation operators to the left and annihilation operators to the right. In case the model contains several kinds of  $s$  bosons, they can all be eliminated from normal-ordered operators, leaving a model of the type  $\ell_1 \dots \ell_p$ -IBM with  $\ell_i \neq 0$ . The assumption  $\ell_i \neq 0$  is not a fundamental restriction but is made for convenience. All formulas given in this paper are valid if some  $\ell_i$  are zero; the recursive algorithm, however, is more efficient if the  $s$  boson(s) is (are) eliminated from the start.

The straightforward elimination of scalar bosons is not possible if operators are not given in normal-ordered form. A particularly important case concerns operators given in multipole form, which is discussed in Section 6. An operator in multipole form can always be reduced to a combination of normal-ordered operators. In Section 7, a generic algorithm is described, which converts a multipole Hamiltonian into its normal-ordered representation for arbitrary order  $k$  of the interaction.

The formalism presented in this paper is capable of dealing with systems consisting of bosons that are intrinsically different but have the same angular momentum. For example, the neutron–proton IBM or IBM-2 [3] can be considered as  $s_\nu s_\pi d_\nu d_\pi$ -IBM, which after the elimination of the  $s_\nu$  and  $s_\pi$  bosons reduces to  $d_\nu d_\pi$ -IBM, the matrix elements of which are quadratic combinations of those in  $d$ -IBM. Likewise, the isospin-invariant IBM or IBM-3 [4] can be treated as  $s_\nu s_\delta s_\pi d_\nu d_\delta d_\pi$ -IBM and can be reduced to a problem in  $d$ -IBM.

The paper starts with a brief review, given in the next section, of the algorithm to calculate coefficients of fractional parentage (CFPs) on a seniority basis.

## 2. Coefficients of Fractional Parentage in a Seniority Basis

The definition of CFPs for fermions can be found in standard textbooks [5] and is easily extended to bosons [6]. All formulas given in this section are valid for fermions as well as for bosons, and the notation  $j$  is used for the angular momentum of either a fermion or a boson. Therefore,  $j$  can be an integer or half-odd-integer. Sections after the present one only deal with bosons with integer angular momenta  $\ell_i$ .

A many-body wave function expanded in terms of CFPs is rotationally invariant; that is, it carries the quantum number of angular momentum  $J$ , and it satisfies the requirement of anti-symmetry for fermions or symmetry for bosons. The definition of CFPs usually applies to single- $j$  systems, when all particles have the same angular momentum  $j$ . This is not a fundamental restriction as the notion of CFPs can be generalized to multi- $j$  systems [7], consisting of particles with several angular momenta  $j_i, i = 1, \dots, p$ .

A many-body calculation in terms of CFPs has the marked advantage that the order of the interaction between the particles is automatically taken care of. This will be illustrated below for single- $j$  CFPs, but the statement is also valid for the multi- $j$  generalization of CFPs appropriate for several angular momenta  $j_i$ . It is indeed possible to formulate the many-body problem to any order of the interaction between the particles in terms of generalized CFPs. This technique can be applied to the shell model in a space with several

single-particle levels or to IBMs with bosons with different angular momenta. Owing to the proliferation of the number of multi- $j$  CFPs, this method, however, is restricted to a relatively small number  $N$  of particles and a limited number  $p$  of angular momenta  $j_i$ .

Numerical calculations are vastly more efficient for single- $j$  than they are for multi- $j$  CFPs. In fact, the matrix elements of a many-body Hamiltonian can always be written in terms of single- $j$  CFPs, but this requires the recoupling of the angular momenta  $j_i$  of the particles. The analytic expressions of such matrix elements become increasingly cumbersome as the order  $k$  of the interaction as well as the number  $p$  of angular momenta  $j_i$  increases. One concludes, therefore, that many-body calculations in terms of multi- $j$  CFPs are convenient if the Hamiltonian contains interactions of higher order  $k$  but become computationally prohibitively expensive as the number  $N$  of particles and/or the number  $p$  of angular momenta increases. On the other hand, analytic expressions for Hamiltonian matrix elements in terms of single- $j$  CFPs become prohibitively complicated with increasing  $k$  or  $p$ .

The method proposed in this paper is based on single- $j$  CFPs, which is computationally more efficient than that based on multi- $j$  CFPs. It avoids the complex analytic expressions of angular-momentum recoupling through a recursive procedure.

The computation of  $(n - 1) \times 1 \rightarrow n$  CFPs relies on the following recursive formula, known as the Redmond relation [5,6]:

$$[j^{n-1}(\alpha_1 J_1), j] \{ j^n[\alpha_2 J_2] J \} = \left[ \delta_{\alpha_1 \alpha_2} \delta_{J_1 J_2} + (n-1)(-)^{J_1+J_2} [J_1][J_2] \sum_{\alpha'_1 J'_1} \left\{ \begin{matrix} J'_1 & j & J_1 \\ J & j & J_2 \end{matrix} \right\} \right. \\ \left. \times [j^{n-2}(\alpha'_1 J'_1), j] \{ j^{n-1} \alpha_1 J_1 \} [j^{n-2}(\alpha'_1 J'_1), j] \{ j^{n-1} \alpha_2 J_2 \} \right], \quad (1)$$

where  $[x]$  stands for  $\sqrt{2x+1}$  and the symbol in curly brackets is a  $6j$  symbol. The recurrence relation (1) expresses an  $(n - 1) \times 1 \rightarrow n$  CFP on the left-hand side of the equation in terms of a sum of  $(n - 2) \times 1 \rightarrow (n - 1)$  CFPs on the right-hand side. The algorithm cannot be applied as such, however, since in the CFP on the left-hand side the  $n$ -particle state refers to a non-orthonormal, overcomplete basis, characterized by the labels  $[\alpha_2 J_2]$  of the  $(n - 1)$ -particle parent state, while in the CFPs on the right-hand side the  $(n - 1)$ -particle states must form an orthonormal basis. To carry out the orthonormalization, one orders states in the overcomplete, non-orthonormal basis in a sequence  $|j^n[\alpha_k J_k] J\rangle$ ,  $k = 1, 2, \dots, q$ , where  $q$  is the number of independent (anti-)symmetric  $n$ -particle states with total angular momentum  $J$ . The sequence and the order of the states are arbitrary but for the fact that the states must be linearly independent. A set of orthonormal states can be obtained from the expansion

$$|j^n \alpha_r J\rangle = \sum_{k=1}^r a_{rk} |j^n[\alpha_k J_k] J\rangle, \quad r = 1, 2, \dots, q, \quad (2)$$

where the coefficients  $a_{rk}$  are identified with the following sequence in a Gram–Schmidt orthonormalisation procedure:

$$\begin{aligned} |j^n \alpha_1 J\rangle &= \frac{1}{\sqrt{o_{11}}} |j^n[\alpha_1 J_1] J\rangle, \\ |j^n \alpha_2 J\rangle &= \frac{1}{\sqrt{o_{22} - (\tilde{o}_{21})^2}} (|j^n[\alpha_2 J_2] J\rangle - \tilde{o}_{21} |j^n[\alpha_1 J_1] J\rangle), \\ &\vdots \\ |j^n \alpha_k J\rangle &= \frac{1}{\sqrt{\mathcal{N}_k}} \left( |j^n[\alpha_k J_k] J\rangle - \sum_{l=1}^{k-1} \tilde{o}_{kl} |j^n[\alpha_l J_l] J\rangle \right), \end{aligned} \quad (3)$$

until  $k = q$ , with

$$\mathcal{N}_k = o_{kk} - \sum_{r=1}^{k-1} (\tilde{o}_{kr})^2, \quad o_{kl} = \langle j^n [\alpha_k J_k] J | j^n [\alpha_l J_l] J \rangle, \quad \tilde{o}_{kr} = \langle j^n [\alpha_k J_k] J | j^n \alpha_r J \rangle. \quad (4)$$

The coefficients  $a_{rk}$  (needed for  $r \geq k$ ) can be calculated recursively from

$$a_{kk} = \frac{1}{\sqrt{\mathcal{N}_k}}, \quad a_{rk} = -\frac{1}{\sqrt{\mathcal{N}_r}} \sum_{s=k}^{r-1} \tilde{o}_{rs} a_{sk}, \quad \tilde{o}_{rs} = \sum_{s'=1}^s a_{ss'} o_{rs'}, \quad (5)$$

which as only input requires the knowledge of the overlap matrix

$$o_{kl} = \sum_{\alpha_i J_i} [j^{n-1}(\alpha_i J_i), j] \{ j^n [\alpha_k J_k] J | j^{n-1}(\alpha_i J_i), j \} \{ j^n [\alpha_l J_l] J \}. \quad (6)$$

With the help of Equation (2), the CFPs in the orthonormal basis can be expressed in terms of those in the non-orthonormal basis,

$$[j^{n-1}(\alpha_1 J_1), j] \{ j^n \alpha_r J \} = \sum_{k=1}^r a_{rk} [j^{n-1}(\alpha_1 J_1), j] \{ j^n [\alpha_k J_k] J \}, \quad r = 1, 2, \dots, q. \quad (7)$$

The Redmond relation (1), together with Equation (7), defines a genuine recursive scheme. The basis defined by the label  $\alpha_r$ , however, depends on the chosen sequence  $[j^n [\alpha_k J_k] J], k = 1, 2, \dots, q$ , which in general is different from a seniority basis. The latter basis is mathematically convenient since it can be obtained from group-theoretical considerations, as will be discussed in Section 3 in the case of bosons. In addition, the seniority label has a useful intuitive interpretation since it corresponds to the number of particles not in pairs coupled to angular momentum  $J = 0$  [8]. For these reasons, it is of interest to adapt the Redmond recursive scheme to include seniority.

To construct basis states  $|j^n v \alpha J\rangle$ , where  $\alpha$  is now a label additional to the seniority  $v$ , one proceeds as follows. As the procedure is recursive, one assumes that the CFPs  $[j^{k-1}(v_1 \alpha_1 J_1) j] \{ j^k v \alpha J \}$  are known for  $k < n$ . The task is to compute them for  $k = n$ . All CFPs  $[j^{n-1}(v_1 \alpha_1 J_1) j] \{ j^n v \alpha J \}$  with  $v = n - 2, n - 4, \dots, 1$  or 0 can be obtained from seniority reduction formulas, which for fermions ( $j$  half-odd-integer) read [5]

$$\begin{aligned} & [j^{n-1}(v - 1, \alpha_1 J_1), j] \{ j^n v \alpha J \} \\ &= \left[ \frac{v(2j + 3 - n - v)}{n(2j + 3 - 2v)} \right]^{1/2} [j^{v-1}(v - 1, \alpha_1 J_1), j] \{ j^v v \alpha J \}, \\ & [j^{n-1}(v + 1, \alpha_1 J_1), j] \{ j^n v \alpha J \} \\ &= (-)^{J+j-J_1} \left[ \frac{(v + 1)(n - v)(2J_1 + 1)}{n(2j + 1 - 2v)(2J + 1)} \right]^{1/2} [j^v(v \alpha J), j] \{ j^{v+1} v + 1, \alpha_1 J_1 \}, \end{aligned} \quad (8)$$

while for bosons ( $j$  integer) they are [6]

$$\begin{aligned} & [j^{n-1}(v - 1, \alpha_1 J_1), j] \{ j^n v \alpha J \} \\ &= \left[ \frac{v(2j - 1 + n - v)}{n(2j - 1 + 2v)} \right]^{1/2} [j^{v-1}(v - 1, \alpha_1 J_1), j] \{ j^v v \alpha J \}, \\ & [j^{n-1}(v + 1, \alpha_1 J_1), j] \{ j^n v \alpha J \} \\ &= (-)^{J+J_1} \left[ \frac{(v + 1)(n - v)(2J_1 + 1)}{n(2j + 1 + 2v)(2J + 1)} \right]^{1/2} [j^v(v \alpha J), j] \{ j^{v+1} v + 1, \alpha_1 J_1 \}. \end{aligned} \quad (9)$$

Equations (8) and (9) determine all  $(n - 1) \times 1 \rightarrow n$  CFPs except those with  $v = n$ ,

$$[j^{n-1}(v_1 = n - 1, \alpha_1 J_1), j] \{ j^n v = n, \alpha J \}. \tag{10}$$

Since a single particle can change seniority by at most one unit, the  $(n - 1)$ -particle state must have  $v_1 = n - 1$  in order to lead to an  $n$ -particle state with  $v = n$ . The CFP (10) must be obtained from the Redmond relation (1) and therefore derived from the CFPs in the non-orthonormal basis,

$$[j^{n-1}(v_1 = n - 1, \alpha_1 J_1), j] \{ j^n [v_2 = n - 1, \alpha_2 J_2] J \}. \tag{11}$$

The state  $|j^n [n - 1, \alpha_2 J_2] J\rangle$ , which has the parent  $|j^{n-1} n - 1, \alpha_2 J_2\rangle$ , is a mixture of  $v = n$  and  $v = n - 2$  states, and the latter components must be projected out in order to arrive at an orthonormal basis. This can be achieved by including the states  $|j^n n - 2, \alpha J\rangle$  in the Gram–Schmidt orthonormalization. The overlap matrix that should be used in this procedure is

$$\begin{pmatrix} \langle j^n n - 2, \alpha_r J | j^n n - 2, \alpha_s J \rangle & \langle j^n n - 2, \alpha_r J | j^n [n - 1, \alpha_l J_l] J \rangle \\ \langle j^n [n - 1, \alpha_k J_k] J | j^n n - 2, \alpha_s J \rangle & \langle j^n [n - 1, \alpha_k J_k] J | j^n [n - 1, \alpha_l J_l] J \rangle \end{pmatrix}, \tag{12}$$

where the upper-left matrix is diagonal,

$$\langle j^n n - 2, \alpha_r J | j^n n - 2, \alpha_s J \rangle = \delta_{rs}, \tag{13}$$

the off-diagonal matrices are

$$\begin{aligned} \langle j^n n - 2, \alpha_r J | j^n [n - 1, \alpha_k J_k] J \rangle &= \langle j^n [n - 1, \alpha_k J_k] J | j^n n - 2, \alpha_r J \rangle \\ &= \sum_{v_i \alpha_i J_i} [j^{n-1}(v_i \alpha_i J_i), j] \{ j^n n - 2, \alpha_r J \} [j^{n-1}(v_i \alpha_i J_i), j] \{ j^n [n - 1, \alpha_k J_k] J \}, \end{aligned} \tag{14}$$

and the lower-right matrix is given by Equation (6).

The above formulas define a recursive procedure to determine the  $(n - 1) \times 1 \rightarrow n$  CFPs  $[j^{n-1}(v_1 \alpha_1 J_1), j] \{ j^n v \alpha J \}$  in an orthonormal basis that includes the seniority quantum number. For the construction of the Hamiltonian matrix of a  $k$ -body interaction in an  $n$ -particle basis, knowledge of  $k \times r \rightarrow n$  CFPs, with  $k + r = n$ , is required. These can be computed from

$$\begin{aligned} &[j^k(v_k \alpha_k J_k), j^r(v_r \alpha_r J_r)] \{ j^n v_n \alpha_n J_n \} \\ &= [J_k] \sum_{v'_n \alpha'_n J'_n} [J'_n] [j, j^{n-1}(v'_n \alpha'_n J'_n)] \{ j^n v_n \alpha_n J_n \} \sum_{v'_k \alpha'_k J'_k} (-)^{j+J'_k+J_n+J_r} \begin{Bmatrix} j & J'_k & J_k \\ J_r & J_n & J'_n \end{Bmatrix} \\ &\times [j, j^{k-1}(v'_k \alpha'_k J'_k)] \{ j^k v_k \alpha_k J_k \} [j^{k-1}(v'_k \alpha'_k J'_k), j^r(v_r \alpha_r J_r)] \{ j^{n-1} v'_n \alpha'_n J'_n \}. \end{aligned} \tag{15}$$

This equation is written in terms of  $1 \times (n - 1) \rightarrow n$  CFPs and defines a recursive procedure ending in  $1 \times (n - 1) \rightarrow n$  CFPs. These can be related to  $(n - 1) \times 1 \rightarrow n$  CFPs through

$$[j, j^{n-1}(v'_n \alpha'_n J'_n)] \{ j^n v_n \alpha_n J_n \} = \phi(-)^{j+J'_n-J_n} [j^{n-1}(v'_n \alpha'_n J'_n), j] \{ j^n v_n \alpha_n J_n \}, \tag{16}$$

where  $\phi = +1$  for bosons and  $\phi = (-)^{n-1}$  for fermions.

### 3. The $s\ell$ -IBM

This section deals with boson models that include a scalar  $s$  boson and another type of boson with angular momentum  $\ell$ . The ensuing model will be referred to as  $s\ell$ -IBM and covers two important cases, namely the vibron model [1] with  $s$  and  $p$  bosons and the IBM [2] with  $s$  and  $d$  bosons. As the dependence of matrix elements on the  $s$  boson can be readily factored out, the problem can be reduced to one in  $\ell$ -IBM. Note that there is, in principle, no restriction on the values of  $\ell$  and therefore the formalism of this section can also be used to deal with models of interacting bosons with high angular momentum, such as those related to aligned neutron–proton pairs [9].

States with  $N$  bosons in  $s\ell$ -IBM can be constructed in the ‘vibrational’ basis

$$\begin{array}{ccccccccc}
 \text{U}(2\ell + 2) & \supset & \text{U}(2\ell + 1) & \supset & \text{SO}(2\ell + 1) & \supset & \text{SO}(3) & \supset & \text{SO}(2) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 [N] & & n & & \nu & & \alpha L & & M
 \end{array}, \tag{17}$$

where  $N$  is the total number of  $s + \ell$  bosons and the number of  $\ell$  bosons is  $n = 0, 1, \dots, N$ . The states in the  $\ell^n$  system associated with the classification  $\text{U}(2\ell + 1) \supset \text{SO}(2\ell + 1) \supset \text{SO}(3)$  coincide with those obtained with the CFP algorithm explained in the previous section. The basis states (17) are characterized by the seniority quantum number  $\nu$ , which can take the values  $\nu = n, n - 2, \dots, 0$  or  $1$ . The allowed values of the angular momentum  $L$  follow from the  $\text{SO}(2\ell + 1) \supset \text{SO}(3)$  reduction, for which no simple, closed formula exists for general  $\ell$  but which can be obtained with group-theoretical methods [10]. The  $\ell$  bosons are possibly characterized by an additional label  $\alpha$ , which specifies how many times the angular momentum  $L$  occurs for a given seniority  $\nu$ ,  $\alpha = 1, 2, \dots, d_\ell(\nu, L)$ . The multiplicity  $d_\ell(\nu, L)$  is known as a complex integral over the characters of  $\text{SO}(2\ell + 1)$  and  $\text{SO}(3)$  [11,12],

$$d_\ell(\nu, L) = \frac{i}{2\pi} \oint_{|z|=1} \frac{(z^{2L+1} - 1)(z^{2\nu+2\ell-1} - 1) \prod_{k=1}^{2\ell-2} (z^{\nu+k} - 1)}{z^{\ell\nu+L+2} \prod_{k=1}^{2\ell-2} (z^{k+1} - 1)} dz. \tag{18}$$

For simplicity of notation, the label  $\alpha$  will be suppressed in the following if it is not needed, that is, if  $d_\ell(\nu, L) = 1$ . The label  $M$  of the final  $\text{SO}(2)$  algebra in Equation (17) is the projection of the angular momentum  $L$  on the  $z$  axis, which satisfies  $M = -L, -L + 1, \dots, +L$ . Throughout this paper rotationally invariant Hamiltonians are considered, for which all such  $M$  states have the same energy. The label  $M$  will be suppressed if it is not needed.

A general number-conserving  $k$ -body interaction between the bosons can be expressed as follows:

$$\sum_L \sum_{m\nu} \sum_{m'\nu'} v_{m\nu, m'\nu'}^{kL} (-)^L B_{km\nu L}^\dagger \cdot \tilde{B}_{km'\nu' L}, \tag{19}$$

where the operator  $B_{km\nu LM}^\dagger$  ( $B_{km\nu LM}$ ) creates (annihilates) a normalized state of  $k$  bosons according to the classification (17), with  $m$  the number of  $\ell$  bosons,  $\nu$  their seniority and  $L$  their angular momentum. Further, the dot indicates a scalar product,  $v_{m\nu, m'\nu'}^{kL}$  is a numerical coefficient that determines the strength of the interaction and  $\tilde{B}_{km\nu LM} \equiv (-)^{L-M} B_{km\nu L-M}$ . The  $s$ -boson part in the operators  $B_{km\nu LM}^\dagger$  and  $\tilde{B}_{km\nu LM}$  can easily be taken care of,

$$B_{km\nu LM}^\dagger = \frac{(s^\dagger)^{k-m}}{\sqrt{(k-m)!}} B_{m\nu LM}^\dagger, \quad \tilde{B}_{km\nu LM} = \frac{(\tilde{s})^{k-m}}{\sqrt{(k-m)!}} \tilde{B}_{m\nu LM}, \tag{20}$$

where  $B_{m\nu LM}^\dagger$  ( $\tilde{B}_{m\nu LM}$ ) creates (annihilates) a normalized state of  $m$   $\ell$  bosons.

Matrix elements of an interaction term in the expansion (19) between basis states (17) can be factored as follows:

$$\begin{aligned} & \langle Nnv\alpha J | (-)^L B_{kmvL}^\dagger \cdot \tilde{B}_{km'v'L} | Nn'v'\alpha' J \rangle \\ &= \frac{\langle s^{N-n} | (s^\dagger)^{k-m} (\tilde{s})^{k-m'} | s^{N-n'} \rangle}{\sqrt{(k-m)!(k-m')!}} \langle nv\alpha J | (-)^L B_{mvL}^\dagger \cdot \tilde{B}_{m'v'L} | n'v'\alpha' J \rangle. \end{aligned} \quad (21)$$

The  $s$ -boson matrix element in this equation follows from

$$\frac{\langle s^{n_s} | (s^\dagger)^{m_s} (\tilde{s})^{m'_s} | s^{n'_s} \rangle}{\sqrt{m_s!m'_s!}} = \left[ \begin{pmatrix} n_s \\ m_s \end{pmatrix} \begin{pmatrix} n'_s \\ m'_s \end{pmatrix} \right]^{1/2} \delta_{n_s-m_s, n'_s-m'_s} \quad (22)$$

while the expression for the  $\ell$ -boson matrix element is

$$\begin{aligned} & \langle nv\alpha J | (-)^L B_{mvL}^\dagger \cdot \tilde{B}_{m'v'L} | n'v'\alpha' J \rangle = \left[ \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n' \\ m' \end{pmatrix} \right]^{1/2} \sum_{n''v''\alpha''L''} \delta_{n'',n-m} \delta_{n'',n'-m'} \\ & \times [\ell^m(vL)\ell^{n''}(v''\alpha''L'') | \ell^n v\alpha J] [\ell^{m'}(v'L)\ell^{n''}(v''\alpha''L'') | \ell^{n'} v'\alpha' J], \end{aligned} \quad (23)$$

in terms of the  $m \times n'' \rightarrow n$  and  $m' \times n'' \rightarrow n'$  CFPs, which are known from Equation (15). Note that the multiplicity label  $\alpha$  is included in the states with  $n$ ,  $n'$  and  $n''$  bosons, where it is frequently needed, but not in the configuration that specifies the interaction, where it is usually not required. For example, in  $sd$ -IBM no multiplicity label is needed up to and including five-body interactions.

Matrix elements of non-scalar operators that do not necessarily conserve boson number can be derived in a similar fashion. One finds

$$\begin{aligned} & \langle Nnv\alpha J | (B_{kmvL}^\dagger \times \tilde{B}_{k'm'v'L'})^{(R)} | N'n'v'\alpha' J' \rangle \\ &= \frac{\langle s^{N-n} | (s^\dagger)^{k-m} (\tilde{s})^{k'-m'} | s^{N'-n'} \rangle}{\sqrt{(k-m)!(k'-m')!}} \langle nv\alpha J | (B_{mvL}^\dagger \times \tilde{B}_{m'v'L'})^{(R)} | n'v'\alpha' J' \rangle. \end{aligned} \quad (24)$$

The  $s$ -boson matrix element is given by Equation (22) while for the  $\ell$ -boson matrix element one obtains

$$\begin{aligned} & \langle nv\alpha J | (B_{mvL}^\dagger \times \tilde{B}_{m'v'L'})^{(R)} | n'v'\alpha' J' \rangle \\ &= [J][R][J'] \left[ \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n' \\ m' \end{pmatrix} \right]^{1/2} \sum_{n''v''\alpha''J''} (-)^{J+R+L'+J''} \left\{ \begin{matrix} L & L' & R \\ J & J & J'' \end{matrix} \right\} \delta_{n'',n-m} \delta_{n'',n'-m'} \\ & \times [\ell^m(vL)\ell^{n''}(v''\alpha''J'') | \ell^n v\alpha J] [\ell^{m'}(v'L)\ell^{n''}(v''\alpha''J'') | \ell^{n'} v'\alpha' J']. \end{aligned} \quad (25)$$

The doubled-barred matrix elements in Equations (24) and (25) are reduced in angular momentum by virtue of the Wigner–Eckart theorem, defined according to the convention of Refs. [5,6].

Equations (21)–(25) are closed expressions for matrix elements between many-boson states in  $s\ell$ -IBM of a  $k$ -body interaction and of a non-scalar operator of arbitrary order. While there is, in principle, no restriction on the order  $k$  of the interaction or on the number of bosons  $N$ , the computational cost for calculating the  $k \times r \rightarrow n$  CFPs increases with  $k$  and  $N$  since these are obtained recursively.

#### 4. The $s\ell_1\ell_2$ -IBM

This section deals with a system consisting of  $N$  bosons that occupy an  $s$  level and two additional levels with angular momenta  $\ell_1$  and  $\ell_2$ . The most relevant application of this case is  $sdg$ -IBM [13] with  $s$ ,  $d$  and  $g$  bosons, that is, if  $\ell_1 = 2$  and  $\ell_2 = 4$ . Throughout

this section it is assumed that labels with index ‘1’ act on the  $\ell_1$  bosons while those with index ‘2’ act on the  $\ell_2$  bosons.

States with  $N$  bosons in  $sl_1\ell_2$ -IBM can be defined in the basis

$$\begin{array}{ccccccc} U(\Omega_{12} + 1) & \supset & U(\Omega_1) & \otimes & U(\Omega_2) & \supset & \cdots \supset SO_{12}(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & n_1 & & n_2 & & L \end{array}, \tag{26}$$

where  $\Omega_i = 2\ell_i + 1$  and  $\Omega_{12} = \Omega_1 + \Omega_2$ . The number of  $\ell_i$  bosons is  $n_i$  and their sum is  $N$ , that is,  $(n_1, n_2) = (N, 0), (N - 1, 1), \dots, (0, N)$ . The intermediate classification in Equation (26) for each unitary algebra  $U(\Omega_i)$  reads

$$\begin{array}{ccccc} U(\Omega_i) & \supset & SO(\Omega_i) & \supset & SO_i(3) \\ \downarrow & & \downarrow & & \downarrow \\ n_i & & v_i & & \alpha_i L_i \end{array}, \tag{27}$$

with labels as defined in the previous section. It is convenient to introduce at this point the short-hand notation  $\Gamma_i$  for the set of labels  $\{n_i v_i \alpha_i J_i\}$  associated with a state and  $\Lambda_i$  for labels  $\{m_i v_i L_i\}$  associated with an operator.

The generalisation of Eq. (19) from  $sl$ -IBM to  $sl_1\ell_2$ -IBM leads to the following expression for a number-conserving  $k$ -body interaction:

$$\sum_{LM} \sum_{\Lambda_1 \Lambda_2} \sum_{\Lambda'_1 \Lambda'_2} v_{\Lambda_1 \Lambda_2, \Lambda'_1 \Lambda'_2}^{kL} \frac{(s^\dagger)^{k-m} (\tilde{s})^{k-m'}}{\sqrt{(k-m)!(k-m')!}} (B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)_M^{(L)} (B_{\Lambda'_1}^\dagger \times B_{\Lambda'_2}^\dagger)_M^{(L)\dagger}, \tag{28}$$

with  $m \equiv m_1 + m_2$  and  $m' \equiv m'_1 + m'_2$ . With the result

$$(B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)_M^{(L)\dagger} = (-)^{L-M} (\tilde{B}_{\Lambda_1} \times \tilde{B}_{\Lambda_2})_{-M}^{(L)}, \tag{29}$$

the interaction can be rewritten as

$$\sum_L \sum_{\Lambda_1 \Lambda_2} \sum_{\Lambda'_1 \Lambda'_2} v_{\Lambda_1 \Lambda_2, \Lambda'_1 \Lambda'_2}^{kL} \frac{(s^\dagger)^{k-m} (\tilde{s})^{k-m'}}{\sqrt{(k-m)!(k-m')!}} (-)^L B_{\Lambda_1 \Lambda_2 L}^\dagger \cdot \tilde{B}_{\Lambda'_1 \Lambda'_2 L}, \tag{30}$$

where the  $B$  operators are defined as

$$B_{\Lambda_1 \Lambda_2 LM}^\dagger \equiv (B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)_M^{(L)}, \quad \tilde{B}_{\Lambda_1 \Lambda_2 LM} \equiv (\tilde{B}_{\Lambda_1} \times \tilde{B}_{\Lambda_2})_M^{(L)}. \tag{31}$$

Matrix elements of a single interaction term in the expansion (30), that is, of the operator

$$\frac{(s^\dagger)^{k-m} (\tilde{s})^{k-m'}}{\sqrt{(k-m)!(k-m')!}} (-)^L B_{\Lambda_1 \Lambda_2 L}^\dagger \cdot \tilde{B}_{\Lambda'_1 \Lambda'_2 L}, \tag{32}$$

between the basis states (26), (27) can be written as the product of two factors. The first one equals

$$\frac{\langle s^{N-n} | (s^\dagger)^{k-m} (\tilde{s})^{k-m'} | s^{N-n'} \rangle}{\sqrt{(k-m)!(k-m')!}}, \tag{33}$$

with  $n \equiv n_1 + n_2$  and  $n' \equiv n'_1 + n'_2$ . This factor is known from Equation (22) while the second is given by

$$\begin{aligned} &\langle N\Gamma_1\Gamma_2; J | (-)^L B_{\Lambda_1\Lambda_2L}^\dagger \cdot \tilde{B}_{\Lambda'_1\Lambda'_2L} | N\Gamma'_1\Gamma'_2; J \rangle \\ &= (-)^{J'_1+J_2+L'_1+L_2+L+J} (2L+1) \sum_R \left\{ \begin{matrix} L_1 & L_2 & L \\ L'_2 & L'_1 & R \end{matrix} \right\} \left\{ \begin{matrix} J_1 & J_2 & J \\ J'_2 & J'_1 & R \end{matrix} \right\} \\ &\times \langle \Gamma_1 || (B_{\Lambda_1}^\dagger \times \tilde{B}_{\Lambda'_1})^{(R)} || \Gamma'_1 \rangle \langle \Gamma_2 || (B_{\Lambda_2}^\dagger \times \tilde{B}_{\Lambda'_2})^{(R)} || \Gamma'_2 \rangle. \end{aligned} \tag{34}$$

The reduced matrix elements on the right-hand side of Equation (34) are known from Equation (25).

Matrix elements of a non-scalar operator can be dealt with in a similar fashion. One introduces the  $B$  operators

$$\begin{aligned} B_{k\Lambda_1\Lambda_2LM}^\dagger &\equiv \frac{(s^\dagger)^{k-m}}{\sqrt{(k-m)!}} (B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)_M^{(L)} = \frac{(s^\dagger)^{k-m}}{\sqrt{(k-m)!}} B_{\Lambda_1\Lambda_2LM}^\dagger \\ \tilde{B}_{k\Lambda_1\Lambda_2LM} &\equiv \frac{(\tilde{s})^{k-m}}{\sqrt{(k-m)!}} (\tilde{B}_{\Lambda_1} \times \tilde{B}_{\Lambda_2})_M^{(L)} = \frac{(\tilde{s})^{k-m}}{\sqrt{(k-m)!}} \tilde{B}_{\Lambda_1\Lambda_2LM}, \end{aligned} \tag{35}$$

and calculates the matrix element of a general non-scalar operator as

$$\begin{aligned} &\langle N\Gamma_1\Gamma_2; J || (B_{k\Lambda_1\Lambda_2L}^\dagger \times \tilde{B}_{k'\Lambda'_1\Lambda'_2L'})^{(R)} || N'\Gamma'_1\Gamma'_2; J' \rangle \\ &= \frac{\langle s^{N-n} | (s^\dagger)^{k-m} (\tilde{s})^{k'-m'} | s^{N'-n'} \rangle}{\sqrt{(k-m)!(k'-m')!}} \langle \Gamma_1\Gamma_2; J || (B_{\Lambda_1\Lambda_2L}^\dagger \times \tilde{B}_{\Lambda'_1\Lambda'_2L'})^{(R)} || \Gamma'_1\Gamma'_2; J' \rangle. \end{aligned} \tag{36}$$

The  $s$ -boson matrix element is given by Equation (33) while for the  $\ell_1\ell_2$ -boson matrix element one obtains

$$\begin{aligned} &\langle \Gamma_1\Gamma_2; J || (B_{\Lambda_1\Lambda_2L}^\dagger \times \tilde{B}_{\Lambda'_1\Lambda'_2L'})^{(R)} || \Gamma'_1\Gamma'_2; J' \rangle \\ &= [L][J][R][L'][J'] \sum_{R_1R_2} [R_1][R_2] \left\{ \begin{matrix} L_1 & L_2 & L \\ L'_1 & L'_2 & L' \\ R_1 & R_2 & R \end{matrix} \right\} \left\{ \begin{matrix} J_1 & J_2 & J \\ J'_1 & J'_2 & J' \\ R_1 & R_2 & R \end{matrix} \right\} \\ &\times \langle \Gamma_1 || (B_{\Lambda_1}^\dagger \times \tilde{B}_{\Lambda'_1})^{(R_1)} || \Gamma'_1 \rangle \langle \Gamma_2 || (B_{\Lambda_2}^\dagger \times \tilde{B}_{\Lambda'_2})^{(R_2)} || \Gamma'_2 \rangle, \end{aligned} \tag{37}$$

where the symbols in curly brackets are  $9j$  symbols [5,6].

Equations (33), (34), (36) and (37) are closed expressions for matrix elements between many-boson states in  $s\ell_1\ell_2$ -IBM of a  $k$ -body interaction and of a non-scalar operator of arbitrary order. As a reminder, in these equations  $\Gamma_i$  stands for the labels  $\{n_i\nu_i\alpha_iJ_i\}$  of a state and  $\Lambda_i$  for the labels  $\{m_i\nu_iL_i\}$  of an operator, both associated with the boson with angular momentum  $\ell_i$ .

### 5. The $s\ell_1 \dots \ell_p$ -IBM

As the number of the different kinds of bosons increases the angular-momentum recoupling becomes more complicated and the resulting expressions increasingly cumbersome. Instead of giving closed expressions in terms of  $3nj$  symbols, it is more appropriate to define a recursive procedure, reducing a calculation in  $s\ell_1 \dots \ell_p$ -IBM to one in  $s\ell_1 \dots \ell_{p-1}$ -IBM, as will be explained in this section.

Basis states in  $s\ell_1 \dots \ell_p$ -IBM can be defined recursively according to

$$\begin{array}{ccccccc}
 U(\Omega_{1\dots p} + 1) & \supset & U(\Omega_{1\dots p-1}) & \otimes & U(\Omega_p) & \supset & \dots \supset SO_{1\dots p}(3) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 [N] & & N - n_p & & n_p & & L
 \end{array} , \tag{38}$$

where  $\Omega_i = 2\ell_i + 1$  and  $\Omega_{1\dots r} = \Omega_1 + \dots + \Omega_r$ . The intermediate algebra  $U(\Omega_{1\dots p-1})$  can be treated in the same fashion, ending in  $SO_{1\dots p-1}(3)$ , which is coupled with  $SO_p(3)$  to the total angular momentum algebra  $SO_{1\dots p}(3)$ . According to this classification an operator can be defined as follows:

$$(\dots ((B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)^{(I_2)} \times B_{\Lambda_3}^\dagger)^{(I_3)} \times \dots \times B_{\Lambda_p}^\dagger)_M^{(L)} , \tag{39}$$

where  $\Lambda_i$  stands for the labels  $\{m_i v_i L_i\}$  and  $I_i$  are intermediate angular momenta, resulting from the coupling of  $I_{i-1}$  and  $L_i$ , in the convention that  $I_1 = L_1$  and that  $I_p = L$  is the total angular momentum of the operator. With the help of Equation (29) it can be shown by induction that the Hermitian conjugate of the operator (39) is

$$\begin{aligned}
 & (\dots ((B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)^{(I_2)} \times B_{\Lambda_3}^\dagger)^{(I_3)} \times \dots \times B_{\Lambda_p}^\dagger)_M^{(L)\dagger} \\
 & = (-)^{L-M} (\dots ((\tilde{B}_{\Lambda_1} \times \tilde{B}_{\Lambda_2})^{(I_2)} \times \tilde{B}_{\Lambda_3})^{(I_3)} \times \dots \times \tilde{B}_{\Lambda_p})_{-M}^{(L)} .
 \end{aligned} \tag{40}$$

Therefore, a general representation of a  $k$ -body interaction in  $s\ell_1 \dots \ell_p$ -IBM is

$$\sum_L \sum_{\bar{\Lambda} \bar{I}} \sum_{\bar{\Lambda}' \bar{I}'} v_{\bar{\Lambda} \bar{I}, \bar{\Lambda}' \bar{I}'}^{kL} \frac{(s^\dagger)^{k-m} (\tilde{s})^{k-m'}}{\sqrt{(k-m)!(k-m')!}} (-)^L B_{\{\bar{\Lambda} \bar{I}\}_p L}^\dagger \cdot \tilde{B}_{\{\bar{\Lambda}' \bar{I}'\}_p L} , \tag{41}$$

where  $k - m$  ( $k - m'$ ) is the number of  $s$ -boson creation (annihilation) operators, which implies  $m \equiv \sum_i m_i$  and  $m' \equiv \sum_i m'_i$ , where the summation is over all  $\ell_i$  bosons,  $i = 1, \dots, p$ . The  $B$  operators are defined as

$$\begin{aligned}
 B_{\{\bar{\Lambda} \bar{I}\}_p LM}^\dagger & \equiv (\dots ((B_{\Lambda_1}^\dagger \times B_{\Lambda_2}^\dagger)^{(I_2)} \times B_{\Lambda_3}^\dagger)^{(I_3)} \times \dots \times B_{\Lambda_p}^\dagger)_M^{(L)} , \\
 \tilde{B}_{\{\bar{\Lambda} \bar{I}\}_p LM} & \equiv (\dots ((\tilde{B}_{\Lambda_1} \times \tilde{B}_{\Lambda_2})^{(I_2)} \times \tilde{B}_{\Lambda_3})^{(I_3)} \times \dots \times \tilde{B}_{\Lambda_p})_M^{(L)} .
 \end{aligned} \tag{42}$$

The labels  $\{\Lambda_1, \dots, \Lambda_p\}$  and  $\{I_1, \dots, I_p\}$  are collectively denoted as  $\{\bar{\Lambda} \bar{I}\}_p$ . As the proposed method is recursive in  $p$ , it is necessary to indicate the number of labels in the sets  $\bar{\Lambda}$  and  $\bar{I}$ . In the interaction matrix elements  $v_{\bar{\Lambda} \bar{I}, \bar{\Lambda}' \bar{I}' }^{kL}$  this index can be dropped since they are defined in  $s\ell_1 \dots \ell_p$ -IBM for a given, fixed  $p$ .

It is straightforward to deal with the  $s$ -boson dependence of any matrix element, which is contained in the factor

$$\frac{\langle s^{N-n} | (s^\dagger)^{k-m} (\tilde{s})^{k'-m'} | s^{N'-n'} \rangle}{\sqrt{(k-m)!(k'-m')!}} , \tag{43}$$

with  $n \equiv \sum_i n_i$  and  $n' \equiv \sum_i n'_i$ . The expression (22) can be used for a general non-scalar operator, with possibly  $N \neq N'$  and  $k \neq k'$ , and for a  $k$ -body interaction, for which  $N = N'$  and  $k = k'$ .

After factoring out the  $s$ -boson dependence, the remaining matrix element is defined in  $\ell_1 \dots \ell_p$ -IBM. A basis state of  $\ell_1 \dots \ell_p$ -IBM can be written as

$$|(\dots ((\Gamma_1 \times \Gamma_2)^{(K_2)} \times \Gamma_3)^{(K_3)} \times \dots \times \Gamma_p)_M^{(J)} \rangle \equiv |\{\bar{\Gamma} \bar{K}\}_p; JM\rangle , \tag{44}$$

where  $\Gamma_i$  stands for the labels  $\{n_i v_i \alpha_i J_i\}$ . Akin to the notation for the operators, the labels  $\{\Gamma_1, \dots, \Gamma_p\}$  and  $\{K_1, \dots, K_p\}$  are collectively denoted as  $\{\bar{\Gamma}\bar{K}\}_p$ . The intermediate angular momenta  $K_i$  result from the coupling of  $K_{i-1}$  with  $J_i$ , in the convention that  $K_1 = J_1$  and that  $K_p = J$  is the total angular momentum of the state. The task is therefore to compute

$$\langle \{\bar{\Gamma}\bar{K}\}_p; J = K_p | (-)^L B_{\{\bar{\Lambda}\bar{I}\}_p}^\dagger \cdot \tilde{B}_{\{\bar{\Lambda}'\bar{I}'\}_p} | \{\bar{\Gamma}'\bar{K}'\}_p; J = K'_p \rangle, \quad (45)$$

which is a generic matrix element of an interaction term in the expansion (41) between basis states (44) of  $\ell_1 \dots \ell_p$ -IBM. With use of the operator representation (39) this matrix element can be rewritten as

$$\begin{aligned} & \langle \{\bar{\Gamma}\bar{K}\}_p | (-)^L B_{\{\bar{\Lambda}\bar{I}\}_p}^\dagger \cdot \tilde{B}_{\{\bar{\Lambda}'\bar{I}'\}_p} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle \\ &= (-)^{K'_{p-1} + J_p + I'_{p-1} + L_p + L} (2L + 1) \sum_R \left\{ \begin{matrix} I_{p-1} & L_p & I_p \\ L'_p & I'_{p-1} & R \end{matrix} \right\} \left\{ \begin{matrix} K_{p-1} & J_p & K_p \\ J'_p & K'_{p-1} & R \end{matrix} \right\} \\ & \times \langle \{\bar{\Gamma}\bar{K}\}_{p-1} | (B_{\{\bar{\Lambda}\bar{I}\}_{p-1}}^\dagger \times \tilde{B}_{\{\bar{\Lambda}'\bar{I}'\}_{p-1}})^{(R)} | \{\bar{\Gamma}'\bar{K}'\}_{p-1} \rangle \langle \Gamma_p | (B_{\Lambda_p}^\dagger \times \tilde{B}_{\Lambda'_p})^{(R)} | \Gamma'_p \rangle. \end{aligned} \quad (46)$$

The interaction matrix element is expressed as a sum of products of two reduced matrix elements, the last one of which is

$$\langle \Gamma_p | (B_{\Lambda_p}^\dagger \times \tilde{B}_{\Lambda'_p})^{(R)} | \Gamma'_p \rangle = \langle n_p v_p \alpha_p J_p | (B_{m_p v_p}^\dagger \times \tilde{B}_{m'_p v'_p})^{(R)} | n'_p v'_p \alpha'_p J'_p \rangle, \quad (47)$$

where, for clarity's sake, on the right-hand side all labels contained in  $\Gamma_p$ ,  $\Gamma'_p$ ,  $\Lambda_p$  and  $\Lambda'_p$  are denoted explicitly. The reduced matrix element (47) is known from Equation (25) while the first reduced matrix element in Equation (46) is defined in  $\ell_1 \dots \ell_{p-1}$ -IBM. Note however that, although one started out with the matrix element of a scalar operator, the expansion involves reduced matrix elements of non-scalar operators. The recursive procedure can be readily extended to the latter operators through the relation

$$\begin{aligned} & \langle \{\bar{\Gamma}\bar{K}\}_p | (B_{\{\bar{\Lambda}\bar{I}\}_p}^\dagger \times \tilde{B}_{\{\bar{\Lambda}'\bar{I}'\}_p})^{(R)} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle \\ &= [I_p][K_p][R][I'_p][K'_p] \sum_{R_{p-1} R_p} [R_{p-1}][R_p] \left\{ \begin{matrix} I_{p-1} & L_p & I_p \\ I'_{p-1} & L'_p & I'_p \\ R_{p-1} & R_p & R \end{matrix} \right\} \left\{ \begin{matrix} K_{p-1} & J_p & K_p \\ K'_{p-1} & J'_p & K'_p \\ R_{p-1} & R_p & R \end{matrix} \right\} \\ & \times \langle \{\bar{\Gamma}\bar{K}\}_{p-1} | (B_{\{\bar{\Lambda}\bar{I}\}_{p-1}}^\dagger \times \tilde{B}_{\{\bar{\Lambda}'\bar{I}'\}_{p-1}})^{(R_{p-1})} | \{\bar{\Gamma}'\bar{K}'\}_{p-1} \rangle \langle \Gamma_p | (B_{\Lambda_p}^\dagger \times \tilde{B}_{\Lambda'_p})^{(R_p)} | \Gamma'_p \rangle. \end{aligned} \quad (48)$$

This result is the generalization of Equation (37) from  $p = 2$  to arbitrary  $p$ . The total angular momenta  $J$ ,  $J'$ ,  $L$  and  $L'$  of the states in bra and ket and of the  $B^\dagger$  and  $\tilde{B}$  operators coincide with the final angular momenta in the series  $\bar{\Gamma}$ ,  $\bar{K}$ ,  $\bar{\Lambda}$  and  $\bar{I}$ , respectively, which implies the equivalences  $J = K_p$ ,  $J' = K'_p$ ,  $L = I_p$  and  $L' = I'_p$ .

## 6. Multipole Interaction to Any Order

The previous sections described the evaluation of matrix elements in  $\ell_1 \dots \ell_p$ -IBM if the Hamiltonian and other operators are specified in normal-ordered form, that is, with all creation operators on the left and all annihilation operators on the right. Alternatively, operators can be written in multipole form in terms of bilinear operators

$$\hat{T}_\mu^\lambda = \sum_{\ell\ell'} t_{\ell\ell'}^\lambda (b_\ell^\dagger \times \tilde{b}_{\ell'})_\mu^{(\lambda)}, \quad (49)$$

where the summation is over the bosons in the model and  $t_{\ell\ell'}^\lambda$  are pre-defined coefficients. For example, the quadrupole operator  $\hat{Q}_\mu$  in the SU(3) limit of  $sd$ -IBM [14] is defined with  $t_{02}^2 = t_{20}^2 = 1$  and  $t_{22}^2 = \pm\sqrt{7}/2$ . If several operators occur with the same angular momentum, they can be distinguished with an additional index  $r$  in  $t_{\ell\ell'}^{\lambda r}$ .

A Hamiltonian with up to  $k$ -body interactions in the bosons can be defined in terms of a multipole expansion of the form

$$(\dots (\hat{T}^{\lambda_1} \times \hat{T}^{\lambda_2})^{(\Lambda_2)} \times \hat{T}^{\lambda_3})^{(\Lambda_3)} \times \dots \times \hat{T}^{\lambda_k})^{(\Lambda_k)}, \quad (50)$$

where for a scalar operator  $\Lambda_k = 0$ . The bosons in the operator (49) can be scalar ( $\ell = 0$  and/or  $\ell' = 0$ ) and, unlike for an operator in normal-ordered representation, the  $s$ -boson dependence of a multipole operator cannot be simply factored out. It is therefore necessary to assume in this section that the angular momenta  $\ell_i$  can take on arbitrary values, including zero.

The task is therefore to evaluate matrix elements of the operator (50) between basis states of  $\ell_1 \dots \ell_p$ -IBM, which, following the notation of the previous section, can be written as  $|\{\bar{\Gamma}\bar{K}\}_p\rangle$ , where  $\bar{\Gamma}$  and  $\bar{K}$  are collective notations that denote  $\{\Gamma_1, \dots, \Gamma_p\}$  and the intermediate angular momenta  $\{K_1, \dots, K_p\}$ , respectively, with  $\Gamma_i$  being the labels  $\{n_i v_i \alpha_i J_i\}$  associated with the  $\ell_i$  boson. The reduced matrix element of an operator in multipole form between basis states  $|\{\bar{\Gamma}\bar{K}\}_p\rangle$  can be calculated recursively from

$$\begin{aligned} & \langle \{\bar{\Gamma}\bar{K}\}_p | (\dots (\hat{T}^{\lambda_1} \times \hat{T}^{\lambda_2})^{(\Lambda_2)} \times \hat{T}^{\lambda_3})^{(\Lambda_3)} \times \dots \times \hat{T}^{\lambda_k})^{(\Lambda_k)} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle \\ &= (-)^{K_p + K'_p + \Lambda_k} [\Lambda_k] \sum_{\{\bar{\Gamma}''\bar{K}''\}_p} \begin{Bmatrix} \Lambda_{k-1} & \lambda_k & \Lambda_k \\ K'_p & K_p & K''_p \end{Bmatrix} \langle \{\bar{\Gamma}''\bar{K}''\}_p | \hat{T}^{\lambda_k} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle \\ & \times \langle \{\bar{\Gamma}\bar{K}\}_p | (\dots (\hat{T}^{\lambda_1} \times \hat{T}^{\lambda_2})^{(\Lambda_2)} \times \hat{T}^{\lambda_3})^{(\Lambda_3)} \times \dots \times \hat{T}^{\lambda_{k-1}})^{(\Lambda_{k-1})} | \{\bar{\Gamma}''\bar{K}''\}_p \rangle, \quad (51) \end{aligned}$$

where it is assumed that, in the final application of this recurrence relation for  $k = 2$ ,  $\Lambda_1 = \lambda_1$ . This relation is recurrent in  $k$ , the number of bilinear operators in the multipole expansion (50), and all matrix elements in Equation (51) are defined in  $\ell_1 \dots \ell_p$ -IBM. The angular momenta  $K_p$ ,  $K'_p$  and  $K''_p$  are the final entries in the series  $\bar{K}$ ,  $\bar{K}'$  and  $\bar{K}''$ , and correspond to the total angular momenta of the bra, ket and intermediate state, that is,  $J = K_p$ ,  $J' = K'_p$  and  $J'' = K''_p$ . The recurrence relation (51) shows that the reduced matrix element of a multipole operator of order  $k$  can be converted into a linear combination of products of  $k$  reduced matrix elements of the type  $\langle \{\bar{\Gamma}\bar{K}\}_p | \hat{T}^{\lambda_i} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle$ . Although the operator  $\hat{T}^{\lambda_i}$  refers to only two bosons with angular momenta  $\ell_i$  and  $\ell'_i$  (which may be zero), the states in bra and ket of the matrix element  $\langle \{\bar{\Gamma}\bar{K}\}_p | \hat{T}^{\lambda_i} | \{\bar{\Gamma}'\bar{K}'\}_p \rangle$  are defined in  $\ell_1 \dots \ell_p$ -IBM. The latter matrix element can be evaluated with the help of Equation (48) with the substitutions  $B_{\{\bar{\Lambda}\bar{I}\}_p}^\dagger \rightarrow b_{\ell_i}^\dagger$  and  $\bar{B}_{\{\bar{\Lambda}'\bar{I}'\}_p} \rightarrow \bar{b}_{\ell'_i}$ .

## 7. From Multipole to Normal-Ordered Interactions

The previous section specified the algorithm to calculate matrix elements in  $\ell_1 \dots \ell_p$ -IBM of a Hamiltonian in multipole form. The algorithm is doubly recursive. First, it reduces a matrix element of a product of  $k$  bilinear operators  $\hat{T}_\mu^\lambda$  to a linear combination of matrix elements of products of  $k - 1$  bilinear operators and, second, it reduces a matrix element of  $\hat{T}_\mu^\lambda$  in  $\ell_1 \dots \ell_p$ -IBM to a linear combination of products of a matrix element in  $\ell_1 \dots \ell_{p-1}$ -IBM times one in  $\ell_p$ -IBM. A more efficient algorithm can be defined by converting the multipole Hamiltonian from the start to its normal-ordered representation.

To achieve the conversion from multipole to normal-ordered form, one calculates all possible matrix elements in the basis (38), diagonal and off-diagonal, for boson numbers  $N = 0, 1, \dots, k_{\max}$ , where  $k_{\max}$  is the maximum order of the multipole expansion. With use of Equation (46) the matrix elements can be expressed analytically in terms of the  $k$ -body interactions  $v_{\bar{\Lambda}\bar{I}, \bar{\Lambda}'\bar{I}'}^{kL}$ . Alternatively, with use of Equation (51), one calculates the same set of matrix elements for the multipole Hamiltonian and equates them to the cor-

responding expressions for the normal-ordered Hamiltonian. The resulting set of linear equations can be solved and gives the interactions  $v_{\Lambda\bar{I},\Lambda'\bar{I}'}^{kL}$  in terms of the parameters of the multipole Hamiltonian.

This procedure makes use of the generic expressions of the matrix elements of a Hamiltonian in its normal-ordered and multipole representations but these are only needed for boson numbers up to and including  $k_{\max}$ . Once the conversion to the Hamiltonian's normal-ordered representation is obtained, the latter can be used for an  $N$ -boson calculation. Equations (48) and (51) provide entirely different recursive schemes to calculate matrix elements in  $\ell_1 \dots \ell_p$ -IBM and therefore the conversion from one to the other representation also provides a rigorous test of the present formalism and its numerical implementation discussed in the next section.

## 8. Numerical Implementation

Several computer codes are available to solve the secular equation of a Hamiltonian matrix for a system of interacting bosons. For applications to nuclei in the framework of the IBM the first such code, named `phint`, was written by Olaf Scholten [15,16]. It solves numerically the eigenvalue problem of a Hamiltonian in  $sd$ -IBM with single-boson energies and two-boson interactions. This code has been the basis for numerous extensions. For example, the version `ibm.f` [17] includes all three-body interactions between  $s$  and  $d$  bosons and incorporates mixing between configurations with different total boson number  $N$  (for an application, see Ref. [18]). A more versatile interacting boson code, named `ArbModel`, was developed by Stefan Heinze [19]. It can deal with bosons with arbitrary angular momentum and CFPs are calculated as the square root of rational numbers, which prevents the loss of precision for large  $N$ . The interactions in `ArbModel` are limited to two-body at most and the Hamiltonian matrix is constructed numerically.

It is worthwhile to compare the available nuclear-physics codes for interacting bosons with those for fermions, that is, with shell-model codes. There exists a panoply of the latter. The most recent ones, still in use, are `nushellx` [20,21], `antoine` [22], `kshell` [23,24] and `bigstick` [25,26]. They can handle a large number of single-particle orbitals (corresponding to  $p$  in the present formalism) but are generally restricted to two-body interactions, which may be made nucleon-number dependent to approximate microscopically calculated three-nucleon interactions. Dimensions of the Hamiltonian matrix in the shell model typically are much larger than those in boson models. For this reason all current shell-model codes are written on an  $m$ -scheme basis, such that matrix elements must not be stored but can be calculated on the fly. To the best of the author's knowledge, there exists no publicly available shell-model code that is able to express symbolically matrix elements in a rotationally invariant  $N$ -nucleon basis in terms of interaction matrix elements of arbitrary order  $k$ .

The formalism presented in this paper has been implemented in a Mathematica code named `ibm.m` [27]. It accepts  $p$  different kinds of bosons with arbitrary angular momenta  $\ell_1, \ell_2, \dots, \ell_p$ , with an interaction of arbitrary order  $k$ . In addition, the Hamiltonian matrix for  $N$  bosons is constructed symbolically; that is, all its elements are known analytically as a linear combination of the  $k$ -body interaction matrix elements.

While the formalism is valid for arbitrary  $p$ ,  $k$  and  $N$ , it is clear that a numerical implementation imposes limitations on their values. In all boson models considered to date,  $p$  is rather low: the most elaborate model in this respect is the  $sdpf$ -IBM, for which  $p = 3$  since the  $s$  boson can be eliminated before the recursive procedure is started. Likewise, in all reasonable applications, the order  $k$  of the interaction is unlikely to cause numerical problems. For  $sd$ -IBM, an example with  $k_{\max} = 4$  is discussed in Section 9, but the order can be further increased to  $k = 5$  or even  $k = 6$ , in which case there are 41 and

176 Hermitian interaction matrix elements, respectively. In more elaborate models, like *sdg*-IBM or *spdf*-IBM, the number of interaction matrix elements steeply increases with  $k$ . For example, in *sdg*-IBM there are 32 two-body Hermitian matrix elements and this number increases to 324 and 3425 for  $k = 3$  and  $k = 4$ , respectively. The corresponding numbers in *spdf*-IBM are 66,976 and 13,038, if parity conservation is imposed. While some method is needed to determine the numerical values of such large sets of interaction matrix elements, either from microscopic considerations or from a physics-based expansion of a Hamiltonian in multipole form, the code `ibm.m` allows the construction of matrix elements in an  $N$ -boson basis for all such values of  $k$ , provided  $N$  is not too large.

It can be concluded, therefore, that neither  $p$  nor  $k$  represents a limiting factor in the numerical implementation of the present formalism. The combination of a high boson angular momentum  $\ell$  and a large boson number  $N$ , however, does lead to numerical difficulties for the following reason. The algorithm outlined in Section 2 calculates a CFP as the square root of a rational number and loss of precision is therefore not an issue. The number of states in the Gram–Schmidt orthonormalization procedure, however, constitutes a limitation of the method. This number is given by the multiplicity  $d_\ell(v, L)$  of Equation (18). For  $\ell = 2$ , as needed in *sd*-IBM, the number is small for all applications to nuclei; for example,  $d_2(v, L) \leq 4$  for  $v \leq 20$  and arbitrary  $L$ . The multiplicity increases dramatically with  $\ell$ ; for example, for a  $g$  boson,  $d_4(v, L)$  increases to 50 for  $v = 10$  and to 778 for  $v = 20$ . The latter number certainly cannot be handled by the Gram–Schmidt orthonormalization procedure of Equations (3)–(6) and therefore an alternative, hitherto unknown, algorithm for the construction of CFPs must be devised. With the current algorithm calculations in *sdg*-IBM, they can be carried out by restricting the number of  $g$  bosons to a lower value than the total boson number  $N$ .

## 9. Examples

### 9.1. The Classical Limit of the *sd*-IBM

Geometric insight into algebraic models can be obtained from their classical limit (i.e., the limit of large boson number  $N$ ) by means of a coherent state [28–30]. The topic of this subsection is the classical limit of *sd*-IBM with the coherent state

$$|N; \beta\gamma\rangle = \frac{1}{\sqrt{N!(1+\beta^2)^N}} \left( s^\dagger + \beta [\cos\gamma d_0^\dagger + \sqrt{\frac{1}{2}} \sin\gamma (d_{-2}^\dagger + d_{+2}^\dagger)] \right)^N |0\rangle, \quad (52)$$

where  $|0\rangle$  is the boson vacuum and  $\beta$  and  $\gamma$  are the quadrupole shape parameters. The calculation of the Hamiltonian's expectation value in this state leads to a function of  $N$ ,  $\beta$  and  $\gamma$ , which has the interpretation of an energy surface in the shape variables and depends on the interaction matrix elements  $v_{nv,n'v'}^{kL}$  defined in Equation (19). More generally, the coherent-state formalism can also be used to calculate off-diagonal matrix elements, which have the interpretation of interactions between intrinsic states. This requires the generalized definition

$$|N; \vec{c}\rangle = \frac{1}{\sqrt{N!(1+c^2)^N}} (B_c^\dagger)^N |0\rangle, \quad B_c^\dagger = s^\dagger + \sum_{\mu=-2}^{+2} c_\mu d_\mu^\dagger, \quad (53)$$

where  $c^2 = \sum_\mu c_\mu^* c_\mu$ . With  $c_0 = \beta \cos\gamma$ ,  $c_{\pm 1} = 0$  and  $c_{\pm 2} = \frac{1}{\sqrt{2}} \beta \sin\gamma$ , the coherent state (52) is obtained but the definition (53) covers other cases as needed, for example, in the calculation of the matrix element between the coherent state (52) and the one obtained after rotation. The generalized matrix element is calculated to be

$$\begin{aligned} \langle N; \bar{c} | (-)^L B_{knvL}^\dagger \cdot \bar{B}_{kn'v'L} | N; \bar{c}' \rangle &= \frac{N!}{(N-k)!} \frac{(1+c^* \cdot c')^{N-k}}{\sqrt{(1+c^2)^N (1+c'^2)^N}} \frac{1}{\sqrt{(k-m)!(k-m')!}} \\ &\times \sum_M \sum_{\{\mu_i\}} \sum_{\{\mu_{i'}\}} a_{nvLM}^{\{\mu_i\}*} a_{n'v'LM}^{\{\mu_{i'}\}} \prod_{i=1}^m c_{\mu_i}^* \prod_{i'=1}^{m'} c_{\mu_{i'}}. \end{aligned} \tag{54}$$

For a diagonal matrix element,  $\bar{c} = \bar{c}'$ , this expression simplifies to

$$\begin{aligned} \langle N; \bar{c} | (-)^L B_{knvL}^\dagger \cdot \bar{B}_{kn'v'L} | N; \bar{c} \rangle &= \frac{N!}{(N-k)!} \frac{1}{(1+c^2)^k} \frac{1}{\sqrt{(k-m)!(k-m')!}} \sum_M \sum_{\{\mu_i\}} \sum_{\{\mu_{i'}\}} a_{nvLM}^{\{\mu_i\}*} a_{n'v'LM}^{\{\mu_{i'}\}} \prod_{i=1}^m c_{\mu_i}^* \prod_{i'=1}^{m'} c_{\mu_{i'}}. \end{aligned} \tag{55}$$

Use is made of the expansion of  $B_{nvLM}^\dagger$  in terms of uncoupled  $d$ -boson creation operators,

$$B_{nvLM}^\dagger = \sum_{\{\mu_i\}} a_{nvLM}^{\{\mu_i\}} d_{\mu_1}^\dagger d_{\mu_2}^\dagger \dots d_{\mu_n}^\dagger, \tag{56}$$

where  $\{\mu_i\}$  refers to all possible sets  $\{\mu_1, \mu_2, \dots, \mu_n\}$  such that  $\mu_1 + \mu_2 + \dots + \mu_n = M$ . The expectation value can then be evaluated by noting that the action of the uncoupled creation and annihilation operators is equivalent to a derivative [31].

The coefficients  $a_{nvLM}^{\{\mu_i\}}$  can be found by diagonalizing a  $d$ -boson Hamiltonian with up to two-body interactions in the  $m$  scheme. The quantum numbers  $n, v$  and  $L$  of the eigenstates for a given value of  $M$  can be obtained from the eigenenergies since these are known analytically. The corresponding eigenvectors determine the coefficients  $a_{nvLM}^{\{\mu_i\}}$ . A difficulty arises as the signs of the coefficients with fixed  $n, v$  and  $L$  but varying  $M$  are related through a Clebsch–Gordan series whereas numerically they are not. To circumvent this problem, one calculates the coefficient  $a_{nvLM}^{\{\mu_i\}}$  for a single value of  $M$ , e.g.,  $M = -L$ , and subsequently applies the raising operator  $\hat{L}_+$  with use of the property  $\hat{L}_+ |LM\rangle = \sqrt{(L-M)(L+M+1)} \hat{L}_+ |L, M+1\rangle$ . To obtain a normalized state, one uses the operator

$$\hat{T}_+ = \frac{\hat{L}_+}{\sqrt{(L-M)(L+M+1)}} = \sum_{m=-\ell}^{\ell-1} r_m b_{\ell, m+1}^\dagger b_{\ell m}, \tag{57}$$

where

$$r_m = (-)^{\ell+m+1} \left[ \frac{2\ell(\ell+1)(2\ell+1)}{3(L-M)(L+M+1)} \right]^{1/2} (\ell m + 1 \ell - m | 1 + 1), \tag{58}$$

with  $\ell = 2$  in the case of  $d$  bosons.

The algorithm outlined above can be used to determine the classical limit of a Hamiltonian in  $sd$ -IBM with up to and including five-body interactions. For interactions of higher order than five, the quantum numbers of seniority and angular momentum no longer suffice to uniquely characterize an operator  $B_{nvLM}^\dagger$  and an additional label  $\alpha$  is sometimes needed. For example, for  $v = 6$  two states occur with angular momentum  $L = 6, d_2(v = L = 6) = 2$ . A possible strategy to deal with such even more complicated interactions is to define the coefficients  $a_{nvLM}^{\{\mu_i\}}$  by diagonalizing in the  $m$  scheme a  $d$ -boson Hamiltonian that includes a three-body interaction. The most logical choice is to consider  $v_{ddd-ddd}^0$ . This particular interaction is expected to be related to  $n_\Delta$ , which is usually introduced as an additional label [2] and is associated with the number of triplets of  $d$  bosons coupled to angular momentum zero (see also the discussion in chapter 8 of Ref. [32]). In fact, it can be shown that the operator  $\bar{B}_{n=v=3, L=M=0}$  annihilates a particular combination of the two states with  $n = v = 6$  and  $L = 6$ , and this provides a possible definition of the state  $|n = v = 6, \alpha = 1, L = 6\rangle$  and therefore of the coefficients  $a_{nvLM}^{\{\mu_i\}}$ .

The energy surface of a generic  $sd$ -IBM Hamiltonian with interactions up to order  $k_{\max}$  can be written in the following manner:

$$\langle N; \beta\gamma | \hat{H}_{\text{IBM}} | N; \beta\gamma \rangle = E_0 + \sum_{k=1}^{k_{\max}} \frac{N(N-1) \cdots (N-k+1)}{(1+\beta^2)^k} \sum_{rt} a_{rt}^{(k)} \beta^{2r+3t} (\cos 3\gamma)^t, \quad (59)$$

where the second summation is over all non-negative integer values of  $r$  and  $t$  such that  $2r + 3t \leq 2k$ . The coefficients  $a_{rt}^{(k)}$  are well known (see, e.g., Ref. [31]) for  $k = 1$ ,

$$a_{00}^{(1)} = \epsilon_s, \quad a_{10}^{(1)} = \epsilon_d, \quad (60)$$

where  $\epsilon_s$  and  $\epsilon_d$  are the  $s$ - and  $d$ -boson energies, and for  $k = 2$ ,

$$\begin{aligned} a_{00}^{(2)} &= \frac{1}{2} v_{ss \cdot ss}^0, & a_{10}^{(2)} &= \frac{1}{\sqrt{5}} v_{ss \cdot dd}^0 + v_{sd \cdot sd}^2, & a_{01}^{(2)} &= -\frac{2}{\sqrt{7}} v_{sd \cdot dd}^2, \\ a_{20}^{(2)} &= \frac{1}{10} v_{dd \cdot dd}^0 + \frac{1}{7} v_{dd \cdot dd}^2 + \frac{9}{35} v_{dd \cdot dd}^4, \end{aligned} \quad (61)$$

which are written in terms of the combinations

$$v_{knv \cdot kn'v'}^L = \frac{1}{2} (v_{knv, kn'v'}^L + v_{kn'v', knv}^L) = v_{knv, kn'v'}^L, \quad (62)$$

where the last equality is valid for a Hermitian interaction.

The algorithm described in this section allows for the systematic calculation of the coefficients  $a_{rt}^{(k)}$  for  $k > 2$  and is part of the code `ibm.m` [27]. The coefficients for  $k = 3$  and 4 are given in the Appendix A. From this extended analysis the following conclusions can be drawn. The number of  $d$  bosons involved in the interaction determines the power  $2r + 3t$  of  $\beta$  in the energy surface (59), that is,  $2r + 3t = n + n'$  where  $n$  and  $n'$  refer to the  $d$ -boson numbers in the interaction (62). A non-zero power of  $\cos 3\gamma$  introduces a  $\gamma$  dependence in the energy surface, which can acquire a triaxial minimum with  $0^\circ < \gamma < 60^\circ$  only if  $t \geq 2$ . A power  $t$  of  $\cos 3\gamma$  requires at least  $3t$   $d$  bosons. In fact, analysis shows that the condition is stronger since it requires at least  $3t$   $d$  bosons not in pairs coupled to angular momentum  $L = 0$ , that is,  $\lfloor (v + v')/3 \rfloor \geq t$ , where  $v$  and  $v'$  refer to the seniorities in the interaction (62) and  $\lfloor x \rfloor$  is the largest integer smaller than or equal to  $x$ . The explicit formula for  $t$  is

$$t_{\max} = \frac{1}{3} \left( v + v' - \text{mod}_3(v + v') [7 - 3 \text{mod}_3(v + v')] \right), \quad t = t_{\max}, t_{\max} - 2, \dots \geq 0. \quad (63)$$

To understand the implications of this formula, one distinguishes the following three cases:

1. If  $\text{mod}_3(v + v') = 0$ , then  $t_{\max} = (v + v')/3$ . This proves that a triaxial shape associated with  $(\cos 3\gamma)^2$  requires at least cubic interactions between the  $d$  bosons [31]. Since the  $3d$ -boson states in  $v_{ddd \cdot ddd}^2$  have seniority  $v = 1$ , this interaction does not contribute to  $(\cos 3\gamma)^2$ , in contrast to the interactions  $v_{ddd \cdot ddd}^L$  with  $L \neq 2$ , which have  $v + v' = 6$ .
2. If  $\text{mod}_3(v + v') = 1$ , then  $t_{\max} = (v + v' - 4)/3$ . As a consequence, quadratic interactions between the  $d$  bosons are independent of  $\gamma$  in the classical limit. Similarly, since the sum of the seniorities in  $v_{ddd_2 \cdot dddd_2}^2$  or  $v_{ddd_2 \cdot dddd_2}^4$  is  $v + v' = 4$ , these four-body interactions are also independent of  $\gamma$  in the classical limit.
3. If  $\text{mod}_3(v + v') = 2$ , then  $t_{\max} = (v + v' - 2)/3$ . Quartic interactions between the  $d$  bosons can have at most  $v + v' = 8$ , in which case  $t_{\max} = 2$ , yielding a term in  $(\cos 3\gamma)^2$  ( $t = 2$ ) and one independent of  $\gamma$  ( $t = 0$ ).

### 9.2. The $\hat{Q}_\mu + \hat{L}_\mu$ Expansion of the $sd$ -Hamiltonian

Given the importance of quadrupole deformation in nuclei, several authors have explored the results of an expansion in the quadrupole operator

$$\hat{Q}_\mu = (s^\dagger \times \tilde{d} + d^\dagger \times \tilde{s})_\mu^{(2)} + \chi(d^\dagger \times \tilde{d})_\mu^{(2)}, \quad (64)$$

with the goal to introduce higher-order interactions in the  $sd$ -IBM, see for example Fortunato et al. [33]. The scope of such an approach can be enlarged by including the angular momentum operator  $\hat{L}_\mu$  in the expansion, in which case eight third-order scalar operators can be constructed, namely

$$\begin{aligned} &(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}, \\ &(\hat{L} \times \hat{Q} \times \hat{Q})^{(0)}, \quad (\hat{Q} \times \hat{L} \times \hat{Q})^{(0)}, \quad (\hat{Q} \times \hat{Q} \times \hat{L})^{(0)}, \\ &(\hat{L} \times \hat{L} \times \hat{Q})^{(0)}, \quad (\hat{L} \times \hat{Q} \times \hat{L})^{(0)}, \quad (\hat{Q} \times \hat{L} \times \hat{L})^{(0)}, \\ &(\hat{L} \times \hat{L} \times \hat{L})^{(0)}, \end{aligned} \quad (65)$$

where in each term the first two operators are coupled to the angular momentum of the third operator to yield an overall scalar.

To answer the question how the operators (65) are related to each other, one applies the algorithm of Section 7 that converts them to a normal-ordered representation. The boson energies and the two-boson interactions of the eight operators are shown in Table 1. The normal-ordered representation shows that

$$(\hat{L} \times \hat{Q} \times \hat{Q})^{(0)} = (\hat{Q} \times \hat{Q} \times \hat{L})^{(0)} \neq (\hat{Q} \times \hat{L} \times \hat{Q})^{(0)}, \quad (66)$$

and that the three-body part of these three operators vanishes for any value of  $\chi$ . Similarly, the three-boson interaction of  $(\hat{L} \times \hat{L} \times \hat{L})^{(0)}$  vanishes identically. The only operators in the list (65) with a non-zero three-boson interaction are those with three  $\hat{Q}_\mu$  operators or with one  $\hat{Q}_\mu$  and two  $\hat{L}_\mu$  operators, see Table 2. In addition, the normal-ordered representation shows that the latter operators are independent of the order of  $\hat{L}_\mu$  and  $\hat{Q}_\mu$ ,

$$(\hat{L} \times \hat{L} \times \hat{Q})^{(0)} = (\hat{L} \times \hat{Q} \times \hat{L})^{(0)} = (\hat{Q} \times \hat{L} \times \hat{L})^{(0)}. \quad (67)$$

One concludes therefore that there are five independent operators in the list (65), two of which have a non-zero interaction of order  $k = 3$ .

**Table 1.** The single-boson energies and the normal-ordered two-boson interactions for the eight cubic operators of Equation (65).

	$(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$	$(\hat{Q} \times \hat{L} \times \hat{Q})^{(0)}$	$(\hat{L} \times \hat{Q} \times \hat{Q})^{(0)}$ $(\hat{Q} \times \hat{Q} \times \hat{L})^{(0)}$	$(\hat{L} \times \hat{L} \times \hat{Q})^{(0)}$ $(\hat{L} \times \hat{Q} \times \hat{L})^{(0)}$ $(\hat{Q} \times \hat{L} \times \hat{L})^{(0)}$	$(\hat{L} \times \hat{L} \times \hat{L})^{(0)}$
$\epsilon_s$	$\sqrt{5}\chi$	$-\sqrt{30}$	0	0	0
$\epsilon_d$	$\frac{\chi(28-3\chi^2)}{14\sqrt{5}}$	$-\frac{\sqrt{3}\chi^2}{\sqrt{10}}$	$-\frac{\sqrt{3}(2+\chi^2)}{\sqrt{10}}$	$\frac{\sqrt{21}\chi}{\sqrt{5}}$	$\sqrt{6}$
$v_{ss:ss}^0$	0	0	0	0	0
$v_{ss:dd}^0$	$6\chi$	$-2\sqrt{6}$	0	0	0
$v_{dd:dd}^0$	$\frac{3\chi(14-3\chi^2)}{7\sqrt{5}}$	$-\frac{\sqrt{6}(2+3\chi^2)}{\sqrt{5}}$	$\frac{\sqrt{6}(2+\chi^2)}{\sqrt{5}}$	$-\frac{2\sqrt{21}\chi}{\sqrt{5}}$	$-2\sqrt{6}$
$v_{sd:sd}^2$	$\frac{6\chi}{\sqrt{5}}$	$-\frac{2\sqrt{6}}{\sqrt{5}}$	0	0	0
$v_{sd:dd}^2$	$\frac{3(14+11\chi^2)}{7\sqrt{10}}$	$-\frac{4\sqrt{3}\chi}{\sqrt{5}}$	0	$\sqrt{\frac{42}{5}}$	0
$v_{dd:dd}^2$	$\frac{9\chi(-14+3\chi^2)}{98\sqrt{5}}$	$-\frac{\sqrt{3}(14+\chi^2)}{7\sqrt{10}}$	$\frac{\sqrt{3}(2+\chi^2)}{\sqrt{10}}$	$-\frac{17\sqrt{3}\chi}{\sqrt{35}}$	$-\sqrt{6}$
$v_{dd:dd}^4$	$\frac{6\chi(14-3\chi^2)}{49\sqrt{5}}$	$\frac{2\sqrt{2}(14+\chi^2)}{7\sqrt{15}}$	$-\frac{2\sqrt{2}(2+\chi^2)}{\sqrt{15}}$	$\frac{68\chi}{\sqrt{105}}$	$4\sqrt{\frac{2}{3}}$

**Table 2.** The normal-ordered three-boson interactions for four of the eight cubic operators of Equation (65).

	$(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$	$(\hat{L} \times \hat{L} \times \hat{Q})^{(0)}$ $(\hat{L} \times \hat{Q} \times \hat{L})^{(0)}$ $(\hat{Q} \times \hat{L} \times \hat{L})^{(0)}$	$(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$	$(\hat{L} \times \hat{L} \times \hat{Q})^{(0)}$ $(\hat{L} \times \hat{Q} \times \hat{L})^{(0)}$ $(\hat{Q} \times \hat{L} \times \hat{L})^{(0)}$
$v_{sss-sss}^0$	0	0	$v_{sd-ssd}^2$	0
$v_{sss-sdd}^0$	0	0	$v_{ssd-sdd}^2$	0
$v_{sss-ddd}^0$	6	0	$v_{ssd-ddd}^2$	0
$v_{sdd-sdd}^0$	$\frac{12\chi}{\sqrt{5}}$	0	$v_{sdd-sdd}^2$	0
$v_{sdd-ddd}^0$	$-\frac{9\sqrt{3}\chi^2}{7\sqrt{5}}$	$-6\sqrt{\frac{7}{5}}$	$v_{sdd-ddd}^2$	$-2\sqrt{\frac{3}{5}}$
$v_{ddd-ddd}^0$	$\frac{123\chi^3}{49\sqrt{5}}$	$\frac{6\sqrt{15}\chi}{\sqrt{7}}$	$v_{ddd-ddd}^2$	$-\frac{4\sqrt{3}\chi}{\sqrt{35}}$
$v_{sdd-sdd}^4$	$\frac{24\chi}{7\sqrt{5}}$	0	$v_{ddd-ddd}^3$	$-\frac{4\sqrt{3}\chi}{\sqrt{35}}$
$v_{sdd-ddd}^4$	$\frac{6\sqrt{11}\chi^2}{7\sqrt{35}}$	$4\sqrt{\frac{11}{15}}$	$v_{ddd-ddd}^6$	$\frac{16\sqrt{3}\chi}{\sqrt{35}}$
$v_{ddd-ddd}^4$	$\frac{108\chi^3}{49\sqrt{5}}$	$-\frac{8\sqrt{5}\chi}{\sqrt{21}}$		

Matrix elements of an operator in multipole form can be calculated in two different ways, either with the recursive Formula (51) or by converting to the operator’s normal-ordered representation and applying Equation (46). Both methods must yield the same result but, as remarked earlier, the latter algorithm is more efficient. This can be illustrated with the example of the  $(\hat{Q} \times \hat{Q} \times \hat{Q})^{(0)}$  operator in the *sd*-IBM. For  $N = 10$  bosons and angular momentum  $J = 0$  the construction of the entire Hamiltonian matrix is about five times faster after conversion to normal order. This factor increases to  $\sim 100$  for  $N = 20$  and  $J = 0$ .

One can extend this analysis to fourth-order multipole operators. If one considers for simplicity’s sake only combinations of  $\hat{Q}_\mu$ , then there are five different operators,

$$(((\hat{Q} \times \hat{Q})^{(\lambda)} \times \hat{Q})^{(2)} \times \hat{Q})^{(0)} = ((\hat{Q} \times \hat{Q})^{(\lambda)} \times (\hat{Q} \times \hat{Q})^{(\lambda)})^{(0)}, \quad \lambda = 0, 1, 2, 3, 4. \quad (68)$$

How the operators (68) are related to each other again can be studied by converting to a normal-ordered representation. The full details of this analysis will not be given here but its results can be summarized as follows. One finds that none of the operators in the set (68) is equivalent to another one. The operators (68) with  $\lambda = 1$  and 3 have vanishing three- and four-boson interactions, and their non-zero single-boson energies and two-boson interactions can be written as follows:

$$\begin{aligned} \lambda = 1: \quad \epsilon_d &= -3f(\chi), \quad v_{dd-dd}^0 = 6f(\chi), \quad v_{dd-dd}^2 = 3f(\chi), \quad v_{dd-dd}^2 = -4f(\chi), \\ \lambda = 3: \quad \epsilon_d &= -7g(\chi), \quad v_{dd-dd}^0 = 14g(\chi), \quad v_{dd-dd}^2 = -8g(\chi), \quad v_{dd-dd}^2 = -g(\chi), \end{aligned} \quad (69)$$

with

$$f(\chi) = \frac{(2 + \chi^2)^2}{20\sqrt{3}}, \quad g(\chi) = \frac{(7 - 4\chi^2)^2}{245\sqrt{7}}. \quad (70)$$

The non-equivalence of the five operators (68) can be illustrated with the example of their expectation value in a  $|0_1^+\rangle$  ground state. For  $\lambda = 0$  this expectation value has an important physics interpretation since it determines fluctuations in the quadrupole deformation parameter  $\beta$  [34]. By way of example one can choose  $|0_1^+\rangle$  to be the ground state of  $\hat{H} = -\kappa\hat{Q} \cdot \hat{Q}$  with  $\kappa > 0$ , which is a frequently used *sd*-IBM Hamiltonian in terms of a single parameter  $\chi$  [35]. For  $\lambda = 1$  and 3 one finds that the expectation value vanishes identically,

$$\langle 0_1^+ | ((\hat{Q} \times \hat{Q})^{(1)} \times (\hat{Q} \times \hat{Q})^{(1)})^{(0)} | 0_1^+ \rangle = \langle 0_1^+ | ((\hat{Q} \times \hat{Q})^{(3)} \times (\hat{Q} \times \hat{Q})^{(3)})^{(0)} | 0_1^+ \rangle = 0, \quad (71)$$

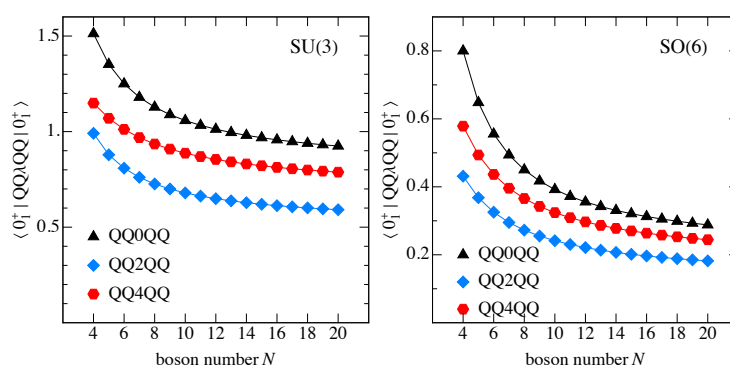
and this is valid for arbitrary values of  $\chi$ . The first of these identities is readily understood since the normal-ordered representation of the  $\lambda = 1$  operator given in Equations (69) and (70) shows that

$$((\hat{Q} \times \hat{Q})^{(1)} \times (\hat{Q} \times \hat{Q})^{(1)})^{(0)} = -\frac{(2 + \chi^2)^2}{40\sqrt{3}} \hat{L}^2. \tag{72}$$

In the three remaining cases with  $\lambda = 0, 2$  and  $4$  the expectation value

$$\frac{1}{N^4} \langle 0_1^+ | ((\hat{Q} \times \hat{Q})^{(\lambda)} \times (\hat{Q} \times \hat{Q})^{(\lambda)})^{(0)} | 0_1^+ \rangle, \tag{73}$$

is shown in Figure 1 for two values of  $\chi$ : its SU(3) value  $\chi = \pm\sqrt{7}/2$  and its SO(6) value  $\chi = 0$ . One observes a clear dependence on  $\lambda$  for all  $N$  and this proves the non-equivalence of the five  $k = 4$  operators (68).



**Figure 1.** The expectation value of the fourth-order operators (68) in the ground state  $|0_1^+\rangle$  of the Hamiltonian  $-\kappa\hat{Q} \cdot \hat{Q}$  in the SU(3) limit ( $\chi = \pm\sqrt{7}/2$ ) and the SO(6) limit ( $\chi = 0$ ). Results are shown for  $\lambda = 0, 2$  and  $4$ . The notation  $QQ\lambda QQ$  stands for  $((\hat{Q} \times \hat{Q})^{(\lambda)} \times (\hat{Q} \times \hat{Q})^{(\lambda)})^{(0)}/N^4$ .

### 10. Concluding Remarks

The formalism presented in this paper enables the solution of the eigenvalue problem for a rotationally invariant and boson-number-conserving Hamiltonian in  $\ell_1 \dots \ell_p$ -IBM. In addition, matrix elements can be evaluated of tensor operators that are not necessarily scalar nor boson-number conserving. There is, in principle, no restriction on  $N$ , the number of bosons, on  $p$ , the number of different kinds of bosons, or on  $k$ , the order of the interaction between the bosons but the overall computational cost mounts with increasing  $N$ ,  $p$  and  $k$ . Specifically, the numerical implementation of the formalism is mostly limited by a combination of a high boson number  $N$  with a high boson angular momentum  $\ell$ .

One of the characteristic advantages of the symbolic method is that it expresses a Hamiltonian matrix element between  $N$ -boson states as a linear combination of the interaction matrix elements. This feature may give insight into the structure of the Hamiltonian matrix. To obtain the eigenspectrum and eigenfunctions, this matrix must be diagonalized. With very rare exceptions, this can only be achieved by inserting appropriate numerical values for the interaction matrix elements.

It is relatively straightforward to extend the present formalism to a system of interacting fermions, with possible applications to the nuclear shell model. Work in this direction is in progress. However, dimensions can be very much larger in the shell model than they are in the IBM, in which case it is more efficient to revert to a standard approach [20–26]. If shell-model dimensions are not too large, the symbolic method may prove to be useful because it reveals the analytic dependence of the many-body Hamiltonian matrix on the interaction matrix elements.

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## Abbreviations

The following abbreviations are used in this manuscript:

CFP	coefficient of fractional parentage
IBM	interacting boson model
$\ell_1 \dots \ell_p$ -IBM	IBM with bosons of $p$ different kinds with angular momenta $\ell_1 \dots \ell_p$
IBM-2	neutron–proton IBM
IBM-3	isopin-invariant IBM

## Appendix A

This appendix lists the coefficients  $a_{rt}^{(k)}$  in Equation (59) for  $k = 3$  and 4. They are expressed in terms of the interaction matrix elements (62) and have been obtained with the code `ibm.m` [27]. For  $k = 3$  the coefficients  $a_{rt}^{(3)}$  have been derived previously [36] and are repeated here for completeness,

$$\begin{aligned}
 a_{00}^{(3)} &= \frac{1}{6} v_{sss\text{-}sss}'^0 & a_{10}^{(3)} &= \frac{1}{\sqrt{15}} v_{sss\text{-}sdd}'^0 + \frac{1}{2} v_{ssd\text{-}ssd}'^2 \\
 a_{01}^{(3)} &= -\frac{1}{3} \sqrt{\frac{2}{35}} v_{sss\text{-}ddd}'^0 - \sqrt{\frac{2}{7}} v_{ssd\text{-}sdd}'^2 \\
 a_{20}^{(3)} &= \frac{1}{10} v_{sdd\text{-}sdd}'^0 + \frac{1}{7} v_{sdd\text{-}sdd}'^2 + \frac{1}{\sqrt{7}} v_{ssd\text{-}ddd}'^2 + \frac{9}{35} v_{sdd\text{-}sdd}'^4 \\
 a_{11}^{(3)} &= -\frac{1}{5} \sqrt{\frac{2}{21}} v_{sdd\text{-}ddd}'^0 - \frac{\sqrt{2}}{7} v_{sdd\text{-}ddd}'^2 - \frac{18}{35} \sqrt{\frac{2}{11}} v_{sdd\text{-}ddd}'^4 \\
 a_{30}^{(3)} &= \frac{1}{14} v_{ddd\text{-}ddd}'^2 + \frac{1}{30} v_{ddd\text{-}ddd}'^3 + \frac{3}{154} v_{ddd\text{-}ddd}'^4 + \frac{7}{165} v_{ddd\text{-}ddd}'^6 \\
 a_{02}^{(3)} &= \frac{1}{105} v_{ddd\text{-}ddd}'^0 - \frac{1}{30} v_{ddd\text{-}ddd}'^3 + \frac{3}{110} v_{ddd\text{-}ddd}'^4 - \frac{4}{1155} v_{ddd\text{-}ddd}'^6.
 \end{aligned} \tag{A1}$$

For  $k = 4$  the coefficients  $a_{rt}^{(4)}$  are

$$\begin{aligned}
 a_{00}^{(4)} &= \frac{1}{24} v_{ssss\text{-}ssss}'^0 & a_{10}^{(4)} &= \frac{1}{2\sqrt{30}} v_{ssss\text{-}ssdd}'^0 + \frac{1}{6} v_{sssd\text{-}sssd}'^2 \\
 a_{01}^{(4)} &= -\frac{1}{3\sqrt{70}} v_{ssss\text{-}sddd}'^0 - \frac{1}{\sqrt{21}} v_{sssd\text{-}ssdd}'^2 \\
 a_{20}^{(4)} &= \frac{1}{20} v_{ssdd\text{-}ssdd}'^0 + \frac{1}{4\sqrt{105}} v_{ssss\text{-}dddd}'^0 + \frac{1}{14} v_{ssdd\text{-}ssdd}'^2 + \frac{1}{\sqrt{21}} v_{sssd\text{-}sddd}'^2 + \frac{9}{70} v_{ssdd\text{-}ssdd}'^4 \\
 a_{11}^{(4)} &= -\frac{1}{5\sqrt{21}} v_{ssdd\text{-}sddd}'^0 - \frac{1}{7} v_{ssdd\text{-}sddd}'^2 - \frac{1}{3\sqrt{21}} v_{sssd\text{-}ddd_2}'^2 - \frac{2}{3\sqrt{231}} v_{sssd\text{-}ddd_4}'^2 - \frac{18}{35\sqrt{11}} v_{ssdd\text{-}sddd}'^4 \\
 a_{30}^{(4)} &= \frac{1}{14} v_{sddd\text{-}sddd}'^2 + \frac{1}{30} v_{sddd\text{-}sddd}'^3 + \frac{3}{154} v_{sddd\text{-}sddd}'^4 + \frac{7}{165} v_{sddd\text{-}sddd}'^6 \\
 &+ \frac{1}{10\sqrt{14}} v_{ssdd\text{-}ddd_4}'^0 + \frac{1}{21} v_{ssdd\text{-}ddd_2}'^2 - \frac{1}{21\sqrt{11}} v_{ssdd\text{-}ddd_4}'^2 + \frac{3}{35} v_{ssdd\text{-}ddd_2}'^4 + \frac{3}{7\sqrt{1430}} v_{ssdd\text{-}ddd_4}'^4 \\
 a_{02}^{(4)} &= \frac{1}{105} v_{sddd\text{-}sddd}'^0 - \frac{1}{30} v_{sddd\text{-}sddd}'^3 + \frac{3}{110} v_{sddd\text{-}sddd}'^4 - \frac{4}{1155} v_{sddd\text{-}sddd}'^6 \\
 &+ \frac{1}{7\sqrt{11}} v_{ssdd\text{-}ddd_4}'^2 - \frac{9}{7\sqrt{1430}} v_{ssdd\text{-}ddd_4}'^4 \\
 a_{21}^{(4)} &= -\frac{1}{35\sqrt{6}} v_{sddd\text{-}ddd_4}'^0 - \frac{1}{21} v_{sddd\text{-}ddd_2}'^2 - \frac{2}{21\sqrt{11}} v_{sddd\text{-}ddd_4}'^2 - \frac{6}{35\sqrt{11}} v_{sddd\text{-}ddd_2}'^4 \\
 &+ \frac{6}{77} \sqrt{\frac{2}{65}} v_{sddd\text{-}ddd_4}'^4 - \frac{6}{77\sqrt{5}} v_{sddd\text{-}ddd_4}'^6 \\
 a_{40}^{(4)} &= \frac{1}{280} v_{ddd_4\text{-}ddd_4}'^0 + \frac{1}{126} v_{ddd_2\text{-}ddd_2}'^2 + \frac{1}{1386} v_{ddd_4\text{-}ddd_4}'^2 + \frac{1}{70} v_{ddd_2\text{-}ddd_2}'^4 + \frac{17}{5005} v_{ddd_4\text{-}ddd_4}'^4 \\
 &+ \frac{1}{210} v_{ddd_4\text{-}ddd_4}'^5 + \frac{2}{1155} v_{ddd_4\text{-}ddd_4}'^6 + \frac{79}{15015} v_{ddd_4\text{-}ddd_4}'^8 - \frac{1}{63\sqrt{11}} v_{ddd_2\text{-}ddd_4}'^2 + \frac{1}{7\sqrt{1430}} v_{ddd_2\text{-}ddd_4}'^4 \\
 a_{12}^{(4)} &= \frac{1}{462} v_{ddd_4\text{-}ddd_4}'^2 - \frac{12}{5005} v_{ddd_4\text{-}ddd_4}'^4 - \frac{1}{210} v_{ddd_4\text{-}ddd_4}'^5 + \frac{1}{165} v_{ddd_4\text{-}ddd_4}'^6 - \frac{16}{15015} v_{ddd_4\text{-}ddd_4}'^8 \\
 &+ \frac{1}{21\sqrt{11}} v_{ddd_2\text{-}ddd_4}'^2 - \frac{3}{7\sqrt{1430}} v_{ddd_2\text{-}ddd_4}'^4.
 \end{aligned} \tag{A2}$$

States with up to three  $d$  bosons are uniquely characterized by their angular momentum  $L$  and there is no need to specify the seniority  $\nu$ . This is no longer the case for  $n = 4$  where for  $L = 2$  and  $4$  the seniority can be  $\nu = 2$  or  $4$ . This is indicated in Equation (A2) with an index, that is,  $dddd_\nu$  stands for a normalized state of four  $d$  bosons with seniority  $\nu$ .

## References

- Iachello, F.; Levine, R.D. *Algebraic Theory of Molecules*; Oxford University Press: Oxford, UK, 1995.
- Iachello, F.; Arima, A. *The Interacting Boson Model*; Cambridge University Press: Cambridge, UK, 1987.
- Arima, A.; Ohtsuka, T.; Iachello, F.; Talmi, I. Collective nuclear states as symmetric couplings of proton and neutron excitations. *Phys. Lett. B* **1977**, *66*, 205–208. [[CrossRef](#)]
- Elliott, J.P.; White, A.P. An isospin invariant form of the interacting boson model. *Phys. Lett. B* **1980**, *97*, 169–172. [[CrossRef](#)]
- de-Shalit, A.; Talmi, I. *Nuclear Shell Theory*; Academic Press: New York, NY, USA, 1963.
- Talmi, I. *Simple Models of Complex Nuclei. The Shell Model and Interacting Boson Model*; Harwood: Chur, Switzerland, 1993.
- Skouras, L.D.; Kossionides, S. Generalized fractional parentage coefficients for shell-model calculations. *Comput. Phys. Commun.* **1986**, *39*, 197–212. [[CrossRef](#)]
- Racah, G. Theory of complex spectra. *Phys. Rev.* **1943**, *63*, 367–382. [[CrossRef](#)]
- Zerguine, S.; Van Isacker, P. Spin-aligned neutron-proton pairs in  $N = Z$  nuclei. *Phys. Rev. C* **2011**, *83*, 064314-1–064314-12. [[CrossRef](#)]
- Wybourne, B.G. *Symmetry Principles and Atomic Spectroscopy*; Wiley-Interscience: New York, NY, USA, 1970.
- Weyl, H. *The Classical Groups*; Princeton University Press: Princeton, NJ, USA, 1939.
- Gheorghe, A.; Raduta, A.A. New results for the missing quantum numbers labelling the quadrupole and octupole boson basis. *J. Phys. A Math. Gen.* **2004**, *37*, 10951–10958. [[CrossRef](#)]
- Devi, Y.D.; Kota, V.K.B. sdg Interacting boson model: hexadecupole degree of freedom in nuclear structure. *Pramana J. Phys.* **1992**, *39*, 413. [[CrossRef](#)]
- Arima, A.; Iachello, F. Interacting boson model of collective nuclear states II. The rotational limit. *Ann. Phys.* **1978**, *111*, 201–238. [[CrossRef](#)]
- Scholten, O. Computer Codes PHINT and FBEM. KVI Report No. 63. 1979. Available online: <https://www.scirp.org/reference/referencespapers?referenceid=3749072> (accessed on 31 January 2026).
- Scholten, O. The Interacting Boson Approximation Model and Applications. Ph.D. Thesis, University of Groningen, Groningen, The Netherlands, 1980.
- Van Isacker, P. Fortran code `ibm.f`. Available on request.
- García-Ramos, J.E.; Heyde, K. Nuclear shape coexistence: A study of the even–even Hg isotopes using the interacting boson model with configuration mixing. *Phys. Rev. C* **2014**, *89*, 014306-1–014306-24. [[CrossRef](#)]
- Heinze, S. Eine Methode zur Lösung Beliebiger Bosonischer und Fermionischer Vielteilchensysteme. Ph.D. Thesis, University of Cologne, Cologne, Germany, 2008.
- Brown, B.A. The nuclear shell model towards the drip lines. *Prog. Part. Nucl. Phys.* **2001**, *47*, 517–599. [[CrossRef](#)]
- Brown, B.A.; Rae, W.D.M. The shell-model code NuShellX@MSU. *Nucl. Data Sheets* **2014**, *120*, 115–118. [[CrossRef](#)]
- Caurier, E.; Martínez-Pinedo, G.; Nowacki, F.; Poves, A.; Zuker, A.P. The shell model as a unified view of nuclear structure. *Rev. Mod. Phys.* **2005**, *77*, 427–488. [[CrossRef](#)]
- Shimizu, N.; Abe, T.; Tsunoda, Y.; Utsuno, Y.; Yoshida, T.; Mizusaki, T.; Honma, M.; Otsuka, T. New-generation Monte Carlo shell model for the K computer era. *Prog. Theor. Exp. Phys.* **2012**, *1*, 01A205. [[CrossRef](#)]
- Shimizu, N.; Mizusaki, T.; Utsuno, Y.; Tsunoda, Y. Thick-restart block Lanczos method for large-scale shell-model calculations. *Comp. Phys. Commun.* **2019**, *244*, 372–384. [[CrossRef](#)]
- Johnson, C.W.; Ormand, W.E.; Krastev, P.G. Factorization in large-scale many-body calculations. *Comp. Phys. Commun.* **2013**, *184*, 2761–2774. [[CrossRef](#)]
- Johnson, C.W.; Ormand, W.E.; McElvain, K.S.; Shan, H. BIGSTICK: A flexible configuration-interaction shell-model code. *arXiv* **2018**, arXiv:1801.08432.
- Van Isacker, P. Mathematica code `ibm.m`. Available on request.
- Ginocchio, J.N.; Kirson, M.W. Relationship between the Bohr collective Hamiltonian and the interacting-boson model. *Phys. Rev. Lett.* **1980**, *44*, 1744–1747. [[CrossRef](#)]
- Dieperink, A.E.L.; Scholten, O.; Iachello, F. Classical limit of the interacting-boson model. *Phys. Rev. Lett.* **1980**, *44*, 1747–1750. [[CrossRef](#)]
- Bohr, A.; Mottelson, B.R. Features of nuclear deformations produced by the alignment of individual particles or pairs. *Phys. Scr.* **1980**, *22*, 468. [[CrossRef](#)]

31. Van Isacker, P.; Chen, J.-Q. Classical limit of the interacting boson Hamiltonian. *Phys. Rev. C* **1981**, *24*, 684–689. [[CrossRef](#)]
32. Frank, A.; Van Isacker, P. *Algebraic Methods in Molecular and Nuclear Structure Physics*; Wiley-Interscience: New York, NY, USA, 1994.
33. Fortunato, L.; Alonso, C.E.; Arias, J.M.; García-Ramos, J.E.; Vitturi, A. Phase diagram for a cubic- $Q$  interacting boson model Hamiltonian: Signs of triaxiality. *Phys. Rev. C* **2011**, *84*, 014326-1–014326-13. [[CrossRef](#)]
34. Poves, A.; Nowacki, F.; Alhassid, Y. Limits on assigning a shape to a nucleus. *Phys. Rev. C* **2020**, *101*, 054307-1–054307-5. [[CrossRef](#)]
35. Warner, D.D.; Casten, R.F. Predictions of the interacting boson approximation in a consistent  $Q$  framework. *Phys. Rev. C* **1983**, *28*, 1798–1806. [[CrossRef](#)]
36. Sorgunlu, B.; Van Isacker, P. Triaxiality in the interacting boson model. *Nucl. Phys. A* **2008**, *808*, 27–46. [[CrossRef](#)]

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