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# Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors?

Anass Bouisaghoulane and Arthemy V Kiselev

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen,  
P.O.Box 407, 9700 AK Groningen, The Netherlands

E-mail: A.V.Kiselev@rug.nl

## Abstract.

From the paper “Formality Conjecture” (Ascona 1996):

*I am aware of only one such a class, it corresponds to simplest good graph, the complete graph with 4 vertices (and 6 edges). This class gives a remarkable vector field on the space of bi-vector fields on  $\mathbb{R}^d$ . The evolution with respect to the time  $t$  is described by the following non-linear partial differential equation: ..., where  $\alpha = \sum_{i,j} \alpha_{ij} \partial/\partial x_i \wedge \partial/\partial x_j$  is a bi-vector field on  $\mathbb{R}^d$ .*

*It follows from general properties of cohomology that 1) **this evolution preserves the class of (real-analytic) Poisson structures**, ...*

*In fact, I cheated a little bit. In the formula for the vector field on the space of bivector fields which one get from the tetrahedron graph, an additional term is present. ... It is possible to prove formally that **if  $\alpha$  is a Poisson bracket, i.e. if  $[\alpha, \alpha] = 0 \in T^2(\mathbb{R}^d)$ , then the additional term shown above vanishes.***

By using twelve Poisson structures with high-degree polynomial coefficients as explicit counterexamples, we show that both the above claims are false: neither does the first flow preserve the property of bi-vectors to be Poisson nor does the second flow vanish identically at Poisson bi-vectors. The counterexamples at hand suggest a correction to the formula for the “exotic” flow on the space of Poisson bi-vectors; in fact, this flow is encoded by the balanced sum involving both the Kontsevich tetrahedral graphs (that give rise to the flows mentioned above). We reveal that it is only the balance 1 : 6 for which the flow does preserve the space of Poisson bi-vectors.

**Introduction.** The Kontsevich graph complex is the language of deformation quantisation on finite-dimensional Poisson manifolds [1, 2]. We consider the class of oriented graphs on two sinks and  $k \geq 1$  internal vertices (of which, each is the tail of two edges and carries a copy of the Poisson bi-vector  $\mathcal{P}$ ). Encoding bi-differential operators, such graphs determine the flows on the space of bi-vectors on a Poisson manifold at hand. The two flows with  $k = 4$  internal vertices in the graphs are provided by the two tetrahedra [1], see Fig. 1 on the next page. By producing 12 counterexamples, we prove that the claim [1, 2] of preservation of the Poisson property is false as stated. Simultaneously, we reveal that the flow which is determined by the second graph is not always vanishing by virtue of the skew-symmetry and Jacobi identity for Poisson bi-vectors  $\mathcal{P}$ .

This paper is structured as follows. First we recall the correspondence between graphs and polydifferential operators [3, 4] and we indicate the mechanism for such an operator to vanish, cf. [5, 6]. In section 2 we recall three constructions of Poisson brackets with polynomial coefficients of arbitrarily high degree (see [7, 8, 9]). In Tables 1–4 on pp. 7–8 we then summarise the properties of all structures from our 12 counterexamples to the claim [1] that



- (i) the flow  $\dot{\mathcal{P}} = \Gamma_1(\mathcal{P})$  which the first graph in Fig. 1 encodes on the space of bi-vectors  $\mathcal{P}$  would preserve their property to be Poisson (in fact, it does not), and that
- (ii) the flow  $\dot{\mathcal{P}} = \Gamma_2(\mathcal{P})$  would always be trivial whenever the bi-vector  $\mathcal{P}$  is Poisson (in fact, this is not true).

In particular, the twelfth counterexample pertains to the infinite-dimensional jet-space geometry of variational Poisson structures [11]. (Quoted from [12], the Hamiltonian differential operator for that variational Poisson bi-vector  $\mathcal{P}$  is processed by using the techniques from [13, 14, 15]).

Finally, we examine at which balance the linear combination of the Kontsevich tetrahedral flows preserves the space of Poisson structures on finite-dimensional manifolds. We argue that the ratio 1 : 6 does the job; this claim has been proved in [6].

### 1. The graphs and operators

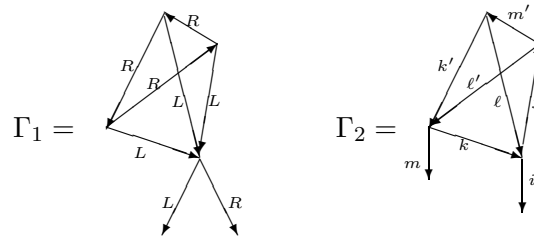
Let us formalise a way to encode polydifferential operators using oriented graphs. Consider the space  $\mathbb{R}^n$  with Cartesian coordinates  $\mathbf{x} = (x_1, \dots, x_n)$ , here  $2 \leq n < \infty$ ; for typographical reasons we use the lower indices to enumerate the variables, so that  $x_1^2 = (x_1)^2$ , etc. By definition, the decorated edge  $\bullet \xrightarrow{i} \bullet$  denotes at once the derivation  $\partial/\partial x_i \equiv \partial_i$  (that acts on the content of the arrowhead vertex) and the summation  $\sum_{i=1}^n$  (over the index  $i$  in the object which is contained in the arrowtail vertex). For example, the graph  $\bullet \xleftarrow{i} \mathcal{P}^{ij}(\mathbf{x}) \xrightarrow{j} \bullet$  encodes the bi-differential operator  $\sum_{i=1}^n (\cdot) \overleftarrow{\partial}_i \mathcal{P}^{ij}(\mathbf{x}) \overrightarrow{\partial}_j (\cdot)$ . If its coefficients  $\mathcal{P}^{ij}$  are antisymmetric, then the graph  $\bullet \xleftarrow{i} \bullet \xrightarrow{j} \bullet$  encodes the bi-vector  $\mathcal{P} = \mathcal{P}^{ij} \partial_i \wedge \partial_j$ , where  $\partial_i \wedge \partial_j = \frac{1}{2}(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i)$ . It then specifies the Poisson bracket  $\{\cdot, \cdot\}_{\mathcal{P}}$  if the  $\frac{n(n-1)}{2}$ -tuple of coefficients solves the system of equations

$$(\mathcal{P}^{ij}) \overleftarrow{\partial}_\ell \cdot \mathcal{P}^{\ell k} + (\mathcal{P}^{jk}) \overleftarrow{\partial}_\ell \cdot \mathcal{P}^{\ell i} + (\mathcal{P}^{ki}) \overleftarrow{\partial}_\ell \cdot \mathcal{P}^{\ell j} = 0, \quad (1)$$

hence the bracket  $\bullet \xleftarrow{i} \mathcal{P}^{ij} \xrightarrow{j} \bullet$  satisfies the Jacobi identity. Clearly,  $\mathcal{P}^{ij}(\mathbf{x}) = \{x_i, x_j\}_{\mathcal{P}}$ .

From now on, let us consider only the oriented graphs whose vertices are either sinks, with no issued edges, or tails for an ordered pair of arrows, each decorated with its own index (see Fig. 1). Allowing the only exception in footnote 1, we shall always assume that there are neither tadpoles, nor double oriented edges, nor two-edge loops.

We also postulate that every vertex which is not a sink carries a copy of a given Poisson bi-vector  $\mathcal{P} = \mathcal{P}^{ij}(\mathbf{x}) \partial_i \wedge \partial_j$ ; the ordering of decorated out-going edges coincides with the ordering “first  $\prec$  second” of the indexes in the coefficients of  $\mathcal{P}$ .



**Figure 1.** These tetrahedral graphs encode flows (2a) and (2b), respectively. Each oriented edge carries a summation index that runs from 1 to the dimension of the Poisson manifold at hand. For each internal vertex (where a copy of the Poisson bi-vector  $\mathcal{P}$  is stored), the pair of out-going edges is ordered,  $L \prec R$ : the left edge (L) carries the first index and the other edge (R) carries the second index in the bi-vector coefficients. (In retrospect, the ordering and labelling of the indexed oriented edges can be guessed from formulas (2) on p. 3.)

**Example 1.** Under all these assumptions, the two tetrahedra which are portrayed in Fig. 1 are, up to a symmetry, the only admissible graphs with  $k = 4$  internal vertices,  $2k = 6 + 2$  edges, and two sinks. The first graph in Fig. 1 encodes the bi-vector

$$\Gamma_1(\mathcal{P}) = \sum_{i,j=1}^n \left( \sum_{k,\ell,m,k',\ell',m'=1}^n \frac{\partial^3 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell \partial x_m} \frac{\partial \mathcal{P}^{kk'}}{\partial x_{\ell'}} \frac{\partial \mathcal{P}^{\ell\ell'}}{\partial x_{m'}} \frac{\partial \mathcal{P}^{mm'}}{\partial x_{k'}} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \quad (2a)$$

Likewise, the second graph in Fig. 1 yields the bi-vector

$$\Gamma_2(\mathcal{P}) = \sum_{i,m=1}^n \left( \sum_{j,k,\ell,k',\ell',m'=1}^n \frac{\partial^2 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell} \frac{\partial^2 \mathcal{P}^{km}}{\partial x_{k'} \partial x_{\ell'}} \frac{\partial \mathcal{P}^{k'\ell}}{\partial x_{m'}} \frac{\partial \mathcal{P}^{m'\ell'}}{\partial x_j} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_m}. \quad (2b)$$

In this paper we examine

- (i) whether the respective flows  $\frac{d}{d\varepsilon}(\mathcal{P}) = \Gamma_\alpha(\mathcal{P})$  at  $\alpha = 1, 2$  preserve or, in fact, destroy the property of bi-vectors  $\mathcal{P}(\varepsilon)$  to be Poisson, provided that the Cauchy datum  $\mathcal{P}|_{\varepsilon=0}$  is such;
- (ii) we also inspect whether the second flow is (actually, it is not) vanishing identically at all  $\varepsilon$ , provided that the Cauchy datum is a Poisson bi-vector.

*Remark 1.* Whenever the bi-vector  $\mathcal{P}$  in every internal vertex of a non-empty graph  $\Gamma$  is Poisson, the bi-differential operator which is encoded by  $\Gamma$  can vanish identically. First, this occurs due to the skew-symmetry of coefficients of the bi-vector.<sup>1</sup> Second, the operators encoded using graphs (with a copy of the Poisson bi-vector  $\mathcal{P}$  at every internal vertex) can vanish by virtue of the Jacobi identity, see (1), or its differential consequences. This mechanism has been illustrated in [5]; making a part of our present argument (see [6]), it is a key to the proof of the fact that the balanced flow  $\frac{d}{d\varepsilon}(\mathcal{P}) = \Gamma_1(\mathcal{P}) + 6\Gamma_2(\mathcal{P})$  does preserve the property of bi-vectors  $\mathcal{P}(\varepsilon)$  to be (infinitesimally) Poisson whenever the Cauchy datum  $\mathcal{P}|_{\varepsilon=0}$  is such.

So, each of the two claims (i–ii) is false if it does not hold for at least one Poisson structure (itself already known to have skew-symmetric coefficients and turn the left-hand side of the Jacobi identity into zero for any triple of arguments of the Jacobiator). To examine both claims, we need a store of Poisson structures such that the coefficients  $\mathcal{P}^{ij}(\mathbf{x})$  are not mapped to zero by the third or second order derivatives in (2a) and (2b), respectively. For that, a regular generator of Poisson structures with polynomial coefficients of arbitrarily high degree would suffice.

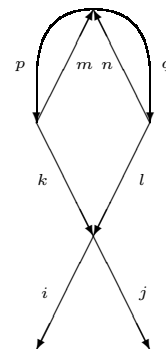
## 2. The generators

Let us recall three regular ways to generate the Poisson brackets or modify a given one, thus obtaining a new such structure. These generators will be used in section 3 to produce the counterexamples to both claims from [1].

<sup>1</sup> For example, consider this oriented graph with ordered pairs of indexed edges ( $i \prec j$ ,  $k \prec \ell$ ,  $m \prec n$ ,  $p \prec q$ ). We claim that due to the antisymmetry of  $\mathcal{P}$  which is contained in each of the four internal vertices, the operator (which this graph encodes) vanishes identically. Indeed, it equals minus itself:

$$\begin{aligned} & \partial_m \partial_n (\mathcal{P}^{pq}) \partial_p (\mathcal{P}^{km}) \partial_q (\mathcal{P}^{\ell n}) \partial_k \partial_\ell (\mathcal{P}^{ij}) \partial_i \wedge \partial_j \\ &= -\partial_m \partial_n (\mathcal{P}^{qp}) \partial_p (\mathcal{P}^{km}) \partial_q (\mathcal{P}^{\ell n}) \partial_k \partial_\ell (\mathcal{P}^{ij}) \partial_i \wedge \partial_j \\ &= -\partial_m \partial_n (\mathcal{P}^{pq}) \partial_q (\mathcal{P}^{\ell n}) \partial_p (\mathcal{P}^{km}) \partial_\ell \partial_k (\mathcal{P}^{ij}) \partial_i \wedge \partial_j = 0. \end{aligned}$$

To establish the second equality, we interchanged the labelling of indices ( $p \rightleftharpoons q$ ,  $k \rightleftharpoons \ell$ , and  $m \rightleftharpoons n$ ) and we recalled that the partial derivatives commute.



### 2.1. The determinant construction

This generator of Poisson bi-vectors is described in [7], cf. [16] and references therein. The construction goes as follows. Let  $x_1, \dots, x_n$  be the Cartesian coordinates on  $\mathbb{R}^{n \geq 3}$ . Let  $\vec{g} = (g_1, \dots, g_{n-2})$  be a fixed tuple of smooth functions in these variables. For any  $a, b \in C^\infty(\mathbb{R}^n)$ , put

$$\{a, b\}_{\vec{g}} = \det(\mathbf{J}(g_1, \dots, g_{n-2}, a, b))$$

where  $\mathbf{J}(\cdot, \dots, \cdot)$  is the Jacobian matrix. Clearly, the bracket  $\{\cdot, \cdot\}_{\vec{g}}$  is bi-linear and skew-symmetric. Moreover, it is readily seen to be a derivation in each of its arguments:  $\{a, b \cdot c\}_{\vec{g}} = \{a, b\}_{\vec{g}} \cdot c + b \cdot \{a, c\}_{\vec{g}}$ . For the validity mechanism of the Jacobi identity for this particular instance of the Nambu bracket we refer to [16] again (see also [17]).

**Example 2** (see entry 3 in Table 2 on p. 7). Fix the functions  $g_1 = x_2^3 x_3^2 x_4$  and  $g_2 = x_1 x_3^4 x_4$ , and insert them in the determinant generator of Poisson bi-vectors. We thus obtain the bi-vector  $\mathcal{P}_0$ ,

$$\mathcal{P}_0^{ij} = \begin{pmatrix} 0 & -2x_1 x_2^3 x_3^5 x_4 & -3x_1 x_2^2 x_3^6 x_4 & 12x_1 x_2^2 x_3^5 x_4^2 \\ 2x_1 x_2^3 x_3^5 x_4 & 0 & -x_2^3 x_3^6 x_4 & 2x_2^3 x_3^5 x_4^2 \\ 3x_1 x_2^2 x_3^6 x_4 & x_2^3 x_3^6 x_4 & 0 & -3x_2^2 x_3^6 x_4^2 \\ -12x_1 x_2^2 x_3^5 x_4^2 & -2x_2^3 x_3^5 x_4^2 & 3x_2^2 x_3^6 x_4^2 & 0 \end{pmatrix}.$$

By construction, the above matrix is skew-symmetric. The validity of Jacobi identity (1) is straightforward: indexed by  $i, j, k$ , all the components  $[\mathcal{P}, \mathcal{P}]^{ijk}$  of the tri-vector vanish.<sup>2</sup> This Poisson bi-vector  $\mathcal{P}$  is used in section 3 in the list of counterexamples to the claims under study.

### 2.2. Pre-multiplication in the 3-dimensional case

Let  $x, y, z$  be the Cartesian coordinates on the vector space  $\mathbb{R}^3$ . For every bi-vector  $\mathcal{P} = \mathcal{P}^{ij} \partial_i \wedge \partial_j$ , introduce the differential one-form  $P = P_1 dx + P_2 dy + P_3 dz$  by setting  $P := -\mathcal{P} \lrcorner dx \wedge dy \wedge dz$ , so that  $P_1 = -\mathcal{P}^{23}$ ,  $P_2 = \mathcal{P}^{13}$ , and  $P_3 = -\mathcal{P}^{12}$ . It is readily seen [8] that the original Jacobi identity for the bi-vector  $\mathcal{P}$  now reads<sup>3</sup>  $dP \wedge P = 0$  for the respective one-form  $P$ . But let us note that the pre-multiplication  $P \mapsto f \cdot P$  of the form  $P$  by a smooth function  $f$  preserves this reading of the Jacobi identity:  $d(fP) \wedge (fP) = f \cdot [df \wedge P \wedge P + f \cdot dP \wedge P] = f^2 \cdot dP \wedge P = 0$ . This shows that the bi-vector  $f\mathcal{P}$  which the form  $fP$  yields on  $\mathbb{R}^3$  is also Poisson.

This pre-multiplication trick provides the examples of Poisson structures of arbitrarily high polynomial degree coefficients (in a manifestly non-symplectic three-dimensional set-up).<sup>4</sup>

### 2.3. The Vanhaecke construction

In [9], Vanhaecke created another construction of high polynomial degree Poisson bi-vectors. Let  $u$  be a monic degree  $d$  polynomial in  $\lambda$  and  $v$  be a polynomial of degree  $d-1$  in  $\lambda$ :

$$u(\lambda) = \lambda^d + u_1 \lambda^{d-1} + \dots + u_{d-1} \lambda + u_d, \quad v(\lambda) = v_1 \lambda^{d-1} + \dots + v_{d-1} \lambda + v_d.$$

<sup>2</sup> Indeed, there are four tuples of distinct values of the indices  $i, j$ , and  $k$  up to permutations; we let  $1 \leq i < j < k \leq n = 4$  so that the check runs over the set of triples  $\{(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)\}$ . For example,  $[\mathcal{P}, \mathcal{P}]^{123} = 6x_1 x_2^5 x_3^{11} x_4^2 - 6x_1 x_2^5 x_3^{11} x_4^2 - 6x_1 x_2^5 x_3^{11} x_4^2 + 6x_1 x_2^5 x_3^{11} x_4^2 - 18x_1 x_2^5 x_3^{11} x_4^2 + 18x_1 x_2^5 x_3^{11} x_4^2 + 12x_1 x_2^5 x_3^{11} x_4^2 - 6x_1 x_2^5 x_3^{11} x_4^2 - 6x_1 x_2^5 x_3^{11} x_4^2 = 0$ . Therefore,  $[\mathcal{P}, \mathcal{P}] = \sum_{1 \leq i < j < k \leq 4} [\mathcal{P}, \mathcal{P}]^{ijk}(\mathbf{x}) \partial_i \wedge \partial_j \wedge \partial_k = 0$ .

<sup>3</sup> The exterior differential  $dP$  is equal to  $dP = (\partial_x \mathcal{P}^{13} + \partial_y \mathcal{P}^{23}) dx \wedge dy + (-\partial_x \mathcal{P}^{12} + \partial_z \mathcal{P}^{23}) dx \wedge dz + (-\partial_y \mathcal{P}^{11} - \partial_z \mathcal{P}^{13}) dy \wedge dz$ . The wedge product is  $dP \wedge P = (\partial_x \mathcal{P}^{31} \mathcal{P}^{12} + \partial_y \mathcal{P}^{23} \mathcal{P}^{21} + \partial_x \mathcal{P}^{12} \mathcal{P}^{13} + \partial_z \mathcal{P}^{23} \mathcal{P}^{31} + \partial_y \mathcal{P}^{12} \mathcal{P}^{23} + \partial_z \mathcal{P}^{31} \mathcal{P}^{32}) dx \wedge dy \wedge dz = (-[\mathcal{P}, \mathcal{P}] \lrcorner dx \wedge dy \wedge dz) dx \wedge dy \wedge dz$ .

<sup>4</sup> In dimension three, this pre-multiplication procedure also provides the examples of Poisson bi-vectors at which the second flow (2b) does not vanish identically.

Consider the space  $\mathbb{k}^{2d}$  (e.g., set  $\mathbb{k} := \mathbb{R}$ ) with Cartesian coordinates  $u_1, \dots, u_n, v_1, \dots, v_d$ . To define the Poisson bracket, fix a bivariate polynomial  $\phi(\cdot, \cdot)$  and for all  $1 \leq i, j \leq d$  set

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \quad \{u_i, v_j\} = \text{coeff. of } \lambda^j \text{ in } \phi(\lambda, v(\lambda)) \cdot \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \mod u(\lambda), \quad (3)$$

where we denote by  $[\dots]_+$  the argument's polynomial part and where the remainder modulo the degree  $d$  polynomial  $u(\lambda)$  is obtained using the Euclidean division algorithm.

Let us emphasise that these Poisson bi-vector are defined on the even-dimensional spaces. Indeed, the coefficients of Poisson bracket (3) are arranged in the block matrix  $\begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}$ , where the components of the matrix  $U$  are  $U^{ij} = \{u_i, v_j\}$ .

#### 2.4. The Hamiltonian differential operators on jet spaces

The variational Poisson brackets  $\{\cdot, \cdot\}_{\mathcal{P}}$  for functionals of sections of affine bundles generalise the notion of Poisson brackets  $\{\cdot, \cdot\}_{\mathcal{P}}$  for functions on finite-dimensional Poisson manifolds  $(N^n, \{\cdot, \cdot\}_{\mathcal{P}})$ . Namely, let us consider the space  $J^\infty(\pi)$  of infinite jets of sections for a given bundle  $\pi$  over a manifold  $M^n$  of positive dimension  $m$ . The variational Poisson brackets  $\{\cdot, \cdot\}_{\mathcal{P}}$  on  $J^\infty(\pi)$  are then specified by using the Hamiltonian differential operators (which we shall denote by  $A$  and the order of which is typically positive).<sup>5</sup> The formalism of variational Poisson bi-vectors  $\mathcal{P} = \frac{1}{2} \langle \xi \cdot \vec{A}(\xi) \rangle$  and the variational Schouten bracket  $[\![\cdot, \cdot]\!]$  is standard (see [11, 19]). The geometry of iterated variations is revealed in [13]; the correspondence between the Kontsevich graphs and local variational polydifferential operators is explained in [14].

**Example 3.** To inspect whether either of the two claims (which we quote from [1] on the title page) would hold in the variational set-up, it is enough to consider a Hamiltonian differential operator with (differential-)polynomial coefficients of degree  $\geq 3$ . Let us take the Hamiltonian operator<sup>6</sup>  $A = u^2 \circ d/dx \circ u^2$  for the Harry Dym equation (see [12]); here  $u$  is the fibre coordinate in the trivial bundle  $\pi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $x$  is the base variable. This operator is obviously skew-adjoint, whence the variational Poisson bracket  $\{\cdot, \cdot\}_{\mathcal{P}}$  is skew-symmetric. The Jacobi identity for  $\{\cdot, \cdot\}_{\mathcal{P}}$  is also easy to check: the variational master-equation  $[\![\mathcal{P}, \mathcal{P}]\!] \cong 0$  does hold for the variational bi-vector  $\mathcal{P} = \frac{1}{2} \langle \xi \cdot \vec{A}(\xi) \rangle$ .

### 3. The counterexamples

We now examine the properties of both tetrahedral flows (2) whenever each of them is evaluated at a given Poisson bi-vector. (Examples of such bi-vectors are produced by using the techniques from section 2.) To motivate the composition of Tables 1–4 and clarify the meaning of their content, let us consider an example: namely, we first take the Poisson bi-vector which was obtained in section 2.1 (see p. 4).

**Example 4** (continued). Rewriting the Poisson bi-vector  $\mathcal{P}_0 \in \Gamma(\wedge^2 TN^4)$  in terms of the parity-odd variables  $\xi$ , we obtain that under the isomorphism  $\Gamma(\wedge^\bullet TN^n) \simeq C^\infty(\Pi T^*N^n)$  the bi-vector  $\mathcal{P}_0^{ij}(x) \partial_i \wedge \partial_j$  becomes  $\frac{1}{2} \mathcal{P}_0^{ij}(x) \xi_i \xi_j$ ; we have that  $\mathcal{P}_0 =$

$$-2x_1x_2^3x_3^5x_4\xi_1\xi_2 - 3x_1x_2^2x_3^6x_4\xi_1\xi_3 + 12x_1x_2^2x_3^5x_4^2\xi_1\xi_4 - x_2^3x_3^6x_4\xi_2\xi_3 + 2x_2^3x_3^5x_4^2\xi_2\xi_4 - 3x_2^2x_3^6x_4^2\xi_3\xi_4.$$

<sup>5</sup> In fact, the Poisson geometry of finite-dimensional affine manifolds  $(N^n, \{\cdot, \cdot\}_{\mathcal{P}})$  is a zero differential order sub-theory in the variational Poisson geometry of infinite jet spaces  $J^\infty(\pi)$ . Indeed, let the fibres in the bundle  $\pi$  be  $N^n$  and proclaim that only *constant* sections are allowed.

<sup>6</sup> More examples of variational Poisson structures, which are relevant for our present purpose, can be found in [20] or, e.g., in [21] (see also the references contained therein).

Now, we calculate the right-hand sides  $\mathcal{P}_1 := \Gamma_1(\mathcal{P}_0)$  and  $\mathcal{P}_2 := \Gamma_2(\mathcal{P}_0)$  of tetrahedral flows (2). The coefficient matrix of the bi-vector  $\mathcal{P}_1$  is

$$\mathcal{P}_1^{ij} = \begin{pmatrix} 0 & -24480 x_1 x_2^9 x_3^{20} x_4^4 & -51840 x_1 x_2^8 x_3^{21} x_4^4 & 12960 x_1 x_2^8 x_3^{20} x_4^5 \\ 24480 x_1 x_2^9 x_3^{20} x_4^4 & 0 & -15480 x_2^9 x_3^{21} x_4^4 & 2448 x_2^9 x_3^{20} x_4^5 \\ 51840 x_1 x_2^8 x_3^{21} x_4^4 & 15480 x_2^9 x_3^{21} x_4^4 & 0 & -18144 x_2^8 x_3^{21} x_4^5 \\ -12960 x_1 x_2^8 x_3^{20} x_4^5 & -2448 x_2^9 x_3^{20} x_4^5 & 18144 x_2^8 x_3^{21} x_4^5 & 0 \end{pmatrix}.$$

In a similar way, the polydifferential operator  $\Gamma_2$  (encoded by the second tetrahedral graph in Fig. 1) yields the matrix

$$\mathcal{P}_2^{ij} = \begin{pmatrix} 16920 x_1^2 x_2^8 x_3^{20} x_4^4 & -12060 x_1 x_2^9 x_3^{20} x_4^4 & -16380 x_1 x_2^8 x_3^{21} x_4^4 & 42840 x_1 x_2^8 x_3^{20} x_4^5 \\ 2700 x_1 x_2^9 x_3^{20} x_4^4 & -7200 x_2^{10} x_3^{20} x_4^4 & 4680 x_2^9 x_3^{21} x_4^4 & -252 x_2^9 x_3^{20} x_4^5 \\ -13140 x_1 x_2^8 x_3^{21} x_4^4 & 5040 x_2^9 x_3^{21} x_4^4 & -12060 x_2^8 x_3^{22} x_4^4 & 13716 x_2^8 x_3^{21} x_4^5 \\ -80280 x_1 x_2^8 x_3^{20} x_4^5 & -18036 x_2^9 x_3^{20} x_4^5 & 21708 x_2^8 x_3^{21} x_4^5 & -58104 x_2^8 x_3^{20} x_4^6 \end{pmatrix}.$$

Notice that this coefficient matrix is not yet antisymmetric, but its *symmetric* counterpart is skipped out in the construction of the bi-vector  $\mathcal{P}_2$  and its transcription by using the anticommuting variables  $\xi$ . Therefore, we antisymmetrise the above matrix at once, the output to be used in what follows. We obtain that the bi-vector is

$$\begin{aligned} \mathcal{P}_2 = & -7380 x_1 x_2^9 x_3^{20} x_4^4 \xi_1 \xi_2 - 1620 x_1 x_2^8 x_3^{21} x_4^4 \xi_1 \xi_3 + 61560 x_1 x_2^8 x_3^{20} x_4^5 \xi_1 \xi_4 \\ & - 180 x_2^9 x_3^{21} x_4^4 \xi_2 \xi_3 + 8892 x_2^9 x_3^{20} x_4^5 \xi_2 \xi_4 - 3996 x_2^8 x_3^{21} x_4^5 \xi_3 \xi_4. \end{aligned}$$

We now see that for the Poisson bi-vector  $\mathcal{P}_0$  from Example 2 on p. 4, **the bi-vector  $\mathcal{P}_2$  does not vanish**, thereby disavowing the second claim from [1].

To check the compatibility of the original Poisson bi-vector  $\mathcal{P}_0$  with the newly obtained bi-vector  $\mathcal{P}_1$ , we calculate their Schouten bracket:

$$\begin{aligned} [\mathcal{P}_0, \mathcal{P}_1] = & 46008 x_1 x_2^{11} x_3^{26} x_4^5 \xi_1 \xi_2 \xi_3 + 852768 x_1 x_2^{11} x_3^{25} x_4^6 \xi_1 \xi_2 \xi_4 \\ & + 1246752 x_1 x_2^{10} x_3^{26} x_4^6 \xi_1 \xi_3 \xi_4 + 340200 x_2^{11} x_3^{26} x_4^6 \xi_2 \xi_3 \xi_4 \neq 0. \end{aligned}$$

The above expression is not identically zero. Therefore, **the leading term  $\mathcal{P}_1$  in the deformation  $\mathcal{P}_0 \mapsto \mathcal{P}(\varepsilon) = \mathcal{P}_0 + \varepsilon \mathcal{P}_1 + \bar{o}(\varepsilon)$  destroys the property of bi-vector  $\mathcal{P}(\varepsilon)$  to be Poisson** at  $\varepsilon \neq 0$  on all of  $\mathbb{R}^4$ .

The same compatibility test for  $\mathcal{P}_0$  and its second flow (2b) yields that

$$\begin{aligned} [\mathcal{P}_0, \mathcal{P}_2] = & -7668 x_1 x_2^{11} x_3^{26} x_4^5 \xi_1 \xi_2 \xi_3 - 142128 x_1 x_2^{11} x_3^{25} x_4^6 \xi_1 \xi_2 \xi_4 \\ & - 207792 x_1 x_2^{10} x_3^{26} x_4^6 \xi_1 \xi_3 \xi_4 - 56700 x_2^{11} x_3^{26} x_4^6 \xi_2 \xi_3 \xi_4. \end{aligned}$$

Again, this expression does not vanish identically on all of the Poisson manifold  $(\mathbb{R}^4, \{\cdot, \cdot\}_{\mathcal{P}_0})$ . We conclude that neither of two flows (2) preserve the property of bi-vector  $\mathcal{P}(\varepsilon)$  to stay (infinitesimally) Poisson at  $\varepsilon \neq 0$  for this example of Poisson bi-vector.<sup>7</sup>

<sup>7</sup> Let us also inspect whether the Jacobi identity holds for any of the bi-vectors  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . For  $\mathcal{P}_1$  we have that the left-hand side of the Jacobi identity is equal to

$$[\mathcal{P}_1, \mathcal{P}_1] = -2963589120 \cdot (x_1 x_2^{17} x_3^{41} x_4^8 \xi_1 \xi_2 \xi_3 + 5 x_1 x_2^{17} x_3^{40} x_4^9 \xi_1 \xi_2 \xi_4 - 2 x_1 x_2^{16} x_3^{41} x_4^9 \xi_1 \xi_3 \xi_4),$$

which does not vanish. For  $\mathcal{P}_2$  the left-hand side of the Jacobi identity equals

$$[\mathcal{P}_2, \mathcal{P}_2] = -262517760 \cdot (x_1 x_2^{17} x_3^{41} x_4^8 \xi_1 \xi_2 \xi_3 + 5 x_1 x_2^{17} x_3^{40} x_4^9 \xi_1 \xi_2 \xi_4 - 2 x_1 x_2^{16} x_3^{41} x_4^9 \xi_1 \xi_3 \xi_4).$$

This expression also does not vanish, so that neither  $\mathcal{P}_1$  nor  $\mathcal{P}_2$  are Poisson bi-vectors.

*Remark 2.* In the above example, the Schouten brackets  $[\![\mathcal{P}_0, \mathcal{P}_1]\!]$  and  $[\![\mathcal{P}_0, \mathcal{P}_2]\!]$  are determined by the same polynomials in the variables  $\mathbf{x}$  and  $\boldsymbol{\xi}$ : we see that  $[\![\mathcal{P}_0, \mathcal{P}_1]\!] = -6 \cdot [\![\mathcal{P}_0, \mathcal{P}_2]\!]$ . This implies that for this example of Poisson bi-vector  $\mathcal{P}_0$ , the leading term  $\mathcal{Q} := \mathcal{P}_1 + 6\mathcal{P}_2$  does (infinitesimally) preserve the property of  $\mathcal{P}(\varepsilon)$  to be Poisson in the course of deformation  $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ .

Moreover, it is readily seen that the ratio  $1 : 6$  is the *only* way to balance the two flows, (2a) vs (2b), such that their nontrivial linear combination  $\mathcal{Q}$  is compatible with the Poisson bi-vector  $\mathcal{P}_0$  from Example 2.<sup>8</sup>

*Remark 3.* In Example 4 the linear combination  $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2 \neq 0$  of two flows (2) is not identically equal to zero. (For other examples this may happen incidentally.) The leading term  $\mathcal{Q}$  in the infinitesimal deformation  $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$  is trivial in the Poisson cohomology with respect to  $\partial_{\mathcal{P}_0}$ , i.e.  $\mathcal{Q} = [\![\mathcal{P}_0, \mathcal{X}]\!]$  for some vector  $\mathcal{X}$  on the four-dimensional space.<sup>9</sup> Hence this  $\mathcal{Q}$  is trivially compatible with the Poisson bi-vector  $\mathcal{P}_0$ : namely,  $[\![\mathcal{P}_0, \mathcal{Q}]\!] \equiv 0$ , see p. 8 below.

In the three tables below we summarise the results about the flows  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , which we evaluate at the examples of Poisson bi-vectors  $\mathcal{P}_0$ . Special attention is paid to the leading deformation term  $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$  in each case: we inspect whether this bi-vector incidentally vanishes and whether it is (indeed, always) compatible with the original Poisson structure  $\mathcal{P}_0$ .

**Table 1.** The Poisson bi-vectors  $\mathcal{P}_0$  are generated using the determinant method from section 2.1 (the dimension is equal to 3, so we specify the fixed argument  $g_1$ ); that generator is combined with the pre-multiplication  $(f \cdot)$  as explained in section 2.2.

Nº	dim	Argument & pre-factor	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0?$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0?$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0?$
1.	3	$[x_1^5 x_2^3 x_3^4 + x_1^2 x_3^5 + x_1 x_2^5 x_3]$ $x_1^3 + x_2^2$	✗	✗	✗	✗	✓
2.	3	$[x_1 x_2 + x_1 x_3 + x_2 x_3]$ $x_1^2 + x_2$	✗	✗	✗	✗	✓

For both examples in Table 1 we have that neither does  $\mathcal{P}_1$  preserve the property of  $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$  to be (infinitesimally) Poisson nor does  $\mathcal{P}_2$  vanish identically — which is in contrast with both the claims from [1].

**Table 2.** In dimensions higher than 3, we generate the Poisson bi-vectors  $\mathcal{P}_0$  by using the determinant method from section 2.1: the auxiliary arguments  $g_1, \dots, g_{n-2}$  are specified.

Nº	dim	Arguments	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0?$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0?$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0?$
3.	4	$[x_2^3 x_3^2 x_4, x_1 x_3^4 x_4]$	✗	✗	✗	✗	✓
4.	4	$[x_1^2 x_2^3 x_3^4 x_4^5, x_1 x_2 x_3 x_4]$	✗	✗	✗	✓	✓
5.	4	$[x_2^2 x_3^2 x_4^2, x_1^2 x_3^2 x_4^2]$	✗	✗	✗	✓	✓
6.	5	$[x_2^3 x_3^2 x_4, x_1 x_3^4 x_4, x_3^3 x_4^2 x_5^4]$	✗	✗	✗	✗	✓

<sup>8</sup> The balance  $1 : \frac{4}{3}$  was considered in [22, §5.2] for the linear combination of flows (2a) and (2b), respectively.

<sup>9</sup> In all the two-dimensional Poisson geometries, the first flow  $\mathcal{P}_1$  is always cohomologically trivial, i.e. it is of the form  $\mathcal{P}_1 = [\![\mathcal{P}_0, \mathcal{X}]\!]$  for some one-vector  $\mathcal{X}$ , see [1].



In Table 2 we again have that neither is the property to be (infinitesimally) Poisson preserved for  $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$  nor is the bi-vector  $\mathcal{P}_2$  vanishing identically.

**Table 3.** The results for the Vanhaecke method from section 2.3: we here specify the bivariate polynomials  $\phi$ .

Nº	dim	$\phi(x, y)$	$[\mathcal{P}_0, \mathcal{P}_1] \stackrel{?}{=} 0$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\mathcal{P}_0, \mathcal{P}_2] \stackrel{?}{=} 0$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\mathcal{P}_0, \mathcal{Q}] \stackrel{?}{=} 0$
7.	4	$[x^2y^2]$	<b>X</b>	<b>X</b>	<b>X</b>	<b>X</b>	✓
8.	4	$[x^2y]$	<b>X</b>	<b>X</b>	<b>X</b>	<b>X</b>	✓
9.	4	$[x^3y^2]$	<b>X</b>	<b>X</b>	<b>X</b>	<b>X</b>	✓
10.	4	$[x^3y^3]$	<b>X</b>	<b>X</b>	<b>X</b>	<b>X</b>	✓
11.	6	$[x^2y^2]$	<b>X</b>	<b>X</b>	<b>X</b>	<b>X</b>	✓

The entries in Table 3 report on the use of the generator from section 2.3: experimentally established, the properties of these Poisson bi-vectors do not match both the claims from [1].

**Table 4.** The results for the infinite-dimensional case.

Nº	dim	Operator	$[\mathcal{P}_0, \mathcal{P}_1] \stackrel{?}{=} 0$	$\mathcal{P}_2 \stackrel{?}{=} 0$
12.	$\infty$	$u^2 \circ d/dx \circ u^2$	<b>X</b>	✓

The variational bi-vector  $\mathcal{P}_1 = \frac{1}{2}\langle \xi \cdot \vec{A}_1(\xi) \rangle$ , which we construct from the variational Poisson bi-vector  $\mathcal{P}_0 = \frac{1}{2}\langle \xi \cdot u^2 \vec{d}/dx(u^2 \xi) \rangle$  by using the geometric technique from [13] (see also [14]), is determined by the (skew-adjoint part of the) first-order differential operator  $A_1 = 192(9u^8u_xu_{xx} - u^9u_{xxx})d/dx$  in total derivatives. Again (see Table 4), the two variational bi-vectors are *not* compatible: we check that  $[\mathcal{P}_0, \mathcal{P}_1] \not\equiv 0$  under the variational Schouten bracket. Remarkably, the variational bi-vector  $\mathcal{P}_2$  is specified by the second-order total differential operator whose skew-adjoint component vanishes, whence the respective variational bi-vector *is* equal to zero (modulo exact terms within its horizontal cohomology class [11]).

## Conclusion

The linear combination  $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$  of the Kontsevich tetrahedral flows preserves the space of Poisson bi-vectors  $\mathcal{P}_0$  under the infinitesimal deformations  $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ . This is manifestly true for all the examples of Poisson bi-vectors on finite-dimensional (vector or affine) spaces  $\mathbb{R}^n$  which we have considered so far. We conjectured that the leading deformation term  $\mathcal{Q} = \mathcal{Q}(\mathcal{P}_0)$  always has this property, that is, **the bi-vector  $\mathcal{Q}$  marks a  $\partial_{\mathcal{P}_0}$ -cohomology class for every Poisson bi-vector  $\mathcal{P}_0$**  on a finite-dimensional affine manifold. (Recall that such class can be  $\partial_{\mathcal{P}_0}$ -trivial; moreover, the bi-vector  $\mathcal{Q}$  can vanish identically — yet the above examples confirm the existence of Poisson geometries where neither of the two options is realised.)

Let us conclude that every claim of an object's vanishing by virtue of the skew-symmetry and Jacobi identity for a given Poisson bi-vector, which that object depends on by construction, must be accompanied with an explicit description of that factorisation mechanism (e.g., see [5]) or at least, with a proof of that mechanism's existence. Apart from the trivial case (here,  $\mathcal{Q} = 0$  so that  $[\mathcal{P}_0, \mathcal{Q}] \equiv 0$ ), such factorisation through the master-equation  $[\mathcal{P}_0, \mathcal{P}_0] = 0$  can be immediate: here, we have that  $[\mathcal{P}_0, \mathcal{Q}] = [\mathcal{P}_0, [\mathcal{P}_0, \mathcal{X}]] = \frac{1}{2}[[\mathcal{P}_0, \mathcal{P}_0], \mathcal{X}] = (\frac{1}{2}[\cdot, \mathcal{X}])([\mathcal{P}_0, \mathcal{P}_0])$  for all  $\partial_{\mathcal{P}_0}$ -exact infinitesimal deformations  $\mathcal{Q} = \partial_{\mathcal{P}_0}(\mathcal{X})$  of the Poisson bi-vectors  $\mathcal{P}_0$ . Elaborated in [5], the Poisson cohomology estimate mechanism of the vanishing  $[\mathcal{P}_0, \mathcal{Q}] \equiv 0$  via  $[\mathcal{P}_0, \mathcal{P}_0] = 0$

works – for the nontrivial cocycles  $\mathcal{Q} \notin \text{im } \partial_{\mathcal{P}_0}$  in the  $\partial_{\mathcal{P}_0}$ -cohomology – due to much more refined principles. That vanishing mechanism is applied to the factorisation problem at hand in the paper [6] (joint with R. Buring), where we prove the above conjecture.

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### Appendix A. The mechanism of vanishing for $[\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})] = 0$ : an example

We wish to recognize the differential consequences of the Jacobi identity in the compatibility equation  $[\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})] = 0$ , to understand why it holds. By a straightforward calculation we learn that  $[\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})] = 0$  for all Poisson bi-vectors on  $\mathbb{R}^3$ . But as soon as the differential consequences of the Jacobi identity are recognized, they can be translated into graphs. Independent of dimension, the language of graphs then answers the question which we started out with. This answer is found in [6].

Let us now illustrate a more analytic approach to the factorization problem for  $[\mathcal{P}, \mathcal{Q}_{1:6}] = 0$  via  $[\mathcal{P}, \mathcal{P}] = 0$  (see [6, App. D] for details). The compatibility equation is a vanishing expression, which is impossible to factorize through the Jacobi identity, which itself is also zero. To make both visible, we perturb a given Poisson bi-vector  $\mathcal{P}$  using  $\tilde{\mathcal{P}} = \mathcal{P} + \epsilon \cdot \Delta$  for a bi-vector  $\Delta$ , in such a way that  $\tilde{\mathcal{P}}$  is no longer Poisson, thereby  $[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}] \neq 0$ . The goal is to perturb the bi-vector  $\mathcal{P}$  such that the left-hand side  $[\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]$  becomes non-zero as well. Now the Jacobi identity’s non-zero differential consequences becomes recognizable in the non-zero expression  $[\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]$ .

**Example 5.** Consider the Poisson bi-vector obtained on  $\mathbb{R}^3$  from the determinant construction using two functions  $g(z)$  and  $f(x)$  as argument and pre-multiplication factor, respectively. Let the perturbation  $\Delta$  be given component-wise by  $\Delta^{12} = f_1(y, z)$ ,  $\Delta^{13} = f_2(y, z)$  and  $\Delta^{23} = 0$ . The perturbed bi-vector then equals

$$\tilde{\mathcal{P}} = \begin{bmatrix} 0 & f \cdot dg/dz & 0 \\ -f \cdot dg/dz & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon \cdot \begin{bmatrix} 0 & f_1 & f_2 \\ -f_1 & 0 & 0 \\ -f_2 & 0 & 0 \end{bmatrix}.$$

The left-hand sides of the Jacobi identity and of the compatibility condition are evaluated to

$$[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}]^{123} = \epsilon f_2 \cdot \frac{df}{dx} \frac{dg}{dz} + \bar{o}(\epsilon), \quad [\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}]^{123} = -\epsilon \cdot \frac{\partial^3 f_2}{\partial y^3} \left( \frac{df}{dx} \right)^4 \left( \frac{dg}{dz} \right)^4 + \bar{o}(\epsilon).$$

There is only one way to recognize a differential consequence of the Jacobiator inside  $[\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]^{123}$ . Namely, the Jacobi identity contains a product of  $f_2$  and derivatives of  $f$  and  $g$ . The same is true for its non-zero differential consequences. Let us extract this product from  $[\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]^{123}$ . The only differential consequences of  $f_2$ ,  $df/dx$ , and  $dg/dy$  in  $[\mathcal{P}, \mathcal{Q}_{1:6}]^{123}$  are  $\partial^3 f_2 / \partial y^3$ ,  $df/dx$  and  $dg/dz$ , respectively. This hints that we have the differential consequence  $[\mathcal{P}, \mathcal{P}]_{yyy}^{123}$ . To understand what its coefficient is, we note that the remaining co-factors in  $[\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]^{123}$  form  $(\mathcal{P}_x^{12})^3$ . We conclude that the left-hand side of the compatibility equation factorizes through the Jacobi identity as follows

$$[\mathcal{P}, \mathcal{Q}_{1:6}]^{123} = \mathcal{P}_x^{12} \mathcal{P}_x^{12} \mathcal{P}_x^{12} [\mathcal{P}, \mathcal{P}]_{yyy}^{123} + \dots$$

Looking at this expression, we construct a list of graphs that can encode it. Such a list fully formed, it is subtracted from  $[[\mathcal{P}, \mathcal{Q}_{1:6}]]$  and resolved with respect to the coefficients of every proposed graph. We keep subtracting the already found graphs from any non-zero perturbations of  $[[\mathcal{P}, \mathcal{Q}_{1:6}]]$  in the future, once the coefficients are known. The example under study gave us the tripod graph, which is the first entry in [6, Eq. (6)]. Proceeding in the same way, we also recognized the 'elephant' graph, which is the sixth entry in that solution (cf. [6, Remarks 10–11]).

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