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Processus de croissance-fragmentation multitypes et excursions planaires

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excursions planaires

Multitype growth-fragmentation processes and planar
excursions

Processus de croissance-fragmentation multitypes et excursions planaires

Résumé

Ce travail est consacré à l'étude des processus de croissance-fragmentation, possiblement avec types, en lien avec les excursions planaires et la gravité quantique de Liouville. Les processus de croissance-fragmentation forment un système branchant de particules introduit par Bertoin [Ber17b], dirigé par un processus de Markov X auto-similaire positif. A chaque instant $s > 0$ où la cellule Ève X fait un saut $-y = \Delta X(s)$ négatif, on introduit une nouvelle particule, dont la *taille* à l'origine est donnée par $y > 0$. Conditionnellement à leurs tailles à la naissance, ces particules évoluent toutes de manière indépendante, en suivant une copie de X également indépendante de la cellule mère. Elles produisent à leur tour une lignée d'individus, et on obtient ainsi par récurrence une collection de cellules qui descendent toutes, d'une manière ou d'une autre, de la cellule Ève initiale. Dans un article fondateur pour notre travail, Bertoin, Budd, Curien et Kortchemski [BBCK18] révèlent la structure spinale d'un tel système branchant de particules, ainsi qu'une famille particulière de processus de croissance-fragmentation qui s'obtient comme limite d'échelle des périmètres dans un processus d'épluchage markovien de grandes cartes aléatoires.

Dans un premier travail en commun avec Élie Aïdékon, nous nous intéressons à une excursion brownienne de 0 à 1 dans le demi-plan supérieur. Nous montrons qu'en coupant une telle excursion à des hauteurs successives, et en enregistrant la *taille* de ces excursions, on retrouve une version signée du processus de croissance-fragmentation découvert par [BBCK18] dans le cas *critique*. Nous démontrons également que la martingale dérivée dans ce cas critique converge presque sûrement vers la durée de l'excursion.

Ces résultats nous amènent par ailleurs à étudier les processus de croissance-fragmentation dans le cadre signé. Dans cette direction, nous établissons dans un travail ultérieur la décomposition spinale du processus de croissance-fragmentation signé. Nous mettons en évidence une telle famille de processus de branchement, reliée à celle de [BBCK18], qui s'obtient en coupant dans le demi-plan des excursions dont la partie réelle est un processus stable.

L'approche de [BBCK18] irrigue d'autre part un troisième projet, en commun avec Juan Carlos Pardo, dans lequel nous présentons une généralisation aux processus avec un nombre fini de *types*. Nos arguments s'appuient sur la théorie des processus de Markov additifs et nous permettent d'établir la décomposition spinale en ces termes. Enfin, nous nous intéressons à une extension au cadre vectoriel isotropique, où les angles peuvent être vus comme des types (infinis) sur la sphère. A la lumière de ces résultats, nous verrons qu'une certaine famille remarquable de processus de croissance-fragmentation vectoriels isotropiques apparaît dans des excursions au-dessus du demi-espace.

La dernière partie de cette thèse présente quelques avancées d'un travail en commun avec Ellen Powell et Alexander Watson portant sur la structure branchante obtenue en explorant un disque quantique de paramètre $\gamma \in (\sqrt{2}, 2)$ avec une courbe $\text{SLE}_{\kappa'}$ remplissante, où $\kappa' = 16/\gamma^2$. Ces considérations s'interprètent au niveau des excursions planaires à travers l'*accouplement d'arbres*, où une telle exploration s'encode naturellement par une excursion brownienne dans un cône. Notre résultat principal dans ce sens caractérise le processus de croissance-fragmentation obtenu dans le cas où $\gamma = \sqrt{8/3}$, aussi appelé *gravité pure*.

Mots-clés : processus de croissance-fragmentation, processus de Markov autosimilaire, décomposition épinale, processus de Lévy stables, excursions, mouvement brownien plan, champ libre gaussien, gravité quantique de Liouville, processus SLE, *mating-of-trees*.

Multitype growth-fragmentation processes and planar excursions

Abstract

This work is devoted to the study of growth-fragmentation processes, possibly with types, in connection with planar excursions and Liouville quantum gravity. Growth-fragmentation processes form a branching particle system introduced by Bertoin [Ber17b] and driven by a positive self-similar Markov process X . At each time $s > 0$ when the Eve cell X makes a negative jump $-y = \Delta X(s)$, we introduce a new particle, whose *size* at birth is given by $y > 0$. Conditional on their sizes at birth, these particles all evolve independently, according to a copy of X also independent of the mother cell. They in turn produce a lineage of individuals, and thus by recursion we get a collection of cells that are all somehow descended from the initial Eve cell. In a seminal paper, Bertoin, Budd, Curien and Kortchemski [BBCK18] revealed the spinal structure of such a branching particle system, as well as a particular family of growth-fragmentation processes showing up in the scaling limit of perimeters in a Markovian peeling process of large random maps.

In a first joint work with Élie Aïdékon, we consider a Brownian excursion from 0 to 1 in the upper half-plane. We show that when slicing the excursion at successive heights, and recording the *size* of these excursions, we recover a signed version of the growth-fragmentation process found by [BBCK18] in the *critical* case. We also prove that, in this critical case, the derivative martingale converges almost surely towards the duration of the excursion.

These results lead us to study growth-fragmentation processes in the signed framework. In this direction, we establish in a subsequent work the spinal decomposition of the signed growth-fragmentation cell system. We reveal such instances of signed growth-fragmentation processes, related to that of [BBCK18], which appear when slicing a distinguished family of half-plane excursions with stable Lévy real part.

The approach of [BBCK18] also pervades a third project, joint with Juan Carlos Pardo, in which we present a generalisation to processes with a finite number of *types*. Our arguments are based on the theory of Markov additive processes and allow us to describe the spinal structure in these terms. Finally, we are interested in an extension to the spatial isotropic framework, where the angles can be seen as (infinite) types on the sphere. In light of these results, we find that a remarkable family of spatial isotropic growth-fragmentation processes appears in excursions away from the half-space.

The final part of this thesis presents some advances in a joint work with Ellen Powell and Alexander Watson towards understanding the branching structure in a space-filling $\text{SLE}_{\kappa'}$ exploration of a quantum disc with parameter $\gamma \in (\sqrt{2}, 2)$, where γ and κ' are related *via* $\kappa' = 16/\gamma^2$. These considerations are interpreted at the level of planar excursions through the *mating-of-trees* theory, where such an exploration is naturally encoded by a Brownian cone excursion. Our main result characterises the growth-fragmentation process obtained in the case where $\gamma = \sqrt{8/3}$, sometimes referred to as *pure gravity*.

Keywords: growth-fragmentation processes, self-similar Markov processes, spine decomposition, stable Lévy processes, excursion theory, planar Brownian motion, Gaussian free field, Liouville quantum gravity, SLE processes, mating-of-trees theory.

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Chapter 1

Introduction

Les processus de croissance-fragmentation forment une famille de processus de branchement introduits par Bertoin [Ber17b]. Ils décrivent l'évolution d'un nuage de particules dont la taille peut varier et qui peuvent se fragmenter au cours du temps en donnant naissance à de nouvelles particules. Ces fragmentations sont binaires et se produisent de manière conservative : à chaque division, la taille de la cellule fille est donnée par le saut de celle de la cellule mère. Les marches aléatoires branchantes et les martingales additives fournissent un puissant arsenal pour étudier ces phénomènes.

Dans la première section, nous exposons les principaux résultats décrivant les processus de croissance-fragmentation autosimilaires. Ensuite, suivant les travaux pionniers de Bertoin, Budd, Curien et Kortchemski [BBCK18], nous décrivons de telles structures branchantes apparaissant dans le processus d'épluchage de grandes cartes aléatoires de Boltzmann. Cette section sera également l'occasion de présenter la bijection de Sheffield (ou bijection de hamburger-cheeseburger) et quelques éléments de gravité quantique de Liouville pour préparer le terrain au cas de l'exploration du disque quantique qui nous intéresse. Enfin, la dernière partie de cette introduction détaille les principaux résultats obtenus dans cette thèse.

1.1 Processus de croissance-fragmentation autosimilaires

Dans cette section, nous décrivons les principaux résultats de cette étude dans le cas où le processus possède une propriété d'invariance d'échelle (autosimilarité). Ces résultats apparaissent pour la plupart dans un article fondateur de Bertoin, Budd, Curien et Kortchemski [BBCK18]. Une partie de notre travail a été d'étendre cette construction dans le cadre où la masse des atomes peut être signée, ou plus généralement avoir un type (une couleur).

Construction du système de particules. On considère un processus de Markov X à valeurs positives, qui sous $\mathbb{P}_x$ commence en $x > 0$. On suppose X *autosimilaire d'indice α* , au sens où pour tous $c, x > 0$, la loi de $(cX(c^{-\alpha}t), t \geq 0)$ sous $\mathbb{P}_x$ est $\mathbb{P}_{cx}$ (on prendra garde au fait que notre indice α est l'opposé de celui de [BBCK18]). On fait de plus l'hypothèse que X converge en l'infini (vers 0), ou que X est absorbé après un temps fini ζ en un état cimetière ∂ . Ce processus modélise le comportement d'une particule de notre nuage. L'idée est ensuite d'introduire une nouvelle particule à chaque temps de saut *négatif* de X , dont la

taille à la naissance est donnée par l'opposé du saut, de sorte que cette nouvelle particule ait une masse positive. Cette cellule fille évoluera alors avec le même comportement que la cellule mère X , donnant elle-même naissance à d'autres cellules. Notons que dans ce modèle les dislocations se font de manière conservative.

Formellement, on construit le processus de croissance-fragmentation dirigé par X de la manière suivante. On introduit l'arbre d'Ulam $\mathbb{U} := \bigcup_{i \geq 0} \mathbb{N}^i$ pour encoder les générations. Nous prenons ici la notation anglo-saxonne $\mathbb{N} = \{1, 2, \dots\}$, et par convention on a noté $\mathbb{N}^0 := \{\emptyset\}$, où $\emptyset$ sera l'étiquette de la première particule du système, appelée Ève. Pour $u \in \mathbb{U}$, on notera $|u|$ la longueur du mot u . On définit donc un processus $\mathcal{X}_\emptyset$ de même loi que X , et on pose $b_\emptyset = 0$. Comme $\mathcal{X}_\emptyset$ converge en l'infini, on peut ordonner les tailles et temps de sauts positifs $(x_i, \beta_i)_{i \geq 1}$ de $-\mathcal{X}_\emptyset$ dans l'ordre lexicographique. Conditionnellement à ces sauts, on définit la première génération $(\mathcal{X}_i, i \geq 1)$ de notre système, comme une suite de processus indépendants de lois respectives $\mathbb{P}_{x_i}$. On introduit l'instant de naissance $b_i := b_\emptyset + \beta_i$ de la i -ème particule, et on note ζ_i sa durée de vie. On procède de même pour les générations suivantes. Ceci détermine un unique système de particules $(\mathcal{X}_u, u \in \mathbb{U})$ indexé par $\mathbb{U}$, de sorte que les enfants de la particule $u \in \mathbb{N}^r$ sont ceux dont l'étiquette est de la forme (u, i) pour $i \in \mathbb{N}$. Nous noterons $\mathcal{P}_x$ la loi de ce système de particules issu de $x > 0$. La figure 1.1 schématise cette construction.

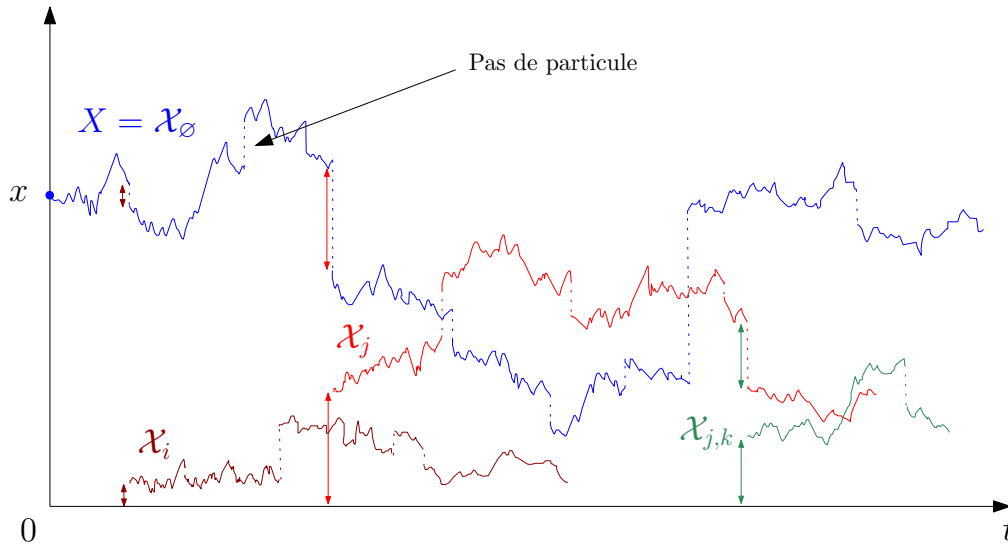


Figure 1.1 – Construction du système de particules dirigé par X . À chaque saut négatif, une nouvelle particule est introduite dans le système, de taille donnée par l'opposé du saut. Le lecteur doit imaginer ces instants de saut comme formant un ensemble dense de temps.

Le processus de croissance-fragmentation est alors la collection

$$\mathbf{X}(t) := \{\{\mathcal{X}_u(t - b_u), b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

de toutes les particules en vie à l'instant t . La construction précédente entraîne naturellement une propriété de branchement qui est au cœur de l'étude des processus de croissance-fragmentation ; voir [Ber17b, Propositions 1 et 2] pour un énoncé généalogique ou temporel.

Les martingales additives et le cumulatif κ . Cette construction est intimement liée aux marches aléatoires branchantes, pour lesquelles on renvoie aux notes de cours [Shi15]. Il n'est pas difficile de voir que le processus

$$\mathcal{Z}_n := \sum_{|u|=n} \delta_{-\log(\mathcal{X}_u(0))}(dz), n \geq 0,$$

est une marche aléatoire branchante. Un objet central à l'étude des marches branchantes est donné par les *martingales additives*, introduites par Mandelbrot et par Kingman [Man74, Kin75], puis exploitées dans un article fondateur de Biggins [Big77]. Dans la traduction précédente en termes de croissance-fragmentation, elles reviennent à sommer la masse des atomes de la génération n , à certaines puissances. Au vu de la propriété de branchement et de l'autosimilarité, il suffit de s'intéresser pour $q \geq 0$ à

$$m(q) := \mathcal{E}_1 \left(\sum_{i=1}^{\infty} X_i(0)^q \right) = \mathbb{E}_1 \left(\sum_{0 < s < \zeta} (-\Delta X(s))^q \mathbb{1}_{\Delta X(s) < 0} \right).$$

Cette quantité se calcule explicitement, faisant intervenir une fonction κ qui jouera un rôle crucial dans l'étude des processus de croissance-fragmentation. Plus précisément, *via* la transformation de Lamperti [Lam72], sous $\mathbb{P}_x$ on peut écrire

$$X(t) := x \exp(\xi(\tau(tx^{-\alpha}))), \quad t \geq 0, \quad (1.1.1)$$

où ξ est un processus de Lévy dont on note ψ l'exposant de Laplace et Λ la mesure de Lévy, et

$$\tau(t) := \inf \left\{ s > 0, \int_0^s \exp(\alpha \xi(u)) du > t \right\}, \quad t \geq 0.$$

Dans l'écriture (1.1.1), on prend la convention que $X(t) = \partial$ lorsque $t \geq x^\alpha \int_0^\infty \exp(\alpha \xi(u)) du$. Après application de la formule de compensation à ξ , on obtient

$$m(q) = \begin{cases} 1 - \frac{\kappa(q)}{\psi(q)} & \text{si } \kappa(q) < \infty \text{ et } \psi(q) < 0, \\ +\infty & \text{sinon,} \end{cases} \quad (1.1.2)$$

où

$$\kappa(q) := \psi(q) + \int_{(-\infty, 0)} \Lambda(dx) (1 - e^x)^q, \quad q \geq 0, \quad (1.1.3)$$

est appelé *cumulant* du processus de croissance-fragmentation $\mathbf{X}$. La relation (1.1.2) implique que les exposants q donnant une martingale additive sont les zéros de κ . Par ailleurs, κ étant convexe, il existe au plus deux tels exposants ; dans le cas où κ admet deux racines, nous les noterons $\omega_- < \omega_+$. En résumé, si ω est une racine de κ , le processus

$$\mathcal{M}(n) = \sum_{|u|=n+1} \mathcal{X}_u(0)^\omega, \quad n \geq 0, \quad (1.1.4)$$

est une martingale sous $\mathcal{P}_x$ pour tout $x > 0$ (et pour la filtration générationnelle $(\mathcal{G}_n, n \geq 0)$). Lorsqu'il n'y a pas de perte de masse, la propriété de martingale générationnelle peut parfois être étendue au cadre temporel en sommant les particules en vie à un instant donné, voir [BBCK18, Corollaires 3.7 et 3.9].

Enfin, soulignons une fois de plus l'importance du cumulants pour caractériser le processus de croissance-fragmentation. Remarquons d'abord que deux cellules Ève distinctes peuvent induire le même processus de croissance-fragmentation : décider, à chaque division, de la cellule mère et de la cellule fille relève *a priori* d'un choix. Ainsi, connaissant la loi de $\mathbf{X}$, il n'est pas possible en général de retrouver la loi de X (c'est-à-dire l'exposant ψ). Néanmoins, une manière canonique de reconstruire une cellule Ève [BBCK18, Théorème 5.1] consiste à s'intéresser au *plus grand fragment local*, obtenu en suivant à chaque instant de division le fragment dont la taille est maximale. En fait, Q. Shi a démontré [Shi17] que le couple (κ, α) détermine complètement la loi du processus de croissance-fragmentation $\mathbf{X}$: deux tels processus de même indice d'autosimilarité ont la même loi si, et seulement si, les cumulants

associés sont égaux. L'idée heuristique est qu'il faut rajouter le deuxième terme dans (1.1.3) pour tenir compte de ce degré de liberté.

Décomposition spinale pour le processus de croissance-fragmentation. Toujours en lien avec les marches aléatoires branchantes, l'article [BBCK18] décrit la décomposition spinale du système de particules précédent. Dans le cas des marches branchantes, l'idée remonte à Kahane et Peyrière [KP76] et consiste à introduire une nouvelle mesure en biaisant la probabilité initiale à l'aide de la martingale additive. Une description du système de particules en termes de *décomposition spinale* est présentée par R. Lyons dans [Lyo97] (voir aussi [LPP95] pour le cas des arbres de Galton-Watson, et [Shi15] pour de nombreuses applications). Sous la nouvelle mesure, on obtient alors une particule spéciale, appelée *épine*, qui se comporte différemment des autres particules, et autour de laquelle sont greffées des copies de la marche branchante originale. De même, les martingales (1.1.4) correspondant aux (éventuelles) racines ω de κ suggèrent d'introduire une nouvelle mesure $\hat{\mathcal{P}}_x$, dont la dérivée de Radon-Nikodym par rapport à $\mathcal{P}_x$ est donnée par $x^{-\omega} \mathcal{M}(n)$ sur $\mathcal{G}_n$. On distingue ensuite une feuille ou *épine* $\mathcal{L} \in \partial \mathbb{U}$ spéciale. Conditionnellement à $\mathcal{G}_n$, la loi du parent $\mathcal{L}(n+1)$ de $\mathcal{L}$ à la génération $n+1$ est donnée par

$$\hat{\mathcal{P}}_x(\mathcal{L}(n+1) = v | \mathcal{G}_n) = \frac{\mathcal{X}_v(0)^\omega}{\mathcal{M}(n)},$$

pour tout $v \in \mathbb{U}$ tel que $|v| = n+1$. Pour décrire le processus de croissance-fragmentation sous la nouvelle mesure $\hat{\mathcal{P}}_x$, on introduit le processus $\hat{\mathcal{X}}$ qui décrit l'évolution du fragment correspondant à cette épine $\mathcal{L}$. On obtient alors [BBCK18, Théorème 4.2] que l'exposant de Lamperti du processus autosimilaire $\hat{\mathcal{X}}$ a pour exposant de Laplace $q \mapsto \kappa(\omega + q)$. Par ailleurs, de manière analogue au cas des marches aléatoires branchantes, on peut reconstruire le processus de croissance-fragmentation sous $\hat{\mathcal{P}}_x$ en introduisant des copies de la loi initiale $\mathbf{X}$ à chaque saut de l'épine. Rappelons qu'il existe au plus deux exposants $\omega_- < \omega_+$ racines de κ , ce qui donne au plus deux épines Y^- et Y^+ associées respectivement aux exposants $\kappa(\omega_- + \cdot)$ et $\kappa(\omega_+ + \cdot)$.

Une famille remarquable de processus de croissance-fragmentation. Nous introduisons ici la famille de processus de croissance-fragmentation découverte par Bertoin, Budd, Curien et Kortchemski dans [BBCK18, Section 5]. Ceci facilitera la description des processus intervenant dans la section suivante pour les cartes de Boltzmann. On fixe $\theta \in (\frac{1}{2}, \frac{3}{2}]$. Soit Υ_θ un processus de Lévy θ -stable. Nous invitons le lecteur à ne pas se soucier des caractéristiques précises de ce processus, qui ne seront pas pertinentes au niveau où on se place. Néanmoins, pour lever toute ambiguïté, précisons qu'on considère ici le θ -stable de paramètre de positivité $\rho := \mathbb{P}(\Upsilon_\theta(1) \geq 0)$ vérifiant $\theta(1 - \rho) = \frac{1}{2}$, normalisé de sorte que sa mesure de Lévy soit

$$\frac{\Gamma(\theta+1)}{\pi} \cos((\theta+1)\pi) \frac{dx}{x^{1+\theta}} \mathbb{1}_{x>0} + \frac{\Gamma(\theta+1)}{\pi} \frac{dx}{|x|^{1+\theta}} \mathbb{1}_{x<0}.$$

Soient $\Upsilon_\theta^\uparrow$ et $\Upsilon_\theta^\downarrow$ les versions de Υ_θ respectivement conditionnée à rester positif et conditionnée à être absorbé continûment en 0, étudiées notamment par Caballero et Chaumont [CC06]. Enfin, définissons

$$\nu_\theta(dx) := \frac{\Gamma(\theta+1)}{\pi} \left(\mathbb{1}_{\frac{1}{2} < x < 1} - \cos(\pi\theta) \mathbb{1}_{x>1} \right) \frac{dx}{|x(1-x)|^{1+\theta}},$$

et $\Lambda_\theta(dx)$ la mesure image de ν_θ par $x \mapsto \ln x$, qui est une mesure sur $(-\ln(2), \infty)$. Le théorème suivant est le résultat de [BBCK18, Proposition 5.2], combiné avec [BBCK18, Theorem 5.1].

Théorème 1.1.1. *Il existe un processus de croissance-fragmentation $\mathbf{X}_\theta$, autosimilaire d'indice θ , dont le cumulante est*

$$\kappa_\theta(q) = \frac{\cos(\pi(q - \theta))}{\sin(\pi(q - 2\theta))} \frac{\Gamma(q - \theta)}{\Gamma(q - 2\theta)}, \quad \theta < q < 2\theta + 1.$$

Ce processus peut être construit à partir de la cellule Ève X_θ dont l'exposant de Lamperti ξ_θ a pour exposant de Laplace

$$\psi_\theta(q) = \kappa_\theta(q) - \int_{-\ln(2)}^0 \Lambda_\theta(dx)(1 - e^x)^q, \quad \theta < q < 2\theta + 1.$$

En particulier, ξ_θ a pour mesure de Lévy Λ_θ . De plus, les racines de κ_θ sont

$$\omega_- = \theta + \frac{1}{2} \quad \text{et} \quad \omega_+ = \theta + \frac{3}{2},$$

et les épines Y_θ^- et Y_θ^+ associées sont respectivement distribuées selon $\Upsilon_\theta^\downarrow$ et $\Upsilon_\theta^\uparrow$.

Concluons cette section par quelques remarques.

- Remarques.** (i) Le Théorème 1.1.1 donne une famille de processus de croissance-fragmentation que nous appellerons *canonique*, au sens où l'épine est reliée à un processus de Lévy θ -stable. L'épine doit être positive, ce qui rend naturel de considérer les versions $\Upsilon_\theta^\downarrow$ et $\Upsilon_\theta^\uparrow$.
- (ii) On peut expliciter l'exposant de Laplace ψ_θ (voir [BBCK18, équation (28)], en prenant garde à une coquille dans le drift). Nous avons choisi de ne pas le faire ici pour simplifier l'exposé.
- (iii) Enfin, la mesure Λ_θ de l'exposant ξ est portée par $(-\ln(2), \infty)$. Ceci signifie que la cellule Ève X_θ correspond en fait au *plus grand fragment local* : pour tout temps t où X a un saut $\Delta X(t)$ négatif, $X(t) > -\Delta X(t)$.

1.2 Liens avec les cartes planaires et la gravité quantique de Liouville

On présente ici quelques modèles de cartes planaires qui sont en lien avec les résultats de cette thèse. Le but n'est pas ici de faire une exposition détaillée, mais de proposer une discussion informelle en guise d'invitation, où on prend la liberté de tracer un chemin parmi d'autres. Le lecteur trouvera une introduction bien plus détaillée dans les notes de cours de Curien [Cur19]. Nous souhaitons également souligner la clarté de l'introduction (en français) de la thèse de L. Chen [Che18]. L'exposition que nous donnons du modèle FK et de la bijection de Sheffield puise son inspiration dans les notes de cours de N. Berestycki et E. Powell [BP21]. Enfin, les deux dernières parties forment une mosaïque d'idées principalement tirées de l'approche de Duplantier, Miller et Sheffield [DMS14].

1.2.1 Les processus de croissance-fragmentation dans les cartes de Boltzmann

On expose ici les résultats de [BBCK18, Section 6]. Après avoir introduit le modèle de cartes planaires considérées, on présentera une version du processus d'exploration (*peeling*) dû à Budd [Bud15]. On verra enfin comment une certaine famille remarquable de processus

de croissance-fragmentation apparaît comme limite d'échelle de périmètres dans une telle exploration markovienne. Nos considérations seront surtout qualitatives : on cache ici une composante combinatoire essentielle, pour laquelle on renvoie le lecteur intéressé à [BBG12], [Bud15] et [BC17].

Cartes de Boltzmann critiques non génériques. On commence par présenter le modèle de cartes planaires étudié dans [BBCK18]. Le but est voir comment certains processus de croissance-fragmentation interviennent naturellement à la limite en explorant de façon markovienne un modèle de cartes planaires. Les résultats présentés ici sont donc issus de [BBCK18] (voir également [BCK18] pour le cas des triangulations). On définit une *carte planaire* de façon informelle comme un graphe fini dessiné dans le plan, vu à déformation continue préservant l'orientation près (voir Figure 1.2). Précisons que pour nous, une carte planaire est par défaut finie, et que dans notre définition de graphe, on autorise un sommet à être relié à lui-même, ou deux sommets à être reliés entre eux par plusieurs arêtes. Si $\mathbf{m}$ est une carte planaire, on appelle respectivement $\text{Vertices}(\mathbf{m})$, $\text{Edges}(\mathbf{m})$ et $\text{Faces}(\mathbf{m})$ l'ensemble des sommets, arêtes et faces de $\mathbf{m}$. D'autre part, on se restreint dans toute la suite au cadre des cartes planaires *enracinées*, c'est-à-dire qu'une carte $\mathbf{m}$ vient implicitement avec une arête particulière, qui est orientée, et qu'on appelle *arête racine*. Le *degré* $\deg f$ d'une face f de $\mathbf{m}$ est le nombre d'arêtes qu'on parcourt en traçant le contour de la face (de cette façon, une arête interne à la face compte double). Par convention, on note alors f_{root} la face à droite de l'arête racine, et on appelle $\deg f_{\text{root}}$ le *périmètre* de la carte $\mathbf{m}$. Enfin, on considère dans cette section que les cartes sont *biparties*, ce qui signifie que chaque face a un degré pair (en particulier, le périmètre est pair). Pour $\ell \geq 0$ et $n \geq 0$, on notera $\text{Map}_n^{(\ell)}$ l'ensemble des cartes planaires à n sommets et de périmètre 2ℓ , $\text{Map}^{(\ell)} := \bigcup_{n \geq 0} \text{Map}_n^{(\ell)}$, et $\text{Map} := \bigcup_{\ell \geq 0} \text{Map}^{(\ell)}$.

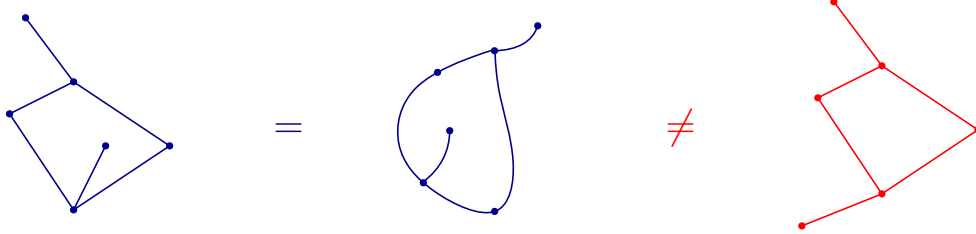


Figure 1.2 – Exemples de cartes planaires : les deux cartes bleues sont identiques, mais la carte rouge est différente.

On va maintenant définir une loi de probabilité de Boltzmann $\mathbb{P}^{(\ell)}$ sur $\text{Map}^{(\ell)}$. Il conviendra également de définir la loi $\mathbb{P}_{\bullet}^{(\ell)}$ de *cartes de Boltzmann pointées*, c'est-à-dire en prenant également un sommet uniformément sur la carte aléatoire biaisée par sa taille. Pour cela, on se donne une famille de poids positifs $\mathbf{q} = (q_k)_{k \geq 1}$ tels qu'il existe $k > 1$ satisfaisant $q_k > 0$, et on introduit la mesure w définie par ces poids :

$$w(\mathbf{m}) := \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_{\text{root}}\}} \frac{q_{\deg f}}{2}, \quad \mathbf{m} \in \text{Map}.$$

Supposons que $\mathbf{q}$ est *admissible*, au sens où la somme $W_{\bullet}^{(\ell)} := \sum_{n \geq 0} n w(\text{Map}_n^{(\ell)})$ est *finie* (on peut montrer que ce critère ne dépend pas de ℓ). On s'intéressera à un éventail classique de poids, qui est appelé :

- *critique* : on suppose que $\sum_{n \geq 0} n^2 w(\text{Map}_n^{(\ell)}) = \infty$,

- *non générique* : $q_k \sim c\gamma^{k-1}k^{-\theta-1}$, pour un certain $\theta \in (\frac{1}{2}, \frac{3}{2})$ et certaines valeurs particulières de $c, \gamma > 0$ (voir [Cur19]).

Pour fixer les idées, on pourra considérer $q_k = c\gamma^{k-1} \frac{\Gamma(k-\theta-\frac{1}{2})}{\Gamma(k+\frac{1}{2})}$, avec $\gamma = \frac{1}{4\theta+2}$ et $c = -\frac{\sqrt{\pi}}{2\Gamma(\frac{1}{2}-\theta)}$. Dans ce cas, on peut définir

$$\mathbb{P}^{(\ell)}(\mathbf{m}) := \frac{w(\mathbf{m})}{W^{(\ell)}}, \quad \mathbf{m} \in \text{Map}^{(\ell)},$$

où $W^{(\ell)}$ est la fonction de partition $W^{(\ell)} := \sum_{n \geq 0} w(\text{Map}_n^{(\ell)})$. On pose également

$$\mathbb{P}_{\bullet}^{(\ell)}((\mathbf{m}, v_{\bullet})) := \frac{w(\mathbf{m})}{W_{\bullet}^{(\ell)}}, \quad \mathbf{m} \in \text{Map}^{(\ell)}, v_{\bullet} \in \text{Vertices}(\mathbf{m}).$$

Une carte aléatoire prise sous $\mathbb{P}^{(\ell)}$ (respectivement $\mathbb{P}_{\bullet}^{(\ell)}$) sera appelée **q**-carte de Boltzmann de périmètre 2ℓ (respectivement **q**-carte de Boltzmann pointée de périmètre 2ℓ), où on omet par souci de concision les adjectifs *critique* et *non générique*. Enfin, on peut définir une version infinie de ces cartes de Boltzmann à périmètre 2ℓ fixé, comme limite (en un sens de *limite locale* [Cur19] qu'on ne définit pas précisément) de **q**-cartes de Boltzmann de périmètre 2ℓ conditionnées à ce que leurs tailles tendent vers l'infini. On appelle $\mathbb{P}_{\infty}^{(\ell)}$ cette loi (on s'autorise seulement à parler de cartes planaires infinies dans ce cas) : heuristiquement, une carte sous $\mathbb{P}_{\infty}^{(\ell)}$, appelée **q**-carte de Boltzmann infinie de périmètre 2ℓ , est une carte infinie qui localement ressemble à une carte de Boltzmann finie.

Épluchage d'une carte de Boltzmann. On introduit maintenant le processus d'épluchage par arêtes ou *edge-peeling* étudié par Bertoin, Budd, Curien et Kortchemski [BBCK18]. On se fixe une carte $\mathbf{m}$ (bipartie enracinée). Nous aurons besoin de considérer des sous-cartes de $\mathbf{m}$. On appelle sous-carte $\mathbf{p} \subset \mathbf{m}$ une carte planaire à trous, qui sont un ensemble $\text{Holes}(\mathbf{p}) \subset \text{Faces}(\mathbf{p})$ de faces distinctes dont le bord est simple, c'est-à-dire sans points de pincement. Ces trous $\mathfrak{h}_1, \dots, \mathfrak{h}_i \in \text{Holes}(\mathbf{p})$ viennent chacun avec une arête orientée distinguée $e_1, \dots, e_i$ de leur bord, et sont tels qu'on peut reconstruire $\mathbf{m}$ à partir de $\mathbf{p}$ en *recollant* des cartes planaires $\mathbf{u}_1, \dots, \mathbf{u}_i$ de mêmes périmètres dans ces trous, en commençant par recoller l'arête racine de $\mathbf{u}_k$ avec e_k (Figure 1.3).

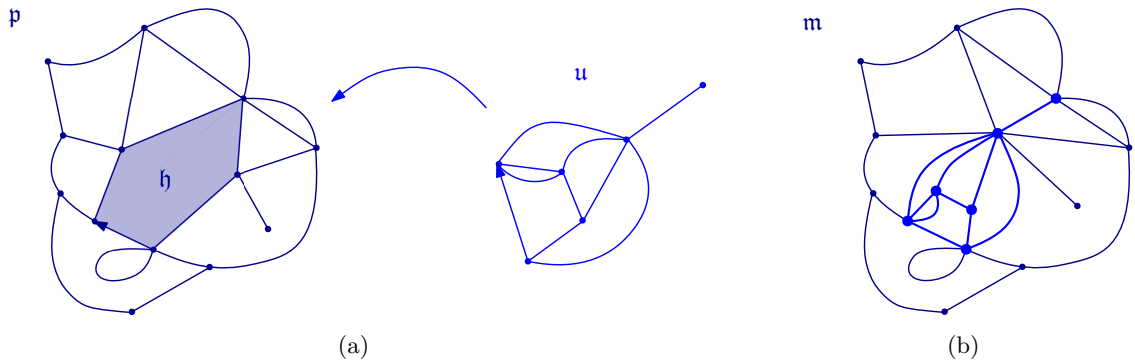


Figure 1.3 – Une carte $\mathbf{m}$ et une sous-carte à trous $\mathbf{p}$. En recollant $\mathbf{u}$ à $\mathbf{p}$ (a), on obtient la carte $\mathbf{m}$ (b).

Le processus d'épluchage de $\mathbf{m}$ est une famille $\mathbf{p}_0(\mathbf{m}) \subset \mathbf{p}_1(\mathbf{m}) \subset \dots \subset \mathbf{m}$ de sous-cartes, obtenu en ouvrant à chaque étape une arête d'un trou de la carte précédente, selon un

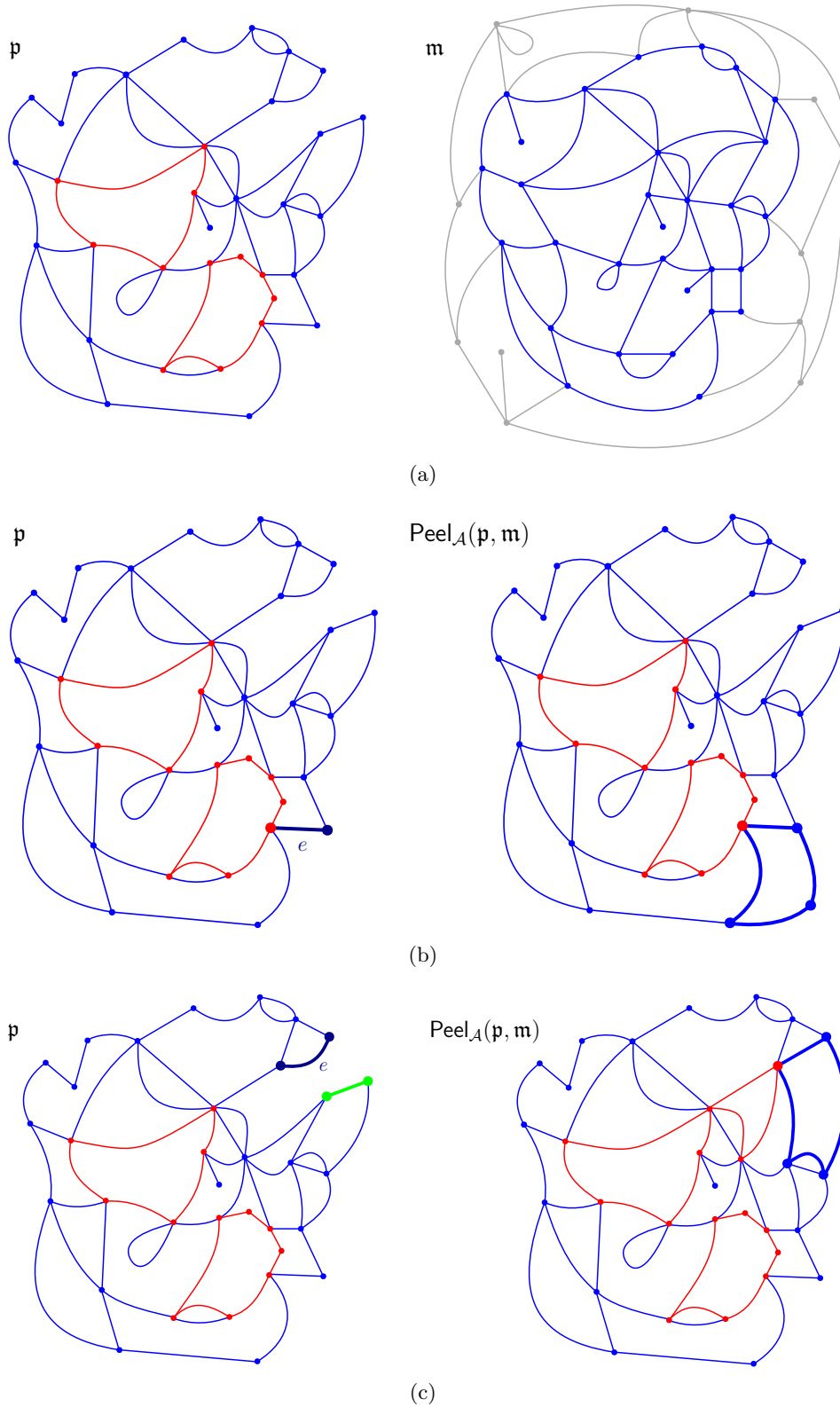


Figure 1.4 – Différents événements d'épluchage : (a) La carte m and la sous-carte p , dont les trous (hormis la face extérieure) sont représentés en rouge ; (b) La nouvelle face f_e dans m découverte en épluchant l'arête e est une nouvelle face pour p (événement P) ; (c) La nouvelle face f_e dans m découverte en épluchant l'arête e est déjà une face de p (événement N).

algorithme déterministe $\mathcal{A}$. On suppose donné un tel algorithme $\mathcal{A}$, c'est-à-dire une fonction qui prend une carte à trous $\mathbf{p}$ et renvoie une arête sur le bord d'un trou dans $\text{Holes}(\mathbf{p})$ (si la carte à trous n'a en fait aucun trou, l'algorithme renvoie une arête cimetièrè $\dagger$). Soit $\mathbf{p} \subset \mathbf{m}$ une sous-carte de $\mathbf{m}$, et supposons que l'algorithme renvoie une arête e sur le bord d'un trou $\mathfrak{h}$ de $\mathbf{p}$. On va construire la carte à trous $\text{Peel}_{\mathcal{A}}(\mathbf{p}, \mathbf{m})$ obtenue à la prochaine étape, comme sur la Figure 1.4. Notons f_e la face de $\mathbf{m}$ qui correspond au trou $\mathfrak{h}$ et est incidente à e . Alors de deux choses l'une :

- ou bien f_e n'est pas une face de $\mathbf{p}$, auquel cas on définit $\text{Peel}_{\mathcal{A}}(\mathbf{p}, \mathbf{m})$ en recollant f_e à $\mathbf{p}$ (on appelle cette situation **P**),
- ou bien f_e est en fait déjà une face de $\mathbf{p}$, c'est-à-dire qu'on découvre en épluchant l'arête e qu'il n'y avait pas de *vrai* trou. Dans ce cas, l'arête e s'identifie à une autre arête e' de $\mathfrak{h}$, qui correspond à la même arête dans $\mathbf{m}$. La carte $\text{Peel}_{\mathcal{A}}(\mathbf{p}, \mathbf{m})$ est la carte obtenue à partir de $\mathbf{p}$ après cette identification. On appelle cette situation **N** : notons que dans ce cas, le trou $\mathfrak{h}$ peut soit disparaître complètement, soit se diviser en deux trous distincts.

Définissons $\mathbf{p}_0(\mathbf{m})$ comme la sous-carte de $\mathbf{m}$ correspondant à la face racine f_{root} vue comme carte à trous. Le processus d'épluchage est alors la famille $\mathbf{p}_0(\mathbf{m}) \subset \mathbf{p}_1(\mathbf{m}) \subset \dots \subset \mathbf{m}$ de sous-cartes obtenues en partant de $\mathbf{p}_0(\mathbf{m})$ puis en épluchant successivement la carte précédente, c'est-à-dire que pour tout $i \geq 0$, $\mathbf{p}_{i+1}(\mathbf{m}) = \text{Peel}_{\mathcal{A}}(\mathbf{p}_i(\mathbf{m}), \mathbf{m})$. Lorsque l'algorithme $\mathcal{A}$ renvoie l'arête cimetièrè $\dagger$ (la carte $\mathbf{p}_i(\mathbf{m})$ n'a plus de trou), le processus d'épluchage s'arrête en stationnant à $\mathbf{p}_i(\mathbf{m})$.

La construction précédente est valable pour une carte $\mathbf{m}$ déterministe fixée, et nous allons maintenant rajouter un aléa en décrivant la loi d'un tel épluchage pour une $\mathbf{q}$ -carte de Boltzmann de périmètre 2ℓ . On se donne un algorithme déterministe $\mathcal{A}$. Si $\mathbf{p}$ est une carte à trous, on note $\text{Holes}(\mathbf{p})$ l'ensemble de ses trous et on pose

$$\tilde{w}(\mathbf{p}) := \prod_{f \in \text{Faces}(\mathbf{p}) \setminus (\text{Holes}(\mathbf{p}) \cup \{f_{\text{root}}\})} q^{\frac{\deg f}{2}},$$

Le résultat suivant est essentiellement dû à Budd [Bud15] ; nous reprenons la formulation de [BBCK18, Proposition 6.3].

Théorème 1.2.1. (Loi du processus d'épluchage pour une carte de Boltzmann de périmètre 2ℓ)

Soient $i \geq 0$, et $\mathbf{p}$ une carte à trous qui peut être obtenue après i épluchages à partir d'une carte à une seule face de périmètre 2ℓ . Soient $2\ell_1, \dots, 2\ell_k$ les périmètres des trous de $\text{Holes}(\mathbf{p})$. Alors

$$\mathbb{P}^{(\ell)}(\mathbf{p}_i(\mathbf{m}) = \mathbf{p}) = \frac{\tilde{w}(\mathbf{p})}{W^{(\ell)}} \prod_{1 \leq j \leq k} W^{(\ell_j)}.$$

De plus, conditionnellement à $\mathbf{p}_i(\mathbf{m}) = \mathbf{p}$, les cartes planaires $\mathbf{u}_1, \dots, \mathbf{u}_i$ qu'il faut recoller dans $\text{Holes}(\mathbf{p})$ pour retrouver $\mathbf{m}$ sont des $\mathbf{q}$ -cartes de Boltzmann indépendantes de périmètres $2\ell_1, \dots, 2\ell_k$.

Une telle description du processus d'épluchage existe également dans le cas des versions infinie $\mathbb{P}_{\infty}^{(\ell)}$ ou pointée $\mathbb{P}_{\bullet}^{(\ell)}$, voir encore [BBCK18, Proposition 6.3]. Dans tous les cas, les énoncés ont la même saveur de *propriété de Markov spatiale*. Sans rentrer dans les détails, notons que dans le cas des versions pointée et infinie, les cartes planaires $\mathbf{u}_1, \dots, \mathbf{u}_i$ à recoller seront distribuées selon $\mathbb{P}^{(\ell_1)}, \dots, \mathbb{P}^{(\ell_k)}$, *sauf pour un certain indice K* , pour lequel $\mathbf{u}_K$ sera distribuée selon $\mathbb{P}_{\infty}^{(\ell)}$ ou $\mathbb{P}_{\bullet}^{(\ell_K)}$ (si $v_{\bullet}$ n'appartient pas déjà à $\mathbf{p}$). Nous prenons la liberté de ne pas discuter de comment l'indice K est choisi pour le moment.

Le processus de croissance-fragmentation comme limite d'échelle des périmètres.

Ce processus d'épluchage en main, nous allons repérer des composantes naturelles sous $\mathbb{P}^{(\ell)}$, $\mathbb{P}_{\bullet}^{(\ell)}$ et $\mathbb{P}_{\infty}^{(\ell)}$ et suivre leurs périmètres au cours du *peeling*. Schématiquement, les composantes considérées seront les suivantes :

- sous $\mathbb{P}_{\infty}^{(\ell)}$, on suivra la composante $(\mathcal{C}_{\infty}(i), i \geq 0)$ qui sépare l'origine de la carte de l'infini,
- sous $\mathbb{P}_{\bullet}^{(\ell)}$, on suivra la composante $(\mathcal{C}_{\bullet}(i), i \geq 0)$ séparant l'origine de $v_{\bullet}$,
- sous $\mathbb{P}^{(\ell)}$, on suivra le *plus grand fragment local* $(\mathcal{C}(i), i \geq 0)$. Rappelons que ce dernier choix est naturel au vu du choix canonique de la cellule Ève pour les processus de croissance-fragmentation (Section 1.1).

Détaillons la construction, sous $\mathbb{P}_{\bullet}^{(\ell)}$, pour une $\mathbf{q}$ -carte de Boltzmann pointée $(\mathbf{m}, v_{\bullet})$ (les autres cas sont analogues). On définit d'abord $\mathcal{C}_{\bullet}(0)$ comme l'unique trou de $\mathbf{p}_0(\mathbf{m})$, et on pose $\sigma_0 = 0$. On construit ensuite $\mathcal{C}_{\bullet}(i+1)$ à partir de $\mathcal{C}_{\bullet}(i)$ suivant l'épluchage de la façon suivante. D'abord, si le trou $\mathcal{C}_{\bullet}(i)$ est vide, alors $\mathcal{C}_{\bullet}(i+1)$ est vide. S'il n'est pas vide, alors on attend le premier instant après $\sigma_{i+1} > \sigma_i$ où l'algorithme d'épluchage sélectionne une arête sur le bord de $\mathcal{C}_{\bullet}(i)$. L'algorithme renvoie alors l'arête $e(\sigma_{i+1}) := \mathcal{A}(\mathbf{p}_{\sigma_{i+1}}(\mathbf{m}))$ sur le bord de $\mathcal{C}_{\bullet}(i)$. On définit alors $\mathcal{C}_{\bullet}(i+1)$ selon les deux événements possibles P et N dans la définition du *peeling* :

- si l'événement P apparaît, alors il se forme un nouveau trou en accolant la nouvelle face $\mathbf{f}_{e(\sigma_{i+1})}$ à $\mathcal{C}_{\bullet}(i)$, qu'on appelle $\mathcal{C}_{\bullet}(i+1)$;
- sinon, on observe N. Dans le cas où le trou $\mathcal{C}_{\bullet}(i)$ disparaît complètement, on définit $\mathcal{C}_{\bullet}(i+1)$ comme étant vide. Sinon, le trou $\mathcal{C}_{\bullet}(i)$ se disloque en deux trous distincts. On définit $\mathcal{C}_{\bullet}(i+1)$ comme celui contenant $v_{\bullet}$.

Remarques. (i) Sous $\mathbb{P}_{\infty}^{(\ell)}$, on procède de même, sauf qu'on remplace $v_{\bullet}$ par l'infini dans le cas où on observe N (notons que ceci est bien défini presque sûrement d'après Stephenson [Ste18a] car une carte de Boltzmann infinie ne peut pas avoir plusieurs branches infinies). Dans ce cas, il n'y a pas de vérification à faire lorsqu'on observe P ($\mathbf{f}_{e(\sigma_{i+1})}$ ne contiendra jamais l'infini) ; de plus, le trou $\mathcal{C}_{\infty}(i)$ ne peut pas disparaître complètement dans N. Néanmoins, il se peut que $\sigma_{i+1} = +\infty$, auquel cas on définit $\mathcal{C}_{\infty}(i+1)$ comme vide.

- (ii) Sous $\mathbb{P}^{(\ell)}$, on répète également la construction précédente, à la différence qu'on définit $\mathcal{C}(i+1)$ comme le trou dont le périmètre est maximal sur l'événement N (en cas d'égalité, on se donne une règle déterministe pour choisir un des trous). Il n'y a pas de dichotomie à faire sur l'événement P.

Dans les trois cas, on définit le périmètre de la composante d'intérêt : notons $P_{\bullet}(i)$, $P_{\infty}(i)$ et $P(i)$ les demi-périmètres respectifs de $\mathcal{C}_{\bullet}(i)$, $\mathcal{C}_{\infty}(i)$ et $\mathcal{C}(i)$. On remarquera que les événements P et N correspondent respectivement à des sauts positifs et négatifs des périmètres. Le théorème suivant reprend [BBCK18, Proposition 6.6] et donne la limite d'échelle des périmètres lors de telles explorations sous $\mathbb{P}^{(\ell)}$, $\mathbb{P}_{\bullet}^{(\ell)}$ et $\mathbb{P}_{\infty}^{(\ell)}$ en faisant apparaître un processus de croissance-fragmentation que nous allons décrire. On rappelle que notre jeu de poids $\mathbf{q}$ dépend notamment d'un exposant $\theta \in (\frac{1}{2}, \frac{3}{2})$ et d'une constante $c > 0$ (non généricité). On utilisera les notations $X_{\theta}, Y_{\theta}^{-}, Y_{\theta}^{+}$ du Théorème 1.1.1 (tous ces processus démarrent en 1). On pose également $c_{\mathbf{q}} = \frac{\pi c}{\Gamma(\theta+1) \cos(\pi(\theta+1))}$.

Théorème 1.2.2. *Pour la topologie de Skorokhod, on a les convergences en distribution suivantes, sous $\mathbb{P}^{(\ell)}$, $\mathbb{P}_{\bullet}^{(\ell)}$ et $\mathbb{P}_{\infty}^{(\ell)}$ respectivement :*

$$\begin{aligned} \left(\frac{1}{\ell} \cdot P(\lfloor \ell^\theta \cdot t \rfloor), t \geq 0 \right) &\xrightarrow{\ell \rightarrow \infty} (X_\theta(c_{\mathbf{q}} \cdot t), t \geq 0), \\ \left(\frac{1}{\ell} \cdot P_{\bullet}(\lfloor \ell^\theta \cdot t \rfloor), t \geq 0 \right) &\xrightarrow{\ell \rightarrow \infty} (Y_\theta^-(c_{\mathbf{q}} \cdot t), t \geq 0), \\ \left(\frac{1}{\ell} \cdot P_{\infty}(\lfloor \ell^\theta \cdot t \rfloor), t \geq 0 \right) &\xrightarrow{\ell \rightarrow \infty} (Y_\theta^+(c_{\mathbf{q}} \cdot t), t \geq 0). \end{aligned}$$

Ce théorème est remarquable à plusieurs titres. D'abord, le premier point donne que le périmètre de la plus grande composante locale au niveau des cartes se comporte en limite d'échelle comme le *plus grand fragment local* du processus de croissance-fragmentation $\mathbf{X}_\theta$. Du point de vue des cartes, le processus de croissance-fragmentation se traduit très visuellement de la façon suivante (on prendra garde au fait que ces considérations ne sont valables qu'à la limite). On considère les sous-cartes *filles* de la plus grande composante locale $\mathcal{C}$, c'est-à-dire tous les *trous* créés aux moment où $\mathcal{C}$ s'est divisée en deux. Ceci correspond, à la limite, à la première génération de $\mathbf{X}_\theta$. Ces trous correspondent à des cartes qui, conditionnellement à la donnée de leurs périmètres (qui sont déterminés par la taille des sauts de P), sont des $\mathbf{q}$ -cartes de Boltzmann indépendantes au vu du Théorème 1.2.1. Dans chacune de ces cartes, on suit alors le plus grand fragment local, et on répète l'opération ainsi de suite. De plus, les deux derniers points donnent également (à la limite) une construction très visuelle des deux épines Y_θ^- et Y_θ^+ . La première revient à considérer un sommet uniforme $\mathbf{v}_\bullet$ sur une $\mathbf{q}$ -carte de Boltzmann de périmètre 2ℓ biaisée par sa taille, et à suivre le périmètre de la région associée. De manière analogue, Y^+ correspond à prendre une $\mathbf{q}$ -carte de Boltzmann *infinie* de périmètre 2ℓ , et à suivre le périmètre de la composante visant l'infini. A cet égard, les lois du processus d'épluchage sous $\mathbb{P}_{\bullet}^{(\ell)}$ et $\mathbb{P}_{\infty}^{(\ell)}$, que nous avons évoquées après le Théorème 1.2.1 sont caractéristiques de la structure d'*épine* : toutes les cartes évoluent comme des $\mathbf{q}$ -cartes de Boltzmann, sauf une, qui à la limite suit l'épine. A la limite toujours, l'indice K mentionné dans la même discussion est pris proportionnellement au poids dans les martingales $\mathcal{M}^-$ et $\mathcal{M}^+$ correspondant à ω_- et ω_+ , c'est-à-dire (d'après le Théorème 1.1.1) proportionnellement au périmètre à la puissance $\theta + \frac{1}{2}$ et $\theta + \frac{3}{2}$ respectivement. Enfin, signalons que dans le cas où $\theta \in (1, \frac{3}{2})$, ces processus de croissance-fragmentation entrent en jeu dans une construction plus géométrique obtenue en coupant les cartes de Boltzmann à des hauteurs. On construit ainsi un processus de même cumulant que $\mathbf{X}_\theta$, mais d'indice d'autosimilarité $\theta - 1$ (voir [BBCK18, Theorem 6.8]).

1.2.2 Modèle de Fortuin-Kasteleyn et bijection de Sheffield

Pour motiver le paragraphe suivant, et en particulier la notion d'*accouplement d'arbres*, nous présentons ici une bijection, due à Sheffield [She16b], qui exprime une façon d'encoder par des mots certaines cartes planaires décorées. Nous restons donc dans ce paragraphe dans le monde discret. On présentera d'abord la bijection de Sheffield (dite *de hamburger-cheeseburger*) qui est déterministe. L'idée est ensuite d'exploiter cette bijection pour montrer un résultat de convergence pour le modèle aléatoire de Fortuin-Kasteleyn.

La bijection de Sheffield. On emprunte les définitions et notations du premier paragraphe de la Section 1.2.1 : on notera $\text{Vertices}(\mathbf{m})$, $\text{Edges}(\mathbf{m})$ et $\text{Faces}(\mathbf{m})$ l'ensemble des sommets, arêtes et faces d'une carte planaire $\mathbf{m}$. Un objet central sera la *carte duale* $\mathbf{m}'$ de $\mathbf{m}$ (par distinction, on emploiera plutôt l'adjectif *primal* pour les objets se rattachant à la carte d'origine $\mathbf{m}$). Pour la définir, on place d'abord au centre de chaque face dans $\text{Faces}(\mathbf{m})$ un sommet : ce sont les sommets de $\mathbf{m}'$. On décrète que deux sommets duaux sont reliés si les

faces correspondantes sont adjacentes dans $\mathbf{m}$, ce qui termine de construire $\mathbf{m}'$. On définit maintenant, au-dessus de $\mathbf{m}$ et $\mathbf{m}'$, la *quadrangulation* $\text{Quad}(\mathbf{m})$ associée (on appelle parfois cette construction la *bijection de Tutte*). Il s'agit d'une carte plane, dont les sommets sont les sommets de $\mathbf{m}$ et les sommets de $\mathbf{m}'$. Les arêtes de $\text{Quad}(\mathbf{m})$ sont formées en joignant chaque sommet dual (chaque face) aux sommets rencontrés lorsqu'on trace le contour de la face correspondante (ces sommets sont les sommets de $\mathbf{m}$ qui entourent la face, mais on peut créer ici des arêtes multiples). Voir Figure 1.5 pour une illustration. On enraine également $\text{Quad}(\mathbf{m})$: un moyen canonique est de considérer l'arête racine de $\mathbf{m}$, issue d'un sommet v , et de décider que l'arête racine de $\text{Quad}(\mathbf{m})$ est la première arête de $\text{Quad}(\mathbf{m})$ rencontrée en tournant dans le sens horaire autour de v .

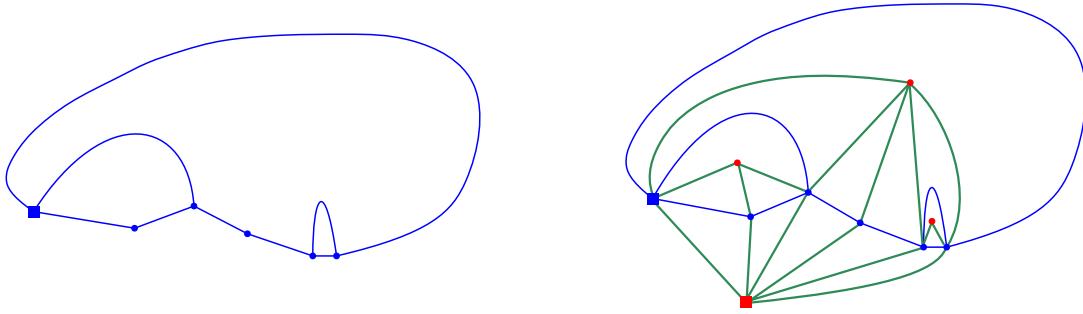


Figure 1.5 – Une carte plane $\mathbf{m}$ et la quadrangulation $\text{Quad}(\mathbf{m})$ associée. L'arête racine de $\text{Quad}(\mathbf{m})$ est l'arête verte entre les deux sommets représentés par des carrés.

Nous introduisons maintenant l'alphabet des mots qui coderont l'exploration de la carte. Soit $\mathcal{A} := \{(\textcircled{\text{H}}), (\textcircled{\text{C}}), (\boxed{\text{H}}), (\boxed{\text{C}}), (\boxed{\text{F}})\}$. Dans cet alphabet, les lettres H peuvent être considérées comme représentant des hamburgers, et les lettres C des cheeseburgers. Les lettres encadrées correspondent aux burgers qui ont été produits, tandis que les lettres encadrées correspondent aux commandes de burgers. Le symbole $\boxed{\text{F}}$ signifie *freshest* : il correspond à la commande du burger le plus récemment produit. Un mot de la forme $W := (\textcircled{\text{H}})(\textcircled{\text{C}})(\textcircled{\text{C}})(\boxed{\text{F}})(\boxed{\text{C}})(\textcircled{\text{H}})(\boxed{\text{H}})(\boxed{\text{H}})$, lu de gauche à droite, peut être vu comme le résumé d'une journée dans un *fast-food*, où les burgers sont mis sur le dessus d'une pile au fur et à mesure qu'ils sont produits, et où on sert toujours au client le burger du type demandé en prenant celui qui se trouve le plus haut dans la pile. Dans le cas particulier précédent, la journée se lit comme suit : le restaurant produit d'abord un hamburger, puis un cheeseburger, puis un autre cheeseburger, un client commande le burger flexible tout en haut de la pile (dans cet exemple, un cheeseburger), un autre client commande un cheeseburger, etc. Lorsqu'un client commande un burger en rupture de stock, on convient que le client part chercher un autre restaurant. Les relations suivantes entre les symboles permettent de *réduire* un mot de façon naturelle :

$$(\textcircled{\text{H}})(\boxed{\text{H}}) = (\textcircled{\text{C}})(\boxed{\text{C}}) = (\textcircled{\text{H}})(\boxed{\text{F}}) = (\textcircled{\text{C}})(\boxed{\text{F}}) = \emptyset,$$

avec les relations de commutativité

$$(\textcircled{\text{H}})(\boxed{\text{C}}) = (\boxed{\text{C}})(\textcircled{\text{H}}) \quad \text{et} \quad (\textcircled{\text{C}})(\boxed{\text{H}}) = (\boxed{\text{H}})(\textcircled{\text{C}}).$$

Partant d'un mot W , le *mot réduit* $\bar{W}$ consiste alors en une suite de commandes, suivie d'une suite de productions de burgers. Les commandes correspondent aux clients qui n'ont pas pu être satisfaits, et les productions restantes correspondent aux burgers qui n'ont pas été commandés (en d'autres termes, c'est l'état de la pile à la fin de la journée). Dans l'exemple précédent $W = (\textcircled{\text{H}})(\textcircled{\text{C}})(\textcircled{\text{C}})(\boxed{\text{F}})(\boxed{\text{C}})(\textcircled{\text{H}})(\boxed{\text{H}})(\boxed{\text{H}})$, on a que $\bar{W} = \emptyset$ est le mot vide : cela

signifie que tous les burgers produits ce jour-là ont été commandés le même jour, et que toutes les commandes ont pu être satisfaites.

Expliquons comment fonctionne la bijection de Sheffield sur un cas simple (celui que Sheffield lui-même présente en premier lieu) – on conseille au lecteur de suivre la construction sur la Figure 1.6. Ce cas est essentiellement dû à Mullin [Mul67], avec les clarifications de Bernardi [Ber07]. On se donne une carte $\mathbf{m}$ ainsi qu'un arbre couvrant $\mathbf{t}$ de $\mathbf{m}$, c'est-à-dire un sous-graphe (planair) de $\mathbf{m}$ qui est un arbre et passe par tous les sommets de $\mathbf{m}$. Nous pouvons alors construire un arbre couvrant *dual* $\mathbf{t}'$ de $\mathbf{m}'$, en joignant deux sommets de $\mathbf{m}'$ lorsque l'arête correspondante ne traverse pas $\mathbf{t}$. De manière équivalente, pour tout quadrilatère de la quadrangulation $\text{Quad}(\mathbf{m})$, on trace la diagonale entre les deux sommets duaux lorsque la diagonale primale n'est pas une arête de $\mathbf{t}$. Une fois $\mathbf{t}'$ tracé, on peut alors former la trajectoire suivante Γ passant entre $\mathbf{t}$ et $\mathbf{t}'$ et visitant toutes les arêtes de la quadrangulation $\text{Quad}(\mathbf{m})$ exactement une fois. On part du milieu de l'arête racine de $\text{Quad}(\mathbf{m})$, et on dessine une boucle en gardant toujours $\mathbf{t}$ à gauche de la trajectoire, et $\mathbf{t}'$ à sa droite. Nous associons à cette trajectoire Γ un mot comme suit. On suit le chemin Γ en partant de l'arête racine. Chaque fois que Γ croise une nouvelle arête de la quadrangulation $\text{Quad}(\mathbf{m})$, un des sommets de l'arête précédente change, tandis que l'autre reste le même. On récupère alors un nouveau sommet (primal ou dual), et on regarde si ce nouveau sommet avait déjà été découvert ou non par le passé. Si c'est la première fois qu'il est découvert, on ajoute un symbole $\textcircled{\text{H}}$ ou $\textcircled{\text{C}}$ selon qu'il s'agit d'un sommet primal ou dual. Sinon, on ajoute un symbole $\boxed{\text{H}}$ ou $\boxed{\text{C}}$ selon qu'il s'agit d'un sommet primal ou dual. Cela forme un mot $W = \mathcal{W}(\mathbf{m}, \mathbf{t})$ dans l'alphabet $\mathcal{B} = \mathcal{A} \setminus \{\boxed{\text{F}}\}$. Remarquons que, dans cette correspondance, un burger est produit chaque fois que le chemin Γ entre dans un nouveau quadrilatère de $\text{Quad}(\mathbf{m})$, et que ce même burger est consommé exactement lorsque la seconde moitié de ce quadrilatère est traversée. Par conséquent, la réduction de W donne $\overline{W} = \emptyset$. On peut montrer que $\mathcal{W}$ est en fait une bijection entre les couples $(\mathbf{m}, \mathbf{t})$ de cartes planaires à n arêtes $\mathbf{m}$ décorées avec arbre couvrant $\mathbf{t}$ et l'ensemble des mots W de longueur $2n$ dans l'alphabet $\mathcal{B}$ tels que $\overline{W} = \emptyset$.

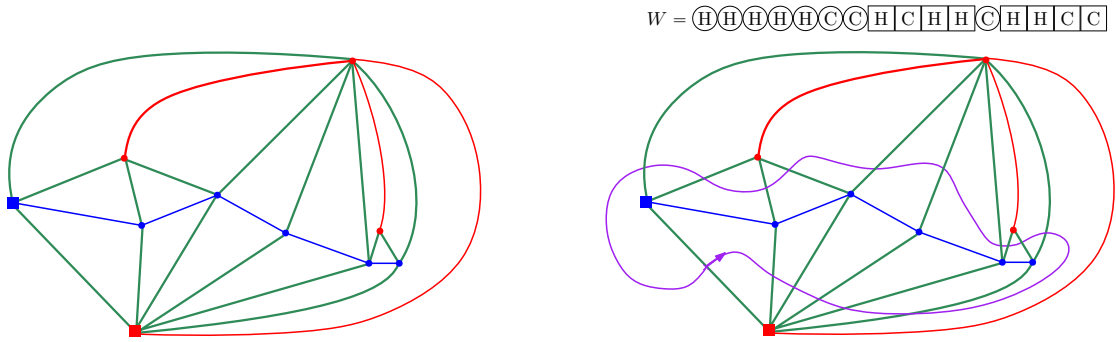


Figure 1.6 – La bijection de Sheffield pour les arbres couvrants. On se donne la carte de la Figure 1.5, dont on considère un arbre couvrant. Le dessin de gauche représente uniquement ce sous-ensemble d'arêtes (en bleu), ainsi que son dual dans la carte (en rouge), qui est un arbre couvrant de la carte duale $\mathbf{m}'$. À droite, on a tracé en violet la trajectoire Γ , qui passe par chaque arête de la quadrangulation $\text{Quad}(\mathbf{m})$ de manière unique.

Pour légitimer la construction précédente, nous ferons une analogie avec les fonctions de contour pour les arbres (Figure 1.8). Ici, on dispose d'un arbre $\mathbf{t}$ et de son dual $\mathbf{t}'$. La bijection n'est rien d'autre qu'une *description du contour des arbres en tenant compte de*

leur interface ¹. En effet, on peut considérer deux marches H et C dans $\mathbb{N}$ qui comptent respectivement les hamburgers et les cheeseburgers. Dans notre construction, les hamburgers sont naturellement associés aux sommets primaux, et les cheeseburgers aux sommets duaux. Les deux marches commencent en 0. A chaque étape, les marches évoluent de $+1$, -1 ou 0 , et ce de manière antagoniste : lorsque l'une d'elle varie, l'autre reste constante. De plus, la contrainte $\bar{W}(\mathbf{m}) = \emptyset$ impose que ces marches restent *positives*, et se terminent en 0. On obtient ainsi deux excursions discrètes. Enfin, de même que les fonctions de contour permettent de retrouver l'arbre sous-jacent, on peut récupérer les arbres $\mathbf{t}$ et $\mathbf{t}'$ en repliant chaque excursion sur elle-même par le même procédé de collage. On perd néanmoins ici la notion d'interface.

Afin de présenter la bijection de Sheffield dans le cas général, on est amené à remplacer l'arbre couvrant $\mathbf{t}$ précédent par des sous-graphes de $\mathbf{m}$ plus généraux. On prend donc maintenant un sous-ensemble d'arêtes de $\mathbf{m}$, et $\mathbf{t}$ le graphe planaire obtenu en ne gardant dans $\mathbf{m}$ que ces arêtes ($\mathbf{t}$ contient tous les sommets de $\mathbf{m}$, dont certains peuvent être isolés). On peut toujours construire le dual $\mathbf{t}'$ de $\mathbf{t}$ en ne gardant dans $\mathbf{m}'$ que les arêtes qui n'intersectent pas $\mathbf{t}$. On peut également construire l'interface entre $\mathbf{t}$ et $\mathbf{t}'$, mais cette fois, on obtient plusieurs boucles au lieu de la trajectoire Γ ; en effet, $\mathbf{t}$ peut maintenant enfermer des composantes car ce n'est plus un arbre (Figure 1.7). Néanmoins on obtient bien plusieurs boucles gardant $\mathbf{t}$ à gauche et $\mathbf{t}'$ à droite, telles que chaque arête de $\text{Quad}(\mathbf{m})$ est traversé par une unique boucle, une et une seule fois, et telles que ces boucles ne traversent jamais $\mathbf{t}$ ni $\mathbf{t}'$. On appellera ces boucles les *boucles FK*. Ces boucles sont l'analogue discret du CLE qui apparaîtra dans la section suivante. La donnée de $\text{Quad}(\mathbf{m})$, $\mathbf{t}$ et $\mathbf{t}'$ construit un ensemble $\mathcal{T}$ de triangles, où chaque triangle est un demi-quadrilatère de $\text{Quad}(\mathbf{m})$. On se propose de donner un moyen canonique d'entrer dans les composantes bloquées. Ceci reviendra à remplacer certaines arêtes de $\mathbf{t}$ ou de $\mathbf{t}'$ par leurs arêtes duales, de telle sorte qu'à la fin on ait construit deux arbres couvrants $\tilde{\mathbf{t}}$ et $\tilde{\mathbf{t}}'$. Pour cela, on commence par repérer la boucle $\mathcal{L}$ passant par l'arête racine de $\text{Quad}(\mathbf{m})$, et on forme $\mathcal{T} \setminus \mathcal{L}$ (où dans cette écriture $\mathcal{L}$ est identifiée à l'ensemble des triangles qu'elle traverse). Cet ensemble peut avoir plusieurs composantes connexes. Pour chacune de ces composantes connexes $\mathcal{C}$, $\mathcal{L}$ passera par au moins un triangle à la frontière extérieure de $\mathcal{C}$ (sinon, $\mathcal{C}$ ne serait pas une composante connexe). On repère alors le dernier tel triangle à être traversé par $\mathcal{L}$. Ce triangle est par définition un demi-quadrilatère de $\text{Quad}(\mathbf{m})$, donc possède une arête qui est une arête de $\mathbf{t}$ ou de $\mathbf{t}'$. On décide de remplacer cette dernière par l'arête duale correspondante. En procédant ainsi, on ouvre la composante $\mathcal{C}$ et on peut maintenant joindre la boucle $\mathcal{L}$ avec une boucle dans $\mathcal{C}$. Après cette étape, on a donc réduit le nombre total de boucles d'autant de composantes $\mathcal{C}$ présentes. On réitère ce procédé jusqu'à obtenir une unique trajectoire Γ passant exactement une fois par toutes les arêtes de $\text{Quad}(\mathbf{m})$. En remplaçant, à chaque étape, des arêtes par leurs duales, on a construit un autre sous-graphe $\tilde{\mathbf{t}}$ et son dual $\tilde{\mathbf{t}}'$, qui sont des arbres couvrants de $\mathbf{m}$ et $\mathbf{m}'$. Les arêtes remplaçantes sont appelées *arêtes fictives*.

Dès lors, comment encoder le couple $(\mathbf{m}, \mathbf{t})$ par un mot ? On forme d'abord le mot $\tilde{W} := \mathcal{W}(\mathbf{m}, \tilde{\mathbf{t}})$ grâce à la bijection précédente pour les arbres couvrants. C'est un mot dans l'alphabet $\mathcal{B}$. Parmi ces symboles, certains correspondent à des *triangles fictifs*, c'est-à-dire des triangles de $\mathcal{T}$ dont l'arête de $\tilde{\mathbf{t}}$ ou $\tilde{\mathbf{t}}'$ est une arête fictive. On fait alors l'observation cruciale suivante : à chaque fois que la trajectoire Γ traverse pour la seconde fois un quadrilatère, cela revient à commander le burger en haut de la pile. En effet, on sait déjà qu'il s'agit d'une commande (c'est la deuxième fois que le quadrilatère est traversé). De plus, à l'intérieur de la composante $\mathcal{C}$ correspondante, on pourra constater qu'aucun burger n'est

¹En ce sens, on peut voir la bijection de Sheffield comme un *accouplement d'arbres* discret : la carte décorée $(\mathbf{m}, \mathbf{t})$ n'est rien d'autre qu'un recollement *planaire* des deux arbres associés aux hamburgers et aux cheeseburgers.

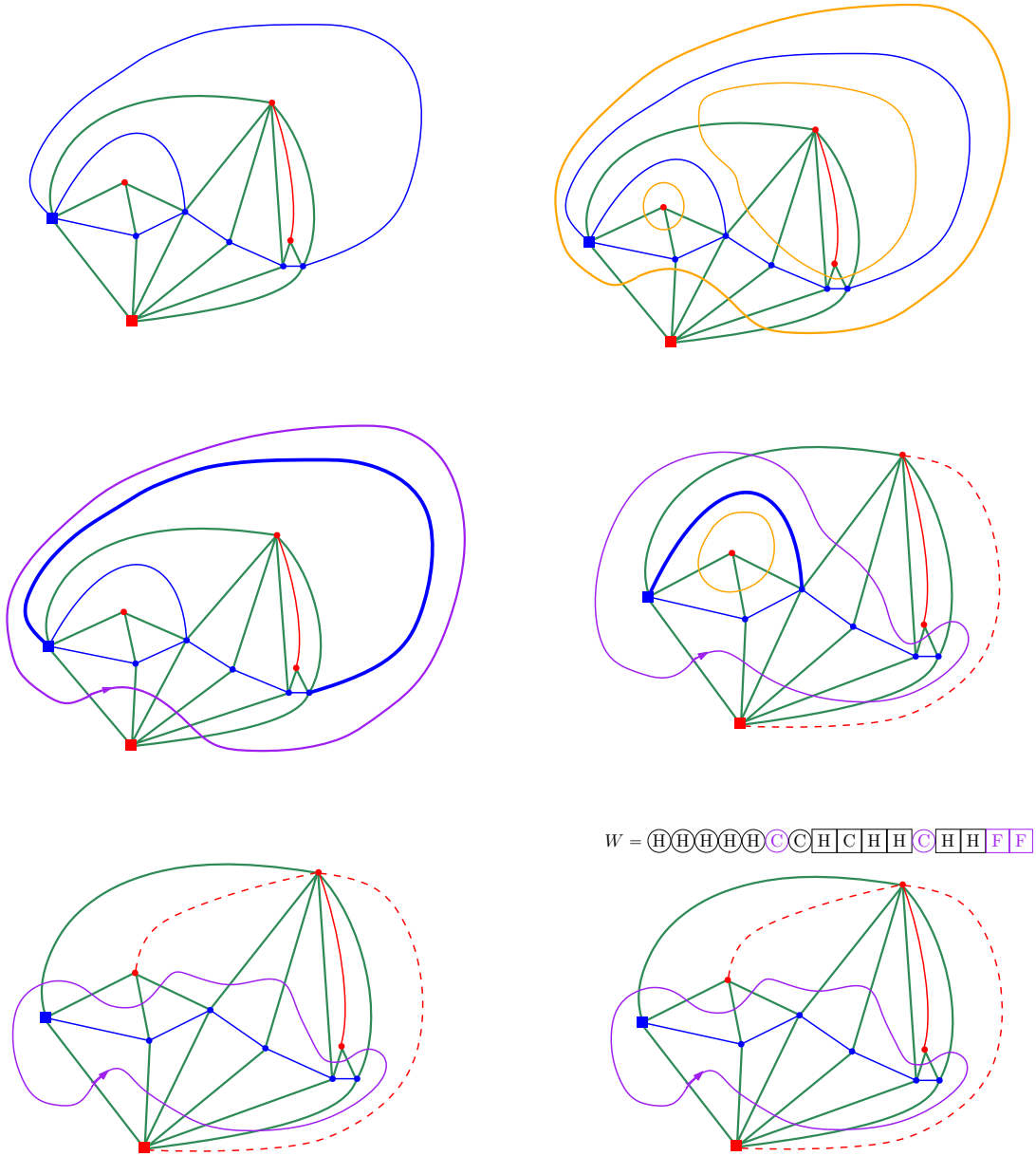


Figure 1.7 – La bijection de Sheffield dans le cas général. On considère la carte de la Figure 1.5, dont on se donne n'importe quel sous-ensemble d'arêtes couvrant les sommets de m . Sur la première figure, on a représenté ce sous-ensemble d'arêtes t (bleu), ainsi que son dual dans m (rouge). Ceci donne lieu à des boucles (en haut à droite, orange), qui décrivent l'interface entre t et le dual t' , et telles que chaque arête de $\text{Quad}(m)$ est parcourue par une et une seule boucle. Pour décrire le mot associé dans la bijection de Sheffield, on considère la boucle orientée $\mathcal{L}$ passant par l'arête racine (milieu gauche, violet). On s'intéresse au dernier triangle de $\mathcal{T}$ visité par Γ , et dont le triangle complémentaire dans la quadrangulation $\text{Quad}(m)$ n'est pas visité par Γ . Ce triangle est constitué de deux arêtes de $\text{Quad}(m)$ et d'une arête de t ou t' . Dans notre cas, il s'agit d'une arête de t (en gras sur le dessin). On la remplace par son arête duale (milieu droite, rouge pointillé). On obtient alors une nouvelle trajectoire (milieu droite, violet). On répète alors la construction précédente : on remplace l'arête bleue en gras (milieu, droite) par l'arête rouge en pointillé (en bas à gauche) qui lui est duale. A la fin, on obtient deux arbres couvrants, auxquels on peut associer un mot. On remplace ensuite les symboles correspondants à des triangles fictifs par un symbole $\boxed{\text{F}}$.

produit s'il n'est pas *consommé* dans la même composante (ceci est dû au fait qu'on a choisi de faire le tour de la composante avant d'y entrer). Pour former le mot W correspondant à $\mathbf{t}$, on remplace alors tous les symboles associés à des *triangles fictifs* par le symbole $\boxed{\mathbf{F}}$. L'argument précédent assure que le mot obtenu correspond à la même suite de burgers produits et consommés que $\widetilde{W}$, d'où en particulier $\overline{W} = 0$. Il apparaît qu'on obtient ainsi une bijection (que l'on notera toujours $\mathscr{W}$) entre les couples $(\mathbf{m}, \mathbf{t})$ de carte planaire $\mathbf{m}$ à n arêtes décorée d'un sous-graphe quelconque $\mathbf{t}$ de $\mathbf{m}$, et les mots W de longueur $2n$ dans l'alphabet $\mathscr{A}$ tels que $\overline{W} = \emptyset$. De W on remonte facilement à $\widetilde{W}$, puis on reconstruit $(\mathbf{m}, \widetilde{\mathbf{t}})$ grâce à la bijection sur les arbres couvrants, et enfin on repère les arêtes fictives grâce aux symboles $\boxed{\mathbf{F}}$. A travers cette bijection, on peut lire quelques propriétés géométriques de la carte. *Il sera important de remarquer que toute boucle (autre que $\mathscr{L}$) correspond à un symbole $\boxed{\mathbf{F}}$ par construction.* En particulier, le nombre de boucles formées par $(\mathbf{m}, \mathbf{t})$ est le nombre de symboles $\boxed{\mathbf{F}}$ plus 1. Ou encore, si on appelle *boucle primale* une boucle dont la frontière extérieure est bordée par des arêtes primales, le nombre de boucles primales est égal au nombre de symboles $\textcircled{\mathbf{C}}$ reliés à des symboles $\boxed{\mathbf{F}}$ (un énoncé analogue est valable pour les boucles *duales*). Enfin, on peut ici encore considérer les fonctions de contour H et C comme dans le cas des arbres couvrants, si l'on remplace le symbole $\boxed{\mathbf{F}}$ par le symbole $\boxed{\mathbf{H}}$ ou $\boxed{\mathbf{C}}$ correspondant. En revanche, dans ce cas il n'y a *a priori* aucun moyen, partant de ces trajectoires, de retrouver les symboles $\boxed{\mathbf{F}}$.

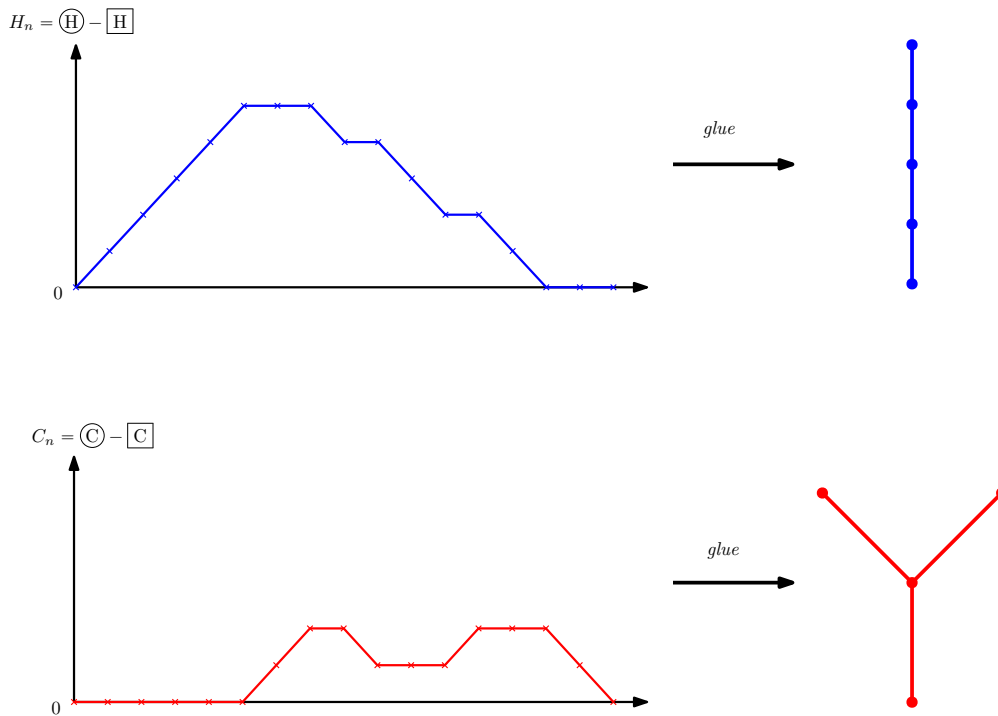


Figure 1.8 – Les trajectoires H_n et C_n comptant respectivement le nombre de hamburgers et de cheeseburgers dans le mot. En repliant les deux fonctions, on obtient les arbres couvrants d'origine (dont H et C forment les fonctions de contour).

Modèle de Fortuin-Kasteleyn. La bijection de Sheffield, décrite dans le paragraphe précédent, est déterministe. Elle donne une façon d'encoder une carte $\mathbf{m}$ décorée de boucles (il est équivalent de se donner $\mathbf{t}$ ou l'ensemble de boucles qu'il induit sur $\mathbf{m}$) par un mot. Le résultat majeur de Sheffield dans [She16b] s'interprète comme un résultat de convergence

via cette bijection sur un modèle aléatoire de cartes planaires décorées, le modèle de Fortuin-Kasteleyn. Soient $n \geq 1$ et $q \in [0, 4)$. Nous allons considérer une variable aléatoire $(\mathfrak{M}, \mathfrak{T})$, qui est un couple formé d'une carte planaire aléatoire $\mathfrak{M}$ à n arêtes, décorée d'un sous-ensemble d'arêtes aléatoire $\mathfrak{T}$. Si $(\mathbf{m}, \mathbf{t})$ est une carte planaire décorée à n arêtes, la probabilité d'occurrence de $(\mathbf{m}, \mathbf{t})$ sous le modèle de Fortuin-Kasteleyn de paramètre q est

$$\mathbb{P}_q^{\text{FK}}((\mathfrak{M}, \mathfrak{T}) = (\mathbf{m}, \mathbf{t})) \propto \sqrt{q}^{\#\text{loops}},$$

où $\#\text{loops}$ est le nombre de boucles formées en traçant l'interface entre $\mathbf{t}$ et $\mathbf{t}'$, comme dans le paragraphe précédent. Autrement dit, conditionnellement à $\mathfrak{M} = \mathbf{m}$, le sous-ensemble d'arêtes $\mathfrak{T}$ a la loi d'une percolation de Fortuin-Kasteleyn (aussi appelée *random-cluster*) auto-duale sur $\mathbf{m}$, pour laquelle on renvoie à la thèse de Duminil-Copin [DC13].

Au vu de la bijection de Sheffield, la question est maintenant : quelle loi $\mathbb{P}_q^{\text{FK}}$ induit-elle sur les mots ? Il n'est pas difficile de voir que la loi peut être décrite comme suit. Posons $p = \frac{\sqrt{q}}{2+\sqrt{q}} \in [0, \frac{1}{2})$. Soient $X_1, \dots, X_{2n}$ des variables *i.i.d.* à valeurs dans $\mathcal{A}$ suivant la loi

$$\mathbb{P}_p(\textcircled{\text{H}}) = \mathbb{P}_p(\textcircled{\text{C}}) = \frac{1}{4}, \quad \mathbb{P}_p(\boxed{\text{F}}) = \frac{p}{4}, \quad \mathbb{P}_p(\boxed{\text{H}}) = \mathbb{P}_p(\boxed{\text{C}}) = \frac{1-p}{4}.$$

Alors la loi de $\mathcal{W}(\mathfrak{M}, \mathfrak{T})$ est celle du mot $X_1 \cdots X_{2n}$ conditionnellement à $\overline{X_1 \cdots X_{2n}} = \emptyset$. Remarquons que sous $\mathbb{P}_p$, une portion p des commandes correspond à des commandes flexibles, et que hamburgers et cheeseburgers apparaissent et disparaissent avec les mêmes probabilités, ce qui implique que la loi de $\mathcal{W}(\mathfrak{M}, \mathfrak{T})$ est invariante sous l'involution qui échange les hamburgers et les cheeseburgers. Pour notre discussion, il sera plus commode de parler d'un modèle infini, qui revient au niveau des cartes à considérer la *limite locale* du modèle précédent d'après Chen [Che17]. Ce modèle est une version infinie naturelle du précédent, où on considère cette fois un mot bi-infini $W = \cdots X_{-1} X_0 X_1 \cdots$ sans contrainte, où les X_i , $i \in \mathbb{Z}$, suivent la loi $\mathbb{P}_p$. Sheffield a montré [She16b, Proposition 2.2] que dans un tel mot, toute commande correspondait bien à une production ; heuristiquement, la réduction de W est le mot vide pour tout p .

Le théorème de Sheffield [She16b, Théorème 2.5] est un résultat de convergence sur le mot. Considérons les deux trajectoires $H := (H_n)_{n \in \mathbb{Z}}$ et $C := (C_n)_{n \in \mathbb{Z}}$ qui comptent respectivement le nombre de hamburgers et de cheeseburgers en fonction du temps. Plus précisément, on pose $H_0 = 0$, et pour $n \geq 1$ on définit H_n comme le nombre de hamburgers produits moins le nombre de hamburgers commandés (y compris lorsqu'ils sont associés à des symboles $\boxed{\text{F}}$) dans $X_1 \cdots X_n$, et H_{-n} comme l'opposé de ce nombre dans $X_{-(n-1)} \cdots X_0$. La raison pour laquelle on prend l'opposé est qu'on veut préserver la même orientation du temps dans H . La trajectoire C est définie de manière analogue. Il est important de remarquer qu'en général, ces trajectoires *ne sont pas* markoviennes : pour satisfaire une commande $\boxed{\text{F}}$, il faut regarder quel type de burger $\textcircled{\text{H}}$ ou $\textcircled{\text{C}}$ se trouve sur la pile à cet instant, et cette information dépend du passé. Par ailleurs, ces trajectoires H et C ne sont qu'une version infinie des excursions discrètes qui encodaient auparavant les arbres couvrants $\mathbf{t}$ et $\mathbf{t}'$. Le théorème de Sheffield s'interprète alors comme un théorème de convergence, en limite d'échelle, sur ces fonctions contour (le cas $p = 0$ étant simplement le théorème de Donsker).

Théorème 1.2.3. (*Sheffield [She16b, Théorème 2.5]*)

Soit $p \in [0, \frac{1}{2})$. Pour tout $n \in \mathbb{N}$, on pose $\sigma_p := \sqrt{\frac{1-p}{2}}$ et

$$Z_n(t) := \frac{1}{\sigma_p \sqrt{n}} \cdot (H_{nt}, C_{nt}), \quad t \in \mathbb{R},$$

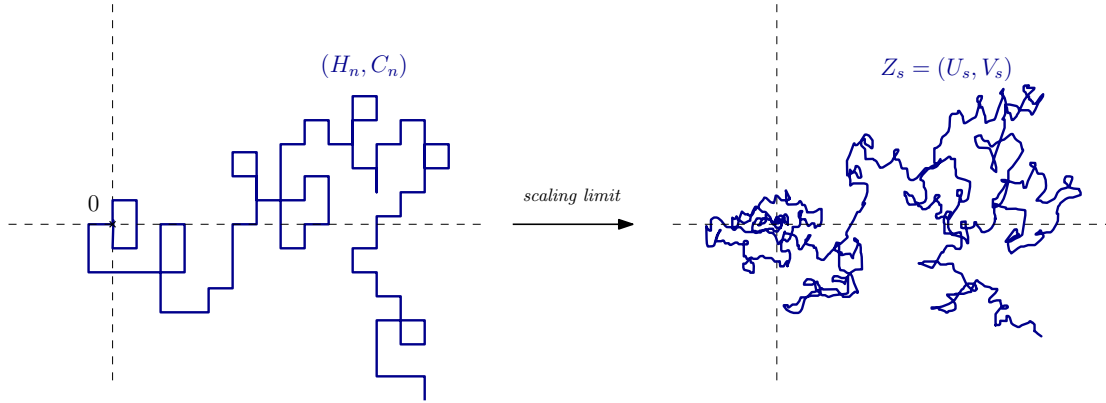


Figure 1.9 – Le résultat de convergence de Sheffield [She16b, Théorème 2.5] (dessin pour des temps positifs) : les marches H et C comptant le nombre de burgers convergent en limite d'échelle vers un mouvement brownien Z corrélé selon (1.2.1).

où on a implicitement prolongé H et C par linéarité. Alors la trajectoire planaire Z_n converge étroitement, pour la topologie de la convergence uniforme sur les compacts, vers le mouvement brownien plan bi-infini $Z = (U, V)$ dont la structure de covariance est donnée par

$$\forall t \in \mathbb{R}, \quad \text{Var}(U(t)) = \text{Var}(V(t)) = |t|, \quad \text{Cov}(U(t), V(t)) = \frac{p}{1-p}|t|. \quad (1.2.1)$$

On notera que dans le cas où $p = 0$, U et V sont indépendants. Plus p se rapproche de la valeur critique $1/2$, plus la corrélation entre U et V tend vers 1².

Considérons maintenant un *temps flexible*, c'est-à-dire un instant s où un symbole $\boxed{\mathbf{F}}$ apparaît dans le mot W . Au niveau de la trajectoire (H, C) , cet instant correspond à un *temps cône*, c'est-à-dire qu'il existe $s' < s$ tel que, pour tout $t \in [s', s]$, $H_s \leq H_t$ et $C_s \leq C_t$. Autrement dit, sur $[s', s]$, (H, C) est contenu dans le quadrant positif de sommet (H_s, C_s) : (H, C) forme une *excursion cône* discrète entre s' et s . Par ailleurs, on peut lire à la fois le *périmètre* et l'*aire* des composantes (primales ou duales) détachées par une boucle FK directement sur cette excursion cône. Si $\mathcal{L}$ est une boucle FK et $\mathcal{C}$ une composante connexe bornée de $\mathcal{T} \setminus \mathcal{L}$ (où $\mathcal{L}$ est encore identifiée à ses triangles), on définit l'aire de $\mathcal{C}$ comme le nombre d'arêtes de $\text{Quad}(\mathbf{m})$ qu'elle contient, et le périmètre de $\mathcal{C}$ comme le nombre d'arêtes primales ou duales (selon le type de composante) sur la frontière de $\mathcal{C}$. Dans le dictionnaire de Sheffield, cette composante $\mathcal{C}$ correspond par construction à un symbole $\boxed{\mathbf{F}}$, et donc à une excursion cône du type précédent. Disons pour simplifier que $\boxed{\mathbf{F}}$ corresponde à un burger de type $\textcircled{\mathbf{H}}$, et appelons le mot entre $\textcircled{\mathbf{H}}$ et $\boxed{\mathbf{F}}$ une boucle $\textcircled{\mathbf{H}}$. On remarquera alors que dans ce dictionnaire, l'aire définie ci-dessus n'est autre que le nombre de symboles dans la boucle $\textcircled{\mathbf{H}}$. D'autre part, le périmètre est déterminé par le nombre de cheeseburgers dans la boucle (ce nombre est négatif). En effet, tout symbole $\textcircled{\mathbf{C}}$ produit dans la boucle $\textcircled{\mathbf{H}}$ sera consommé avant d'atteindre $\boxed{\mathbf{F}}$, et les symboles $\boxed{\mathbf{C}}$ correspondant à des burgers *en-dehors* de la boucle sont ceux qui proviennent des triangles à la frontière de la composante $\mathcal{C}$, car un burger associé à un triangle est consommé lorsque la trajectoire traverse le deuxième triangle du même quadrilatère. En d'autres termes, dans le langage des excursions, on voit que l'aire et le périmètre de $\mathcal{C}$ correspondent respectivement à la *durée* et à la *longueur*

²On retrouvera cette même structure de covariance en gravité quantique de Liouville dans la section suivante ; le théorème précédent se traduira alors en un théorème de convergence sur la carte aléatoire à boucles $(\mathfrak{M}, \mathfrak{T})$ vers une certaine *surface quantique* décorée d'un CLE_κ .

de l'excursion cône associée à la boucle $(\textcircled{H})$ ³. Cette excursion va du bord au sommet du quadrant positif, et la longueur est définie simplement comme la différence entre le point de départ et le point d'arrivée. Par ailleurs, on notera que le type de la composante (primal ou dual) est également déterminé par le *type* d'excursion cône, défini selon qu'elle commence sur la demi-droite horizontale ou verticale (ce qui signifie que le mot associé est une boucle $(\textcircled{H})$ ou $(\textcircled{C})$).

Nous avons vu qu'en considérant la trajectoire (H, C) , on perd de l'information sur la carte décorée, car on ne peut pas retrouver les lettres $\boxed{F}$. Cependant, ce phénomène disparaît en limite d'échelle. En utilisant la bijection de Sheffield, Gwynne, Mao et Sun [GMS19] ont montré que les temps flexibles où les symboles $\boxed{F}$ apparaissent convergent après renormalisation vers les temps cônes de Z . Cet énoncé demande des précisions car Z a un nombre indénombrable de temps cônes, tandis qu'il n'y a qu'un nombre dénombrable de symboles $\boxed{F}$; on renvoie à [GMS19, Théorème 1.8] pour davantage de précisions. L'étape principale de la preuve est en fait de montrer qu'il existe dans le mot des boucles (associées à une lettre $\boxed{F}$) *macroscopiques* avec une bonne probabilité lorsque n est grand. Un des intérêts du travail de Gwynne, Mao et Sun est d'augmenter la topologie pour laquelle la convergence a lieu : combiné avec le Théorème 1.2.3, leur résultat donne une convergence pour une topologie qui encode maintenant toute l'information du modèle FK (on a rajouté les temps flexibles). Par ailleurs, la convergence en limite d'échelle des temps flexibles vers les temps cônes de Z entraîne une cascade de résultats de convergence sur des observables naturelles du modèle FK. Tel est par exemple le cas du périmètre et de l'aire des composantes primales ou duales bornées macroscopiques qui sont détachées par les boucles FK autour d'une arête prescrite.

1.2.3 Gravité quantique de Liouville et *mating-of-trees*

La principale conjecture [KN04, She16b] sur le modèle de Fortuin-Kasteleyn de paramètre $q \in (0, 4)$ décrit dans la Section 1.2.2 précédente est qu'il devrait converger en limite d'échelle vers une *surface quantique* de paramètre γ décoré d'un $\text{CLE}_{\kappa'}$, où $\gamma \in (\sqrt{2}, 2)$ et $\kappa' \in (4, 8)$ sont reliés à q par

$$q = 2 + 2 \cos(8\pi/\kappa'), \quad \gamma = 4/\sqrt{\kappa'}. \quad (1.2.2)$$

Nous présentons ici le premier lien rigoureux entre les deux modèles. On verra notamment que le Théorème 1.2.3 dû à Sheffield peut se voir comme un théorème de convergence sur le modèle de Fortuin-Kasteleyn. Par souci de concision, nous ne détaillerons par la construction des objets continus (champ libre gaussien, surfaces quantiques, $\text{CLE}_{\kappa'}$, etc.) considérés – on pourra pour cela consulter [BP21], [PW20] ou [DMS14]. Nous nous contenterons de donner une définition heuristique, le but étant surtout ici de souligner les similitudes avec la Section 1.2.2.

Champ libre gaussien et mesure de Liouville. Soit $D \subset \mathbb{C}$ un domaine borné du plan. On définit $p_D(t, x, y)$ comme la densité au temps t du mouvement brownien plan issu de x et tué hors de D . La fonction de Green (voir par exemple le livre de Lawler [Law08]) est définie par

$$G_D(x, y) = \pi \int_0^\infty p_D(t, x, y) dt.$$

Intuitivement, cette quantité correspond à normalisation près à la moyenne du temps passé en y avant de sortir du domaine D . Cette quantité a également un sens dans le cas de

³Cette idée se retrouve dans le continu, où la durée et la longueur d'une excursion cône pour le mouvement brownien corrélé du Théorème 1.2.3 correspondra respectivement à l'*aire quantique* et à la *longueur quantique* des composantes connexes bornées détachées par une boucle CLE_{κ} .

domaines plus généraux. L'invariance conforme du mouvement brownien se transmet à G_D , et permet de montrer que $G_D(x, \cdot)$ est harmonique sur $D \setminus \{x\}$.

Le *champ libre gaussien* est alors le processus gaussien dont la covariance est donnée par G_D , si l'on précise le sens de cette définition. Le problème est que G_D explose sur la diagonale $y = x$, donc la variance d'un tel processus serait infinie. Néanmoins, on peut donner un sens à ce processus *au sens des distributions*. Formellement, on définit le champ libre gaussien $h = (h_\rho, \rho \in \mathcal{M}_D)$ comme le processus, indexé par l'ensemble $\mathcal{M}_D$ des mesures signées sur D , tel que pour tous $\rho_1, \dots, \rho_r \in \mathcal{M}_D$, $(h_{\rho_1}, \dots, h_{\rho_r})$ est un vecteur gaussien de matrice de covariance donnée par

$$\text{Cov}(h_{\rho_i}, h_{\rho_j}) = \int_D \int_D G(x, y) \rho_i(dx) \rho_j(dy), \quad 1 \leq i, j \leq r.$$

Naturellement, h possède également la propriété d'invariance conforme. Signe de l'universalité du champ libre gaussien, Berestycki, Powell et Ray [BPR20] en ont montré une réciproque dont les hypothèses fondamentales sont l'invariance conforme et une propriété de Markov spatiale. Le champ libre gaussien est un objet issu de la physique statistique qui apparaît comme une généralisation naturelle du mouvement brownien, lorsqu'on remplace le temps par un paramètre multidimensionnel (ici bidimensionnel). Il intervient à la limite de nombreuses fonctions de hauteur en physique statistique [Ken01], ou encore dans les polynômes caractéristiques de grandes matrices aléatoires [RV07]. On pourra consulter pour plus de détails l'introduction générale de Sheffield [She07], ou encore les notes de cours de Berestycki et Powell [BP21] ou de Powell et Werner [PW20].

On peut faire remonter l'histoire du *chaos multiplicatif gaussien* (c'est-à-dire des mesures définies comme l'exponentielle d'un champ gaussien) à Kahane [Kah85]. On souhaite ici considérer la *mesure de Liouville* μ_h^γ (ou plus simplement μ_h lorsqu'il n'y aura pas d'ambiguïté) définie comme l'exponentielle de γ fois le champ libre gaussien h . Il y a cependant une obstruction majeure : h n'est pas une fonction mais une distribution, si bien que $e^{\gamma h}$ n'est pas bien défini *a priori*. Néanmoins, on peut donner un sens à cet objet pour $\gamma \in (0, 2)$ en considérant des approximations du champ libre gaussien [DS11, RV14]. Cette construction a été obtenue de façon plus élémentaire par Berestycki [Ber17a]. Par ailleurs, on peut aussi donner un sens à la *mesure de longueur quantique* de ∂D , ou de certaines courbes dans D (ce sera le cas notamment des SLE_κ qu'on considérera dans la suite d'après Sheffield [She16a]) : on l'appellera dans la suite ν_h^γ (ou ν_h selon le contexte). On dispose ainsi d'une mesure d'aire μ_h et d'une mesure de longueur ν_h aléatoires, qui correspondent conjecturalement, en limite d'échelle, aux mesures d'aire et de longueur pour les cartes planaires aléatoires (qui sont définies par la mesure de comptage des arêtes de la carte ou du bord de la carte respectivement). Il est parfois aussi utile, pour définir les surfaces quantiques générales, d'ajouter au champ libre gaussien certaines fonctions (éventuellement singulières en quelques points), ou bien d'en considérer une perturbation localement absolument continue ; dans la suite, nous noterons tout de même h le processus considéré.

De manière informelle, une *surface quantique* (D, h) désignera la surface aléatoire paramétrée par D et encodée par la mesure d'aire μ_h . En d'autres termes, sur la surface (D, h) on a déformé les distances et les aires avec l'exponentielle du champ libre : un élément d'aire dz correspondra informellement à une aire $e^{\gamma h(z)} dz$ sur la surface quantique. Notamment, un point typique sur la surface sera un point autour duquel le champ libre h aura tendance à prendre de très grandes "valeurs". Néanmoins, on désire également encoder la *structure conforme* de (D, h) . Si $D' \subset \mathbb{C}$ est un autre domaine simplement connexe et $f : D \rightarrow D'$ une application conforme, alors il s'ensuit de [DS11] que $\mu_h \circ f^{-1} = \mu_{h'}$ presque sûrement, où

$$h' := h \circ f^{-1} + Q \log |(f^{-1})'|, \quad (1.2.3)$$

avec $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$. L'égalité $\mu_h \circ f^{-1} = \mu_{h'}$ a en fait lieu presque sûrement pour tout domaine D' et toute application conforme f simultanément d'après un résultat de Sheffield et Wang [SW16]. Notons que le sens de $\mu_{h'}$ est clair au vu de la formule de changement de coordonnées (1.2.3) car $h \circ f^{-1}$ est un champ libre gaussien sur D' . À l'instar de Duplantier et Sheffield, il est naturel de définir la relation d'équivalence suivante entre deux couples (D, h) et (D', h') : on décide que ces deux couples sont équivalents si, et seulement si, h et h' satisfont la relation (1.2.3) pour une application conforme $f : D \rightarrow D'$. Cette relation d'équivalence nous permet finalement de définir une véritable notion de *surface aléatoire* : on appelle ainsi toute paire (D, h) , où D est un domaine du plan et h une distribution aléatoire sur D , vue *modulo* l'identification précédente. En réalité, on souhaitera souvent considérer des *surfaces marquées* avec un ou plusieurs points de D ou de ∂D , ou d'autres décorations comme des courbes. Dans ce cas, on requiert en plus de (1.2.3) que f envoie ces points (ou ces décorations) sur ceux de la surface marquée paramétrée par D' . Bien sûr, on voudra prendre h comme étant le champ libre gaussien, et on parlera alors de surface quantique (éventuellement marquée).

L'accouplement d'arbres ou *mating-of-trees*. Des résultats profonds surgissent lorsque, sur une surface quantique (D, h) donnée de paramètre γ , on trace une courbe $\text{SLE}_{\kappa'}$ [Sch00, RS05] indépendante de paramètre $\kappa' = \frac{16}{\gamma^2}$ (ou un SLE_{κ} avec $\kappa = \gamma^2$, qui est en quelque sorte dual). On considérera dans toute la suite que $\gamma \in (\sqrt{2}, 2)$. Dans l'univers discret, l'analogue de ces modèles revient à considérer une carte aléatoire, décorée par un modèle de physique statistique dont la fonction de partition se trouve correspondre en un sens au modèle de carte planaire (la percolation FK est un exemple). L'*accouplement d'arbres* [DMS14] décrit précisément cette relation étroite entre SLE et gravité quantique de Liouville, et donne un sens, dans le cadre continu, à la bijection de Sheffield ou accouplement d'arbres discret déjà évoqué en Section 1.2.2.

Sans entrer dans les détails, nous souhaitons donner un aperçu de cette théorie dans le cas d'une surface particulière, le *cône quantique* (*quantum cone*) de paramètre γ . Cette surface est celle paramétrée par le plan $D = \mathbb{C}$ tout entier, qui s'obtient intuitivement en zoomant en un point typique pour la mesure de Liouville. Il s'agit d'une surface de volume infini, qui peut être vue comme une version continue des cartes FK infinies. On considère en fait la surface marquée $\mathcal{S} := (\mathbb{C}, h, 0, \infty)$. Sur cette surface, on trace un $\text{SLE}_{\kappa'}$ *remplissant* [MS17] η' allant de ∞ à ∞ et indépendant de $\mathcal{S}$. Rappelons qu'il est possible d'y donner un sens en considérant d'abord un $\text{SLE}_{\kappa'}$, puis en remplissant les régions qu'il déconnecte de l'infini. Cette courbe η' jouera le rôle du processus d'exploration de Sheffield Γ dans le modèle de Fortuin-Kasteleyn infini. Elle trace des boucles $\text{CLE}_{\kappa'}$ sur son chemin, de la même façon que Γ trace les boucles FK. On supposera η' paramétré de sorte que $\eta'(0) = 0$ et pour tout $s < t$, $\mu_h(\eta'([s, t])) = t - s$. On peut définir les longueurs quantiques à gauche L_t et à droite R_t au temps t de la frontière de η' par rapport à 0 (la Figure 1.13 montre une illustration dans le cas du disque quantique). On prendra garde au fait que L_t et R_t sont définis pour tout temps $t \in \mathbb{R}$ éventuellement négatif. Ces longueurs L et R sont l'analogue continu des trajectoires H et C comptant respectivement le nombre de hamburgers et de cheeseburgers dans la bijection de Sheffield. On peut les voir comme des fonctions de contour qui encodent la surface $\mathcal{S}$ décorée par η' . Le résultat suivant constitue une partie de l'accouplement d'arbres, dans le cas d'un cône quantique, tel qu'exposé dans [DMS14, Théorème 1.9].

Théorème 1.2.4. *Le processus $(L_t, R_t)_{t \in \mathbb{R}}$ est un mouvement brownien plan bi-infini corrélé, dont la structure de covariance est donnée par :*

$$\forall t \in \mathbb{R}, \quad \text{Var}(L_t) = \text{Var}(R_t) = |t|, \quad \text{Cov}(L_t, R_t) = -\cos\left(\frac{4\pi}{\kappa'}\right)|t|. \quad (1.2.4)$$

Remarquons la proximité des covariances (1.2.1) et (1.2.4). Les deux quantités sont égales lorsque

$$-\cos\left(\frac{4\pi}{\kappa'}\right) = \frac{p}{p-1},$$

ce qui équivaut à la conjecture que nous avons mentionnée en (1.2.2). A cet égard, le Théorème 1.2.4 permet de réinterpréter la convergence du Théorème 1.2.3 au niveau des cartes. Pour une certaine topologie où on décrète que deux surfaces sont *proches* si leurs fonctions de contour sont proches, le Théorème 1.2.4 s'interprète comme la convergence en limite d'échelle du modèle de Fortuin-Kasteleyn vers un cône quantique décoré d'un $\text{CLE}_{\kappa'}$. Une telle convergence est dite au sens *peanosphère*. D'autre part, le couple (L, R) détermine la surface décorée $(\mathcal{S}, \eta')$ modulo le changement de coordonnées (1.2.3). Signalons également que le Théorème 1.2.4 a été depuis [DMS14] étendu au cas du *quantum wedge* pour $\gamma \in (0, 2)$ [GHMS17], et du *disque quantique* [AG21], qui est une surface quantique à bord.

Au vu du dernier paragraphe de la Section 1.2.2, il sera intéressant d'étudier les *temps cônes* du mouvement brownien corrélé (L, R) . Rappelons que pour un tel temps cône s , il existe $s' < s$ tel que, pour tout $t \in [s', s]$, $H_s \leq H_t$ et $C_s \leq C_t$. Lorsque η' est paramétré par sa masse quantique μ_h comme dans le Théorème 1.2.4 précédent, un temps cône correspond à un instant où η' termine de remplir une *bulle* qu'il déconnecte de l'infini. Une telle *bulle* est obtenue comme composante connexe bornée détachée par une boucle $\text{CLE}_{\kappa'}$. Dans cette image, le temps $s' < s$ maximal vérifiant la propriété de temps cône (ce temps est associé à un point à la frontière du cône) correspond à l'instant où η' finit de tracer le contour extérieur de la bulle, qu'il va remplir avant d'en sortir au temps s . De plus, l'aire et le périmètre quantiques de la bulle ainsi formée se lit directement sur la trajectoire brownienne (L, R) : ce sont respectivement la durée et la longueur de l'excursion cône définie par l'intervalle (s', s) ⁴. De plus, l'orientation de l'excursion cône, c'est-à-dire l'axe sur lequel (L, R) se situe au temps s' , détermine si la bulle est déconnectée à *gauche* ou à *droite* de η' .

Remarquons également qu'on peut appliquer une transformation linéaire au couple (L, R) du Théorème 1.2.4 pour obtenir un mouvement brownien standard. Plus précisément, pour $\theta := \frac{4\pi}{\kappa'}$, la matrice

$$\Lambda := \begin{pmatrix} \frac{1}{\sin \theta} & \frac{1}{\tan \theta} \\ 0 & 1 \end{pmatrix}, \quad (1.2.5)$$

envoie un mouvement brownien Z de covariance (1.2.4) sur un mouvement brownien standard plan W , et le quadrant $\mathbb{R}_+^2$ sur l'adhérence du cône $\mathcal{C}_\theta := \Lambda(\mathbb{R}_+^2) = \{z \in \mathbb{C}, \arg(z) \in (0, \theta)\}$ d'angle θ . Ceci permet de voir les temps cônes précédents comme de vrais temps cônes pour $\mathcal{C}_\theta$. Dans le cas du disque quantique, un théorème d'accouplement d'arbres analogue peut être énoncé, faisant intervenir un mouvement brownien corrélé dans le quadrant, qui se traduit *via* (1.2.5) par une excursion brownienne dans le cône $\mathcal{C}_\theta$ [AG21].

Ainsi, les composantes connexes bornées détachées par une boucle FK sont décrites par les *temps flexibles* dans la bijection de Sheffield, de la même manière que celles d'une boucle $\text{CLE}_{\kappa'}$ sont décrites par les *temps cônes* du mouvement brownien corrélé (L, R) . Le résultat de [GMS19] déjà mentionné à la fin de la Section 1.2.2 s'interprète donc comme un résultat de convergence de ces boucles FK vers les boucles $\text{CLE}_{\kappa'}$ ⁵.

⁴On constatera l'analogie avec les *temps flexibles* dans le modèle de hamburger-cheeseburger. On invite ici le lecteur à relire la discussion suivant le Théorème 1.2.3.

⁵On insiste là encore sur le fait que dans le discret, la trajectoire (H, C) peut présenter des temps cônes à des instants autres que les temps flexibles. Néanmoins, ces instants arrivent avec une trop faible probabilité, si bien qu'ils disparaissent dans la limite d'échelle.

1.3 Motivations et principaux résultats de la thèse

Nous décrivons ici dans leur contexte les résultats obtenus au cours de cette thèse. Ils seront détaillés et démontrés dans les chapitres suivants.

Le processus de croissance-fragmentation dans le mouvement brownien plan.

Dans un travail en commun avec Élie Aïdékon [AS20], on se propose d'étudier un processus branchant qui survient lorsqu'on coupe une excursion brownienne à des hauteurs. On montre que l'objet ainsi obtenu est le processus de croissance-fragmentation découvert à la limite par Bertoin, Budd, Curien et Kortchemski [BBCK18] dans le cas critique $\theta = 1$. Notre construction n'est pas sans rappeler celle de Bertoin [Ber02], qui retrouve, en considérant les durées dans une excursion brownienne en une dimension, le processus de fragmentation pure d'Aldous-Pitman [AP98] de l'arbre réel continu (voir également [Ber00] pour une construction similaire, aussi due à Bertoin).

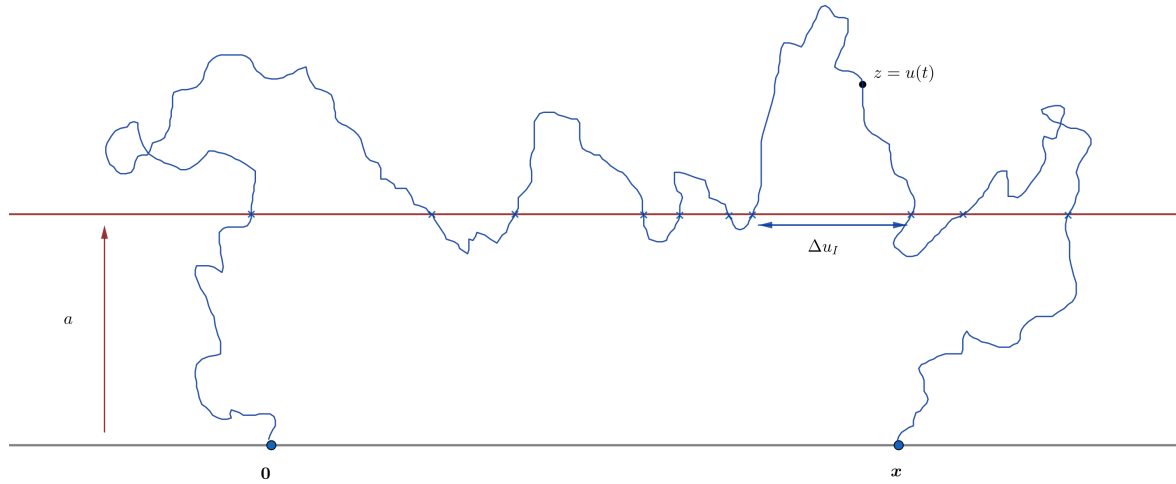


Figure 1.10 – Dessin naïf d’une excursion brownienne u dans le demi-plan entre 0 et x , coupée à hauteur a . On s’intéresse aux tailles des sous-excursions au-dessus du niveau a . A chaque temps t de la trajectoire brownienne, tel que $z = u(t)$ est au-dessus du niveau a , correspond une certaine sous-excursion repérée par l’intervalle I . On enregistre sa *taille* Δu_I , dans le sens de parcours de l’excursion (ici, Δu_I est positif).

Plus précisément, considérons une excursion brownienne de 0 à 1 dans le demi-plan $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$. Il est bien connu que ce conditionnement a un sens, dans le cadre de domaines d’ailleurs plus généraux (voir par exemple le chapitre 5 de [Law08]). Dans le cas du demi-plan, il existe en fait une façon très concrète de construire une telle excursion u (ce qui se transfère ensuite à des domaines du plan plus généraux par invariance conforme). On commence par choisir une variable R , qui sera la durée de notre excursion u , distribuée comme l’inverse d’une exponentielle de paramètre 1. Ensuite, on considère un pont de Bessel de dimension 3 de 0 à 0 de durée R pour la partie imaginaire de u , et un pont brownien indépendant de 0 à 1 de durée R pour la partie réelle de u . Cette description résulte essentiellement de la *description d’Itô* de la mesure d’Itô des excursions browniennes (en dimension 1), pour laquelle on renvoie le lecteur au chapitre 12 de [RY99]. Pour chaque hauteur $a > 0$, l’excursion u fait (potentiellement) un nombre dénombrable de sous-excursions au-dessus de la ligne horizontale $\{\Im(z) = a\}$ de hauteur a . On enregistre la *taille* de ces sous-excursions, définie comme la différence entre le point final et le point

initial : ceci donne une famille $\mathbf{Z}(a)$ de réels indexée par la hauteur a .

Lorsque a augmente, la famille $\mathbf{Z}(a)$ révèle une structure branchante : une sous-excursion au niveau a peut se scinder en deux excursions à un niveau ultérieur (correspondant à un minimum local de l'excursion u). Notre résultat principal dans [AS20] est le Théorème 1.1 décrivant précisément cette structure branchante en termes d'un processus de croissance-fragmentation autosimilaire. Notons cependant que les éléments de $\mathbf{Z}(a)$ peuvent être négatifs, contrairement à la construction de Bertoin résumée en section 1.1. Plus précisément, soit $Z = (Z_a)_{0 \leq a < \zeta}$ un processus de Markov positif issu de $z_0 > 0$, autosimilaire d'indice 1, dont la transformée de Lamperti s'écrit

$$Z_a = z_0 \exp(\xi(\tau(z_0^{-1}a))),$$

où ξ est un processus de Lévy d'exposant de Laplace

$$\Psi(q) = -\frac{4}{\pi}q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y} dy}{(e^y - 1)^2}, \quad q < 3, \quad (1.3.1)$$

τ est le changement de temps

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\xi(u)} du > a \right\},$$

et $\zeta = \inf\{a \geq 0, Z_a = 0\}$. On peut alors légèrement modifier la construction des processus de croissance-fragmentation de [Ber17b] pour prendre aussi en compte les sauts positifs de Z : ceux-ci donneront naissance à des particules de masses négatives, qui évolueront selon une copie de $-Z$. Notre résultat est alors que le processus de croissance-fragmentation (signé) issu de Z de cette manière a la même loi que le processus $\mathbf{Z}$. De plus, lorsqu'on tue dans la collection $\mathbf{Z}(a)$ toutes les particules de masses négatives (ainsi que leurs descendants), il s'avère qu'on obtient le même processus de croissance-fragmentation que celui découvert dans [BBCK18] en épluchant une carte de Boltzmann dans le cas $\theta = 1$. En particulier, notre construction brownienne prouve que l'arbre de fragmentation derrière le cas $\theta = 1$ est l'arbre brownien (les fragmentations ont lieu aux minima locaux de la partie imaginaire), ce qui ne semble pas du tout évident du point de vue des cartes planaires. Insistons enfin sur le fait que la mesure de Lévy dans (1.3.1) est portée par $(-\ln(2), \infty)$: cela provient de ce que le processus Z considéré correspond en réalité au *plus grand fragment local* du processus de croissance-fragmentation.

Notre approche met également en évidence une martingale remarquable pour ce processus de croissance-fragmentation signé, qui revient à sommer la taille des sous-excursions au carré. On établit alors une décomposition spinale dans l'esprit de [BBCK18], où l'épine se comporte comme un processus de Cauchy. Par ailleurs, il s'avère que la martingale générationnelle sous-jacente correspond à la martingale *critique* dans le cadre des marches branchantes (voir [Shi15]). Dans ce cas, la martingale tend vers 0 et on la remplace par la *martingale dérivée* pour avoir une limite non triviale. La martingale dérivée a d'abord été introduite par Lalley et Sellke dans le contexte du mouvement brownien branchant [LS87], puis utilisée pour les marches avec branchement par Kyprianou [Kyp98] et Biggins et Kyprianou [BK04]. Elle joue un rôle crucial *via* sa limite [BK04], notamment dans le théorème d'Aïdékon [Aïd13] donnant la loi limite de la position extrême. Nous nous sommes ainsi intéressés à la martingale dérivée associée au processus de croissance-fragmentation dans l'excursion brownienne. On prouve que la martingale dérivée converge presque sûrement vers (deux fois) la durée R de l'excursion.

Ces résultats se traduisent naturellement dans le langage des cartes de Boltzmann de [BBCK18]. En effet, on peut voir une carte de Boltzmann comme le *gasket* d'un

modèle $O(n)$ (voir [LGM11, BBG11]). Dans ce contexte, un saut positif dans la croissance-fragmentation correspond à la découverte d'une boucle, qu'on pourrait alors choisir d'explorer. Notre processus de croissance-fragmentation $\mathbf{Z}$ dans l'excursion brownienne correspondrait (en limite d'échelle) à l'exploration d'un modèle $O(2)$, dans lequel on choisit d'explorer les boucles, et où le signe représente la parité du nombre de boucles qui entourent la région considérée. L'analogue dans le continu reviendrait à considérer un SLE_4 remplissant dans un disque quantique pour le paramètre critique $\gamma = 2$. Le *mating-of-trees* énoncé ci-dessus (Théorème 1.2.4) dégénère dans ce cas limite, mais on observe qu'après transformation par (1.2.5), le cône d'angle θ se rapproche du demi-plan lorsque $\gamma \rightarrow 2$. Signalons que le cas critique de l'accouplement d'arbres a fait l'objet d'un travail récent de Aru, Holden, Powell et Sun [AHPS21]. Notre résultat de convergence sur la martingale dérivée prend également un sens dans ce cadre. Il est prouvé dans [BBCK18] que la limite de la martingale additive $\mathcal{M}^-$ (associée à la racine ω_- de κ dans (1.1.3)) peut être interprétée comme l'aire de la carte de Boltzmann sous-jacente. Dans le cas critique, il faut considérer la martingale dérivée, et celle-ci converge bien vers la durée de l'excursion brownienne, qui encode l'aire quantique dans l'accouplement d'arbres. Au passage, notons que cela donne une conjecture pour l'aire d'un modèle $O(2)$ comme (deux fois) l'inverse d'une exponentielle de paramètre 1. Le cas du modèle $O(n)$ non critique $n \neq 2$ est déjà connu et a été conjecturé par Chen, Curien et Maillard [CCM20]. Enfin, ces résultats ne sont pas sans rappeler le processus d'épluchage de Budd [Bud18] d'une carte de Boltzmann couplé avec un modèle $O(n)$. Cette exploration fait naturellement intervenir une classe de processus de Markov dits *ricochets*, qui sont des processus positifs avec une certaine condition de réflexion au bord. Ces processus sont directement reliés aux processus de croissance-fragmentation mentionnés ici (voir [Wat21]) ; dans le cas du processus de croissance-fragmentation brownien, ce lien s'obtient en considérant la valeur absolue des tailles dans notre système de particules.

Ces résultats seront exposés dans le Chapitre 2.

Processus de croissance-fragmentation signés. Les résultats du paragraphe précédent, issus de [AS20], mettent en lumière un système de particules qui possède une structure branchante. Ce système est un avatar du processus de croissance-fragmentation de Bertoin [Ber17b], mais où la masse des particules peut être *négative*. Dans le travail [Sil21], nous nous sommes donc intéressés à étendre le cadre des processus de croissance-fragmentation au cas signé. Cette extension apporte essentiellement deux modifications au modèle d'origine : le processus X peut désormais être à valeurs dans $\mathbb{R}^*$, et on s'autorise à prendre en compte *tous* les sauts de X dans la généalogie. Hormis la prise en compte des sauts positifs dans la généalogie, la construction du processus de croissance-fragmentation signé est similaire à celle de [Ber17b]. Elle repose sur un processus Ève qui appartient à la classe des processus de Markov autosimilaires à valeurs dans $\mathbb{R}^*$ définis par la propriété : pour tous $c > 0$ et $z \neq 0$, la loi de $(cX(c^{-\alpha}t), t \geq 0)$ sous $\mathbb{P}_z$ est $\mathbb{P}_{cz}$. Notons que cette propriété n'exclut pas que X puisse avoir un comportement différent pour les valeurs positives et négatives. La difficulté principale dans l'extension à ce cadre provient de la structure des processus de Markov autosimilaires à valeurs dans $\mathbb{R}^*$. Celle-ci est plus complexe que dans le cas positif, où X s'écrit comme l'exponentielle d'un processus de Lévy changé en temps. Dans le cas signé, cette description est similaire en esprit, mais tient compte des changements de signe, qui induisent des sauts spéciaux : la transformation de Lamperti [Lam72] est remplacée par la transformation de Lamperti-Kiu [Kiu80, CPR13], et le royaume des processus de Lévy par celui des *processus de Markov additifs* [Asm08]. Un processus de Markov additif pour une filtration $(\mathcal{G}_t)_{t \geq 0}$ est un processus càdlàg (ξ, J) à valeurs dans $\mathbb{R} \times E$, de loi $\mathbb{P}$, tel que $(J(t), t \geq 0)$ est une chaîne de Markov à temps continu, et qui vérifie la propriété suivante : pour tous $i \in E$, $t \geq 0$,

Conditionnellement à $J(t) = i$, le processus $(\xi(t+s) - \xi(t), J(t+s))_{s \geq 0}$ est indépendant de $\mathcal{G}_t$ et est distribué comme $(\xi(s) - \xi(0), J(s))_{s \geq 0}$ sachant $J(0) = i$.

Autrement dit, ξ évolue grossièrement comme un processus de Lévy, *modulo* le type qui est gouverné par J . La *représentation de Lamperti-Kiu* [CPR13] établit qu'un processus de Markov autosimilaire dans $\mathbb{R}^*$ s'écrit comme l'exponentielle d'un processus de Markov additif ξ , modulo les changements de signe, qui correspondent aux sauts de la chaîne J (dans ce cas, $E = \{-1, +1\}$). Une bonne introduction aux processus de Markov additifs et leurs interactions avec les processus de Markov autosimilaires se trouve dans le livre de Kyprianou et Pardo [KP21]. Par ailleurs, nous avons vu qu'un calcul central dans l'étude des processus de croissance-fragmentation passe par l'évaluation de sommes de puissances, ce qui requiert des précisions dans le cas signé. Par exemple, le sens de la martingale (1.1.4) n'est *a priori* pas clair lorsqu'on autorise la masse des particules à être signée.

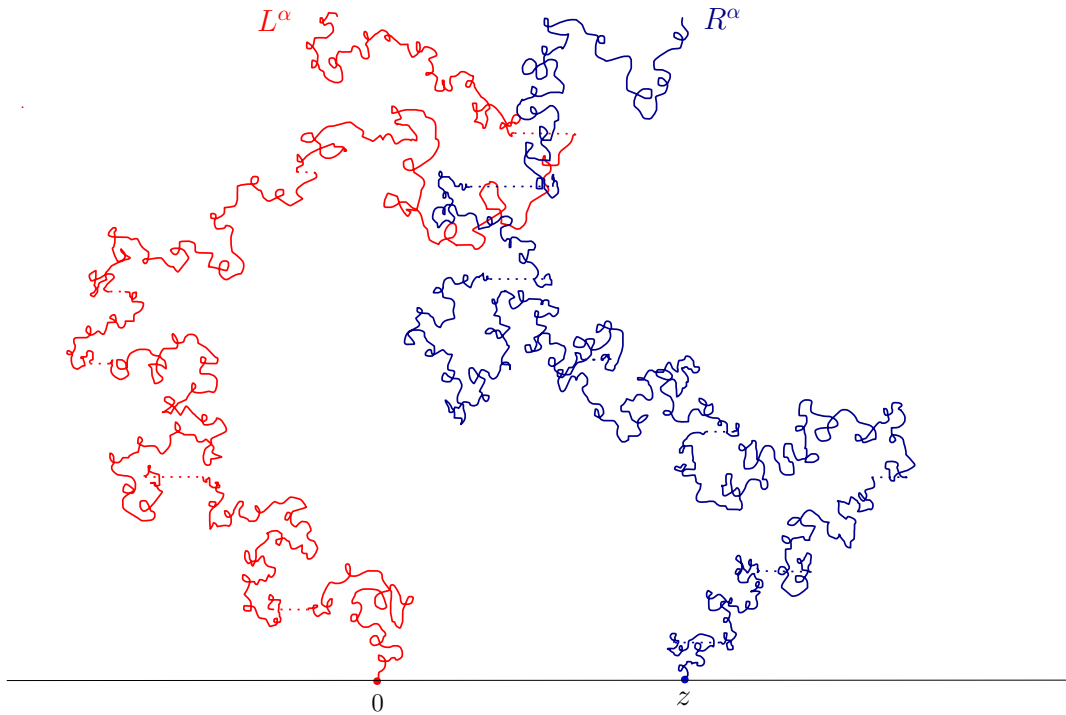


Figure 1.11 – Changement de mesure pour l'excursion à partie réelle α -stable (cas spectralement négatif). Sous la nouvelle mesure, l'excursion se scinde en deux parties indépendantes L^α et R^α (en rouge et bleu). L^α évolue comme un couple indépendant (X^α, Y) , où X est un processus de Lévy α -stable spectralement négatif et Y est un processus de Bessel de dimension 3 commençant en 0. R^α évolue comme une copie indépendante de $(z - X^\alpha, Y)$.

L'approche menée dans cette thèse met en avant des martingales remarquables dans le cas des processus de croissance-fragmentation signés. Nous prouvons qu'il existe une paire $(\mathcal{K}_+, \mathcal{K}_-)$ de *cumulants signés*, qui s'expriment en fonction des caractéristiques du processus de Markov additif sous-jacent, et qui jouent le rôle du cumulatif défini par Bertoin dans [Ber17b]. Les cumulants signés $(\mathcal{K}_+, \mathcal{K}_-)$ dépendent d'un vecteur (v_+, v_-) et d'un réel q , de telle façon que si $\mathcal{K}_+$ et $\mathcal{K}_-$ ont un zéro simultané en (v_+, v_-, ω) , alors le processus

$$\mathcal{M}(n) := \sum_{|u|=n+1} v_{\text{sgn}(\mathcal{X}_u(0))} |\mathcal{X}_u(0)|^\omega, \quad n \geq 0, \quad (1.3.2)$$

est une martingale pour la filtration générationnelle $(\mathcal{G}_n, n \geq 0)$. En comparant à (1.1.4), on

voit que ce sont les constantes (v_+, v_-) qui prennent en compte le signe dans la croissance-fragmentation. Ceci est naturel au vu de la forme de la martingale de Wald ou transformée d'Esscher pour les processus de Markov additifs (voir [KP21, Proposition 11.6], ou le panorama offert par Pardo et Rivero dans [PR13]). De plus, lorsque le processus de Markov sous-jacent est *symétrique*, c'est-à-dire lorsque X partant de $-x < 0$ a même loi que $-X$, X partant de x , on verra que nécessairement $v_+ = v_-$ (cette sous-classe spécifique de processus de Markov autosimilaires a été étudiée en détail par Chybiryakov [Chy06]). Le résultat principal de [Sil21] est le Théorème 5.4, où nous établissons la décomposition spinale pour le processus signé. La même image persiste au niveau du processus de croissance-fragmentation que dans le cas positif : sous la nouvelle mesure biaisée, l'épine dessine le tronc d'un arbre, qui est ensuite décoré par des copies du processus de croissance-fragmentation initial. En particulier, nous décrivons entièrement la loi de l'épine sous la nouvelle mesure : l'épine évolue comme un processus de Markov autosimilaire (de même indice), dont nous déterminons l'exposant de Lamperti-Kiu.

Enfin, nous introduisons une famille particulière de processus de croissance-fragmentation signés. Celle-ci s'inspire du modèle de l'excursion brownienne dans le demi-plan, à la différence qu'on considère ici une excursion où la partie réelle est remplacée par un processus de Lévy α -stable. La partie imaginaire, elle, reste inchangée. Autrement dit, cela revient à garder la structure d'arbre brownien (les fragmentations auront toujours lieu aux minima locaux de la partie imaginaire), mais à changer les étiquettes (correspondant aux *tailles*) sur cet arbre. On montre qu'en coupant cette excursion dans le demi-plan à des hauteurs successives et en enregistrant la taille des sous-excursions formées, on obtient un processus de croissance-fragmentation signé. De la même manière que dans [AS20], on exhibe également une martingale, qui consiste à sommer la *valeur absolue* des tailles des excursions au niveau a à la puissance $1 + \frac{\alpha}{2}$, *pondérées* par des coefficients qui dépendent du signe, comme en (1.3.2). On étudie la décomposition spinale du processus de croissance-fragmentation signé en biaisant l'excursion originale par cette martingale : l'image obtenue sous la nouvelle mesure est celle présentée en Figure 1.11. En particulier, on obtient que l'épine suit la loi d'un processus de Lévy $\frac{\alpha}{2}$ -stable. On pousse plus loin l'analyse dans le cas où la partie réelle est un processus stable spectralement négatif : en tuant toutes les particules de tailles négatives (y compris leurs descendants), on montre que la famille de processus obtenus correspond aux processus de croissance-fragmentation $\mathbf{X}_\theta$ introduits par Bertoin, Budd, Curien et Kortchemski [BBCK18], pour $\theta \in (\frac{1}{2}, 1)$.

Ces résultats seront exposés dans le Chapitre 3.

Processus de croissance-fragmentation multi-types. Dans un travail en commun avec Juan Carlos Pardo, nous nous sommes intéressés à une généralisation de ces résultats à un ensemble plus général de types $\mathcal{I}$. On suppose que $\mathcal{I}$ est fini : dans ce cas, la structure des processus de Markov additifs est beaucoup mieux connue [Asm08, KP21]. Néanmoins, l'extension du cas signé au cas d'un espace de types (fini ou dénombrable) n'est pas immédiate. En effet, l'argument majeur de [Sil21] repose sur un changement de cellule Ève pour se ramener au cadre classique d'une cellule Ève positive. Par ailleurs, la construction exige de donner un sens aux types des descendants, là où le signe dans [Sil21] des particules filles provient naturellement du signe du saut de la particule mère. Ce cadre abstrait nous force à raisonner directement sur les processus de Markov additifs eux-mêmes. L'image du processus de croissance-fragmentation reste cependant la même que dans le cas signé. On définit une famille $(\mathcal{K}_i)_{i \in \mathcal{I}}$, indicée par l'ensemble de types, de *cumulants multi-types* dépendant d'un vecteur $(v_i)_{i \in \mathcal{I}}$ et d'un exposant q , tel que si $((v_i)_{i \in \mathcal{I}}, \omega)$ est une

racine commune des $\mathcal{K}_i, i \in \mathcal{I}$, alors

$$\mathcal{M}(n) := \sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega, \quad n \geq 0, \quad (1.3.3)$$

où $\mathcal{J}_u(0)$ désigne le type de la particule u à la naissance, est une martingale pour la filtration générationnelle $(\mathcal{G}_n, n \geq 0)$. Dans le cas où le type désigne le signe, on retrouve bien sûr l'expression (1.3.2). Après le changement de mesure associé à (1.3.3), on décrit en utilisant la structure des processus de Markov additifs la décomposition spinale du système de particules. On donne la loi du processus de Markov additif sous-jacent à l'épine, en explicitant son exposant de Lamperti-Kiu.

Nous nous intéressons ensuite à un modèle de croissance-fragmentation vectoriel, où le trait d'intérêt est un vecteur de $\mathbb{R}^d$, au lieu d'être une masse (positive [Ber17b, BBCK18], ou signée [Sil21]). Ce point de vue nous amène à étendre le modèle de Bertoin dans le cas où la cellule Ève est un processus de Markov autosimilaire à valeurs dans $\mathbb{R}^d$. Dans ce contexte, la représentation de Lamperti-Kiu, attribuée à Kiu [Kiu80] (voir également les discussions de Alili, Chaumont, Graczyk et Żak [ACGŻ17] qui étendent les considérations de Kiu au cas non symétrique), s'énonce comme suit. Soient $\alpha > 0$, et X un processus de Markov autosimilaire d'indice α , à valeurs dans $\mathbb{R}^d$. Alors il existe un processus de Markov additif (ξ, Θ) à valeurs dans $\mathbb{R} \times \mathbb{S}^{d-1}$, éventuellement tué en un temps ς , tel que

$$X(t) = e^{\xi(\varphi(t))} \Theta(\varphi(t)), \quad t \leq I_\varsigma, \quad (1.3.4)$$

où φ est le changement de temps

$$\varphi(t) := \inf\{s > 0, \int_0^s e^{\alpha\xi(u)} du > t\},$$

et $I_\varsigma := \int_0^\varsigma e^{\alpha\xi(s)} ds$ est le temps de vie de X . Réciproquement, tout processus X défini par (1.3.4) est un processus de Markov à valeurs dans $\mathbb{R}^d$, autosimilaire d'indice α . En toute rigueur, nous n'avons pas défini les processus de Markov additifs à espace de types non dénombrable, mais on peut facilement étendre leur construction par la propriété :

$$\mathbb{E}_{x,\theta}[f(\xi(t+s) - \xi(t), \Theta(t+s)) \mathbb{1}_{t+s < \varsigma} | \mathcal{G}_t] = \mathbb{1}_{t < \varsigma} \mathbb{E}_{0, \Theta(t)}[f(\xi(s), \Theta(s)) \mathbb{1}_{s < \varsigma}],$$

pour toute fonction mesurable bornée $f : \mathbb{R} \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$, $s, t \geq 0$ et $(x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$.

La représentation (1.3.4) se lit comme une décomposition polaire. Comme dans le cas réel $d = 1$, la composante ξ décrit la distance à l'origine, tandis que Θ remplace la chaîne de Markov J et décrit l'orientation du vecteur. En général, l'analyse d'une telle décomposition est extrêmement complexe, car les processus de Markov additifs à valeurs dans $\mathbb{R} \times \mathbb{S}^{d-1}$ n'ont pas une structure explicite. Néanmoins, il existe une classe de processus de Markov autosimilaires dont la structure polaire est plus simple, les processus *isotropes*. Comme le nom l'indique, il s'agit de processus qui, en plus de l'autosimilarité, jouissent d'une propriété d'invariance selon la direction. Dans le cas unidimensionnel, l'isotropie correspond à la catégorie des processus symétriques. Les processus isotropes partagent une propriété centrale : la composante radiale ξ est alors un processus de Lévy (autrement dit, $|X|$ est un processus de Markov autosimilaire positif, et ξ est son exposant de Lamperti). Cette propriété donne accès aux outils provenant des processus de Lévy. Observons cependant qu'il n'y a pas d'indépendance entre ξ et Θ en toute généralité.

On poursuit donc l'étude des processus de croissance-fragmentation multi-types dans ce cadre vectoriel isotrope. Dans ce contexte, on définit un *cumulant isotrope*, dont les racines produisent des martingales, puis on étudie la décomposition spinale. On verra

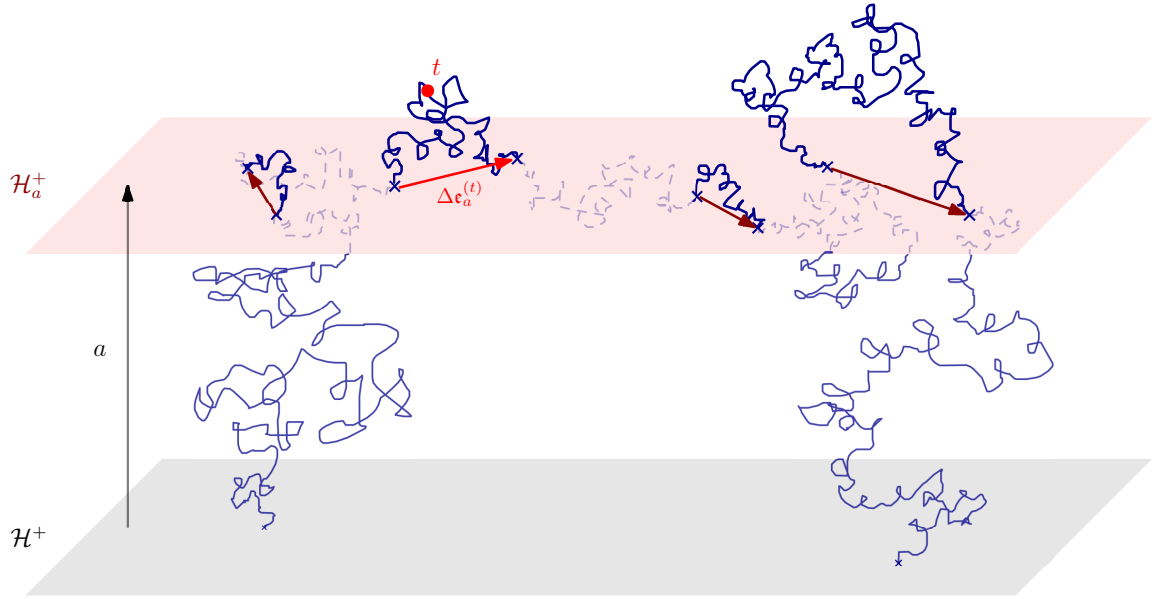


Figure 1.12 – Une excursion u au-dessus de l’hyperplan $\mathcal{H}$, coupée à hauteur a . La trajectoire bleue représente l’excursion (ici, $N = 3$). Pour chaque hauteur $a > 0$ fixée, on forme l’hyperplan de hauteur a et on enregistre les sous-excursions $\mathcal{H}_a^+$ au-dessus de cet hyperplan. Les quatre plus grandes sont ici représentées en bleu foncé (le lecteur doit bien sûr imaginer beaucoup d’excursions infinitésimales). Les flèches rouges indiquent la *taille* de ces sous-excursions, comptées avec l’orientation de u .

que l’isotropie de la cellule Ève garantit que la martingale qui étend naturellement (1.3.3) ne dépend pas de la partie angulaire, et que l’épine est elle-même isotrope. Ensuite, on montre qu’un tel objet apparaît naturellement dans une excursion brownienne au-dessus de l’hyperplan $\mathcal{H} := \{(x_1, \dots, x_N) \in \mathbb{R}^N, x_N = 0\}$ de $\mathbb{R}^N$. Pour cela, on considère l’analogue du cas planaire [AS20], où on coupe une excursion dans le demi-espace, tel qu’étudié par Burdzy [Bur86], par des hyperplans. Pour chaque hauteur, ceci donne une famille (éventuellement vide, mais au plus dénombrable) de sous-excursions au-dessus des hyperplans, dont on définit la *taille* comme la différence entre le point d’arrivée et le point de départ. Cette taille est maintenant un vecteur de $\mathbb{R}^{N-1}$; on pourra trouver une tentative d’illustration en Figure 1.12.

Notre résultat est alors que la collection de vecteurs ainsi obtenue forme un processus de croissance-fragmentation vectoriel (isotrope) dans $\mathbb{R}^{N-1}$, dont l’épine est donnée par un processus de Cauchy multidimensionnel. Suivant [Sil21], on étend ce résultat lorsque les $(N - 1)$ premières coordonnées forment un processus stable (isotrope).

Ces résultats seront exposés dans le Chapitre 4.

Un processus de croissance-fragmentation relié à une exploration SLE_6 du disque quantique de paramètre $\sqrt{8/3}$. Le processus de croissance-fragmentation obtenu dans une excursion brownienne plane est reliée à travers le *mating-of-trees* critique [AHPS21] aux explorations d’un SLE_4 remplissant sur le disque quantique de paramètre $\gamma = 2$. Dans un travail en cours avec Ellen Powell et Alex Watson, on s’intéresse à une version non critique de cette question, qui implique des explorations $\text{SLE}_{\kappa'}$ remplissantes du disque quantique de paramètre γ , où $\gamma \in (\sqrt{2}, 2)$ et $\kappa' = \frac{16}{\gamma^2} \in (4, 8)$. Dans ce cadre, l’accouplement d’arbres est dû dans sa version la plus générale à Ang et Gwynne [AG21], et s’énonce comme suit (encore une fois, nous ne définissons pas tous les objets par souci de concision).

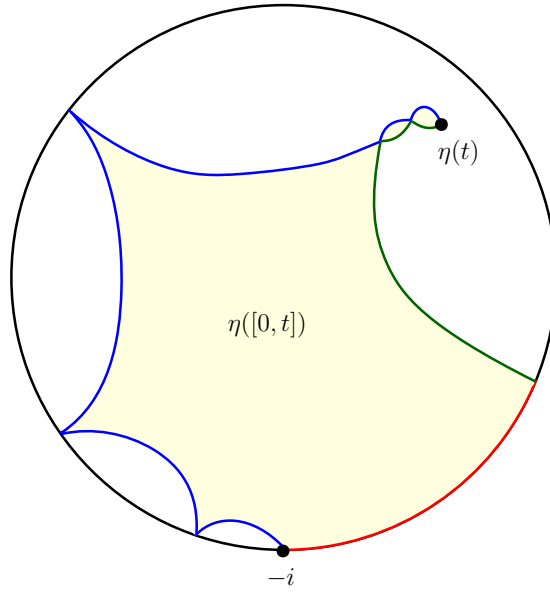


Figure 1.13 – Un disque quantique $(\mathbb{D}, \psi, -i)$ de paramètre $\sqrt{8/3}$ décoré par un SLE_6 indépendant anti-horaire remplissant η de $-i$ à $-i$, paramétré par l'aire quantique. La longueur quantique à gauche L_t correspond à la longueur quantique de la courbe bleue. La longueur quantique à droite R_t correspond à la longueur quantique de la partie verte *moins* celle de la partie rouge.

Théorème 1.3.1. Soient $\gamma \in (0, 2)$ et $(\mathbb{D}, \psi, -i)$ un disque quantique marqué de paramètre γ , de périmètre quantique égal à 1, et d'aire quantique $\mu_\psi^\gamma(\mathbb{D})$ aléatoire. On considère un $\text{SLE}_{\kappa'}$ remplissant anti-horaire $\eta : [0, \mu_\psi^\gamma(\mathbb{D})] \rightarrow \overline{\mathbb{D}}$ de $-i$ à $-i$, indépendant du champ libre ψ , mais reparamétré par l'aire quantique. On note L_t et R_t le changement de longueur quantique à gauche et à droite de $\eta([0, t])$ relativement au temps 0 comme sur la Figure 1.13, normalisé par $(L_0, R_0) = (0, 1)$. Alors $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ est un mouvement brownien corrélé conditionné à rester dans le quadrant positif $\mathbb{R}_+ \times \mathbb{R}_+$, partant de $(0, 1)$ sur la frontière, et conditionné à sortir du quadrant en l'origine. La structure de covariance de $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ est donnée par

$$\text{Var}(L_t) = \text{Var}(R_t) = \mathfrak{a}^2 t, \quad \text{Cov}(L_t, R_t) = -\cos\left(\frac{4\pi}{\kappa'}\right) \mathfrak{a}^2 t, \quad (1.3.5)$$

où $\mathfrak{a}$ est une constante qui dépend de γ .

Dans cette thèse, nous considérons essentiellement le cas $\gamma = \sqrt{8/3}$, parfois appelé *gravité quantique pure*. Le cas général présente des difficultés toujours en cours de résolution et principalement dues à l'absence de la propriété de *target invariance* que nous utilisons crucialement dans le cas $\gamma = \sqrt{8/3}$. On considère un disque quantique $(\mathbb{D}, \psi, -i)$ de paramètre $\sqrt{8/3}$ marqué par un point de la frontière. On décore ce disque quantique d'une trajectoire indépendante η qui est un SLE_6 remplissant anti-horaire de $-i$ à $-i$. On peut alors définir la *branche* du SLE η vers un point z du disque, reparamétrée par son *temps local quantique*, au sens où l'on n'explore aucune composante détachée de z par η . Ce procédé définit des branches vers tout point du disque, de sorte que si x et y sont deux points du disque, les branches vers x et y coïncident jusqu'à ce que η déconnecte x et y (Figure 1.14). Pour $t \geq 0$, on considère alors toutes les branches encore en vie à l'instant t , et on enregistre le périmètre quantique du domaine restant à explorer au temps t (voir Figure 1.15) : ceci donne une famille de réels positifs qu'on dénote $\mathbf{Z}(t)$.

Notre résultat décrit la structure branchante de $\mathbf{Z}$ en termes du processus de croissance-fragmentation $\mathbf{X}_{3/2}$ de [BBCK18]. Plus précisément, soit ν la mesure sur $(\frac{1}{2}, 1)$ définie par

$$\nu(dx) = \frac{3}{4\sqrt{\pi}} \cdot \frac{dx}{x^{5/2}(1-x)^{5/2}} \mathbb{1}_{1/2 < x < 1},$$

et Λ sa mesure image par $x \mapsto \log(x)$. Introduisons le processus de Lévy ξ d'exposant de Laplace

$$\Psi(q) := -\frac{2}{\sqrt{\pi}}cq + c \int_{-\log(2)}^0 (e^{qy} - 1 - q(e^y - 1))\Lambda(dy), \quad q > \frac{3}{2}, \quad (1.3.6)$$

où c est une certaine constante. On construit le processus de Markov autosimilaire positif X d'indice $\frac{3}{2}$ dont la transformée de Lamperti est donné par ξ . On montre alors le résultat suivant.

Théorème 1.3.2. *Le processus $\mathbf{Z}$ a la même loi que le processus de croissance-fragmentation $\mathbf{X}$ dirigé par X .*

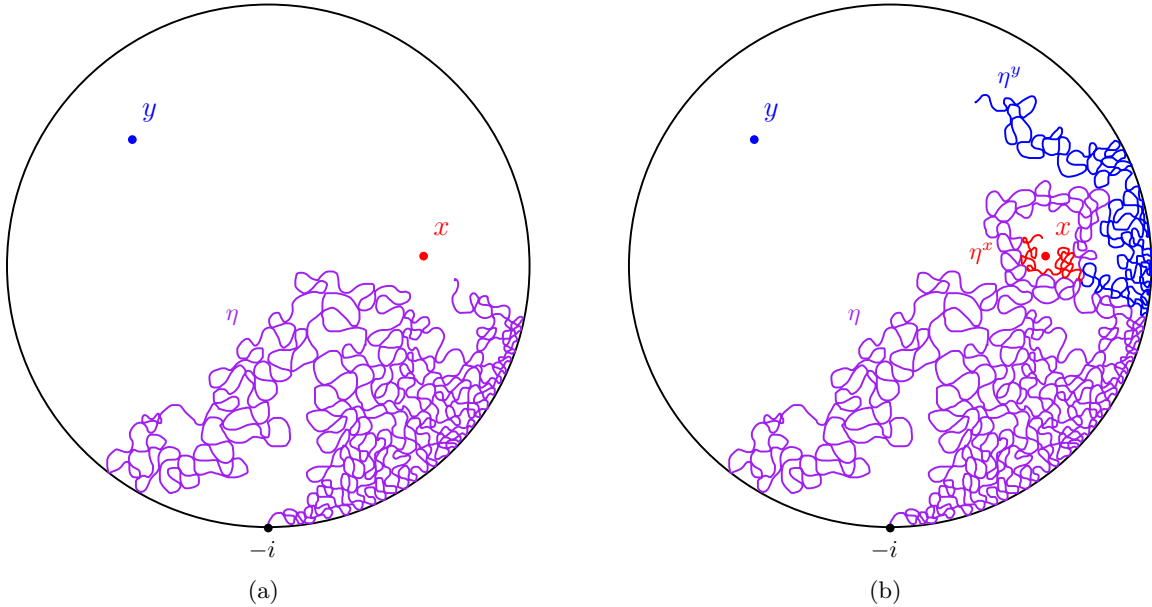


Figure 1.14 – Branches vers x et y du SLE_6 remplissant η dessinées sur le disque quantique de paramètre $\sqrt{8/3}$. (a) Les branches de η vers x et y (violet) sont les mêmes. (b) Les deux branches sont déconnectées : ici, η a tracé une boucle autour de x . La branche η^x vers x est représentée en (violet puis) rouge, et la branche η^y vers y en (violet puis) bleu.

Ce théorème peut être reformulé en termes d'excursions browniennes dans le cône d'angle $\frac{2\pi}{3}$ via le *mating-of-trees*. Nous ne détaillerons pas ici cette correspondance, mais signalons que le processus de croissance-fragmentation $\mathbf{Z}$ se traduit vaguement en considérant les *sous-excursions cônes* d'une excursion cône d'origine, dans un esprit similaire au cas du demi-plan étudié dans [AS20]. Ce point de vue nous permet de proposer une autre voie (partiellement complète) pour la démonstration du Théorème 1.3.2, uniquement fondée sur des arguments de théorie des excursions pour le mouvement brownien dans un cône. Dans cette direction, nous établissons une décomposition à la Bismut de la mesure d'excursion brownienne dans le cône. Ce résultat sera démontré en toute généralité, quel que soit l'angle d'ouverture du cône entre $\frac{\pi}{2}$ et π . Nous pensons qu'il pourrait avoir des applications en gravité quantique de Liouville.

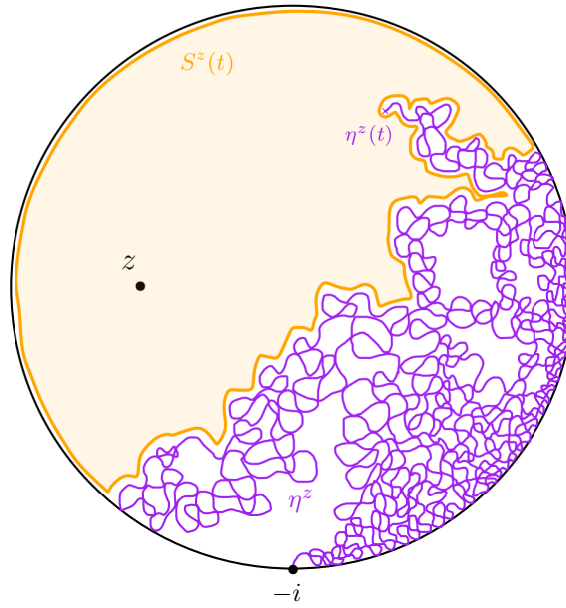


Figure 1.15 – Périmètre quantique de la région contenant z au temps t . Le dessin montre la branche η^z vers un point z du disque jusqu'au temps local quantique t (violet). Le domaine restant à explorer au temps t est la région orangée : on enregistre le périmètre quantique $S^z(t)$ de cette région.

Le Théorème 1.3.2 s'inscrit naturellement dans une lignée de travaux et doit être comparé à la littérature existante. D'abord, il constitue un pendant continu aux considérations de [BCK18] puis [BBCK18]. Nous retrouvons en effet le processus de croissance-fragmentation obtenu comme limite d'échelle des périmètres dans l'épluchage de grandes triangulations de Boltzmann pour $\theta = 3/2$, tel que décrit dans la Section 1.2.1. Ensuite, notre résultat peut être vu comme un complément à la construction par Miller, Sheffield et Werner [MSW20] des processus de croissance-fragmentation obtenus à la limite par [BBCK18], pour $\theta \in (1, \frac{3}{2})$, où il correspond à la limite $\theta \rightarrow \frac{3}{2}$, c'est-à-dire $\gamma \rightarrow \sqrt{8/3}$. Dans ce cas, le CLE_κ considéré par [MSW20] disparaît, et on est ramené au cadre d'un SLE remplissant comme dans la Section 1.2.1. Notons que la disparition du CLE se reflète du point de vue du processus de croissance-fragmentation dans l'absence de sauts positifs dans le cas $\theta = \frac{3}{2}$ de [BBCK18] (ce qui est cohérent avec la forme de (1.3.6)). Enfin, signalons qu'un processus de croissance-fragmentation très relié à $\mathbf{X}_\theta$ apparaît également dans la description, par Le Gall et Riera [LGR20], du disque brownien coupé à des hauteurs successives : on obtient alors un processus de croissance-fragmentation de même cumulant, mais d'indice d'autosimilarité $\frac{1}{2}$ (voir également [BBCK18, Section 6.5]). Ces résultats sont naturels à la fois au vu de l'équivalence de certaines surfaces quantiques de paramètre $\sqrt{8/3}$ équipées d'une structure métrique avec des surfaces browniennes [MS20, MS16, MS21], et au vu de la convergence en limite d'échelle de grandes quadrangulations de Boltzmann vers le disque brownien [Bet15, BM17, GM19].

Ces résultats seront exposés dans le Chapitre 5.

Chapter 2

Growth-fragmentation process embedded in a planar Brownian excursion

Abstract

This chapter contains the results of [AS20], to appear in *Probability Theory and Related Fields*. The aim of this paper is to present a self-similar growth-fragmentation process linked to a Brownian excursion in the upper half-plane $\mathbb{H}$, obtained by cutting the excursion at horizontal levels. We prove that the associated growth-fragmentation is related to one of the growth-fragmentation processes introduced by Bertoin, Budd, Curien and Kortchemski in [BBCK18].

2.1 Introduction

We consider a Brownian excursion in the upper half-plane $\mathbb{H}$ from 0 to a positive real number z_0 . For $a > 0$, if the excursion hits the set $\{z \in \mathbb{C} : \Im(z) = a\}$ of points with imaginary part a , it will make a countable number of excursions above it, that we denote by $(e_i^{a,+}, i \geq 1)$. For any such excursion, we let $\Delta e_i^{a,+}$ be the difference between the endpoint of the excursion and its starting point, which we will refer to as the *size* or *length* of the excursion. Since both points have the same imaginary part, the collection $(\Delta e_i^{a,+}, i \geq 1)$ is a collection of real numbers and we suppose that they are ranked in decreasing order of their magnitude. Our main theorem describes the law of the process $(\Delta e_i^{a,+}, i \geq 1)_{a \geq 0}$ indexed by a in terms of a self-similar growth-fragmentation. We refer to [Ber17b] and [BBCK18] for background on growth-fragmentations. Let us describe the growth-fragmentation process involved in our case.

Let $Z = (Z_a)_{0 \leq a < \zeta}$ be the positive self-similar Markov process of index 1 whose Lamperti representation is

$$Z_a = z_0 \exp(\xi(\tau(z_0^{-1}a))),$$

where ξ is the Lévy process with Laplace exponent

$$\Psi(q) = -\frac{4}{\pi}q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y} dy}{(e^y - 1)^2}, \quad q < 3, \quad (2.1.1)$$

τ is the time change

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\xi(u)} du > a \right\},$$

and $\zeta = \inf\{a \geq 0, Z_a = 0\}$. The *cell system* driven by Z can be roughly constructed as follows. The size of the so-called *Eve cell* is z_0 at time 0 and evolves according to Z . Then, conditionally on Z , we start at times a when a jump $\Delta Z_a = Z_a - Z_{a-}$ occurs independent processes starting from $-\Delta Z_a$, distributed as Z when $\Delta Z_a < 0$ and as $-Z$ when $\Delta Z_a > 0$. These processes represent the sizes of the daughters of the Eve particle. Then repeat the process for all the daughter cells: at each jump time of the cell process, start an independent copy of the process Z if the jump is negative, $-Z$ if the jump is positive, with initial value the negative of the corresponding jump. This defines the sizes of the cells of the next generation and we proceed likewise. We then define, for $a \geq 0$, $\bar{\mathbf{X}}(a)$ as the collection of sizes of cells alive at time a , ranked in decreasing order of their magnitude.

Growth-fragmentation processes were introduced in [Ber17b]. Beware that the growth-fragmentation process we just defined is not included in the framework of [Ber17b] or [BBCK18] because we allow cells to be created at times corresponding to positive jumps, giving birth to cells with negative size. Therefore, the process $\bar{\mathbf{X}}$ is not a true growth-fragmentation process. The formal construction of the process $\bar{\mathbf{X}}$ is done in Section 2.4. The following theorem is the main result of the paper.

Theorem 2.1.1. *The process $(\Delta e_i^{a,+}, i \geq 1)_{a \geq 0}$ is distributed as $\bar{\mathbf{X}}$.*

Remarks.

- The fact that there is no local explosion (in the sense that there is no compact of $\mathbb{R} \setminus \{0\}$ with infinitely many elements of $\bar{\mathbf{X}}$) can be seen as a consequence of the theorem.
- From the skew-product representation of planar Brownian motion, this theorem has an analog in the radial setting. It can be stated as follows. Take a Brownian excursion in the unit disc from boundary to boundary, with continuous determination of its argument (i.e., its winding number around the origin) $z_0 > 0$. Then, for each $a \geq 0$, record for each excursion made in the disc of radius e^{-a} the corresponding winding number. The collection of these winding numbers, ranked in decreasing order of their magnitude and indexed by a is distributed as $\bar{\mathbf{X}}$.
- One could finally look at the growth-fragmentation associated to the Brownian bubble measure in $\mathbb{H}$ (we refer to [Law08], Chapter 5.5 for the definition of the bubble measure). It would give an infinite measure on the space of (signed) growth-fragmentation processes starting from 0. In the non-critical case (i.e. when the natural martingale associated to the intrinsic area converges in L^1), a σ -finite measure on growth-fragmentation processes starting from 0 has been constructed by Bertoin, Curien and Kortchemski [BCK18], see Section 4.3 there. Their construction hinges on the spine decomposition, using a special distinguished cell under some pseudo-excursion measure. We believe that a similar representation should hold in our case, when the bubble measure is tilted by the duration of the bubble.

Related works. A pure fragmentation process was identified by Bertoin [Ber02] in the case of the linear Brownian excursion where the *size* of an excursion was there its duration. Le Gall and Riera [LGR20] identified a growth-fragmentation process in the Brownian motion indexed by the Brownian tree. We will follow the strategy of this paper, making use of excursion theory to prove our theorem.

When killing in $\bar{\mathbf{X}}$ all cells with negative size (and their progeny), one recovers a genuine self-similar (positive) growth-fragmentation driven by Z , call it $\mathbf{X}$. The process $\mathbf{X}$ appears in the work of Bertoin et al. [BBCK18], compare Proposition 5.2 in [BBCK18] with Proposition

2.4.3 below. In Section 3.3 of [BBCK18], the authors exhibit remarkable martingales associated to growth-fragmentation processes and describe the corresponding changes of measure. In the case of $\mathbf{X}$, the martingale consists in summing the sizes raised to the power $5/2$ of all cells alive at time a . Under the change of measure, the process $\mathbf{X}$ has a spinal decomposition: the size of the tagged particle is a Cauchy process conditioned on staying positive, while other cells behave normally. In the case of $\overline{\mathbf{X}}$, where we also include cells with negative size, a similar martingale appears, substituting 2 for $5/2$, while the tagged particle will now follow a Cauchy process (with no conditioning). It is the content of Section 2.3.2. This martingale is related to the one appearing in [AHS20], where a change of measure was also specified. In that paper, the authors exhibit a martingale in the radial case, see Section 7.1 there. The martingale in our setting can be viewed as a limit case, where one conformally maps the unit disc to the upper half-plane, then sends the image of the origin towards infinity.

Connection with random planar maps. In [BBCK18], the authors relate a distinguished family of growth-fragmentation processes to the exploration of a Boltzmann planar map, see Proposition 6.6 there. The mass of a particle in the growth-fragmentation represents the perimeter of a region in the planar map which is currently explored, a negative jump the splitting of the region into two smaller regions to be explored, and a positive jump the discovery of a face with large degree. In this setting, only a negative jump is a birth event. The area of the map is identified as the limit of a natural martingale associated to the underlying branching random walk, see Corollary 6.7 there.

On the other hand, a Boltzmann random map can also be seen as the gasket of a $O(n)$ loop model, see Section 8 of [LGM11]. From this point of view, a positive jump of the growth-fragmentation stands for the discovery of a loop which still has to be explored, so that positive jumps will be birth events too. The signed growth-fragmentation $\overline{\mathbf{X}}$ of our paper would represent the exploration of a planar map decorated with the $O(n)$ model with $n = 2$, where the sign depends on the parity of the number of loops which surrounds the explored region. One could wonder whether we would have an intrinsic area as in [BBCK18]. Actually, the natural martingale associated to the branching random walk converges to 0: it is the so-called critical martingale in the branching random walk literature. The martingale to consider is then the derivative martingale, see Section 2.5, whose limit is proved to be twice the duration of the Brownian excursion (i.e. the inverse of an exponential random variable, see (2.2.7)). This gives a conjectured limit of the area of a $O(2)$ decorated planar map properly renormalized, see [CCM20], Theorem 9, for the analogous results in the $O(n)$ model for $n \neq 2$.

The paper is organized as follows. In Section 2.2, we recall some excursion theory for the planar Brownian motion. Among others, we will define the locally largest fragment, which will be our Eve particle. In Section 2.3, we show the branching property, identify the law of the Eve particle with that of Z and exhibit the martingale in our context. Theorem 2.1.1 will be proved in Section 2.4, where we also show the relation with [BBCK18]. Finally, we identify the limit of the derivative martingale in Section 2.5.

Acknowledgements: We are grateful to Jean Bertoin and Bastien Mallein for stimulating discussions, and to Juan Carlos Pardo for a number of helpful discussions regarding self-similar processes. After a first version of this article appeared online, Nicolas Curien pointed to us the connection with random planar maps, and the link between the duration of the excursion and the area of the map. We warmly thank him for his explanations. We also learnt that Timothy Budd in an unpublished note had already predicted the link between growth-fragmentations of [BBCK18] and planar excursions. Finally, we thank two anonymous referees for their careful reading and valuable comments.

2.2 Excursions of Brownian motion in $\mathbb{H}$

2.2.1 The excursion process of Brownian motion in $\mathbb{H}$

In this section, we recall some basic facts from excursion theory. Let (X, Y) be a planar Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t \geq 0}$ be the usual augmented filtration.

In addition, we call $\mathcal{X}$ the space of real-valued continuous functions w defined on an interval $[0, R(w)] \subset [0, \infty)$, endowed with the usual σ -fields generated by the coordinate mappings $w \mapsto w(t \wedge R(w))$. Let also $\mathcal{X}_0$ be the subset of functions in $\mathcal{X}$ vanishing at their endpoint $R(w)$. We set $U := \{u = (x, y) \in \mathcal{X} \times \mathcal{X}_0, u(0) = 0 \text{ and } R(x) = R(y)\}$ and $U_\diamond := U \cup \{\diamond\}$, where $\diamond$ is a cemetery function and write $U^\pm$ for the set of such functions in U with nonnegative and nonpositive imaginary part respectively. These sets are endowed with the product σ -field denoted $\mathcal{U}_\diamond$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ adapted to the coordinate process on U . For $u \in U$, we take the obvious notation $R(u) := R(x) = R(y)$. Finally, let $(L_s)_{s \geq 0} = (L_s^Y)_{s \geq 0}$ denote the local time at 0 of Y and $\tau_s = \tau_s^Y$ its inverse defined by $\tau_s := \inf\{r > 0, L_r > s\}$. Recall that the set of zeros of Y is almost surely equal to the set of τ_s, τ_{s-} ; we refer to [RY99] for more details on local times.

Definition 2.2.1. *The excursion process is the process $e = (e_s, s > 0)$ with values in $(U_\diamond, \mathcal{U}_\diamond)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by*

(i) if $\tau_s - \tau_{s-} > 0$, then

$$e_s : r \mapsto (X_{r+\tau_{s-}} - X_{\tau_{s-}}, Y_{r+\tau_{s-}}), \quad r \leq \tau_s - \tau_{s-},$$

(ii) if $\tau_s - \tau_{s-} = 0$, then $e_s = \diamond$.

Figure 2.1 is a (naive) drawing of such an excursion.

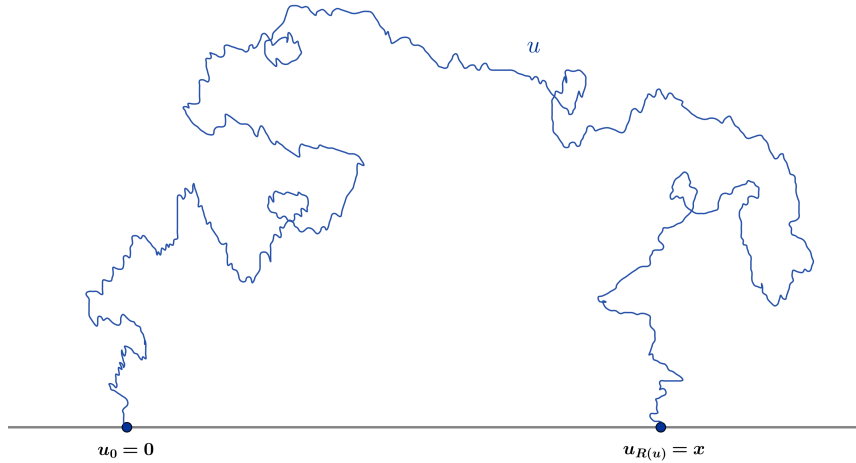


Figure 2.1 – Drawing of an excursion in the upper half-plane $\mathbb{H}$.

The next proposition follows from the one-dimensional case, see [RY99], Chapter XII.2.

Proposition 2.2.2. *The excursion process $(e_s)_{s>0}$ is a $(\mathcal{F}_{\tau_s})_{s>0}$ -Poisson point process.*

We write $\mathbf{n}$ for the intensity measure of this Poisson point process. It is a measure on U , and we shall denote by $\mathbf{n}_+$ and $\mathbf{n}_-$ its restrictions to U^+ and U^- . For $T \geq 0$, we let $X^T := (X_t, t \in [0, T])$. Following Chapter XII.2 in [RY99], we denote by n the one-dimensional Itô measure on $\mathcal{X}_0$, and by n_+ its restriction to the space of positive excursions.

Proposition 2.2.3. *We have the following expression for $\mathbf{n}$: $\mathbf{n}(dx, dy) = n(dy)\mathbb{P}(X^{R(y)} \in dx)$. Similarly, $\mathbf{n}_+(dx, dy) = n_+(dy)\mathbb{P}(X^{R(y)} \in dx)$.*

2.2.2 The Markov property under $\mathbf{n}$

For any $u \in U$ and any $a > 0$, let $T_a := \inf\{0 \leq t \leq R(u), y(t) = a\}$ be the hitting time of a by y . Then we have the following kind of Markov property under $\mathbf{n}_+$.

Lemma 2.2.4. *(Markov property under $\mathbf{n}$)*

Under $\mathbf{n}_+$, on the event $\{T_a < \infty\}$, the process $(u(T_a + t) - u(T_a))_{0 \leq t \leq R(u) - T_a}$ is independent of $\mathcal{F}_{T_a}$ and has the law of a Brownian motion killed at the time ρ when it reaches $\{\mathfrak{S}(z) = -a\}$.

Proof. This results from the fact that under the one-dimensional Itô's measure n_+ , the coordinate process $t \mapsto y(t)$ has the transition of a Brownian motion killed when it reaches 0 (cf. Theorem 4.1, Chap. XII in [RY99]).

Let f, g, h_1, h_2 be nonnegative measurable functions defined on $\mathcal{X}$. For simplicity, write for $w \in \mathcal{X}$ or $w \in C([0, \infty))$, $w(\theta_r) = w(r + \cdot) - w(r)$ and for $T > 0$, $w^T := (w(t), t \in [0, T])$. We want to compute

$$\begin{aligned} & \int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbf{n}_+(dx, dy) \\ &= \int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}n_+(dy)\mathbb{P}(X^{R(y)} \in dx) \\ &= \int_{\mathcal{X}_0} g(y(\theta_{T_a}))h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbb{E}\left[f\left(\tilde{X}^{R(y)-T_a(y)}\right)h_1\left(X^{T_a(y)}\right)\right]n_+(dy) \end{aligned}$$

where $\tilde{X} = X(\theta_{T_a(y)})$, and for $y \in \mathcal{X}_0$, $T_a = T_a(y)$ is the hitting time of a by y . Using the simple Markov property at time $T_a(y)$ in the above expectation gives

$$\begin{aligned} & \int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbf{n}_+(dx, dy) \\ &= \int_{\mathcal{X}_0} g(y(\theta_{T_a}))h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbb{E}\left[f\left(X^{R(y)-T_a(y)}\right)\right]\mathbb{E}\left[h_1\left(X^{T_a(y)}\right)\right]n_+(dy). \end{aligned}$$

Then we can use the Markov property under n_+ stated in Theorem 4.1, Chap. XII in [RY99]:

$$\begin{aligned} & \int_U f(x(\theta_{T_a}))g(y(\theta_{T_a}))h_1(x^{T_a})h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbf{n}_+(dx, dy) \\ &= \int_{\mathcal{X}_0} \mathbb{E}\left[h_1\left(X^{T_a(y)}\right)\right]h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}n_+(dy)\mathbb{E}\left[g\left(Y^{T-a}\right)f\left(X^{T-a}\right)\right] \\ &= \int_U h_1(x^{T_a})h_2(y^{T_a})\mathbb{1}_{\{T_a < \infty\}}\mathbf{n}_+(dx, dy)\mathbb{E}\left[g\left(Y^{T-a}\right)f\left(X^{T-a}\right)\right]. \end{aligned}$$

This concludes the proof of Lemma 2.2.4. □

2.2.3 Excursions above horizontal levels

We next set some notation for studying the excursions above a given level. Let $a \geq 0$ and $u = (x, y) \in U^+$. In the following list of definitions, one should think of u as a Brownian excursion in the sense of Definition 2.2.1.

Define

$$\mathcal{I}(a) = \{s \in [0, R(u)], y(s) > a\}. \quad (2.2.1)$$

Then by continuity $\mathcal{I}(a)$ is a countable (possibly empty) union of disjoint open intervals $I_1, I_2, \dots$. For any such interval $I = (i_-, i_+)$, take $u_I(s) = u(i_- + s) - u(i_-)$, $0 \leq s \leq i_+ - i_-$, for the restriction of u to I , and $\Delta u_I = x(i_+) - x(i_-)$ for the *size* or *length* of u_I . Note that $u_I \in U$.

If now $0 \leq t \leq R(u)$ and $0 \leq a < \mathfrak{S}(u(t))$, we define

$$e_a^{(t)} = e_a^{(t)}(u) = u_I,$$

where I is the unique open interval in the above partition of $\mathcal{I}(a)$ such that $t \in I$. By convention, we also set for $a = \mathfrak{S}(u(t))$, $e_a^{(t)} = u(t)$ and $\Delta e_a^{(t)} = 0$. This is represented in an excessively naive way in Figure 2.2 below.

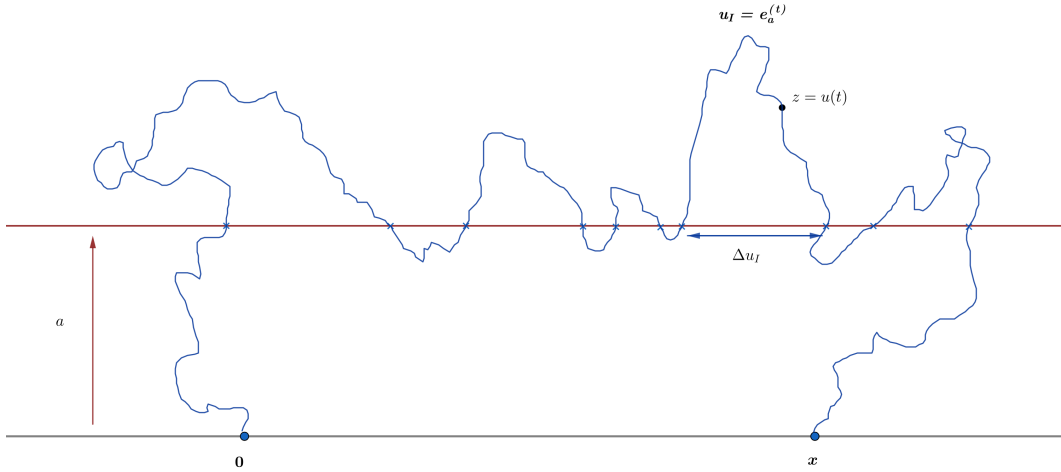


Figure 2.2 – Excursions above the level t .

Let $F^{(t)} : a \in [0, \mathfrak{S}(u(t))] \mapsto \Delta e_a^{(t)}$. Define

$$u^{t, \leftarrow} := (u(t-s) - u(t))_{0 \leq s \leq t}, \quad (2.2.2)$$

$$u^{t, \rightarrow} := (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}. \quad (2.2.3)$$

If we set for $a \in [0, y(t)]$,

$$T_a^{t, \leftarrow} := \inf\{s \geq 0, y(t-s) = a\}, \quad (2.2.4)$$

$$T_a^{t, \rightarrow} := \inf\{s \geq 0, y(t+s) = a\}, \quad (2.2.5)$$

then $t - T_a^{t, \leftarrow}$ and $t + T_a^{t, \rightarrow}$ are the extremities i_- and i_+ of the interval excursion straddling t and we can rewrite $F^{(t)}(a)$ as the first coordinate of $u^{t, \rightarrow}(T_a^{t, \rightarrow}) - u^{t, \leftarrow}(T_a^{t, \leftarrow})$.

Lemma 2.2.5. *For any $u \in U^+$, for all $0 \leq t \leq R(u)$, the function $F^{(t)}$ is càdlàg.*

Proof. Fix $t \in [0, R(u)]$. We want to show that $F^{(t)}$ is càdlàg on $[0, y(t)]$. By usual properties of inverse of continuous functions (see Lemma 4.8 and the remark following it in Chapter 0 of Revuz-Yor [RY99]), $a \mapsto T_a^{t, \leftarrow}$ and $a \mapsto T_a^{t, \rightarrow}$ are càdlàg (in a). Hence $F^{(t)}$ is càdlàg since u is continuous. $\square$

2.2.4 Bismut's description of Itô's measure in $\mathbb{H}$

In the case of one-dimensional Itô's measure n , Bismut's description roughly states that if we pick an excursion u at random according to n , and some time $0 \leq t \leq R(u)$ according to the Lebesgue measure, then the "law" of $u(t)$ is the Lebesgue measure and conditionally on $u(t) = \alpha$, the left and right parts of u (seen from $u(t)$) are independent Brownian motions killed at $-\alpha$ (see Theorem 4.7, Chap. XII in [RY99]). We deduce an analogous result in the case of Itô's measure in $\mathbb{H}$ and we apply it to show that for $\mathbf{n}_+$ -almost every excursion, there is no loop remaining above any horizontal level.

Proposition 2.2.6. (*Bismut's description of Itô's measure in $\mathbb{H}$*)

Let $\bar{\mathbf{n}}_+$ be the measure defined on $\mathbb{R}_+ \times U^+$ by

$$\bar{\mathbf{n}}_+(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u)\}} dt \mathbf{n}_+(du).$$

Then under $\bar{\mathbf{n}}_+$ the "law" of $(t, (x, y)) \mapsto y(t)$ is the Lebesgue measure $d\alpha$ on $\mathbb{R}_+$ and conditionally on $y(t) = \alpha$, $u^{t, \leftarrow} = (u(t-s) - u(t))_{0 \leq s \leq t}$ and $u^{t, \rightarrow} = (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}$ are independent Brownian motions killed when reaching $\{\Im(z) = -\alpha\}$.

See Figure 2.3. Proposition 2.2.6 is a direct consequence of the one-dimensional analogous result, for which we refer to [RY99] (see Theorem 4.7, Chapter XII).

The next proposition ensures that for almost every excursion under $\mathbf{n}_+$, there is no loop growing above any horizontal level. Let

$$\mathcal{L} := \{u \in U^+, \exists 0 \leq t \leq R(u), \exists 0 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\},$$

be the set of excursions u having a loop remaining above some level a . Then we have :

Proposition 2.2.7.

$$\mathbf{n}_+(\mathcal{L}) = 0.$$

Proof. We first prove the result under $\bar{\mathbf{n}}_+$, namely

$$\bar{\mathbf{n}}_+\left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}\right) = 0.$$

Recall the notation (2.2.2)-(2.2.5). From Bismut's description of $\bar{\mathbf{n}}_+$ we get

$$\begin{aligned} & \bar{\mathbf{n}}_+\left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}\right) \\ &= \bar{\mathbf{n}}_+\left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < y(t), u^{t, \rightarrow}(T_a^{t, \rightarrow}) = u^{t, \leftarrow}(T_a^{t, \leftarrow})\}\right) \\ &= \int_0^\infty d\alpha \mathbb{P}\left(\exists 0 < a \leq \alpha, X_{T_a} = X'_{T'_a}\right), \end{aligned}$$

where X and X' are independent linear Brownian motions, and T_a and T'_a are hitting times of a of other independent Brownian motions (corresponding to the imaginary parts). Now, X_{T_a} and $X'_{T'_a}$ are independent symmetric Cauchy processes, and therefore $X_{T_a} - X'_{T'_a}$ is again a Cauchy process (see Section 4, Chap. III of [RY99]). Since points are polar for the symmetric Cauchy process (see [Ber96], Chap. II, Section 5), we obtain $\mathbb{P}(\exists 0 < a \leq \alpha, X_{T_a} = X'_{T'_a}) = 0$ and under $\bar{\mathbf{n}}_+$ the result is proved.

To extend the result to $\mathbf{n}_+$, we notice that if $u \in \mathcal{L}$, then the set of t 's satisfying the definition of $\mathcal{L}$ has positive Lebesgue measure: namely, it contains all the times until the loop comes back to itself. This translates into

$$\mathcal{L} \subset \left\{u \in U^+, \int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\}} dt > 0\right\}.$$

But

$$\begin{aligned}
& \mathbf{n}_+ \left(\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < y(t), \Delta e_a^{(t)}(u)=0\}} dt \right) \\
&= \int_{U^+} \int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < y(t), \Delta e_a^{(t)}(u)=0\}} dt \mathbf{n}_+(du) \\
&= \bar{\mathbf{n}}_+ \left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < y(t), \Delta e_a^{(t)}(u)=0\} \right).
\end{aligned}$$

Hence, by the first step of the proof,

$$\mathbf{n}_+ \left(\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < y(t), \Delta e_a^{(t)}(u)=0\}} dt \right) = 0,$$

which gives $\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < y(t), \Delta e_a^{(t)}(u)=0\}} dt = 0$ for $\mathbf{n}_+$ -almost every excursion, and the desired result. $\square$

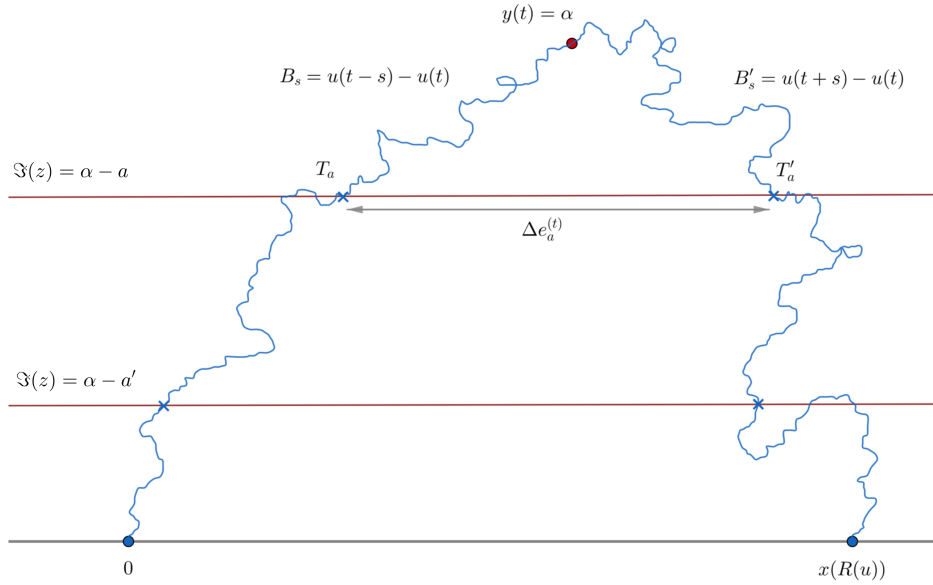


Figure 2.3 – Bismut's description of $\mathbf{n}_+$

2.2.5 The locally largest excursion

In [BBCK18], the authors give a canonical way to construct the growth-fragmentation, through the so-called locally largest fragment. We want to mimic this construction in our case.

In order to define the locally largest excursion, we set for $u \in U^+$ and $0 \leq t \leq R(u)$,

$$S(t) := \sup \left\{ a \in [0, y(t)], \quad \forall 0 \leq a' \leq a, \quad |F^{(t)}(a')| \geq |F^{(t)}(a'^-) - F^{(t)}(a')| \right\}.$$

Observe that the supremum is taken over a non-empty set by Lemma 2.2.5 as soon as $y(t) > 0$ and $u(R(u)) \neq 0$. Let

$$S := \sup_{0 \leq t \leq R(u)} S(t).$$

In the case of Brownian excursions, the following proposition holds.

Proposition 2.2.8. *For almost every u under $\mathbf{n}_+$, there exists a unique $0 \leq t^\bullet \leq R(u)$ such that $S(t^\bullet) = S$. Moreover, $S = \mathfrak{S}(z^\bullet)$ where $z^\bullet = u(t^\bullet)$.*

We call $(e_a^{(t^\bullet)})_{0 \leq a \leq \mathfrak{S}(z^\bullet)}$ the locally largest excursion and $(\Xi(a) = \Delta e_a^{(t^\bullet)})_{0 \leq a \leq \mathfrak{S}(z^\bullet)}$ the locally largest fragment.

Thus Ξ is the length of the excursion which is locally the largest, meaning that at any level a where the locally largest excursion splits, Ξ is larger (in absolute value) than the length of the other excursion. See Figure 2.4 for a picture of $z^\bullet$. Following [BBCK18], we will see it as the Eve particle of our growth-fragmentation process.

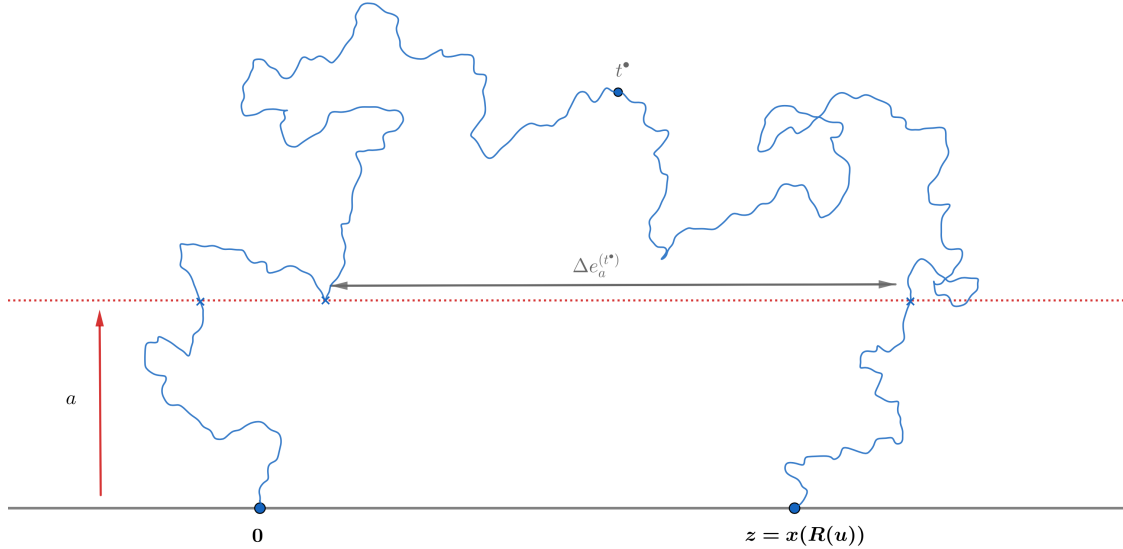


Figure 2.4 – The locally largest excursion.

Proof. Existence. We deal with the excursions u satisfying the following properties, which happen $\mathbf{n}_+$ -almost everywhere : u has no loop above any horizontal level (see Proposition 2.2.7) and y has distinct local minima. Take a convergent sequence $(t_n, n \geq 1)$ such that $S(t_n)$ converges to S , and denote by $t^\bullet$ the limit of t_n . We have necessarily, by definition of $S(t)$, that $y(t_n) \geq S(t_n)$. By continuity of y , we get that $y(t^\bullet) \geq S$. Take $a < S$. For n large enough, since $a < y(t^\bullet)$, we observe that t_n and $t^\bullet$ are in the same excursion above a , i.e. $e_a^{(t^\bullet)} = e_a^{(t_n)}$. For such n , $F^{(t_n)}(a') = F^{(t^\bullet)}(a')$ for all $a' \leq a$. Moreover, for n large enough, $S(t_n) > a$ hence for all $a' \leq a$, $|F^{(t^\bullet)}(a')| = |F^{(t_n)}(a')| \geq |F^{(t_n)}(a'^-) - F^{(t_n)}(a')| = |F^{(t^\bullet)}(a'^-) - F^{(t^\bullet)}(a')|$. It implies that $S(t^\bullet) \geq a$, hence $S(t^\bullet) \geq S$ by taking a arbitrarily close to S . We found $t^\bullet$ such that $S(t^\bullet) = S$.

We show that $y(t^\bullet) = S$. Notice that, for all $0 \leq t \leq R(u)$, by right-continuity of $F^{(t)}$, the set

$$A(t) := \left\{ 0 \leq a \leq y(t), \quad \forall 0 \leq a' \leq a, \quad |F^{(t)}(a')| \geq |F^{(t)}(a'^-) - F^{(t)}(a')| \right\}$$

is open in $[0, y(t)]$. Indeed, for $a < y(t)$, $e_a^{(t)}$ cannot be an excursion with size 0 by assumption, and so by right-continuity, we can take $\delta > 0$ such that on $[a, a + \delta]$, $F^{(t)}$ takes values in $\left(\frac{3}{4}F^{(t)}(a), \frac{3}{2}F^{(t)}(a)\right)$ (in the case $F^{(t)}(a) > 0$, without loss of generality). For such a δ , and for any $a' \in [a, a + \delta]$, $F^{(t)}(a') > \frac{3}{4}F^{(t)}(a) > \frac{3}{4}\frac{2}{3}F^{(t)}(a'^-) = \frac{1}{2}F^{(t)}(a'^-)$, and $F^{(t)}(a'^-) \geq 0$. These two inequalities imply that $|F^{(t)}(a')| \geq |F^{(t)}(a'^-) - F^{(t)}(a')|$.

Now suppose that $S < y(t^\bullet)$ and let us find a contradiction. We have $A(t^\bullet) = [0, S)$, hence $|F^{(t^\bullet)}(S)| < |F^{(t^\bullet)}(S^-) - F^{(t^\bullet)}(S)|$. Write $e_a^{(t^\bullet)} = u_I$ with $I = (i_{a,-}, i_{a,+})$, so that $F^{(t^\bullet)}(a) = x(i_{a,+}) - x(i_{a,-})$. Since $F^{(t^\bullet)}$ jumps at S , either $i_{\cdot,-}$ or $i_{\cdot,+}$ jumps at S . Both cases cannot happen at the same time because local minima of y are all distinct. Suppose for example that $i_{S,-,-} < i_{S,-}$. Take $t \in (i_{S,-,-}, i_{S,-})$ (see Figure 2.5). We have $F^{(t)}(a) = F^{(t^\bullet)}(a)$ for all $a < S$ and

$$\begin{aligned} F^{(t)}(S) &= x(i_{S,-}) - x(i_{S,-,-}) = x(i_{S,-,+}) - x(i_{S,-,-}) - (x(i_{S,+}) - x(i_{S,-})) \\ &= F^{(t^\bullet)}(S^-) - F^{(t^\bullet)}(S) \\ &= F^{(t)}(S^-) - F^{(t^\bullet)}(S). \end{aligned}$$

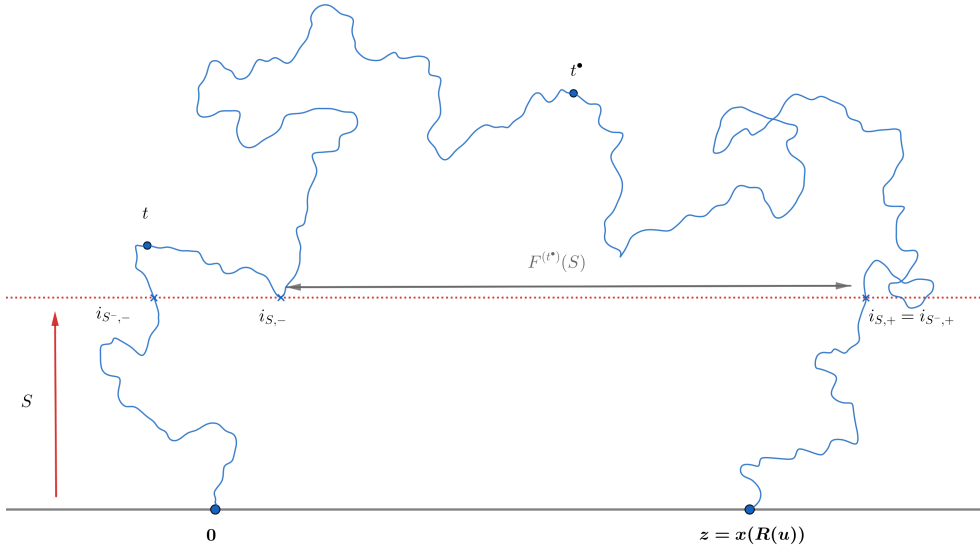


Figure 2.5 – Construction of the locally largest excursion.

We deduce that $|F^{(t)}(S)| = |F^{(t^\bullet)}(S^-) - F^{(t^\bullet)}(S)| > |F^{(t^\bullet)}(S)| = |F^{(t)}(S^-) - F^{(t)}(S)|$. Then $A(t)$ is open in $[0, y(t)]$, contains S , and we have $y(t) > S$. Hence $\sup A(t) > S$ which gives the desired contradiction.

Uniqueness. Suppose that $S(t) = S(t') = S$ with $t < t'$ and let us find again a contradiction. We showed that necessarily, $y(t) = y(t') = S$. Let $t_m \in [t, t']$ such that $y(t_m) = \min\{y(r), r \in [t, t']\}$. Set $a_m := y(t_m)$. Observe that t and t' cannot be starting times or ending times of an excursion of y (otherwise we could have extended the locally

largest fragment inside this excursion for some positive height). Hence $a_m < S$. At level a_m , there must be a splitting into two excursions (one straddling time t , the other t') with equal size. It happens on a negligible set under $\mathbf{n}_+$. To see it, we can restrict to $t < t'$ rationals and use the Markov property at time t' . $\square$

2.2.6 Disintegration of Itô's measure over the size of the excursions

We are interested in conditioning Itô's measure of excursions in $\mathbb{H}$ on their initial size, *i.e.* in fixing the value of $x(R(u)) = z$. This will allow us to define probability measures γ_z which disintegrate $\mathbf{n}_+$ over the value of the endpoint z . Properties will simply transfer from $\mathbf{n}_+$ to γ_z via the disintegration formula. Define $\mathbb{P}_r^{a \rightarrow b}$ as the law of the one-dimensional Brownian bridge of length r between a and b , and Π_r as the law of a three-dimensional Bessel (BES^3) bridge of length r from 0 to 0. It is known ([RY99], Chap. XII, Theorem 4.2) that Π_r is the law of a one-dimensional excursion under $\mathbf{n}_+$ conditioned on having duration r .

Proposition 2.2.9. *We have the following disintegration formula*

$$\mathbf{n}_+ = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} \gamma_z, \quad (2.2.6)$$

where for $z \neq 0$,

$$\gamma_z = \int_{\mathbb{R}_+} dv \frac{e^{-1/(2v)}}{2v^2} \mathbb{P}_{vz^2}^{0 \rightarrow z} \otimes \Pi_{vz^2}. \quad (2.2.7)$$

Proof. Let f and g be two nonnegative measurable functions defined on $\mathcal{X}$ and $\mathcal{X}_0$ respectively. Thanks to Itô's description of $\mathbf{n}_+$ (see [RY99], Chap. XII, Theorem 4.2), we have

$$\begin{aligned} \int_U f(x)g(y) \mathbf{n}_+(dx, dy) &= \int_U f(x)g(y) \mathbf{n}_+(dy) \mathbb{P}(X^{R(y)} \in dx) \\ &= \int_{\mathbb{R}_+} \frac{dr}{2\sqrt{2\pi r^3}} \int_{\mathcal{X}} f(x) \Pi_r[g] \mathbb{P}(X^r \in dx). \end{aligned}$$

Now, decomposing on the value of the Gaussian r.v. X_r yields

$$\int_U f(x)g(y) \mathbf{n}_+(dx, dy) = \int_{\mathbb{R}_+} \frac{dr}{2\sqrt{2\pi r^3}} \int_{\mathbb{R}} dz \frac{e^{-z^2/(2r)}}{\sqrt{2\pi r}} \Pi_r[g] \mathbb{E}_r^{0 \rightarrow z}[f].$$

We finally perform the change of variables $v(r) = r/z^2$ to get

$$\int_U f(x)g(y) \mathbf{n}_+(dx, dy) = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} \int_{\mathbb{R}_+} dv \frac{e^{-1/(2v)}}{2v^2} \mathbb{E}_{vz^2}^{0 \rightarrow z}[f] \Pi_{vz^2}[g].$$

$\square$

Lemma 2.2.10. *Let z be a nonzero real number. The image measure of γ_z by the function which sends (x, y) to*

$$\left(\frac{x(tz^2)}{z}, \frac{y(tz^2)}{|z|} \right), \quad 0 \leq t \leq \frac{R(u)}{z^2},$$

is γ_1 .

Proof. It comes from the definition of γ_z and the scaling property of BES^3 bridge and Brownian bridge. $\square$

2.2.7 The metric space of excursions in $\mathbb{H}$

Very often, results under γ_z can be obtained by proving the analog under the Itô's measure $\mathbf{n}_+$, and then disintegrating over $z = x(R(u))$. This usually provides results under γ_z for Lebesgue-almost every $z > 0$, and so we would like to study the continuity of $z \mapsto \gamma_z$. This requires to define a topology on the space of excursions U^+ . All these results will be stated for $z > 0$ because the scaling depends on the sign of the endpoint (Lemma 2.2.10), but they all extend to the general case.

We therefore introduce the usual distance

$$d(u, v) = |R(u) - R(v)| + \sup_{t \geq 0} |u(t \wedge R(u)) - v(t \wedge R(v))|,$$

where we identified $\diamond$ with the excursion with lifetime 0. The distance d makes U^+ into a Polish space. The following lemmas may come in useful.

Lemma 2.2.11. *The map $\Delta : u \in U^+ \mapsto \Delta u = x(R(u))$ is continuous.*

Proof. This is straightforward since $|x(R(u)) - x'(R(u'))| = |u(R(u)) - u'(R(u'))| \leq d(u, u')$ for $u = (x, y)$ and $u' = (x', y')$. $\square$

Lemma 2.2.12. *Let $u \in U^+$. Then $z \in \mathbb{R}_+^* \mapsto u^{(z)} := zu(\cdot/z^2) = (zu(t/z^2), 0 \leq t \leq R(u)z^2)$ is a continuous function.*

Proof. Let $z_0 > 0$. Then for all $z > 0$

$$d(u^{(z)}, u^{(z_0)}) = |R(u)z^2 - z_0^2| + \sup_{t \geq 0} \left| zu\left(\frac{t}{z^2} \wedge R(u)\right) - z_0 u\left(\frac{t}{z_0^2} \wedge R(u)\right) \right|.$$

The second term is

$$\begin{aligned} & \sup_{t \geq 0} \left| zu\left(\frac{t}{z^2} \wedge R(u)\right) - z_0 u\left(\frac{t}{z_0^2} \wedge R(u)\right) \right| \\ & \leq z \sup_{t \geq 0} \left| u\left(\frac{t}{z^2} \wedge R(u)\right) - u\left(\frac{t}{z_0^2} \wedge R(u)\right) \right| + \sup_{t \geq 0} \left| (z - z_0) u\left(\frac{t}{z^2} \wedge R(u)\right) \right| \\ & \leq z \sup_{t \geq 0} \left| u\left(\frac{t}{z^2} \wedge R(u)\right) - u\left(\frac{t}{z_0^2} \wedge R(u)\right) \right| + |z - z_0| \sup_{t \geq 0} |u(t)|. \end{aligned}$$

We conclude by using the uniform continuity of u . $\square$

If we equip the set $\mathcal{P}(U^+)$ of probability measures on U^+ with the topology of weak convergence, we have the following result.

Proposition 2.2.13. *The map $z \in \mathbb{R}_+^* \mapsto \gamma_z$ is continuous.*

Proof. Let G be a continuous bounded function on U^+ . Then by scaling (Lemma 2.2.10), for all $z > 0$,

$$\gamma_z(G) = \gamma_1[G(u^{(z)})].$$

Applying Lemma 2.2.12 together with the dominated convergence theorem yields the desired result. $\square$

Also, we will use the continuity of the excursions cut at horizontal levels. Recall from Section 2.2.3 that $\mathcal{I}(a)$ is the set of times when the excursion $u \in U^+$ lies above a , and for each connected component I of $\mathcal{I}(a)$, u_I denotes the associated excursion above a . The path u_I is an excursion above a , I is the *time interval* of u_I , and the *size* or *length* of u_I is the difference between its endpoint and its starting point.

On $\{T_a < \infty\}$, we rank the excursions above a according to the absolute value of their size. Write $z_1^{a,+} = z_1^{a,+}(u), z_2^{a,+} = z_2^{a,+}(u), \dots$ for the sizes, ranked in descending order of their absolute value, and $e_1^{a,+} = e_1^{a,+}(u), e_2^{a,+} = e_2^{a,+}(u), \dots$ for the corresponding excursions. This is possible since for any fixed $\varepsilon > 0$ there are only finitely many excursions with length larger than ε in absolute value.

Proposition 2.2.14. *Let $a > 0$ and $z > 0$. For any $i \geq 1$, the function $e_i^{a,+}$ is continuous on U^+ on the event $\{T_a < \infty\}$ outside a γ_z -negligible set.*

Proof. We consider the set $\mathcal{E}$ of trajectories $u = (x, y)$ such that $T_a < \infty$ and satisfying the following conditions, which occur with γ_z -probability one when conditioned on touching a : the level a is not a local minimum for y , there exist infinitely many excursions above a , all excursions touch a only at their starting point and endpoint, the sizes $(z_i^{a,+}, i \geq 1)$ of the excursions are all distinct. Let $i \geq 1$ and $u = (x, y) \in \mathcal{E}$. We want to show that $e_i^{a,+}$ is continuous at u .

Let t be a time in the excursion $e_i^{a,+}$, i.e. such that $y(t) > a$ and $e_a^{(t)} = e_i^{a,+}$. We restrict our attention to $u' = (x', y') \in \mathcal{E}$ close enough to u so that $y'(t) > a$ and we will write $e_a'^{(t)}$ for the excursion of u' corresponding to t . Let $\varepsilon > 0$.

- *First, we want to find $\delta > 0$ such that, whenever $d(u, u') < \delta$, the durations of the excursions $e_a^{(t)}$ and $e_a'^{(t)}$ are close, namely $|R(e_a^{(t)}) - R(e_a'^{(t)})| < \varepsilon$. Write $(i_-(a), i_+(a))$, and $(i'_-(a), i'_+(a))$, for the excursion time intervals corresponding to $e_a^{(t)}$ and $e_a'^{(t)}$ respectively. For simplicity, we take the notation $R = R(e_a^{(t)})$ and $R' = R(e_a'^{(t)})$. Since a is not a local minimum for y , there exist times $t_1 \in (i_-(a) - \frac{\varepsilon}{2}, i_-(a))$ and $t_2 \in (i_+(a), i_+(a) + \frac{\varepsilon}{2})$ when y is strictly below a . Take $\delta_1 \in (0, a)$ such that $y(t_1)$ and $y(t_2)$ are in $(0, a - \delta_1)$. Let $u' = (x', y') \in \mathcal{E}$ such that $d(u, u') < \frac{\delta_1}{2}$. We deduce that $y'(t_1) < y(t_1) + \frac{\delta_1}{2} < a$ and similarly $y'(t_2) < a$. This implies that $i'_-(a) \geq t_1 > i_-(a) - \frac{\varepsilon}{2}$ and $i'_+(a) \leq t_2 < i_+(a) + \frac{\varepsilon}{2}$. Likewise, pick two times $t_3 \in (i_-(a), i_-(a) + \frac{\varepsilon}{2})$ and $t_4 \in (i_+(a) - \frac{\varepsilon}{2}, i_+(a))$ such that $t_3 < t < t_4$. Since the excursion $e_a^{(t)}$ touches level a only at its extremities, the distance between the compact $u([t_3, t_4])$ and the closed set $\{\mathfrak{F}(z) = a\}$ is positive, and so, on the interval $[t_3, t_4]$, y remains above, say, $a + \delta_2$ where $\delta_2 > 0$. Then when $d(u, u') < \frac{\delta_2}{2}$, the excursion $e_a'^{(t)}$ will satisfy $i'_-(a) < t_3 < i_-(a) + \frac{\varepsilon}{2}$ and $i'_+(a) > t_4 > i_+(a) - \frac{\varepsilon}{2}$. Therefore, when $d(u, u') < \delta = \min(\frac{\delta_1}{2}, \frac{\delta_2}{2})$, we get that $|i'_-(a) - i_-(a)| < \frac{\varepsilon}{2}$ and $|i'_+(a) - i_+(a)| < \frac{\varepsilon}{2}$, so in particular $|R' - R| < \varepsilon$. Observe that we not only proved that the durations are close, but also that the times i_-, i'_- (and i_+, i'_+) are close, and this will be useful in the remainder of the proof.*

- *Secondly, we show that we can take $\delta' > 0$ small enough so that*

$$\sup_{s \geq 0} |e_a^{(t)}(s \wedge R) - e_a'^{(t)}(s \wedge R')| < \varepsilon,$$

whenever $d(u, u') < \delta'$.

Take $\eta = \eta(\varepsilon) > 0$ some modulus of uniform continuity of u with respect to ε . The previous paragraph gives the existence of $\delta > 0$ such that when $u' \in \mathcal{E}$ and $d(u, u') < \delta$,

$|i'_-(a) - i_-(a)| < \eta/3$ and $|i'_+(a) - i_+(a)| < \eta/3$. Without loss of generality, we can assume that $\delta < \varepsilon$. Define $\delta' = \min(\delta, \eta)$, and let $u' \in \mathcal{E}$ such that $d(u, u') < \delta'$. For all $s \geq 0$, we have

$$\begin{aligned} & |e_a^{(t)}(s \wedge R) - e_a'^{(t)}(s \wedge R')| \\ &= |u(i_-(a) + (s \wedge R)) - u(i_-(a)) - u'(i'_-(a) + (s \wedge R')) + u'(i'_-(a))| \\ &\leq |u(i_-(a)) - u'(i'_-(a))| + |u(i_-(a) + (s \wedge R)) - u(i'_-(a) + (s \wedge R'))|. \end{aligned} \quad (2.2.8)$$

Now,

$$|u(i_-(a)) - u'(i'_-(a))| \leq |u(i_-(a)) - u(i'_-(a))| + |u(i'_-(a)) - u'(i'_-(a))|,$$

and so by uniform continuity of u and because $d(u, u') < \delta' < \varepsilon$, we obtain

$$|u(i_-(a)) - u'(i'_-(a))| \leq 2\varepsilon. \quad (2.2.9)$$

Similarly, the second term of (2.2.8) is

$$\begin{aligned} & |u(i_-(a) + (s \wedge R)) - u'(i'_-(a) + (s \wedge R'))| \\ &\leq |u(i_-(a) + (s \wedge R)) - u(i'_-(a) + (s \wedge R'))| \\ &\quad + |u(i'_-(a) + (s \wedge R')) - u'(i'_-(a) + (s \wedge R'))|, \end{aligned}$$

and since $|i_-(a) + (s \wedge R) - i'_-(a) - (s \wedge R')| < \eta$, we can conclude in the same way that

$$|u(i_-(a) + (s \wedge R)) - u'(i'_-(a) + (s \wedge R'))| \leq 2\varepsilon. \quad (2.2.10)$$

Inequalities (2.2.8), (2.2.9) and (2.2.10) give

$$|e_a^{(t)}(s \wedge R) - e_a'^{(t)}(s \wedge R')| \leq 4\varepsilon,$$

which is the desired result.

So far, we proved that $e_a^{(t)}$ is continuous at u . To conclude, we need an argument to say that this is the i -th excursion above a for u' sufficiently close to u .

- Finally, we show that we can take $\delta'' > 0$ small enough so that $e_i^{a,+} = e_a'^{(t)}$ whenever $d(u, u') < \delta''$.

This is derived in two steps.

- Step 1: Let $\eta > 0$, and introduce, for $u' \in \mathcal{E}$, the number $N_\eta(u')$ of time intervals (i_-, i_+) of excursions of u' above a such that $i_+ - i_- > \eta$. Note that $N_\eta(u') \leq \frac{R(u')}{\eta} < \infty$. We take η such that u has no excursion time interval above a satisfying $i_+ - i_- = \eta$. The first step consists in proving that for $u' \in \mathcal{E}$ sufficiently close to u , $N_\eta(u') = N_\eta(u)$. From the first point (applied $N_\eta(u)$ times), we know that for $\delta > 0$ small enough, $N_\eta(u') \geq N_\eta(u)$ whenever $d(u, u') < \delta$. To prove that $N_\eta(u') \leq N_\eta(u)$ holds as well when δ is sufficiently small, we use an argument by contradiction and we consider a sequence $(u_n)_{n \geq 1}$ of elements in $\mathcal{E}$ such that $d(u, u_n) \rightarrow 0$ and $N_\eta(u_n) \geq N_\eta(u) + 1$. Consider $N_\eta(u) + 1$ distinct excursion time intervals $(i_{j,-}^{(n)}, i_{j,+}^{(n)})$, $1 \leq j \leq N_\eta(u) + 1$, of u_n above a such that $i_{j,+}^{(n)} - i_{j,-}^{(n)} > \eta$.

We can write the corresponding excursions $e_a^{(t_j^{(n)})}(u_n)$ for some $t_j^{(n)}$'s. Moreover, we may take $t_j^{(n)}$ such that $|i_{j,+}^{(n)} - t_j^{(n)}| > \eta/2$ and $|i_{j,-}^{(n)} - t_j^{(n)}| > \eta/2$. Since $|R(u) - R(u_n)| \rightarrow 0$, we can assume (up to some extraction) that when n

goes to infinity, $i_{j,+}^{(n)} \rightarrow i_{j,+}$, $i_{j,-}^{(n)} \rightarrow i_{j,-}$ and $t_j^{(n)} \rightarrow t_j \in [0, R(u)]$, for some $i_{j,+}, i_{j,-}, t_j \in [0, R(u)]$. From $u_n \rightarrow u$, we deduce that for all j , $y(i_{j,-}) = a$ and $y(i_{j,+}) = a$. For n large enough, because $i_{j,+}^{(n)} - i_{j,-}^{(n)} > \eta$ and $|i_{j,\pm}^{(n)} - t_j^{(n)}| > \eta/2$, we have $e_a^{(t_j^{(n)})}(u_n) = e_a^{(t_j)}(u_n)$. Now consider $e_a^{(t_j)}(u)$. From the two previous points, $e_a^{(t_j)}(u_n) \rightarrow e_a^{(t_j)}(u)$. For any time $s \in (i_{j,-}, i_{j,+})$, we have $y(s) > a$ (otherwise a would be a local minimum of y). Hence $(i_{j,-}, i_{j,+})$ is an excursion time interval for u and $i_{j,+} - i_{j,-} > \eta$. Therefore we constructed $N_\eta(u) + 1$ distinct excursion time intervals above a for u , which gives the desired contradiction.

- Step 2: Suppose for example that $z_i^{a,+} > 0$. Take $\delta < \frac{z_i^{a,+}}{6}$ and $\eta = \eta(\delta) > 0$ some modulus of uniform continuity for u with respect to δ . We can assume in order to apply Step 1 that η is such that u has no excursion above a satisfying $|i_+ - i_-| = \eta$. We look at the $N := N_\eta(u)$ excursions $e_1, \dots, e_N$ of u above a (ranked by decreasing order of the absolute value of their sizes) such that $|i_+ - i_-| > \eta$, and denote their sizes by $z_1, \dots, z_N$. Observe that the first i excursions among these are the excursions $e_1^{a,+}, \dots, e_i^{a,+}$. Indeed, if $|i_+ - i_-| \leq \eta$, then by uniform continuity,

$$|u(i_+) - u(i_-)| \leq \delta < z_i^{a,+}.$$

Let $\varepsilon' = \frac{1}{2}(\min_{1 \leq k \leq N-1} |z_{k+1} - z_k| \wedge z_i)$ (this is positive since all the sizes are assumed to be distinct in $\mathcal{E}$). Take times $t_1, \dots, t_N$ in the excursion time intervals of $e_1, \dots, e_N$. Thanks to Step 1 and the first point of the proof (applied N times), there exists $\delta' > 0$ such that for $d(u, u') < \delta'$, if we denote by $(i_-^{(t_k)}, i_+^{(t_k)})$ the excursion time interval of $e_a^{(t_k)}$, $1 \leq k \leq N$, then

- (i) $N_\eta(u') = N$,
- (ii) the excursions $e_a^{(t_k)}$, $1 \leq k \leq N$, are distinct,
- (iii) $\forall 1 \leq k \leq N$, $|i_+^{(t_k)} - i_-^{(t_k)}| > \eta$,
- (iv) $\forall 1 \leq k \leq N$, $|z_k^{a,+} - \Delta e_a^{(t_k)}| \leq \varepsilon'$.

An easy calculation shows that by our choice of ε' and (iv), the $\Delta e_a^{(t_k)}$, $1 \leq k \leq N$, are ranked in decreasing order, and that

$$\forall 1 \leq k \leq i, \quad \Delta e_a^{(t_k)} > \frac{z_i^{a,+}}{2}. \quad (2.2.11)$$

In addition, by (i), (ii) and (iii), the $e_a^{(t_k)}$, $1 \leq k \leq N$, are the excursions of u' above a satisfying $|i_+ - i_-| > \eta$.

Now set $\delta'' = \min(\delta, \delta')$ and assume that $d(u, u') < \delta''$. Then for all $1 \leq k \leq i$, $e_a^{(t_k)} = e_k^{a,+}(u')$. Indeed, if (i_-, i_+) is an excursion time interval of u' such that $|i_+ - i_-| \leq \eta$, then

$$|u'(i_+) - u'(i_-)| \leq |u'(i_+) - u(i_+)| + |u(i_+) - u(i_-)| + |u(i_-) - u'(i_-)| \leq 3\delta,$$

and so in particular $|u'(i_+) - u'(i_-)| < \frac{z_i^{a,+}}{2}$. This proves that the first i excursions of u' are among the N previous excursions satisfying $|i_+ - i_-| > \eta$. Since these are ranked in decreasing order, necessarily $e_a^{(t_k)} = e_k^{a,+}(u')$ for all $1 \leq k \leq i$, which concludes the proof.

Putting these three points together, we proved that $e_i^{a,+}$ is continuous on $\mathcal{E}$ which has full probability under $\gamma_z(\cdot | T_a < \infty)$, hence Proposition 2.2.14. $\square$

Remark 2.2.15. A byproduct of the proof of Proposition 2.2.14 is that the extremities of the time interval corresponding to the excursions $e_k^{a,+}$, $k \geq 1$, are continuous. More precisely, if we denote by $(i_-^k(a), i_+^k(a))$ the excursion interval corresponding to $e_k^{a,+}$ in the partition (2.2.1), then $i_-^k(a)$ and $i_+^k(a)$ are continuous on U^+ and on the event $\{T_a < \infty\}$ outside a negligible set.

2.3 Markovian properties

In this section, we are interested in Markovian properties of excursions cut at horizontal levels. Time will therefore be indexed by the height a of the cutting.

2.3.1 The branching property for excursions in $\mathbb{H}$

Consider an excursion under the measure γ_z . Then cutting it at some height $a > 0$ yields a family of excursions above a as defined in Section 2.2.3. Our aim is to show that conditionally on what happens below a , these are independent and distributed according to the measures γ_z , where z is the size of the corresponding excursion. We shall first consider the case when the original excursion is taken under the Itô's measure $\mathbf{n}_+$ in $\mathbb{H}$, and then transfer the property to γ_z by the previous disintegration result (2.2.6).

We consider the trajectory u once you cut out the excursions above a , and close up the time gaps. A formal definition of this process is $u_t^{<a} := u_{\tau_t^{<a}}$ if $t < A_{R(u)}$ and $u_t^{<a} := u(R(u))$ if $t = A_{R(u)}$ where

$$A_t := \int_0^t \mathbf{1}_{\{y(s) \leq a\}} ds \quad \text{and} \quad \tau_t^{<a} := \inf\{s > 0 : A_s > t\}.$$

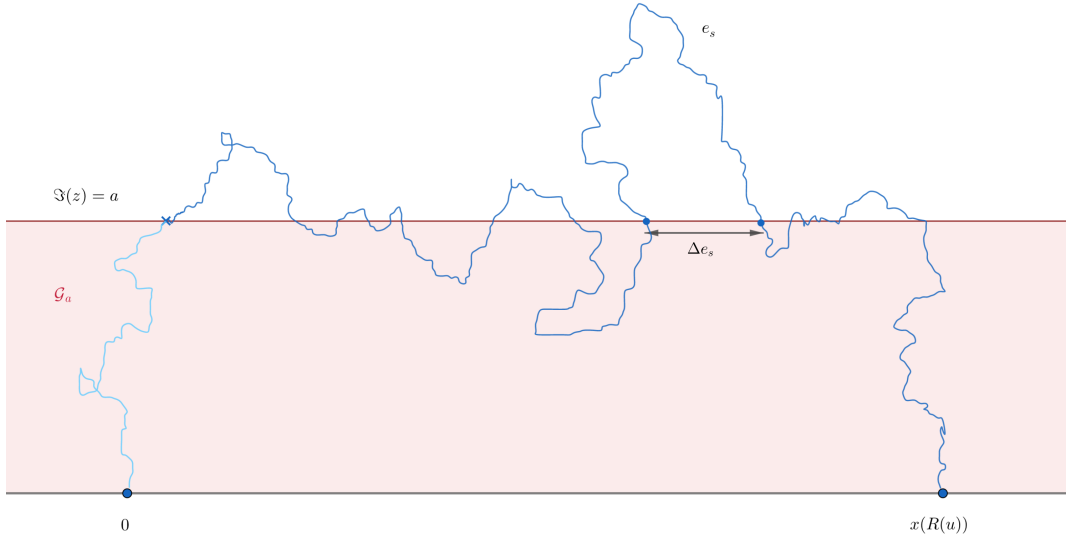


Figure 2.6 – The excursion process above a .

The process $u^{<a}$ is a random variable in the space of $\mathbb{C}$ -valued càdlàg paths with finite lifetime, equipped with the Borel σ -field induced by the Skorokhod topology. We let $\mathcal{G}_a$ be the σ -field generated by $u^{<a}$ completed with the $\mathbf{n}_+$ -negligible sets. In other words, it is the σ -field containing all the information of the trajectory below level a .

From now on, we suppose that we are on the event $\{T_a < \infty\}$. Recall from Section 2.2.7 that $z_1^{a,+}, z_2^{a,+}, \dots$ are the sizes of the excursions above a , ranked in decreasing order of their absolute value, and $e_1^{a,+}, e_2^{a,+}, \dots$ are the corresponding excursions.

Let $(L_t^a)_{t \in [0, R(u)]}$ be the local time process of u at level a and let $(e_s^a, s \in (0, L_{R(u)}^a))$ be the excursion process at level a of u in the sense of Definition 2.2.1 (its existence under $\mathbf{n}_+$ comes from the Markov property at time T_a , see Lemma 2.2.4). We set e_0^a and $e_{L_{R(u)}^a}^a$ to be respectively the first and last parts of the trajectory u between $\{\mathfrak{S} = 0\}$ and $\{\mathfrak{S} = a\}$, which we also consider as excursions of u below a , and we write $\mathfrak{G}_a := (e_s^a, s \in [0, L_{R(u)}^a])$. Denote by $\mathfrak{G}_a^-$ the point process when we replace each excursion $e_i^{a,+} \in U^+$ above a appearing in $\mathfrak{G}_a$ by its size $\Delta e_i^{a,+} = z_i^{a,+}$. Because $u^{<a}$ is obtained by concatenation of the excursions and jumps in $\mathfrak{G}_a^-$, we see that the σ -field $\mathcal{G}_a$ is generated by $\mathfrak{G}_a^-$.

Proposition 2.3.1. (*Branching property for excursions in $\mathbb{H}$ under $\mathbf{n}_+$*)

For any $A \in \mathcal{G}_a$, and for all nonnegative measurable functions $G_1, \dots, G_k : U^+ \rightarrow \mathbb{R}_+$, $k \geq 1$,

$$\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right). \quad (2.3.1)$$

Proof. Lemma 2.2.4 ensures that on the event $\{T_a < \infty\}$, the trajectory u after time T_a has the law of a killed Brownian motion. The Markov property at time T_a and excursion theory tell us that given the excursions below a , the excursions above a form a Poisson point process on U^+ with intensity $L_{R(u)}^a \mathbf{n}_+(du)$, see Figure 2.6. Finally, conditionally on the sizes $(z_i^{a,+})_{i \geq 1}$, these excursions are independent with law $\gamma_{z_i^{a,+}}$. In other words, conditionally on $\mathfrak{G}_a^-$, the excursions $(e_i^{a,+})$ are independent with law $\gamma_{z_i^{a,+}}$. We deduce the proposition since $\mathfrak{G}_a^-$ generates $\mathcal{G}_a$. $\square$

We can now transfer this property to the probability measures γ_z .

Proposition 2.3.2. (*Branching property for excursions in $\mathbb{H}$ under γ_z*)

Let $z \in \mathbb{R} \setminus \{0\}$. For any $A \in \mathcal{G}_a$, and for all nonnegative measurable functions $G_1, \dots, G_k : U^+ \rightarrow \mathbb{R}_+$, $k \geq 1$,

$$\gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right).$$

Proof. It suffices to prove the proposition for bounded continuous functions $G_1, \dots, G_k : U^+ \rightarrow \mathbb{R}_+$, $k \geq 1$. Take a nonnegative measurable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ and a bounded continuous function $h : U^+ \rightarrow \mathbb{R}_+$ which is $\mathcal{G}_a$ -measurable. Observe that $x(R(u))$ is $\mathcal{G}_a$ -measurable as a function of u . From Proposition 2.3.1, we know that

$$\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} h(u) f(x(R(u))) \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} h(u) f(x(R(u))) \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right).$$

Thanks to the disintegration formula (2.2.6), we can split $\mathbf{n}_+$ over the size:

$$\int_{\mathbb{R}} \frac{dz}{2\pi z^2} f(z) \gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \int_{\mathbb{R}} \frac{dz}{2\pi z^2} f(z) \gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right).$$

Since this holds for any f , it entails for Lebesgue-almost every $z \in \mathbb{R}$,

$$\gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right). \quad (2.3.2)$$

To prove that this holds for all z , we need a continuity argument. We first treat the case $z = 1$. Using the scaling property 2.2.10 of the measures γ_z , for $z > 0$ the left-hand side of (2.3.2) is

$$\gamma_1 \left(\mathbb{1}_{\{T_{a/z} < \infty\}} h(u^{(z)}) \prod_{i=1}^k G_i(e_i^{a,+}(u^{(z)})) \right)$$

where we recall from Lemma 2.2.12 that $u^{(z)} = zu(\cdot/z^2)$. The right-hand side term, on the other hand, is

$$\gamma_z \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right) = \gamma_1 \left(\mathbb{1}_{\{T_{a/z} < \infty\}} h(u^{(z)}) \prod_{i=1}^k \gamma_{z_i^{a,+}(u^{(z)})}[G_i] \right),$$

and so (2.3.2) translates into

$$\gamma_1 \left(\mathbb{1}_{\{T_{a/z} < \infty\}} h(u^{(z)}) \prod_{i=1}^k G_i(e_i^{a,+}(u^{(z)})) \right) = \gamma_1 \left(\mathbb{1}_{\{T_{a/z} < \infty\}} h(u^{(z)}) \prod_{i=1}^k \gamma_{z_i^{a,+}(u^{(z)})}[G_i] \right), \quad (2.3.3)$$

for Lebesgue-almost every $z > 0$. In particular this is true for a dense set of z . Taking $z \searrow 1$ along some decreasing sequence, we first get that $u^{(z)} \rightarrow u$ by Lemma 2.2.12 and $T_{a/z} \rightarrow T_a$ by left-continuity of the stopping times. In addition, for all $1 \leq i \leq k$, $z_i^{a,+}(u^{(z)}) \rightarrow z_i^{a,+}(u)$ γ_1 -almost surely because $z \rightarrow z_i^{a,+}(u^{(z)}) = \Delta e_i^{a,+}(u^{(z)})$ is a continuous function (outside a negligible set) by Lemmas 2.2.11, 2.2.12 and Proposition 2.2.14. Finally, by continuity of $z \mapsto \gamma_z$ (Lemma 2.2.13), for all $1 \leq i \leq k$, $\gamma_{z_i^{a,+}(u^{(z)})}[G_i] \rightarrow \gamma_{z_i^{a,+}}[G_i]$. Applying the dominated convergence theorem to both sides of equation (2.3.3) triggers

$$\gamma_1 \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k G_i(e_i^{a,+}) \right) = \gamma_1 \left(\mathbb{1}_{\{T_a < \infty\}} h \prod_{i=1}^k \gamma_{z_i^{a,+}}[G_i] \right),$$

and concludes the proof of Proposition 2.3.2 for $z = 1$. The general case follows by scaling. $\square$

2.3.2 A change of measures

We begin by calling attention to a natural martingale associated to the growth-fragmentation process.

Proposition 2.3.3. *Let $z \in \mathbb{R} \setminus \{0\}$. Under γ_z , the process*

$$\mathcal{M}_a = \mathbb{1}_{\{T_a < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,+}|^2, \quad a \geq 0,$$

is a $(\mathcal{G}_a)_{a \geq 0}$ -martingale.

Recall from Section 2.3.1 that $(L_t^a)_{t \in [0, R(u)]}$ denotes the local time process of u at level a and $\mathfrak{G}_a = (e_s^a)_{s \in [0, L_{R(u)}^a]}$ its excursion process at level a . We use the shorthand $s^+ \in [0, L_{R(u)}^a]$ to denote times $0 \leq s \leq L_{R(u)}^a$ such that $e_s^a \in U^+$, i.e. when the excursion e_s^a is an excursion of u above a . Then we can rewrite the martingale $\mathcal{M}_a$ as

$$\mathcal{M}_a = \mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} |\Delta e_s^a|^2,$$

where by convention $\Delta e = 0$ when e is the cemetery function $\diamond$.

Proof of Proposition 2.3.3. The branching property 2.3.1 shows that it is enough to prove that $\gamma_z[\mathcal{M}_a] = z^2$ for all $a \geq 0$. Let $g : \mathbb{R} \rightarrow \mathbb{R}_+$ be a nonnegative measurable function. By the master formula,

$$\begin{aligned} \mathbf{n}_+(\mathcal{M}_a g(x(R(u)))) &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} dL_s^a \int_{-\infty}^{+\infty} \frac{dz}{2\pi z^2} z^2 \mathbb{E}[g(z' + X_{T-a})] \Big|_{z'=z+X_s} \right) \\ &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} dL_s^a \int_{-\infty}^{+\infty} \frac{dz'}{2\pi} \mathbb{E}[g(z' + X_{T-a})] \right) \\ &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} L_{R(u)}^a \right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} g, \end{aligned}$$

since the Lebesgue measure is an invariant measure for the Brownian motion. A similar application of the master formula shows that $\mathbf{n}_+(T_a < \infty) = \mathbf{n}_+(\mathbb{1}_{\{T_a < \infty\}} L_{R(u)}^a) \mathbf{n}_-(T-a < \infty)$ hence $\mathbf{n}_+(\mathbb{1}_{\{T_a < \infty\}} L_{R(u)}^a) = 1$ since $\mathbf{n}_+(T_a < \infty) = \mathbf{n}_-(T-a < \infty)$. So finally

$$\mathbf{n}_+(\mathcal{M}_a g(x(R(u)))) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g.$$

Disintegrating $\mathbf{n}_+$ over z as in Proposition 2.2.9 yields

$$\int_{-\infty}^{+\infty} \frac{dz}{2\pi z^2} g(z) \gamma_z[\mathcal{M}_a] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g.$$

This holds for all nonnegative measurable function g , and thus for Lebesgue-almost every $z \in \mathbb{R}$,

$$\gamma_z[\mathcal{M}_a] = z^2.$$

Recall the notation $u^{(z)} = zu(\cdot/z^2)$ for $z > 0$ from Lemma 2.2.12. By scaling, this means for Lebesgue-almost every $z > 0$,

$$\gamma_1 \left[\mathbb{1}_{\{z^2 T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,+}(u^{(z)})|^2 \right] = z^2,$$

which yields

$$\gamma_1 \left[\mathbb{1}_{\{T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,+}(u^{(z)})|^2 \right] = z^2. \quad (2.3.4)$$

Again, this must hold on a dense set of endpoints z , and thus taking z according to some sequence, Lemma 2.2.12 and Proposition 2.2.14 together with Fatou's lemma imply that $\gamma_1[\mathcal{M}_a] \leq 1$. This holds for all a , and so by scaling we deduce that for all $z \neq 0$, $\gamma_z[\mathcal{M}_a] \leq z^2$. On the other hand, notice that $\Delta e_i^{a,+}(u^{(z)}) = z \Delta e_i^{a/z,+}(u)$. By the branching property under γ_1 (Proposition 2.3.2), for a $z < 1$ such that equation (2.3.4) holds,

$$\begin{aligned} 1 &= \gamma_1 \left[\mathbb{1}_{\{T_{a/z} < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a/z,+}|^2 \right] = \gamma_1 \left[\mathbb{1}_{\{T_a < \infty\}} \sum_{i \geq 1} \gamma_{\Delta e_i^{a,+}} \left(\mathbb{1}_{T_{\frac{a}{z}-a} < \infty} \sum_{j \geq 1} |\Delta e_j^{\frac{a}{z}-a,+}|^2 \right) \right] \\ &\leq \gamma_1 \left[\mathbb{1}_{\{T_a < \infty\}} \sum_{i \geq 1} |\Delta e_i^{a,+}|^2 \right]. \end{aligned}$$

Finally combining the two inequalities, we have $\gamma_1[\mathcal{M}_a] = 1$, and $\gamma_z[\mathcal{M}_a] = z^2$ by scaling. $\square$

Recall the definition of $u^{<a}$ in Section 2.3.1. By Kolmogorov extension theorem, we can define on the same probability space a process $(\mathfrak{U}_a^z, a > 0)$ such that for any $a > 0$, the law of $\mathfrak{U}_a^z$ is that of $u^{<a}$ under the measure $\frac{\mathcal{M}_a}{z^2} d\gamma_z$.

We aim at making explicit the law of the process $(\mathfrak{U}_a^z, a > 0)$. Following Chap. 5.3 of [Law08], we call $\mathbb{H}$ -excursion a process in the upper half-plane whose real part is a Brownian motion and whose imaginary part is an independent three-dimensional Bessel process starting at 0. For $a > 0$, we let $\tilde{\mathfrak{U}}_a^z$ be the process obtained by running two independent $\mathbb{H}$ -excursions, one starting from the origin, the other starting from z , concatenating them “at infinity”, and considering only the part of the concatenated trajectory seen below level a . Let us define this process explicitly. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2 = \mathfrak{h}_2^z$ be two independent $\mathbb{H}$ -excursions with $\mathfrak{h}_1(0) = 0$ and $\mathfrak{h}_2(0) = z$ (we will write $\mathfrak{h}_2^z$ when we want to stress the dependence on z). Define for $i \in \{1, 2\}$,

$$A_i(t) := \int_0^t \mathbb{1}_{\{\Im(\mathfrak{h}_i(s)) \leq a\}} ds, \quad \tau_i(t) := \inf\{s > 0 : A_i(s) > t\}.$$

Let $A_i(\infty) := \lim_{t \rightarrow \infty} A_i(t)$ which is the total time spent by $\mathfrak{h}_i$ below level a . Then define $\tilde{\mathfrak{U}}_a^z$ by

$$\tilde{\mathfrak{U}}_a^z(t) := \begin{cases} \mathfrak{h}_1(\tau_1(t)) & \text{if } t \in [0, A_1(\infty)), \\ \mathfrak{h}_2^z(\tau_2(A_1(\infty) + A_2(\infty) - t)) & \text{if } t \in [A_1(\infty), A_1(\infty) + A_2(\infty)], \end{cases}$$

with the convention that $\mathfrak{h}_2^z(\tau_2(A_2(\infty))) := \mathfrak{h}_2^z(\tau_2(A_2(\infty))^-)$. Observe that $\tilde{\mathfrak{U}}_a^z$ follows the trajectory of $\mathfrak{h}_1$ below level a until it leaves $\{\Im \leq a\}$ forever, makes a jump to the exit point of $\{\Im \leq a\}$ by $\mathfrak{h}_2^z$, then follows the trajectory of $\mathfrak{h}_2^z$ below a in reversed time and finally ends up at z .

Theorem 2.3.4. *Let $z \in \mathbb{R} \setminus \{0\}$. The process $(\mathfrak{U}_a^z, a > 0)$ is distributed as $(\tilde{\mathfrak{U}}_a^z, a > 0)$.*

We deduce that through the change of measures associated to the martingale $\mathcal{M}_a$, we can consider that u splits into two independent $\mathbb{H}$ -excursions starting at 0 and z respectively.

The theorem is actually a corollary of the following statement. Let $(\tau_s^a)_{s \in [0, L_{R(u)}^a]}$ denote the inverse local time at level a . Let also $\mathbf{u}_{1,a}^s = \mathbf{u}_1^s$ and $\mathbf{u}_{2,a}^s = \mathbf{u}_2^s$ (again, we write $\mathbf{u}_{i,a}$ when we want to explicit the dependence on a) denote respectively the trajectories

$$\begin{aligned} \mathbf{u}_1^s &:= (u(t), t \in [0, \tau_s^a]), \\ \mathbf{u}_2^s &:= (u(R(u) - s), t \in [0, R(u) - \tau_s^a]), \end{aligned}$$

which stand respectively for the trajectory of u before the excursion e_s^a and for the time-reversed trajectory of u after the excursion e_s^a . Let $F : U \times U \rightarrow \mathbb{R}_+$ be a nonnegative measurable function. Then

$$\begin{aligned} & \gamma_z \left[\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} F(\mathbf{u}_1^s, \mathbf{u}_2^s) |\Delta e_s^a|^2 \right] \\ &= z^2 \mathbb{E} \left[F((\mathfrak{h}_1(t), t \in [0, S_1^a]), (\mathfrak{h}_2^z(t), t \in [0, S_2^a])) \right] \end{aligned} \quad (2.3.5)$$

where on the right-hand side, $S_i^a := \sup\{t \geq 0 : \Im(\mathfrak{h}_i(t)) \leq a\}$ is the exit time of $\{\Im \leq a\}$ by $\mathfrak{h}_i$. The theorem follows by taking for $F(\mathbf{u}_1^s, \mathbf{u}_2^s)$ and all s some measurable function of $u^{<a}$.

Proof of equation (2.3.5). It suffices to take $f, g : U \rightarrow \mathbb{R}_+$ two bounded continuous functions and prove (2.3.5) for $F(u, v) = f(u)g(v)$. We first consider the left-hand side under

the measure $\mathbf{n}_+$. By the master formula,

$$\begin{aligned} \mathbf{n}_+ & \left(\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} f(u_1^s) g(u_2^s) |\Delta e_s^a|^2 \right) \\ &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} f(u_1^{L_r^a}) dL_r^a \right. \\ & \quad \left. \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E}[g(x + x' + X_{T-a-s}, a + Y_{T-a-s}, 0 \leq s \leq T-a)]_{x'=X_r} \right). \end{aligned} \quad (2.3.6)$$

The change of variables $x + X_r \mapsto x$ shows that it is also

$$\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} f(u_1^{L_r^a}) dL_r^a \right) \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E}[g(x + X_{T-a-s}, a + Y_{T-a-s}, 0 \leq s \leq T-a)].$$

The path $(X_s, 0 \leq s \leq T-a)$ is conditionally on Y distributed as a linear Brownian motion stopped at time $T-a$ (recall that $T-a$ is a measurable function of Y). Since the Lebesgue measure is a reversible measure for the Brownian motion, by time-reversal, the "law" of $(x + X_{T-a-s}, 0 \leq s \leq T-a)$ for x chosen with the Lebesgue measure is the "law" of a linear Brownian motion with initial measure the Lebesgue measure, stopped at time $T-a$ ($T-a$ still referring to the hitting time of Y). Then we use that the process $(a + Y_{T-a-s}, 0 \leq s \leq T-a)$ has the law of a 3-dimensional Bessel process starting from 0 and run until its exit time of $[0, a]$, see Corollary 4.6, Chap. VII of [RY99]. In other words, we proved that

$$\int_{-\infty}^{+\infty} \frac{dx}{2\pi} \mathbb{E}[g(x + X_{T-a-s}, a + Y_{T-a-s}, 0 \leq s \leq T-a)] = \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E}[g(\mathfrak{h}_2^z(t), t \in [0, S_2^a])].$$

Going back to (2.3.6), we get that

$$\begin{aligned} \mathbf{n}_+ & \left(\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} f(u_1^s) g(u_2^s) |\Delta e_s^a|^2 \right) \\ &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} f(u_1^{L_r^a}) dL_r^a \right) \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E}[g(\mathfrak{h}_2^z(t), t \in [0, S_2^a])]. \end{aligned}$$

On the other hand, by another application of the master formula,

$$\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} f(u_1^{L_{R(u)}^a}) \right) = \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} f(u_1^{L_r^a}) dL_r^a \right) \mathbf{n}_+(T-a < \infty).$$

Therefore $\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \int_0^{R(u)} f(u_1^{L_r^a}) dL_r^a \right) = \mathbf{n}_+ \left(f(u_1^{L_{R(u)}^a}) \mid T_a < \infty \right)$. We observe that under $\mathbf{n}_+(\cdot \mid T_a < \infty)$, u up to the exit time of $\{\mathfrak{S} \leq a\}$ has the law of $\mathfrak{h}_1$ up to S_1^a . Consequently,

$$\begin{aligned} \mathbf{n}_+ & \left(\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} f(u_1^s) g(u_2^s) |\Delta e_s^a|^2 \right) \\ &= \mathbb{E}[f(\mathfrak{h}_1(t), t \in [0, S_1^a])] \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E}[g(\mathfrak{h}_2^z(t), t \in [0, S_2^a])]. \end{aligned}$$

Finally, we disintegrate $\mathbf{n}_+$ over $x(R(u))$ to get

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{dz}{2\pi z^2} \gamma_z \left[\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} f(u_1^s) g(u_2^s) |\Delta e_s^a|^2 \right] \\ = \mathbb{E}[f(\mathfrak{h}_1(t), t \in [0, S_1^a])] \int_{-\infty}^{+\infty} \frac{dz}{2\pi} \mathbb{E}[g(\mathfrak{h}_2^z(t), t \in [0, S_2^a])]. \end{aligned}$$

Now multiply g by any measurable function $h : \mathbb{R} \rightarrow \mathbb{R}_+$ of $x(R(u))$ to see that for Lebesgue-almost every $z \in \mathbb{R}$,

$$\begin{aligned} & \gamma_z \left[\mathbb{1}_{\{T_a < \infty\}} \sum_{s^+ \in [0, L_{R(u)}^a]} f(u_1^s) g(u_2^s) |\Delta e_s^a|^2 \right] \\ &= z^2 \mathbb{E} \left[f(\mathbf{h}_1(t), t \in [0, S_1^a]) \right] \mathbb{E} \left[g(\mathbf{h}_2^z(t), t \in [0, S_2^a]) \right]. \end{aligned}$$

A continuity argument that we feel free to skip for concision shows that it is actually valid for all z . □

Remark 2.3.5. Theorem 2.3.4 indicates that a particular Cauchy process will be of paramount importance in the description of the excursions cut at heights: under the tilted measure, u splits into two independent $\mathbb{H}$ -excursions and so the size at some level a of the spine going to infinity is just the difference of two Brownian motions started from infinity taken at their hitting time of $\{\mathfrak{S}(z) = a\}$. This also gives an insight on why the Cauchy process should be hidden in some sense in the law of Ξ , as described in the following subsection.

2.3.3 The locally largest evolution

Recall that Proposition 2.2.8 gives a canonical choice of excursion at level $a > 0$, which is the locally largest excursion $e_a^{(t^\bullet)}$. One may wonder whether the locally largest fragment $\Xi(a) = \Delta e_a^{(t^\bullet)}$ still exhibits some kind of Markovian behavior. The following theorem answers this question.

Theorem 2.3.6. *Let $z > 0$. Under γ_z , $(\Xi(a))_{0 \leq a < \mathfrak{S}(z^\bullet)}$ is distributed as the positive self-similar Markov process $(Z_a)_{0 \leq a < \zeta}$ with index 1 starting from z whose Lamperti representation is*

$$Z_a = z \exp(\xi(\tau(z^{-1}a))),$$

where ξ is the Lévy process starting at 0 with Laplace exponent $\Psi(q) := \ln \gamma_z[e^{q\xi(1)}]$ given by

$$\Psi(q) = -\frac{4}{\pi}q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y} dy}{(e^y - 1)^2}, \quad q < 3, \quad (2.3.7)$$

τ is the time change

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\xi(u)} du > a \right\},$$

and $\zeta = \inf\{a \geq 0, Z_a = 0\}$.

The following martingale will come into play. Recall from [CC06] that a symmetric Cauchy process starting from $z > 0$ killed when entering the negative half-line can be written using its Lamperti representation as $ze^{\xi^0(\tau^0(a))}$ where

$$\tau^0(a) := \inf \left\{ s \geq 0, \int_0^s ze^{\xi^0(u)} du \geq a \right\}, \quad (2.3.8)$$

and $(\xi^0(a), a \geq 0)$ is a Lévy process killed at an exponential time of parameter $\frac{2}{\pi}$, starting from 0 with Laplace exponent

$$\Psi^0(q) = \frac{2}{\pi} \int_{\mathbb{R}} (e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1}) e^y (e^y - 1)^{-2} dy - \frac{2}{\pi}, \quad -1 < q < 1. \quad (2.3.9)$$

Let $\Delta\xi_b^0$ denote the jump of ξ^0 at time b , i.e. $\Delta\xi_b^0 := \xi_b^0 - \xi_{b-}^0$. The following lemma is the analog of Lemma 17 in [LGR20], and exhibits a martingale on the event that the Cauchy process never more than halves itself making a jump. When proving Theorem 2.3.6, we shall use this martingale stopped at the specific time $\tau^0(a)$. We denote by P a probability measure associated to ξ^0 .

Lemma 2.3.7. *For every $a \geq 0$, set*

$$M_a = e^{-2\xi_a^0} \mathbf{1}_{\{\forall b \in [0, a], \Delta\xi_b^0 > -\ln(2)\}}.$$

Then $(M_a)_{a \geq 0}$ is a martingale with respect to the canonical filtration of the process ξ^0 . Under the tilted probability measure $e^{-2\xi_a^0} \mathbf{1}_{\{\forall b \in [0, a], \Delta\xi_b^0 > -\ln(2)\}} \cdot P$, the process $(\xi^0(b))_{b \in [0, a]}$ is a Lévy process with Laplace exponent Ψ introduced in (2.3.7) in Theorem 2.3.6.

Proof. We compute

$$E[e^{(q-2)\xi_a^0} \mathbf{1}_{\{\forall b \in [0, a], \Delta\xi_b^0 > -\ln(2)\}}].$$

Indeed, that $(M_a)_{a \geq 0}$ is a martingale will come from the fact that ξ^0 is a Lévy process and that the expectation above is 1 when $q = 0$. To compute this expectation, we decompose ξ^0 into its small and large jumps parts:

$$\xi_a^0 = \xi'_a + \xi''_a,$$

where $\xi''_a = \sum_{0 \leq b \leq a} \Delta\xi_b^0 \mathbf{1}_{\Delta\xi_b^0 \leq -\ln(2)}$. Notice that ξ' and ξ'' are independent. Then by independence, the above expectation is

$$\begin{aligned} E[e^{(q-2)\xi_a^0} \mathbf{1}_{\{\forall b \in [0, a], \Delta\xi_b^0 > -\ln(2)\}}] &= E[\mathbf{1}_{\{\xi''_a = 0\}} e^{(q-2)\xi'_a}] \\ &= P(\xi''_a = 0) E[e^{(q-2)\xi'_a}]. \end{aligned} \quad (2.3.10)$$

Thus, we need to compute the Laplace exponents of ξ' and ξ'' (under P), that we denote respectively by Ψ' and Ψ'' . Because ξ'' is the pure-jump process given by the jumps of ξ^0 smaller than $-\ln(2)$, its Laplace exponent is given by the Lévy measure of ξ^0 restricted to $(-\infty, -\ln(2)]$, namely

$$\Psi''(q) = \frac{2}{\pi} \int_{y \leq -\ln(2)} (e^{qy} - 1) \frac{e^y}{(e^y - 1)^2} dy. \quad (2.3.11)$$

It results from the independence of ξ' and ξ'' that the Laplace exponent of ξ' is $\Psi' = \Psi^0 - \Psi''$, hence by equations (2.3.9) and (2.3.11), for all $-1 < q < 1$,

$$\begin{aligned} \Psi'(q) &= \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1) \mathbf{1}_{\{|e^y - 1| < 1\}}) e^y (e^y - 1)^{-2} dy \\ &\quad - \frac{2}{\pi} q \int_{y \leq -\ln(2)} (e^y - 1) \mathbf{1}_{|e^y - 1| < 1} e^y (e^y - 1)^{-2} dy - \frac{2}{\pi}. \end{aligned} \quad (2.3.12)$$

The middle term in this expression (2.3.12) is

$$\begin{aligned} \frac{2}{\pi} q \int_{y \leq -\ln(2)} (e^y - 1) e^y (e^y - 1)^{-2} dy &= -\frac{2}{\pi} q \int_{y \leq -\ln(2)} \frac{e^y}{1 - e^y} dy \\ &= -\frac{2}{\pi} q \int_0^{1/2} \frac{dx}{1 - x} \\ &= -\frac{2}{\pi} q \ln(2). \end{aligned}$$

Hence

$$\begin{aligned} \Psi'(q) &= \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{|e^y - 1| < 1}) e^y (e^y - 1)^{-2} dy \\ &\quad + \frac{2}{\pi} q \ln(2) - \frac{2}{\pi}. \end{aligned} \quad (2.3.13)$$

This extends analytically to all $q < 1$. Let us come back to (2.3.10). We have for $q < 3$

$$\begin{aligned} E[e^{(q-2)\xi_a^0} \mathbb{1}_{\forall b \in [0, a], \Delta \xi_b^0 > -\ln(2)}] &= P(\xi_a'' = 0) E[e^{(q-2)\xi_a'}] \\ &= e^{a\Psi''(\infty)} e^{a\Psi'(q-2)} \\ &= \exp\left(-\frac{2}{\pi} a \int_{y \leq -\ln(2)} \frac{e^y}{(e^y - 1)^2} dy\right) e^{a\Psi'(q-2)} \\ &= e^{a(\Psi'(q-2) - \frac{2}{\pi})}, \end{aligned}$$

by a change of variables.

This essentially concludes the calculation of the new Laplace exponent $\tilde{\Psi}$ of ξ^0 under the tilted measure $e^{-2\xi_a^0} \mathbb{1}_{\{\forall b \in [0, a], \Delta \xi_b^0 \geq -\ln(2)\}} \cdot P$, which is simply

$$\tilde{\Psi}(q) = \Psi'(q-2) - \frac{2}{\pi}, \quad q < 3. \quad (2.3.14)$$

Still we can put it in a Lévy-Khintchin form. Replacing q by $q-2$ in the integral in (2.3.13), we get

$$\begin{aligned} &\int_{y > -\ln(2)} (e^{-2y} e^{qy} - 1 - (q-2)(e^y - 1)\mathbb{1}_{|e^y - 1| < 1}) e^y (e^y - 1)^{-2} dy \\ &= \int_{y > -\ln(2)} (e^{qy} - e^{2y} - (q-2)(e^{3y} - e^{2y})\mathbb{1}_{|e^y - 1| < 1}) e^{-y} (e^y - 1)^{-2} dy \\ &= \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{|e^y - 1| < 1}) e^{-y} (e^y - 1)^{-2} dy \\ &\quad + \int_{y > -\ln(2)} \left[1 - e^{2y} + (q(e^y - 1) - (q-2)(e^{3y} - e^{2y}))\mathbb{1}_{|e^y - 1| < 1}\right] \frac{e^{-y}}{(e^y - 1)^2} dy. \end{aligned}$$

After simplifications, we find that the last integral is equal to

$$\begin{aligned} &\int_{y > -\ln(2)} \left[1 - e^{2y} + (q(e^y - 1) - (q-2)(e^{3y} - e^{2y}))\mathbb{1}_{|e^y - 1| < 1}\right] \frac{e^{-y}}{(e^y - 1)^2} dy \\ &= 2 + 2\ln(2) - q\left(2\ln(2) + \frac{3}{2}\right). \end{aligned} \quad (2.3.15)$$

From equations (2.3.14), (2.3.12) and (2.3.15), we deduce

$$\tilde{\Psi}(q) = -\frac{2}{\pi} \left(\ln(2) + \frac{3}{2}\right) q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)\mathbb{1}_{|e^y - 1| < 1}) \frac{e^{-y} dy}{(e^y - 1)^2}. \quad (2.3.16)$$

Finally, we can remove the indicator using simple calculations. One finds that

$$\int_{y > -\ln(2)} (1 - e^y) \frac{e^{-y}}{(e^y - 1)^2} \mathbb{1}_{\{|e^y - 1| \geq 1\}} dy = \frac{1}{2} - \ln(2),$$

and therefore

$$\tilde{\Psi}(q) = -\frac{4}{\pi} q + \frac{2}{\pi} \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) \frac{e^{-y} dy}{(e^y - 1)^2}, \quad q < 3.$$

Hence $\tilde{\Psi} = \Psi$ with the notation of Theorem 2.3.6. Since $\tilde{\Psi}(0) = 0$, this gives both the martingale property and the law of ξ^0 under the change of measure described in Lemma 2.3.7. $\square$

Let us now turn to the proof of Theorem 2.3.6.

Proof. Let H be a bounded measurable function on càdlàg paths. On the event $\{a < \mathfrak{S}(z^\bullet)\}$, we have

$$H(\Xi(b), b \in [0, a]) = \sum_{s^+ \in [0, L_{R(u)}^a]} F(u_1^s, u_2^s) |\Delta e_s^a|^2$$

where

$$F(u_1^s, u_2^s) = |\Delta e_s^a|^{-2} H(\Xi(b), b \in [0, a]) \mathbb{1}_{\{e_a^{(t^\bullet)} = e_s^a\}}. \quad (2.3.17)$$

The right-hand side is indeed a function of (u_1^s, u_2^s) since $(\Delta e_b^{(t^\bullet)}, b \in [0, a])$ is a measurable function of $u^{<a}$, which is itself a measurable function of (u_1^s, u_2^s) . We apply equation (2.3.5):

$$\gamma_z \left[H(\Xi(b), b \in [0, a]) \mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \right] = z^2 \mathbb{E} \left[F((h_1(t), t \in [0, S_1^a]), (h_2^z(t), t \in [0, S_2^a])) \right].$$

We need to understand the measurable function $F((h_1(t), t \in [0, S_1^a]), (h_2^z(t), t \in [0, S_2^a]))$.

Let η be the process defined by $\eta_b := h_2(S_2^b) - h_1(S_1^b)$ for $b \geq 0$. For $i \in \{1, 2\}$, the process $\mathfrak{R}(h_i(S_i^b))$, $b \geq 0$ is a Cauchy process (for example, use that it is a Lévy process, that S_i^a is distributed as the hitting time of a by a Brownian motion by Corollary 4.6, Chap. VII of [RY99], then Proposition 3.11, Chap. III of [RY99]). Therefore η is a (càdlàg) symmetric Cauchy process of Laplace exponent $\psi(\lambda) = -2|\lambda|$. Let $\Delta\eta_b$ be the jump of η at time b . From the definition of F in (2.3.17), we have

$$F((h_1(t), t \in [0, S_1^a]), (h_2^z(t), t \in [0, S_2^a])) = \eta_a^{-2} H(\eta_b, b \in [0, a]) \mathbb{1}_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta\eta_b|\}}.$$

Therefore

$$\gamma_z \left[H(\Xi(b), b \in [0, a]) \mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \right] = \mathbb{E} \left[\frac{z^2}{\eta_a^2} H(\eta_b, b \in [0, a]) \mathbb{1}_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta\eta_b|\}} \right]. \quad (2.3.18)$$

Notice that, almost surely, on the event $\{\forall b \in [0, a], |\eta_b| \geq |\Delta\eta_b|\}$, if $\eta_0 > 0$, then η_b is positive for all $b \in [0, a]$. Therefore, on this event we can write $\eta_b = \exp(\xi^0(\tau^0(b)))$ for $0 \leq b \leq a$ with the notation of (2.3.8). We finish the proof of Theorem 2.3.6 by appealing to Lemma 2.3.7, together with the arguments of [LGR20] that we reproduce here to be self-contained. With this notation at hand, identity (2.3.18) now reads

$$\gamma_z \left[H(\Xi(b), b \in [0, a]) \mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \right] = E \left[M_{\tau^0(a)} H \left(z \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \right].$$

The optional stopping theorem implies that for any $c > 0$,

$$\begin{aligned} & E \left[M_{\tau^0(a)} H \left(z \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau^0(a)\}} \right] \\ &= E \left[M_c H \left(z \exp(\xi^0(\tau^0(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau^0(a)\}} \right]. \end{aligned}$$

By Lemma 2.3.7, the right-hand side is, with the notation ξ of the theorem,

$$E \left[H \left(z \exp(\xi(\tau(b))), b \in [0, a] \right) \mathbb{1}_{\{c > \tau(a)\}} \right].$$

Making $c \mapsto \infty$ and using dominated convergence completes the proof. $\square$

In addition, in order to study the genealogy of the growth-fragmentation process linked to Brownian excursions in the next section, we need to clarify the behavior of the *offspring* of Ξ . By offspring we mean all the excursions that were created at times a when the excursion $e_a^{(t^\bullet)}$ divided into two excursions (*i.e.* at jump times of Ξ). We rank these excursions in descending order of the absolute value of their sizes. This way we get a sequence $(z_i, a_i)_{i \geq 1}$ of jump sizes and times for Ξ , associated to excursions $(e_i)_{i \geq 1}$, of size z_i above a_i .

Theorem 2.3.8. *Let $z \in \mathbb{R} \setminus \{0\}$. Under γ_z , conditionally on the jump sizes and jump times $(z_i, a_i)_{i \geq 1}$ of Ξ , the excursions $(e_i)_{i \geq 1}$, are independent and each e_i has law γ_{z_i} .*

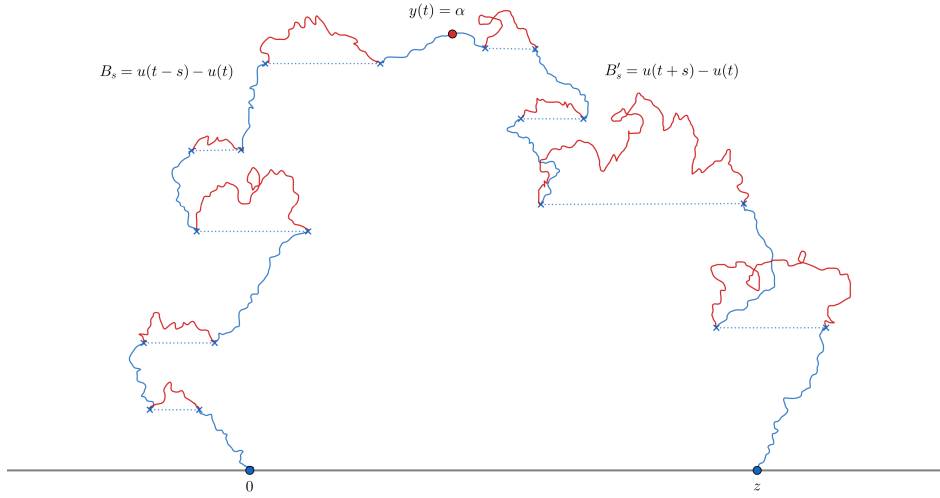


Figure 2.7 – Excursions of $B = (X, Y)$ and $B' = (X', Y')$ above their past infimum. The past infimum process is depicted in blue, and by Lévy's theorem the excursions above it form a Poisson point process represented in red.

Proof. Let $(f_i)_{1 \leq i \leq n}$, $(g_i)_{1 \leq i \leq n}$, $n \in \mathbb{N}$, be nonnegative measurable functions respectively on U and $\mathbb{R} \times \mathbb{R}_+$. Let $a > 0$. Denote by $(e_i^{(a)})_{i \geq 1}$ the offspring of Ξ created before a , ranked in descending order of the absolute value of their sizes $z_i^{(a)}$, and let b_i the jump time. We write

$$\prod_{i=1}^n f_i(e_i^{(a)}) g_i(z_i^{(a)}, b_i) = \sum_{0 < s^+ < L_{R(u)}^a} F(u_1^s, u_2^s) |\Delta e_s^a|^2$$

where

$$F(u_1^s, u_2^s) = |\Delta e_s^a|^{-2} \prod_{i=1}^n f_i(e_i^{(a)}) g_i(z_i^{(a)}, b_i) \mathbf{1}_{\{e_a^{(t^\bullet)} = e_s^a\}}.$$

In this case, $F((h_1(t), t \in [0, S_1^a]), (h_2(t), t \in [0, S_2^a]))$ is

$$\eta_a^{-2} \prod_{i=1}^n f_i(\varepsilon_i) g_i(z(\varepsilon_i), b(\varepsilon_i)) \mathbf{1}_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}}$$

where $\varepsilon_1, \dots, \varepsilon_n$ are the n largest excursions of h_1 and h_2 above the future infimum of their imaginary parts (which is respectively $b(\varepsilon_i)$) before leaving $\{\Im \leq a\}$ and $z(\varepsilon_i)$ is the size of

the excursion ε_i . We get by equation (2.3.5)

$$\gamma_z \left[\mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \prod_{i=1}^n f_i(e_i^{(a)}) g_i(z_i^{(a)}, b_i) \right] = \mathbb{E} \left[\frac{z^2}{\eta_a^2} \prod_{i=1}^n f_i(\varepsilon_i) g_i(z(\varepsilon_i), b(\varepsilon_i)) \mathbb{1}_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}} \right].$$

Consider the collection $\{(b, \mathfrak{e}_b), b \geq 0\}$ where $\mathfrak{e}_b$ is an excursion of $\mathfrak{h}_1$ or $\mathfrak{h}_2$ above the future infimum (set $\mathfrak{e}_b = \diamond$ if no such excursion exists) and b is the corresponding value of the future infimum. By time-reversal (Corollary 4.6, Chap. VII of [RY99]) and Lévy's Theorem, (Theorem 2.3, Chap. VI of [RY99]), the collection $\{(b, \mathfrak{e}_b), b \geq 0\}$ is a Poisson point process of intensity $2\mathbb{1}_{\mathbb{R}_+} db \mathbf{n}_+(du)$. Write $z(e)$ for the size of an excursion e . Conditionally on the sizes $\{(b, z(\mathfrak{e}_b)), b \geq 0\}$, the excursions $\mathfrak{e}_b$ are distributed as independent excursions with law $\gamma_{z(\mathfrak{e}_b)}$, hence

$$\begin{aligned} & \gamma_z \left[\mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \prod_{i=1}^n f_i(e_i^{(a)}) g_i(z_i^{(a)}, b_i) \right] \\ &= \mathbb{E} \left[\frac{z^2}{\eta_a^2} \prod_{i=1}^n \gamma_{z(\varepsilon_i)}(f_i(\varepsilon_i)) g_i(z(\varepsilon_i), b(\varepsilon_i)) \mathbb{1}_{\{\forall b \in [0, a], |\eta_b| \geq |\Delta \eta_b|\}} \right]. \end{aligned}$$

Using equation (2.3.5) backwards, we see that

$$\gamma_z \left[\mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \prod_{i=1}^n f_i(e_i^{(a)}) g_i(z_i^{(a)}, b_i) \right] = \gamma_z \left[\mathbb{1}_{\{a < \mathfrak{S}(z^\bullet)\}} \prod_{i=1}^n \gamma_{z_i^{(a)}}(f_i(e_i^{(a)})) g_i(z_i^{(a)}, b_i) \right].$$

The statement follows. $\square$

2.4 The growth-fragmentation process of excursions in $\mathbb{H}$

In this section, we summarize the previous results in the language of the self-similar growth-fragmentations introduced by Bertoin in [Ber17b]. The main reference here is [BBCK18], but for the sake of completeness we shall recall in the first paragraph the bulk of the construction of such processes. At the heart of this section lies the calculation of the cumulant function. We recover the cumulant function of [BBCK18], formula (19), in the specific case when $\theta = 1$. Recall the definition of Z in Theorem 2.3.6. The process Z starting at $z < 0$ is defined to be the negative of the process Z starting at $-z$.

2.4.1 Construction of $\overline{\mathbf{X}}$

We explain how one can define the *cell system* driven by Z . We use the Ulam tree $\mathbb{U} = \cup_{i=0}^\infty \mathbb{N}^i$, where $\mathbb{N} = \{1, 2, \dots\}$, to encode the genealogy of the cells (we write $\mathbb{N}^0 = \{\emptyset\}$, and $\emptyset$ is called the Eve cell). A node $u \in \mathbb{U}$ is a list $(u_1, \dots, u_i)$ of positive integers where $|u| = i$ is the *generation* of u . The children of u are the lists in $\mathbb{N}^{i+1}$ of the form $(u_1, \dots, u_i, k)$, with $k \in \mathbb{N}$. A *cell system* is a family $\mathcal{X} = (\mathcal{X}_u, u \in \mathbb{U})$ indexed by $\mathbb{U}$, where $\mathcal{X}_u = (\mathcal{X}_u(a))_{a \geq 0}$ is meant to describe the evolution of the size or mass of the cell u with its age a .

To define the cell system driven by Z , we first define $\mathcal{X}_\emptyset$ as Z , started from some initial mass $z \neq 0$, and set $b_\emptyset = 0$. Observe the realization of $\mathcal{X}_\emptyset$ and its jumps. Since Z hits 0 in finite time, we may rank the sequence of jump sizes and times $(x_1, \beta_1), (x_2, \beta_2), \dots$ of $-\mathcal{X}_\emptyset$ by decreasing order of the $|x_i|$'s. Conditionally on these jump sizes and times, we define the first generation of our cell system $\mathcal{X}_i, i \in \mathbb{N}$, to be independent with $\mathcal{X}_i$ distributed as Z , starting from x_i . We also set $b_i = b_\emptyset + \beta_i$ for the birth time of the particle $i \in \mathbb{N}$. By recursion, one defines the law of the n -th generation given generations $1, \dots, n-1$ in the same way. Hence the cell labelled by $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ is born from $u' = (u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1}$ at time

$b_u = b_{u'} + \beta_{u_n}$, where β_{u_n} is the time of the u_n -th largest jump of $\mathcal{X}_{u'}$, and conditionally on $\mathcal{X}_{u'}(\beta_{u_n}) - \mathcal{X}_{u'}(\beta_{u_n}^-) = -y$, $\mathcal{X}_u$ has the law of Z with initial value y and is independent of the other daughter cells at generation n . We write ζ_u for the lifetime of the particle u . We may then define, for $a \geq 0$,

$$\overline{\mathbf{X}}(a) := (\mathcal{X}_u(a - b_u), u \in \mathbb{U} \text{ and } b_u \leq a < b_u + \zeta_u), \quad (2.4.1)$$

as the family of the sizes of all the cells alive at time a . We arrange the elements in $\overline{\mathbf{X}}(a)$ in descending order of their absolute values.

2.4.2 The growth-fragmentation process of excursions in $\mathbb{H}$

We restate Theorem 2.1.1. Beware that the signed growth-fragmentation $\overline{\mathbf{X}}$ in this section starts from z .

Theorem 2.4.1. *Let $z \in \mathbb{R} \setminus \{0\}$. Under γ_z ,*

$$(\overline{\mathbf{X}}(a), a \geq 0) \stackrel{\text{law}}{=} \left((\Delta e_i^{a,+}, i \geq 1), a \geq 0 \right).$$

Proof. Let $u \in U^+$ be such that the locally largest excursion described in Subsection 2.3.3 is well-defined, i.e. u has no loop above any level, has distinct local minima, and no splitting in two equal sizes (this set of excursions has full probability under γ_z). This gives our Eve cell process. The independence of the daughter excursions given their size at birth has already been proved in Theorem 2.3.8, and we have taken Z according to the law of the largest fragment in Theorem 2.3.6, so it remains to prove that every excursion can be found in the genealogy of $\overline{\mathbf{X}}$ as constructed in the former section.

For $a \geq 0$, we denote by $\overline{\mathbf{X}}^{exc}(a)$ the set of all excursions associated to the sizes in $\overline{\mathbf{X}}(a)$. Let $0 \leq t \leq R(u)$ such that $\Im(u(t)) > a$. We want to show that $e_a^{(t)} \in \overline{\mathbf{X}}^{exc}(a)$. Set

$$\mathcal{A} = \left\{ a' \in [0, a], e_{a'}^{(t)} \in \overline{\mathbf{X}}^{exc}(a') \right\}.$$

Then $\mathcal{A}$ is an interval containing 0.

- $\mathcal{A}$ is open in $[0, a]$. Let $a' \in \mathcal{A}$ with $a' < a$. Write $e_b^{(\tau^\bullet)}, b \geq a'$, for the locally largest excursion inside $e_{a'}^{(t)}$. Then for small enough $\varepsilon > 0$, $e_{a'+\varepsilon}^{(t)} = e_{a'+\varepsilon}^{(\tau^\bullet)}$. Indeed, the first height $b \geq a'$ when $e_b^{(t)} \neq e_b^{(\tau^\bullet)}$ is equal to the minimum of $y(s)$ for s between t and $\tau^\bullet$, and so it is strictly above a' . This implies that $a' + \varepsilon \in \mathcal{A}$ since $e_{a'+\varepsilon}^{(\tau^\bullet)} \in \overline{\mathbf{X}}^{exc}(a' + \varepsilon)$ as the locally largest excursions are in the genealogy.
- $\mathcal{A}$ is closed in $[0, a]$. Let a_n be a sequence of elements in $\mathcal{A}$ increasing to a_∞ . For all $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\forall a' \in (a_\infty - \delta, a_\infty), \quad |\Delta e_{a'}^{(t)} - \Delta e_{a_\infty}^{(t)}| < \varepsilon.$$

Then for all $a_1, a_2 \in (a_\infty - \delta, a_\infty)$,

$$|\Delta e_{a_1}^{(t)} - \Delta e_{a_2}^{(t)}| \leq |\Delta e_{a_1}^{(t)} - \Delta e_{a_\infty}^{(t)}| + |\Delta e_{a_2}^{(t)} - \Delta e_{a_\infty}^{(t)}| < 2\varepsilon.$$

Take $\varepsilon = |\Delta e_{a_\infty}^{(t)}|/4$ and N large enough so that $a_N \in (a_\infty - \delta, a_\infty)$. Then the excursion $e_{a_N}^{(t)}$ is such that for all $a' \in [a_N, a_\infty)$, $e_{a'}^{(t)}$ is taken along the locally largest excursion inside $e_{a_N}^{(t)}$. Indeed, it follows from these inequalities that for all $a_1, a_2 \in (a_\infty - \delta, a_\infty)$, $|\Delta e_{a_1}^{(t)} - \Delta e_{a_2}^{(t)}| \leq \frac{1}{2} |\Delta e_{a_\infty}^{(t)}| < |\Delta e_{a_1}^{(t)}|$, then take $a_1 = a'$ and $a_2 \nearrow a'$. This entails that $a_\infty \in \mathcal{A}$.

By connectedness $\mathcal{A}$ must be $[0, a]$. This concludes the proof. $\square$

Remark 2.4.2. In the parlance of [Sil21], Theorem 2.4.1 states that the collection of excursion sizes $(\Delta e_i^{a,+}, i \geq 1)$ is a *self-similar signed growth-fragmentation process*, and identifies Z as a driving cell process. Notice that, in our setting, positive and negative particles evolve symmetrically, hence the constants v_+ and v_- in [Sil21] are equal. Furthermore, using the description in Theorem 2.3.6, one can see from the compensation formula that

$$\gamma_z \left(\sum_{t>0} |\Delta Z(t)|^2 \right) = 1,$$

which justifies that the exponent appearing in the genealogical martingale is 2. Combining Theorem 5.4 in [Sil21] and Corollary 11 in [CPR13] enables to retrieve the spine of the signed growth-fragmentation cell system as a Cauchy process.

2.4.3 The cumulant function

The process $\bar{\mathbf{X}}$ is not a growth-fragmentation in the sense of [BBCK18] because it carries negative masses. We show in this section that if one discards all cells with negative masses together with their progeny, one obtains one of the growth-fragmentation processes studied in [BBCK18].

Formally, let $\mathbf{X}$ defined by (2.4.1) where we only consider the u 's such that $\mathcal{X}_v(b_v) > 0$ for all ancestors v of u (including itself) in the Ulam tree. The process $\mathbf{X}$ is a growth-fragmentation in the sense of [BBCK18]. It is characterized by its self-similarity index $\alpha = -1$ and its cumulant function defined for $q \geq 0$, by

$$\kappa(q) := \Psi(q) + \int_{-\infty}^0 (1 - e^y)^q \Lambda(dy),$$

where Λ denotes the Lévy measure of the Lévy process ξ .

The following proposition is Proposition 5.2 of [BBCK18] in the case $\theta = 1$, $\hat{\beta} = 1$ and $\gamma = \hat{\gamma} = 1/2$ with the additional factor 2 (corresponding to a time change).

Proposition 2.4.3. *Let $\omega_+ = 5/2$, and $\Phi^+(q) = \kappa(q + \omega_+)$ for $q \geq 0$. Then Φ^+ is the Laplace exponent of a symmetric Cauchy process conditioned to stay positive, namely*

$$\Phi^+(q) = -2 \frac{\Gamma(\frac{1}{2} - q) \Gamma(\frac{3}{2} + q)}{\Gamma(-q) \Gamma(1 + q)}, \quad -\frac{3}{2} < q < \frac{1}{2}. \quad (2.4.2)$$

Furthermore, the associated growth-fragmentation X has no killing and its cumulant function is

$$\kappa(q) = -2 \frac{\cos(\pi q)}{\pi} \Gamma(q - 1) \Gamma(3 - q), \quad 1 < q < 3. \quad (2.4.3)$$

Remark 2.4.4. In [BBCK18], the roots of κ pave the way for remarkable martingales. It should not come as a surprise that in our case these roots happen to be $\omega_- = \frac{3}{2}$ and $\omega_+ = \frac{5}{2}$. Indeed, the h -transform for the symmetric Cauchy process conditioned to stay positive (resp. conditioned to hit 0 continuously) is given by $x \mapsto x^{1/2}$ (resp. $x \mapsto x^{-1/2}$). This turns the martingale in Proposition 2.3.3 into the sum over all masses in $\mathbf{X}$ to the power $\omega_+ = 2 + \frac{1}{2}$, and $\omega_- = 2 - \frac{1}{2}$ respectively, which are exactly the quantities considered in [BBCK18].

Proof. For simplicity, we shall write ω instead of ω_+ in the proof. The strategy is as follows. In view of Theorem 5.1 in [BBCK18], we first compute $\kappa(q + \omega) - \kappa(\omega)$ and we put it in a Lévy-Khintchin form so as to retrieve the Laplace exponent of the Lévy process involved

in the Lamperti representation of a Cauchy process conditioned to stay positive, which is known from [CC06]. We then show that $\kappa(\omega) = 0$, and therefore deduce the expression of κ .

Recall first that by definition

$$\kappa(q) = \Psi(q) + \int_{-\infty}^0 (1 - e^y)^q \Lambda(dy),$$

with Ψ given by (2.3.7). In fact, we rather use formula (2.3.16), which is closer to [CC06]:

$$\Psi(q) = -\frac{2}{\pi} \left(\ln(2) + \frac{3}{2} \right) q + \frac{2}{\pi} \int_{y > -\ln(2)} \left(e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2}.$$

Let $-\frac{3}{2} < q < \frac{1}{2}$. Then

$$\begin{aligned} & \frac{\pi}{2} (\kappa(q + \omega) - \kappa(\omega)) \\ &= -\left(\ln(2) + \frac{3}{2} \right) q + \int_{y > -\ln(2)} \left(e^{(q+\omega)y} - e^{\omega y} - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2} \\ & \quad + \int_{-\ln(2)}^0 \left((1 - e^y)^{q+\omega} - (1 - e^y)^\omega \right) \frac{e^{-y} dy}{(e^y - 1)^2}. \end{aligned}$$

Performing the change of variables $e^x = 1 - e^y$ in the second integral entails

$$\begin{aligned} & \frac{\pi}{2} (\kappa(q + \omega) - \kappa(\omega)) \\ &= -\left(\ln(2) + \frac{3}{2} \right) q + \int_{y > -\ln(2)} \left(e^{(q+\omega)y} - e^{\omega y} - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2} \\ & \quad + \int_{-\infty}^{-\ln(2)} \left(e^{(q+\omega)x} - e^{\omega x} \right) \frac{e^{-x} dx}{(e^x - 1)^2}. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{\pi}{2} (\kappa(q + \omega) - \kappa(\omega)) \\ &= -\left(\ln(2) + \frac{3}{2} \right) q + \int_{-\infty}^{+\infty} \left(e^{(q+\omega)y} - e^{\omega y} - q e^{\omega y} (e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{-y} dy}{(e^y - 1)^2} \\ & \quad + q \int_{y > -\ln(2)} (e^{\omega y} (e^y - 1) - (e^y - 1)) \mathbb{1}_{|e^y - 1| < 1} \frac{e^{-y} dy}{(e^y - 1)^2} \\ & \quad + q \int_{-\infty}^{-\ln(2)} e^{\omega y} (e^y - 1) \underbrace{\mathbb{1}_{|e^y - 1| < 1}}_{=1} \frac{e^{-y} dy}{(e^y - 1)^2} \\ &= -\left(\ln(2) + \frac{3}{2} \right) q + \int_{-\infty}^{+\infty} \left(e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1} \right) \frac{e^{(\omega-1)y} dy}{(e^y - 1)^2} \\ & \quad + q \int_{-\ln(2)}^{\ln(2)} (e^{\omega y} - 1) \frac{e^{-y} dy}{e^y - 1} + q \int_{-\infty}^{-\ln(2)} e^{\omega y} \frac{e^{-y} dy}{e^y - 1}. \end{aligned}$$

Because $\omega = 5/2$, this has the form of $\Phi^\dagger$ of Corollary 2 in [CC06] for the symmetric Cauchy process ($\alpha = 1$ and $\rho = 1/2$), apart from a possible extra drift. We now show that the drifts do in fact coincide. Let I and J denote the last two integrals in the above expression. Using the change of variables $x = e^y$, we get

$$\begin{aligned} I &= \int_{1/2}^2 \frac{x^{5/2} - 1}{x^2(x - 1)} dx, \\ J &= \int_0^{1/2} \frac{\sqrt{x}}{x - 1} dx. \end{aligned}$$

Now

$$I = \int_{1/2}^2 \frac{x^{5/2} - x^2}{x^2(x-1)} dx + \int_{1/2}^2 \frac{x^2 - 1}{x^2(x-1)} dx = \int_{1/2}^2 \frac{\sqrt{x} - 1}{x-1} dx + \underbrace{\int_{1/2}^2 \frac{x+1}{x^2} dx}_{:=I_1},$$

and

$$J = \int_0^{1/2} \frac{\sqrt{x} - 1}{x-1} dx + \underbrace{\int_0^{1/2} \frac{1}{x-1} dx}_{:=J_1}.$$

One can check that $I_1 + J_1 = \ln(2) + \frac{3}{2}$. Therefore the linear term in the above expression of $\kappa(q + \omega) - \kappa(\omega)$ is precisely

$$a_+ = \frac{2}{\pi} \int_0^2 \frac{\sqrt{x} - 1}{x-1} dx = \frac{2}{\pi} \int_0^1 \frac{\sqrt{1+u} - 1}{u} du - \frac{2}{\pi} \int_0^1 \frac{\sqrt{1-u} - 1}{u} du,$$

which is exactly $a_+ = 2a^\dagger$ as defined in Corollary 2 of [CC06] for the symmetric Cauchy process. Note that there is a sign error in formula (17) of the latter paper. Hence Corollary 2 of [CC06] triggers that $\kappa(q + \omega) - \kappa(\omega)$ is twice the Laplace exponent of a Cauchy process conditioned to stay positive, and now by [KP21], we deduce

$$\kappa(q + \omega) - \kappa(\omega) = -2 \frac{\Gamma(\frac{1}{2} - q)\Gamma(\frac{3}{2} + q)}{\Gamma(-q)\Gamma(1 + q)}, \quad -\frac{3}{2} < q < \frac{1}{2}.$$

Taking $q = -1/2$ in this formula, one sees that $\kappa(2) - \kappa(5/2) = -\frac{2}{\pi}$. Yet one can easily compute $\kappa(2)$ from the definition of κ . Simple calculations left to the reader actually lead to $\kappa(2) = -\frac{2}{\pi}$, and thus $\kappa(5/2) = 0$. Finally, we recovered the expression of Φ^+ , and using Euler's reflection formula

$$\kappa(q) = -2 \frac{\cos(\pi q)}{\pi} \Gamma(q-1)\Gamma(3-q), \quad 1 < q < 3.$$

□

2.5 Convergence of the derivative martingale

Recall the construction of the cell system in Section 2.4 and that for $u \in \mathbb{U}$, $|u|$ denotes its generation. The collection $(\ln(|\mathcal{X}_u(0)|), u \in \mathbb{U})$ defines a branching random walk, see [Shi15] for a general reference on branching random walks. We will work under the associated filtration

$$\mathcal{G}_n := \sigma(\mathcal{X}_u, |u| \leq n), \quad n \geq 0.$$

Observe that any initial size of the form $\mathcal{X}_u(0)$ with $|u| = n+1$ comes as a jump of some $-\mathcal{X}_v$, where $|v| = n$, and is therefore $\mathcal{G}_n$ -measurable. By construction, with the notation of Theorem 2.3.6, one has for all suitable measurable function f such that $f(0) = 0$, under γ_z ,

$$\sum_{|u|=1} f(\mathcal{X}_u(0)) = \sum_{a \geq 0} f\left(ze^{\xi(a)} - ze^{\xi(a-)}\right).$$

From there, one can check by computations (making use of the expression of the cumulant function found in (2.4.3)) that

$$\gamma_z \left[\sum_{|u|=1} |\mathcal{X}_u(0)|^2 \right] = z^2, \quad \gamma_z \left[\sum_{|u|=1} |\mathcal{X}_u(0)|^2 \ln(|\mathcal{X}_u(0)|) \right] = 0$$

which implies that the martingale $\mathcal{M}(n) := \sum_{|u|=n+1} |\mathcal{X}_u(0)|^2$ is the critical martingale for the branching random walk. In this case, $\mathcal{M}(n)$ converges to 0. In order to have a non-trivial limit, one needs to consider the so-called derivative martingale defined by

$$\mathcal{D}(n) := - \sum_{|u|=n+1} \ln(|\mathcal{X}_u(0)|) |\mathcal{X}_u(0)|^2, \quad n \geq 0.$$

Remark that the temporal analogue of $\mathcal{D}(n)$ would be $\mathcal{D}_a := - \sum_{i \geq 1} \ln(|\Delta e_i^{a,+}|) |\Delta e_i^{a,+}|^2$, which is not a martingale. Indeed, the many-to-one formula yields $-z^{-2} \gamma_z[\mathcal{D}_a] = E_z[\ln |Y_a|]$, where Y is a Cauchy process, and this quantity is not constant in a . The aim of this section is to show that the limit of $\mathcal{D}(n)$ is twice the duration of the excursion.

First notice that the duration of the excursion is measurable with respect to the cell system, or equivalently to the growth-fragmentation $\bar{\mathbf{X}}$. Indeed, the number of excursions above level a with height greater than $\varepsilon > 0$ is measurable with respect to $\bar{\mathbf{X}}$, hence the local time of the excursion at level a is also measurable, and so is the total duration of the excursion by the occupation times formula. Then, by Lévy's martingale convergence theorem, one would only have to show that the derivative martingale is the conditional expectation of the duration with respect to the filtration $\mathcal{G}_n$. But the duration is not integrable, hence this strategy cannot work. Instead, one needs to use a truncation procedure introduced in [BK04].

Let $C > 0$ and denote by $\mathbb{U}^{(C)}$ the set of labels obtained from $\mathbb{U}$ by killing all the cells (with their progeny) when their size is larger than C in absolute value.

Lemma 2.5.1. *Let $z \neq 0$ and*

$$T_C := \int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| < C\}} dt,$$

be the amount of time spent by excursions with size between $-C$ and C . Then

$$\gamma_z(T_C) = \pi z^2 \mathcal{R}_C(z/2) \mathbf{1}_{\{|z| < C\}}$$

where $\mathcal{R}_C(z) := -\frac{1}{2\pi} \left(\ln \left| \frac{\sqrt{(1+\tilde{z})/(1-\tilde{z})} - 1}{\sqrt{(1+\tilde{z})/(1-\tilde{z})} + 1} \right| - \ln \left| \frac{\sqrt{(1+\tilde{z})/(1-\tilde{z})} + 1}{\sqrt{(1+\tilde{z})/(1-\tilde{z})} - 1} \right| \right)$, with $\tilde{z} = \frac{2z}{C}$, is the Green function at 0 of the Cauchy process in $(-\frac{C}{2}, \frac{C}{2})$, see [BGR61].

Proof. Lemma 2.5.1 follows from an application of Bismut's description of $\mathbf{n}_+$. Indeed, if $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is a nonnegative measurable function, then by Proposition 2.2.6,

$$\begin{aligned} \mathbf{n}_+ \left(\int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| < C\}} f(x(R(u))) dt \right) \\ = \int_0^\infty da \mathbb{E} \left[\mathbf{1}_{\{\forall 0 \leq b \leq a, |X'(T'_b) - X(T_b)| < C\}} f(X'(T'_a) - X(T_a)) \right] \end{aligned}$$

where under $\mathbb{P}$, (X, Y) and (X', Y') are independent planar Brownian motions starting from 0, and T_{-a} and T'_{-a} denote their respective hitting times of $\{\Im(z) = -a\}$. Observe that $a \mapsto X'(T'_a) - X(T_a)$ is the double of a Cauchy process (see for instance Proposition 3.11 of [RY99], Chap. III). By definition of the Green's function, we get

$$\mathbf{n}_+ \left(\int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| < C\}} f(x(R(u))) dt \right) = \int_{-C/2}^{C/2} f(2z) \mathcal{R}_C(z) dz.$$

We deduce that

$$\mathbf{n}_+(T_C f(x(R(u)))) = \frac{1}{2} \int_{-C}^C f(z) \mathcal{R}_C(z/2) dz = \int_{-C}^C \frac{f(z)}{2\pi z^2} \pi z^2 \mathcal{R}_C(z/2) dz.$$

Therefore, by disintegration over $x(R(u))$, for Lebesgue-almost every z , we get

$$\gamma_z(T_C) = \pi z^2 \mathcal{R}_C(z/2) \mathbf{1}_{\{|z| < C\}}.$$

The fact that it holds for all z is obtained through the usual continuity arguments. $\square$

Let

$$\mathcal{D}^{(C)}(n) := \pi \sum_{u \in \mathbb{U}^{(C)}: |u|=n+1} \mathcal{R}_C(\mathcal{X}_u(0)/2) |\mathcal{X}_u(0)|^2, \quad n \geq 0.$$

Corollary 2.5.2. *The following identity holds for all $n \geq 0$ and $z \neq 0$:*

$$\gamma_z(T_C \mid \mathcal{G}_n) = \mathcal{D}^{(C)}(n).$$

Consequently, $(\mathcal{D}^{(C)}(n), n \geq 0)$ is a uniformly integrable $(\mathcal{G}_n)_{n \geq 0}$ -martingale.

Proof. For simplicity, we prove this for $n = 0$: the identity then follows by induction from the branching property. Splitting the integral over the children of Ξ and using Theorem 2.3.8, we have for $|z| < C$,

$$\gamma_z \left(\int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| \leq C\}} dt \mid \mathcal{G}_1 \right) = \sum_{i \geq 1} \gamma_{\Delta e_i} \left(\int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| \leq C\}} dt \right),$$

where the $e_i, i \geq 1$ denote the excursions created by the jumps of Ξ . Now, using Lemma 2.5.1, we immediately get

$$\gamma_z \left(\int_0^{R(u)} \mathbf{1}_{\{\forall 0 \leq b \leq y(t), |\Delta e_b^{(t)}| \leq C\}} dt \mid \mathcal{G}_1 \right) = \sum_{i \geq 1} \mathcal{R}_C(\Delta e_i/2) \pi |\Delta e_i|^2 \mathbf{1}_{\{|\Delta e_i| < C\}},$$

which is the desired equality. $\square$

Theorem 2.5.3. *Under γ_z , the derivative martingale $(\mathcal{D}(n), n \geq 0)$ converges almost surely towards twice the duration $R(u)$ of the Brownian excursion.*

Proof. The proof is standard in the branching random walk literature, see [BK04] or [Shi15]. It suffices to prove it on the event where all excursions have size smaller than C , this for any $C > 0$. Let then $C > 0$ and suppose the corresponding event holds. Using that $\mathcal{R}_C(z) = -\frac{1}{2\pi} \ln(z) + O_z(1)$ when $z \rightarrow 0$ and that the martingale $\mathcal{M}(n)$ converges to 0, we get that

$$\mathcal{D}^{(C)}(n) \underset{n \rightarrow \infty}{\sim} \frac{1}{2} \mathcal{D}(n).$$

Since on our event, $T_C = R(u)$, Lévy's martingale convergence theorem together with Corollary 2.5.2 imply the result. $\square$

Chapter 3

Self-similar signed growth-fragmentations

Abstract

This chapter presents the results obtained in [Sil21]. The aim of this paper is twofold. First, we extend the theory of growth-fragmentation processes to allow signed mass. Among others, we introduce genealogical martingales and establish a spinal decomposition for the associated cell system, following [BBCK18]. Then, we study a particular family of such self-similar signed growth-fragmentation processes which arise when cutting half-planar excursions at horizontal levels. When restricting this process to the positive masses, we recover part of the family introduced by Bertoin, Budd, Curien and Kortchemski in [BBCK18].

3.1 Introduction

Markovian growth-fragmentation processes were first introduced in [Ber17b]. They describe a system of positive masses, starting from a unique cell called the Eve cell, which may evolve over time, and suddenly split into a mother cell and a daughter cell. This happens with conservation of mass at splittings: the total of the mother and daughter masses after dislocation is equal to the mass of the mother cell right before. The latter daughter cells then evolve independently one of the others, with the same stochastic evolution as the mother cell, thereafter dividing in the same way, and giving birth to granddaughter cells, and so on. Thus, newborn particles arise according to the *negative* jumps of the mass of the mother cell.

Self-similar growth-fragmentation processes form a rich subclass of such models and have been extensively studied in the seminal article [BBCK18]. Natural genealogical martingales in particular arise from the so-called additive martingales in the branching random walk setting (see the Lecture notes [Shi15]). These martingales depend on exponents which can be found as the roots of the growth-fragmentation *cumulant function* κ . Performing the corresponding change of measure, [BBCK18] then completely describes the *spinal decomposition* of the growth-fragmentation cell system. Under the new tilted probability measure, all the cells roughly behave as in the original cell system, except for the Eve cell, which behaves as some tweaked version of it.

The article [BBCK18] further introduces a remarkable family of such processes that are closely linked to θ -stable Lévy processes for $\frac{1}{2} < \theta \leq \frac{3}{2}$, and relates them to the scaling limit of the exploration of a Boltzmann planar map. The case $\theta = \frac{3}{2}$ was later recovered up to a time-change in [LGR20] when studying, among others, the boundary sizes of superlevel sets

of Brownian motion indexed by the Brownian tree, whereas the critical case $\theta = 1$ appears in [AS20] when cutting a half-planar Brownian excursion at horizontal levels. The growth-fragmentation processes associated with parameters $\theta \in (1, \frac{3}{2})$ were also retrieved directly in the continuum in [MSW20] by exploring a conformal loop ensemble on an independent LQG surface.

In [AS20], the masses of the cells correspond to the sizes of the excursions above horizontal levels, defined as the difference between the endpoint and the starting point. Note that, for these to fall in the growth-fragmentation framework, one has to remove all the negative excursions of the system. Moreover, a Boltzmann planar map can be seen as the gasket of a loop $O(n)$ model (see [LGM11]), and from this standpoint, a positive jump in the growth-fragmentation represent the discovery of a loop which has not yet been explored.

The present work extends the study in [BBCK18] to the case when the masses may be *signed*. We therefore allow positive jumps to be birth events and give rise to negative cells, so that the conservation rule still holds at splittings. Markov additive processes and the Lamperti-Kiu representation provide a very natural framework for this.

We illustrate this with the following examples, which are a slight generalisation of the growth-fragmentation embedded in the half-planar Brownian excursions. For $z_0 > 0$, we consider a half-planar excursion from 0 to z_0 , where the imaginary part Y is a one-dimensional Brownian excursion, but the real part X^α is some instance of an α -stable process, with $\alpha \in (1, 2)$. If $a > 0$ and the excursion hits the horizontal level $\{z \in \mathbb{C}, \Im(z) = a\}$ at height a , it will make a countable number of excursions $(e_i^{a,+}, i \geq 1)$ above this level. We record the *sizes* $(\Delta e_i^{a,+}, i \geq 1)$ of these excursions, defined as the difference between the endpoint and the starting point. This is a collection of real numbers which exhibits a branching structure as the height a evolves. Our main result in this direction is that this collection of sizes behaves as a signed self-similar growth-fragmentation process. Moreover, we show that X^α can be tailored so that, when removing the negative excursions in the system, the previous construction gives back the family of growth-fragmentation processes introduced in [BBCK18] for $\frac{1}{2} < \theta < 1$. In this case, X^α is simply a *spectrally negative* α -stable Lévy process.

Related work. In the pure fragmentation framework, multitype self-similar fragmentation processes have been introduced and their structure described, in terms of the underlying Markov additive process, in [Ste18b].

The paper is organised as follows. In Section 3.2, we provide some background on real-valued self-similar Markov processes and their Lamperti-Kiu representation. In Section 3.3, we make use of a connection with multitype branching random walks to introduce genealogical martingales similar to the ones in [BBCK18]. Section 3.4 is devoted to proving that the form of these martingales only depend on the growth-fragmentation itself. Along the way, we will define *signed cumulant functions* that are the analogs of the cumulant function in the positive case. The spinal decomposition will be described in Section 3.5. Finally, we will investigate in Section 3.6 a distinguished family of self-similar signed growth-fragmentations constructed by cutting half-planar excursions at horizontal levels.

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3.2 Real-valued self-similar Markov processes

We first recall some aspects of the Lamperti-Kiu theory for real-valued self-similar Markov processes. The Lamperti representation [Lam72] reveals a correspondence between positive self-similar Markov processes and Lévy processes. In the real-valued case, there is a more general correspondence, called the Lamperti-Kiu representation, between self-similar Markov processes and Markov additive processes, which are needed to take into account the sign changes.

3.2.1 Markov additive processes

Let E be a finite set, endowed with the discrete topology, and $(\mathcal{G}_t)_{t \geq 0}$ a standard filtration. A *Markov additive process* (MAP) with respect to $(\mathcal{G}_t)_{t \geq 0}$ is a càdlàg process (ξ, J) in $\mathbb{R} \times E$ with law $\mathbb{P}$, such that $(J(t), t \geq 0)$ is a continuous-time Markov chain, and the following property holds: for all $i \in E$, $t \geq 0$,

Conditionally on $J(t) = i$, the process $(\xi(t+s) - \xi(t), J(t+s))_{s \geq 0}$ is independent of $\mathcal{G}_t$ and is distributed as $(\xi(s) - \xi(0), J(s))_{s \geq 0}$ given $J(0) = i$.

We shall write $\mathbb{P}_i := \mathbb{P}(\cdot \mid \xi(0) = 0 \text{ and } J(0) = i)$ for $i \in E$. Details on MAPs can be found in [Asm08]. In particular, their structure is known to be given by the following proposition.

Proposition 3.2.1. *The process (ξ, J) is a Markov additive process if, and only if, there exist independent sequences $(\xi_i^n, n \geq 0)_{i \in E}$ and $(U_{i,j}^n, n \geq 0)_{i,j \in E}$, all independent of J , such that:*

- for $i \in E$, $(\xi_i^n, n \geq 0)$ is a sequence of i.i.d. Lévy processes,
- for $i, j \in E$, $(U_{i,j}^n, n \geq 0)$ are i.i.d.,
- if $(T_n)_{n \geq 0}$ denotes the sequence of jump times of the chain J (with the convention $T_0 = 0$), then for all $n \geq 0$,

$$\xi(t) = \left(\xi(T_n^-) + U_{J(T_n^-), J(T_n)}^n \right) \mathbf{1}_{n \geq 1} + \xi_{J(T_n)}^n(t - T_n), \quad T_n \leq t < T_{n+1}. \quad (3.2.1)$$

Proposition 3.2.1 describes $(\xi(t), t \geq 0)$ as a concatenation of independent Lévy processes with law depending on the current state of J , with additional random jumps occurring whenever the chain J has a jump.

We now turn to defining the *matrix exponent* of a MAP, which replaces the Laplace exponent in the setting of Lévy processes. For simplicity, we assume that $E = \{1, \dots, N\}$ and that J is irreducible. We write $Q = (q_{i,j})_{1 \leq i,j \leq N}$ for its intensity matrix. Also, we denote for all $i, j \in E$, all on the same probability space, by ξ_i a Lévy process distributed as the ξ_i^n 's, and by $U_{i,j}$ a random variable distributed as the $U_{i,j}^n$'s, with the convention $U_{i,i} = 0$ and $U_{i,j} = 0$ if $q_{i,j} = 0$. Finally, we introduce the Laplace exponent ψ_i of ξ_i and the Laplace transform $G_{i,j}(z) := \mathbb{E}[e^{zU_{i,j}}]$ of $U_{i,j}$ (this defines a matrix $G(z)$ with entries $G_{i,j}(z)$). Then the matrix exponent F of (ξ, J) is defined as

$$F(z) := \text{diag}(\psi_1(z), \dots, \psi_N(z)) + Q \circ G(z), \quad (3.2.2)$$

where $\circ$ denotes pointwise multiplication of the entries. Then the following equality holds for all $i, j \in E$, $z \in \mathbb{C}$, $t \geq 0$, whenever one side of the equality is defined:

$$\mathbb{E}_i \left[e^{z\xi(t)} \mathbf{1}_{J(t)=j} \right] = (e^{F(z)t})_{i,j}.$$

3.2.2 The Lamperti-Kiu representation

In [Lam72], Lamperti proved that positive self-similar Markov processes can be expressed as the exponential of a time-changed Lévy process. In the real-valued case, one has to track the sign changes, but the same kind of representation holds. Let X be a real-valued Markov process, which under $\mathbb{P}_z$ starts from $z \neq 0$, and denote by T_0 its first hitting time of 0. We assume that X is *self-similar with index α* in the following sense: for any $c > 0$ and for all $z \neq 0$, the law of $(cX(c^{-\alpha}t), t \geq 0)$ under $\mathbb{P}_z$ is $\mathbb{P}_{cz}$. The next theorem summarises the main result of [CPR13]. Though it may appear intricate at first glance, we insist that the gist of it is intrinsically simple. It should be streamlined as follows. As long as X remains positive (*resp.* negative), it evolves as the exponential (*resp.* minus the exponential) of a time-changed Lévy process. The Lévy processes keeping track of the positive and negative parts must not necessarily be equal. In addition, an exponential clock (modulo time-change) rings every time the sign of X changes, and at these times a special jump occurs (again, the two exponential clocks and the law of the jumps may be different depending on the current sign of X).

Theorem 3.2.2. (*Lamperti-Kiu representation, [CPR13]*)

There exist independent sequences $(\xi^{\pm,k})_{k \geq 0}$, $(\zeta^{\pm,k})_{k \geq 0}$, $(U^{\pm,k})_{k \geq 0}$ of i.i.d. variables fulfilling the following properties:

- (i) *The $\xi^{\pm,k}$ are Lévy processes, the $\zeta^{\pm,k}$ are exponential random variables with parameter $q_{\pm}$, and the $U^{\pm,k}$ are real-valued random variables.*
- (ii) *For $z \neq 0$ and $k \geq 0$, if we define:*

$$\begin{aligned} \bullet \quad (\xi^{z,k}, \zeta^{z,k}, U^{z,k}) &:= \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}) & \text{if } \text{sgn}(z)(-1)^k = 1 \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}) & \text{if } \text{sgn}(z)(-1)^k = -1 \end{cases} \\ \bullet \quad \mathcal{T}_0^{(z)} &:= 0 \text{ and } \mathcal{T}_n^{(z)} := \sum_{k=0}^{n-1} \zeta^{(z,k)} \text{ for } n \geq 1 \\ \bullet \quad N_t^{(z)} &:= \max\{n \geq 0, \mathcal{T}_n^{(z)} \leq t\} \text{ and } \sigma_t^{(z)} := t - \mathcal{T}_{N_t^{(z)}}^{(z)}, \end{aligned}$$

then, under $\mathbb{P}_z$, X has the representation:

$$X(t) = z \exp(\mathcal{E}_{\tau(t)}^{(z)}), \quad 0 \leq t < T_0,$$

where

$$\mathcal{E}_t^{(z)} := \xi_{\sigma_t^{(z)}}^{N_t^{(z)}} + \sum_{k=0}^{N_t^{(z)}-1} \left(\xi_{\zeta^{(z,k)}}^{(z,k)} + U^{(z,k)} \right) + i\pi N_t^{(z)}, \quad t \geq 0,$$

and

$$\tau(t) := \inf \left\{ s > 0, \int_0^s |\exp(\alpha \mathcal{E}_u^{(z)})| du > t|z|^{-\alpha} \right\}, \quad t < T_0.$$

Conversely, any process of this form is a self-similar Markov process with index α .

This can be rephrased in the language of Markov additive processes, as was pointed out in [KKPW14].

Proposition 3.2.3. *Let X be a real-valued self-similar Markov process, with Lamperti-Kiu exponent $\mathcal{E}$. Introduce for $z \neq 0$,*

$$(\xi^{(z)}(t), J^{(z)}(t)) := \left(\Re(\mathcal{E}_t^{(z)}), \left[\frac{\Im(\mathcal{E}_t^{(z)})}{\pi} + \mathbf{1}_{z>0} \right] \right), \quad t \geq 0,$$

where $[\cdot]$ denotes reduction modulo 2.

Then $(\xi^{(z)}(t), J^{(z)}(t))$ is a MAP with state space $\{0, 1\}$ and under $\mathbb{P}_z$ for all $z \neq 0$,

$$X(t) = z \exp\left(\xi^{(z)}(\tau(t)) + i\pi(J^{(z)}(\tau(t)) + 1)\right), \quad 0 \leq t < T_0,$$

where, in terms of $\xi^{(z)}$,

$$\tau(t) := \inf\left\{s > 0, \int_0^s \exp(\alpha \xi^{(z)}(u)) du > t|z|^{-\alpha}\right\}, \quad t < T_0.$$

3.3 Martingales in self-similar growth-fragmentation with signs

We follow closely [Ber17b] and [BBCK18] to extend the construction to real-valued driving processes. Let X be a càdlàg Feller process which is self-similar in the sense of Section 3.2.2, with values in $\mathbb{R}^*$. Denote by $\mathbb{P}_z$ the law of X started from $z \neq 0$, and assume that X is either absorbed at a cemetery point ∂ after a finite time ζ or converges (to 0) as $t \rightarrow \infty$ under $\mathbb{P}_z$ for all z . Introduce the MAP (ξ, J) associated to X via the Lamperti-Kiu representation in Proposition 3.2.3, and denote by F its matrix exponent. Recall that this matrix exponent is determined by the law of the Lévy processes $\xi_{\pm}$, special jumps $U_{\pm}$, and random clocks $\zeta_{\pm}$ (which are exponential with parameter $q_{\pm}$) dealing with the parts of the trajectory where X is positive or negative. Recall also the notation $\mathbb{P}_{\pm}$ to denote the law of X starting from ± 1 (and $\mathbb{E}_{\pm}$ for the corresponding expectation). We further write $\Delta X(r) = X(r) - X(r^-)$ for the jump of X at time r .

3.3.1 Self-similar signed growth-fragmentation processes

We now explain how to construct the *cell system* driven by X . As usual, we label the cells using the Ulam tree $\mathbb{U} = \bigcup_{i=0}^{\infty} \mathbb{N}^i$, where in our notation $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{N}^0 := \{\emptyset\}$ refers to the *Eve cell*. A node $u \in \mathbb{U}$ is a list $(u_1, \dots, u_i)$ of positive integers where $|u| = i$ is the *generation* of u . Then the offspring of u is labelled by the lists $(u_1, \dots, u_i, k)$, with $k \in \mathbb{N}$.

We then define the cell system $(\mathcal{X}_u(t), u \in \mathbb{U})$ driven by X by recursion. Again, we repeat the procedure in [BBCK18], except that we include the positive jumps in the genealogy. First, set $b_{\emptyset} = 0$ and $\mathcal{X}_{\emptyset}$ to be distributed as X started from some mass $z \neq 0$. Then at each jump of $\mathcal{X}_{\emptyset}$, we will create a new particle with mass given by the opposite of this jump (so that the mass is conserved at each splitting). Since X converges at infinity, one can rank the sequence of jump sizes and times $(x_1, \beta_1), (x_2, \beta_2), \dots$ of $-\mathcal{X}_{\emptyset}$ by descending lexicographical order for the absolute value of the x_i . Given this sequence of jumps, we define the first generation $\mathcal{X}_i, i \in \mathbb{N}$, of our cell system as independent processes with respective law $\mathbb{P}_{x_i}$, and we set $b_i = b_{\emptyset} + \beta_i$ for the *birth time* of i and ζ_i for its lifetime. The law of the n -th generation is constructed given generations $1, \dots, n-1$ following the same procedure. Therefore, a cell $u' = (u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1}$ gives birth to the cell $u = (u_1, \dots, u_{n-1}, i)$, with lifetime ζ_u , at time $b_u = b_{u'} + \beta_i$ where β_i is the i -th jump of $\mathcal{X}_{u'}$ (in terms of the same ranking as before). Moreover, conditionally on the jump sizes and times of $\mathcal{X}_{u'}$, $\mathcal{X}_u$ has law $\mathbb{P}_y$ with $-y = \Delta \mathcal{X}_{u'}(\beta_i)$ and is independent of the other daughter cells at generation n . See Figure 3.1.

Beware that, in this construction, the cells are not labelled chronologically. Nonetheless, exactly as in [BBCK18], this uniquely defines the law $\mathcal{P}_z$ of the cell system driven by X and started from z . The cell system $(\mathcal{X}_u(t), u \in \mathbb{U})$ is meant to describe the evolution of a population of atoms u with size $\mathcal{X}_u(t)$ evolving with its age t and fragmenting in a binary way. We stress once more that the difference with [BBCK18] is that the masses can

be negative and that the genealogy also carries the positive jumps (which corresponds to negative newborn masses).

Then we may define for $t \geq 0$

$$\mathbf{X}(t) := \{\{\mathcal{X}_u(t - b_u), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\},$$

where the double brackets denote multisets. In other words, the *signed growth-fragmentation process* $\mathbf{X}(t)$ is the collection of the sizes of all the cells in the system alive at time t . We denote by $\mathbf{P}_z$ the law of $\mathbf{X}$ started from z .

Remark 3.3.1. This construction does not require X to be self-similar.

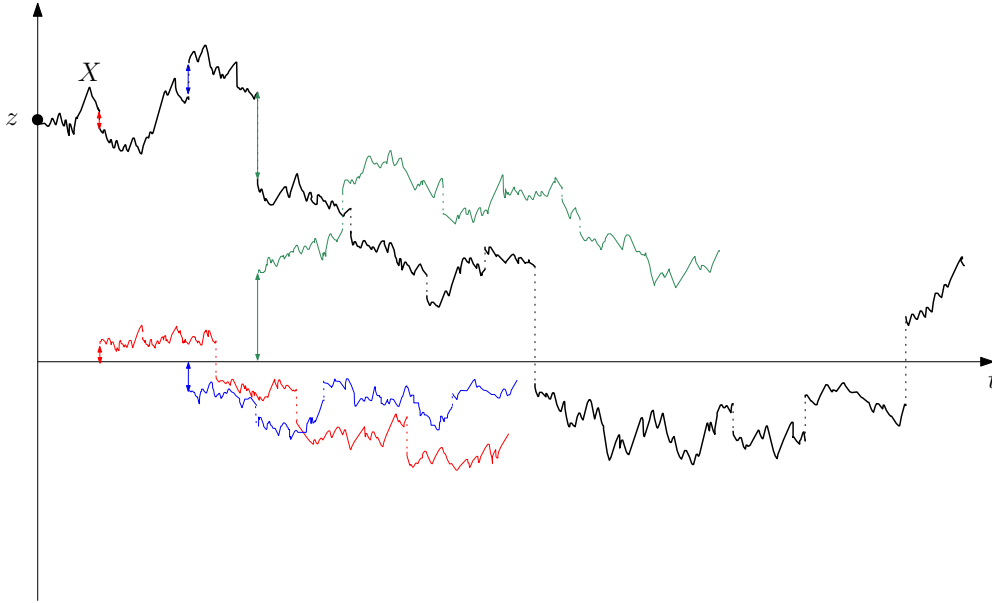


Figure 3.1 – Construction of the cell system from the driving process X . Each jump $\Delta X(s)$ of X gives birth to a new particle (in colours), with size given by the intensity $-\Delta X(s)$ of the jump. These particles, in turn, give rise to the second generation (not shown in this figure).

We now state a temporal version of the branching property. Introduce the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of $(\mathbf{X}(t), t \geq 0)$. As we shall need a stronger version of the branching property, we also record the generations by setting

$$\overline{\mathbf{X}}(t) := \{\{(\mathcal{X}_u(t - b_u), |u|), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}\}, \quad t \geq 0,$$

and we denote by $(\overline{\mathcal{F}}_t)_{t \geq 0}$ the filtration associated to $\overline{\mathbf{X}}$. We assume that $\mathbf{X}$ admits an *excessive function*, that is a measurable function $f : \mathbb{R}^* \rightarrow [0, +\infty)$ such that $\inf_{|x| > a} f(x) > 0$ for all $a > 0$, and

$$\forall z \in \mathbb{R}^*, \forall t \geq 0, \quad \mathbb{E}_z \left(\sum_{x \in \mathbf{X}(t)} f(x) \right) \leq f(z).$$

In this case, we may rank the elements $X_1(t), X_2(t), \dots$ of $\mathbf{X}(t)$ in non-increasing order of their absolute value. For self-similar processes, the existence of such excessive functions will result from the assumptions that we will make later on. Then we have the following branching property, analogous to Lemma 3.2 in [BBCK18].

Theorem 3.3.2. (*Branching property of $\bar{\mathbf{X}}$*)

For every $t \geq 0$, conditionally on $\bar{\mathbf{X}}(t) = \{\{(x_i, n_i), i \geq 1\}\}$, the process $(\bar{\mathbf{X}}(t+s), s \geq 0)$ is independent of $\bar{\mathcal{F}}_t$ and is distributed as

$$\bigsqcup_{i \geq 1} \bar{\mathbf{X}}_i(s) \circ \theta_{n_i}, \quad s \geq 0,$$

where the $\bar{\mathbf{X}}_i$ are independent processes distributed as $\bar{\mathbf{X}}$ under $\mathbf{P}_{x_i}$, and θ_n is the shift operator $\{\{(y_j, k_j), j \geq 1\}\} \circ \theta_n := \{\{(y_j, k_j + n), j \geq 1\}\}$.

This theorem follows from the arguments given in Proposition 2 in [Ber17b] which holds as soon as $\mathbf{X}$ has an excessive function.

3.3.2 Multitype branching random walks and a genealogical martingale

We use a connection with branching random walks to exhibit a genealogical martingale as in [BBCK18]. Recall that the main difference with [BBCK18] is that one has to deal with types in the branching structure.

Multitype branching random walks. We start by recalling the main features of multitype branching random walks. Let $I = \{1, \dots, K\}$ be a set of *types* with $K \geq 1$. The branching mechanism is then governed by K random sequences of displacements $\Xi^{(k)} = (\xi_1^{(k)}, \dots, \xi_\nu^{(k)})$, and types $\eta^{(k)} = (\eta_1^{(k)}, \dots, \eta_\nu^{(k)})$, $k \in I$, where ν is also random and can be infinite. Denote by $\mathbb{P}_k$ the law of $(\Xi^{(k)}, \eta^{(k)})$. Start from a particle $\emptyset$ at position $X_\emptyset = x \in \mathbb{R}$ and with initial type $k \in I$. At time $n = 1$ this particle dies, giving birth to a random cloud of particles whose displacements from their parent and types are distributed as $\Xi^{(k)}$ and $\eta^{(k)}$ respectively. At time $n = 2$, all these particles die and give birth in the same fashion to children of their own, independently of one another and independently of the past. This construction is repeated indefinitely as long as there are particles in the system. Therefore, if $u \in \mathbb{U}$ has type $i \in I$, it gives birth at the next generation to $\nu(u)$ particles with displacements $(X_{u1}, \dots, X_{u\nu(u)})$ and types $(i_{u1}, \dots, i_{u\nu(u)})$ according to $\mathbb{P}_i$. Also, we write $[\emptyset, u]$ for the unique shortest path connecting the root to the node u , so that

$$V(u) := \sum_{v \in [\emptyset, u]} X_v,$$

is the position of particle u . We denote by $(\mathcal{F}_n)_{n \geq 0}$ the natural filtration of the multitype branching random walk, *i.e.*

$$\mathcal{F}_n := \sigma((X_u, i_u), |u| \leq n).$$

For $q \in \mathbb{R}$ define $m(q)$ to be the $K \times K$ -matrix with entries

$$m_{ij}(q) := \mathbb{E}_i \left[\sum_{|v|=1} e^{-qX_v} \mathbf{1}_{i_v=j} \right].$$

We then make the following assumption.

Assumption (A0): $\forall i, j, \quad \mathbb{P}_i(\exists 1 \leq l \leq \nu, \eta_l^{(i)} = j) > 0$.

Then the matrix $m(q)$ is positive and we may apply Perron-Frobenius theory. Let $e^{\lambda(q)}$ be its largest eigenvalue and $v(q) = (v_1(q), \dots, v_K(q))$ an associated positive eigenvector. Then we have the following result.

Theorem 3.3.3. *For any $1 \leq i \leq K$, under $\mathbb{P}_i$, the process*

$$M_n := \sum_{|u|=n} v_{i_u}(q) e^{-qV(u)-n\lambda(q)}, \quad n \geq 0,$$

is a martingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$.

Proof. $(M_n)_{n \geq 0}$ is $(\mathcal{F}_n)_{n \geq 0}$ -adapted, and for $n \geq 0$,

$$\begin{aligned} \mathbb{E}_i[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}_i \left[\sum_{|u|=n} \sum_{|w|=1} v_{i_{uw}}(q) e^{-qV(u)-qX_{uw}-(n+1)\lambda(q)} \mid \mathcal{F}_n \right] \\ &= \sum_{|u|=n} e^{-qV(u)-(n+1)\lambda(q)} \mathbb{E}_i \left[\sum_{|w|=1} v_{i_{uw}}(q) e^{-qX_{uw}} \mid \mathcal{F}_n \right]. \end{aligned} \quad (3.3.1)$$

By the branching property, for all $|u| = n$,

$$\mathbb{E}_i \left[\sum_{|w|=1} v_{i_{uw}}(q) e^{-qX_{uw}} \mid \mathcal{F}_n \right] = \mathbb{E}_{i_u} \left[\sum_{|w|=1} v_{i_w}(q) e^{-qX_w} \right].$$

Since $v(q)$ is an eigenvector for $m(q)$, this is

$$\mathbb{E}_i \left[\sum_{|w|=1} v_{i_{uw}}(q) e^{-qX_{uw}} \mid \mathcal{F}_n \right] = e^{\lambda(q)} v_{i_u}(q).$$

Coming back to (3.3.1), we get

$$\mathbb{E}_i[M_{n+1} \mid \mathcal{F}_n] = \sum_{|u|=n} v_{i_u}(q) e^{-qV(u)-n\lambda(q)} = M_n.$$

□

The genealogical martingale of self-similar signed growth-fragmentations. It is easily seen from the self-similarity of the cell processes and the branching structure of growth-fragmentations that if $\text{sgn}(x)$ denotes the sign function, the process $(-\log |\mathcal{X}_u(0)|, \text{sgn}(\mathcal{X}_u(0)))_{u \in \mathbb{U}}$ is a multitype branching random walk, where the set of types is just $I = \{+, -\}$. Define

$$\mathcal{G}_n := \sigma(\mathcal{X}_u, |u| \leq n), \quad n \geq 0.$$

Note that in this setting, for any $u \in \mathbb{U}$ with $|u| = n \geq 1$, $\mathcal{X}_u(0)$ is $\mathcal{G}_{n-1}$ -measurable. For $q \in \mathbb{R}$, $m(q)$ is now a 2×2 -matrix with entries

$$m_{ij}(q) = \mathbb{E}_i \left[\sum_{s>0} |\Delta X(s)|^q \mathbf{1}_{\text{sgn}(-\Delta X(s))=j} \right]. \quad (3.3.2)$$

We work under the following assumption, analogous to Assumption (A0).

Assumption (A): *The process X admits positive and negative jumps.*

Again, for q such that $m(q)$ is finite, we denote by $e^{\lambda(q)}$ the Perron-Frobenius eigenvalue of m and by $v(q)$ an associated positive eigenvector. We make the following additional assumption.

Assumption (B): *There exists $\omega \in \mathbb{R}$ such that $\lambda(\omega) = 0$.*

We say that (v_+, v_-, ω) is *admissible* for X if $m(\omega)$ has Perron-Frobenius eigenvalue 1 (i.e. assumption (B) is satisfied) and (v_+, v_-) is an associated positive eigenvector. Then theorem 3.3.3 translates to

Theorem 3.3.4. For $u \in \mathbb{U}$, write $v_u := v_{sgn(\mathcal{X}_u(0))}(\omega)$ for simplicity. For any $z \neq 0$, the process

$$\mathcal{M}(n) := \sum_{|u|=n+1} v_u |\mathcal{X}_u(0)|^\omega, \quad n \geq 0,$$

is a $(\mathcal{G}_n)_{n \geq 0}$ -martingale under $\mathcal{P}_z$.

Assumption (B) will be studied later on in section 3.4.2. We conclude this paragraph by a very simple but typical calculation leading to a first temporal martingale for X .

Proposition 3.3.5. Under $\mathbb{P}_z$, the process

$$M(s) := v_{sgn(X(s))}(\omega) |X(s)|^\omega + \sum_{0 < r \leq s \wedge \zeta} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega,$$

is a uniformly integrable martingale for the natural filtration $(F_t^X)_{t \geq 0}$ of X , with terminal value $\sum_{r > 0} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega$.

Proof. X is clearly adapted to the filtration. Let us prove that for $s \geq 0$,

$$\mathbb{E}_i \left[\sum_{r > 0} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \middle| F_s^X \right] = v_{sgn(X(s))}(\omega) |X(s)|^\omega + \sum_{0 < r \leq s \wedge \zeta} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega.$$

Indeed,

$$\begin{aligned} & \mathbb{E}_i \left[\sum_{r > 0} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \middle| F_s^X \right] \\ &= \mathbb{E}_i \left[\sum_{r > s} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \middle| F_s^X \right] + \sum_{0 \leq r \leq s \wedge \zeta} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega. \end{aligned} \quad (3.3.3)$$

Then by the Markov property at time s and self-similarity of X , the first term is

$$\mathbb{E}_i \left[\sum_{r > s} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \middle| F_s^X \right] = |X(s)|^\omega \mathbb{E}_{sgn(X(s))} \left(\sum_{r > 0} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \right).$$

By definition of $(v_+(\omega), v_-(\omega))$,

$$\mathbb{E}_{sgn(X(s))} \left(\sum_{r > 0} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \right) = v_{sgn(X(s))}(\omega),$$

and so

$$\mathbb{E}_i \left[\sum_{r > s} v_{sgn(-\Delta X(r))}(\omega) |\Delta X(r)|^\omega \middle| F_s^X \right] = v_{sgn(X(s))}(\omega) |X(s)|^\omega.$$

Finally, equation (3.3.3) gives the desired result. $\square$

Remark 3.3.6. In particular, this implies that the quantity $f(x) = v_{sgn(x)}(\omega) |x|^\omega$ defines an excessive function for the signed growth-fragmentation $\mathbf{X}$. See [Ber17b], Theorem 1, which can be extended in the signed case.

3.3.3 A change of measure

We first define a new probability measure $\widehat{\mathcal{P}}_z$ for $z \neq 0$ thanks to the martingale $(\mathcal{M}(n))_{n \geq 0}$ in Theorem 3.3.4. This new measure is the analog of the one defined in Section 4.1 in [BBCK18]. It describes the law of a new cell system $(\mathcal{X}_u)_{u \in \mathbb{U}}$ together with an infinite *ray*, or *leaf*, $\mathcal{L} \in \partial \mathbb{U} = \mathbb{N}^{\mathbb{N}}$. On $\mathcal{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative with respect to $\mathcal{P}_z$ given by $\mathcal{M}(n)$, normalized so that we get a probability measure, *i.e.* for all $G_n \in \mathcal{G}_n$,

$$\widehat{\mathcal{P}}_z(G_n) := \frac{1}{v_{sgn(z)}(\omega)|z|^\omega} \mathcal{E}_z(\mathcal{M}(n)\mathbb{1}_{G_n}).$$

Moreover, the law of the particular leaf $\mathcal{L}$ under $\widehat{\mathcal{P}}_z$ is determined for all $n \geq 0$ and all $u \in \mathbb{U}$ such that $|u| = n + 1$ by

$$\widehat{\mathcal{P}}_z(\mathcal{L}(n+1) = u \mid \mathcal{G}_n) := \frac{v_u |\mathcal{X}_u(0)|^\omega}{\mathcal{M}(n)},$$

where for any $\ell \in \partial \mathbb{U}$, $\ell(n)$ denotes the ancestor of ℓ at generation n . In other words, to define the next generation of the spine, we select one of its jumps proportionally to its size to the power ω (the spine at generation 0 being the Eve cell). One can check from the martingale property and the branching structure of the system that these definitions are compatible, and therefore this defines a probability measure by an application of the Kolmogorov extension theorem.

We now introduce the *tagged cell*, that is the size of the cell associated to the leaf $\mathcal{L}$. First, we write $b_\ell = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial \mathbb{U}$. Then, we define $\widehat{\mathcal{X}}$ by $\widehat{\mathcal{X}}(t) := \partial$ if $t \geq b_\mathcal{L}$ and

$$\widehat{\mathcal{X}}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}), \quad t < b_\mathcal{L},$$

where n_t is the unique integer n such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$.

Observe from the definition of $\widehat{\mathcal{P}}_z$ that we have for all nonnegative measurable function f and all $\mathcal{G}_n$ -measurable nonnegative random variable B_n ,

$$v_{sgn(z)}(\omega)|z|^\omega \widehat{E}_z(f(\mathcal{X}_{\mathcal{L}(n+1)}(0))B_n) = \mathcal{E}_z\left(\sum_{|u|=n+1} v_u |\mathcal{X}_u(0)|^\omega f(\mathcal{X}_u(0))B_n\right).$$

This extends to a temporal identity in the following way. Recall that we have enumerated $\mathbf{X}(t) = \{\{X_i(t), i \geq 1\}\}$, $t \geq 0$ (this is possible since, according to the remark following Proposition 3.3.5, we know that $\mathbf{X}$ has an excessive function).

Proposition 3.3.7. *For every $t \geq 0$, every nonnegative measurable function f vanishing at ∂ , and every $\overline{\mathcal{F}}_t$ -measurable nonnegative random variable B_t , we have*

$$v_{sgn(z)}(\omega)|z|^\omega \widehat{E}_z(f(\widehat{\mathcal{X}}(t))B_t) = \mathcal{E}_z\left(\sum_{i \geq 1} v_{sgn(X_i(t))}(\omega)|X_i(t)|^\omega f(X_i(t))B_t\right).$$

Proof. We reproduce the proof of [BBCK18], though the ideas are the same, because the martingale used in the change of measure is slightly different.

Let $t \geq 0$. We restrict to proving the result for B_t which is $\overline{\mathcal{F}}_t \cap \mathcal{G}_k$ -measurable for some $k \in \mathbb{N}$. First, observe that since $f(\partial) = 0$, almost surely,

$$f(\widehat{\mathcal{X}}(t))B_t \mathbb{1}_{b_{\mathcal{L}(n+1)} > t} \xrightarrow{n \rightarrow \infty} f(\widehat{\mathcal{X}}(t))B_t.$$

Therefore, by monotone convergence,

$$\widehat{E}_z(f(\widehat{\mathcal{X}}(t))B_t) = \lim_{n \rightarrow \infty} \widehat{E}_z(f(\widehat{\mathcal{X}}(t))B_t \mathbb{1}_{b_{\mathcal{L}(n+1)} > t}).$$

Now if we condition on $\mathcal{G}_n$ and decompose $\mathcal{L}(n+1)$ over the cells at generation $n+1$, provided $n > k$ so that B_t is $\mathcal{G}_n$ -measurable, we get

$$\hat{E}_z\left(f(\hat{\mathcal{X}}(t))B_t\mathbb{1}_{b_{\mathcal{L}(n+1)}>t}\right) = \frac{1}{v_{sgn(z)}(\omega)|z|^\omega}\mathcal{E}_z\left(\sum_{|u|=n+1} v_u|\mathcal{X}_u(0)|^\omega\mathbb{1}_{b_u>t}f(\mathcal{X}_{u(t)}(t-b_{u(t)}))B_t\right).$$

Here we wrote $u(t)$ for the most recent ancestor of u at time t . We now decompose the sum over the ancestor $u(t)$ at time t . This gives

$$\begin{aligned} \mathcal{E}_z\left(\sum_{|u|=n+1} v_u|\mathcal{X}_u(0)|^\omega\mathbb{1}_{b_u>t}f(\mathcal{X}_{u(t)}(t-b_{u(t)}))B_t\right) \\ = \mathcal{E}_z\left(\sum_{|u'|\leq n}\sum_{|u|=n+1} v_u|\mathcal{X}_u(0)|^\omega\mathbb{1}_{b_u>t}f(\mathcal{X}_{u'}(t-b_{u'}))B_t\mathbb{1}_{u(t)=u'}\right), \end{aligned} \quad (3.3.4)$$

and by conditioning on $\bar{\mathcal{F}}_t$ and applying the temporal branching property stated in Theorem 3.3.2,

$$\begin{aligned} \mathcal{E}_z\left(\sum_{|u|=n+1} v_u|\mathcal{X}_u(0)|^\omega\mathbb{1}_{b_u>t}f(\mathcal{X}_{u(t)}(t-b_{u(t)}))B_t\right) \\ = \mathcal{E}_z\left(\sum_{|u'|\leq n} f(\mathcal{X}_{u'}(t-b_{u'}))B_t\mathcal{E}_z\left[\sum_{|u|=n+1} v_u|\mathcal{X}_u(0)|^\omega\mathbb{1}_{b_u>t}\mathbb{1}_{u(t)=u'}\middle|\bar{\mathcal{F}}_t\right]\right) \\ = \mathcal{E}_z\left(\sum_{|u'|\leq n} f(\mathcal{X}_{u'}(t-b_{u'}))B_t\mathcal{E}_{\mathcal{X}_{u'}(t-b_{u'})}\left[\sum_{|u|=n+1-|u'|} v_{u'u}|\mathcal{X}_{u'u}(0)|^\omega\right]\mathbb{1}_{b_{u'}\leq t<b_{u'}+\zeta_{u'}}\right) \\ = \mathcal{E}_z\left(\sum_{|u'|\leq n} f(\mathcal{X}_{u'}(t-b_{u'}))B_t\mathbb{1}_{b_{u'}\leq t<b_{u'}+\zeta_{u'}}v_{sgn(\mathcal{X}_{u'}(t-b_{u'}))}(\omega)|\mathcal{X}_{u'}(t-b_{u'})|^\omega\right). \end{aligned}$$

Finally, taking $n \rightarrow \infty$ and using monotone convergence, we obtain the desired result. $\square$

Corollary 3.3.8. *The process*

$$\mathcal{M}_t := \sum_{i=1}^{\infty} v_{sgn(X_i(t))}(\omega)|X_i(t)|^\omega, \quad t \geq 0,$$

is a supermartingale with respect to $(\mathcal{F}_t, t \geq 0)$.

Proof. Proposition 3.3.7 with $f := \mathbb{1}_{x \neq \emptyset}$ gives that $\mathcal{E}_z(\mathcal{M}_t) \leq v_{sgn(z)}(\omega)|z|^\omega$ and the supermartingale property follows readily from the branching property. $\square$

3.4 Universality of $\mathcal{M}(n)$ and the signed cumulant functions

The construction in section 3.3.2 produces martingales $\mathcal{M}(n)$ depending on X . We now aim at proving that actually, these do not depend on the choice of the Eve process, in the sense that any admissible triplet (v_+, v_-, ω) for X will also lead to a martingale for any other cell process driving the same growth-fragmentation process. The strategy is as follows. First, we prove universality for all constant sign driving cell processes by defining *signed cumulant functions* which characterise the couple $(v_+/v_-, \omega)$. Then, starting from a possibly signed Eve process X , we flip it every time its sign changes and reduce to the previous case. Along the way, we extend the definition of signed cumulant functions to signed processes. Using the constant sign case, this will provide universality for all cell processes driving the same growth-fragmentation.

3.4.1 Signed cumulants and universality in the constant sign case

We introduce two key players in the study of self-similar signed growth-fragmentation processes. We focus on the case when X has no sign change: in this case, particles born with a positive mass will continue to have a positive mass until they die, and those with negative mass will remain negative. Then the law of X under $\mathbb{P}_z$ is determined by its self-similarity index α , and the Laplace exponents ψ_+ and ψ_- of the Lamperti exponents ξ_+ and ξ_- underlying X , depending on the sign of z . It is convenient to consider

$$F(q) := \begin{pmatrix} \psi_+(q) & 0 \\ 0 & \psi_-(q) \end{pmatrix}, \quad q \geq 0,$$

as the matrix exponent of X (this is the analog of (3.2.2) in the simple case when there is no sign change). In the constant sign case, because the Lamperti representation holds, it is easy to compute $\lambda(q)$ and to define *signed cumulant functions* which are analogs of the cumulants in [BBCK18].

Indeed, let us compute $\mathbb{E}_+ \left[\sum_{0 < r < \zeta} v_{\text{sgn}(-\Delta X(r))}(q) |\Delta X(r)|^q \right]$, for $q \geq 0$. We will assume that $m(q)$ is finite all the way, so that in particular $(v_+(q), v_-(q))$ is well-defined. Let Λ_+ denote the Lévy measure of ξ_+ . Since we are summing over all times, we can omit the Lamperti time-change. In addition, note that, because X is positive, the sign of any jump of X is the same as the one of the corresponding jump of ξ_+ , so that the previous expectation boils down to

$$\mathbb{E}_+ \left[\sum_{0 < r < \zeta} v_{\text{sgn}(-\Delta X(r))}(q) |\Delta X(r)|^q \right] = \mathbb{E}_+ \left[\sum_{r > 0} v_{\text{sgn}(-\Delta \xi_+(r))}(q) \left| e^{\xi_+(r)} - e^{\xi_+(r^-)} \right|^q \right].$$

From there, we can use the compensation formula for Lévy processes, *i.e.*

$$\begin{aligned} \mathbb{E}_+ \left[\sum_{0 < r < \zeta} v_{\text{sgn}(-\Delta X(r))}(q) |\Delta X(r)|^q \right] &= \mathbb{E}_+ \left[\sum_{r > 0} v_{\text{sgn}(-\Delta \xi_+(r))}(q) e^{q\xi_+(r^-)} \left| e^{\Delta \xi_+(r)} - 1 \right|^q \right] \\ &= \int_0^\infty dr \mathbb{E}_+[e^{q\xi_+(r)}] \int_{\mathbb{R}} \Lambda_+(dx) v_{-\text{sgn}(x)}(q) |e^x - 1|^q. \end{aligned}$$

Since we assumed that $m(q)$ was finite, we get that $\psi_+(q) < 0$ and $\int_{\mathbb{R}} \Lambda_+(dx) |e^x - 1|^q < \infty$. We obtain

$$\mathbb{E}_+ \left[\sum_{0 < r < \zeta} \frac{v_{\text{sgn}(-\Delta X(r))}(q)}{v_+(q)} |\Delta X(r)|^q \right] = -\frac{1}{\psi_+(q)} \int_{\mathbb{R}} \Lambda_+(dx) \frac{v_{-\text{sgn}(x)}(q)}{v_+(q)} |e^x - 1|^q.$$

Therefore, if we set

$$\mathcal{K}_+(q) = \psi_+(q) + \int_{\mathbb{R}} \Lambda_+(dx) \frac{v_{-\text{sgn}(x)}(q)}{v_+(q)} |e^x - 1|^q,$$

we see that

$$\mathbb{E}_+ \left[\sum_{0 < r < \zeta} \frac{v_{\text{sgn}(-\Delta X(r))}(q)}{v_+(q)} |\Delta X(r)|^q \right] = \begin{cases} 1 - \frac{\mathcal{K}_+(q)}{\psi_+(q)} & \text{if } \psi_+(q) < 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (3.4.1)$$

Equation (3.4.1) is reminiscent of Lemma 3 in [Ber17b].

Likewise, under symmetrical assumptions on q (or applying the previous calculations to $-X$),

$$\mathbb{E}_- \left[\sum_{0 < r < \zeta} \frac{v_{\text{sgn}(-\Delta X(r))}(q)}{v_-(q)} |\Delta X(r)|^q \right] = \begin{cases} 1 - \frac{\mathcal{K}_-(q)}{\psi_-(q)} & \text{if } \psi_-(q) < 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (3.4.2)$$

where, with obvious notations,

$$\mathcal{K}_-(q) = \psi_-(q) + \int_{\mathbb{R}} \Lambda_-(dx) \frac{v_{\text{sgn}(x)}(q)}{v_-(q)} |e^x - 1|^q.$$

Then Assumption (B) translates to $\mathcal{K}_+(\omega) = \mathcal{K}_-(\omega) = 0$. Therefore the roots of $(\mathcal{K}_+, \mathcal{K}_-)$ give rise to martingales, as explained in Theorem 3.3.4. We call $\mathcal{K}_+$ and $\mathcal{K}_-$ the *signed cumulant functions*. We will also use the term *cumulant functions* to refer to the one defined in [BBCK18]. More precisely, let $\mathbf{X}^+$ (*resp.* $\mathbf{X}^-$) be the growth-fragmentation process obtained from $\mathbf{X}$ by killing all the cells with negative mass (*resp.* positive mass) together with their progeny, under $\mathbf{P}_+$ (*resp.* $\mathbf{P}_-$). We define κ_+ and κ_- as the cumulant functions, in the sense of [BBCK18], of $\mathbf{X}^+$ and $\mathbf{X}^-$ respectively. Recall that they are given by

$$\kappa_+(q) := \psi_+(q) + \int_{-\infty}^0 \Lambda_+(dx) |e^x - 1|^q, \quad q \geq 0,$$

and

$$\kappa_-(q) := \psi_-(q) + \int_{-\infty}^0 \Lambda_-(dx) |e^x - 1|^q, \quad q \geq 0,$$

so that for instance

$$\mathcal{K}_+(q) = \kappa_+(q) + \frac{v_-(q)}{v_+(q)} \int_0^\infty \Lambda_+(dx) |e^x - 1|^q, \quad q \geq 0.$$

Moreover, it is well-known that κ_+ and κ_- are invariants of $\mathbf{X}^+$ and $\mathbf{X}^-$, and characterise them respectively (see [Shi17] for more details). We now prove the universality of $(\mathcal{M}(n))_{n \geq 0}$ in the constant sign case. Let X and X' be two driving Markov processes with constant sign defining the growth-fragmentation processes $\mathbf{X}$ and $\mathbf{X}'$ respectively. We write $m(q)$ and $m'(q)$ for the corresponding matrices introduced in (3.3.2). Recall that (v_+, v_-, ω) is said *admissible* for X if $m(\omega)$ has Perron-Frobenius eigenvalue 1 and (v_+, v_-) is an associated positive eigenvector, so that the triplet (v_+, v_-, ω) defines a martingale as explained in section 3.3.2.

Proposition 3.4.1. *Suppose that $\mathbf{X} \stackrel{\mathcal{L}}{=} \mathbf{X}'$. If (v_+, v_-, ω) is admissible for X , then it is also admissible for X' .*

Proof. By definition of $(\mathcal{K}_+, \mathcal{K}_-)$, the triplet (v_+, v_-, ω) is admissible for X if, and only if, $\mathcal{K}_+(\omega) = \mathcal{K}_-(\omega) = 0$. This, in turn, is equivalent to the system

$$\begin{cases} \frac{v_-}{v_+} = - \frac{\kappa_+(\omega)}{\int_0^\infty \Lambda_+(dx) |e^x - 1|^\omega}, \\ 0 = \kappa_-(\omega) \kappa_+(\omega) - \int_0^\infty \Lambda_+(dx) |e^x - 1|^\omega \cdot \int_0^\infty \Lambda_-(dx) |e^x - 1|^\omega. \end{cases}$$

Note that all the fractions are well defined because (v_+, v_-) is a positive eigenvector. Since κ_+ and κ_- only depend on the growth-fragmentation $\mathbf{X}$ induced by X , the result follows

if we show that $\Lambda_{+|(0,+\infty)}$ and $\Lambda_{-|(0,+\infty)}$ also depend only on $\mathbf{X}$. In fact, since $\mathbf{X}$ and $\mathbf{X}'$ share the same positive cumulant, for instance, we have for $q \geq 0$, with obvious notations

$$\kappa_+(q) = \psi_+(q) + \int_{-\infty}^0 \Lambda_+(dx) |e^x - 1|^q = \psi'_+(q) + \int_{-\infty}^0 \Lambda'_+(dx) |e^x - 1|^q.$$

Up to translation, κ_+ is a Laplace exponent, and therefore uniqueness in the Lévy-Khintchine formula triggers that $\Lambda_{+|(0,+\infty)} = \Lambda'_{+|(0,+\infty)}$. Similarly, using invariance of κ_- , one has $\Lambda_{-|(0,+\infty)} = \Lambda'_{-|(0,+\infty)}$, and this concludes the proof of Proposition 3.4.1. $\square$

Corollary 3.4.2. *Suppose that $\mathbf{X} \stackrel{\mathcal{L}}{=} \mathbf{X}'$. If (v_+, v_-, ω) is admissible for X , then under $\mathbb{P}_z$, the process*

$$M'(s) := v_{\text{sgn}(X'(s))}(\omega) |X'(s)|^\omega + \sum_{0 < r \leq s \wedge \zeta} v_{\text{sgn}(-\Delta X'(r))}(\omega) |\Delta X'(r)|^\omega,$$

is a uniformly integrable martingale for the natural filtration $(F_t^{X'})_{t \geq 0}$ of X' , with terminal value $\sum_{r > 0} \frac{v_{\text{sgn}(-\Delta X'(r))}(\omega)}{v_i(\omega)} |\Delta X'(r)|^\omega$.

3.4.2 Universality of $\mathcal{M}(n)$ in the general case

We now move from the constant sign case to the general case. To this end, we construct from any Eve cell process X a constant sign process $X^\uparrow$ driving the same growth-fragmentation. We then prove that the triplets (v_+, v_-, ω) are simultaneously admissible for X and $X^\uparrow$.

Constructing a constant sign driving process from a signed Eve cell. To study signed growth-fragmentation, it is reasonable to reduce to the constant sign case in [BBCK18]. One natural choice for this, starting from a signed Eve process X with positive mass at time 0, is to follow it until it jumps to the negatives, and then select the particle this jump creates (which has positive mass, since the jump is negative). If we continue by induction, this constructs some process $X^\uparrow$ which, under $\mathbb{P}_z$ for $z > 0$, remains positive at all times. Likewise, we can construct $X^\uparrow$ starting from a negative mass. A first observation is that the branching structure, Markov property and self-similarity of X ensure that $X^\uparrow$ is a self-similar Markov process under $\mathbb{P}_z$ for all $z \neq 0$. Moreover, it is plain that $X^\uparrow$ carries the same growth-fragmentation as X itself. In short, X and $X^\uparrow$ are two driving cell processes for the same growth-fragmentation. If we manage to make explicit the law of $X^\uparrow$ (or rather, its Lamperti exponent) in terms of X (or its Lamperti-Kiu characteristics), then we are in good shape to reduce the study to constant sign driving cell processes. Let us focus on the case when X starts from a positive mass $z > 0$, say, and recall the notation ξ_+ , q_+ and U_+ of section 3.2.1. Then the independence and stationarity of increments of the Lévy process ξ_+ imply that, at the exponential level, going from X to $X^\uparrow$ amounts to adding jumps to ξ_+ at times ζ_+ , which is exponential with parameter q_+ . Call H_1 the first time when X crosses 0. Then the intensity δ of these jumps is exactly what it takes, at the exponential level, to go from $X(H_1^-)$ to $X^\uparrow(H_1)$, i.e. $\delta = \log\left(\frac{-\Delta X(H_1)}{X(H_1^-)}\right) = \log(1 + e^{U_+})$. Therefore, the Lamperti exponent $\xi_+^\uparrow$ of $X^\uparrow$ started from $z > 0$ results in the superposition of ξ_+ and an independent compound Poisson process with rate q_+ and jumps distributed as the image $\tilde{\Lambda}_{U_+}(dx)$ of the law $\Lambda_{U_+}(dx)$ of U_+ , by the mapping $x \mapsto \log(1 + e^x)$. Hence its Laplace exponent is

$$\psi_+^\uparrow(q) = \psi_+(q) + q_+ \left(\int_{\mathbb{R}} (1 + e^x)^q \Lambda_{U_+}(dx) - 1 \right), \quad q \geq 0, \quad (3.4.3)$$

and, in particular, its Lévy measure is

$$\Lambda_+^\uparrow(dx) := \Lambda_+(dx) + q_+ \tilde{\Lambda}_{U_+}(dx). \quad (3.4.4)$$

Note that these expressions only depend on the positive characteristics of X , and this is coherent with the construction of $X^\uparrow$. The same calculations can be carried out in the case when $z < 0$, and finally, we obtain

$$\psi_-^\uparrow(q) = \psi_-(q) + q_- \left(\int_{\mathbb{R}} (1 + e^x)^q \Lambda_{U_-}(dx) - 1 \right), \quad q \geq 0. \quad (3.4.5)$$

See Figure 3.2 for a drawing of $X^\uparrow$.

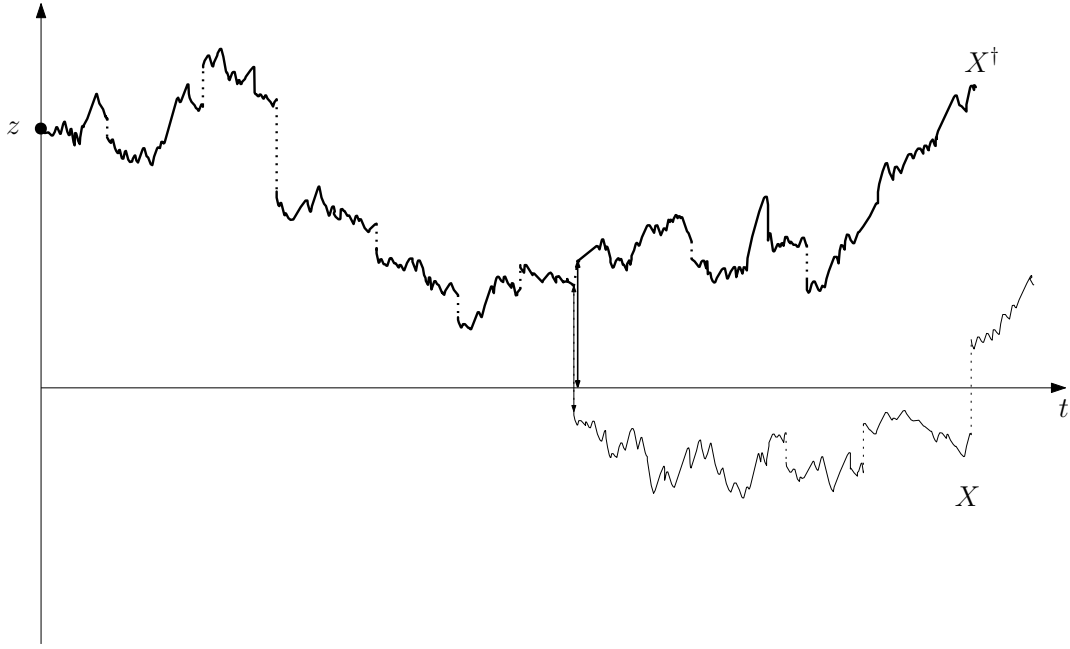


Figure 3.2 – Constructing a positive Eve cell process from X . The process $X^\uparrow$ (in bold) is constructed from X by selecting the complementary positive cell created when X jumps below 0, and then by induction.

Universality of $\mathcal{M}(n)$ and the signed cumulant functions. We want to extend the result of Proposition 3.4.1 to general signed driving processes. To do this, we resort to $X^\uparrow$ and link admissible triplets (v_+, v_-, ω) for X and $X^\uparrow$. First, we state a technical lemma, that is probably superfluous, but simplifies calculations.

Lemma 3.4.3. *The following points are equivalent:*

- (i) (v_+, v_-, ω) is admissible for X .
- (ii) The process M defined in Proposition 3.3.5 with parameters (v_+, v_-, ω) is a uniformly integrable martingale.
- (iii) Let H_1 be the first time X crosses 0. Then,

$$\mathbb{E}_\pm[M(H_1)] = v_\pm.$$

Proof. The implication (i) $\Rightarrow$ (ii) has already been proved in Proposition 3.3.5, and (ii) $\Rightarrow$ (iii) follows from an application of the optional stopping theorem and the uniform integrability

of M . Therefore only (iii) $\Rightarrow$ (i) remains to be proved. Assume that we know (iii). Denote by $H_0 = 0 < H_1 < H_2 < \dots$ the sequence of times when X crosses 0. Then

$$\mathbb{E}_+ \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_+ \left[\sum_{H_k < s \leq H_{k+1}} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right].$$

By the Markov property and the self-similarity of X , this entails

$$\begin{aligned} \mathbb{E}_+ \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] &= \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k})|^\omega] \mathbb{E}_+ \left[\sum_{0 < s \leq H_1} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] \\ &\quad + \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k+1})|^\omega] \mathbb{E}_- \left[\sum_{0 < s \leq H_1} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right]. \end{aligned}$$

Making use of (iii), this means

$$\begin{aligned} &\mathbb{E}_+ \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] \\ &= \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k})|^\omega] (v_+ - v_- \mathbb{E}_+ [|X(H_1)|^\omega]) + \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k+1})|^\omega] (v_- - v_+ \mathbb{E}_- [|X(H_1)|^\omega]). \end{aligned}$$

Using the Markov property backwards, we have $\mathbb{E}_+ [|X(H_{2k})|^\omega] \mathbb{E}_+ [|X(H_1)|^\omega] = \mathbb{E}_+ [|X(H_{2k+1})|^\omega]$, and likewise $\mathbb{E}_+ [|X(H_{2k+1})|^\omega] \mathbb{E}_- [|X(H_1)|^\omega] = \mathbb{E}_+ [|X(H_{2k+2})|^\omega]$ for all $k \geq 0$. This gives

$$\mathbb{E}_+ \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] = v_+ \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k})|^\omega] - v_+ \sum_{k \geq 0} \mathbb{E}_+ [|X(H_{2k+2})|^\omega].$$

Therefore,

$$\mathbb{E}_+ \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] = v_+.$$

Similarly,

$$\mathbb{E}_- \left[\sum_{s>0} v_{-sgn(\Delta X(s))} |\Delta X(s)|^\omega \right] = v_-,$$

and so (v_+, v_-, ω) is admissible for X . $\square$

We may now bridge the gap between X and $X^\uparrow$.

Proposition 3.4.4. *A triplet (v_+, v_-, ω) is admissible for X if, and only if, it is admissible for $X^\uparrow$.*

Proof. We use (iii) in Lemma 3.4.3 above. Define $M^\uparrow$ as the process in Proposition 3.3.5 associated with $X^\uparrow$ and with parameters (v_+, v_-, ω) . The key remark is that $M(H_1) = M^\uparrow(H_1)$ a.s. Indeed, under $\mathbb{P}_+$ say, $-\Delta X(H_1) = X^\uparrow(H_1) > 0$, and $-\Delta X^\uparrow(H_1) = X(H_1) < 0$, a.s. Therefore, (v_+, v_-, ω) is admissible for X if and only if,

$$\mathbb{E}_\pm [M^\uparrow(H_1)] = v_\pm. \quad (3.4.6)$$

It remains to prove that this is equivalent to (v_+, v_-, ω) being admissible for $X^\uparrow$. First, the optional stopping theorem gives that if (v_+, v_-, ω) is admissible for $X^\uparrow$, then (3.4.6) holds. Conversely, we can more or less run the same arguments as in the proof of (iii) $\Rightarrow$ (i) in Lemma 3.4.3. For example, if we denote by $T_0 = 0 < T_1 < T_2 < \dots$ the times corresponding

to those special jumps of $X^\uparrow$ that correspond to sign changes for X , then using the Markov property and self-similarity of $X^\uparrow$

$$\mathbb{E}_+ \left[\sum_{t>0} v_{-sgn(\Delta X^\uparrow(t))} |\Delta X^\uparrow(t)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_+ [|X^\uparrow(T_k)|^\omega] \mathbb{E}_+ \left[\sum_{0 < t \leq T_1} v_{-sgn(\Delta X^\uparrow(t))} |\Delta X^\uparrow(t)|^\omega \right]$$

By (3.4.6), this is

$$\mathbb{E}_+ \left[\sum_{t>0} v_{-sgn(\Delta X^\uparrow(t))} |\Delta X^\uparrow(t)|^\omega \right] = v_+ \sum_{k \geq 0} \mathbb{E}_+ [|X^\uparrow(T_k)|^\omega] (1 - \mathbb{E}_+ [|X^\uparrow(T_1)|^\omega]).$$

Yet by applying the Markov property backwards,

$$\sum_{k \geq 0} \mathbb{E}_+ [|X^\uparrow(T_k)|^\omega] (1 - \mathbb{E}_+ [|X^\uparrow(T_1)|^\omega]) = \sum_{k \geq 0} \mathbb{E}_+ [|X^\uparrow(T_k)|^\omega] - \sum_{k \geq 0} \mathbb{E}_+ [|X^\uparrow(T_{k+1})|^\omega] = 1.$$

Therefore

$$\mathbb{E}_+ \left[\sum_{t>0} v_{-sgn(\Delta X^\uparrow(t))} |\Delta X^\uparrow(t)|^\omega \right] = v_+,$$

and similarly

$$\mathbb{E}_- \left[\sum_{t>0} v_{-sgn(\Delta X^\uparrow(t))} |\Delta X^\uparrow(t)|^\omega \right] = v_-.$$

Thus (v_+, v_-, ω) is admissible for $X^\uparrow$. □

Proposition 3.4.4, in turn, enables us to define general *signed cumulant functions*. Recall from section 3.2.1 the notation $G_{+,-}(q) := \mathbb{E}[e^{qU^+}]$ and $G_{-,+}(q) := \mathbb{E}[e^{qU^-}]$ for the Laplace transforms of the special jumps.

Corollary 3.4.5. *Let*

$$\begin{aligned} \mathcal{K}_+(q) &= \psi_+^\uparrow(q) + \int_{\mathbb{R}} \Lambda_+^\uparrow(dx) \frac{v_{-sgn(x)}(q)}{v_+(q)} |e^x - 1|^q \\ &= \kappa_+(q) + \frac{v_-(q)}{v_+(q)} \left(\int_0^\infty \Lambda_+(dx) |e^x - 1|^q + q_+ G_{+,-}(q) \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_-(q) &= \psi_-^\uparrow(q) + \int_{\mathbb{R}} \Lambda_-^\uparrow(dx) \frac{v_{sgn(x)}(q)}{v_-(q)} |e^x - 1|^q \\ &= \kappa_-(q) + \frac{v_+(q)}{v_-(q)} \left(\int_0^\infty \Lambda_-(dx) |e^x - 1|^q + q_- G_{-,+}(q) \right), \end{aligned}$$

be the signed cumulant functions associated with $X^\uparrow$ – which we rephrased in terms of X thanks to (3.4.3) and (3.4.5). Then the suitable martingale exponents ω for X are the roots of $(\mathcal{K}_+, \mathcal{K}_-)$.

The final end to the universality of $\mathcal{M}(n)$ is provided by the next theorem.

Theorem 3.4.6. *(Universality of $\mathcal{M}(n)$)*

Let X and X' be two possibly signed cell processes, driving the same growth-fragmentation $\mathbf{X} = \mathbf{X}'$. Then (v_+, v_-, ω) is admissible for X if, and only if, it is admissible for X' .

Proof. This is a corollary of Propositions 3.4.1 and 3.4.4. We have the following equivalences: (v_+, v_-, ω) is admissible for X if and only if it is admissible for $X^\uparrow$ (Proposition 3.4.4), i.e. if and only if it is admissible for $(X')^\uparrow$ (Proposition 3.4.1), i.e. if and only if it is admissible for X' (Proposition 3.4.4). □

3.5 The spinal decomposition

This section is devoted to the study of self-similar signed growth-fragmentations under the change of measure given in section 3.3.3. In particular, we aim at describing the law of the tagged cell under $\widehat{\mathcal{P}}_z$. Roughly speaking, we shall see that by changing the measure according to section 3.3.3, the tagged cell $\widehat{\mathcal{X}}$ evolves as an explicit self-similar Markov process Y , and conditionally on its evolution, the growth-fragmentations induced by the jumps of $\widehat{\mathcal{X}}$ are independent with law $\mathbf{P}_x$ where $-x$ is the jump size.

3.5.1 Description and results

Description of the Markov process Y . We first introduce a Markov process that will describe the law of the *spine* in the next paragraph. Remember the couple of Lévy measures $(\Lambda_+^\uparrow, \Lambda_-^\uparrow)$ for the constant sign process constructed in paragraph 3.4.2. Let us set some notation and write

$$\begin{aligned}\sigma_+(q) &:= \frac{v_-(\omega)}{v_+(\omega)} \int_0^\infty |e^x - 1|^q \Lambda_+^\uparrow(dx) \\ &= \frac{v_-(\omega)}{v_+(\omega)} \left(\int_0^\infty \Lambda_+(dx) |e^x - 1|^q + q_+ G_{+,-}(q) \right), \quad q \geq 0,\end{aligned}$$

and symmetrically,

$$\begin{aligned}\sigma_-(q) &:= \frac{v_+(\omega)}{v_-(\omega)} \int_0^\infty |e^x - 1|^q \Lambda_-^\uparrow(dx) \\ &= \frac{v_+(\omega)}{v_-(\omega)} \left(\int_0^\infty \Lambda_-(dx) |e^x - 1|^q + q_- G_{-,+}(q) \right), \quad q \geq 0.\end{aligned}$$

Recall the notation $(\mathcal{K}_+, \mathcal{K}_-)$ for the signed cumulant functions and (κ_+, κ_-) for the cumulant functions (see section 3.4.1 and Corollary 3.4.5). Define the following matrix

$$\widehat{F}(q) := \begin{pmatrix} \kappa_+(\omega + q) & \sigma_+(\omega + q) \\ \sigma_-(\omega + q) & \kappa_-(\omega + q) \end{pmatrix}, \quad q \geq 0.$$

Lemma 3.5.1. *Let $(\widehat{\xi}_+, \widehat{\xi}_-)$ be a pair of independent Lévy processes with Laplace exponents*

$$\widehat{\psi}_+(q) := \kappa_+(\omega + q) - \kappa_+(\omega), \quad q \geq 0,$$

and

$$\widehat{\psi}_-(q) := \kappa_-(\omega + q) - \kappa_-(\omega), \quad q \geq 0.$$

Furthermore, let $\widehat{q}_\pm := \sigma_\pm(\omega)$, and $(\widehat{U}_{+,-}, \widehat{U}_{-,+})$ be a pair of random variables with respective Laplace transforms $G_{+,-}(q) := \frac{\sigma_+(\omega+q)}{\sigma_+(\omega)}$ and $G_{-,+}(q) := \frac{\sigma_-(\omega+q)}{\sigma_-(\omega)}$ for $q \geq 0$. Then the Markov additive process $(\widehat{\xi}, \widehat{J})$ defined piecewise as in (3.2.1) with these characteristics has matrix exponent $\widehat{F}$.

Remark 3.5.2. Note that for instance

$$\kappa_+(\omega + q) - \kappa_+(\omega) = \psi_+^\uparrow(\omega + q) - \psi_+^\uparrow(\omega) + \int_{(-\infty, 0)} \left((1 - e^x)^{\omega+q} - (1 - e^x)^\omega \right) \Lambda_+^\uparrow(dx).$$

Therefore $\widehat{\xi}_+$ can be obtained by the Lévy-Itô decomposition as a superposition of a Lévy process η_+ with Laplace exponent $q \mapsto \psi_+^\uparrow(\omega + q) - \psi_+^\uparrow(\omega)$, and a compound Poisson process ν_+ with Lévy measure $e^{\omega x} \widetilde{\Lambda}_+|_{(-\infty, 0)}(dx)$, where $\widetilde{\Lambda}_+$ is the pushforward measure of $\Lambda_+^\uparrow$ by $x \mapsto \log |1 - e^x|$. In this decomposition, ν_+ will in fact stand for special jumps of the spine corresponding to changes in the *generation* of the spine (when we select a negative jump), whereas η_+ stems from biasing ξ_+ according to its exponential martingale.

Notation 3.5.3. We shall denote by Y the real-valued self-similar Markov process with Lamperti-Kiu characteristics $(\alpha, \hat{F})$.

Proof. The only point is to prove that $\hat{F}$ is indeed the matrix exponent of this MAP. This follows from straightforward calculations, using $\mathcal{K}_+(\omega) = \mathcal{K}_-(\omega) = 0$. For example, the first entry of the matrix exponent should be

$$\hat{\psi}_+(q) - \hat{q}_+ = \kappa_+(\omega+q) - \kappa_+(\omega) - \frac{v_-(\omega)}{v_+(\omega)} \int_0^\infty |e^x - 1|^\omega \Lambda_+^\uparrow(dx) = \kappa_+(\omega+q) - \mathcal{K}_+(\omega) = \kappa_+(\omega+q).$$

□

Rebuilding the growth-fragmentation from the spine. To give a precise statement on the law of the growth-fragmentation under $\hat{\mathcal{P}}_z$, we need to rebuild the growth-fragmentation from the spine. As in section 3.3.1, the first step is to label the jumps of $\hat{\mathcal{X}}$. In general, we do not know if we can rank those in lexicographical order, and thus we use the following procedure. Jumps of the tagged cell $\hat{\mathcal{X}}$ will be labelled by couples (n, j) , $n \geq 0$ denoting the generation of the tagged cell immediately before the jump, and $j \geq 1$ being the rank (in the usual lexicographical sense) of the jump among those of the tagged cell at generation n (we also count the final jump, when the generation changes to $n+1$). For each such (n, j) , we can define the growth-fragmentation $\hat{\mathbf{X}}_{n,j}$ stemming from the corresponding jump. More precisely, if the generation stays the same during the (n, j) -jump, then we set

$$\hat{\mathbf{X}}_{n,j}(t) := \{\{\mathcal{X}_{uw}(t - b_{uw} + b_u), w \in \mathbb{U} \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\}\},$$

where u is the label of the cell born at the (n, j) -jump. On the contrary, if the (n, j) -jump corresponds to a jump for the generation of the tagged cell, then the tagged cell jumps from label u to label uk say, and we set

$$\hat{\mathbf{X}}_{n,j}(t) := \{\{\mathcal{X}_{uw}(t - b_{uw} + b_{uk}), w \in \mathbb{U} \setminus \{k\} \text{ and } b_{uw} \leq t + b_{uk} < b_{uw} + \zeta_{uw}\}\}.$$

Finally, we agree that $\hat{\mathbf{X}}_{n,j} := \partial$ when the (n, j) -jump does not exist, and this sets $\hat{\mathbf{X}}_{n,j}$ for all $n \geq 0$ and all $j \geq 1$.

Description of the growth-fragmentation under $\hat{\mathcal{P}}_z$. We are now set to describe the law of $\mathbf{X}$ under $\hat{\mathcal{P}}_z$. Recall the definition of Y from Notation 3.5.3, and that n_t denotes the generation of the spine at time t .

Theorem 3.5.4. *Under $\hat{\mathcal{P}}_1$, $(\hat{\mathcal{X}}(t), 0 \leq t < b_{\mathcal{L}})$ is distributed as $(Y(t), 0 \leq t < I)$. Moreover, conditionally on $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$, the processes $\hat{\mathbf{X}}_{n,j}$, $n \geq 0$, $j \geq 1$, are independent and each $\hat{\mathbf{X}}_{n,j}$ has law $\mathbf{P}_{x(n,j)}$ where $-x(n, j)$ is the size of the (n, j) -th jump.*

Before we come to the proof, let us make some comments on this result.

Remark 3.5.5. (i) We can give the joint law of $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$. Note that, unlike $\hat{\mathcal{X}}$, the law of n_t depends on the choice of the Eve cell. For example, in the case when the Eve cell is $X^\uparrow$, the joint law of $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$ is the same as $(Y(t), N(\tau_t))_{0 \leq t < I}$, where $(N(t), t \geq 0)$ is the Poisson process arising from the superposition of ν_+ and the compound Poisson process corresponding to the sign changes of Y (modulo Lamperti time-change τ_t).

(ii) We can rephrase the theorem perhaps more tellingly by clarifying the characteristics $\hat{\xi}_\pm, \hat{q}_\pm, \hat{U}_{\pm, \mp}$ describing the MAP. Let us do this for the positive part (the negative one being analogous). As explained in Remark 3.5.2, the Lévy process $\hat{\xi}_+$ is the

result of a superposition between a biased version of ξ_+ , and a compound Poisson process. This compound Poisson process takes care of special jumps of the spine: namely, it takes care of the eventuality that the spine selects a *negative* jump of the driving process, so that the spine remains positive at the next generation. The variable $\hat{q}_+$ is an exponential random variable which has parameter $\sigma_+(\omega)$, that is to say it corresponds to the first time the spine becomes negative. This happens either because the driving process it follows does, or because the spine jumps to a negative cell, and this is conspicuous in the two terms of σ_+ . Finally, the variable U_+ is the intensity of the jumps of the spine when it crosses 0 (again, both cases can happen).

- (iii) The signed growth-fragmentation $\mathbf{X}$ is characterised by (κ_+, κ_-) . Theorem 3.5.4 shows that the law of the spine also characterises $\mathbf{X}$.
- (iv) One can retrieve from the first entry of $\hat{F}$ the description of the spine for *unsigned* growth-fragmentation presented in [BBCK18], Theorem 4.2. Note, however, that the exponent ω differs, and this is because of the h -transform used to condition the spine to remain positive. We refer to [DDK17], Appendix 8, for details on these harmonic functions for self-similar real-valued Markov processes. We will give details of this for a particular family of signed growth-fragmentation processes in the next section.
- (v) The process $(\mathcal{M}_t, t \geq 0)$ in Corollary 3.3.8 is a supermartingale, but when is it a martingale? Proposition 3.3.7 gives that

$$\forall t \geq 0, \quad \mathcal{E}_z[\mathcal{M}_t] = v_{\text{sgn}(z)}(\omega) |z|^\omega \hat{\mathcal{P}}_z(\hat{\mathcal{X}}(t) \in \mathbb{R}^*).$$

Therefore $(\mathcal{M}_t, t \geq 0)$ is a martingale if, and only if, for all $t \geq 0$, $\hat{\mathcal{P}}_z(\hat{\mathcal{X}}(t) \in \mathbb{R}^*) = 1$. This, in turn, is equivalent to Y having infinite lifetime. In particular, if $\alpha\kappa'_+(\omega) > 0$ and $\alpha\kappa'_-(\omega) > 0$, then $\alpha\hat{\xi}_+$ and $\alpha\hat{\xi}_-$ both drift to $+\infty$ ($\hat{\xi}_+$ and $\hat{\xi}_-$ both drift to $+\infty$ or $-\infty$ depending on the sign of $\hat{\psi}'_\pm(0) = \kappa'_\pm(\omega)$), and by Lamperti time-change Y has infinite lifetime and $(\mathcal{M}_t, t \geq 0)$ is a martingale. On the other hand, if $\alpha\kappa'_+(\omega) < 0$ or $\alpha\kappa'_-(\omega) < 0$, then for symmetric reasons $(\mathcal{M}_t, t \geq 0)$ is not a martingale.

3.5.2 Proof of Theorem 3.5.4

Proof in the constant sign case. We look at the specific example when the Eve cell X has no sign change. In this case, the Lamperti representation holds, and so the compensation formula for Lévy processes makes it simpler to determine the law of the spine $\hat{\mathcal{X}}$. This paragraph is therefore an extension of [BBCK18], when we also take into account the *positive* jumps.

Let us prove the first claim. First of all, we can restrict to the homogeneous case $\alpha = 0$: for a general index α , the result then stems from Lamperti time-change. Furthermore, the definition of $\hat{\mathcal{X}}$ and the branching property ensure that $(\hat{\mathcal{X}}(t), t \geq 0)$ is an homogeneous Markov process, and therefore can be written as the exponential of a MAP. The claim now boils down to finding its characteristics $(\Xi_\pm, Q_\pm, V_{\pm, \mp})$, and for obvious reasons of symmetry, we restrict to finding $(\Xi_+, Q_+, V_{+, -})$.

- ▷ DETERMINING THE LAW OF Ξ_+ . This is essentially done in [BBCK18], but we recall the main ideas for the sake of completeness. The branching structure enables us to focus on the law of $(\Xi_+(t), 0 \leq t \leq b_{\mathcal{L}(1)})$. Let f, g be two nonnegative measurable functions defined on the space of finite càdlàg paths and on $\mathbb{R}$ respectively. Therefore, we want to

compute

$$\begin{aligned}
& \hat{E}_1 \left[f\left(\log(\hat{\mathcal{X}}(s)), 0 \leq s < b_{\mathcal{L}(1)}\right) g\left(\log \frac{\hat{\mathcal{X}}(b_{\mathcal{L}(1)})}{\hat{\mathcal{X}}(b_{\mathcal{L}(1)}^-)}\right) \mathbb{1}_{\forall 0 \leq s \leq b_{\mathcal{L}(1)}, \hat{\mathcal{X}}(s) > 0} \right] \\
&= \mathbb{E}_+ \left[\sum_{t>0} |\Delta X(t)|^\omega f(\log(X(s)), 0 \leq s < t) g\left(\log \frac{-\Delta X(t)}{X(t^-)}\right) \mathbb{1}_{\Delta X(t) < 0} \right] \\
&= \mathbb{E}_+ \left[\sum_{t>0} e^{\omega \xi_+(t^-)} f(\xi_+(s), 0 \leq s < t) g\left(\log |1 - e^{\Delta \xi_+(t)}|\right) |1 - e^{\Delta \xi_+(t)}|^\omega \mathbb{1}_{\Delta \xi_+(t) < 0} \right] \\
&= \mathbb{E}_+ \left[\int_0^\infty dt e^{\omega \xi_+(t)} f(\xi_+(s), 0 \leq s < t) \right] \int_{-\infty}^0 \Lambda_+(dx) |1 - e^x|^\omega g(\log |1 - e^x|) \\
&= \mathbb{E}_+ \left[\int_0^\infty dt e^{\omega \xi_+(t)} f(\xi_+(s), 0 \leq s < t) \right] \int_{-\infty}^0 \tilde{\Lambda}_+(dx) e^{\omega x} g(x),
\end{aligned}$$

where we used the compensation formula. Thus, under $\hat{\mathcal{P}}_1$, on the event that $\hat{\mathcal{X}}(s) > 0$ for all $s \in [0, b_{\mathcal{L}(1)}]$, the two processes $\log \frac{\hat{\mathcal{X}}(b_{\mathcal{L}(1)})}{\hat{\mathcal{X}}(b_{\mathcal{L}(1)}^-)}$ and $(\log(\hat{\mathcal{X}}(s)), 0 \leq s < b_{\mathcal{L}(1)})$ are independent. The former has the law $-\psi_+(\omega)^{-1} e^{\omega x} \tilde{\Lambda}_+|_{(-\infty, 0)}(dx)$, and the latter is distributed as ξ_+ killed according to $\exp(\omega \xi_+(t))$, so that it gives a Lévy process with Laplace exponent $q \mapsto \psi_+(\omega + q)$. Note that, in particular, $b_{\mathcal{L}(1)}$ is an exponential random variable with parameter $-\psi_+(\omega)$. On the second hand, we can do the same for $\hat{\xi}_+$. Recall the notation (η_+, ν_+) from Remark 3.5.2. Denote by T_1 the first time when the compound Poisson process ν_+ has a jump: T_1 is exponential with parameter $-\psi_+(\omega)$. Since these jumps arise according to $e^{\omega x} dt \cdot \tilde{\Lambda}_+|_{(-\infty, 0)}(dx)$, its first jump $\Delta \nu_+(T_1)$ is distributed according to $-\psi_+(\omega)^{-1} e^{\omega x} \tilde{\Lambda}_+|_{(-\infty, 0)}(dx)$, and is independent of the process $(\eta_+(s), 0 \leq s < T_1)$. The latter, in turn, is η_+ killed at an independent exponential time with parameter $-\psi_+(\omega)$. Since the Laplace exponent of η_+ is by definition

$$\psi_{\eta_+}(q) := \psi_+(\omega + q) - \psi_+(\omega),$$

we get that $(\eta_+(s), 0 \leq s < T_1)$ has Laplace exponent $q \mapsto \psi_+(\omega + q)$. Therefore, we obtain the same description, and this entails that Ξ_+ and $\hat{\xi}_+$ have the same distribution.

▷ DETERMINATION OF Q_+ . Call $\hat{H}_1$ the first time when $\hat{\mathcal{X}}$ becomes negative. Since X always remains positive when started from a positive mass, $\hat{H}_1$ corresponds to the first time when the spine picks a positive jump in the change of measure 3.3.3. Therefore, $\hat{H}_1$ can be written

$$\hat{H}_1 = \sum_{i=1}^G \tau_i,$$

where G is a random variable corresponding to the generation of the spine at which a negative particle is selected, and $\tau_i = b_{\mathcal{L}(i)} - b_{\mathcal{L}(i-1)}$, $i \geq 1$. Since on the event that the spine selects a negative jump, we have seen that $b_{\mathcal{L}(1)}$ is exponential with parameter $-\psi_+(\omega)$, we may deduce from the branching property that the τ_i 's form an independent family of exponential variables with parameter $-\psi_+(\omega)$. Moreover, G is a geometric variable on $\mathbb{N}^*$ with probability of success p given by

$$p := \hat{\mathcal{P}}_1(\hat{\mathcal{X}}(b_{\mathcal{L}(1)}) < 0) = \mathbb{E}_+ \left(\sum_{t>0} \frac{v_-(\omega)}{v_+(\omega)} |\Delta X(t)|^\omega \mathbb{1}_{\Delta X(t) > 0} \right).$$

Again, the compensation formula for ξ_+ yields

$$p = -\frac{1}{\psi_+(\omega)} \cdot \frac{v_-(\omega)}{v_+(\omega)} \int_0^\infty \Lambda_+(dx) |e^x - 1|^\omega = -\frac{\sigma_+(\omega)}{\psi_+(\omega)}.$$

As a sum of a geometric number of independent exponential variables, $\widehat{H}_1$ is an exponential random variable with parameter

$$Q_+ = -\psi_+(\omega) \cdot p = \sigma_+(\omega).$$

Therefore $Q_+ = \widehat{q}_+$.

▷ DETERMINATION OF $V_{+,-}$. For $q \geq 0$, we have

$$\widehat{E}_1[e^{qV_{+,-}}] = \sum_{i=1}^\infty \widehat{E}_1 \left[\left(\frac{|\widehat{\mathcal{X}}(\widehat{H}_1)|}{\widehat{\mathcal{X}}(\widehat{H}_1^-)} \right)^q \mathbf{1}_{G=i} \right]$$

Let $a_i := \widehat{E}_1 \left[\left(\frac{|\widehat{\mathcal{X}}(\widehat{H}_1)|}{\widehat{\mathcal{X}}(\widehat{H}_1^-)} \right)^q \mathbf{1}_{G=i} \right]$, $i \geq 1$. Then, for $i \geq 2$, conditioning on the spine at time $b_{\mathcal{L}(1)}$ and using the Markov property yields

$$\begin{aligned} a_i &= \mathbb{E}_+ \left(\sum_{t>0} |\Delta X(t)|^\omega \mathbf{1}_{\Delta X(t)<0} \cdot \widehat{E}_{-\Delta X(t)} \left[\left(\frac{|\widehat{\mathcal{X}}(\widehat{H}_1)|}{\widehat{\mathcal{X}}(\widehat{H}_1^-)} \right)^q \mathbf{1}_{G=i-1} \right] \right) \\ &= \mathbb{E}_+ \left(\sum_{t>0} |\Delta X(t)|^\omega \mathbf{1}_{\Delta X(t)<0} \right) \cdot a_{i-1}, \end{aligned}$$

by self-similarity. Hence, $(a_i)_{i \geq 1}$ is a geometric progression with common ratio

$$\mathbb{E}_+ \left(\sum_{t>0} |\Delta X(t)|^\omega \mathbf{1}_{\Delta X(t)<0} \right) = -\frac{1}{\psi_+(\omega)} \int_{-\infty}^0 |e^x - 1|^\omega \Lambda_+(dx),$$

by an application of the compensation formula. Moreover, by another use of the compensation formula, the first term is

$$\begin{aligned} a_1 &= \widehat{E}_1 \left[\left(\frac{|\widehat{\mathcal{X}}(\widehat{H}_1)|}{\widehat{\mathcal{X}}(\widehat{H}_1^-)} \right)^q \mathbf{1}_{G=1} \right] \\ &= \mathbb{E}_+ \left[\sum_{t>0} \mathbf{1}_{\Delta X(t)>0} \frac{v_-(\omega)}{v_+(\omega)} |\Delta X(t)|^\omega \left(\frac{|\Delta X(t)|}{X(t^-)} \right)^q \right] \\ &= \frac{v_-(\omega)}{v_+(\omega)} \mathbb{E}_+ \left[\sum_{t>0} \mathbf{1}_{\Delta \xi_+(t)>0} e^{\omega \xi_+(t^-)} |e^{\Delta \xi_+(t)} - 1|^{q+\omega} \right] \\ &= -\frac{v_-(\omega)}{v_+(\omega)} \cdot \frac{1}{\psi_+(\omega)} \int_0^\infty |e^x - 1|^{q+\omega} \Lambda_+(dx) \\ &= -\frac{\sigma_+(q+\omega)}{\psi_+(\omega)}. \end{aligned}$$

Finally, we get that

$$\widehat{E}_1[e^{qV_{+,-}}] = -\frac{\sigma_+(\omega+q)}{\psi_+(\omega) + \int_{-\infty}^0 |e^x - 1|^\omega \Lambda_+(dx)}.$$

Using that $\mathcal{K}_+(\omega) = 0$, we come to the conclusion that

$$\widehat{E}_1[e^{qV_{+,-}}] = \frac{\sigma_+(\omega+q)}{\sigma_+(\omega)}.$$

We have proved that $(\Xi_+, Q_+, V_{+,-}) \stackrel{\mathcal{L}}{=} (\hat{\xi}_+, \hat{q}_+, \hat{U}_{+,-})$. Therefore the first claim of Theorem 3.5.4 follows readily from Lemma 3.5.1.

The spinal decomposition in the general case. We now prove the spinal decomposition under the tilted measure $\hat{\mathcal{P}}_1$ by restricting to the previous case. More precisely, we want to prove that the law of $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$ under $\hat{\mathcal{P}}_1$ is the same as under $\hat{\mathcal{P}}_1^\uparrow$, where $\hat{\mathcal{P}}_1^\uparrow$ is the change of probability induced by $X^\uparrow$ via section 3.3.3. Indeed, since $X^\uparrow$ is nonnegative, the case of $\hat{\mathcal{P}}_1^\uparrow$ comes under the previous paragraph, for which the spinal decomposition has just been established.

The definition of $\hat{\mathcal{P}}_1$ clearly depends on the Eve cell. Note however that we have proved in Theorem 3.4.6 that the exponent ω and the constants $(v_-(\omega), v_+(\omega))$ depend only on the growth-fragmentation (see also Proposition 3.4.4 for the relation between X and $X^\uparrow$). Therefore, Proposition 3.3.7 entails that the marginal law of $\hat{\mathcal{X}}$ only depends on the growth-fragmentation $\mathbf{X}$. In order to prove that the law of $\hat{\mathcal{X}}$ itself is invariant within the same growth-fragmentation, we need to extend Proposition 3.3.7 to finite-dimensional distributions. To avoid cumbersome notation, we state and prove the result for two times $s < t$. We want to show that for $z \neq 0$ and nonnegative measurable functions f, g such that $f(\partial) = g(\partial) = 0$,

$$\begin{aligned} & v_{sgn(z)}(\omega) |z|^\omega \hat{E}_z \left(f(\hat{\mathcal{X}}(t)) g(\hat{\mathcal{X}}(s)) \right) \\ &= \mathcal{E}_z \left(\sum_{j \geq 1} g(X_j(s)) \mathcal{E}_{X_j(s)} \left[\sum_{i \geq 1} v_{sgn(X_i(t-s))}(\omega) |X_i(t-s)|^\omega f(X_i(t-s)) \right] \right). \end{aligned} \quad (3.5.1)$$

If one is willing to accept that $\hat{\mathcal{X}}$ is a Markov process, then this follows readily from Proposition 3.3.7. Otherwise, we can prove this directly. Let us mimic the proof of Proposition 3.3.7. Splitting over $u(t)$ as in equation (3.3.4) and then conditioning on $\mathcal{F}_t$ and using the branching property, we get

$$\begin{aligned} & v_{sgn(z)}(\omega) |z|^\omega \hat{E}_z \left(f(\hat{\mathcal{X}}(t)) g(\hat{\mathcal{X}}(s)) \mathbb{1}_{b_{\mathcal{L}(n+1)} > t} \right) \\ &= \mathcal{E}_z \left(\sum_{|w| \leq n} g(\mathcal{X}_w(s) (s - b_w(s))) v_{sgn(\mathcal{X}_w(t-b_w))}(\omega) |\mathcal{X}_w(t-b_w)|^\omega f(\mathcal{X}_w(t-b_w)) \mathbb{1}_{b_w \leq t} \right). \end{aligned}$$

We may then split this again over $w(s) = w'$. Using the branching property, this gives

$$\begin{aligned} & v_{sgn(z)}(\omega) |z|^\omega \hat{E}_z \left(f(\hat{\mathcal{X}}(t)) g(\hat{\mathcal{X}}(s)) \mathbb{1}_{b_{\mathcal{L}(n+1)} > t} \right) = \mathcal{E}_z \left(\sum_{|w'| \leq n} g(\mathcal{X}_{w'}(s - b_{w'})) \mathbb{1}_{b_{w'} < s} \right. \\ & \times \mathcal{E}_{\mathcal{X}_{w'}(s-b_{w'})} \left[\sum_{|w| \leq n-|w'|} v_{sgn(\mathcal{X}_w(t-s-b_w))}(\omega) |\mathcal{X}_w(t-s-b_w)|^\omega f(\mathcal{X}_w(t-s-b_w)) \mathbb{1}_{b_w < t-s} \right] \Big). \end{aligned}$$

Now taking $n \rightarrow \infty$ yields the desired identity (3.5.1).

Proof of the second assertion. We finally prove the second assertion of Theorem 3.5.4 directly in the general setting. We will limit ourselves to proving the statement for the first generation (this is easily extended using the branching property). Let f be a nonnegative measurable functional on the space of càdlàg trajectories, and $g_j, j \geq 1$, be nonnegative measurable functionals on the space of multiset-valued paths. For $t > 0$, denote by $(\Delta_j(t), j \geq 1)$ the sequence consisting of the value of $\mathcal{X}_\emptyset(t)$, and all those jumps of $\mathcal{X}_\emptyset$

that happened strictly before time t , ranked in descending order of their absolute value. Our goal is to show that

$$\widehat{E}_1 \left(f(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} g_j(\widehat{\mathbf{X}}_{0,j}) \right) = \widehat{E}_1 \left(f(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} \mathbf{E}_{\Delta_j(b_{\mathcal{L}(1)})} [g_j(\mathbf{X})] \right).$$

But,

$$\begin{aligned} & \widehat{E}_1 \left(f(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} g_j(\widehat{\mathbf{X}}_{0,j}) \right) \\ &= \mathcal{E}_1 \left(\sum_{t > 0} \frac{v_{\text{sgn}(-\Delta \mathcal{X}_\emptyset(t))}(\omega)}{v_+(\omega)} |\Delta \mathcal{X}_\emptyset(t)|^\omega f(\mathcal{X}_\emptyset(s), 0 \leq s \leq t) \prod_{j \geq 1} g_j(\widehat{\mathbf{X}}_{0,j}) \right), \end{aligned}$$

and the definition of the $\widehat{\mathbf{X}}_{0,j}$ together with the branching property give

$$\begin{aligned} & \widehat{E}_1 \left(f(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} g_j(\widehat{\mathbf{X}}_{0,j}) \right) \\ &= \mathcal{E}_1 \left(\sum_{t > 0} \frac{v_{\text{sgn}(-\Delta \mathcal{X}_\emptyset(t))}(\omega)}{v_+(\omega)} |\Delta \mathcal{X}_\emptyset(t)|^\omega f(\mathcal{X}_\emptyset(s), 0 \leq s \leq t) \prod_{j \geq 1} \mathbf{E}_{\Delta_j(t)} [g_j(\mathbf{X})] \right). \end{aligned}$$

Applying the change of measure backwards, we get the desired identity. Therefore Theorem 3.5.4 is proved.

3.6 A distinguished family of signed growth-fragmentations

Following [AS20], we construct a particular family of signed growth-fragmentations. These can be seen in the upper half-plane by cutting at heights a path with real part given by a stable Lévy process, and imaginary part a positive Brownian excursion. This can be done for any self-similarity index α in $(0, 2)$, but for reasons that will be clarified later on, we take α to be in $(1, 2)$.

3.6.1 Notation and setup

We recall from [AS20] the following framework. All the definitions and results basically extend directly from the half-planar Brownian case.

The excursion measure $\mathbf{n}^\alpha$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, on which X^α is an α -stable Lévy process, and Y an independent Brownian motion. Call $(\mathcal{F}_t)_{t \geq 0}$ the standard filtration associated with (X^α, Y) . Write $\mathcal{X}$ for the space of càdlàg functions x with finite duration $R(x)$, equipped with the standard σ -field generated by the coordinates. Let $\mathcal{X}_0$ be the subset of such functions in $\mathcal{X}$ that are *continuous* and vanish at $R(x)$. Finally, let

$$U := \{u = (x, y) \in \mathcal{X} \times \mathcal{X}_0, u(0) = 0 \text{ and } R(x) = R(y)\} \quad \text{and} \quad U_\partial := U \cup \{\partial\},$$

where ∂ is a cemetery state. We endow this set with the product σ -field $\mathcal{U}_\delta$ and the filtration $(\mathcal{F}_t)_{t \geq 0}$ adapted to the coordinate process on U . Also, we write $(L_s, s \geq 0)$ for the local time at 0 of the Brownian motion Y and $\mathcal{T}_s$ its inverse.

We define on $(\Omega, \mathcal{F}, \mathbb{P})$ the *excursion process* $(e_s^\alpha, s > 0)$ as in the case of planar Brownian motion in [AS20], except that we take for the real part X the α -stable Lévy process X^α (which has discontinuities), namely:

(i) if $\mathcal{T}_s - \mathcal{T}_{s-} > 0$, then

$$e_s^\alpha : r \mapsto (X_{r+\mathcal{T}_{s-}}^\alpha - X_{\mathcal{T}_{s-}}^\alpha, Y_{r+\mathcal{T}_{s-}}), \quad r \leq \mathcal{T}_s - \mathcal{T}_{s-},$$

(ii) if $\mathcal{T}_s - \mathcal{T}_{s-} = 0$, then $e_s^\alpha = \partial$.

Then it is not difficult to see that the excursion process $(e_s^\alpha)_{s>0}$ is a $(\mathcal{F}_{\mathcal{T}_s})_{s>0}$ -Poisson point process (see [RY99], Chap. XII, Theorem 2.4, for the one-dimensional case). We denote by $\mathbf{n}^\alpha$ its intensity, which is a measure on U , and we denote by $\mathbf{n}_+^\alpha$ and $\mathbf{n}_-^\alpha$ its restrictions to $U^+ := \{u = (x, y) \in U, y \geq 0\}$ and $U^- := \{u = (x, y) \in U, y \leq 0\}$. An easy calculation gives the following expression for $\mathbf{n}^\alpha$.

Proposition 3.6.1. $\mathbf{n}^\alpha(dx, dy) = n(dy)\mathbb{P}((X^\alpha)^{R(y)} \in dx)$, where n denotes the one-dimensional (Brownian) Itô measure on $\mathcal{X}_0$, and $(X^\alpha)^T := (X_t, t \in [0, T])$.

Figure 3.3 shows a drawing of such an excursion.

The locally largest excursion. We recall the following facts from [AS20]. Let $u = (x, y) \in U^+$ and $a \geq 0$. Write $R(u) := R(x) = R(y)$.

The set

$$\mathcal{I}(a) = \{s \in [0, R(u)], y(s) > a\},$$

is a countable (possibly empty) union of disjoint open intervals, and for any such interval $I = (i_-, i_+)$, we write $u_I(s) := u(i_- + s) - u(i_-)$, $0 \leq s \leq i_+ - i_-$, for the restriction of u to I , and $\Delta u_I = x(i_+) - x(i_-)$. We call Δu_I the *size* or *length* of u_I , which may be negative.

For $z = u(t)$, $0 \leq t \leq R(u)$, and $0 \leq a < \Im(z)$, we define $e_a^{(t)} = u_I$, where I is the unique open interval in the above partition of $\mathcal{I}(a)$ containing t .

Moreover, set

$$\begin{aligned} u^{t, \leftarrow} &:= (u(t-s) - u(t))_{0 \leq s \leq t}, \\ u^{t, \rightarrow} &:= (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}. \end{aligned}$$

Define

$$F^{(t)} : a \in [0, \Im(z)] \mapsto \Delta e_a^{(t)} = u^{t, \rightarrow}(T_a^{t, \rightarrow}) - u^{t, \leftarrow}(T_a^{t, \leftarrow}),$$

where

$$T_a^{t, \leftarrow} := \inf\{s \geq 0, y(t-s) = a\} \quad \text{and} \quad T_a^{t, \rightarrow} := \inf\{s \geq 0, y(t+s) = a\}.$$

Following the strategy of [AS20], Proposition 2.8, one can establish the following result.

Proposition 3.6.2. For $u \in U^+$ and $0 \leq t \leq R(u)$, let

$$S(t) := \sup\{a \in [0, y(t)], \quad \forall 0 \leq a' \leq a, |F^{(t)}(a')| \geq |F^{(t)}(a'^-) - F^{(t)}(a')|\},$$

and $S := \sup_{0 \leq t \leq R(u)} S(t)$. For almost every u under $\mathbf{n}_+^\alpha$, there exists a unique $0 \leq t^\bullet \leq R(u)$ such that $S(t^\bullet) = S$. Moreover, $S = \Im(z^\bullet)$ where $z^\bullet = u(t^\bullet)$.

By definition of S , $(e_a^{(t^\bullet)})_{0 \leq a \leq \Im(z^\bullet)}$ is the excursion which is locally the largest, meaning that at any level a where the locally largest excursion splits, its size is larger (in absolute value) than the length of the other excursion. For this reason we refer to $(e_a^{(t^\bullet)})_{0 \leq a \leq \Im(z^\bullet)}$ as the *locally largest excursion* and $(\Xi(a) = \Delta e_a^{(t^\bullet)})_{0 \leq a \leq \Im(z^\bullet)}$ as the *locally largest fragment*. See Figure 3.3.

Bismut's description of $\mathbf{n}_+^\alpha$. We now recall Bismut's description of $\mathbf{n}_+^\alpha$, which also extends easily from the Brownian case.

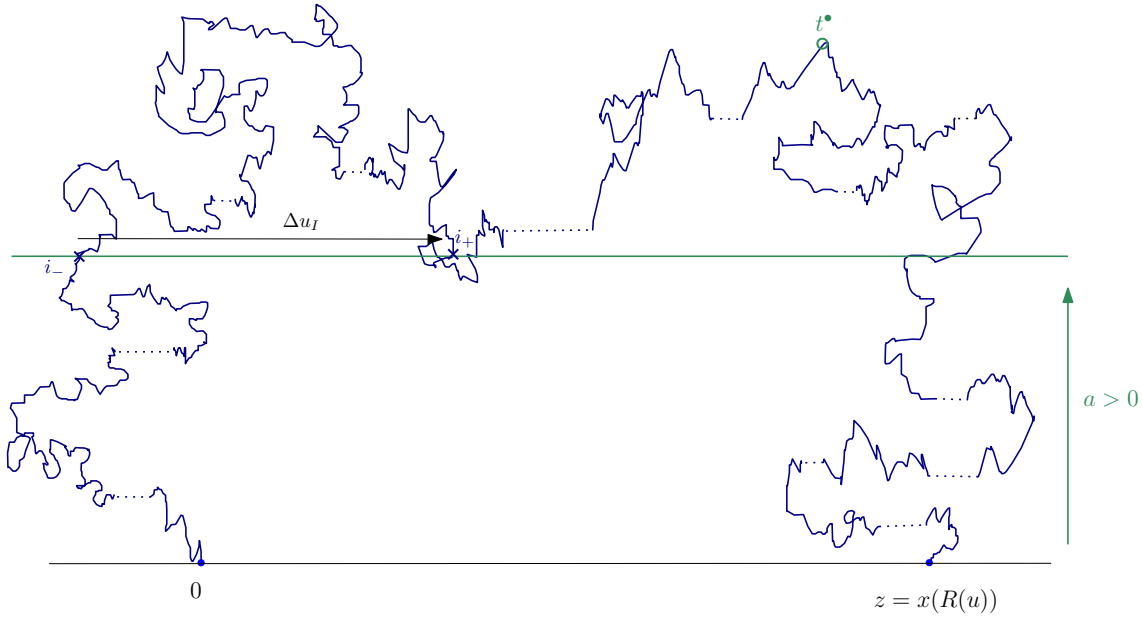


Figure 3.3 – Drawing of an excursion in the upper half-plane $\mathbb{H}$ and the locally largest excursion.

Proposition 3.6.3. (*Bismut's description of $\mathbf{n}_+^\alpha$*)

Let $\overline{\mathbf{n}}_+^\alpha$ be the measure defined on $\mathbb{R}_+ \times U^+$ by

$$\overline{\mathbf{n}}_+^\alpha(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u)\}} dt \mathbf{n}_+^\alpha(du).$$

Then under $\overline{\mathbf{n}}_+^\alpha$ the "law" of $(t, (x, y)) \mapsto y(t)$ is the Lebesgue measure da and conditionally on $y(t) = a$, $u^{t, \leftarrow} = (u(t-s) - u(t))_{0 \leq s \leq t}$ and $u^{t, \rightarrow} = (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}$ are independent with respective laws $(-X^\alpha, Y)$ and (X^α, Y) killed when reaching $\{\mathfrak{F}(z) = -a\}$.

Note that, unlike the planar Brownian case, there is a minus sign for the left part of the trajectory: this is because of the time-reversal, which involves the dual of X^α . See Figure 3.4 for an illustration.

As in [AS20], this enables to prove that excursions under $\mathbf{n}_+^\alpha$ have no loop above any level.

Proposition 3.6.4. *Let*

$$\mathcal{L} := \{u \in U^+, \exists 0 \leq t \leq R(u), \exists 0 \leq a < y(t), \Delta e_a^{(t)}(u) = 0\},$$

be the set of excursions u having a loop remaining above some level a . Then $\mathbf{n}_+^\alpha(\mathcal{L}) = 0$.

Remark 3.6.5. We remark that, in the Brownian case, the proof of Proposition 3.6.4 relies in particular on the fact that points are polar for the Cauchy process. This argument still holds in our case because points are also polar for the $\frac{\alpha}{2}$ -stable Lévy process (see [Ber96], Chapter II, section 5).

Disintegration of $\mathbf{n}_+^\alpha$. Finally, we disintegrate the infinite measure $\mathbf{n}_+^\alpha$ over the endpoint $z = x(R(u))$. This defines probability measures γ_z^α , $z \neq 0$, which are the laws of excursions (X^α, Y) conditionally on having initial size z . Introduce $\mathbb{P}_{\alpha, r}^{a \rightarrow b}$ as the law of an α -stable Lévy bridge of length r between a and b , and Π_r as the law of a three-dimensional Bessel (BES^3) bridge of length r from 0 to 0. Moreover, we denote by $(p_t^\alpha)_{t \geq 0}$ the transitions of the α -stable Lévy process.

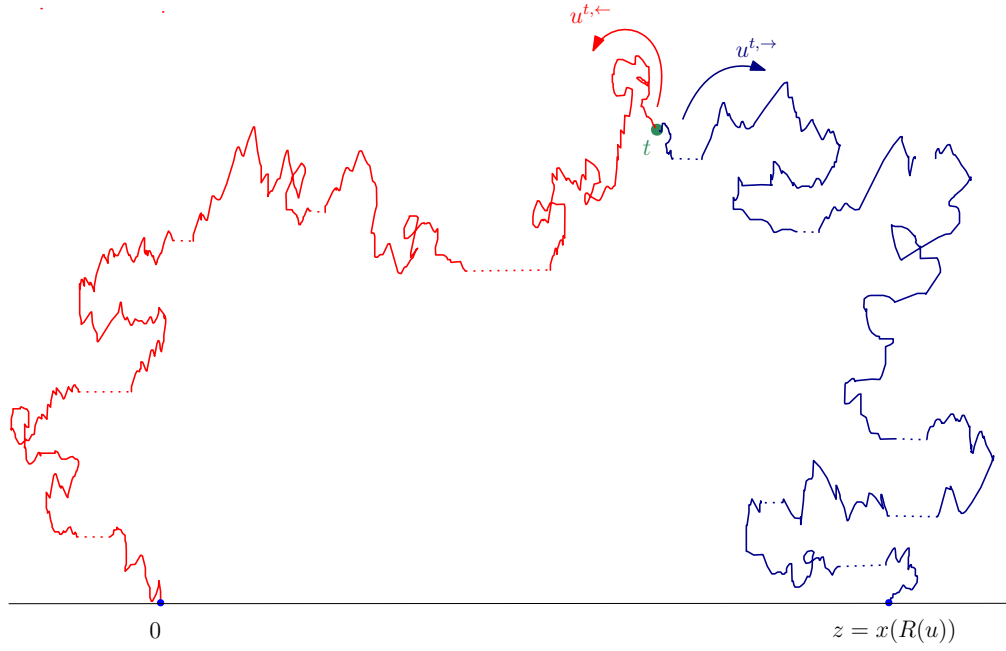


Figure 3.4 – An illustration of Bismut's description of $\mathbf{n}_+^\alpha$. Conditionally on $y(t)$, the left part (in red) and right part (in blue) evolve independently and are identically distributed up to time-reversal.

Proposition 3.6.6. *We have the following disintegration formula*

$$\mathbf{n}_+^\alpha = \int_{\mathbb{R}} dz \frac{C_{\text{sgn}(z)}}{|z|^{1+\alpha/2}} \gamma_z^\alpha, \quad (3.6.1)$$

where

$$C_\pm = \frac{\alpha}{2\sqrt{2\pi}} \int_0^\infty r^{\alpha/2} p_1^\alpha(\pm r) dr,$$

and for $z \neq 0$, γ_z^α is the probability measure defined by

$$\gamma_z^\alpha = \int_{\mathbb{R}_+} dv \frac{p_1^\alpha(\text{sgn}(z)v^{-1/\alpha})}{2\sqrt{2\pi}C_{\text{sgn}(z)}v^{3/2+1/\alpha}} \mathbb{P}_{\alpha,|z|^\alpha v}^{0 \rightarrow z} \otimes \Pi_{|z|^\alpha v}. \quad (3.6.2)$$

Remark 3.6.7. The constants $C_\pm$ can be calculated (see [KP21], Section 1). For example, if X^α is the so-called *normalised* stable process of index α , then

$$C_+ = \frac{\alpha}{2\sqrt{2\pi}} \Gamma\left(\frac{\alpha}{2}\right) \sin\left(\frac{\pi\alpha(1-\rho)}{2}\right),$$

where $\rho = \mathbb{P}(X^\alpha(1) < 0)$.

Proof. Although the proof follows exactly the same lines as in [AS20], we include it here to highlight the importance of the sign of z , which do not show up in the Brownian case for symmetry reasons. Let f and g be two nonnegative measurable functions defined on $\mathcal{X}$ and $\mathcal{X}_0$ respectively. Applying Itô's description of n_+ in terms of Bessel bridge (see [RY99], Chap. XII, Theorem 4.2), we get

$$\begin{aligned} \int_U f(x)g(y) \mathbf{n}_+^\alpha(dx, dy) &= \int_U f(x)g(y) n_+(dy) \mathbb{P}\left((X^\alpha)^{R(y)} \in dx\right) \\ &= \int_{\mathbb{R}_+} \frac{dr}{2\sqrt{2\pi}r^3} \int_{\mathcal{X}} f(x) \Pi_r[g] \mathbb{P}((X^\alpha)^r \in dx). \end{aligned}$$

Now, decomposing on the value of $X^\alpha(r)$ yields

$$\int_U f(x)g(y) \mathbf{n}_+^\alpha(dx, dy) = \int_{\mathbb{R}_+} \frac{dr}{2\sqrt{2\pi r^3}} \int_{\mathbb{R}} dz p_r^\alpha(z) \Pi_r[g] \mathbb{E}_{\alpha, r}^{0 \rightarrow z}[f].$$

Using scale invariance, we have $p_r^\alpha(z) = r^{-1/\alpha} p_1^\alpha(r^{-1/\alpha} z)$. We finally perform the change of variables $v(r) = r/|z|^\alpha$ to get

$$\int_U f(x)g(y) \mathbf{n}_+^\alpha(dx, dy) = \int_{\mathbb{R}} \frac{dz}{|z|^{1+\alpha/2}} \int_{\mathbb{R}_+} dv \frac{p_1^\alpha(\text{sgn}(z)v^{-1/\alpha})}{2\sqrt{2\pi}v^{3/2+1/\alpha}} \mathbb{E}_{\alpha, v|z|^\alpha}^{0 \rightarrow z}[f] \Pi_{v|z|^\alpha}[g].$$

The constants C_+ and C_- are then the normalisation constants needed for γ_z^α to be a probability measure. $\square$

3.6.2 The law of the locally largest evolution

We introduce $\theta := \frac{\alpha}{2}$, so that $\frac{1}{2} < \theta < 1$, and we define η^θ under P_z as the θ -stable Lévy process starting at $z \in \mathbb{R}$. Recall that the Laplace exponent of η^θ is given by

$$\psi_\theta(q) := \frac{c_+ - c_-}{1 - \theta} q + \int_{\mathbb{R}} (e^{qy} - 1 - qe^y \mathbf{1}_{|y|<1}) \nu_\theta(y) dy,$$

where the density of the Lévy measure

$$\nu_\theta(y) := c_{\text{sgn}(y)} |y|^{-\theta-1}, \quad (3.6.3)$$

depends on constants c_+, c_- such that $c_+ + c_- > 0$. An important feature is the positivity parameter $\rho := P_0(\eta_1^\theta > 0)$ which can be fixed by choosing c_+, c_- to equal

$$c_- = \frac{\Gamma(1+\theta)}{\pi} \sin(\pi\theta(1-\rho)) \quad \text{and} \quad c_+ = \frac{\Gamma(1+\theta)}{\pi} \sin(\pi\theta\rho). \quad (3.6.4)$$

See [CC06] and [KP21]. Moreover, in order to retrieve the family with no killing introduced in [BBCK18] for $\theta < 1$, we will in this subsection choose the following explicit constants. First, we fix c_+, c_- so that $\theta\rho = 1/2$, which gives $c_+ = \frac{\Gamma(1+\theta)}{\pi}$ and $c_- = -\frac{\Gamma(1+\theta)}{\pi} \cos(\pi\theta)$. Notice that this implies $\alpha \in (1, 2)$, which justifies our choice. Finally, we take X^α to be an α -stable Lévy process with positivity parameter ρ . It is important to note that, since $\alpha\rho = 1$, X^α is *spectrally negative*, meaning that it only has negative jumps, as can be seen from (3.6.4) with α replacing θ (see [Ber96, Chapter VIII] for more background). The process X^α being fixed, we now claim the following result.

Theorem 3.6.8. *Fix $z > 0$. Let $\Xi = (\Xi(a), 0 \leq a \leq \mathfrak{S}(z^\bullet))$ denote the size of the locally largest fragment. Under γ_z^α , $(\Xi(a))_{0 \leq a < \mathfrak{S}(z^\bullet)}$ is distributed as the positive self-similar Markov process $(Z_a)_{0 \leq a < \zeta}$ with index θ starting from z whose Lamperti representation is*

$$Z_a = z \exp(\xi(\tau(z^{-\theta}a))),$$

where ξ is the Lévy process with Laplace exponent

$$\Psi(q) = dq + \int_{y > -\ln(2)} (e^{qy} - 1 - q(e^y - 1)) e^{-\theta y} \nu_\theta(-(e^y - 1)) dy, \quad q < 2\theta + 1, \quad (3.6.5)$$

τ is the Lamperti time-change

$$\tau(a) = \inf \left\{ s \geq 0, \int_0^s e^{\theta \xi(u)} du > a \right\},$$

and $\zeta = \inf\{a > 0, Z_a = 0\}$. Moreover, $d = c_- B(1 - \theta, 2\theta) - c_+ B_{\frac{1}{2}}(1 - \theta, -\theta)$, where

$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt$ and $B_{\frac{1}{2}}(x, y) := \int_0^{1/2} t^{x-1} (1-t)^{y-1} dt$ are respectively the beta function and the incomplete beta function at $\frac{1}{2}$.

Remark 3.6.9. One can give a similar description, starting from a negative $z < 0$, for the locally largest (negative) evolution. In this case one would obtain a killing parameter in (3.6.5).

The rest of this subsection is devoted to the proof of Theorem 3.6.8. We start by recalling the main ingredients of the proof of Theorem 3.3 in [AS20], which extend naturally to our case thanks to Bismut's description of $\mathbf{n}^\alpha$ (Proposition 3.6.3). Let H be a bounded continuous nonnegative function defined on the space of finite càdlàg paths, and $a \geq 0$. We denote by T_b the hitting time of $\{\mathfrak{Z}(z) < b\}$ of the planar trajectory (X^α, Y) . We also set $\hat{\eta}_b = \eta_{(a-b)^-}^\theta, 0 \leq b \leq a$. Extending Lemma 3.4, and using that $b \mapsto X^\alpha(T_{-b})$ has the law of the θ -stable Lévy process η^θ , we get the identity

$$\mathbf{n}_+^\alpha \left[H(\Xi(b), 0 \leq b \leq a) \mathbb{1}_{a < \mathfrak{Z}(z \bullet)} \right] = \int_{\mathbb{R}} dz \frac{C_{\text{sgn}(z)}}{|z|^{1+\theta}} h(-a, z), \quad (3.6.6)$$

where $h(-a, z) := E_z[H(\hat{\eta}_b, 0 \leq b \leq a) \mathbb{1}_{\forall 0 \leq b \leq a, |\hat{\eta}_b| > |\Delta \hat{\eta}_b|}]$. We now reverse equation (3.6.6) in time. More precisely,

$$\int_{\mathbb{R}} dz \frac{C_{\text{sgn}(z)}}{|z|^{1+\theta}} h(-a, z) = \int_{\mathbb{R}} dz E_z \left(\frac{C_{\text{sgn}(\hat{\eta}_a)}}{|\hat{\eta}_a|^{1+\theta}} H(\hat{\eta}_b, 0 \leq b \leq a) \mathbb{1}_{\forall 0 \leq b \leq a, |\hat{\eta}_b| > |\Delta \hat{\eta}_b|} \right).$$

Since $\hat{\eta}$ and $-\eta^\theta$ are in duality with respect to the Lebesgue measure (see [Ber96], II.1), identity (3.6.6) becomes

$$\mathbf{n}_+^\alpha \left[H(\Xi(b), 0 \leq b \leq a) \mathbb{1}_{a < \mathfrak{Z}(z \bullet)} \right] = \int_{\mathbb{R}} dz E_z \left(\frac{C_{-\text{sgn}(\eta_a^\theta)}}{|\eta_a^\theta|^{1+\theta}} H(-\eta_b^\theta, 0 \leq b \leq a) \mathbb{1}_{\forall 0 \leq b \leq a, |\eta_b^\theta| > |\Delta \eta_b^\theta|} \right).$$

Now observe that on the event $\mathcal{A} = \{\forall 0 \leq b \leq a, |\eta_b^\theta| > |\Delta \eta_b^\theta|\}$, η^θ has no sign change. Therefore

$$\mathbf{n}_+^\alpha \left[H(\Xi(b), 0 \leq b \leq a) \mathbb{1}_{a < \mathfrak{Z}(z \bullet)} \right] = \int_{\mathbb{R}} dz \frac{C_{-\text{sgn}(z)}}{|z|^{1+\theta}} E_z \left(\frac{|z|^{1+\theta}}{|\eta_a^\theta|^{1+\theta}} H(-\eta_b^\theta, 0 \leq b \leq a) \mathbb{1}_{\forall 0 \leq b \leq a, |\eta_b^\theta| > |\Delta \eta_b^\theta|} \right).$$

Disintegrating $\mathbf{n}^\alpha$ over z (Proposition 3.6.6) and using continuity arguments brings

$$\gamma_z^\alpha \left[H(\Xi(b), 0 \leq b \leq a) \mathbb{1}_{a < \mathfrak{Z}(z \bullet)} \right] = E_{-z} \left(\frac{|z|^{1+\theta}}{|\eta_a^\theta|^{1+\theta}} H(-\eta_b^\theta, 0 \leq b \leq a) \mathbb{1}_{\forall 0 \leq b \leq a, |\eta_b^\theta| > |\Delta \eta_b^\theta|} \right),$$

for all $z \neq 0$. Assume that $z > 0$ say. Observe that on the event $\mathcal{A}$, $-\eta^\theta$ remains positive when η^θ is started from $-z$. Therefore, on this event, $-\eta^\theta$ can be written using the Lamperti representation of a θ -stable Lévy process killed when entering the negative half-line, found in [CC06] (we will take the form presented in [KP21]). More precisely, under $P_{-z}, z > 0$, we can write $-\eta_b^\theta = ze^{\xi^0(\tau^0(b))}$ on $\mathcal{A}$, where

$$\tau^0(b) := \inf \left\{ s \geq 0, \int_0^s z^\theta e^{\theta \xi^0(u)} du \geq b \right\} = \int_0^b \frac{ds}{(-\eta_s^\theta)^\theta},$$

and ξ^0 is a Lévy process with Laplace exponent

$$\Psi^0(q) := -\frac{c_+}{\theta} + \frac{c_- - c_+}{1 - \theta} q + \int_{\mathbb{R}} (e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1}) e^y \nu_\theta(-(e^y - 1)) dy, \quad -1 < q < \theta. \quad (3.6.7)$$

Note that compared to [KP21] we have inverted the constants c_+ and c_- , because we are considering the Lamperti representation of $-\eta^\theta$. Furthermore, observe that in this correspondence, the event E is $\{\forall 0 \leq b \leq \tau^0(a), \Delta \xi^0(b) > -\log(2)\}$. Thus Theorem 3.6.8 is proved as soon as we have established the following lemma. The result then follows by a simple application of the optional stopping theorem. See also Lemma 17 in [LGR20] and Lemma 3.5 in [AS20].

Lemma 3.6.10. *Under P_{-z} , the process*

$$M_a^{(\theta)} := |z|^{1+\theta} e^{-(1+\theta)\xi^0(a)} \mathbb{1}_{\forall 0 \leq b \leq a, \Delta\xi^0(b) > -\log(2)}, \quad a \geq 0,$$

is a martingale with respect to the canonical filtration of ξ^0 . Under the tilted probability measure $|z|^{1+\theta} e^{-(1+\theta)\xi^0(a)} \mathbb{1}_{\forall 0 \leq b \leq a, \Delta\xi^0(b) > -\log(2)} \cdot P_{-z}$, the process $(\xi^0(b), 0 \leq b \leq a)$ is a Lévy process with Laplace exponent Ψ given by equation (3.6.5).

Proof. By self-similarity, we may focus on the case $z = 1$, and we write $P = P_{-1}$ for simplicity. We aim at computing the quantity

$$E \left[e^{(q-1-\theta)\xi^0(a)} \mathbb{1}_{\forall 0 \leq b \leq a, \Delta\xi^0(b) > -\log(2)} \right].$$

To do this, we write $\xi_b^0 = \xi_b' + \xi_b''$, where $\xi_b'' := \sum_{0 \leq b \leq a} \Delta\xi_b^0 \mathbb{1}_{\Delta\xi^0(b) \leq -\log(2)}$ is the Poisson point process of the small jumps of ξ^0 . Then ξ' and ξ'' are independent, and so the previous expectation is

$$E \left[e^{(q-1-\theta)\xi^0(a)} \mathbb{1}_{\forall 0 \leq b \leq a, \Delta\xi^0(b) > -\log(2)} \right] = P(\xi_a'' = 0) E[e^{(q-1-\theta)\xi_a'}].$$

If we denote by Ψ' and Ψ'' the Laplace exponents of ξ' and ξ'' , then we have

$$E \left[e^{(q-1-\theta)\xi^0(a)} \mathbb{1}_{\forall 0 \leq b \leq a, \Delta\xi^0(b) > -\log(2)} \right] = e^{a(\Psi''(\infty) + \Psi'(q-1-\theta))}. \quad (3.6.8)$$

Therefore the calculation boils down to computing $\Psi''(\infty) + \Psi'(q-1-\theta)$. First of all, we know that the Lévy measure of ξ'' is the one of ξ^0 restricted to $(-\infty, -\log(2)]$, so that

$$\Psi''(q) = \int_{y \leq -\log(2)} (e^{qy} - 1) e^y \nu_\theta(-(e^y - 1)) dy, \quad q > -1. \quad (3.6.9)$$

Hence, by the expression of ν_θ in (3.6.3),

$$\Psi''(\infty) = -c_+ \int_{y \leq -\log(2)} \frac{e^y}{(1 - e^y)^{1+\theta}} dy = \frac{c_+}{\theta} (1 - 2^\theta).$$

It remains to compute $q \mapsto \Psi'(q-1-\theta)$. By independence of ξ' and ξ'' , we have for all $-1 < q < \theta$, $\Psi'(q) = \Psi^0(q) - \Psi''(q)$. Equations (3.6.7) and (3.6.9) provide

$$\begin{aligned} \Psi'(q) &= -\frac{c_+}{\theta} + q \left(\frac{c_- - c_+}{1 - \theta} - \frac{c_+}{1 - \theta} (1 - 2^{\theta-1}) \right) \\ &\quad + \int_{y > -\log(2)} (e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1}) e^y \nu_\theta(-(e^y - 1)) dy, \quad -1 < q < \theta, \end{aligned} \quad (3.6.10)$$

This extends analytically to all $q < \theta$. We now fix $q < 2\theta + 1$, and we want to put $\Psi'(q-1-\theta)$ in a Lévy-Khintchine form. Replacing q by $q-1-\theta$ in (3.6.10), we see that

$$\Psi'(q-1-\theta) = -\mathbf{k}' + \mathbf{d}'q + \int_{y > -\log(2)} (e^{qy} - 1 - q(e^y - 1) \mathbb{1}_{|e^y - 1| < 1}) e^{-\theta y} \nu_\theta(-(e^y - 1)) dy,$$

with

$$\mathbf{k}' := -\Psi'(-1-\theta),$$

and

$$\begin{aligned} \mathbf{d}' &:= \frac{c_- - c_+}{1 - \theta} + \frac{c_+}{1 - \theta} (1 - 2^{\theta-1}) + c_+ \int_{-\log(2)}^0 \left(e^y - 1 - (e^{(\theta+2)y} - e^{(\theta+1)y}) \right) \frac{e^{-\theta y}}{(1 - e^y)^{1+\theta}} dy \\ &\quad + c_- \int_0^{\log(2)} \left(e^y - 1 - (e^{(\theta+2)y} - e^{(\theta+1)y}) \right) \frac{e^{-\theta y}}{(e^y - 1)^{1+\theta}} dy. \end{aligned} \quad (3.6.11)$$

In order to retrieve equation (3.6.5), it remains to prove that $\mathbf{k}' = \Psi''(\infty)$ and $\mathbf{d}' = d$. Let us start with the latter identity. The first integral in the expression of $\mathbf{d}'$ is

$$\begin{aligned} \int_{-\log(2)}^0 \left(e^y - 1 - (e^{(\theta+2)y} - e^{(\theta+1)y}) \right) \frac{e^{-\theta y}}{(1 - e^y)^{1+\theta}} dy &= \int_{-\log(2)}^0 \frac{e^y}{(1 - e^y)^\theta} dy - \int_{-\log(2)}^0 \frac{e^{-\theta y}}{(1 - e^y)^\theta} dy \\ &= \int_{1/2}^1 \frac{1}{(1 - x)^\theta} dx - \int_{1/2}^1 \frac{1}{x^{\theta+1}(1 - x)^\theta} dx, \end{aligned}$$

with the change of variables $x = e^y$. Hence

$$\int_{-\log(2)}^0 \left(e^y - 1 - (e^{(\theta+2)y} - e^{(\theta+1)y}) \right) \frac{e^{-\theta y}}{(1 - e^y)^{1+\theta}} dy = \frac{2^{\theta-1}}{1 - \theta} - B_{1/2}(1 - \theta, -\theta),$$

where $B_{1/2}$ is the incomplete beta function. Likewise, we can compute the last integral in (3.6.11) as

$$\int_0^{\log(2)} \left(e^y - 1 - (e^{(\theta+2)y} - e^{(\theta+1)y}) \right) \frac{e^{-\theta y}}{(e^y - 1)^{1+\theta}} dy = -\frac{1}{1 - \theta} + B_{1/2}(1 - \theta, 2\theta).$$

These two computations yield $\mathbf{d}' = c_- B_{1/2}(1 - \theta, 2\theta) - c_+ B_{1/2}(1 - \theta, -\theta)$, which is d . Finally, we prove that $\mathbf{k}' = \Psi''(\infty)$. Thanks to [KP21], Theorem 1, we know that ξ^0 belongs to the class of so-called hypergeometric Lévy processes, so that $\Psi^0(q)$ is given by

$$\Psi^0(q) = -\frac{\Gamma(\theta - q)\Gamma(1 + q)}{\Gamma(\frac{1}{2} - q)\Gamma(\frac{1}{2} + q)}.$$

This can be extended analytically to all $q \notin \mathbb{Z} + \frac{1}{2}$, hence

$$\Psi^0(-1 - \theta) = -\frac{\Gamma(1 + 2\theta)\Gamma(-\theta)}{\Gamma(\frac{3}{2} + \theta)\Gamma(-\frac{1}{2} - \theta)} = -\frac{\Gamma(-\theta)}{2\Gamma(-2\theta)\sin(\pi\theta)}.$$

In addition, recall from (3.6.9) that for $q > -1$,

$$\Psi''(q) = c_+ \int_{y \leq -\log(2)} (e^{qy} - 1) \frac{e^y dy}{(1 - e^y)^{1+\theta}}.$$

The change of variables $x = e^y$ gives

$$\Psi''(q) = c_+ B_{1/2}(q + 1, -\theta) - \frac{c_+}{\theta} (1 - 2^\theta),$$

whence $\Psi''(q) = c_+ B_{1/2}(q + 1, -\theta) - \Psi''(\infty)$. The two-variable function $B_{1/2}(a, b)$ can be extended analytically to all $a, b \notin -\mathbb{N}$ via the identity $aB_{1/2}(a, b+1) - bB_{1/2}(a+1, b) = 2^{-a-b}$ obtained by straightforward integration by parts. We then need to evaluate $B_{1/2}$ at $(-\theta, -\theta)$. But for $q > 0$, $B_{1/2}(q, q) = \frac{1}{2}B(q, q) = \frac{\Gamma(q)^2}{2\Gamma(2q)}$ by symmetry. Uniqueness of analytic continuation implies that this must still hold for all $q \notin -\mathbb{N}$. This allows to write that $B_{1/2}(-\theta, -\theta) = \frac{\Gamma(-\theta)^2}{2\Gamma(-2\theta)}$. Since $c_+ = \frac{\Gamma(1+\theta)}{\pi}$, we have $c_+ B_{1/2}(-\theta, -\theta) = -\frac{\Gamma(-\theta)}{2\Gamma(-2\theta)\sin(\pi\theta)} = \Psi^0(-1 - \theta)$, and therefore

$$\Psi'(-1 - \theta) = \Psi^0(-1 - \theta) - \Psi''(-1 - \theta) = \Psi''(\infty).$$

Combining these expressions of $\mathbf{k}'$ and $\mathbf{d}'$ with equation (3.6.8), we retrieve the Laplace exponent Ψ of (3.6.5). $\square$

3.6.3 The temporal martingale

The results in this section hold in full generality beyond the spectrally negative case. We first point out a temporal martingale for excursions cut at heights. For $a \geq 0$, we will write $(e_i^{a,+})_{i \geq 1}$ for the possible excursions that u makes above a , and we let $(\mathcal{G}_a)_{a \geq 0}$ be the complete filtration of events occurring below level a , which can be defined formally by indexing the path u by the inverse of $t \mapsto \int_0^t \mathbf{1}_{\{y(s) \leq a\}} ds$. Recall the definition of the constants $C_{\pm}$ introduced in the disintegration property 3.6.6.

Proposition 3.6.11. *The process*

$$\mathcal{M}_a^\alpha = \sum_{i \geq 1} C_{sgn(\Delta e_i^{a,+})}^{-1} \cdot |\Delta e_i^{a,+}|^{1+\theta}, \quad a \geq 0,$$

is a $(\mathcal{G}_a)$ -martingale.

The proof follows from the same arguments as in [AS20]. The previous martingale points at a natural change of measure. Define μ_z^α for $z \neq 0$ as

$$\left. \frac{d\mu_z^\alpha}{d\gamma_z^\alpha} \right|_{\mathcal{G}_a} := \frac{\mathcal{M}_a^\alpha}{|z|^{1+\theta}}, \quad a \geq 0.$$

This new measure can be defined more rigourously as in [AS20]. We introduce two independent half-planar trajectories L^α, R^α as follows. The process L^α has a real part distributed as X^α and a three-dimensional Bessel imaginary part starting at 0. The process R^α has real part distributed as $-X^\alpha$ and a similar Bessel imaginary part. Also, for $u \in U$, we set $T_a := \inf\{s > 0, y(s) = a\}$ and $S_a := \inf\{s > 0, y(R(u) - s) = a\}$. Then one can make the following description of μ_z^α , similar to Theorem 3.8, [AS20].

Theorem 3.6.12. *Under μ_z^α , for any $a > 0$, the processes $(u(s), 0 \leq s \leq T_a)$ and $(u(R(u) - s), 0 \leq s \leq S_a)$ evolve independently as L^α and R^α stopped at their hitting times of $\{\Im(z) = a\}$.*

Under μ_z^α , the path u therefore splits into two infinite trajectories from 0 and z to ∞ . See Figure 3.5 below (the picture shows the spectrally negative case).

Remark 3.6.13. (i) If we believe that the sizes of the excursions cut at heights form a signed growth-fragmentation process $\mathbf{X}$ (this will be stated in the following section), then Theorem 3.6.12 describes the law of the *spine* defined in section 3.3.3. Namely, it is given by the time-reversal of the difference of (the real part of) trajectories R^α and L^α coming down from infinity, taken at a Brownian hitting time. Since the latter are subordinators of index $\frac{1}{2}$ and X^α is an α -stable Lévy process, this yields a stable Lévy process of index $\theta = \frac{\alpha}{2}$, and with positivity parameter $\rho' := 1 - \rho$.

(ii) We remark that, in the spectrally negative case, this is consistent with the martingales appearing in [BBCK18]. Indeed, they have the same form with a power given by $\omega_+ = \theta + \frac{3}{2} = (\theta + 1) + \frac{1}{2}$. Hence ω_+ is the power appearing in M_a plus one half. Moreover, since we have chosen c_+ and c_- such that $\theta(1 - \rho') = 1/2$, the h -transform used to condition the θ -stable process to remain positive is given by $x \mapsto x^{\theta(1-\rho')} = \sqrt{x}$ (see [CC06]). Therefore, under γ_z^α with $z > 0$, if $\overline{\mathbf{X}}^+$ denotes under the family of *positive* excursions obtained by removing from $\mathbf{X}$ the negative sizes (together with their progeny), then for any nonnegative measurable function f ,

$$\gamma_z^\alpha \left(\sum_{e \in \overline{\mathbf{X}}^+(a)} |\Delta e|^{1+\theta} \sqrt{|\Delta e|} f(\Delta e) \right) = E_z \left(f(Y_0^\theta(a)) \sqrt{|Y_0^\theta(a)|} \right) = E_z \left(f(Y_+^\theta(a)) \right)$$

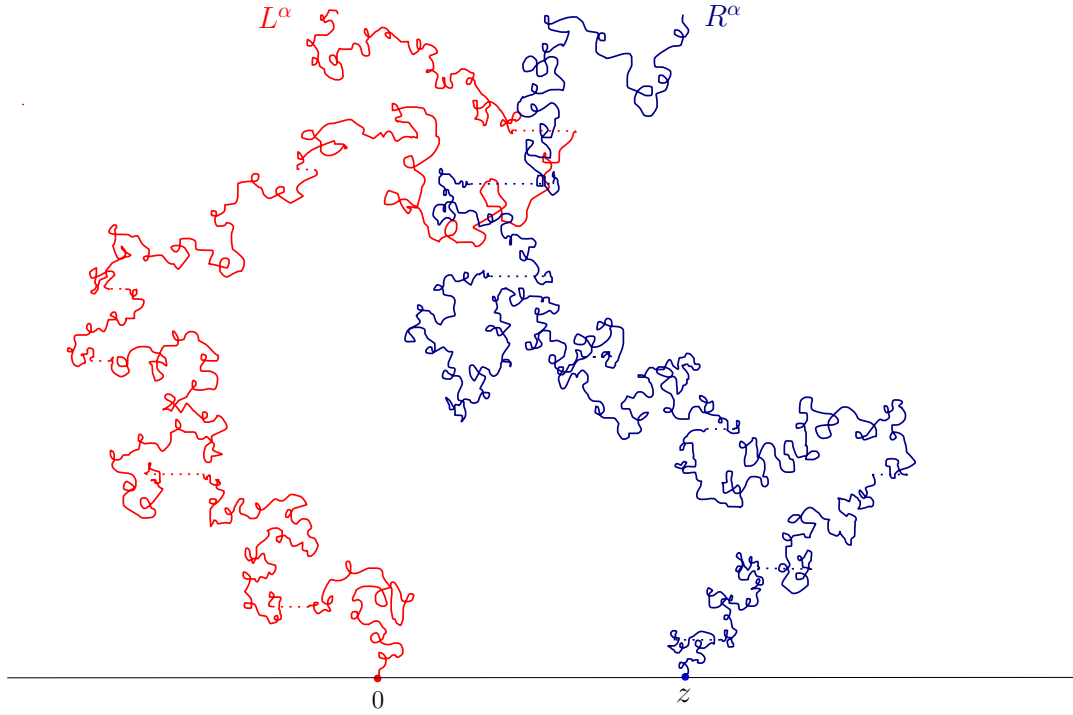


Figure 3.5 – Splitting the excursion according to the change of measure μ_z^α (in the case when X^α is spectrally negative). The red and blue trajectories are independent and evolve as L^α and R^α respectively.

where under P_z , Y_0^θ and Y_+^θ are respectively a θ -stable Lévy process killed below 0 and a θ -stable Lévy process conditioned to remain positive, both started at z . This gives both the martingale and the spine in the positive growth-fragmentation induced by $\mathbf{X}$.

3.6.4 The growth-fragmentation embedded in half-planar excursions

We now turn to the description of the cell system in terms of a growth-fragmentation process. The main results hold in general, but in order to retrieve the growth-fragmentation processes with no killing introduced by [BBCK18] for $\frac{1}{2} < \theta < 1$, we focus on the case when X^α is *spectrally negative*, where the law of the locally largest fragment was explicated in Theorem 3.6.8. Recall that this amounts to set the positivity parameter ρ of X^α so that $\theta\rho = \frac{1}{2}$. Building on the previous constructions, we have the following growth-fragmentation process.

Theorem 3.6.14. *The process*

$$\mathbf{X}(a) := \left\{ \left\{ \Delta e_i^{a,+}, i \geq 1 \right\} \right\}, \quad a \geq 0,$$

is a signed growth-fragmentation process.

The proof is a simple extension of the Brownian case [AS20]. Note that the discontinuities of u do not conflict with conservativity at times when the growth-fragmentation cells divide: indeed, by independence, they almost surely occur at levels which are not local minima for the Brownian motion. We now determine the spine under the change of measure $\hat{\mathcal{P}}_z$.

Theorem 3.6.15. *The vector $(C_+^{-1}, C_-^{-1}, \theta + 1)$ is admissible for the locally largest evolution Ξ . Under $\hat{\mathcal{P}}_z$, the spine $\hat{\mathcal{X}}$ defined in section 3.3.3 evolves as a θ -stable Lévy process with Lévy measure $2\nu_\theta(-y)dy$ and hence positivity parameter $\rho' = 1 - \rho$.*

In particular, the positive growth-fragmentation $\mathbf{X}^+$ obtained by removing from $\mathbf{X}$ all the negative cells and their descendants is the same that of [BBCK18], for the appropriate self-similarity index $\frac{1}{2} < \theta < 1$. Indeed, by the many-to-one lemma, the law of the spine $\hat{\mathcal{X}}^+$ for the cell system of *positive* masses is that of $\hat{\mathcal{X}}$ conditioned to stay positive, hence is distributed as the spine appearing in [BBCK18] for $\frac{1}{2} < \theta < 1$. Yet [BBCK18, Theorem 5.1] entails that the spine characterises the law of the growth-fragmentation, and thus $\mathbf{X}^+$ has the law of the growth-fragmentation process described in [BBCK18] for $\frac{1}{2} < \theta < 1$.

Proof. There are several ways to prove admissibility. For example, we use that $(\mathcal{M}_a, a \geq 0)$ is a martingale (Proposition 3.6.11), and we condition on the first generation (the offspring of Ξ) to obtain

$$C_{sgn(z)}^{-1}|z|^{1+\theta} = \mathcal{E}_z[\mathcal{M}_a] = C_+^{-1}E_z[|\Xi(a)|^{1+\theta}\mathbf{1}_{a < \Im(z\bullet)}] + E_z\left[\sum_{s < a} C_{sgn(\Delta\Xi(s))}^{-1}|\Delta\Xi(s)|^{1+\theta}\right].$$

We then let a tend to infinity and get that

$$E_z\left[\sum_{s > 0} C_{sgn(\Delta\Xi(s))}^{-1}|\Delta\Xi(s)|^{1+\theta}\right] = C_{sgn(z)}^{-1}|z|^{1+\theta}.$$

The Lamperti-Kiu representation of stable processes was established in [CPR13] (see Corollary 11): it is then a simple check to see that Theorem 3.5.4 gives the same matrix exponent. Alternatively, we can use the description of Theorem 3.6.12 of the spine as the difference of two θ -stable processes, together with a version of Proposition 3.3.7. $\square$

Chapter 4

Self-similar growth-fragmentation processes with types

Abstract

This chapter is based on joint work with Juan Carlos Pardo. It consists roughly of two parts. In the first part, we investigate multitype versions of the self-similar growth-fragmentation processes introduced in [Ber17b] with finitely many types, therefore extending the signed case of [Sil21]. Our main result in this direction describes the law of the spine in the multitype setting. We stress that our arguments only rely on the structure of the underlying Markov additive processes, and hence is more general than [Sil21]. In the second part, we study $\mathbb{R}^d$ -valued self-similar growth-fragmentation processes driven by an isotropic process. These can be seen as *multitype* growth-fragmentation processes, where the set of types is the sphere $\mathbb{S}^{d-1}$. We give the spinal description in this setting. Finally, we prove that such a family of processes shows up when slicing half-space excursions with hyperplanes.

4.1 Introduction

Self-similar growth-fragmentation processes first appeared in [Ber17b] to describe the evolution of a cloud of atoms which may grow and dislocate in a binary way. More precisely, these atoms are assumed to have a specific one-dimensional trait of interest, which we can think of as its *mass* or *size*. Initially, the cloud starts from one particle (the common ancestor of all future particles) whose size is a positive quantity evolving in time in a Markovian way. This size will have jumps, and at each negative jump $y < 0$ we wish to add to the cloud a new particle, whose size at birth will be given by $-y$, at the time when the jump occurs. This creates children of the original ancestor in such a way that the divisions are conservative, that is summing the size of the child and the size of the parent just after division exactly gives the size of the parent before division. Then, the newborn particles evolve independently of the parent, and independently of one another, in the same Markovian way as the parent. We proceed similarly creating the offspring of those particles, thereby introducing the grandchildren, great grandchildren, and so on, of the original ancestor.

Such growth-fragmentation models have been given a striking geometric flavour, in the context of random planar maps. This originated from [BCK18] and [BBCK18], where a remarkable class of self-similar growth-fragmentations shows up in the scaling limit of perimeter processes (see [Bud15]) in Markovian explorations of Boltzmann planar maps. These growth-fragmentation processes are closely related to stable Lévy processes with stability parameter $\theta \in (\frac{1}{2}, \frac{3}{2}]$. Since then, the same growth-fragmentation processes were

directly constructed in the continuum [MSW20] for $1 < \theta \leq \frac{3}{2}$ by drawing a CLE exploration on a quantum disc. Moreover, the boundary case $\theta = \frac{3}{2}$, corresponding to the random triangulations in [BCK18], already appeared in [LGR20] as the collection of perimeters obtained when slicing a Brownian disc at heights, up to a time-change which already appears in the geometric construction of [BBCK18, Section 6.5]. The critical Cauchy case $\theta = 1$, in turn, corresponds to slicing a Brownian half-plane excursion at heights [AS20]. This approach was recently extended to $\frac{1}{2} < \theta < 1$ [Sil21] by considering other half-plane excursions.

Let us point out that in [AS20] (and subsequently in [Sil21]), *negative* mass is taken into account in the system, whereas the aforementioned construction of growth-fragmentation processes deals with positive mass only. In those examples, the sign depends on the time orientation of the excursions. In particular, slicing a half-plane Brownian excursion only yields the critical case $\theta = 1$ in [BBCK18] provided one discards the negative cells. On a related note, the driving cell processes in the distinguished family of growth-fragmentations in [BBCK18] also have positive jumps (except for $\theta = \frac{3}{2}$). This has a geometric meaning: Boltzmann planar maps correspond to the gasket of a loop $O(n)$ -model, and the positive jumps occur when discovering a loop, which could then be explored. In the continuum, positive jumps also arise in [MSW20] when hitting a CLE loop for the first time. This prompted [Sil21] to provide a framework for self-similar *signed* growth-fragmentations.

Adding negative mass to the system presents some technical issues. The analysis of the positive case carried out in [Ber17b] and [BBCK18] relies heavily on the Lamperti representation [Lam72] for positive self-similar Markov processes, allowing for a large toolbox of Lévy techniques. This breaks down if one is willing to deal with signed processes: in other words, the effect of introducing a sign is to move from the class of Lévy processes to the one of Markov additive processes, see for instance [CPR13], [KKPW14], [KP21] and [PR13]. Part of this paper aims at extending the framework to a general (finite) set of types. This has a counterpart in the *pure* fragmentation setting, see for instance [Ste18b]. In this case, we show that natural martingales arise, in connection to the additive martingales appearing in the context of multitype branching random walks (Section 4.3). These martingales have the same form as in [Ber17b], except that they are weighted by the types. Following the same lines as [BBCK18, Theorem 4.2], our main theorem in the multitype setting (Theorem 4.4.3) describes the cell system under the change of measures with respect to these martingales (Section 4.4). We stress that, although the framework developed here includes the signed case which was already treated in [Sil21], our methodology is completely different. Indeed, [Sil21] hinges upon a change of driving cell process to reduce to the positive case, whereas in this paper we directly work with Markov additive processes.

Next, we were interested in extending the growth-fragmentation framework to $\mathbb{R}^d$ -valued Markov processes (Section 4.5). In this case, we take advantage of the interplay between self-similar Markov processes and Markov additive processes on the sphere $\mathbb{S}^{d-1}$, see [KP21]. This can be considered as a multitype growth-fragmentation model as described in the previous paragraph, where the types are the angles, so that they live in the (uncountable) set $\mathbb{S}^{d-1}$. Because of the complexity of the Markov additive process structure on uncountable state space, we mainly restrict to the *isotropic* setting, which forms a nicer subclass of self-similar Markov processes in $\mathbb{R}^d$. In this case, we provide martingales, and prove that the corresponding exponents can be found as the roots of a convex function reminiscent of the cumulant function defined in [Ber17b]. We prove that a similar spine decomposition holds under the associated change of measures (Section 4.6).

This lays the groundwork for the construction in Section 4.7 of a distinguished family of spatial growth-fragmentations. In light of [AS20] and [Sil21], we consider some excursions in the half-space $\{(x_1, \dots, x_d) \in \mathbb{R}^d, x_d > 0\}$. Slicing these excursions at heights along

the hyperplanes $\{x_d = a\}$, we obtain a collection of excursions which exhibit a branching structure. We define the *size* of such an excursion as the difference between the endpoint and the starting point (this is a vector in $\mathbb{R}^{d-1}$). We show that considering the collection of these sizes at varying heights constructs a special growth-fragmentation in $\mathbb{R}^{d-1}$. Finally, we specify the spine obtained in this context.

4.2 Self-similar Markov processes with types

We start by presenting some shared features of the Markov processes we will be interested in, revolving around the notion of self-similarity. We explain how to deal with *types* for self-similar processes, in the cases when the set of types is finite, countable, or the sphere $\mathbb{S}^{d-1}$. As a common thread and a key ingredient of our analysis, we will use the Lamperti-Kiu representation, which gives a bijection between these self-similar processes and a class of Markov additive processes. We refer to [KP21] for a detailed treatment of these questions.

Markov additive processes. Let E be a finite set, or more generally a locally compact, complete and separable metric space, endowed with a cemetery state $\dagger$. Let $(\xi(t), \Theta(t), t \geq 0)$ be a regular Feller process in $\mathbb{R} \times E$ with probabilities $\mathbb{P}_{x,\theta}$, $x \in \mathbb{R}$, $\theta \in E$, on $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $(\mathcal{G}_t)_{t \geq 0}$ the natural standard filtration associated with (ξ, Θ) . We say that (ξ, Θ) is a *Markov additive process* (MAP for short) if for every bounded measurable $f : \mathbb{R} \times E \rightarrow \mathbb{R}$, $s, t \geq 0$ and $(x, \theta) \in \mathbb{R} \times E$,

$$\mathbb{E}_{x,\theta} \left[f(\xi(t+s) - \xi(t), \Theta(t+s)) \mathbf{1}_{\{t+s < \varsigma\}} \middle| \mathcal{G}_t \right] = \mathbf{1}_{\{t < \varsigma\}} \mathbb{E}_{0,\Theta(t)} \left[f(\xi(s), \Theta(s)) \mathbf{1}_{\{s < \varsigma\}} \right],$$

where $\varsigma := \inf\{t > 0, \Theta(t) = \dagger\}$. Observe that the process Θ is itself a regular Feller process in E . We call ξ the *ordinate* and Θ the *modulator* of the MAP. The notation $\mathbb{P}_\theta := \mathbb{P}(\cdot \mid \xi(0) = 0 \text{ and } \Theta(0) = \theta)$ for $\theta \in E$ will be in force throughout the paper. Whilst MAPs have found a prominent role in e.g. classical applied probability models for queues and dams when Θ is a Markov chain (see for instance [Asm08] and [Iva11]), the case that Θ is a general Markov process has received somewhat less attention. However, this case has been treated in the literature before, see for instance [Çin75] and references therein.

Informally, one should think of a MAP as a natural extension of a Lévy process in the sense that Θ is an arbitrary well-behaved Markov process and $((\xi(t), \Theta(t))_{t \geq 0}, \mathbb{P}_{x,\theta})$ is equal in law to $((\xi(t) + x, \Theta(t))_{t \geq 0}, \mathbb{P}_\theta)$. Moreover when Θ is a Markov chain a more natural description can be given for the ordinate process ξ . Indeed it can be thought as the concatenation of Lévy processes which depend on the current type in E given by Θ . Here we are interested in two specific cases which will be developed below: the case when $E = \mathcal{I}$ is finite and the case when $E = \mathbb{S}^{d-1}$ which describes the angles of a process in $\mathbb{R}^d$.

Markov additive processes with finite type set $\mathcal{I}$. One particularly important situation is when the set of types $E = \mathcal{I}$ is finite, in which case Θ is a continuous-time Markov chain with values in $\mathcal{I}$. As we mentioned before, this case has been deeply studied, see for instance [Asm08], [Iva11] and the references therein.

An important property in this case, which in particular describes the structure of MAPs, is given by the following proposition, see [Iva11], [KKPW14], [KP21] or the survey [PR13].

Proposition 4.2.1. *The process (ξ, Θ) is a Markov additive process if and only if there exist independent sequences $(\xi_i^n, n \geq 0)_{i \in \mathcal{I}}$ and $(U_{i,j}^n, n \geq 0)_{i,j \in \mathcal{I}}$, all independent of Θ , such that:*

- for $i \in \mathcal{I}$, $(\xi_i^n, n \geq 0)$ is a sequence of i.i.d. Lévy processes,

- for $i, j \in \mathcal{I}$, $(U_{i,j}^n, n \geq 0)$ are i.i.d. random variables,
- if $(T_n)_{n \geq 0}$ denotes the sequence of jump times of the chain Θ (with the convention $T_0 = 0$), then for all $n \geq 0$,

$$\xi(t) = \left(\xi(T_n^-) + U_{\Theta(T_n^-), \Theta(T_n)}^n \right) \mathbf{1}_{\{n \geq 1\}} + \xi_{\Theta(T_n)}^n(t - T_n), \quad T_n \leq t < T_{n+1}. \quad (4.2.1)$$

We now turn to defining the *matrix exponent* of a MAP, which is the analogue of the Laplace exponent in the setting of Lévy processes. Without loss of generality, we assume that $\mathcal{I} = \{1, \dots, N\}$ where $N \in \mathbb{N} \cup \{\infty\}$, and that Θ is irreducible. We write $Q = (q_{i,j})_{1 \leq i,j \leq N}$ for its intensity matrix, and ρ_i , $i \in \mathcal{I}$ for the exponential time Θ takes to jump from state i to some other state. Also, we denote for all $i, j \in \mathcal{I}$, all on the same probability space, by ξ_i a Lévy process distributed as the ξ_i^n 's, and by $U_{i,j}$ a random variable distributed as the $U_{i,j}^n$'s, with the convention $U_{i,i} = 0$ and $U_{i,j} = 0$ if $q_{i,j} = 0$. Finally, we introduce the Laplace exponent ψ_i of ξ_i and the Laplace transform $G_{i,j}(z) := \mathbb{E}[e^{zU_{i,j}}]$ of $U_{i,j}$ (this defines a matrix $G(z)$ with entries $G_{i,j}(z)$). Then the matrix exponent F of (ξ, Θ) is defined as

$$F(z) := \text{diag}(\psi_1(z), \dots, \psi_N(z)) + Q \circ G(z), \quad (4.2.2)$$

where $\circ$ denotes pointwise multiplication of the entries. Then the following equality holds for all $i, j \in \mathcal{I}$, $z \in \mathbb{C}$, $t \geq 0$, whenever one side of the equality is defined:

$$\mathbb{E}_{0,i} \left[e^{z\xi(t)} \mathbf{1}_{\{\Theta(t)=j\}} \right] = (e^{F(z)t})_{i,j}.$$

Spectral properties of MAPs with finite type set $\mathcal{I}$. We also state for future reference the following classical results (see [Asm08, Iva11], or the survey [PR13]) about the leading eigenvalue of a MAP, often dubbed *Perron-Frobenius* eigenvalue. We consider a MAP (ξ, Θ) on $\mathbb{R} \times \mathcal{I}$ with matrix exponent F .

Proposition 4.2.2. *Let F denote the matrix exponent of some Markov additive process, and $z \in \mathbb{R}$ such that $F(z)$ is well-defined. Then the matrix $F(z)$ has a real simple eigenvalue $\chi(z)$, which is larger than the real parts of all its other eigenvalues. In addition, $\chi(z)$ is associated to a positive eigenfunction $w(z)$.*

The leading eigenvalue enables to identify the following *Wald martingale*, which is a multitype version of the exponential martingale for Lévy processes.

Proposition 4.2.3. *Fix γ such that $F(\gamma)$ is well-defined. With the notation of Proposition 4.2.2, let*

$$\mathcal{W}^{(\gamma)}(t) := \frac{w_{\Theta(t)}(\gamma)}{w_{\Theta(0)}(\gamma)} e^{\gamma\xi(t) - t\chi(\gamma)}, \quad t \geq 0.$$

Then $\mathcal{W}^{(\gamma)}$ is a martingale with respect to the natural filtration of the MAP, and under any initial distribution of $(\xi(0), \Theta(0))$. Moreover, the law of (ξ, Θ) under the corresponding change of measure is that of a Markov additive process with matrix exponent

$$F^{(\gamma)}(z) := \text{diag}(w_i(\gamma), i \in \mathcal{I})^{-1} (F(\gamma + z) - \chi(\gamma)\text{Id}) \text{diag}(w_i(\gamma), i \in \mathcal{I}).$$

In particular, the leading eigenvalue of $F^{(\gamma)}(z)$ is given by $\chi^{(\gamma)}(z) := \chi(\gamma + z) - \chi(\gamma)$.

The following property will also come in useful.

Proposition 4.2.4. *We take the notation of Proposition 4.2.2. Let D be an interval of $\mathbb{R}$ on which F is defined. Then the leading eigenvalue χ is smooth and convex on D .*

An important quantity associated to a MAP (ξ, Θ) on $\mathbb{R} \times \mathcal{I}$, particularly in view of the lifetime (4.2.6) appearing in the next paragraph in relation to the Lamperti-Kiu transform, is the so-called *exponential functional*, namely

$$I(\xi) := \int_0^\infty e^{\xi(s)} ds.$$

This quantity has been studied in great detail, first for Lévy processes (see, notably, [BY05, CPY97]), and then more recently for MAPs (see in particular [KKPW14, Ste18b]). We stress that the study of $I(\xi)$ usually involves the spectral properties of the MAP, and in particular the leading eigenvalue χ . We state for future reference the following lemma, giving a finiteness criterion for the moments of the exponential functional of Markov additive processes. This result does not seem to be contained in the existing literature, although it partially overlaps [KKPW14, Proposition 3.6]. The case of Lévy processes is also fully understood [Riv12, Lemma 3]. We say that (ξ, Θ) satisfies Cramér's hypothesis if there exists $\gamma_0 > 0$ and $\Upsilon \in (0, \gamma_0)$ such that F is defined on $(0, \gamma_0)$ and $\chi(\Upsilon) = 0$.

Proposition 4.2.5. *Assume that (ξ, Θ) satisfies Cramér's hypothesis, with Cramér number Υ . Then $I(\xi)$ has finite moment of order γ for all $\gamma < \Upsilon$, under any initial distribution of (ξ, Θ) .*

Proof. We remark that by convexity of χ (Proposition 4.2.4), $\chi'(0^-) < 0$ and $\chi'(\Upsilon) > 0$. This in turn implies by convexity that $\chi(\gamma) < 0$ for all $\gamma < \Upsilon$. Now fix $\gamma \in (0, \Upsilon)$ and $\varepsilon \in (0, \Upsilon)$. Jensen's inequality provides

$$\begin{aligned} \left(\int_0^\infty e^{\xi(s)} ds \right)^\gamma &= \left(\int_0^\infty e^{\xi(s)} (-\chi(\varepsilon))^{-1} e^{-s\chi(\varepsilon)} (-\chi(\varepsilon) e^{s\chi(\varepsilon)}) ds \right)^\gamma \\ &\leq (-\chi(\varepsilon))^{1-\gamma} \int_0^\infty e^{\gamma\xi(s) - (\gamma-1)s\chi(\varepsilon)} ds. \end{aligned} \quad (4.2.3)$$

Now write $C = \max_{i,j \in \mathcal{I}} \frac{w_j(\gamma)}{w_i(\gamma)} \cdot (-\chi(\varepsilon))^{1-\gamma} > 0$, and let $i \in \mathcal{I}$. Taking the $\mathbb{P}_{0,i}$ -expectation of (4.2.3), a rough estimate yields

$$\begin{aligned} \mathbb{E}_{0,i} \left[\left(\int_0^\infty e^{\xi(s)} ds \right)^\gamma \right] &\leq C \int_0^\infty \mathbb{E}_{0,i} [\mathcal{W}^{(\gamma)}(s) e^{s\chi(\gamma)}] e^{-(\gamma-1)s\chi(\varepsilon)} ds \\ &= C \int_0^\infty e^{s(\chi(\gamma) - (\gamma-1)\chi(\varepsilon))} ds, \end{aligned}$$

by the martingale property of $\mathcal{W}^{(\gamma)}$ in Proposition 4.2.3. Noting that $\chi(\gamma) < 0$ and $\chi(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we get that $\chi(\gamma) - (\gamma-1)\chi(\varepsilon) < 0$ for small enough ε , which completes the proof. $\square$

Self-similar Markov processes with types in $\mathcal{I}$. Similarly to the construction of positive self-similar Markov processes through Lévy processes, it is possible to build a more general family of self-similar Markov processes using MAPs. More precisely, let (ξ, Θ) be a MAP on $\mathbb{R} \times \mathcal{I}$, and fix $\alpha \in \mathbb{R}$. We construct the following process (X, J) with values in $\mathbb{R}_+ \times \mathcal{I}$ with a possible cemetery state ∂ via a Lamperti-type procedure. First, introduce

$$\varphi(t) := \inf \left\{ s > 0, \int_0^s \exp(\alpha\xi(u)) du > t \right\}, \quad t \geq 0. \quad (4.2.4)$$

Then, for $x > 0$, let

$$X(t) := x \exp(\xi(\varphi(tx^{-\alpha}))), \quad J(t) := \Theta(\varphi(tx^{-\alpha})), \quad t \geq 0, \quad (4.2.5)$$

with the convention that $(X(t), J(t)) = \partial$ when $t \geq \zeta$ where

$$\zeta := x^\alpha \int_0^\infty \exp(\alpha \xi(u)) du. \quad (4.2.6)$$

We write $\mathbb{P}_{x,i}$, $x > 0$, with $i \in \mathcal{I}$, for the law of (X, J) started from (x, i) , and $\mathbb{P}_i = \mathbb{P}_{1,i}$. It is plain from this construction that (X, J) is a Markov process, and that X is a *self-similar* process, that is to say for any $c > 0$ and for all $x > 0$, $i \in \mathcal{I}$,

$$\left((cX(c^{-\alpha}s), s \geq 0), \mathbb{P}_{x,i} \right) \stackrel{d}{=} \left((X(s), s \geq 0), \mathbb{P}_{cx,i} \right). \quad (4.2.7)$$

Conversely, if (X, J) is a Markov process in $\mathbb{R}_+ \times \mathcal{I}$, such that X is self-similar with index α in the sense of (4.2.7), then one can find a MAP such that (4.2.5) holds, with the time change (4.2.4). This construction is reminiscent of the Lamperti or Lamperti-Kiu representations [Lam72, CPR13, KKPW14] for positive or real-valued self-similar Markov processes respectively. In the latter case, the type J is the sign, see [CPR13, KKPW14]. We call this process (X, J) , or sometimes just X , a *self-similar Markov process with types*.

Self-similar Markov processes in $\mathbb{R}^d$ and isotropy. We now describe the other important case for our purposes, that is $E = \mathbb{S}^{d-1}$. Let $\alpha \in \mathbb{R}$. The Lamperti representation of α -self-similar $\mathbb{R}^d$ -valued Markov processes is the content of the following proposition which is attributed to [Kiu80] with additional clarification from [ACGŻ17], building on the original work of [Lam72].

Proposition 4.2.6. *Let X be a self-similar $\mathbb{R}^d$ -valued Markov process with index α . Then there exists a Markov additive process (ξ, Θ) in $\mathbb{R} \times \mathbb{S}^{d-1}$ such that*

$$X(t) = e^{\xi(\varphi(t))} \Theta(\varphi(t)), \quad t \leq I_\zeta := \int_0^\zeta e^{\alpha \xi(s)} ds, \quad (4.2.8)$$

where

$$\varphi(t) := \inf \left\{ s > 0, \int_0^s e^{\alpha \xi(u)} du > t \right\},$$

and I_ζ is the lifetime of X . Conversely, any process X satisfying (4.2.8) is a self-similar Markov process in $\mathbb{R}^d$ with index α .

In the previous statement we implicitly took the convention that $0 \times \dagger = 0$. The integral $\zeta = I_\zeta$ is the lifetime of X until it eventually hits 0, which acts as an absorbing state. For $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, we denote by $\mathbb{P}_{\mathbf{x}}$ for the law of X issued from $\mathbf{x}$.

The analysis of MAPs with infinite (and uncountable above all) state space is much more intricate. One way to capture their properties is using the celebrated compensation formula. It was proved in [Cin75] that any MAP (ξ, Θ) on $\mathbb{R} \times \mathbb{S}^{d-1}$ is associated with a so-called *Lévy system* (H, L) , made up of an increasing additive functional $t \mapsto H_t$ of Θ and a transition kernel L from $\mathbb{S}^{d-1}$ to $\mathbb{R}^* \times \mathbb{S}^{d-1}$ such that, for all $\theta \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{R}^*} (1 \wedge |x|^2) L_\theta(dx \times \{\theta\}) < \infty.$$

More importantly, this Lévy system satisfies the following *compensation formula* for all bounded measurable $F : \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$, and all $(x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$,

$$\begin{aligned} \mathbb{E}_{x,\theta} \left[\sum_{s>0} F(s, \xi(s^-), \Delta \xi(s), \Theta(s^-), \Theta(s)) \mathbf{1}_{\{\xi(s^-) \neq \xi(s) \text{ or } \Theta(s^-) \neq \Theta(s)\}} \right] \\ = \mathbb{E}_{x,\theta} \left[\int_0^\infty dH_s \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\Theta(s)}(dx, d\Phi) F(s, \xi(s), x, \Theta(s), \Phi) \right]. \end{aligned} \quad (4.2.9)$$

For the remainder of the paper we restrict ourselves to the usual setting $H_t = t$. Because of the bijection in Proposition 4.2.6, this naturally puts us in a restricted class of self-similar Markov processes through the underlying driving MAP. Observe how (4.2.9) compares with the compensation formula for Lévy processes: L essentially plays the role of a Lévy measure, albeit now depending on the current angle from which the process jumps.

A nice subclass of MAPs is provided by *isotropic* self-similar Markov processes, and we shall mainly restrict ourselves to this setting. We say that a self-similar Markov process X is isotropic if, for all isometry U , and all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, the law of $(U \cdot X(t), \mathbb{P}_{\mathbf{x}})$ is $\mathbb{P}_{U \cdot \mathbf{x}}$. Equivalently, this means [KP21, Theorem 11.14] that for all $(x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$, the law of $((\xi, U \cdot \Theta), \mathbb{P}_{x, \theta})$ is $\mathbb{P}_{x, U \cdot \theta}$. The key advantage of restricting to isotropic processes is the following proposition, which is [KP21, Corollary 11.15].

Proposition 4.2.7. *If X is an isotropic self-similar Markov process, then the underlying ordinate ξ is a Lévy process.*

Let us briefly mention that the proof of Proposition 4.2.7 relies on the fact that by isotropy, $|X|$ is a positive self-similar Markov process, for which we can apply the classical Lamperti theory. This result opens the way to many useful Lévy tools, such as the Lévy-Itô description of ξ , the compensation or exponential formulas, or the existence of an exponential martingale and the corresponding change of measures. We will make heavy use of these additional properties when describing growth-fragmentations driven by isotropic processes in Section 4.5. Note that this notion of isotropy in particular covers the α -stable isotropic Lévy case [?, Theorem 3.13], for which the Lévy system is given by $H_t = t$ and

$$L_\theta(dx, d\Phi) = \frac{c(\alpha)e^{dx}}{|e^x\Phi - \theta|^{\alpha+d}} dx \sigma_{d-1}(d\Phi),$$

where $c(\alpha) = 2^{\alpha-1} \pi^{-d} \frac{\Gamma((d+\alpha)/2) \Gamma(d/2)}{|\Gamma(-\alpha/2)|}$, and $\sigma_{d-1}(d\Phi)$ is the surface measure on the sphere $\mathbb{S}^{d-1}$. See also [BW96] for the planar case. Numerous applications of Lévy systems can be found in [KRS18, KSY20] to name but a few.

4.3 Multitype growth-fragmentation processes

In this section, we present an extension of the growth-fragmentation framework [Ber17b] to particles with finitely many types in $\mathcal{I}$. We point out that the approach here presented is completely different than the treatment [Sil21] of the signed case, which relies on a change of Eve cell to go back to the positive setting. We shall describe the martingales appearing in this context, and how they can be found in the roots of *multitype cumulants*.

4.3.1 Construction of the multitype growth-fragmentation cell system

Following section 4.2, we will consider either a càdlàg self-similar Markov process with types (X, J) . For technical reasons, we further assume that (X, J) is either absorbed at the cemetery state ∂ after a finite time ζ , or that X converges to 0 under $\mathbb{P}_{x, i}$ for all $x \in \mathbb{R}_+^*, i \in \mathcal{I}$. We write $\Delta X(s) := (X(s) - X(s^-)) \mathbf{1}_{\{X(s) < X(s^-)\}}$ for the jump of X , when it is negative.

We now construct a cell system whose building block is the self-similar Markov process with types (X, J) . This cell system will start from a single particle whose size and type are given by the process (X, J) , that will split in a binary way whenever X has a negative jump. This will create new particles with initial size given by the intensity of the jump, and which will then evolve as (X, J) independently of the mother cell, and independently of

one another, conditionally on their sizes at birth. This construction takes the viewpoint presented in [Ber17b], but note that in our context we need to clarify the types of the offspring. To this end, we introduce some preliminary notation. Call (ξ, Θ) the MAP underlying (X, J) via the Lamperti-Kiu transform (4.2.5). We assume throughout the paper that for all $i \in \mathcal{I}$, the Lévy measure Λ_i of ξ_i can be decomposed as a sum of Lévy measures

$$\Lambda_i(dx) := \sum_{k \in \mathcal{I}} \Lambda_i^k(dx), \quad (4.3.1)$$

satisfying the following integrability condition

$$\int_{\mathbb{R}} (1 \wedge |x|^2) \Lambda_i(dx) < \infty.$$

Likewise, for $i, j \in \mathcal{I}$ we give ourselves a decomposition of the law $\Lambda_{U_{i,j}}$ of $U_{i,j}$ as

$$\Lambda_{U_{i,j}}(dx) := \sum_{k \in \mathcal{I}} \Lambda_{U_{i,j}}^k(dx). \quad (4.3.2)$$

Equations (4.3.1) and (4.3.2) can be interpreted as a thinning of ξ_i and $U_{i,j}$ respectively: the jumps of ξ_i and $U_{i,j}$ should be understood as the result of a superimposition of jumps coming with a type $k \in \mathcal{I}$. Through the Lamperti time change (4.2.4), we see that any jump $\Delta X(s)$ of X now also comes with some type, that we denote $J_\Delta(s)$.

We may now construct the cell system associated with (X, J) and indexed by the tree $\mathbb{U} := \bigcup_{i \geq 0} \mathbb{N}^i$, with $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^0 := \{\emptyset\}$ is the label of the *Eve cell*. For $u := (u_1, \dots, u_i) \in \mathbb{U}$, we denote by $|u| = i$ the *generation* of u . In this tree, the offspring of u will be labelled by the lists $(u_1, \dots, u_i, k)$, with $k \in \mathbb{N}$.

We then define the law $\mathcal{P}_{x,i}$, $x > 0$, $i \in \mathcal{I}$, of the cell system $((\mathcal{X}_u(t), \mathcal{J}_u(t)), u \in \mathbb{U})$ driven by X similarly to [Ber17b]. Let $b_\emptyset = 0$ and take a copy $(\mathcal{X}_\emptyset, \mathcal{J}_\emptyset)$ of (X, J) started from (x, i) . We can rank the sequence of positive jumps and times $(x_1, \beta_1), (x_2, \beta_2), \dots$ of $-\mathcal{X}_\emptyset$ by descending lexicographical order of the x_k 's (this is possible because in any case X converges at infinity). We write $j_1, j_2, \dots$ for the corresponding types. Given this sequence $(x_k, j_k, \beta_k, k \in \mathbb{N})$, we define the first generation $(\mathcal{X}_k, \mathcal{J}_k)$, $k \in \mathbb{N}$, of our cell system as independent processes with respective law $\mathbb{P}_{x_k, j_k}$, and we set $b_k = b_\emptyset + \beta_k$ for the *birth time* of k and ζ_k for its *lifetime*. Likewise, we define the n -th generation given generations $1, \dots, n-1$. A cell $u' = (u_1, \dots, u_{n-1}) \in \mathbb{N}^{n-1}$ gives birth to the cell $u = (u_1, \dots, u_{n-1}, k)$, with lifetime ζ_u , at time $b_u = b_{u'} + \beta_k$ where β_k is the k -th jump of $\mathcal{X}_{u'}$ (in terms of the same ranking as before). Moreover, conditionally on the jump sizes, types and times of $\mathcal{X}_{u'}$, $(\mathcal{X}_u, \mathcal{J}_u)$ has law $\mathbb{P}_{y,j}$ and is independent of the other daughter cells at generation n , where $-y = \Delta \mathcal{X}_{u'}(\beta_k)$ comes with type j . Note that division events are *conservative* in the sense that the sum of the size of a particle born at time t and of its mother cell at time t exactly equals the size of the mother cell before dislocation.

Although by construction the cells are not labelled chronologically, this uniquely defines the law $\mathcal{P}_{x,i}$ of the cell system driven by (X, J) and started from (x, i) . The cell system $((\mathcal{X}_u(t), \mathcal{J}_u(t)), u \in \mathbb{U})$ is meant to describe the evolution of a population of atoms u with size $\mathcal{X}_u(t)$ and type $\mathcal{J}_u(t)$ evolving with its age t and fragmenting in a binary way.

Finally, we define the *multitype growth-fragmentation process*

$$\mathbf{X}(t) := \{(\mathcal{X}_u(t - b_u), \mathcal{J}_u(t - b_u)), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}, \quad t \geq 0,$$

where the double brackets denote multisets: $\mathbf{X}(t)$ is the collection of all the particles alive in the system at time t . We denote by $\mathbf{P}_{x,i}$ the law of $\mathbf{X}$ started from (x, i) and $(\mathcal{F}_t, t \geq 0)$ the natural filtration of $\mathbf{X}$.

Remark 4.3.1. We emphasize that only the *negative* jumps of X give birth to new cells. One could also be willing to create new particles at the positive jump times, corresponding to cells with negative mass, so that the conservation rule still holds at splittings, similarly as in [Sil21]. This will simply result in doubling the number of types of the chain J , by considering the sign itself as a type. Hence we can restrict without loss of generality to considering only positive cells, *i.e.* negative jumps.

It is clear from the definition of growth-fragmentation processes that the cell system enjoys a genealogical branching structure. Under mild assumptions, this extends to a temporal branching property. Construct

$$\bar{\mathbf{X}}(t) := \{(\mathcal{X}_u(t - b_u), \mathcal{J}_u(t - b_u), |u|), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u\}, \quad t \geq 0,$$

by adjunction of the generations to the growth-fragmentation process, and consider its associated filtration $(\bar{\mathcal{F}}_t, t \geq 0)$. A measurable function $f : \mathbb{R}_+ \rightarrow [0, +\infty)$ is called *excessive* for $\mathbf{X}$ if $\inf_{x > a} f(x) > 0$ for all $a > 0$, and for all $(x, i) \in \mathbb{R}_+ \times \mathcal{I}$ and all $t \geq 0$,

$$\mathbf{E}_{x,i} \left[\sum_{(z,j) \in \mathbf{X}(t)} f(z) \right] \leq f(x). \quad (4.3.3)$$

If such an excessive function exists, then one can rank the elements $(X_1(t), J_1(t)), (X_2(t), J_2(t)), \dots$ of $\mathbf{X}(t)$ by descending order of their size for any fixed t .

Proposition 4.3.2. *Assume that $\mathbf{X}$ has an excessive function. Then for any $t \geq 0$, conditionally on $\bar{\mathbf{X}}(t) = \{ \{(x_i, j_i, n_i)\} \}$, the process $(\bar{\mathbf{X}}(t + s), s \geq 0)$ is independent of $\bar{\mathcal{F}}_t$ and distributed as*

$$\bigsqcup_{i \geq 1} \bar{\mathbf{X}}_i(s) \circ \theta_{n_i},$$

where the $\mathbf{X}_i, i \geq 1$, are independent processes distributed as $\bar{\mathbf{X}}$ under $\mathbf{P}_{x_i, j_i}$, θ_n is the shift operator, *i.e.* $\{ \{(z_i, y_i, k_i), i \geq 1\} \} \circ \theta_n := \{ \{(z_i, y_i, k_i + n), i \geq 1\} \}$, and $\sqcup$ denotes union of multisets.

Proof. We refer to [Ber17b, Proposition 2] for a proof of the statement in the classical framework, which is then easily extended to the multitype case. $\square$

4.3.2 Martingales in multitype growth-fragmentation processes

We continue the study of martingales for multitype growth-fragmentation processes initiated in [Sil21] in the signed case. The fact that $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u \in \mathbb{U}}$ forms a multitype branching random walk provides several tools, including genealogical martingales for the growth-fragmentation cell system. The key feature is the following matrix $m(q)$ indexed by the type set $\mathcal{I}$, with entries

$$m_{i,j}(q) := \mathbb{E}_i \left[\sum_{0 < s < \zeta} |\Delta X(s)|^q \mathbf{1}_{\{J_\Delta(s)=j\}} \right], \quad q \in \mathbb{R}.$$

This matrix has only nonnegative entries. We make the following two assumptions throughout the paper.

Assumption A : *For $q \in \mathbb{R}$ such that $m(q)$ has finite entries, the matrix $m(q)$ is irreducible.*

In other words, Assumption A means that all the types communicate in the growth-fragmentation cell system (this is not too restrictive, since we could restrict to communication

classes otherwise). When $\mathcal{I}$ is finite, it enables us to consider the Perron-Frobenius eigenvalue $e^{\lambda(q)}$ and an associated positive eigenvector.

Assumption B : *There exists $\omega \in \mathbb{R}$ such that $\lambda(\omega) = 0$.*

We shall give a criterion for Assumption B later on in section 4.3.3. If $(v_i)_{i \in \mathcal{I}}$ has positive entries and $\omega \geq 0$, we say that $((v_i)_{i \in \mathcal{I}}, \omega)$ is *admissible* for X if $\lambda(\omega) = 0$, and $(v_i)_{i \in \mathcal{I}}$ is an associated eigenvector of $m(\omega)$. In other words, $((v_i)_{i \in \mathcal{I}}, \omega)$ is admissible for X if, and only if,

$$\forall i \in \mathcal{I}, \quad \mathbb{E}_i \left[\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] = v_i.$$

This invariance property extends to a genealogical martingale as follows. Define

$$\mathcal{G}_n := \sigma(\mathcal{X}_u, |u| \leq n),$$

noting that by definition, if $u \in \mathbb{U}$ is such that $|u| = n + 1$, then $\mathcal{X}_u(0)$ is $\mathcal{G}_n$ -measurable.

Proposition 4.3.3. *For all $(x, i) \in \mathbb{R}_+^* \times \mathcal{I}$, the process*

$$\mathcal{M}(n) := \sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega, \quad n \geq 0,$$

defines a $(\mathcal{G}_n, n \geq 0)$ -martingale under $\mathbb{P}_{x,i}$.

Proof. The process $\mathcal{M}$ is obtained as the genealogical martingale of the multitype branching random walk $(-\log \mathcal{X}_u(0), \mathcal{J}_u(0))_{u \in \mathbb{U}}$, see [Sil21, Theorem 3.3]. $\square$

Moreover, the following martingale for X will turn out useful in the next section. In particular, it implies the existence of an excessive function by extending [Ber17b, Theorem 1] to the multitype case.

Proposition 4.3.4. *For all $(x, i) \in \mathbb{R}_+^* \times \mathcal{I}$, under $\mathbb{P}_{x,i}$ the process*

$$M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0 < s \leq t \wedge \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega, \quad t \geq 0,$$

is a uniformly integrable martingale for the filtration $(F_t^X, t \geq 0)$ of X , with terminal value $\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega$.

Proof. We omit the proof as it essentially follows from [Sil21, Proposition 3.5]. $\square$

4.3.3 Multitype cumulant functions

For any sequence $((v_i)_{i \in \mathcal{I}}, \omega)$, define

$$M(t) := v_{J(t)} |X(t)|^\omega + \sum_{0 < s \leq t \wedge \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega, \quad s \geq 0,$$

where we omit the dependence on ω and $(v_i)_{i \in \mathcal{I}}$ in the notation of M for simplicity. Proposition 4.3.4 states that when the underlying sequence is admissible, M is a martingale under $\mathbb{P}_i$ for all $i \in \mathcal{I}$ (see the signed case). A converse statement also holds, providing a more tractable characterisation of admissibility.

Proposition 4.3.5. *Let H be the first jump time of J . Then $((v_i)_{i \in \mathcal{I}}, \omega)$ is admissible for X if and only if, for all $i \in \mathcal{I}$,*

$$\mathbb{E}_i[M(H)] = v_i.$$

Proof. The implication $(\Rightarrow)$ follows easily from the optional stopping theorem applied to the martingale M in Proposition 4.3.4. Conversely, if we denote $(H_k, k \geq 0)$, the successive jump times of J (with $H_0 = 0$), then for any $i \in \mathcal{I}$, by the Markov property of (X, J) and self-similarity of X ,

$$\begin{aligned} \mathbb{E}_i \left[\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] &= \sum_{k \geq 0} \mathbb{E}_i \left[\sum_{H_k < s \leq H_{k+1}} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] \\ &= \sum_{k \geq 0} \mathbb{E}_i \left[|X(H_k)|^\omega \mathbb{E}_{J(H_k)} \left[\sum_{s \leq H} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] \right]. \end{aligned}$$

Because we have assumed $\mathbb{E}_j[M(H)] = v_j$ for all $j \in \mathcal{I}$, this is

$$\mathbb{E}_i \left[\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_i \left[|X(H_k)|^\omega \left(v_{J(H_k)} - \mathbb{E}_{J(H_k)}[v_{J(H)} |X(H)|^\omega] \right) \right].$$

Hence, using again the Markov property and self-similarity of X backwards, we find ourselves with

$$\mathbb{E}_i \left[\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] = \sum_{k \geq 0} \mathbb{E}_i \left[v_{J(H_k)} |X(H_k)|^\omega \right] - \sum_{k \geq 0} \mathbb{E}_i \left[v_{J(H_{k+1})} |X(H_{k+1})|^\omega \right],$$

which ultimately cancels out, leaving $\mathbb{E}_i \left[\sum_{0 < s < \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] = v_i$. $\square$

Next, we identify multitype cumulant functions $\mathcal{K}_i$, $i \in \mathcal{I}$, whose common roots correspond to the admissible exponents ω . To do so, we compute $\mathbb{E}_i[M(H)]$ in terms of the underlying MAP characteristics, for any (not necessarily admissible) sequence $((v_i)_{i \in \mathcal{I}}, \omega)$. The expectation can be written as $\mathbb{E}_i[M(H)] = A + B$, where

$$A := \mathbb{E}_i \left[\sum_{0 < s \leq H \wedge \zeta} v_{J_\Delta(s)} |\Delta X(s)|^\omega \right] \quad \text{and} \quad B := \mathbb{E}_i \left[v_{J(H)} |X(H)|^\omega \right].$$

Let us start with the term A . For $s > 0$, we write as in (4.2.5), $X(\varphi^{-1}(s)) = e^{\xi(s)}$ and $J(\varphi^{-1}(s)) = \Theta(s)$ under $\mathbb{P}_i$, where φ is the usual time-change (4.2.4). From this standpoint,

$$A = \mathbb{E}_i \left[\sum_{0 < s < \rho_i} v_{\iota_\Delta(s)} e^{\omega \xi_i(s^-)} (1 - e^{\Delta \xi_i(s)})^\omega \right] + \mathbb{E}_i \left[v_{\iota_\Delta(\rho_i)} e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_{i, \Theta(\rho_i)}})^\omega \right], \quad (4.3.4)$$

where $\iota_\Delta(s)$ stands for the type corresponding to the jump of ξ at time s . By independence and the compensation formula for ξ_i , the first term of (4.3.4) is

$$\begin{aligned} &\mathbb{E}_i \left[\sum_{s < \rho_i} v_{\iota_\Delta(s)} e^{\omega \xi_i(s^-)} (1 - e^{\Delta \xi_i(s)})^\omega \right] \\ &= \int_0^\infty dt (-q_{i,i}) e^{q_{i,i}t} \sum_{k \in \mathcal{I}} v_k \mathbb{E}_i \left[\int_0^t ds e^{\omega \xi_i(s)} \right] \int_{-\infty}^0 \Lambda_i^k(dx) (1 - e^x)^\omega \\ &= \sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Lambda_i^k(dx) (1 - e^x)^\omega \cdot \frac{1}{\psi_i(\omega)} \int_0^\infty dt (-q_{i,i}) e^{q_{i,i}t} (e^{\psi_i(\omega)t} - 1) \\ &= -\frac{1}{\psi_i(\omega) + q_{i,i}} \cdot \sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Lambda_i^k(dx) (1 - e^x)^\omega, \end{aligned}$$

provided $\psi_i(\omega) + q_{i,i} < 0$ (otherwise the expectation blows up). Now let $\iota^* = \Theta(\rho_i)$ be the type to which the Markov chain jumps at time ρ_i . Then ι^* is independent of ρ_i , and for all $j \in \mathcal{I} \setminus \{i\}$, $\iota^* = j$ with probability $-\frac{q_{i,j}}{q_{i,i}}$. By conditioning on ρ_i , we obtain that the second term of (4.3.4) is

$$\begin{aligned} & \mathbb{E}_i \left[v_{\iota_\Delta(\rho_i)} e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_{i,\iota^*}})^\omega \right] \\ &= \int_0^\infty dt (-q_{i,i}) e^{q_{i,i}t} \sum_{j \in \mathcal{I} \setminus \{i\}} \frac{q_{i,j}}{(-q_{i,i})} \mathbb{E}_i \left[e^{\omega \xi_i(t)} \right] \sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Lambda_{U_{i,j}}^k(dx) (1 - e^x)^\omega \\ &= \int_0^\infty dt e^{(\psi_i(\omega) + q_{i,i})t} \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} \sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Lambda_{U_{i,j}}^k(dx) (1 - e^x)^\omega \\ &= -\frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{k \in \mathcal{I}} v_k \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} \int_{-\infty}^0 \Lambda_{U_{i,j}}^k(dx) (1 - e^x)^\omega, \end{aligned}$$

provided again that $\psi_i(\omega) + q_{i,i} < 0$. Therefore, we end up with

$$A = -\frac{1}{\psi_i(\omega) + q_{i,i}} \sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Pi_{i,k}(dx) (1 - e^x)^\omega,$$

with $\Pi_{i,k}(dx) := \Lambda_i^k(dx) + \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} \Lambda_{U_{i,j}}^k(dx)$.

We now compute

$$B = \mathbb{E}_i \left[v_{\iota^*} e^{\omega(\xi_i(\rho_i) + U_{i,\iota^*})} \right].$$

As before, we condition on ρ_i and decompose over the possible values $j \in \mathcal{I} \setminus \{i\}$ for ι^* :

$$\begin{aligned} B &= \sum_{j \in \mathcal{I} \setminus \{i\}} \int_0^\infty ds (-q_{i,i}) e^{q_{i,i}s} \frac{q_{i,j}}{(-q_{i,i})} v_j \mathbb{E}_i \left[e^{\omega(\xi_i(s) + U_{i,j})} \right] \\ &= \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} v_j \int_0^\infty ds e^{q_{i,i}s} e^{\psi_i(\omega)s} G_{i,j}(\omega) \\ &= -\frac{1}{\psi_i(\omega) + q_{i,i}} \cdot \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} v_j G_{i,j}(\omega), \end{aligned}$$

as long as $\psi_i(\omega) + q_{i,i} < 0$. We come to the conclusion that

$$\mathbb{E}_i[M(H)] = -\frac{1}{\psi_i(\omega) + q_{i,i}} \cdot \left(\sum_{k \in \mathcal{I}} v_k \int_{-\infty}^0 \Pi_{i,k}(dx) (1 - e^x)^\omega + \sum_{j \in \mathcal{I} \setminus \{i\}} q_{i,j} v_j G_{i,j}(\omega) \right).$$

This is equal to v_i if, and only if,

$$\mathcal{K}_i(\omega) := (\psi_i(\omega) + q_{i,i}) + \sum_{k \in \mathcal{I}} \frac{v_k}{v_i} \int_{-\infty}^0 \Pi_{i,k}(dx) (1 - e^x)^\omega + \sum_{j \in \mathcal{I} \setminus \{i\}} \frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) = 0,$$

and, thanks to Proposition 4.3.5, Assumption A in Section 4.3.2 boils to the existence of $\omega \in \mathbb{R}$ and a sequence $(v_i)_{i \in \mathcal{I}}$ of positive numbers such that, for all $i \in \mathcal{I}$, $\mathcal{K}_i(\omega) = 0$. We will call the family $(\mathcal{K}_i, i \in \mathcal{I})$ the *multitype cumulant functions*. We also write

$$\kappa_i(q) := (\psi_i(q) + q_{i,i}) + \int_{-\infty}^0 \Pi_{i,i}(dx) (1 - e^x)^q, \quad q \geq 0, \quad (4.3.5)$$

for the *cumulant function* corresponding to type i , so that for $q \geq 0$,

$$\mathcal{K}_i(q) := \kappa_i(q) + \sum_{j \in \mathcal{I} \setminus \{i\}} \frac{v_j}{v_i} \left(\int_{-\infty}^0 \Pi_{i,j}(dx) (1 - e^x)^q + q_{i,j} G_{i,j}(q) \right). \quad (4.3.6)$$

4.4 The spine decomposition of multitype growth-fragmentation processes

4.4.1 Description of the spine under the change of measure

A change of measure. The martingale $(\mathcal{M}(n), n \geq 0)$ in Proposition 4.3.3 enables us to introduce a new probability measure $\widehat{\mathcal{P}}_{x,i}$ for $x > 0, i \in \mathcal{I}$. Under this change of measure, the cell system has a spine decomposition that we aim to describe (see [BBCK18, Section 4.1]). The measure $\widehat{\mathcal{P}}_{x,i}$ singles out a particular *leaf* $\mathcal{L} \in \partial\mathbb{U} = \mathbb{N}^{\mathbb{N}}$. On $\mathcal{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative $\mathcal{M}(n)$ with respect to $\mathcal{P}_{x,i}$, up to normalisation, *viz.* for all $G_n \in \mathcal{G}_n$,

$$\widehat{\mathcal{P}}_{x,i}(G_n) := \frac{1}{v_i|x|^\omega} \mathcal{E}_{x,i}[\mathcal{M}(n)\mathbf{1}_{G_n}].$$

Moreover, conditionally on $\mathcal{G}_n$, the parent of the particular leaf $\mathcal{L}$ at generation $n+1$ is chosen under $\widehat{\mathcal{P}}_{x,i}$ proportionally to its weight in the martingale $\mathcal{M}(n)$. More precisely, let $\ell(n)$ denote the ancestor of the leaf $\ell \in \partial\mathbb{U}$ at generation n . Then for all $n \geq 0$ and all $u \in \mathbb{U}$ such that $|u| = n+1$,

$$\widehat{\mathcal{P}}_{x,i}(\mathcal{L}(n+1) = u \mid \mathcal{G}_n) := \frac{v_{\mathcal{J}_u(0)}|\mathcal{X}_u(0)|^\omega}{\mathcal{M}(n)}. \quad (4.4.1)$$

The consistency of formula (4.4.1) stems from the martingale property of $(\mathcal{M}(n), n \geq 0)$ and the branching structure of the system, thus defining a unique probability measure by an application of the Kolmogorov extension theorem.

One key player is provided by the so called *tagged cell* or *spine*, which consists in following the evolution of the cell associated with the leaf $\mathcal{L}$. The tagged cell will have the role of a backbone in the *spine decomposition* of the cell system under $\widehat{\mathcal{P}}_{x,i}$. Let $b_\ell = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial\mathbb{U}$. Then, we define $\widehat{\mathcal{X}}$ by $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) := \partial$ if $t \geq b_\mathcal{L}$ and

$$\widehat{\mathcal{X}}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{and} \quad \widehat{\mathcal{J}}(t) := \mathcal{J}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}) \quad \text{if} \quad t < b_\mathcal{L},$$

where n_t is the unique integer n such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$. From the very definition of $\widehat{\mathcal{P}}_{x,i}$, for all nonnegative measurable function f and all $\mathcal{G}_n$ -measurable nonnegative random variable B_n ,

$$v_i|x|^\omega \widehat{E}_{x,i} \left[f(\mathcal{X}_{\mathcal{L}(n+1)}(0), \mathcal{J}_{\mathcal{L}(n+1)}(0)) B_n \right] = \mathcal{E}_{x,i} \left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega f(\mathcal{X}_u(0), \mathcal{J}_u(0)) B_n \right].$$

This extends to a temporal identity in the following way. Recall that

$$\mathbf{X}(t) = \{ \{(X_k(t), J_k(t)), k \geq 1\} \}, \quad t \geq 0,$$

has been enumerated by descending order of the $X_k(t)$.

Proposition 4.4.1. *For every $t \geq 0$, every nonnegative measurable function f vanishing at ∂ , and every $\overline{\mathcal{F}}_t$ -measurable nonnegative random variable B_t , we have*

$$v_i|x|^\omega \widehat{E}_{x,i} \left[f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t)) B_t \right] = \mathcal{E}_{x,i} \left[\sum_{k \geq 1} v_{J_k(t)} |X_k(t)|^\omega f(X_k(t), J_k(t)) B_t \right].$$

Proof. The proof essentially follows from the arguments presented in the proof of [BBCK18, Proposition 4.1].

Let $t \geq 0$. Consider the case when B_t is $\overline{\mathcal{F}}_t \cap \mathcal{G}_k$ -measurable for some $k \in \mathbb{N}$ (the result would then be readily extended by a monotone class argument). Since $f(\partial) = 0$, almost surely,

$$f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}} \xrightarrow{n \rightarrow \infty} f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))B_t.$$

Therefore, by monotone convergence,

$$\widehat{E}_{x,i} [f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))B_t] = \lim_{n \rightarrow \infty} \widehat{E}_{x,i} [f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}}].$$

Now, we want to condition on $\mathcal{G}_n$ and decompose $\mathcal{L}(n+1)$ over the cells at generation $n+1$, provided $n > k$ so that B_t is $\mathcal{G}_n$ -measurable. For u such that $b_u > t$, write $u(t)$ for the most recent ancestor of u at time t . Then

$$\begin{aligned} & \widehat{E}_{x,i} [f(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))B_t \mathbf{1}_{\{b_{\mathcal{L}(n+1)} > t\}}] \\ &= \frac{1}{v_i(\omega)|x|^\omega} \mathcal{E}_{x,i} \left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t \right]. \end{aligned}$$

Splitting over the value of $u(t)$ yields

$$\begin{aligned} & \mathcal{E}_{x,i} \left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t \right] \\ &= \mathcal{E}_{x,i} \left[\sum_{|u'| \leq n} \sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \mathbf{1}_{\{u(t)=u'\}} \right], \end{aligned} \tag{4.4.2}$$

and by conditioning on $\overline{\mathcal{F}}_t$ and applying the temporal branching property stated in Proposition 4.3.2,

$$\begin{aligned} & \mathcal{E}_{x,i} \left[\sum_{|u|=n+1} v_{\mathcal{J}_u(0)} |\mathcal{X}_u(0)|^\omega \mathbf{1}_{\{b_u > t\}} f(\mathcal{X}_{u(t)}(t - b_{u(t)}), \mathcal{J}_{u(t)}(t - b_{u(t)})) B_t \right] \\ &= \mathcal{E}_{x,i} \left[\sum_{|u'| \leq n} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \right. \\ & \quad \times \mathcal{E}_{\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})} \left[\sum_{|u|=n+1-|u'|} v_{u'u} |\mathcal{X}_{u'u}(0)|^\omega \mathbf{1}_{\{b_{u'} \leq t < b_{u'} + \zeta_{u'}\}} \right] \\ & \quad \left. \right] \\ &= \mathcal{E}_{x,i} \left[\sum_{|u'| \leq n} f(\mathcal{X}_{u'}(t - b_{u'}), \mathcal{J}_{u'}(t - b_{u'})) B_t \mathbf{1}_{\{b_{u'} \leq t < b_{u'} + \zeta_{u'}\}} v_{\mathcal{J}_{u'}(t - b_{u'})}(\omega) |\mathcal{X}_{u'}(t - b_{u'})|^\omega \right]. \end{aligned}$$

Finally, taking $n \rightarrow \infty$ and using monotone convergence, we obtain the desired result. $\square$

Remark 4.4.2. Proposition 4.4.1 applied with $f := \mathbf{1}_{\{x \neq \partial\}}$ yields that the temporal analogue of $\mathcal{M}(n)$ in Proposition 4.3.3,

$$\mathcal{M}_t := \sum_{i=1}^{\infty} v_{J_i(t)} |X_i(t)|^\omega, \quad t \geq 0,$$

is a supermartingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

The law of the growth-fragmentation under $\widehat{\mathcal{P}}_{x,i}$. We now describe the law of $\widehat{\mathbf{X}}$ under $\widehat{\mathcal{P}}_{x,i}$. Loosely speaking, the tagged cell will serve as a backbone evolving as some explicit self-similar multitype Markov process, to which we attach independent copies of the original growth-fragmentation process. We must first reconstruct the whole cell system from the spine by recording the negative jumps of $\widehat{\mathcal{X}}$, as detailed in [BBCK18, Section 4.1]. We will label these by couples (n, j) , where $n \geq 0$ is the generation of the tagged cell immediately before the jump, and $j \geq 1$ is the rank (for the usual ranking) of the jump among those of the tagged cell at generation n (including the final jump when the generation changes from n to $n+1$). To each such (n, j) corresponds a growth-fragmentation $\widehat{\mathbf{X}}_{n,j}$ stemming from the corresponding jump: if the generation does not change during the (n, j) -jump, then we set

$$\widehat{\mathbf{X}}_{n,j}(t) := \{(\mathcal{X}_{uw}(t - b_{uw} + b_u), \mathcal{J}_{uw}(t - b_{uw} + b_u)), w \in \mathbb{U} \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\},$$

where u is the label of the cell born at the (n, j) -jump. Otherwise, the (n, j) -jump corresponds to a jump for the generation of the tagged cell and the tagged cell jumps from label u to label uk say, in which case

$$\begin{aligned} & \widehat{\mathbf{X}}_{n,j}(t) \\ &:= \{(\mathcal{X}_{uw}(t - b_{uw} + b_{uk}), \mathcal{J}_{uw}(t - b_{uw} + b_{uk})), w \in \mathbb{U} \setminus \{k\} \text{ and } b_{uw} \leq t + b_{uk} < b_{uw} + \zeta_{uw}\}. \end{aligned}$$

We agree that $\widehat{\mathbf{X}}_{n,j} := \partial$ when the (n, j) -jump does not exist, and this completely defines $\widehat{\mathbf{X}}_{n,j}$ for all $n \geq 0$ and all $j \geq 1$.

Let $\widehat{F}(q) := (\widehat{F}_{i,j}(q))_{i,j \in \mathcal{I}}$ be the matrix with entries

$$\widehat{F}_{i,j}(q) = \begin{cases} \frac{v_j}{v_i} \left(\int_{\mathbb{R}^*} \Pi_{i,j}(dx) |e^x - 1|^{q+\omega} + q_{i,j} G_{i,j}(q + \omega) \right) & \text{if } i \neq j, \\ \kappa_i(\omega + q) & \text{if } i = j. \end{cases} \quad (4.4.3)$$

Theorem 4.4.3. *Under $\widehat{\mathcal{P}}_{x,i}$, $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t), 0 \leq t < b_{\mathcal{L}})$ is a self-similar Markov process with types in $\mathcal{I}$, whose underlying Markov additive process has the matrix exponent $\widehat{F}$ in (4.4.3).*

Moreover, conditionally on $(\widehat{\mathcal{X}}(t), \widehat{\mathcal{J}}(t))_{0 \leq t < b_{\mathcal{L}}}$, the processes $\widehat{\mathbf{X}}_{n,j}$, $n \geq 0$, $j \geq 1$, are independent and each $\widehat{\mathbf{X}}_{n,j}$ has law $\mathbf{P}_{x(n,j)}$ where $-x(n, j)$ is the size of the (n, j) -th jump.

Remark 4.4.4. (i) The law of the generation n_t of the spine at time t is not so explicit as in [BBCK18] or [Sil21] in the constant sign case. In fact, $b_{\mathcal{L}(1)}$ may not be exponential because of the current type of the spine before it jumps.

(ii) The proof of Theorem 4.4.3 goes through determining all three components $\widehat{\psi}_i$, $\widehat{q}_{i,j}$, and $\widehat{G}_{i,j}$ of the MAP in (4.2.2). This sheds light on the structure of the MAP under (4.4.3).

4.4.2 Martingale exponents in multitype growth-fragmentation processes

We postpone the proof of Theorem 4.4.3 until Section 4.4.3, and discuss instead some applications, which are also new for the signed case [Sil21]. First, we prove that there exist at most two exponents ω satisfying **Assumption B** of Section 4.3.2. In the usual framework, this is straightforward from the convexity of the cumulant function, see [Ber17b, Section 3.2]. Because of the convoluted expression of the multitype cumulant functions (4.3.6), which involves the positive eigenvector v , our arguments are different, although also relying on

some convexity. Recall from Section 4.3.2 the notation $m(q), q \in \mathbb{R}$, for the matrix with nonnegative entries

$$m_{i,j}(q) := \mathbb{E}_i \left[\sum_{0 < s < \zeta} |\Delta X(s)|^q \mathbf{1}_{\{J_\Delta(s)=j\}} \right],$$

and $e^{\lambda(q)}$ for the Perron-Frobenius eigenvalue of $m(q)$. We will also make use of the spectral properties of Markov additive processes listed in Section 4.2. In particular, Proposition 4.2.2 applies for the matrix exponent $\widehat{F}$ of some spine in our growth-fragmentation process (see (4.4.3)): in this case, we shall denote by $\widehat{\chi}(q)$ the leading eigenvalue.

Proposition 4.4.5. *There exist at most two exponents ω such that $\lambda(\omega) = 0$.*

Proof. Suppose that ω is a given root of λ , and denote the associated positive eigenvector of $m(\omega)$ by v . Let ω' be another root of λ associated to a positive eigenvector v' , and set $\Delta\omega := \omega' - \omega$. Write $\widehat{F}$ for the matrix corresponding to ω and v via (4.4.3) (so that $\widehat{F}$ describes the spinal Markov additive process associated with ω), and c for the vector with entries $c_i := \frac{v'_i}{v_i}$, $i \in \mathcal{I}$. Then for all $i \in \mathcal{I}$,

$$\sum_{j \in \mathcal{I}} \frac{c_j}{c_i} \widehat{F}_{i,j}(\Delta\omega) = \kappa_i(\omega') + \sum_{j \neq i} \frac{v'_j}{v'_i} \left(\int_{\mathbb{R}^*} \Pi_{i,j}(dx) |e^x - 1|^{\omega'} + q_{i,j} G_{i,j}(\omega') \right) = \mathcal{K}_i(\omega'), \quad (4.4.4)$$

where the $\mathcal{K}_i$ are the multitype cumulant functions defined in (4.3.6). Now since ω' is also a root of λ , we must have $\mathcal{K}_i(\omega') = 0$ for all $i \in \mathcal{I}$. Identity (4.4.4) therefore yields that the vector c is an eigenvector of $\widehat{F}(\Delta\omega)$ associated to the eigenvalue 0.

The proof of Proposition 4.4.5 will follow from an application of Proposition 4.2.4 as soon as we prove that 0, as a matter of fact, is the *leading* eigenvalue of $\widehat{F}(\Delta\omega)$. Let $\mu = \widehat{\chi}(\Delta\omega)$ be the leading eigenvalue, and $y = (y_i)_{i \in \mathcal{I}}$ a positive eigenvector associated to μ through Proposition 4.2.2. By definition, for all $i \in \mathcal{I}$, we have

$$\mu y_i = \sum_{j \in \mathcal{I}} \widehat{F}_{i,j}(\Delta\omega) y_j.$$

We may rewrite this identity as

$$\mu y_i - \kappa_i(\Delta\omega) y_i = \sum_{j \neq i} \widehat{F}_{i,j}(\Delta\omega) \frac{y_j}{c_j} c_j. \quad (4.4.5)$$

Now let i_0 be an index such that $\frac{y_{i_0}}{c_{i_0}} = \max_{k \in \mathcal{I}} \frac{y_k}{c_k}$. Since the $\widehat{F}_{i,j}(\Delta\omega)$, $j \neq i$, are nonnegative, and c is a positive eigenvector, (4.4.5) yields the inequality

$$\mu y_i - \kappa_i(\Delta\omega) y_i \leq \frac{y_{i_0}}{c_{i_0}} \sum_{j \neq i} \widehat{F}_{i,j}(\Delta\omega) c_j,$$

for all $i \in \mathcal{I}$. Taking $i = i_0$ entails

$$\mu y_{i_0} \leq \frac{y_{i_0}}{c_{i_0}} \left(\kappa_{i_0}(\Delta\omega) c_{i_0} + \sum_{j \neq i_0} \widehat{F}_{i_0,j}(\Delta\omega) c_j \right).$$

But since c is an eigenvector associated to the eigenvalue 0 (see (4.4.4)), we must have

$$\kappa_{i_0}(\Delta\omega) c_{i_0} + \sum_{j \neq i_0} \widehat{F}_{i_0,j}(\Delta\omega) c_j = 0.$$

Because y is a positive vector, this implies that $\mu \leq 0$. And since 0 is known to be an eigenvalue, we must have that the leading eigenvalue is $\mu = 0$. We conclude by convexity of $\widehat{\chi}$ (Proposition 4.2.4). $\square$

The proof of Proposition 4.4.5 shows that, given one martingale exponent (with spine associated with some exponent $\widehat{F}$), the other one can be deduced from the non-trivial root of the leading eigenvalue $\widehat{\chi}$ of $\widehat{F}$. Moreover, if ω and ω' are two martingale exponents, the corresponding spines $(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ and $(\widehat{\mathcal{X}}', \widehat{\mathcal{J}}')$ are explicitly related. More precisely, one is obtained from the other one upon tilting the measure by the so-called Wald martingale (see Proposition 4.2.3) for the underlying Markov additive process. Write $(\widehat{\xi}, \widehat{\Theta})$ and $(\widehat{\xi}', \widehat{\Theta}')$ for the Markov additive processes with respective laws $\widehat{P}$ and $\widehat{P}'$ appearing in the Lamperti-Kiu representations of $(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ and $(\widehat{\mathcal{X}}', \widehat{\mathcal{J}}')$ respectively, and let $\widehat{F}$ be the matrix exponent of $(\widehat{\xi}, \widehat{\Theta})$. Denote by v and v' the eigenvectors of m corresponding to ω and ω' , and write c for the vector with entries $c_i := \frac{v'_i}{v_i}$, $i \in \mathcal{I}$. The proof of Proposition 4.4.5 gives that the leading eigenvalue of $\widehat{F}(\omega' - \omega)$ is $\widehat{\chi} = 0$ and is associated to the positive eigenvector c . Hence the Wald martingale at $\omega' - \omega$ for $(\widehat{\xi}, \widehat{\Theta})$ is

$$\mathcal{W}(t) = \frac{c_{\widehat{\Theta}(t)}}{c_{\widehat{\Theta}(0)}} e^{(\omega' - \omega)\widehat{\xi}(t)}, \quad t \geq 0.$$

The law of $(\widehat{\xi}, \widehat{\Theta})$ under the probability measure biased by $\mathcal{W}$ is also given by Proposition 4.2.3, and one can check that it coincides with the law of $(\widehat{\xi}', \widehat{\Theta}')$ given by Theorem 4.4.3. In other words, for any nonnegative measurable function f and all $x \in \mathbb{R}$, $i \in \mathcal{I}$, $t \in \mathbb{R}_+$,

$$\widehat{\mathbb{E}}'_{x,i}[f(\widehat{\xi}'(s), \widehat{\Theta}'(s), s \leq t)] = \widehat{\mathbb{E}}_{x,i}\left[f(\widehat{\xi}(s), \widehat{\Theta}(s), s \leq t) \cdot \frac{c_{\widehat{\Theta}(t)}}{c_i} e^{(\omega' - \omega)\widehat{\xi}(t)}\right]. \quad (4.4.6)$$

Alternatively (and perhaps more tellingly), one can apply the many-to-one lemma (Proposition 4.4.1) to relate the two spines.

4.4.3 Proof of Theorem 4.4.3

It is plain that the spine $(\widehat{\mathcal{X}}, \widehat{\mathcal{J}})$ inherits the Markov property and self-similarity of (X, J) , and therefore it can be described in terms of a MAP *via* the representation (4.2.5). Without loss of generality, we may restrict to the homogeneous case when $\alpha = 0$. The result is then easily extended thanks to Lamperti time-change. We aim at finding the characteristics $(\widehat{\psi}_i, \widehat{q}_{i,j}, \widehat{G}_{i,j})$ of the matrix exponent of this MAP.

Description of the spine. Let $\widehat{H}$ be the first time when the type of the spine changes, and $\widehat{\mathcal{J}}(\widehat{H})$ denote the corresponding type. Fix $q \geq 0$, and $i, j \in \mathcal{I}$.

- ▷ DETERMINING THE LAPLACE EXPONENT $\widehat{\psi}_i$. This part is similar to the proof of [BBCK18, Theorem 4.2]. We denote by $\widehat{\xi}$ the first component of the MAP corresponding to $\widehat{\mathcal{X}}$, that is

$$\widehat{\xi}(s) = \log \widehat{\mathcal{X}}(s), \quad s \geq 0,$$

and $\widehat{\xi}_k$ the underlying Lévy processes depending on type $k \in \mathcal{I}$. We want to show that the Lévy process $\widehat{\xi}_i$ has Laplace exponent $\widehat{\psi}_i(q) = \kappa_i(q + \omega) - \kappa_i(\omega)$. Notice that a process η with Laplace exponent $\widehat{\psi}_i$ can be written as the superposition $\eta = \eta^1 + \eta^2$ of independent Lévy processes η^1 and η^2 , with respective Laplace exponents $\psi^1(q) := \psi_i(q + \omega) - \psi_i(\omega)$ and

$$\psi^2(q) := \int_{-\infty}^0 \left((1 - e^x)^{q+\omega} - (1 - e^x)^\omega \right) \Pi_{i,i}(dx).$$

In particular, η^2 is a compound Poisson process with Lévy measure $e^{\omega x} \widetilde{\Pi}_{i,i}(dx)$, where $\widetilde{\Pi}_{i,i}(dx)$ is the image measure of $\Pi_{i,i}(dx) \mathbf{1}_{\{x < 0\}}$ through $x \mapsto \log(1 - e^x)$. Let T be the first time when η^2 has a jump. The branching property of the cell system and the Markov

property of η ensures that the result will hold if we manage to prove that the distribution of $(\hat{\xi}_i(t), t \leq b_{\mathcal{L}(1)})$ is the same as that of $(\eta(t), t \leq T)$. Let f, g be two nonnegative measurable functions defined respectively on the space of càdlàg trajectories and on $(-\infty, 0)$. Let

$$\Delta \hat{\xi}(s) = \log \frac{\hat{\mathcal{X}}(s)}{\hat{\mathcal{X}}(s-)}, \quad s \geq 0,$$

then

$$\begin{aligned} & \hat{E}_i \left[f(\hat{\xi}(s), s < b_{\mathcal{L}(1)}) g(\Delta \hat{\xi}(b_{\mathcal{L}(1)})) \mathbb{1}_{\{b_{\mathcal{L}(1)} < \hat{H}\}} \right] \\ &= \mathbf{E}_i \left[\sum_{t>0} \frac{v_{\Delta}(t)}{v_i} e^{\omega \xi(t^-)} (1 - e^{\Delta \xi(t)})^\omega \mathbb{1}_{\{\iota_{\Delta}(t)=i\}} \mathbb{1}_{\{t \leq \rho_i\}} f(\xi(s), s < t) g(\log(1 - e^{\Delta \xi(t)})) \right] \\ &= \mathbf{E}_i \left[\sum_{0 < t < \rho_i} e^{\omega \xi_i(t^-)} (1 - e^{\Delta \xi_i(t)})^\omega \mathbb{1}_{\{\iota_{\Delta}(t)=i\}} f(\xi_i(s), s < t) g(\log(1 - e^{\Delta \xi_i(t)})) \right] \\ & \quad + \mathbf{E}_i \left[e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_{i,\Theta(\rho_i)}})^\omega \mathbb{1}_{\{\iota_{\Delta}(\rho_i)=i\}} f(\xi_i(s), s < \rho_i) g(\log(1 - e^{U_{i,\Theta(\rho_i)}})) \right]. \end{aligned}$$

The compensation formula for ξ_i entails that the first term is

$$\begin{aligned} & \mathbf{E}_i \left[\sum_{0 < t < \rho_i} e^{\omega \xi_i(t^-)} (1 - e^{\Delta \xi_i(t)})^\omega \mathbb{1}_{\{\iota_{\Delta}(t)=i\}} f(\xi_i(s), s < t) g(\log(1 - e^{\Delta \xi_i(t)})) \right] \\ &= \int_0^\infty dt e^{q_{i,i}t} \mathbf{E}_i \left[f(\xi_i(s), s < t) e^{\omega \xi_i(t)} \right] \int_{-\infty}^0 g(\log(1 - e^x)) (1 - e^x)^\omega \Lambda_i^i(dx). \quad (4.4.7) \end{aligned}$$

The second term can be computed as follows:

$$\begin{aligned} & \mathbf{E}_i \left[e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_{i,\Theta(\rho_i)}})^\omega \mathbb{1}_{\{\iota_{\Delta}(\rho_i)=i\}} f(\xi_i(s), s < \rho_i) g(\log(1 - e^{U_{i,\Theta(\rho_i)}})) \right] \\ &= \int_0^\infty dt (-q_{i,i}) e^{q_{i,i}t} \mathbf{E}_i \left[f(\xi_i(s), s < t) e^{\omega \xi_i(t)} \right] \sum_{k \in \mathcal{I} \setminus \{i\}} \frac{q_{i,k}}{(-q_{i,i})} \int_{-\infty}^0 g(\log(1 - e^x)) (1 - e^x)^\omega \Lambda_{U_{i,k}}^i(dx) \\ &= \int_0^\infty dt e^{q_{i,i}t} \mathbf{E}_i \left[f(\xi_i(s), s < t) e^{\omega \xi_i(t)} \right] \sum_{k \in \mathcal{I} \setminus \{i\}} q_{i,k} \int_{-\infty}^0 g(\log(1 - e^x)) (1 - e^x)^\omega \Lambda_{U_{i,k}}^i(dx) \quad (4.4.8) \end{aligned}$$

Combining (4.4.7) and (4.4.8), we finally obtain

$$\begin{aligned} & \hat{E}_i \left[f(\hat{\xi}(s), s < b_{\mathcal{L}(1)}) g(\Delta \hat{\xi}(b_{\mathcal{L}(1)})) \mathbb{1}_{\{b_{\mathcal{L}(1)} < \hat{H}\}} \right] \\ &= \int_0^\infty dt e^{q_{i,i}t} \mathbf{E}_i \left[f(\xi_i(s), s < t) e^{\omega \xi_i(t)} \right] \int_{-\infty}^0 g(y) e^{\omega y} \tilde{\Pi}_{i,i}(dy). \end{aligned}$$

This proves that $(\hat{\xi}_i(s), s < b_{\mathcal{L}(1)})$ and $\Delta \hat{\xi}(b_{\mathcal{L}(1)})$ are independent. The latter is distributed as $-\mathbb{1}_{\{y < 0\}} \frac{1}{q_{i,i} + \psi_i(\omega)} e^{\omega y} \tilde{\Pi}_{i,i}(dy)$, which is the law of $\Delta \eta^2(T)$. The former is ξ_i biased by the exponential martingale, and killed at an independent exponential time with parameter $-q_{i,i} - \psi_i(\omega)$, hence has Laplace exponent $\psi_i(q + \omega) + q_{i,i}$. We retrieve the Laplace exponent of η^1 killed at T , and conclude that $(\hat{\xi}_i(t), t \leq b_{\mathcal{L}(1)})$ evolves as $(\eta(t), t \leq T)$.

▷ DETERMINING THE LAPLACE TRANSFORM $\hat{G}_{i,j}$ OF THE SPECIAL JUMPS. We want to compute

$$\hat{E}_i \left[\left| \frac{\hat{\mathcal{X}}(\hat{H})}{\hat{\mathcal{X}}(\hat{H}-)} \right|^q \mathbb{1}_{\{\hat{\mathcal{J}}(\hat{H})=j\}} \right].$$

We first split over the possible current generations for this special jump to occur:

$$\widehat{E}_i \left[\left| \frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)} \right|^q \mathbb{1}_{\{\widehat{\mathcal{T}}(\widehat{H})=j\}} \right] = \underbrace{\sum_{k=0}^{\infty} \widehat{E}_i \left[\left| \frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)} \right|^q \mathbb{1}_{\{\widehat{\mathcal{T}}(\widehat{H})=j\}} \cdot \mathbb{1}_{\{b_{\mathcal{L}(k)} < \widehat{H} \leq b_{\mathcal{L}(k+1)}\}} \right]}_{:=a_k}. \quad (4.4.9)$$

For $k \geq 1$, the definition of $\widehat{\mathcal{P}}_i$ and the Markov property at time $\widehat{H}$ yield

$$a_k = \mathbb{E}_i \left[\sum_{0 < s < \zeta} \frac{v_{J_{\Delta}(s)}}{v_i} |\Delta X(s)|^{\omega} \mathbb{1}_{\{H \geq s\}} \mathbb{1}_{\{J_{\Delta}(s)=i\}} \right] \cdot a_{k-1},$$

where recall that H denotes the first jump time of J . Therefore $a_k = \mu_{i,i}(\omega) \cdot a_{k-1} = \mu_{i,i}(\omega)^k \cdot a_0$, with

$$\mu_{i,i}(\omega) := \mathbb{E}_i \left[\sum_{0 < s < \zeta} |\Delta X(s)|^{\omega} \mathbb{1}_{\{H \geq s\}} \mathbb{1}_{\{J_{\Delta}(s)=i\}} \right].$$

Then, provided $\mu_{i,i}(\omega) < 1$, identity (4.4.9) triggers

$$\widehat{E}_i \left[\left| \frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)} \right|^q \mathbb{1}_{\{\widehat{\mathcal{T}}(\widehat{H})=j\}} \right] = \frac{a_0}{1 - \mu_{i,i}(\omega)}. \quad (4.4.10)$$

It remains to compute a_0 and $\mu_{i,i}(\omega)$. We begin with the latter:

$$\begin{aligned} \mu_{i,i}(\omega) &= \mathbb{E}_i \left[\sum_{0 < s \leq \rho_i} e^{\omega \xi(s^-)} (1 - e^{\Delta \xi(s)})^{\omega} \mathbb{1}_{\{\iota_{\Delta}(s)=i\}} \right] \\ &= \mathbb{E}_i \left[\sum_{0 < s < \rho_i} e^{\omega \xi_i(s^-)} (1 - e^{\Delta \xi_i(s)})^{\omega} \mathbb{1}_{\{\iota_{\Delta}(s)=i\}} \right] + \mathbb{E}_i \left[e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_i, \Theta(\rho_i)})^{\omega} \mathbb{1}_{\{\iota_{\Delta}(\rho_i)=i\}} \right]. \end{aligned}$$

A computation similar to (4.3.4) gives

$$\mu_{i,i}(\omega) = -\frac{1}{q_{i,i} + \psi_i(\omega)} \int_{-\infty}^0 \Pi_{i,i}(dx) (1 - e^x)^{\omega},$$

provided $\psi_i(\omega) + q_{i,i} < 0$. On the other hand, in a_0 the type of the spine can either change because J jumps to j (*i.e.* $\widehat{H} < b_{\mathcal{L}(1)}$), or because one picks a jump of type j at time $b_{\mathcal{L}(1)}$ (*i.e.* $\widehat{H} = b_{\mathcal{L}(1)}$). This writes

$$a_0 = A + B,$$

with

$$A = \widehat{E}_i \left[\left| \frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)} \right|^q \mathbb{1}_{\{\widehat{\mathcal{T}}(\widehat{H})=j\}} \cdot \mathbb{1}_{\{\widehat{H} < b_{\mathcal{L}(1)}\}} \right] \quad \text{and} \quad B = \widehat{E}_i \left[\left| \frac{\widehat{\mathcal{X}}(\widehat{H})}{\widehat{\mathcal{X}}(\widehat{H}^-)} \right|^q \mathbb{1}_{\{\widehat{\mathcal{T}}(\widehat{H})=j\}} \cdot \mathbb{1}_{\{\widehat{H} = b_{\mathcal{L}(1)}\}} \right].$$

Performing the change of measure, we first get

$$A = \mathbb{E}_i \left[\sum_{H < t < \zeta} \frac{v_{J_{\Delta}(t)}}{v_i} |\Delta X(t)|^{\omega} \left| \frac{X(H)}{X(H^-)} \right|^q \mathbb{1}_{\{J(H)=j\}} \right],$$

with $J(H)$ being the type to which J first jumps. We now apply the Markov property at time H and self-similarity of X :

$$A = \mathbb{E}_i \left[\left| \frac{X(H)}{X(H^-)} \right|^q \mathbb{1}_{\{J(H)=j\}} |X(H)|^{\omega} \mathbb{E}_j \left[\sum_{0 < t < \zeta} \frac{v_{J_{\Delta}(t)}}{v_i} |\Delta X(t)|^{\omega} \right] \right].$$

By admissibility of $((v_i)_{i \in \mathcal{I}}, \omega)$, $\mathbb{E}_j \left[\sum_{0 < t < \zeta} v_{J_\Delta(t)} |\Delta X(t)|^\omega \right] = v_j$. Hence,

$$\begin{aligned} A &= \frac{v_j}{v_i} \mathbb{E}_i \left[\left| \frac{X(H)}{X(H^-)} \right|^{q+\omega} |X(H^-)|^\omega \mathbf{1}_{\{J(H)=j\}} \right] \\ &= \frac{v_j}{v_i} \mathbf{E}_i \left[e^{(q+\omega)U_{i,j}} e^{\omega \xi_i(\rho_i^-)} \mathbf{1}_{\{\Theta(\rho_i)=j\}} \right], \end{aligned}$$

and by independence we obtain

$$\begin{aligned} A &= \frac{v_j}{v_i} \frac{q_{i,j}}{(-q_{i,i})} G_{i,j}(q+\omega) \int_0^\infty ds (-q_{i,i}) e^{q_{i,i}s} \mathbf{E}_i[e^{\omega \xi_i(s)}] \\ &= -\frac{v_j}{v_i} \frac{q_{i,j}}{q_{i,i}} \frac{G_{i,j}(q+\omega)}{\psi_i(\omega) + q_{i,i}}. \end{aligned}$$

Besides,

$$\begin{aligned} B &= \mathbb{E}_i \left[\sum_{0 < t < \zeta} \mathbf{1}_{\{t \leq H\}} \mathbf{1}_{\{J_\Delta(t)=j\}} \frac{v_{J_\Delta(t)}}{v_i} |\Delta X(t)|^\omega \left| \frac{\Delta X(t)}{X(t^-)} \right|^q \right] \\ &= \frac{v_j}{v_i} \mathbf{E}_i \left[\sum_{0 < t < \rho_i} e^{\omega \xi_i(t^-)} (1 - e^{\Delta \xi_i(t)})^{q+\omega} \mathbf{1}_{\{\iota_\Delta(t)=j\}} \right] + \frac{v_j}{v_i} \mathbf{E}_i \left[e^{\omega \xi_i(\rho_i^-)} (1 - e^{U_{i,\Theta(\rho_i)}})^\omega \mathbf{1}_{\{\iota_\Delta(\rho_i)=j\}} \right]. \end{aligned}$$

By the compensation formula as in (4.3.4), we finally get

$$B = -\frac{v_j}{v_i} \frac{1}{q_{i,i} + \psi_i(\omega)} \int_{-\infty}^0 \Pi_{i,j}(dx) (1 - e^x)^{q+\omega}.$$

We can now come back to (4.4.10) and deduce that

$$\hat{E}_i \left[\left| \frac{\hat{\mathcal{X}}(\hat{H})}{\hat{\mathcal{X}}(\hat{H}^-)} \right|^q \mathbf{1}_{\{\hat{\mathcal{J}}(\hat{H})=j\}} \right] = -\frac{\frac{v_j}{v_i} \left(\int_{-\infty}^0 \Pi_{i,j}(dx) (1 - e^x)^{q+\omega} + q_{i,j} G_{i,j}(q+\omega) \right)}{(\psi_i(\omega) + q_{i,i}) + \int_{-\infty}^0 \Pi_{i,i}(dx) (1 - e^x)^\omega}.$$

Recalling (4.3.5), we are left with

$$\hat{E}_i \left[\left| \frac{\hat{\mathcal{X}}(\hat{H})}{\hat{\mathcal{X}}(\hat{H}^-)} \right|^q \mathbf{1}_{\{\hat{\mathcal{J}}(\hat{H})=j\}} \right] = -\frac{\frac{v_j}{v_i} \left(\int_{-\infty}^0 \Pi_{i,j}(dx) (1 - e^x)^{q+\omega} + q_{i,j} G_{i,j}(q+\omega) \right)}{\kappa_i(\omega)}.$$

Note that, because $\mathcal{K}_i(\omega) = 0$, we get

$$\kappa_i(\omega) = - \sum_{j \in \mathcal{I} \setminus \{i\}} \frac{v_j}{v_i} \left(\int_{-\infty}^0 \Pi_{i,j}(dx) |e^x - 1|^\omega + q_{i,j} G_{i,j}(\omega) \right),$$

which upon taking $q = \omega$ already gives $\hat{q}_{i,j}$ up to a multiplicative constant.

▷ DETERMINING THE EXPONENTIAL PARAMETERS $\hat{q}_{i,j}$. Recall that we have assumed homogeneity. Thus, for $q \geq 0$, we wish to compute,

$$\hat{E}_i \left[e^{q\hat{H}} \mathbf{1}_{\{\hat{\mathcal{J}}(\hat{H})=j\}} \right] = \sum_{k=0}^{\infty} \underbrace{\hat{E}_i \left[e^{q\hat{H}} \mathbf{1}_{\{\hat{\mathcal{J}}(\hat{H})=j\}} \mathbf{1}_{\{b_{\mathcal{L}(k)} < \hat{H} \leq b_{\mathcal{L}(k+1)}\}} \right]}_{:=a'_k}.$$

Again, using the definition of $\widehat{\mathcal{P}}_i$ and the Markov property just as we did with a_k , we end up with

$$a'_k = r a'_{k-1}, \quad k \geq 1,$$

where

$$\begin{aligned} r &= \mathbb{E}_i \left[\sum_{0 < s \leq \rho_i} e^{qs} e^{\omega \xi(s^-)} \left(1 - e^{\Delta \xi(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}} \right] \\ &= \mathbb{E}_i \left[\sum_{0 < s < \rho_i} e^{qs} e^{\omega \xi_i(s^-)} \left(1 - e^{\Delta \xi_i(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}} \right] + \mathbb{E}_i \left[e^{q\rho_i} e^{\omega \xi_i(\rho_i^-)} \left(1 - e^{U_{i,\Theta(\rho_i)}}\right)^\omega \mathbf{1}_{\{\iota_\Delta(\rho_i)=i\}} \right]. \end{aligned} \quad (4.4.11)$$

Then, we use the compensation formula and we obtain that the first term is

$$\begin{aligned} \mathbb{E}_i \left[\sum_{0 < s < \rho_i} e^{qs} e^{\omega \xi_i(s^-)} \left(1 - e^{\Delta \xi_i(s)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(s)=i\}} \right] &= \int_0^\infty ds e^{(q+q_{i,i})s} e^{\psi_i(\omega)s} \int_{-\infty}^0 \Lambda_i^i(dx) (1 - e^x)^\omega \\ &= -\frac{1}{q + q_{i,i} + \psi_i(\omega)} \int_{-\infty}^0 \Lambda_i^i(dx) (1 - e^x)^\omega. \end{aligned} \quad (4.4.12)$$

By independence, the second term of (4.4.11) is

$$\begin{aligned} &\mathbb{E}_i \left[e^{q\rho_i} e^{\omega \xi_i(\rho_i^-)} \left(1 - e^{U_{i,\Theta(\rho_i)}}\right)^\omega \mathbf{1}_{\{\iota_\Delta(\zeta)=i\}} \right] \\ &= \sum_{k \in \mathcal{I} \setminus \{i\}} \frac{q_{i,k}}{(-q_{i,i})} \int_0^\infty ds (-q_{i,i}) e^{(q+q_{i,i})s} e^{\psi_i(\omega)s} \int_{-\infty}^0 \Lambda_{U_{i,k}}^i(dx) (1 - e^x)^\omega \\ &= -\frac{1}{q + q_{i,i} + \psi_i(\omega)} \sum_{k \in \mathcal{I} \setminus \{i\}} q_{i,k} \int_{-\infty}^0 \Lambda_{U_{i,k}}^i(dx) (1 - e^x)^\omega. \end{aligned} \quad (4.4.13)$$

Thanks to (4.4.12) and (4.4.13), equation (4.4.11) boils down to

$$r = -\frac{1}{q + q_{i,i} + \psi_i(\omega)} \int_{-\infty}^0 \Pi_i^i(dx) (1 - e^x)^\omega.$$

On the other hand,

$$a'_0 = \widehat{E}_i \left[e^{q\widehat{H}} \mathbf{1}_{\{\widehat{\mathcal{J}}(\widehat{H})=j\}} \mathbf{1}_{\{\widehat{H} \leq b_{\mathcal{L}(1)}\}} \right],$$

and we may split the indicator over $\{\widehat{H} < b_{\mathcal{L}(1)}\}$ and $\{\widehat{H} = b_{\mathcal{L}(1)}\}$. We therefore get $a'_0 = A' + B'$, where

$$A' = \mathbb{E}_i \left[\sum_{0 < t < \zeta} e^{qH} \mathbf{1}_{\{J(H)=j\}} \mathbf{1}_{\{H < t\}} \frac{v_{J_\Delta(t)}}{v_i} |\Delta X(t)|^\omega \right],$$

and

$$B' = \mathbb{E}_i \left[\sum_{0 < t \leq \rho_i} e^{qt} \frac{v_j}{v_i} e^{\omega \xi(t^-)} \left(1 - e^{\Delta \xi(t)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(t)=j\}} \right].$$

First of all, B' can be rewritten as follows

$$B' = \frac{v_j}{v_i} \mathbb{E}_i \left[\sum_{0 < t < \rho_i} e^{qt} e^{\omega \xi(t^-)} \left(1 - e^{\Delta \xi(t)}\right)^\omega \mathbf{1}_{\{\iota_\Delta(t)=j\}} \right] + \frac{v_j}{v_i} \mathbb{E}_i \left[e^{q\rho_i} e^{\omega \xi_i(\rho_i^-)} \left(1 - e^{U_{i,\Theta(\rho_i)}}\right)^\omega \mathbf{1}_{\{\iota_\Delta(\rho_i)=j\}} \right].$$

Continuing along the lines of (4.4.12), (4.4.13), we eventually get to

$$B' = -\frac{v_j}{v_i} \frac{1}{q + q_{i,i} + \psi_i(\omega)} \int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x) (1 - e^x)^\omega.$$

Moreover, by using the Markov property at time H , self-similarity of X , and by admissibility of $((v_i)_{i \in \mathcal{I}}, \omega)$, we have

$$\begin{aligned} A' &= \mathbb{E}_i \left[e^{qH} \mathbb{1}_{\{J(H)=j\}} \mathbb{E}_{X(H)} \left[\sum_{0 < t < \zeta} \frac{v_{J_\Delta(t)}}{v_i} |\Delta X(t)|^\omega \right] \right] \\ &= \mathbb{E}_i \left[e^{qH} \mathbb{1}_{\{J(H)=j\}} |X(H)|^\omega \mathbb{E}_j \left[\sum_{0 < t < \zeta} \frac{v_{J_\Delta(t)}}{v_i} |\Delta X(t)|^\omega \right] \right] \\ &= \frac{v_j}{v_i} \mathbb{E}_i \left[e^{qH} \mathbb{1}_{\{J(H)=j\}} |X(H)|^\omega \right]. \end{aligned}$$

Now, on the event that $J(H) = j$, we have $X(H) = e^{\xi_i(\rho_i^-) + U_{i,j}}$ under $\mathbb{P}_i$. This entails

$$\begin{aligned} A' &= -\frac{v_j}{v_i} \frac{q_{i,j}}{q_{i,i}} \mathbf{E}_i[e^{(q+\psi_i(\omega))\rho_i}] G_{i,j}(\omega) \\ &= -\frac{v_j}{v_i} \frac{q_{i,j} G_{i,j}(\omega)}{q + q_{i,i} + \psi_i(\omega)}. \end{aligned}$$

Therefore,

$$a_0 = -\frac{\frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) + \frac{v_j}{v_i} \int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x) (1 - e^x)^\omega}{q + q_{i,i} + \psi_i(\omega)}.$$

We finally conclude that

$$\widehat{E}_i \left[e^{q\widehat{H}} \mathbb{1}_{\{\widehat{J}(\widehat{H})=j\}} \right] = \frac{a_0}{1-r} = -\frac{\frac{v_j}{v_i} q_{i,j} G_{i,j}(\omega) + \frac{v_j}{v_i} \int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x) (1 - e^x)^\omega}{q + q_{i,i} + \psi_i(\omega) + \int_{-\infty}^0 \Pi_{i,i}(\mathrm{d}x) (1 - e^x)^\omega}.$$

This shows that, for all $i, j \in \mathcal{I}$, with $j \neq i$, the jump time of the chain $\widehat{\mathcal{J}}$ from state i to state j is an exponential random variable, with parameter

$$\widehat{q}_{i,j} = \frac{v_j}{v_i} \left(q_{i,j} G_{i,j}(\omega) + \int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x) |1 - e^x|^\omega \right).$$

▷ THE MATRIX EXPONENT. The previous calculations determine $\widehat{F}(q) = (\widehat{F}_{i,j}(q))_{i,j \in \mathcal{I}}$, the matrix exponent of the spine as the matrix with entries:

$$\forall i \in \mathcal{I}, \quad \widehat{F}_{i,i}(q) = \kappa_i(\omega + q)$$

and

$$\forall i, j \in \mathcal{I}, i \neq j, \quad \widehat{F}_{i,j}(q) = \frac{v_j}{v_i} \left(\int_{-\infty}^0 \Pi_{i,j}(\mathrm{d}x) (1 - e^x)^{q+\omega} + q_{i,j} G_{i,j}(q + \omega) \right).$$

4.5 Spatial isotropic growth-fragmentation processes

Now, we extend the framework of [Ber17b] to isotropic $\mathbb{R}^d$ -valued Markov processes for $d \geq 2$. We exclude the case $d = 1$, since it can be deduced from the previous construction by considering a symmetric self-similar Markov process or from [Sil21]. It is important to note that the construction in the previous sections does not consider the isotropy assumption as well as in [Sil21].

In what follows X will be an isotropic $\mathbb{R}^d$ -valued self-similar Markov process with index α , as defined in the last paragraph of Section 4.2, which under $\mathbb{P}_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, starts from $\mathbf{x}$. For technical reasons, we shall assume that X is either absorbed after time ζ at some cemetery state ∂ , or that X converges to 0 at infinity, for all starting points. We also recall that (ξ, Θ) denotes the MAP associated with X .

4.5.1 Construction of spatial growth-fragmentation processes

The construction of the cell system in this case is similar to (and simpler than) the multitype case, so that we will only outline the construction. This construction actually holds without the self-similarity or isotropy assumptions. We alter a bit the previous notation by now letting $\Delta X(t) := X(t) - X(t^-)$ for $t \geq 0$ denote the possible jump of X at time t . At any jump time t of X , one places a new particle in the system and, conditionally given their size $-\Delta X(t)$ at birth, each of these newborn particles evolves independently as $\mathbb{P}_{-\Delta X(t)}$. Then, one repeats this construction for any such child, thus creating the second generation, and so on. As in the multitype case (Section 4.3.1), a more formal construction goes through iteratively defining variables $\mathcal{X}_u, u \in \mathbb{U}$, modelling the evolution of particles indexed by the Ulam tree.

In this construction, the cells are not labelled chronologically. However, it still uniquely defines the law $\mathcal{P}_{\mathbf{x}}$ of the cell system $(\mathcal{X}_u(t), u \in \mathbb{U}, t \geq 0)$ started from $\mathbf{x}$. Finally, we introduce the (*spatial*) *growth-fragmentation process*

$$\mathbf{X}(t) := \{ \{ \mathcal{X}_u(t - b_u), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u \} \}, \quad t \geq 0,$$

describing the collection of cells alive at time $t \geq 0$ (the double brackets here denote multisets). We define $\mathbf{P}_{\mathbf{x}}$ to be the law of the growth-fragmentation $\mathbf{X}$ started at $\mathbf{x}$.

We point out that one can view this construction as a *multitype growth-fragmentation* process, where the types correspond to the directions (in the $d = 1$ case, it is the sign). The set of types is therefore the sphere $\mathbb{S}^{d-1}$, which is uncountable, so that the construction does not quite fall into the framework developed in Section 4.3. From this standpoint, note that the type corresponding to the daughter cell created by the jump $\Delta X(t)$ is, up to time-change,

$$\Theta_{\Delta}(t) := \frac{\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)}{|\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)|}.$$

Similarly as in the multitype case, we have a temporal version of the branching property, see Proposition 4.3.2. Let

$$\overline{\mathbf{X}}(t) := \{ \{ (\mathcal{X}_u(t - b_u), |u|), u \in \mathbb{U} \text{ and } b_u \leq t < b_u + \zeta_u \} \}, \quad t \geq 0.$$

We shall denote by $(\mathcal{F}_t, t \geq 0)$ the natural filtration associated with $\mathbf{X}$, and $(\overline{\mathcal{F}}_t, t \geq 0)$ the one associated with $\overline{\mathbf{X}}$. Under the existence of an excessive function (see (4.3.3) for its definition) for $\mathbf{X}$, one can rank the elements $X_1(t), X_2(t), \dots$ of $\mathbf{X}(t)$ by descending order of their norm for any fixed t . Under the same assumption, we have the following.

Proposition 4.5.1. *Assume that $\mathbf{X}$ has an excessive function. Then for any $t \geq 0$, conditionally on $\bar{\mathbf{X}}(t) = \{\{\mathbf{x}_i, n_i\}\}$, the process $(\bar{\mathbf{X}}(t+s), s \geq 0)$ is independent of $\bar{\mathcal{F}}_t$ and distributed as*

$$\bigsqcup_{i \geq 1} \bar{\mathbf{X}}_i(s) \circ \theta_{n_i},$$

where the $\mathbf{X}_i, i \geq 1$, are independent processes distributed as $\bar{\mathbf{X}}$ under $\mathcal{P}_{\mathbf{x}_i}$, θ_n is the shift operator $\{\{\mathbf{z}_i, k_i\}, i \geq 1\} \circ \theta_n := \{\{\mathbf{z}_i, k_i + n\}, i \geq 1\}$, and $\sqcup$ denotes union of multisets.

4.5.2 The isotropic cumulant function and genealogical martingales

We are first of all interested in pointing out martingales as in Section 4.3.2 in the spatial growth-fragmentation setting. It turns out that the exponents ω corresponding to these martingales will be found as the roots of an *isotropic cumulant function* which generalises the cumulant function κ in [Ber17b, BBCK18]. Recall that, as readily seen from the rotational invariance property, the radial part of X is a positive self-similar Markov process, so that the ordinate ξ is in fact a Lévy process. We will extensively make use of this argument and its consequences.

Let us start with a simple but typical calculation: for $q \geq 0$ and $\theta \in \mathbb{S}^{d-1}$, we aim at computing the quantity $\mathbb{E}_\theta \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right]$ in terms of the MAP characteristics of X . We will now consider the Lévy system (L, H) of (ξ, Θ) (see Section 4.2), and we take as usual $H_t = t$ to avoid notational clutter. Since we want to sum over all t 's, we can omit the Lamperti-Kiu time-change between X and (ξ, Θ) , so that

$$\mathbb{E}_\theta \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \mathbb{E}_{0, \theta} \left[\sum_{0 < t < \zeta} e^{q\xi(t^-)} |\Theta(t^-) - e^{\Delta\xi(t)} \Theta(t)|^q \right].$$

The compensation formula (4.2.9) for Markov additive processes then yields

$$\mathbb{E}_\theta \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \mathbb{E}_{0, \theta} \left[\int_0^\infty dt e^{q\xi(t)} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\Theta(t)}(dx, d\Phi) |\Theta(t) - e^x \Phi|^q \right]. \quad (4.5.1)$$

Remark that the integral

$$\int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_\theta(dx, d\Phi) |\theta - e^x \Phi|^q,$$

does not depend on the angle θ , since isotropy entails that if $\theta, \theta' \in \mathbb{S}^{d-1}$, and U is an isometry mapping θ to θ' , then $L_{\theta'}(dx, d\Phi) = L_\theta(dx, U^{-1}d\Phi)$. More generally, the image measures $\tilde{L}_\theta(dy, d\phi)$ of $L_\theta(dx, d\Phi)$ through the mapping $(y, \phi) = (\log |\theta - e^x \Phi|, \frac{\theta - e^x \Phi}{|\theta - e^x \Phi|})$ satisfy the same relationship. Indeed, for any nonnegative measurable function F , since $|U| = 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\theta'}(dx, d\Phi) F\left(\log |\theta' - e^x \Phi|, \frac{\theta' - e^x \Phi}{|\theta' - e^x \Phi|}\right) \\ &= \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_\theta(dx, U^{-1}d\Phi) F\left(\log |U\theta - e^x \Phi|, \frac{U\theta - e^x \Phi}{|U\theta - e^x \Phi|}\right) \\ &= \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_\theta(dx, d\varphi) F\left(\log |U\theta - e^x U\varphi|, \frac{U\theta - e^x U\varphi}{|U\theta - e^x U\varphi|}\right) \\ &= \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_\theta(dx, d\varphi) F\left(\log |\theta - e^x \varphi|, U \frac{\theta - e^x \varphi}{|\theta - e^x \varphi|}\right). \end{aligned} \quad (4.5.2)$$

Hence $\tilde{L}_{\theta'}(dx, d\Phi) = \tilde{L}_{\theta}(dx, U^{-1}d\Phi)$. Singling out the image measure $\tilde{L}(dy, d\phi)$ of $L_{\theta}(dx, d\Phi)$ when $\theta = (1, 0, \dots, 0)$ say, (4.5.1) boils down to

$$\mathbb{E}_{\theta} \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \mathbb{E}_{0, \theta} \left[\int_0^{\infty} dt e^{q\xi(t)} \right] \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \tilde{L}(dy, d\phi) e^{qy}.$$

Recall that ξ is a Lévy process and assume that its Laplace exponent ψ satisfies $\psi(q) < 0$, otherwise the first integral blows up. Then we are left with

$$\mathbb{E}_{\theta} \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = 1 - \frac{\kappa(q)}{\psi(q)},$$

where we have set

$$\kappa(q) = \psi(q) + \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \tilde{L}(dy, d\phi) e^{qy}. \quad (4.5.3)$$

We stress once more that κ can be calculated using any of the measures $\tilde{L}_{\theta}$ in place of $\tilde{L}$. The previous calculations finally show that

$$\mathbb{E}_{\theta} \left[\sum_{0 < t < \zeta} |\Delta X(t)|^q \right] = \begin{cases} 1 - \frac{\kappa(q)}{\psi(q)} & \text{if } \kappa(q) < \infty \text{ and } \psi(q) < 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (4.5.4)$$

We call the function κ the *isotropic cumulant function*. Its roots will lead to martingales for the growth-fragmentation cell system through the identity (4.5.4). Thus, throughout the paper we make the following assumption

(H) *There exists $\omega \geq 0$ such that $\kappa(\omega) = 0$.*

Notice that, as readily seen from (4.5.3), κ is a convex function, so that there exist at most two such roots. For such a root ω , we obtain by self-similarity and (4.5.4) that for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$,

$$\mathbb{E}_{\mathbf{x}} \left[\sum_{0 < t < \zeta} |\Delta X(t)|^{\omega} \right] = |\mathbf{x}|^{\omega}. \quad (4.5.5)$$

Following the strategy of Section 4.3.2, we now show that the roots of κ pave the way for remarkable martingales. The proof of the following result follows exactly from the same arguments as those used in Proposition 4.3.4.

Proposition 4.5.2. *Under $\mathbb{P}_{\mathbf{x}}$, for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$, the process*

$$M(t) := |X(t)|^{\omega} + \sum_{0 < s \leq t \wedge \zeta} |\Delta X(s)|^{\omega},$$

is a martingale for the filtration $(F_t^X, t \geq 0)$ associated with X .

Moreover, the definition of ω and the branching structure of growth-fragmentation processes entail the existence of the following genealogical martingale, which will be crucial for the spine decomposition. Let $\mathcal{G}_n := \sigma(\mathcal{X}_u, |u| \leq n)$, $n \geq 0$.

Theorem 4.5.3. *The process*

$$\mathcal{M}(n) := \sum_{|u|=n+1} |\mathcal{X}_u(0)|^{\omega}, \quad n \geq 0,$$

is a $(\mathcal{G}_n, n \geq 0)$ -martingale under $\mathcal{P}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$.

The arguments used to deduce the previous result are the same as those presented in Proposition 4.3.3.

4.5.3 A change of measures

We introduce a new probability measure $\widehat{\mathcal{P}}_{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$ using the martingale $(\mathcal{M}(n))_{n \geq 0}$ in Theorem 4.5.3. This is the analogue of [BBCK18, Section 4.1] in the positive case or [Sil21, Section 3.3] in the $d = 1$ case. It describes the law of a new cell system $(\mathcal{X}_u)_{u \in \mathbb{U}}$ together with an infinite distinguished *ray*, or *leaf*, $\mathcal{L} \in \partial \mathbb{U} = \mathbb{N}^{\mathbb{N}}$. On $\mathcal{G}_n$, for $n \geq 0$, it has Radon-Nikodym derivative with respect to $\mathcal{P}_{\mathbf{x}}$ given by $\mathcal{M}(n)$, normalized to be a probability measure, *i.e.* for all $G_n \in \mathcal{G}_n$,

$$\widehat{\mathcal{P}}_{\mathbf{x}}(G_n) := |\mathbf{x}|^{-\omega} \mathcal{E}_{\mathbf{x}}[\mathcal{M}(n) \mathbf{1}_{G_n}].$$

The law of the particular leaf $\mathcal{L}$ under $\widehat{\mathcal{P}}_{\mathbf{x}}$ is chosen so that, for all $n \geq 0$ and all $u \in \mathbb{U}$ such that $|u| = n + 1$

$$\widehat{\mathcal{P}}_{\mathbf{x}}(\mathcal{L}(n+1) = u | \mathcal{G}_n) := \frac{|\mathcal{X}_u(0)|^{\omega}}{\mathcal{M}(n)}, \quad (4.5.6)$$

where for any $\ell \in \partial \mathbb{U}$, $\ell(n)$ denotes the ancestor of ℓ at generation n . In words, to define the next generation of the spine, we select one of its jumps proportionally to its size to the power ω (the spine at generation 0 being the Eve cell). By an application of the Kolmogorov extension theorem, the martingale property and the branching structure of the system ensure that these definitions are compatible, and therefore this uniquely defines the probability measure $\widehat{\mathcal{P}}_{\mathbf{x}}$.

We will be interested in the evolution of the *tagged cell*, which is the cell associated with the distinguished leaf $\mathcal{L}$. More precisely, set $b_{\ell} = \lim \uparrow b_{\ell(n)}$ for any leaf $\ell \in \partial \mathbb{U}$. Then, define $\widehat{\mathcal{X}}$ by $\widehat{\mathcal{X}}(t) := \partial$ if $t \geq b_{\mathcal{L}}$ and

$$\widehat{\mathcal{X}}(t) := \mathcal{X}_{\mathcal{L}(n_t)}(t - b_{\mathcal{L}(n_t)}), \quad t < b_{\mathcal{L}}, \quad (4.5.7)$$

where n_t is the unique integer n such that $b_{\mathcal{L}(n)} \leq t < b_{\mathcal{L}(n+1)}$.

By construction of $\widehat{\mathcal{P}}_{\mathbf{x}}$, we have the following genealogical *many-to-one* formula: for all nonnegative measurable function f and all $\mathcal{G}_n$ -measurable nonnegative random variable B_n ,

$$|\mathbf{x}|^{\omega} \widehat{E}_{\mathbf{x}}[f(\mathcal{X}_{\mathcal{L}(n+1)}(0))B_n] = \mathcal{E}_{\mathbf{x}} \left[\sum_{|u|=n+1} |\mathcal{X}_u(0)|^{\omega} f(\mathcal{X}_u(0))B_n \right].$$

This may be extended to a temporal many-to-one formula. The existence of (v, ω) ensures that we may rank the elements in $\mathbf{X}(t) = \{\{X_i(t), i \geq 1\}\}$, $t \geq 0$, by decreasing order of the norms.

Proposition 4.5.4. *For every $t \geq 0$, every nonnegative measurable function f vanishing at ∂ , and every $\overline{\mathcal{F}}_t$ -measurable nonnegative random variable B_t , we have*

$$|\mathbf{x}|^{\omega} \widehat{E}_{\mathbf{x}}[f(\widehat{\mathcal{X}}(t))B_t] = \mathcal{E}_{\mathbf{x}} \left[\sum_{i \geq 1} |X_i(t)|^{\omega} f(X_i(t))B_t \right].$$

Proof. See Proposition 4.4.1 for the multitype case, which is easily extended. $\square$

4.6 The spine decomposition of spatial isotropic growth - fragmentation processes

4.6.1 The spine decomposition for isotropic growth-fragmentation processes

In this section, we describe the law of the growth-fragmentation process under the change of measures $\widehat{\mathcal{P}}_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d$, and in particular the law of the tagged cell $\widehat{\mathcal{X}}$ (4.5.7). In order to make

sense of this, we need to rebuild the growth-fragmentation along the spine, and so we must first label the jumps of $\hat{\mathcal{X}}$. In general, one cannot rank those in lexicographical order. Instead, they will be labelled by couples (n, j) , where $n \geq 0$ stands for the generation of the tagged cell immediately before the jump, and $j \geq 1$ is the rank (in the usual lexicographical sense) of the jump among those of the tagged cell at generation n (including the final jump, when the generation changes to $n + 1$). For each such (n, j) , we define the growth-fragmentation $\hat{\mathbf{X}}_{n,j}$ induced by the corresponding jump. More precisely, if the generation stays constant during the (n, j) -jump, then we set

$$\hat{\mathbf{X}}_{n,j}(t) := \{\{\mathcal{X}_{uw}(t - b_{uw} + b_u), w \in \mathbb{U} \text{ and } b_{uw} \leq t + b_u < b_{uw} + \zeta_{uw}\}\},$$

where u is the label of the cell born at the (n, j) -jump. Otherwise the (n, j) -jump corresponds to a jump for the generation of the tagged cell so that the tagged cell jumps from label u to label uk say. In this case, we set

$$\hat{\mathbf{X}}_{n,j}(t) := \{\{\mathcal{X}_{uw}(t - b_{uw} + b_{uk}), w \in \mathbb{U} \setminus \{k\} \text{ and } b_{uw} \leq t + b_{uk} < b_{uw} + \zeta_{uw}\}\}.$$

Finally, we agree that $\hat{\mathbf{X}}_{n,j} := \partial$ when the (n, j) -jump does not exist, and this sets $\hat{\mathbf{X}}_{n,j}$ for all $n \geq 0$ and all $j \geq 1$.

Recall also that n_t was defined in (4.5.7) and stands for the generation of the spine at time t . We can now state our main theorem describing the law of the growth-fragmentation under $\hat{\mathcal{P}}_{\mathbf{x}}$.

Theorem 4.6.1. *Under $\hat{\mathcal{P}}_{\mathbf{x}}$, $\hat{\mathcal{X}}$ is a self-similar Markov process with values in $\mathbb{R}^d$ and index α . The Lévy system of the underlying Markov additive process (ξ, Θ) is given by $(\hat{H}, \hat{L})$ where $\hat{H}_t = t$ and*

$$\hat{L}_\theta(dy, d\phi) = e^{\omega y} (L_\theta(dy, d\phi) + \tilde{L}_\theta(dy, d\phi)). \quad (4.6.1)$$

Besides, $\hat{\mathcal{X}}$ is isotropic, and the ordinate $\hat{\xi}$ is a Lévy process with Laplace exponent $\hat{\psi}(q) = \kappa(\omega + q)$. Moreover, conditionally on $(\hat{\mathcal{X}}(t), n_t)_{0 \leq t < b_{\mathcal{L}}}$, the processes $\hat{\mathbf{X}}_{n,j}$, $n \geq 0$, $j \geq 1$, are independent and each $\hat{\mathbf{X}}_{n,j}$ has law $\mathbf{P}_{x(n,j)}$ where $-x(n, j)$ is the size of the (n, j) -th jump.

Remark 4.6.2. (i) Observe that we have the following description of the MAP $(\hat{\xi}, \hat{\Theta})$.

Let (η^0, Φ^0) be a MAP with Lévy system (H^0, L^0) given by $H_t^0 := t$ and $L_\theta^0(dy, d\phi) := e^{\omega y} L_\theta(dy, d\phi)$. Consider an independent compound Poisson process $D = (D_1, D_2)$ on $\mathbb{R}^* \times \mathbb{S}^{d-1}$ with intensity measure $e^{\omega y} \tilde{L}(dy, d\phi)$. This definition makes sense because, since $\kappa(\omega) = 0$,

$$\int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \tilde{L}(dy, d\phi) e^{\omega y} = -\psi(\omega) < \infty.$$

Then $(\hat{\xi}, \hat{\Theta})$ is the *superimposition* of (η^0, Φ^0) and D , in the following sense. Let T_1 the first jump time of D , which is exponential with parameter $-\psi(\omega)$. Then $(\hat{\xi}(s), \hat{\Theta}(s), s < b_{\mathcal{L}(1)})$ evolves as $(\eta^0(s), \Phi^0(s), s < T_1)$, and $(\hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}))$ is distributed as

$$(\eta^0(T_1) + D_1(T_1), U_{\Phi^0(T_1)} \cdot D_2(T_1)),$$

where U_θ is an isometry mapping $(1, 0, \dots, 0)$ to θ .

- (ii) The proof actually provides a more precise statement describing the law of $(\hat{\mathcal{X}}(t), n_t, t \geq 0)$. The process n_t is then the Poisson process counting the jumps arising in D up to the usual Lamperti time change.
- (iii) The MAP (ξ^0, η^0) is exactly the so-called *Esscher transform* $(\xi^{[\omega]}, \Theta^{[\omega]})$ of (ξ, Θ) . More precisely, recall that in the isotropic setting, ξ is itself a Lévy process, so that we can consider the usual exponential martingale $(e^{\omega \xi(t) - t\psi(\omega)}, t \geq 0)$. Then the law of (ξ, Θ) under the exponential change of measures is $(\xi^{[\omega]}, \Theta^{[\omega]})$. This will appear in the proof.

- (iv) Combining these two remarks casts light on equation (4.6.1). Loosely speaking, it is a decomposition of $\widehat{L}$ in terms of the jumps of the Esscher transform of (ξ, Θ) and the special jumps when the spine picks one of the jumps according to (4.5.6).
- (v) We deduce from Theorem 4.6.1 that the temporal version of $(\mathcal{M}(n), n \geq 0)$, namely

$$\mathcal{M}_t := \sum_{i=1}^{\infty} |X_i(t)|^\omega, \quad t \geq 0,$$

is a $(\mathcal{F}_t)$ -martingale if, and only if, $\alpha\kappa'(\omega) < 0$. Indeed, by taking $f = \mathbb{1}_\partial$ the many-to-one formula (Proposition 4.5.4) yields that $(\mathcal{M}_t, t \geq 0)$ is a supermartingale, and that it is a martingale if, and only if, X has infinite lifetime. From the Lamperti representation of $|X|$, and the expression $\widehat{\psi}(q) = \kappa(\omega + q)$ of the Laplace exponent of $\widehat{\xi}$, this happens exactly when $\alpha\kappa'(\omega) < 0$.

4.6.2 Proof of Theorem 4.6.1

The proof will roughly follow the same lines as the one of Theorem 4.4.3, although the structure of the modulator is more involved.

The law of the spine $\widehat{\mathcal{X}}$. The definition of $\widehat{\mathcal{X}}$ readily shows that $\widehat{\mathcal{X}}$ is an α -self-similar Markov process. By Lamperti time change, we may place ourselves in the homogeneous case $\alpha = 0$. In this case, note that there is no time change between $\widehat{\mathcal{X}}$ and $(\widehat{\xi}, \widehat{\Theta})$. For this reason, and to avoid notational clutter, we will sometimes make an abuse of notation by considering them on the same probability space. Likewise, we will use expressions involving both X and its MAP (ξ, Θ) as a shorthand. Moreover, the Markov property implies that we only need to check the compensation formula up to the first time $b_{\mathcal{L}(1)}$ when the spine selects another generation. More precisely, we want to show that

$$\begin{aligned} \widehat{E}_\theta \left[\sum_{s>0} F(s, \widehat{\xi}(s^-), \Delta\widehat{\xi}(s), \widehat{\Theta}(s^-), \widehat{\Theta}(s)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] \\ = \widehat{E}_\theta \left[\int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \widehat{L}_{\widehat{\Theta}(s)}(dx, d\varphi) F(s, \widehat{\xi}(s), x, \widehat{\Theta}(s), \varphi) \right]. \end{aligned} \quad (4.6.2)$$

We may split the sum into two parts:

$$\begin{aligned} \widehat{E}_\theta \left[\sum_{s>0} F(s, \widehat{\xi}(s^-), \Delta\widehat{\xi}(s), \widehat{\Theta}(s^-), \widehat{\Theta}(s)) \mathbb{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] = \\ \widehat{E}_\theta \left[\sum_{s < b_{\mathcal{L}(1)}} F(s, \widehat{\xi}(s^-), \Delta\widehat{\xi}(s), \widehat{\Theta}(s^-), \widehat{\Theta}(s)) \right] + \widehat{E}_\theta [F(b_{\mathcal{L}(1)}, \widehat{\xi}(b_{\mathcal{L}(1)}^-), \Delta\widehat{\xi}(b_{\mathcal{L}(1)}), \widehat{\Theta}(b_{\mathcal{L}(1)}^-), \widehat{\Theta}(b_{\mathcal{L}(1)}))]. \end{aligned} \quad (4.6.3)$$

We compute the first term of (4.6.3). By definition of $b_{\mathcal{L}(1)}$,

$$(\widehat{\xi}(s), \widehat{\Theta}(s), s < b_{\mathcal{L}(1)}) = (\xi(s), \Theta(s), s < b_{\mathcal{L}(1)}).$$

Applying the change of measure (4.5.6), and recalling that we are in the homogeneous case, we get

$$\begin{aligned} \widehat{E}_\theta \left[\sum_{s < b_{\mathcal{L}(1)}} F(s, \widehat{\xi}(s^-), \Delta\widehat{\xi}(s), \widehat{\Theta}(s^-), \widehat{\Theta}(s)) \right] \\ = \mathbb{E}_\theta \left[\sum_{s>0} \sum_{t>s} F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right]. \end{aligned} \quad (4.6.4)$$

Now, the Markov property of X at fixed time $s > 0$ yields that

$$\begin{aligned} \mathbb{E}_\theta \left[\sum_{t>s} F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right] \\ = \mathbb{E}_\theta \left[F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) \mathbb{E}_{X(s)} \left[\sum_{t>0} |\Delta X(t)|^\omega \right] \right], \end{aligned}$$

and using the definition of ω in identity (4.5.5),

$$\mathbb{E}_\theta \left[\sum_{t>s} F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) |\Delta X(t)|^\omega \right] = \mathbb{E}_{0,\theta} \left[F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) e^{\omega\xi(s)} \right].$$

Coming back to (4.6.4), this means

$$\hat{E}_\theta \left[\sum_{s < b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta\hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right] = \mathbb{E}_{0,\theta} \left[\sum_{s>0} F(s, \xi(s^-), \Delta\xi(s), \Theta(s^-), \Theta(s)) e^{\omega\xi(s)} \right].$$

Using the compensation formula entails

$$\begin{aligned} \hat{E}_\theta \left[\sum_{s < b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta\hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right] \\ = \mathbb{E}_{0,\theta} \left[\int_0^\infty ds e^{\omega\xi(s)} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\Theta(s)}(dx, d\varphi) e^{\omega x} F(s, \xi(s), x, \Theta(s), \varphi) \right]. \quad (4.6.5) \end{aligned}$$

We now tilt the measure using the classical Esscher transform (see for example [KP21]). Recall from Remark 4.6.2 that the process obtained has the law $\mathbb{P}_{0,\theta}^0$ of (η^0, Φ^0) . Thus equation (4.6.5) rewrites

$$\begin{aligned} \hat{E}_\theta \left[\sum_{s < b_{\mathcal{L}(1)}} F(s, \hat{\xi}(s^-), \Delta\hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \right] \\ = \mathbb{E}_{0,\theta}^0 \left[\int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\Phi^0(s)}(dx, d\varphi) e^{\omega x} F(s, \eta^0(s), x, \Phi^0(s), \varphi) \right]. \quad (4.6.6) \end{aligned}$$

Note that, since $L_\theta(dx, d\varphi)e^{\omega x}$ is the jump measure of the Lévy system associated with (η^0, Φ^0) , this shows that $(\hat{\xi}(s), \hat{\Theta}(s), s < b_{\mathcal{L}(1)})$ behaves as $(\eta^0(s), \Phi^0(s), s < T_1)$, where T_1 is an independent exponential time with parameter $-\psi(\omega)$, a fact that could have been derived directly.

Let us now compute the second term of (4.6.3). Changing the measure according to (4.5.6) again, one obtains

$$\begin{aligned} \hat{E}_\theta \left[F(b_{\mathcal{L}(1)}, \hat{\xi}(b_{\mathcal{L}(1)}^-), \Delta\hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}^-), \hat{\Theta}(b_{\mathcal{L}(1)})) \right] \\ = \mathbb{E}_\theta \left[\sum_{s>0} |\Delta X(s)|^\omega F \left(s, \xi(s^-), \log |\Delta X(s)| - \xi(s^-), \Theta(s^-), \frac{\Delta X(s)}{|\Delta X(s)|} \right) \right] \\ = \mathbb{E}_{0,\theta} \left[\sum_{s>0} e^{\omega\xi(s^-)} |\Theta(s^-) - e^{\Delta\xi(s)} \Theta(s)|^\omega F(s, \xi(s^-), \log |\Theta(s^-) - e^{\Delta\xi(s)} \Theta(s)|, \Theta(s^-), \Theta_\Delta(s)) \right], \end{aligned}$$

where as usual

$$\Theta_\Delta(s) = \frac{\Theta(s^-) - e^{\Delta\xi(s)} \Theta(s)}{|\Theta(s^-) - e^{\Delta\xi(s)} \Theta(s)|}.$$

Using the compensation formula for (ξ, Θ) , this is

$$\begin{aligned} & \hat{E}_\theta \left[F(b_{\mathcal{L}(1)}, \hat{\xi}(b_{\mathcal{L}(1)}^-), \Delta \hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}^-), \hat{\Theta}(b_{\mathcal{L}(1)})) \right] \\ &= \mathbb{E}_{0,\theta} \left[\int_0^\infty ds e^{\omega \xi(s)} \int L_{\Theta(s)}(dx, d\varphi) |\Theta(s) - e^x \varphi|^\omega \right. \\ & \quad \left. \times F\left(s, \xi(s), \log |\Theta(s) - e^x \varphi|, \Theta(s), \frac{\Theta(s) - e^x \varphi}{|\Theta(s) - e^x \varphi|}\right) \right] \end{aligned}$$

We want to perform the change of variables $(y, \phi) = (\log |\theta - e^x \varphi|, \frac{\theta - e^x \varphi}{|\theta - e^x \varphi|})$ for fixed θ in the second integral. Recall that we have defined $\tilde{L}_\theta$ as the image measure of L_θ through this mapping, and that these measures satisfy the isotropy relationship (4.5.2). Therefore,

$$\begin{aligned} & \hat{E}_\theta \left[F(b_{\mathcal{L}(1)}, \hat{\xi}(b_{\mathcal{L}(1)}^-), \Delta \hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}^-), \hat{\Theta}(b_{\mathcal{L}(1)})) \right] \\ &= \mathbb{E}_{0,\theta} \left[\int_0^\infty ds e^{\omega \xi(s)} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \tilde{L}_{\Theta(s)}(dy, d\phi) e^{\omega y} F(s, \xi(s), y, \Theta(s), \phi) \right]. \end{aligned}$$

Tilting with the exponential martingale of ξ finally provides

$$\begin{aligned} & \hat{E}_\theta \left[F(b_{\mathcal{L}(1)}, \hat{\xi}(b_{\mathcal{L}(1)}^-), \Delta \hat{\xi}(b_{\mathcal{L}(1)}), \hat{\Theta}(b_{\mathcal{L}(1)}^-), \hat{\Theta}(b_{\mathcal{L}(1)})) \right] \\ &= \mathbb{E}_{0,\theta}^0 \left[\int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \tilde{L}_{\Phi^0(s)}(dy, d\phi) e^{\omega y} F(s, \eta^0(s), y, \Phi^0(s), \phi) \right]. \quad (4.6.7) \end{aligned}$$

Putting together (4.6.3), (4.6.6) and (4.6.7), we end up with

$$\begin{aligned} & \hat{E}_\theta \left[\sum_{s>0} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \mathbf{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] \\ &= \mathbb{E}_{0,\theta}^0 \left[\int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \hat{L}_{\Phi^0(s)}(dx, d\varphi) F(s, \eta^0(s), x, \Phi^0(s), \varphi) \right], \end{aligned}$$

and since $(\eta^0(s), \Phi^0(s), s < T_1)$ has the same law as $(\hat{\xi}(s), \hat{\Theta}(s), s < b_{\mathcal{L}(1)})$, we can rewrite this as

$$\begin{aligned} & \hat{E}_\theta \left[\sum_{s>0} F(s, \hat{\xi}(s^-), \Delta \hat{\xi}(s), \hat{\Theta}(s^-), \hat{\Theta}(s)) \mathbf{1}_{\{s \leq b_{\mathcal{L}(1)}\}} \right] \\ &= \hat{E}_\theta \left[\int_0^\infty ds e^{\psi(\omega)s} \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \hat{L}_{\hat{\Theta}(s)}(dx, d\varphi) F(s, \hat{\xi}(s), x, \hat{\Theta}(s), \varphi) \right]. \quad (4.6.8) \end{aligned}$$

This completes the proof of (4.6.1).

The second assertion of the theorem is then a straightforward consequence. First, it is clear that since X is isotropic, so is $\hat{\mathcal{X}}$ by construction. Hence, by Proposition 4.2.7, $\hat{\xi}$ must be a Lévy process. The expression for $\hat{\psi}$ can be found using a particular case of the compensation formula (4.6.8). Alternatively, for any nonnegative measurable functionals F and G defined respectively on the space of finite càdlàg paths and on $\mathbb{R}$, we may compute

$$\begin{aligned} & \hat{E}_\theta \left[F(\hat{\xi}(s), s < b_{\mathcal{L}(1)}) G(\Delta \hat{\xi}(b_{\mathcal{L}(1)})) \right] \\ &= \mathbb{E}_\theta \left[\sum_{t>0} |\Delta X(t)|^\omega F(\log |X(s)|, s < t) G\left(\log \frac{|\Delta X(t)|}{|X(t^-)|}\right) \right] \\ &= \mathbb{E}_{0,\theta} \left[\sum_{t>0} e^{\omega \xi(t^-)} |\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)|^\omega F(\xi(s), s < t) G\left(\log |\Theta(t^-) - e^{\Delta \xi(t)} \Theta(t)|\right) \right] \\ &= \mathbb{E}_{0,\theta} \left[\int_0^\infty dt e^{\omega \xi(t)} F(\xi(s), s < t) \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} L_{\Theta(t)}(dx, d\varphi) |\Theta(t) - e^x \varphi|^\omega G(\log |\Theta(t) - e^x \varphi|) \right]. \end{aligned}$$

By isotropy of X , the second integral does not depend on the angle $\Theta(t)$ (see (4.5.2)). Hence by applying the change of variables $(y, \phi) = \left(\log |\Theta(t) - e^x \varphi|, \frac{\Theta(t) - e^x \varphi}{|\Theta(t) - e^x \varphi|} \right)$, we end up with

$$\begin{aligned} \widehat{E}_\theta \left[F(\widehat{\xi}(s), s < b_{\mathcal{L}(1)}) G(\Delta \widehat{\xi}(b_{\mathcal{L}(1)})) \right] \\ = \mathbb{E}_{0,\theta} \left[\int_0^\infty dt e^{\omega \xi(t)} F(\xi(s), s < t) \right] \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \widetilde{L}(dy, d\phi) e^{\omega y} G(y). \end{aligned}$$

In words, this proves that $(\widehat{\xi}(s), s < b_{\mathcal{L}(1)})$ and $\Delta \widehat{\xi}(b_{\mathcal{L}(1)})$ are independent. The former has the law of ξ killed according to its exponential martingale, leading to a Lévy process with Laplace exponent $q \mapsto \psi(\omega + q)$. On the other hand, the latter is distributed as $(\psi(\omega))^{-1} \int_{\phi \in \mathbb{S}^{d-1}} \widetilde{L}(dy, d\phi) e^{\omega y}$, which is the law of the first jump of a compound Poisson process with intensity measure $\int_{\phi \in \mathbb{S}^{d-1}} \widetilde{L}(dy, d\phi) e^{\omega y}$. By removing the killing, this entails that $\widehat{\xi}$ has Laplace exponent

$$\widehat{\psi}(q) = \psi(\omega + q) - \psi(\omega) + \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \widetilde{L}(dy, d\phi) e^{\omega y} (e^{qy} - 1), \quad q \geq 0.$$

Using that $\kappa(\omega) = 0$, this is

$$\widehat{\psi}(q) = \psi(\omega + q) + \int_{\mathbb{R}^* \times \mathbb{S}^{d-1}} \widetilde{L}(dy, d\phi) e^{(\omega+q)y}, \quad q \geq 0,$$

whence $\widehat{\psi}(q) = \kappa(\omega + q)$.

The law of the growth-fragmentations $\widehat{\mathbf{X}}_{n,j}$. We prove the last assertion of Theorem 4.6.1. It actually follows from the same arguments as in [BBCK18], but we provide the proof for the sake of completeness. To avoid cumbersome notation, we will restrict to proving the statement for the first generation. This is then easily extended thanks to the branching property. Let F be a nonnegative measurable functional on the space of càdlàg trajectories, and G_j , $j \geq 1$, be nonnegative measurable functionals on the space of multiset-valued paths. For $t > 0$, denote by $(\Delta_j(t), j \geq 1)$ the sequence consisting of all the jumps of $\mathcal{X}_\emptyset$ that happened strictly before time t , and the extra value of $\mathcal{X}_\emptyset(t)$, all ranked in descending order of their absolute value. We are after the identity

$$\widehat{E}_1 \left[F(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} G_j(\widehat{\mathbf{X}}_{0,j}) \right] = \widehat{E}_1 \left[F(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} \mathbf{E}_{\Delta_j(b_{\mathcal{L}(1)})} [G_j(\mathbf{X})] \right].$$

We start from the left-hand side, and apply the change of measure (4.5.6):

$$\begin{aligned} \widehat{E}_1 \left[F(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} G_j(\widehat{\mathbf{X}}_{0,j}) \right] \\ = \mathcal{E}_1 \left[\sum_{t>0} |\Delta \mathcal{X}_\emptyset(t)|^\omega F(\mathcal{X}_\emptyset(s), 0 \leq s \leq t) \prod_{j \geq 1} G_j(\widehat{\mathbf{X}}_{0,j}) \right]. \end{aligned}$$

Using the definition of the $\widehat{\mathbf{X}}_{0,j}$ together with the branching property under $\mathcal{P}_1$ give

$$\begin{aligned} \widehat{E}_1 \left[F(\mathcal{X}_\emptyset(s), 0 \leq s \leq b_{\mathcal{L}(1)}) \prod_{j \geq 1} G_j(\widehat{\mathbf{X}}_{0,j}) \right] \\ = \mathcal{E}_1 \left[\sum_{t>0} |\Delta \mathcal{X}_\emptyset(t)|^\omega F(\mathcal{X}_\emptyset(s), 0 \leq s \leq t) \prod_{j \geq 1} \mathbf{E}_{\Delta_j(t)} [G_j(\mathbf{X})] \right]. \end{aligned}$$

Applying the change of measure backwards, we get the desired identity. This concludes the proof of Theorem 4.6.1.

4.6.3 Comments on the isotropy assumption

The previous analysis of $\mathbb{R}^d$ -valued growth-fragmentations relies heavily on the isotropy assumption. Because of the complications caused by the underlying MAP structure, describing growth-fragmentations driven by anisotropic processes is a much more challenging task. We stress the importance of the isotropy assumption and comment on possible extensions to anisotropic growth-fragmentation processes.

First, we expect that in the anisotropic case, there should be an angular component in all the (super-)martingales, appearing in particular in Theorem 4.5.3. This already takes place in the $d = 1$ case [Sil21], for asymmetric signed growth-fragmentation processes, where the angular component is nothing but the sign. Remember in addition that, in analogy with the discrete multitype case (Section 4.3), the *types* in the spatial framework are the angles, and that the martingales in the multitype setting also involve the types (Section 4.3.2). If X is a $\mathbb{R}^d \setminus \{0\}$ -valued self-similar Markov process, this actually prompts us to define for $q \geq 0$ the linear operator

$$T_q : f \in \mathcal{C} \mapsto \left(\theta \in \mathbb{S}^{d-1} \mapsto \mathbb{E}_\theta \left[\sum_{t>0} f(\Theta_\Delta(\varphi(t))) |\Delta X(t)|^q \right] \right),$$

where φ is the Lamperti-Kiu time-change. This is the analogue of the matrix m appearing in the multitype case. Assume that X has jumps (otherwise the construction is irrelevant), and that $M_q := \sup_{\theta \in \mathbb{S}^{d-1}} \mathbb{E}_\theta [\sum_{t>0} |\Delta X(t)|^q] < \infty$. Then $T_q(f)$ is well-defined for all $f \in \mathcal{C}$, and for $f \in \mathcal{C}$,

$$\|T_q(f)\|_\infty \leq M_q \|f\|_\infty,$$

whence T_q is a continuous operator. Note also that, at least under the assumption that X jumps with positive probability to any open set $D \subset \mathbb{S}^{d-1}$ of directions, T_q is strongly positive, in the sense that for all nonnegative $f \neq 0$, $T_q(f) > 0$. Assume moreover that T_q takes values in $\mathcal{C}$, and that it is a compact operator. Then, by the Krein-Rutman theorem [Dei10], it must have positive spectral radius $r(q)$, which is moreover a simple eigenvalue associated to a positive eigenfunction v . In the spirit of Assumption (H), Section 4.5.2, one could impose the additional assumption

$$(H') \quad \text{There exists } \omega \geq 0 \text{ such that } r(\omega) = 1.$$

Then by definition, we have

$$\forall \theta \in \mathbb{S}^{d-1}, \quad \mathbb{E}_\theta \left[\sum_{t>0} v(\Theta_\Delta(\varphi(t))) |\Delta X(t)|^\omega \right] = v(\theta).$$

This generalises to vectors in $\mathbb{R}^d$ by self-similarity of X :

$$\forall (r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}, \quad \mathbb{E}_{r\theta} \left[\sum_{t>0} v(\Theta_\Delta(\varphi(t))) |\Delta X(t)|^\omega \right] = v(\theta) r^\omega. \quad (4.6.9)$$

Remark 4.6.3. When X is *isotropic* in the sense of Section 4.2, one can show that $v(\theta) = 1$ for all $\theta \in \mathbb{S}^{d-1}$ up to normalisation, and one therefore retrieves the cumulant approach presented in Section 4.5.2. Indeed, isotropy entails that if v is an eigenfunction associated with $r(q)$, then for all isometries U , $v(U \cdot)$ is also an eigenfunction associated with $r(q)$, and we conclude by simplicity of the eigenvalue that $v(U \cdot) = v$, so that v is constant.

Once (4.6.9) holds for some positive function v , then modulo these adjustments one can carry through the arguments for the genealogical martingale (Theorem 4.5.3) and

the many-to-one formula (Proposition 4.5.4). However, the description of the spine in Theorem 4.6.1 is more involved. This is mainly due to the fact that the jump intensity at time $b_{\mathcal{L}(1)}$ depends on the current angle of the spine. In the isotropic case, one can more or less get rid of this dependency. The proof of Theorem 4.6.1 hinges upon the existence of an *Esscher transform*. In the isotropic case, this readily comes from the fact that the ordinate ξ of X is a Lévy process, which does not hold anymore for anisotropic processes. This in particular yielded that $b_{\mathcal{L}(1)}$ (up to Lamperti time change) is an exponential random variable. This last feature should not hold in general, as already indicated by the discrete multitype case.

4.7 The growth-fragmentation embedded in Brownian excursions from hyperplanes

4.7.1 The excursion measure

Construction of the excursion measure $\mathbf{n}_+$. We fix $N \geq 3$ and recall from [Bur86] how one may define the Brownian excursion measure from hyperplanes in $\mathbb{R}^N$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a complete filtered probability space, on which is defined a N -dimensional Brownian motion B^N . We single out the last coordinate and write $B^N = (B^{N-1}, Z)$. Introduce the set $\mathcal{X}$ of càdlàg functions x defined on some finite time interval $[0, R(x)]$, and the set $\mathcal{X}_0$ of such functions x in $\mathcal{X}$ that are continuous and vanish at $R(x)$. Moreover, we define

$$U := \left\{ u := (x_1, \dots, x_{N-1}, z) \in \mathcal{X}^{N-1} \times \mathcal{X}_0, R(x_1) = \dots = R(x_{N-1}) = R(z) \text{ and } u(0) = 0 \right\}.$$

For $u \in U$, we shall write $R(u)$ for the common value of the lifetimes. All these sets are equipped with their usual σ -fields. Finally, in order to study the excursions of B^N from the hyperplane $\mathcal{H} = \{x_N = 0\}$, we introduce the local time $(\ell_s, s \geq 0)$ at 0 of the Brownian motion Z , as well as its inverse $(\tau_s, s \geq 0)$.

The *excursion process* $(\mathbf{e}_s, s > 0)$ of our interest is easily defined following the one-dimensional case (see [RY99], Chapter XII), by

(i) if $\tau_s - \tau_{s-} > 0$, then

$$\mathbf{e}_s : r \mapsto \left(B_{r+\tau_{s-}}^{N-1} - B_{\tau_{s-}}^{N-1}, Z_{r+\tau_{s-}} \right), \quad r \leq \tau_s - \tau_{s-},$$

(ii) if $\tau_s - \tau_{s-} = 0$, then $\mathbf{e}_s = \partial$,

where ∂ is some cemetery state. The following proposition directly stems from the one-dimensional case.

Proposition 4.7.1. *The excursion process $(\mathbf{e}_s, s > 0)$ is a $(\mathcal{F}_{\tau_s})_{s>0}$ -Poisson point process of excursions in U . Its intensity measure is*

$$\mathbf{n}(du', dz) := n(dz) \mathbb{P}((B^{N-1})^{R(z)} \in du'),$$

where n denotes the one-dimensional Itô measure on $\mathcal{X}_0$, and for any process X , and any time T , $X^T := (X_t, t \in [0, T])$.

We shall denote by $\mathbf{n}_+$ and $\mathbf{n}_-$ the restrictions of $\mathbf{n}$ to $U^+ := \{(u', z) \in U, z \geq 0\}$ and $U^- := \{(u', z) \in U, z \leq 0\}$ respectively. In [Bur86], excursion measures from hyperplanes in $\mathbb{R}^N$ are rather constructed using Bessel processes. More precisely, one first samples

the duration of the excursion with density $r \mapsto (2\pi r^3)^{-1/2} \mathbb{1}_{\{r \geq 0\}}$ with respect to Lebesgue measure, and then for the last coordinate, one samples a 3-dimensional Bessel bridge from 0 to 0 over $[0, r]$. This is equivalent to $\mathbf{n}_+$ in our representation (up to a multiplicative factor) thanks to Itô's description of n , for which we refer again to [RY99]. We conclude this paragraph with the following Markov property under $\mathbf{n}_+$.

Proposition 4.7.2. *On the event that $T_a := \inf\{0 \leq t \leq R(u), z(t) = a\} < \infty$, the process $(u(T_a + t) - u(T_a), 0 \leq t \leq R(u) - T_a)$ is independent of $(u(t), 0 \leq t \leq T_a)$ and is a d -dimensional Brownian motion stopped when hitting $\{x_N = -a\}$.*

Disintegration of $\mathbf{n}_+$. We now construct measures $\gamma_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^{N-1}$, for Brownian excursions from the hyperplane $\{x_N = 0\}$ conditioned on ending at $(v, 0)$, by disintegrating $\mathbf{n}_+$ over its endpoint. Whenever $r \geq 0$ and $\mathbf{x} \in \mathbb{R}^{N-1}$, we write Π_r for the law of a Bessel bridge from 0 to 0 over $[0, r]$, and $\mathbb{P}_r^{0 \rightarrow \mathbf{x}}$ for the law of a $(N-1)$ -dimensional Brownian bridge from 0 to $\mathbf{x}$ with duration r . See [AS20] for the case $N = 2$.

Proposition 4.7.3. *The following disintegration formula holds:*

$$\mathbf{n}_+ = \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(\frac{N}{2})}{2\pi^{N/2} |\mathbf{x}|^N} \cdot \gamma_{\mathbf{x}},$$

where $\gamma_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, are probability measures. In addition, for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$,

$$\gamma_{\mathbf{x}} = \int_0^\infty dr \frac{e^{-\frac{1}{2r}}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2}) r^{\frac{N}{2}+1}} \mathbb{P}_r^{0 \rightarrow \mathbf{x}} \otimes \Pi_r|_{|\mathbf{x}|^2}.$$

Proof. The proposition follows from Theorem 3.3 in [Bur86], but we rephrase it in our framework for completeness. Let $f : \mathcal{X}^{N-1} \rightarrow \mathbb{R}_+$ and $g : \mathcal{X}_0 \rightarrow \mathbb{R}_+$ be two nonnegative measurable functions. Then by Proposition 4.7.1,

$$\int_{U^+} f(u') g(z) \mathbf{n}_+(du', dz) = \int_{U^+} f(u') g(z) \mathbf{n}_+(dz) \mathbb{P}((B^{N-1})^{R(z)} \in du').$$

Then by Itô's description of $\mathbf{n}_+$ (see Chap. XII, Theorem 4.2 in [RY99]), we may split this integral over the duration $R(z)$:

$$\int_{U^+} f(u') g(z) \mathbf{n}_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi r^3}} \Pi_r[g] \mathbb{E}[f((B^{N-1})^r)].$$

We now condition on B_r^{N-1} , and we obtain

$$\int_{U^+} f(u') g(z) \mathbf{n}_+(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi r^3}} \int_{\mathbb{R}^{N-1}} d\mathbf{x} \frac{e^{-\frac{|\mathbf{x}|^2}{2r}}}{(2\pi r)^{\frac{N-1}{2}}} \Pi_r[g] \mathbb{E}_r^{0 \rightarrow \mathbf{x}}[f].$$

Finally, we perform the change of variables $r \mapsto t = r/|\mathbf{x}|^2$:

$$\int_{U^+} f(u') g(z) \mathbf{n}_+(du', dz) = \int_{\mathbb{R}^{N-1}} \frac{d\mathbf{x}}{|\mathbf{x}|^N} \int_0^\infty dt \frac{e^{-\frac{1}{2t}}}{2(2\pi)^{\frac{N}{2}} t^{\frac{N}{2}+1}} \Pi_{t|\mathbf{x}|^2}[g] \mathbb{E}_{t|\mathbf{x}|^2}^{0 \rightarrow \mathbf{x}}[f].$$

Since

$$\int_0^\infty dt \frac{e^{-\frac{1}{2t}}}{2(2\pi)^{\frac{N}{2}} t^{\frac{N}{2}+1}} = \frac{1}{2} \pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right),$$

this gives that the $\gamma_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, are probability measures, and the disintegration claim holds. $\square$

Bismut's description of $\mathbf{n}_+$. The following decomposition of $\mathbf{n}_+$ describes the left and right parts of the trajectory seen from a point chosen uniformly at random on the Brownian excursion weighted by its lifetime. Proposition 4.7.4 is a straightforward consequence of Bismut's description of the one-dimensional Itô measure n . The following picture illustrates how the excursion splits when seen from a uniform point.

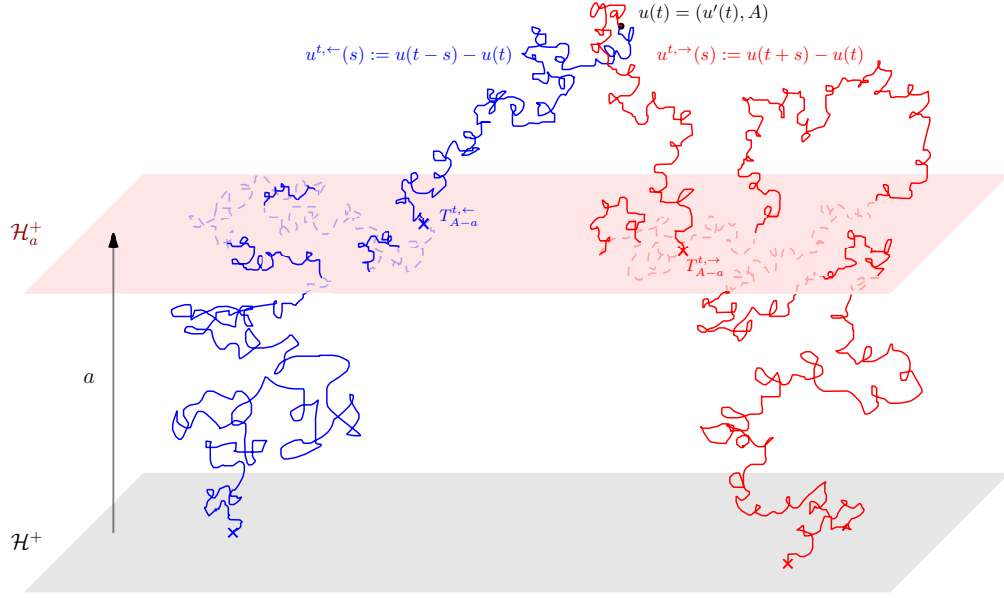


Figure 4.1 – Bismut's description of $\mathbf{n}_+$ in dimension $N = 3$. The height of a uniformly chosen point t on the excursion weighted by its duration is distributed according to the Lebesgue measure dA . Moreover, conditionally on the height, the excursion splits into two independent trajectories depicted in blue and red. Both are distributed as Brownian motion killed when hitting the bottom half-plane (in grey).

Proposition 4.7.4. (*Bismut's description of $\mathbf{n}_+$*)

Let $\bar{\mathbf{n}}_+$ be the measure defined on $\mathbb{R}_+ \times U^+$ by

$$\bar{\mathbf{n}}_+(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u)\}} dt \mathbf{n}_+(du).$$

Then under $\bar{\mathbf{n}}_+$ the "law" of $(t, (u', z)) \mapsto z(t)$ is the Lebesgue measure dA on $\mathbb{R}_+$, and conditionally on $z(t) = A$, $u^{t,\leftarrow} = (u(t-s) - u(t))_{0 \leq s \leq t}$ and $u^{t,\rightarrow} = (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}$ are independent Brownian motions killed when reaching the hyperplane $\{x_N = -A\}$.

4.7.2 Slicing excursions with hyperplanes

This section is an easy extension of the framework introduced in [AS20]. Let $u \in U^+$, and $a \geq 0$. We may write $u := (u', z)$ with $u' \in \mathcal{X}^{N-1}$ and $z \in \mathcal{X}_0, z \geq 0$.

Notation and setup. Define the superlevel set

$$\mathcal{I}(a) = \{s \in [0, R(u)], z(s) > a\}. \quad (4.7.1)$$

This is a countable (possibly empty) union of disjoint open intervals, and for any such interval $I = (i_-, i_+)$, we write $u_I(s) := u(i_- + s) - u(i_-)$, $0 \leq s \leq i_+ - i_-$, for the restriction of u to I , and $\Delta u_I := x(i_+) - x(i_-)$. Remark that Δu_I is a vector in the hyperplane

$\mathcal{H}_a := \{x_N = a\}$, which we call the *size* or *length* of the excursion u_I , see Figure 4.2. If $0 \leq t \leq R(u)$, we denote by $e_a^{(t)}$ the excursion u_I corresponding to the unique such interval I which contains t . Moreover, we define $\mathcal{H}_a^+$ as the set of excursions above $\mathcal{H}_a$ corresponding to the previous partition of $\mathcal{I}(a)$.

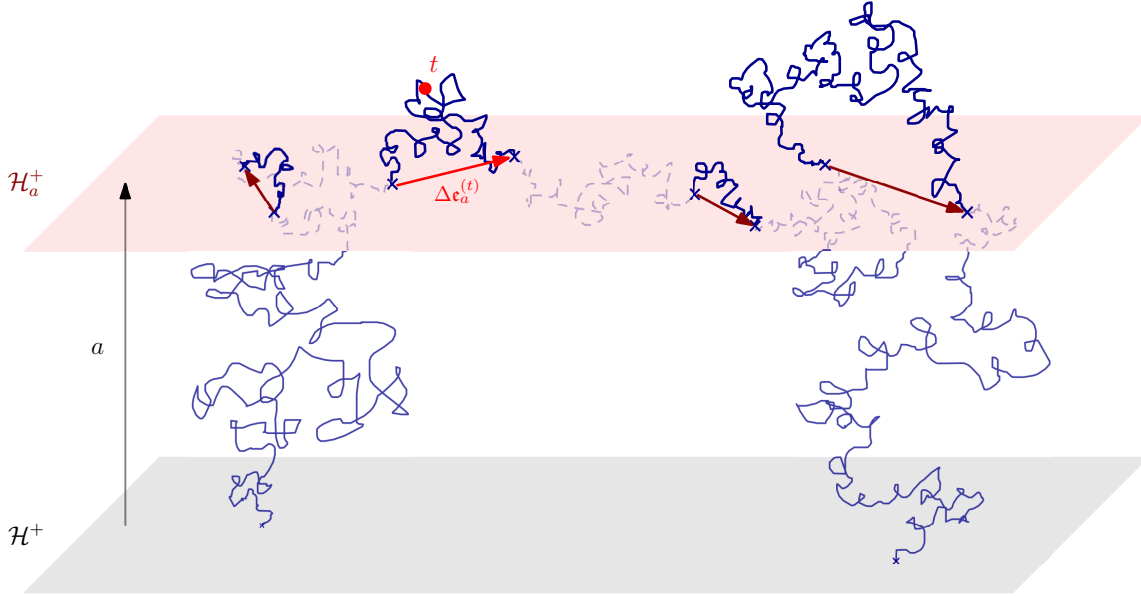


Figure 4.2 – Slicing at height a of an excursion u away from $\mathcal{H}$. The blue trajectory represents an excursion in the half-space $\{x_N > 0\}$, $N = 3$. For some fixed height $a > 0$ we draw the hyperplane $\mathcal{H}_a$ and record the sub-excursions above $\mathcal{H}_a$. The four largest of them are represented in dark blue (the reader should imagine many infinitesimal excursions). The red arrows indicate the size of the sub-excursions, counted with respect to the orientation of u .

We may now present an application of Proposition 4.7.4, which is similar to Proposition 2.7 in [AS20]. We show that, almost surely, excursions cut at heights do not make bubbles above any hyperplane. More precisely, we set

$$\mathcal{L} := \{u \in U^+, \exists 0 \leq t \leq R(u), \exists 0 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}. \quad (4.7.2)$$

This is the set of $u \in U^+$ making above some level an excursion which comes back to itself. Then

Proposition 4.7.5.

$$\mathbf{n}_+(\mathcal{L}) = 0.$$

Proof. We first notice that if $u \in \mathcal{L}$, then the set of t 's such that $\Delta e_a^{(t)}(u) = 0$ for some $0 \leq a < z(t)$ has positive Lebesgue measure. Therefore

$$\mathcal{L} \subset \left\{ u \in U^+, \int_0^{R(u)} \mathbf{1}_{\{\exists 0 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\}} dt > 0 \right\}. \quad (4.7.3)$$

Now using the notation in Proposition 4.7.4, and defining

$$T_a^{t, \leftarrow} := \inf\{s > 0, z(t-s) = a\} \quad \text{and} \quad T_a^{t, \rightarrow} := \inf\{s > 0, z(t+s) = a\},$$

we get

$$\begin{aligned} \mathbf{n}_+ \left(\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < z(t), \Delta e_a^{(t)}(u)=0\}} dt \right) \\ = \bar{\mathbf{n}}_+ \left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < z(t), \Delta e_a^{(t)}(u) = 0\} \right) \\ = \bar{\mathbf{n}}_+ \left(\{(t, u) \in \mathbb{R}_+ \times U^+, \exists 0 \leq a < z(t), u^{t, \leftarrow}(T_a^{t, \leftarrow}) = u^{t, \rightarrow}(T_a^{t, \rightarrow})\} \right). \end{aligned}$$

Bismut's description 4.7.4 of $\mathbf{n}_+$ (see Figure 4.1) finally gives

$$\mathbf{n}_+ \left(\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < z(t), \Delta e_a^{(t)}(u)=0\}} dt \right) = \int_0^{+\infty} dA \mathbb{P}(\exists 0 < a \leq A, B_1^{N-1}(T_a^1) = B_2^{N-1}(T_a^2)),$$

where B_1^{N-1}, B_2^{N-1} are independent $(N-1)$ -dimensional Brownian motions, and T_a^1, T_a^2 are independent Brownian hitting times. It is now well-known that the entries of $B_1^{N-1}(T_a^1)$ and $B_2^{N-1}(T_a^2)$ are symmetric Cauchy processes in a . By independence, the entries of $B_1^{N-1}(T_a^1) - B_2^{N-1}(T_a^2)$ are also Cauchy processes, for which points are polar (see [Ber96], Chap. II, Section 5). Hence

$$\mathbf{n}_+ \left(\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < z(t), \Delta e_a^{(t)}(u)=0\}} dt \right) = 0.$$

This yields that for $\mathbf{n}_+$ -almost every excursion u ,

$$\int_0^{R(u)} \mathbb{1}_{\{\exists 0 \leq a < z(t), \Delta e_a^{(t)}(u)=0\}} dt = 0,$$

and given the inclusion (4.7.3), we infer that $\mathbf{n}_+(\mathcal{L}) = 0$. $\square$

The branching property of excursions in $\mathcal{H}_a^+$. When cutting excursions with the hyperplanes $\mathcal{H}_a$, the natural filtration is the one carrying the information below these hyperplanes. We call $(\mathcal{G}_a, a \geq 0)$ this filtration, completed with the $\mathbf{n}_+$ -negligible sets. Recall that we have set $T_a := \inf\{0 \leq t \leq R(u), z(t) = a\}$. Finally, we let $a > 0$ and rank the excursions $(e_i^{a,+}, i \geq 1)$ in $\mathcal{H}_a^+$ by descending order of the norm of their sizes $(\mathbf{x}_i^{a,+}, i \geq 1)$. Then the following branching property holds.

Proposition 4.7.6. *For all $A \in \mathcal{G}_a$, and all nonnegative measurable functions $F_1, \dots, F_k : U^+ \rightarrow \mathbb{R}_+, k \geq 1$,*

$$\mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k F_i(e_i^{a,+}) \right) = \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k \gamma_{\mathbf{x}_i^{a,+}}[F_i] \right),$$

and the same also holds under $\gamma_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$.

Proof. We refer to [AS20] for the proof in the planar case, which is easily extended to higher dimensions. $\square$

4.7.3 A temporal martingale

Recall from (4.7.1) the notation $\mathcal{H}_a^+$ for the set of excursions above $\mathcal{H}_a$.

Theorem 4.7.7. *Under $\gamma_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, the process*

$$\mathcal{M}_a := \mathbb{1}_{\{T_a < \infty\}} \cdot \sum_{e \in \mathcal{H}_a^+} |\Delta e|^N, \quad a \geq 0,$$

is a martingale with respect to $(\mathcal{G}_a, a \geq 0)$.

The proof is again an adaptation of [AS20], Proposition 3.7, which is the planar case $N = 2$.

Proof. By the branching property (Proposition 4.7.6), we may restrict ourselves to proving that $\gamma_{\mathbf{x}}[\mathcal{M}_a] = |\mathbf{x}|^N$ for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$ and all $a \geq 0$. Let $f : \mathbb{R}^{N-1} \setminus \{0\} \rightarrow \mathbb{R}_+$ a nonnegative measurable function. We aim at computing $\mathbf{n}_+(\mathcal{M}_a f(u'(R(u))))$, where as usual we write $u = (u', z) \in \mathcal{X}^{N-1} \times \mathcal{X}_0$ for $u \in U^+$. By the Markov property 4.7.2 under $\mathbf{n}_+$, we have

$$\mathbf{n}_+(\mathcal{M}_a f(u'(R(u)))) = \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{E} \left[\sum_{s \leq \ell_{T-a}} \mathbb{1}_{\mathbf{e}_s \in U^+} |\Delta \mathbf{e}_s|^N f(B^{N-1}(T-a)) \right] \right). \quad (4.7.4)$$

Using the Master formula for the excursion process $(\mathbf{e}_s, s > 0)$ and the density of the endpoint under $\mathbf{n}_+$ given by Proposition 4.7.3 yields

$$\begin{aligned} & \mathbb{E} \left[\sum_{s \leq \ell_{T-a}} \mathbb{1}_{\mathbf{e}_s \in U^+} |\Delta \mathbf{e}_s|^N f(B^{N-1}(T-a)) \right] \\ &= \mathbb{E} \left[\int_0^{T-a} d\ell_s \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(N/2)}{2\pi^{N/2} |\mathbf{x}|^N} |\mathbf{x}|^N \mathbb{E}[f(\mathbf{x}' + B^{N-1}(T-a))] \Big|_{\mathbf{x}' = B^{N-1}(s) + \mathbf{x}} \right]. \end{aligned}$$

A change of variables then gives

$$\begin{aligned} & \mathbb{E} \left[\sum_{s \leq \ell_{T-a}} \mathbb{1}_{\mathbf{e}_s \in U^+} |\Delta \mathbf{e}_s|^N f(B^{N-1}(T-a)) \right] \\ &= \mathbb{E} \left[\int_0^{T-a} d\ell_s \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(N/2)}{2\pi^{N/2}} \mathbb{E}[f(\mathbf{x} + B^{N-1}(T-a))] \right], \end{aligned}$$

and since the Lebesgue measure is an invariant measure for Brownian motion,

$$\mathbb{E} \left[\sum_{s \leq \ell_{T-a}} \mathbb{1}_{\mathbf{e}_s \in U^+} |\Delta \mathbf{e}_s|^N f(B^{N-1}(T-a)) \right] = \mathbb{E}[\ell_{T-a}] \times \left(\frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x}) d\mathbf{x} \right).$$

Recall from [RY99], Chap. VI, Section 4, that ℓ_{T-a} is an exponential random variable with mean $2a$, so that we end up with

$$\mathbb{E} \left[\sum_{s \leq \ell_{T-a}} \mathbb{1}_{\mathbf{e}_s \in U^+} |\Delta \mathbf{e}_s|^N f(B^{N-1}(T-a)) \right] = 2a \times \left(\frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x}) d\mathbf{x} \right).$$

Coming back to (4.7.4), we have finally come to the formula

$$\mathbf{n}_+(\mathcal{M}_a f(u'(R(u)))) = 2a \times \left(\frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x}) d\mathbf{x} \right) \times \mathbf{n}_+(T_a < \infty).$$

We now use Proposition 3.6, Chapter XII, of [RY99] to get that $\mathbf{n}_+(T_a < \infty) = \frac{1}{2a}$. Therefore

$$\mathbf{n}_+(\mathcal{M}_a f(u'(R(u)))) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x}) d\mathbf{x}.$$

We then disintegrate $\mathbf{n}_+$ thanks to Proposition 4.7.3:

$$\int_{\mathbb{R}^{N-1} \setminus \{0\}} \frac{\Gamma(N/2)}{2\pi^{N/2} |\mathbf{x}|^N} f(\mathbf{x}) \gamma_{\mathbf{x}}(\mathcal{M}_a) d\mathbf{x} = \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} f(\mathbf{x}) d\mathbf{x}.$$

This holds for all nonnegative measurable functions f , and so we may deduce by using a continuity argument that for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, $\gamma_{\mathbf{x}}(\mathcal{M}_a) = |\mathbf{x}|^N$. $\square$

4.7.4 A change of measures

We fix $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$. To the martingale in Theorem 4.7.7, we can associate the change of measure $\mu_{\mathbf{x}}$ such that

$$\left. \frac{d\mu_{\mathbf{x}}}{d\gamma_{\mathbf{x}}} \right|_{\mathcal{G}_a} := \frac{\mathcal{M}_a}{|\mathbf{x}|^N}, \quad a \geq 0.$$

This new measure can be defined more rigourously as in [AS20]. The martingale property of $\mathcal{M}$ makes this definition consistent, and the existence of $\mu_{\mathbf{x}}$ then follows by Kolmogorov's extension theorem. Performing this change of measure will result in splitting the excursion into two independent excursions in the half-space $\mathcal{H}^+ := \{x_N > 0\}$ going to infinity, as in Figure 4.3. To describe the law $\mu_{\mathbf{x}}$, we call $\mathcal{H}^+$ -excursion a process in $\mathbb{R}^N$ whose first $(N-1)$ entries are independent Brownian motions, and whose last entry is an independent 3-dimensional Bessel process starting at 0 (so that this process actually remains in $\mathcal{H}^+$). We recall that $T_a := \inf\{0 \leq t \leq R(u), z(t) = a\}$, and set $S_a := \inf\{0 \leq t \leq R(u), z(R(u)-t) = a\}$.

Theorem 4.7.8. *Under $\mu_{\mathbf{x}}$, for all $a > 0$, the processes $(u(s), s \leq T_a)$ and $(u(R(u)-s), s \leq S_a)$ are independent $\mathcal{H}^+$ -excursions started respectively from 0 and $(\mathbf{x}, 0)$ and stopped when hitting $\mathcal{H}_a$.*

Proof. The proof is taken almost *verbatim* from [AS20]. Let f, g two nonnegative measurable functions defined on $\mathcal{X}^N$. By the Markov property 4.7.2,

$$\begin{aligned} & \mathbf{n}_+ \left(f(u(s), 0 \leq s \leq T_a) g(u(R(u)-s), 0 \leq s \leq S_a) \mathcal{M}_a \right) \\ &= \mathbf{n}_+ \left(\mathbb{1}_{\{T_a < \infty\}} f(u(s), 0 \leq s \leq T_a) \right) \times \mathbb{E} \left[\sum_{r < \ell_{T-a}} \mathbb{1}_{\mathbf{e}_r \in U^+} |\Delta \mathbf{e}_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right], \end{aligned}$$

where we have set $S := \inf\{0 \leq t \leq T-a, Z(R(u)-t) = 0\}$. Yet another application of the Master formula triggers that the second expectation is

$$\begin{aligned} & \mathbb{E} \left[\sum_{r < \ell_{T-a}} \mathbb{1}_{\mathbf{e}_r \in U^+} |\Delta \mathbf{e}_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right] \\ &= \mathbb{E} \left[\int_0^{T-a} d\ell_r \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{\Gamma(N/2)}{2\pi^{N/2} |\mathbf{x}|^N} |\mathbf{x}|^N \mathbb{E}[g((\mathbf{x}', a) + B^N(T-a-s), s \leq S)]_{|\mathbf{x}' = B^{N-1}(r) + \mathbf{x}} \right]. \end{aligned}$$

Therefore, by a change of variables, and using that $\mathbb{E}[\ell(T-a)] = 2a$, we are left with

$$\begin{aligned} & \mathbb{E} \left[\sum_{r < \ell_{T-a}} \mathbb{1}_{\mathbf{e}_r \in U^+} |\Delta \mathbf{e}_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right] \\ &= 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \mathbb{E}[g((\mathbf{x}, a) + B^N(T-a-s), s \leq S)]. \end{aligned}$$

As usual, we write $B^N = (B^{N-1}, Z)$. Given Z , $(B^{N-1}(s), s \leq T-a)$ is a $(N-1)$ -dimensional Brownian motion, stopped at $T-a$. Since the Lebesgue measure is reversible for Brownian

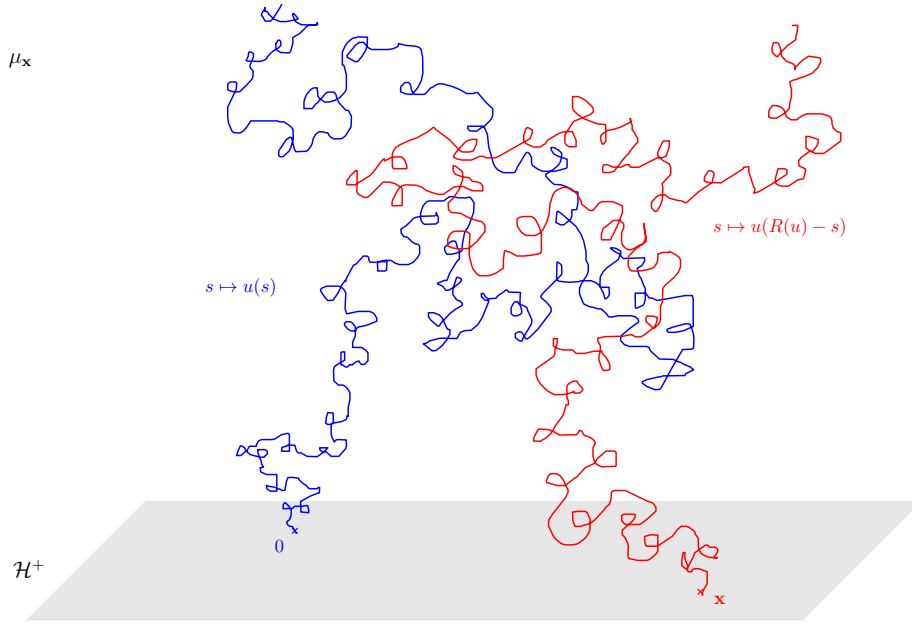


Figure 4.3 – Splitting of the excursion under μ_x . Under the change of measure, the excursion splits into two independent $\mathcal{H}^+$ -excursions (blue and red). For fixed height $a > 0$, the sub-excursion above a straddling the point at infinity is obtained by running two independent N -dimensional Brownian motions started from infinity and stopped when hitting the hyperplane $\mathcal{H}_a$.

motion, we get

$$\begin{aligned} & \mathbb{E} \left[\sum_{r < \ell_{T-a}} \mathbf{1}_{\mathbf{e}_r \in U+} |\Delta \mathbf{e}_r|^N g(B^N(T-a-s), 0 \leq s \leq S) \right] \\ &= 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \mathbb{E}[g((\mathbf{x} + B^{N-1}(s), a + Z(T-a-s)), s \leq S)]. \end{aligned}$$

Finally, recall from [RY99], Chap. VII, Corollary 4.6, that $(a + Z(T-a-s), s \leq S)$ has the law of a 3-dimensional Bessel process V run until time T_a^V when it first hits a . Thus

$$\begin{aligned} & \mathbf{n}_+ \left(f(u(s), 0 \leq s \leq T_a) g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \right) \\ &= \mathbf{n}_+ \left(\mathbf{1}_{\{T_a < \infty\}} f(u(s), 0 \leq s \leq T_a) \right) \times 2a \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \mathbb{E}[g((\mathbf{x} + B^{N-1}(s), V(s)), s \leq T_a^V)], \end{aligned}$$

and since $\mathbf{n}_+(T_a < \infty) = \frac{1}{2a}$, we obtain

$$\begin{aligned} & \mathbf{n}_+ \left(f(u(s), 0 \leq s \leq T_a) g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \right) \\ &= \mathbf{n}_+ \left(f(u(s), 0 \leq s \leq T_a) | T_a < \infty \right) \times \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \mathbb{E}[g((\mathbf{x} + B^{N-1}(s), V(s)), s \leq T_a^V)]. \end{aligned}$$

On the other hand, one can prove using Williams' description of n_+ that conditionally on

$T_a < \infty$, $(u(s), 0 \leq s \leq T_a)$ is distributed as $((B^{N-1}(s), V(s)), 0 \leq s \leq T_a^V)$. Hence,

$$\begin{aligned} & \mathbf{n}_+ \left(f(u(s), 0 \leq s \leq T_a) g(u(R(u) - s), 0 \leq s \leq S_a) \mathcal{M}_a \right) \\ &= \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \mathbb{E}[g(\mathbf{x} + B^{N-1}(s), V(s)), s \leq T_a^V] \times \mathbb{E}[f((B^{N-1}(s), V(s)), 0 \leq s \leq T_a^V)]. \end{aligned}$$

Disintegrating over the endpoint as in Proposition 4.7.3, and using a continuity argument, this concludes the proof of Theorem 4.7.8 $\square$

Remark 4.7.9. We reformulate the previous results in the parlance of Section 4.5. Setting

$$\mathbf{Z}(a) := \left\{ \left\{ \Delta e, e \in \mathcal{H}_a^+ \right\} \right\}, \quad a \geq 0,$$

it follows from Proposition 4.7.6 that $\mathbf{Z}$ enjoys a branching property akin to Proposition 4.5.1. We could have pointed out an Eve cell in the spirit of [AS20, Theorem 3.3] by considering the locally largest excursion. Together with an avatar of [AS20, Theorem 3.6], this proves that under $\gamma_{\mathbf{x}}$, $\mathbf{Z}$ is a *spatial growth-fragmentation process*. Actually, one should first check that the evolution of the Eve cell generates all the excursions, but this is a simple consequence of the arguments presented in [AS20, Theorem 4.1]. In the previous exposition, we chose to rather dwell on the spine description. More specifically, the martingale in Theorem 4.7.7 is a temporal version of the martingale in Theorem 4.5.3. Then, Theorem 4.7.8 determines the law of the spine without reference to Theorem 4.6.1. The spine is described as the Brownian motion B^{N-1} taken at the hitting times of another independent linear Brownian motion, and hence is a $(N-1)$ -dimensional isotropic Cauchy process.

4.7.5 Extension to isotropic stable Lévy processes

As in [Sil21], we can extend the previous construction to stable processes with index $\alpha \in (0, 2)$. We recall that we have set $N \geq 3$, and that the case $N = 2$ was already treated in [Sil21]. We will not provide all the details of the proofs since the arguments are similar to the Brownian case.

The excursion measure $\mathbf{n}^\alpha$. We shall consider the following excursions, which consist in replacing the first $(N-1)$ entries of the previous setting by an isotropic α -stable Lévy process in $\mathbb{R}^{N-1}$. We keep the notation in Section 4.7.1, except that now is defined on the probability space a $(N-1)$ -dimensional isotropic stable Lévy process X^{N-1} , and we consider the process $Z^N := (X^{N-1}, Z)$ with Brownian last coordinate. Then, we introduce the excursion process $(\mathbf{e}_s^\alpha, s > 0)$ as

(i) if $\tau_s - \tau_{s-} > 0$, then

$$\mathbf{e}_s^\alpha : r \mapsto \left(X_{r+\tau_{s-}}^{N-1} - X_{\tau_{s-}}^{N-1}, Z_{r+\tau_{s-}} \right), \quad r \leq \tau_s - \tau_{s-},$$

(ii) if $\tau_s - \tau_{s-} = 0$, then $\mathbf{e}_s^\alpha = \partial$.

As in Proposition 4.7.1, this defines a Poisson point process with intensity measure

$$\mathbf{n}^\alpha(du', dz) := n(dz) \mathbb{P}((X^{N-1})^{R(z)} \in du').$$

Let $\mathbf{n}_+^\alpha$ be the restriction of $\mathbf{n}^\alpha$ to positive excursions. We now want to condition $\mathbf{n}_+^\alpha$ on the endpoint of the excursion. For $\mathbf{x} \in \mathbb{R}^{N-1}$ and $r > 0$, let $\mathbb{P}_r^{\alpha, 0 \rightarrow \mathbf{x}}$ denote the law of an α -stable bridge from 0 to $\mathbf{x}$ over $[0, r]$. In addition, we write $(p_r^\alpha, r \geq 0)$ for the transition densities of X^{N-1} . Throughout this section, we fix $\omega_N := N - 1 + \frac{\alpha}{2}$.

Proposition 4.7.10. *The following disintegration formula holds:*

$$\mathbf{n}_+^\alpha = \int_{\mathbb{R}^{N-1} \setminus \{0\}} d\mathbf{x} \frac{C_N}{|\mathbf{x}|^{\omega_N}} \cdot \gamma_{\mathbf{x}}^\alpha, \quad (4.7.5)$$

where $\gamma_{\mathbf{x}}^\alpha$, $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, are probability measures, and

$$C_N = \frac{\alpha}{2\sqrt{2\pi}} \int_{\mathbb{R}_+} dv p_1^\alpha(v) v^{\omega_N-1}.$$

In addition, for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$,

$$\gamma_{\mathbf{x}}^\alpha = \int_0^\infty dr \frac{p_1^\alpha(r^{-1/\alpha})}{2\sqrt{2\pi} r^{1+\frac{\omega_N}{\alpha}}} \mathbb{P}_{r|\mathbf{x}|^2}^{\alpha,0 \rightarrow \mathbf{x}} \otimes \Pi_{r|\mathbf{x}|^2}.$$

Proof. Let f and g be two nonnegative measurable functions, respectively defined on $\mathcal{X}^{N-1}$ and $\mathcal{X}_0$. Following the proof of Proposition 4.7.3, we end up with

$$\begin{aligned} \int_{U^+} f(u') g(z) \mathbf{n}_+^\alpha(du', dz) &= \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^3} \Pi_r[g] \mathbb{E}[f((X^{N-1})^r)] \\ &= \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^3} \int_{\mathbb{R}^{N-1}} d\mathbf{x} p_r^\alpha(\mathbf{x}) \Pi_r[g] \mathbb{E}_r^{\alpha,0 \rightarrow \mathbf{x}}[f(X^{N-1})]. \end{aligned}$$

Note that, by self-similarity, for all $r > 0$ and $\mathbf{x} \in \mathbb{R}^{N-1}$,

$$p_r^\alpha(\mathbf{x}) = r^{-\frac{N-1}{\alpha}} p_1^\alpha(r^{-1/\alpha} \mathbf{x}).$$

Hence

$$\int_{U^+} f(u') g(z) \mathbf{n}_+^\alpha(du', dz) = \int_0^\infty \frac{dr}{2\sqrt{2\pi} r^3} \int_{\mathbb{R}^{N-1}} d\mathbf{x} r^{-\frac{N-1}{\alpha}} p_1^\alpha(r^{-1/\alpha} \mathbf{x}) \Pi_r[g] \mathbb{E}_r^{\alpha,0 \rightarrow \mathbf{x}}[f(X^{N-1})],$$

and by the change of variables $u(r) := \frac{r}{|\mathbf{x}|^\alpha}$, this is

$$\begin{aligned} \int_{U^+} f(u') g(z) \mathbf{n}_+^\alpha(du', dz) &= \int_{\mathbb{R}^{N-1}} \frac{d\mathbf{x}}{|\mathbf{x}|^{\omega_N}} \int_0^\infty \frac{du}{2\sqrt{2\pi} u^3} u^{-\frac{N-1}{\alpha}} p_1^\alpha\left(u^{-1/\alpha} \frac{\mathbf{x}}{|\mathbf{x}|}\right) \Pi_{u|\mathbf{x}|^\alpha}[g] \mathbb{E}_{u|\mathbf{x}|^\alpha}^{\alpha,0 \rightarrow \mathbf{x}}[f(X^{N-1})]. \end{aligned}$$

Observe that the isotropy of X^{N-1} yields the relationship $p_1^\alpha\left(u^{-1/\alpha} \frac{\mathbf{x}}{|\mathbf{x}|}\right) = p_1^\alpha(u^{-1/\alpha})$, so that

$$\int_{U^+} f(u') g(z) \mathbf{n}_+^\alpha(du', dz) = \int_{\mathbb{R}^{N-1}} \frac{d\mathbf{x}}{|\mathbf{x}|^{\omega_N}} \int_0^\infty du \frac{p_1^\alpha(u^{-1/\alpha})}{2\sqrt{2\pi} u^{1+\frac{\omega_N}{\alpha}}} \Pi_{u|\mathbf{x}|^\alpha}[g] \mathbb{E}_{u|\mathbf{x}|^\alpha}^{\alpha,0 \rightarrow \mathbf{x}}[f(X^{N-1})].$$

The proposition follows. $\square$

Remark 4.7.11. We emphasize that the proof of Proposition 4.7.10 uses the isotropy assumption on X^{N-1} , and indeed formula (4.7.5) shows that the excursion measure $\mathbf{n}_+^\alpha$ assigns a weight to the endpoint $\mathbf{x}$ which only depends on its radial part $|\mathbf{x}|$. If X^{N-1} were not isotropic, then one would have to deal with the angular part of $\mathbf{x}$ in the disintegration.

The following proposition is a Bismut description of $\mathbf{n}_+^\alpha$, which is easily extended from Proposition 4.7.4. The picture looks roughly the same as in Figure 4.1, albeit the two trajectories have their first $(N-1)$ entries distributed as an isotropic stable process in $\mathbb{R}^{N-1}$.

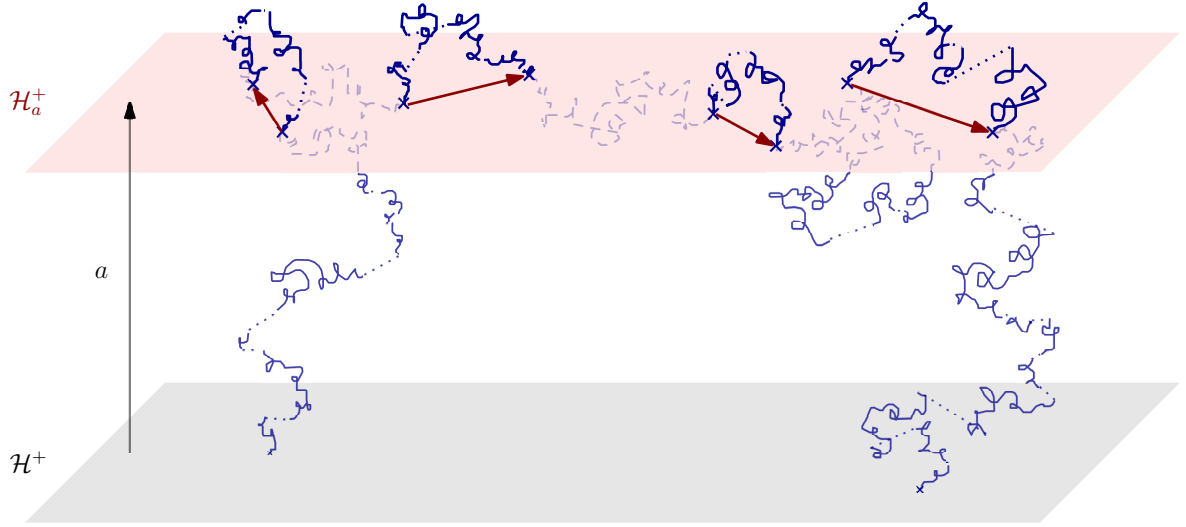


Figure 4.4 – Slicing of an excursion in $\mathcal{H}_+$ with stable first two coordinates, in dimension $N = 3$. The excursion is drawn in blue. The trajectory is càdlàg but jumps never occur for the height. We record the *length* (in red) of the sub-excursions (in dark blue) made above $\mathcal{H}_a$.

Proposition 4.7.12. (*Bismut's description of $\mathbf{n}_+^\alpha$*)

Let $\overline{\mathbf{n}}_+^\alpha$ be the measure defined on $\mathbb{R}_+ \times U^+$ by

$$\overline{\mathbf{n}}_+^\alpha(dt, du) = \mathbb{1}_{\{0 \leq t \leq R(u)\}} dt \mathbf{n}_+^\alpha(du).$$

Then under $\overline{\mathbf{n}}_+^\alpha$ the "law" of $(t, (u', z)) \mapsto z(t)$ is the Lebesgue measure dA on $\mathbb{R}_+$, and conditionally on $z(t) = A$, $u^{t, \leftarrow} = (u(t-s) - u(t))_{0 \leq s \leq t}$ and $u^{t, \rightarrow} = (u(t+s) - u(t))_{0 \leq s \leq R(u)-t}$ are independent and evolve as Z^N killed when reaching the hyperplane $\{x_N = -A\}$.

One of the consequences of this decomposition is that for $\mathbf{n}_+^\alpha$ -almost every excursion, there is no loop above any level. More precisely, recall the definition of $\mathcal{L}$ in (4.7.2). Then $\mathbf{n}_+^\alpha(\mathcal{L}) = 0$. The proof can be taken *verbatim* from Proposition 4.7.5, using that a stable process in dimension $N - 1 \geq 2$ does not hit points (see [Ber96, II, Corollary 17]).

The branching property under $\mathbf{n}_+^\alpha$. We will be interested in cutting excursions with hyperplanes at varying heights, and study the *length* of the subexcursions above these hyperplanes (Figure 4.4). As in Proposition 4.7.6, this exhibits a branching structure that we summarise in the next result, in the language introduced in Section 4.7.2.

Proposition 4.7.13. For all $A \in \mathcal{G}_a$, and all nonnegative measurable functions $F_1, \dots, F_k : U^+ \rightarrow \mathbb{R}_+, k \geq 1$,

$$\mathbf{n}_+^\alpha \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k F_i(e_i^{a,+}) \right) = \mathbf{n}_+^\alpha \left(\mathbb{1}_{\{T_a < \infty\}} \mathbb{1}_A \prod_{i=1}^k \gamma_{\mathbf{x}_i^{a,+}}^\alpha [F_i] \right),$$

and the same also holds under $\gamma_{\mathbf{x}}^\alpha$ for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$.

Martingale and spine decomposition under $\gamma_{\mathbf{x}}^\alpha$. In line with Theorem 4.7.7 and Theorem 4.7.8, we reveal the martingale in the stable setting and describe the law after the change of measure. The notation is implicitly taken from the Brownian case. All the proofs are omitted because they are simple extensions of their Brownian analogues. Recall that $\omega_N = N - 1 + \frac{\alpha}{2}$.

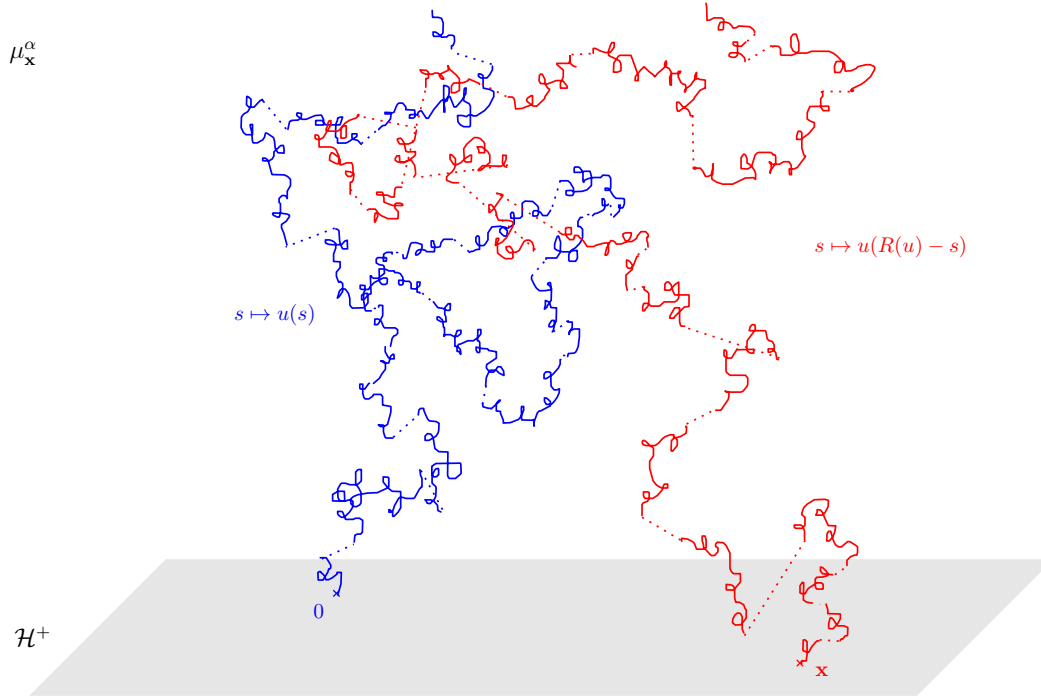


Figure 4.5 – The excursion u seen under $\mu_{\mathbf{x}}^\alpha$. Under the change of measure, u splits into two independent $(\alpha, \mathcal{H}_+)$ -excursions (blue and red), which are the analogues of the Brownian half-space excursions appearing in Figure 4.3, when the first $(N - 1)$ coordinates are replaced with an isotropic stable process. The length of the sub-excursion above some height a straddling the point at infinity is obtained by subordinating the isotropic process at the Brownian hitting time of level a . Let us stress once more that the last coordinate is continuous, so that this length is well defined for all positive height $a > 0$.

Theorem 4.7.14. Under $\gamma_{\mathbf{x}}^\alpha$ for all $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$, the process

$$\mathcal{M}_a^\alpha := \mathbb{1}_{\{T_a < \infty\}} \cdot \sum_{e \in \mathcal{H}_a^+} |\Delta e|^{\omega_N}, \quad a \geq 0,$$

is a martingale with respect to $(\mathcal{G}_a, a \geq 0)$.

Let $\mathbf{x} \in \mathbb{R}^{N-1} \setminus \{0\}$. Consider the change of measure $\mu_{\mathbf{x}}^\alpha$ such that

$$\left. \frac{d\mu_{\mathbf{x}}^\alpha}{d\gamma_{\mathbf{x}}^\alpha} \right|_{\mathcal{G}_a} := \frac{\mathcal{M}_a^\alpha}{|\mathbf{x}|^{\omega_N}}, \quad a \geq 0.$$

We now come to the description of the excursion under $\mu_{\mathbf{x}}^\alpha$. Call $(\alpha, \mathcal{H}^+)$ -excursion a process in $\mathbb{R}^N$ whose first $(N - 1)$ entries form an isotropic α -stable Lévy process, and whose last entry is an independent 3-dimensional Bessel process starting at 0 (so that this process actually remains in $\mathcal{H}^+$). We set $T_a := \inf\{0 \leq t \leq R(u), z(t) = a\}$, and $S_a := \inf\{0 \leq t \leq R(u), z(R(u) - t) = a\}$.

Theorem 4.7.15. Under $\mu_{\mathbf{x}}^\alpha$, for all $a > 0$, the processes $(u(s), s \leq T_a)$ and $(u(R(u) - s), s \leq S_a)$ are independent $(\alpha, \mathcal{H}^+)$ -excursions started respectively from 0 and $(\mathbf{x}, 0)$ and stopped when hitting $\mathcal{H}_a$.

Figure 4.5 illustrates the theorem.

Remark 4.7.16. Let us notice that, similarly to Remark 4.7.9, the process

$$\mathbf{Z}(a) := \left\{ \left\{ \Delta e, e \in \mathcal{H}_a^+ \right\} \right\}, \quad a \geq 0,$$

is a spatial growth-fragmentation process under $\gamma_{\mathbf{x}}^\alpha$. One could fiddle with the ideas of [Sil21, Theorem 6.8] in order to define an Eve cell process driving $\mathbf{Z}$, but beware that the (signed) growth-fragmentation process described therein is not isotropic as such (one needs to adjust the constants c_+ and c_- to recover an isotropic process). Theorem 4.7.15 provides the law of the spine as an isotropic $(N - 1)$ -dimensional $\frac{\alpha}{2}$ -stable process.

Chapter 5

A growth-fragmentation process connected to space-filling explorations of pure gravity $\sqrt{8/3}$ -quantum disc

Abstract

This chapter is based on joint work with Ellen Powell and Alexander Watson. We consider a space-filling SLE_6 η on an independent $\sqrt{8/3}$ -quantum gravity disc. We first show that, when looking at the branches η^z targeted at all points z in the quantum disc, the underlying tree is a growth-fragmentation tree. The branch towards z in the growth-fragmentation process accounts for the total quantum boundary length of the connected component of $\mathbb{D} \setminus \eta([0, t])$ containing $z \in \mathbb{D}$. Our result therefore corresponds to the boundary case $\gamma \rightarrow \sqrt{8/3}$ of [MSW20]. As expected from the convergence in the scaling limit of Boltzmann triangulations towards $\sqrt{8/3}$ -LQG, we retrieve the growth-fragmentation revealed by [BCK18], and elaborated upon in [BBCK18], for $\theta = 3/2$. In this respect, our work also constitutes a quantum equivalent to the Brownian disc point of view of [LGR20].

Secondly, we make use of the mating-of-trees theory [AG21] to cast another light on the growth-fragmentation process involved. We show that it can alternatively be constructed in a Brownian excursion from boundary to apex in a cone with angle $\frac{2\pi}{3}$. As it turns out, this provides a construction of an instance of the spectrally positive $\frac{3}{2}$ -stable process arising from cone excursions. We also establish along the way a general Bismut description of Brownian cone excursions, which is of independent interest.

5.1 Introduction

5.1.1 Overview

Background on Liouville measure and quantum surfaces. Suppose that we are given a planar simply connected domain $D \subset \mathbb{C}$, together with an instance of the Gaussian free field (GFF) h in D . The construction of the *Liouville quantum gravity (LQG) area measure* μ_h can be traced back to the pioneering work of Kahane [Kah85] on Gaussian multiplicative chaos, or to the work of Høegh-Krohn [HK71] on quantum field theory. Informally, one would like to make sense of the measure

$$\mu_h^\gamma(dz) = e^{\gamma h(z)} dz, \quad (5.1.1)$$

where $\gamma \geq 0$ is a parameter. The issue with this definition is that h is not a random function but a random distribution, so that making sense of its exponential requires some

clarifications. Nevertheless, one can give a meaning to (5.1.1) for $\gamma \in (0, 2)$ by using multiple approximation procedures [Kah85, DS11, Ber17a]. Likewise, one can construct the *LQG boundary length measure* ν_h^γ on ∂D or more generally on some curves in D , including SLE_κ or $\text{SLE}_{\kappa'}$ type of curves, for $\kappa := \gamma^2$ and $\kappa' = 16/\gamma^2$ [She16a].

The γ -quantum surface (D, h) can be defined in an informal way as the surface parametrised by D , with area measure μ_h^γ . In other words, on this surface, the distances and areas have been distorted by the measure (5.1.1): an infinitesimal element of area dz now corresponds to an area $e^{\gamma h(z)} dz$, which favours points around which the free field h is large (we stress once again that h is not defined pointwise). Moreover, we want the notion of quantum surface to encode the *conformal structure* of (D, h) . Indeed, the measures μ_h^γ and ν_h^γ enjoy the following conformal covariance property. Let $D' \subset \mathbb{C}$ be another simply connected domain, and $f : D \rightarrow D'$ a conformal map taking D onto D' . Then it follows from [DS11] that $\mu_h \circ f^{-1} = \mu_{h'}$ almost surely, where

$$h' := h \circ f^{-1} + Q \log |(f^{-1})'|, \quad (5.1.2)$$

with $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$. In fact, the identity $\mu_h^\gamma \circ f^{-1} = \mu_{h'}^\gamma$ holds almost surely whatever the domain D' and the conformal map f [SW16]. This prompts us to regard two pairs (D, h) and (D', h') as equivalent if h and h' satisfy the change of coordinates formula (5.1.2) for some conformal map $f : D \rightarrow D'$. A γ -quantum surface is then an equivalence class of such pairs (D, h) [DS11, She16a, DMS14]. In reality, we will often want to consider quantum surfaces with some distinguished points on $D \cup \partial D$ or some extra decoration. In this case we introduce equivalent classes as in (5.1.2), except that we also require that f maps the decorations of D (e.g. the marked points) onto those of D' .

Quantum surfaces conjecturally correspond to the scaling limits of random planar maps. In this setting, the measures μ_h^γ and ν_h^γ are expected to be the scaling limits of the counting measures on vertices and on boundary vertices respectively. This is already known for a few models of planar maps conformally embedded in the plane via the Tutte embedding [GMS17] or the Cardy embedding [HS19]. For several models of uniform planar maps, this has also been proved in the so-called Gromov-Hausdorff-Prokhorov topology: see [LG13, Mie13, BM17, GM17, BMR19]. The present work focuses on the case $\gamma = \sqrt{8/3}$, sometimes called *pure gravity*, and associated with uniform random planar maps. We denote $\mu_h = \mu_h^\gamma$ and $\nu_h = \nu_h^\gamma$ for $\gamma = \sqrt{8/3}$.

SLE on LQG and the mating-of-trees. In the discrete world, one often considers, on top of a random planar map model, a statistical mechanics model. These two level of randomness are finely tuned so that the partition function of the latter matches the distribution of the planar maps. Such is the case, for instance, for FK-weighted planar maps. This is a model on loop-decorated planar maps, or equivalently on planar maps decorated with a subset of edges. The probability of occurrence of such a pair $(\mathbf{m}, \mathbf{t})$ describing a planar map together with a distinguished set of edges is proportional to

$$\mathbb{P}_q^{\text{FK}}((\mathbf{m}, \mathbf{t})) \propto \sqrt{q}^{\#\text{loops}},$$

for some parameter $q > 0$, and where $\#\text{loops}$ is the *number of loops* formed when drawing the interface between $\mathbf{t}$ and its dual $\mathbf{t}'$. Improving on Mullin's bijection in the spanning tree case [Mul67], Sheffield [She16b] presented a way to encode such a model by a non-Markovian random walk on $\mathbb{Z}^2$, or equivalently a pair of discrete random trees. Such a bijection goes under the name of *hamburger-cheeseburger bijection* or *Sheffield's bijection*.

The continuum analogue of considering a statistical model on top of a random planar map is to consider an independent SLE_κ or $\text{SLE}_{\kappa'}$ [Sch00, RS05] decoration on top of a Liouville

quantum gravity surface, for $\kappa = \gamma^2$ or $\kappa' = 16/\gamma^2$. There are many deep connections between SLE and LQG surfaces. At the heart of these connections, the so-called *mating-of-trees* theory is of paramount importance [DMS14, MS19]. As in the hamburger-cheeseburger bijection, it gives a way to encode SLE on LQG in the continuum by *mating* two continuum random trees. The continuum version of the FK-weighted planar map model amounts to drawing an independent space-filling $\text{SLE}_{\kappa'} \eta$ with $\kappa' = 16/\gamma^2$ on top of a particular infinite-volume surface called the γ -quantum cone, which is parametrised by the whole plane. Loosely speaking, the mating-of-trees theorem in this case [DMS14, Theorem 1.9] states that, provided we parametrise η according to γ -LQG area, the change in quantum boundary lengths of the left and right outer boundaries (L, R) of η evolve as a pair of correlated Brownian motions, with correlation $-\cos(4\pi/\kappa')$. The global picture is that the γ -quantum cone decorated with η can be constructed by *mating* the two continuum random trees associated with the Brownian motions L and R .

Besides constructing the hamburger-cheeseburger bijection, [She16b] provides a landmark scaling limit result, showing that the non-Markovian random walk on $\mathbb{Z}^2$ encoding the FK-weighted planar map model for $q \in (0, 4)$ scales to a pair of correlated Brownian motions. The correlation depends on q in such a way that, if one defines κ' via

$$q = 2 + 2\cos(8\pi/\kappa'), \quad \gamma = 4/\sqrt{\kappa'}, \quad (5.1.3)$$

then one retrieves the same covariance structure as in the mating-of-trees theorem for the space-filling $\text{SLE}_{\kappa'}$ -decorated γ -quantum cone. This can be interpreted as a scaling limit result for FK-weighted planar maps towards a γ -quantum cone, $\gamma \in (\sqrt{2}, 2)$, decorated with a collection of $\text{SLE}_{\kappa'}$ type of loops known as the *conformal loop ensemble* $\text{CLE}_{\kappa'}$, $\kappa' \in (4, 8)$, in the sense that the encoding walks converge. Such a convergence statement is said to hold in the *peanosphere* sense. Let us mention that the identity (5.1.3) matches the previous conjectures of [She09, SW12]. Furthermore, the convergence result of [She16b] has also been improved by [GMS19], where it was proved that many quantities (*e.g.* length or area) associated with the discrete FK loops converge to the corresponding quantum ones of the $\text{CLE}_{\kappa'}$ loops on the quantum cone. Such a problem is naturally related to the study of *cone points* of Brownian motion (this can already be seen at the discrete level from the hamburger-cheeseburger encoding).

For reasons that will become clear later on, our work deals with another type of quantum surface, a γ -quantum disc. A *unit boundary quantum disc* is a specific kind of quantum surface, with finite (but random) quantum area and quantum boundary length equal to 1. This is a natural quantum surface with boundary, for which a similar in spirit mating-of-trees theorem was proved [DMS14, MS19], and then extended more recently by [AG21]. This mating-of-trees theorem considers the *counterclockwise* version of space-filling $\text{SLE}_{\kappa'}$ (see Section 5.2.2 for details), and can be stated as follows (in the form of [AG21, Theorem 1.1]).

Theorem 5.1.1. *Let $\gamma \in (0, 2)$ and $(\mathbb{D}, \psi, -i)$ be a unit boundary marked quantum disc with random quantum area $\mu_\psi^\gamma(\mathbb{D})$. Consider a counterclockwise space-filling $\text{SLE}_{\kappa'} \eta : [0, \mu_\psi^\gamma(\mathbb{D})] \rightarrow \overline{\mathbb{D}}$ from $-i$ to $-i$, independent of ψ , but parametrised by quantum area. Denote by L_t and R_t the change in quantum boundary lengths of the left and right sides of $\eta([0, t])$ relative to time 0 as in Figure 5.1, normalised so that $(L_0, R_0) = (0, 1)$. Then $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ is a correlated Brownian motion conditioned to remain in the positive quadrant $\mathbb{R}_+ \times \mathbb{R}_+$, started from $(0, 1)$ on the boundary, and run until it hits the origin. The covariance structure of $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ is given by*

$$\text{Var}(L_t) = \text{Var}(R_t) = \mathfrak{a}^2 t, \quad \text{Cov}(L_t, R_t) = -\cos\left(\frac{4\pi}{\kappa'}\right) \mathfrak{a}^2 t, \quad (5.1.4)$$

where $\mathfrak{a}$ is a constant depending on γ .

Remark that the Brownian conditioning in Theorem 5.1.1 is singular; we will make its meaning more precise later on in Section 5.3.1. Besides, the constant a appearing in (5.1.4) has been identified recently in [ARS21].

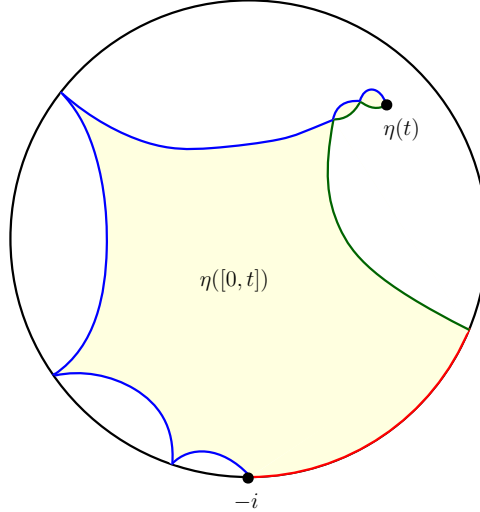


Figure 5.1 – A unit boundary $\sqrt{8/3}$ -quantum disc $(\mathbb{D}, \psi, -i)$ decorated with an independent counterclockwise space-filling SLE_6 η from $-i$ to $-i$, parametrised by quantum area. The left quantum boundary length L_t corresponds to the ν_ψ -length of the blue curve. The right quantum boundary length R_t corresponds to the green ν_ψ -length *minus* the red one.

CLE explorations of a quantum disc and growth-fragmentation processes. The point of view of Miller, Sheffield and Werner [MSW20] will be very relevant to the present work. As such, we review their results, which reveal some growth-fragmentation processes in the context of CLE explorations of the quantum disc. We fix $\kappa := \gamma^2$ and $\kappa' := 16/\gamma^2$, and assume that $\gamma \in (\sqrt{8/3}, 2)$.

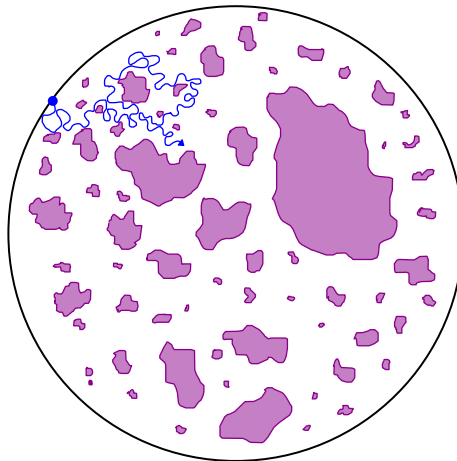


Figure 5.2 – Drawing of the CPL on the CLE_κ carpet in the γ -quantum disc. The CLE_κ loops are shown in purple, and the CPL in blue (the target point is implicitly fixed).

First, we consider on top of the quantum disc an independent *conformal loop ensemble* CLE_κ [She09, SW16], which is a random collection of non-crossing loops in the disc. The

other key player is the *conformal percolation interface* (CPI) in the CLE_κ carpet between boundary points x and y [MSW17], as represented in Figure 5.2. Roughly speaking, this is an SLE type curve which is the only curve staying in the CLE_κ carpet, enjoying conformal invariance and locality properties, such that anytime it hits a CLE_κ loop, it leaves it to its right. It is also possible to make sense simultaneously of all the CPI branches towards any point in the disc.

The paper [MSW20] constructs through these CPI branches a growth-fragmentation process, which is the continuum analogue of the one arising from *peeling* explorations of Boltzmann planar maps [BBCK18]. Consider some CPI branch (parametrised by quantum length) towards some point z in the CLE_κ carpet. The branch in the growth-fragmentation towards z corresponds to recording the quantum boundary length of the connected component containing z as the CPI evolves. To see how this growth-fragmentation process comes into play, it is important to understand when jumps occur for the boundary length of the domain containing z . There are two types of jumps, corresponding to the following events (see Figure 5.3 and 5.4):

- (i) The CPI hits a new CLE_κ loop. Denote by ℓ the quantum boundary length of the loop. Then at this time the quantum boundary length of the domain containing z goes from L to $L + \ell$. Such times are therefore associated to *positive jumps* for the quantum boundary length of the component containing z .

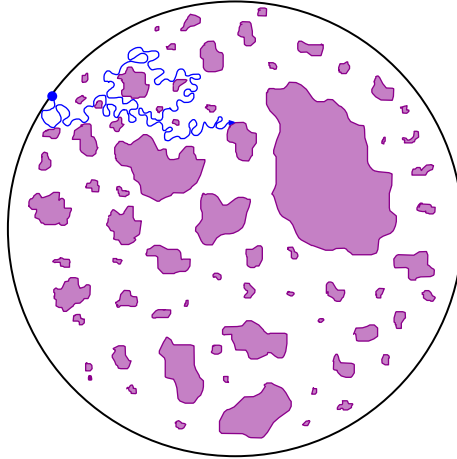


Figure 5.3 – The CPI discovers a new CLE_κ loop (event (i)). The to-be-explored domain swallows the loop at once and its total boundary length has a positive jump.

- (ii) The CPI splits the remaining-to-be-explored domain into two smaller subdomains. In this case, the quantum boundary length of the domain containing z drops from L to $L - \ell$, where ℓ is the quantum boundary length of the other subdomain. Hence these times correspond to *negative jumps* for the quantum boundary length of the component containing z .

Moreover, if x and y are any two points in the CLE_κ carpet, the branches targeting x and y respectively will coincide up to some time when they will get disconnected by the CPI.

The main result of [MSW20] (Theorem 1.1) describes this branching structure in terms of the growth-fragmentation processes introduced in [BBCK18]. Let $t \geq 0$. Consider all the above CPI branches up to time t which did not vanish yet, and record the total quantum boundary lengths of the remaining-to-be explored domains. This gives a countable family

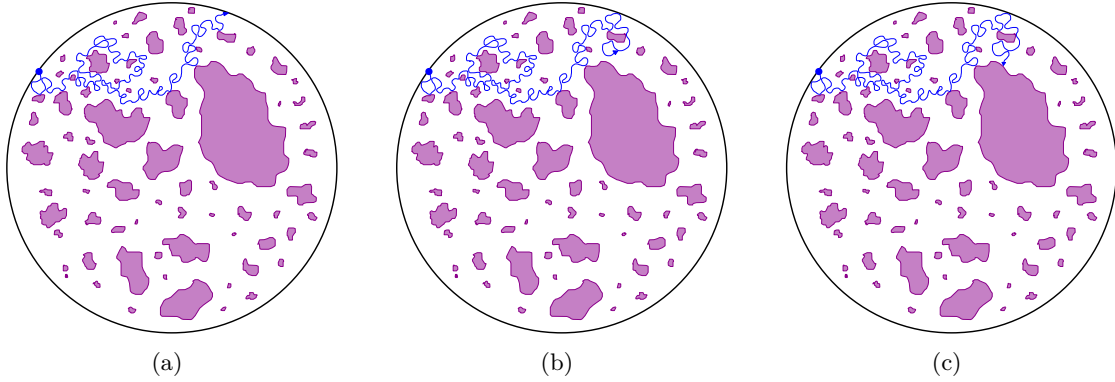


Figure 5.4 – Different situations when event (ii) may occur: (a) the CPI hits the boundary of the disc (b) the CPI hits itself (c) the CPI hits a previously visited CLE_κ loop.

$\mathbf{Z}(t)$ of positive real numbers (where the dependence on γ is implicit), which can be given the following description. Fix $\theta := \frac{\kappa'}{4}$, and define the measure ν_θ on $(\frac{1}{2}, \infty)$ by

$$\nu_\theta(dx) := \frac{\Gamma(\theta + 1)}{\pi} \left(\frac{1}{x^{\theta+1}(1-x)^{\theta+1}} \mathbf{1}_{1/2 < x < 1} + \sin(\pi(\theta - 1/2)) \cdot \frac{1}{x^{\theta+1}(x-1)^{\theta+1}} \mathbf{1}_{x > 1} \right) dx.$$

Let Λ_θ be the image measure of ν_θ through $x \mapsto \log x$, and define the Laplace exponent

$$\Psi(q) := d_\theta q + \int_{\mathbb{R}} (e^{qy} - 1 - q(e^y - 1)) \Lambda_\theta(dy), \quad \theta < q < 2\theta + 1, \quad (5.1.5)$$

where d_θ is some explicit drift that we do not specify, but which can be found in [BBCK18, Section 5.2]. Take ξ^θ to be a Lévy process with Laplace exponent (5.1.5), and consider under P_x the positive self-similar Markov process X^θ with index θ started at $x > 0$, whose Lamperti exponent is ξ^θ , namely

$$X^\theta(t) := x \exp(\xi^\theta(\tau(x^{-\theta}t))), \quad 0 \leq t < \zeta, \quad (5.1.6)$$

where

$$\tau(t) := \inf\{s \geq 0, \int_0^s e^{\theta \xi^\theta(u)} du > t\}, \quad t \geq 0,$$

and $\zeta := \inf\{t > 0, X^\theta(t) = 0\}$. Observe that the measure ν_θ is supported on $(\frac{1}{2}, \infty)$, which after Lamperti transformation means that X^θ never more than halves itself during a jump. This is known to be a canonical choice of particle from the growth-fragmentation perspective, called the *locally largest evolution* in [BBCK18]. The *growth-fragmentation process* driven by X^θ can be roughly constructed as follows. At time $t = 0$, the system starts from one particle X^θ with initial size $x > 0$, which then evolves according to P_x . Conditionally on X^θ , one starts a new particle at times t when X^θ has a negative jump, which starts from $y = -\Delta X^\theta(t) := X^\theta(t^-) - X^\theta(t)$ and whose behaviour is governed by P_y . This constructs the children of X^θ , for which we repeat the same procedure, thus creating the second generation, and so on. For $t \geq 0$, we let $\mathbf{X}^\theta(t)$ denote the collection of sizes of the cells alive at time t . Then the main result of [MSW20] states that $\mathbf{Z}$ and $\mathbf{X}^\theta$ have the same law (actually up to a multiplicative constant in (5.1.5)).

This is expected since the processes $\mathbf{X}^\theta$ introduced by Bertoin, Curien, Budd and Kortchemski [BBCK18] are obtained in the scaling limit of a discrete version of the exploration presented above, defined through a *peeling* procedure [Bud15]. To conclude this section, we mention that the analysis conducted in [BBCK18] paves the way for remarkable martingales,

which have been given a geometric flavour (either in terms of random planar maps or of CLE on LQG). This in particular enabled [MSW20] to define the natural LQG measure living on the CLE carpet corresponding to the *intrinsic area measure* obtained in [BBCK18] as the limit of a natural martingale.

5.1.2 Main results

The present work deals with the boundary case $\gamma = \sqrt{8/3}$ in the approach of [MSW20]. As before, we set $\kappa = \gamma^2 = 8/3$ and $\kappa' = 16/\kappa = 6$. Note that in this case, $\text{CLE}_{8/3}$ is almost surely empty [She09]. Hence the natural analogous framework is the following. Consider a $\sqrt{8/3}$ -quantum disc $(\mathbb{D}, \psi, -i)$ marked with one boundary point. Draw an independent counterclockwise space-filling $\text{SLE}_{\kappa'} \eta$ in the quantum disc. One can then consider the *branch* of η towards any point z in the disc, in the sense that it does not explore the components that it detaches along its way to z , and reparametrise it accordingly by *quantum natural time* (see Section 5.2.2). This defines the branches towards all the points in the disc, such that if one considers any two points, the branches coincide until η disconnects them (Figure 5.8). Now, for $t \geq 0$, consider all the branches alive at time t , and record the total boundary length of the remaining-to-be-explored domains: this yields a countable collection of positive real numbers, that we denote by $\mathbf{Z}(t)$.

Our goal is to describe the branching structure of $\mathbf{Z}$ as in [MSW20]. Let ν be the measure on $(\frac{1}{2}, 1)$ defined by

$$\nu(dx) = \frac{3}{4\sqrt{\pi}} \cdot \frac{dx}{x^{5/2}(1-x)^{5/2}} \mathbb{1}_{1/2 < x < 1},$$

and Λ be the image measure of ν through the mapping $x \mapsto \log(x)$. Introduce the Lévy process ξ with Laplace exponent

$$\Psi(q) := -\frac{2}{\sqrt{\pi}}cq + c \int_{-\log(2)}^0 (e^{qy} - 1 - q(e^y - 1))\Lambda(dy), \quad q > \frac{3}{2}, \quad (5.1.7)$$

where c is some (non-explicit) constant appearing in the chordal exploration of the disc (see the discussion following Theorem 5.2.6). This corresponds to taking $\theta \rightarrow \frac{3}{2}$ in (5.1.5). Construct the positive self-similar Markov process X with index $\frac{3}{2}$ whose Lamperti exponent is given by ξ , as in (5.1.6), which again will consist in following the locally largest fragment in the growth-fragmentation. Our main theorem goes as follows.

Theorem A. *The process $\mathbf{Z}$ has the law of the growth-fragmentation process $\mathbf{X}$ driven by X .*

At this point, one should compare Theorem A with the results of [LGR20]. Indeed, the growth-fragmentation $\mathbf{X}$ already appears (up to a time-change) in the description of the Brownian disc sliced at heights. Since the Brownian disc is obtained as the scaling limit of rescaled Boltzmann quadrangulations when taking the boundary size to infinity [Bet15, BM17, GM19], the results of [LGR20] constitute a natural continuum analogue to the geometric construction involved in [BBCK18, Section 6.5]. On the other hand, certain $\sqrt{8/3}$ -surfaces when endowed with the appropriate metric structure are known to be equivalent to *Brownian surfaces* [MS20, MS16, MS21]. The exploration that we here consider is closer in spirit to [BBCK18, Section 6.4], and indeed one can see Theorem A as a construction of the growth-fragmentation process appearing in the limit in [BCK18] (for $\theta = \frac{3}{2}$) *via* Liouville quantum gravity.

The first purpose of the present work is to derive Theorem A using arguments from Liouville quantum gravity theory. The structure of the proof follows the same lines as in [MSW20]. The core idea is to relate the branch of the space-filling exploration process to

a chordal exploration mechanism in the quantum disc. We make use of the description of the latter in [GM18] to describe the jumps of the left and right quantum boundary lengths (L, R) targeting a boundary point, and then transfer the results to the total boundary length in the *locally largest* exploration.

Our second motivation is different in nature, but closely linked to Theorem A. In fact, the mating-of-trees theorem (recall Theorem 5.1.1) relates space-filling SLE explorations of the quantum disc to a Brownian excursion in the upper right quadrant. Making use of this connection, we pave the way for a proof of Theorem A relying only excursion theory. The proof is complete up to a conjecture stating that the spectrally positive $3/2$ -stable process conditioned to remain positive can be constructed in a nice way from Brownian motion using cone excursions with angle $\frac{2\pi}{3}$. It is well-known that planar Brownian motion makes excursions in any cone with apex angle $\theta > \frac{\pi}{2}$ [Shi85, Bur85].

First, remark that one can apply a linear transformation to the correlated Brownian motion pair in Theorem 5.1.1 and obtain standard Brownian motion in some cone with opening angle $\theta := \frac{\pi\gamma^2}{4}$. More precisely, the matrix

$$\Lambda := \frac{1}{a} \begin{pmatrix} \frac{1}{\sin \theta} & \frac{1}{\tan \theta} \\ 0 & 1 \end{pmatrix}, \quad (5.1.8)$$

sends a pair $\widetilde{W}$ of correlated Brownian motions with covariance structure (5.1.4) onto standard planar Brownian motion W , and maps the quadrant $\mathbb{R}_+^2$ onto the closure of the cone $\mathcal{C}_\theta := \Lambda(\mathbb{R}_+^2) = \{z \in \mathbb{C}, \arg(z) \in (0, \theta)\}$ with apex angle θ . One can therefore see Theorem 5.1.1 as a statement for a boundary-to-apex Brownian cone excursion in $\mathcal{C}_\theta$, with no correlation (see [AG21, Section 4.2]). As $\theta = \frac{2\pi}{3}$ when $\gamma = \sqrt{8/3}$, this provides evidence that Theorem A ought to have a natural translation in terms of a Brownian excursion in a cone with angle $\frac{2\pi}{3}$.

This naturally leads us to discuss Brownian motion in cones. Such excursions have already been studied in a number of remarkable papers in relation to Brownian windings, see [Shi85, Bur85, Eva85, LG87]. The point of view of Le Gall [LG87] turns out to be very relevant to the present work. We review some of the results which are of particular interest to us – they may all be found in [LG87] up to a rotation. Consider a planar Brownian motion $(W_t, t \geq 0)$ started from the origin, and let $\theta \in [0, 2\pi]$ and $\lambda := \frac{\pi}{\theta}$. Call *backward cone time* a time t such that, for all $s \in [0, t]$, $W_s \in W_t + \mathcal{C}_\theta$, i.e. backward cone times correspond to times t for which the cone with origin W_t and angle θ contains the whole past trajectory. Results of Shimura and Burdzy [Shi85, Bur85] state that the set H_θ of backward cone times almost surely contains non-zero times if, and only if, $\theta > \frac{\pi}{2}$. Le Gall constructs a local time $(\mathfrak{l}_\theta(s), s \geq 0)$ on the set H_θ , and determines its Hausdorff dimension in terms of θ . Moreover, if we introduce the inverse local time

$$\mathfrak{t}_\theta(t) := \inf\{s \geq 0, \mathfrak{l}_\theta(s) > t\},$$

then the process $(W(\mathfrak{t}_\theta(t)), t \geq 0)$ is a two-dimensional stable Lévy process with index $2 - \lambda$, and $(\mathfrak{t}_\theta(t), t \geq 0)$ is a stable subordinator with index $1 - \frac{\lambda}{2}$. In particular, this extends the classical construction of Spitzer of the Cauchy process [Spi58], which corresponds to $\theta = \pi$.

But one could also be interested in other types of cone excursions. The other type of cone excursions which is relevant to our work already appears in the LQG considerations of Duplantier, Miller and Sheffield [DMS14, Section 1.4.2], although their construction stems from independent works of Burdzy and Shimura [Bur85, Shi85]. In this respect, we call *forward cone time* of the Brownian motion W , any time $t \geq 0$ for which there exists $\varepsilon > 0$ such that, for all $s \in (t, t + \varepsilon)$, $W_s \in W_t + \mathcal{C}_\theta$. We then define a *cone-free time* (or *ancestor-free time* in the parlance of [DMS14]) as a time contained in none of the forward

cone excursions: namely, t is cone-free if there exists no forward cone time $t' \in (0, t)$ such that $W_s \in W_{t'} + \mathcal{C}_\theta$ for all $s \in (t', t]$. These points have a regenerative property, so that one may define a local time $(\ell_\theta(t), t \geq 0)$ supported on the set of cone-free times, as in [DMS14]. Finally, introduce the cone-free inverse local time

$$\tau_\theta(t) := \inf\{s \geq 0, \ell_\theta(s) > t\}. \quad (5.1.9)$$

Duplantier, Miller and Sheffield [DMS14, Proposition 1.13] also expressed the law of the time-changed Brownian motion $(W(\tau_\theta(t)), t \geq 0)$ in terms of a pair of independent spectrally positive λ -stable Lévy processes.

Theorem A should be a consequence of some decomposition of the cone excursion appearing in the mating-of-trees (Theorem 5.1.1). More precisely, we will be interested in Brownian *excursions* in the cone, in the sense of Itô [Itô72]. In Section 5.3.1, we show that the backward cone excursions, labelled by the inverse local time t_θ , form a Poisson point process of cone excursions. We define the *backward cone excursion measure* $\mathbf{n}$ as the intensity of the Poisson point process, and obtain a Bismut-type of description for $\mathbf{n}$. This enables us to relate the exploration targeting a uniformly chosen point on a size-biased cone excursion to the locally largest evolution, and eventually to infer its law, up to a conjecture. We stress that the Bismut description proved in this paper holds for any $\theta > \frac{\pi}{2}$, and we believe that it could be of independent interest in this more general setting.

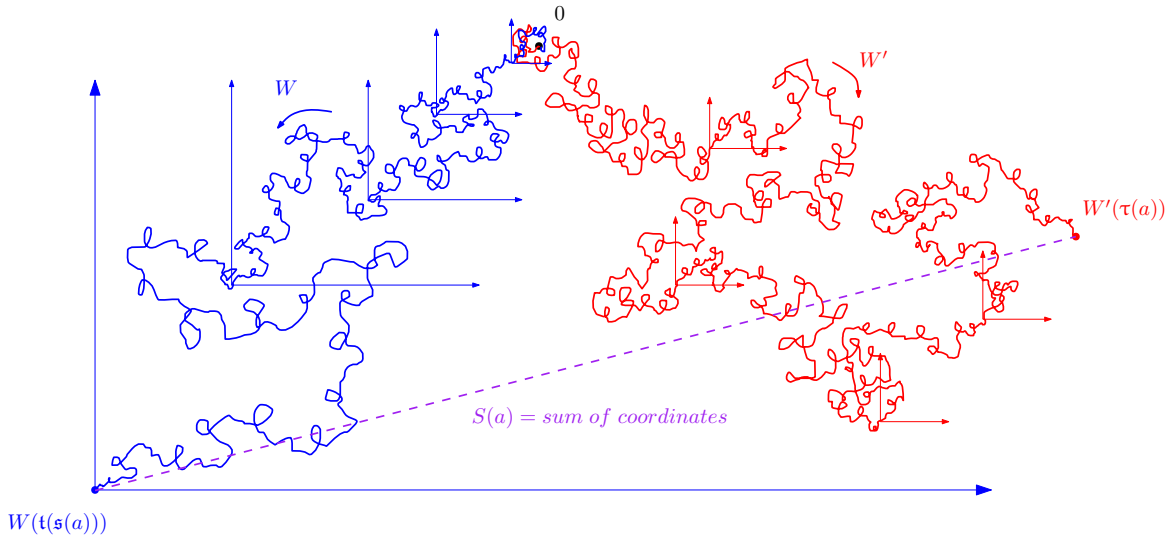


Figure 5.5 – The spectrally positive $\frac{3}{2}$ -stable Lévy process conditioned to stay positive embedded in Brownian motion. The picture represents a pair of correlated Brownian motion in the positive quadrant, with correlation given by (5.1.4). One can map this picture back onto planar Brownian motion conditioned to stay in a cone using the transformation Λ in (5.1.8).

Our conjecture plays with the two notions of cone excursions when $\theta = \frac{2\pi}{3}$ (i.e. $\gamma = \sqrt{8/3}$), and outputs a spectrally positive λ -stable Lévy process conditioned to remain positive. We drop the subscript θ as θ is fixed in the theorem. Notice that $\lambda = \frac{3}{2}$ in this case.

Conjecture B. Let W and W' be two independent planar Brownian motions starting from the origin. For $a \geq 0$, consider the $\frac{3}{2}$ -stable process $(W'(\tau(t)), t \leq a)$ corresponding to the forward $\frac{2\pi}{3}$ -cone times up to time a , where τ is the inverse local time attached to W' as in

(5.1.9). *Introduce the first passage time*

$$\mathfrak{s}(a) := \inf\{s \geq 0, W'(\tau(t)) \in W(\mathfrak{t}(s)) + \mathcal{C}_\theta \text{ for all } t \leq a\},$$

of the $\frac{1}{2}$ -stable process $(W(\mathfrak{t}(t)), t \geq 0)$ below the path $(W'(\tau(t)), t \leq a)$. Let $S(a)$ be the sum of the coordinates of $W' \circ \tau(a) - W \circ \mathfrak{t}(\mathfrak{s}(a))$. Then S evolves as a spectrally positive $\frac{3}{2}$ -stable Lévy process conditioned to remain positive.

See Figure 5.5 for an illustration. Note that Conjecture B can also be expressed solely in terms of stable processes without reference to Brownian motion. Moreover, the spectrally positive $\frac{3}{2}$ -stable Lévy process conditioned to remain positive (or its dual) is known to be closely related to the growth-fragmentation $\mathbf{X}$ above, and more precisely to its spine decomposition, see [BBCK18].

5.2 The growth-fragmentation embedded in space-filling explorations of the $\sqrt{8/3}$ -quantum disc

The goal of this section is to prove our main theorem (Theorem A). We first recall some of the basic features of SLE₆ on the $\sqrt{8/3}$ -quantum disc. As in [MSW20], we first relate the law of a branch of the space-filling SLE to a chordal exploration of the quantum disc [GM18], and then prove that this enables one to describe the growth-fragmentation process.

5.2.1 Background on the GFF and quantum surfaces

We recall for completeness in this section the definitions of the main objects of concern in this work: an instance of the Gaussian free field, quantum discs, and several versions of SLE.

Neumann Gaussian free field. Let $D \in \mathbb{C}$ any proper, simply connected domain of the complex plane. Denote by $\overline{\mathcal{D}}(D)$ the space of $\mathcal{C}^\infty$ functions in $\overline{D}$, with finite Dirichlet energy

$$(f, f)_\nabla := \frac{1}{2\pi} \int_D \nabla f \cdot \nabla f < \infty,$$

and which are defined *modulo constants*. That is, we identify two such functions if their difference is constant. The operation $(\cdot, \cdot)_\nabla$ defines an inner product on $\overline{\mathcal{D}}(D)$, and we consider the closure $\overline{H}^1(D)$ of $\overline{\mathcal{D}}(D)$ with respect to this inner product. A *distribution modulo constants* is a continuous linear functional on the space of smooth compactly supported test functions f on D such that $\int_D f = 0$. The space of distributions modulo constants is then endowed with the natural weak- $\star$ topology.

Definition 5.2.1. (*Neumann GFF on D*).

Let $(\varphi_i)_{i \geq 1}$ be an orthonormal basis of $\overline{H}^1(D)$, and $(X_i)_{i \geq 1}$ be independent $\mathcal{N}(0, 1)$ -distributed random variables. Then, the random sum

$$h_n := \sum_{i=1}^n X_i \varphi_i, \quad n \geq 1,$$

converges almost surely in the space of distributions modulo constants. Moreover, the law of the limit is independent of the choice of orthonormal basis. We call this limit the Neumann GFF.

It is well-known (see for example the lectures notes [BP21]) that the Neumann GFF enjoys a conformal invariance property, and a nice domain Markov property. On the other hand, we stress that a Neumann GFF is only defined up to a constant. When working with a Neumann GFF, one may want to fix the additive constant using various normalisation procedures. For the Neumann GFF on the strip $\mathcal{S} := \mathbb{R} \times i(0, \pi)$ that we will consider, one canonical choice is to require that its average on $[0, i\pi]$ is zero.

Suppose that the additive constant of a Neumann GFF $\tilde{h}$ has been fixed in some way, and consider $h = \tilde{h} + g$, where g is a random continuous function on D . Then one can define, for $\gamma \in (0, 2)$, the γ -LQG area measure by

$$\mu_h^\gamma(dz) := \lim_{\varepsilon \rightarrow 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz,$$

along a dyadic subsequence, where $h_\varepsilon(z)$ denotes the average of the field h on the circle with radius ε centered at z [DS11]. Likewise, one can define the γ -LQG boundary length measure ν_h^γ of a segment of ∂D where g extends continuously. These constructions can be seen as instances of so-called *Gaussian multiplicative chaos* [Kah85], where one tries to construct measures defined as the exponential of a log-correlated Gaussian field.

Quantum discs. Recall from Section 5.1.1 the definition of quantum surfaces, and more precisely of *marked* quantum surfaces. These are an *equivalent class* of surfaces parametrised by some domain, equipped with a metric given by the GFF h , and with a certain number of marked points. In these equivalent classes, we identify two surfaces satisfying the change of coordinates formula for some conformal mapping preserving the marked points. The present work deals with specific quantum surfaces known as *quantum discs* [DMS14]. There are several ways to define such quantum surfaces: we will here stick to the definition in terms of Bessel excursions. We refer to [MS19, DMS14] for other constructions, such as a limiting procedure, or the characterisation of the bubbles cut out by an $\text{SLE}_{\kappa'}$, or the construction arising from conformal field theory [HRV18] (the equivalence of which was proved in [Cer19]).

We first construct the infinite measure on doubly marked quantum discs. The strip $\mathcal{S} = \mathbb{R} \times (0, \pi)$ turns out to be convenient as a parametrisation (for other domains, one should apply the coordinate change formula (5.1.2)). We set $\delta := 3 - \frac{4}{\gamma^2}$, and introduce the Hilbert spaces $\mathcal{H}^0(\mathcal{S})$ and $\mathcal{H}^\dagger(\mathcal{S})$ of mean-zero functions on $\mathcal{S}$ with finite Dirichlet energy which are constant or have mean zero on each vertical line segment of $\mathcal{S}$ respectively (not viewed modulo constants). By [DMS14, Lemma 4.3], the whole space of mean-zero functions on $\mathcal{S}$ with finite Dirichlet energy decomposes over the orthogonal sum of $\mathcal{H}^0(\mathcal{S})$ and $\mathcal{H}^\dagger(\mathcal{S})$.

Definition 5.2.2. (*Infinite quantum disc measure*).

For $\gamma \in (0, 2)$, the infinite measure $\mathcal{M}_{\text{disc}}^\gamma$ on doubly marked quantum discs is the “law” of the surface $(\mathcal{S}, h, -\infty, \infty)$, where $h = h_0 + h^\dagger \in \mathcal{H}^0(\mathcal{S}) \oplus \mathcal{H}^\dagger(\mathcal{S})$ is defined as follows:

- Let e be a Bessel excursion of dimension δ . Define h^0 as $2\gamma^{-1} \log e$ reparametrised to have quadratic variation $2dx$.
- Take $h^\dagger$ to be the projection onto $\mathcal{H}^\dagger(\mathcal{S})$ of an independent Neumann GFF in $\mathcal{S}$.

The fact that the Bessel excursion measure gives finite mass to excursions with duration at least $t > 0$, say, entails that $\mathcal{M}_{\text{disc}}^\gamma$ gives finite mass to surfaces with quantum area and/or quantum boundary length larger than $a > 0$ and/or $\ell > 0$ respectively. The law of the (doubly marked) *quantum disc with boundary length ℓ* is then defined as $\mathcal{M}_{\text{disc}}^\gamma$ conditioned on $\nu_h^\gamma(\partial \mathcal{S}) = \ell$. Similarly, one can define the law of the *quantum disc with left and right*

boundary lengths (ℓ_L, ℓ_R) as $\mathcal{M}_{\text{disc}}^\gamma$ conditioned on $\nu_h^\gamma(\mathbb{R} \times \{\pi\}) = \ell_L$ and $\nu_h^\gamma(\mathbb{R} \times \{0\}) = \ell_R$. Results on Bessel excursions provide the disintegration

$$\mathcal{M}_{\text{disc}}^\gamma(\cdot) = c \int_{\mathbb{R}_+} \mathcal{M}_{\text{disc}}^\gamma(\cdot \mid \nu_h^\gamma(\partial\mathcal{S}) = \ell) \ell^{-4/\gamma^2} d\ell,$$

for some constant $c > 0$. Finally, one may obtain *singly marked* quantum discs by forgetting one marked point, see [DMS14, Section 4.5].

5.2.2 Space-filling SLE_6 explorations of the $\sqrt{8/3}$ -quantum disc and the mating-of-trees

Next, we introduce some variants of SLE on quantum discs. We will be interested in two types of SLE explorations: the *space-filling* one, for which we review the so-called *peanosphere construction* of [DMS14], and the *chordal* one, which was made explicit in [GM18] for $\gamma = \sqrt{8/3}$. The two results will be used later on to prove Theorem A and to relate it to the cone excursion setting. Recall that we have set

$$\gamma = \sqrt{\frac{8}{3}}, \quad \kappa = \gamma^2, \quad \kappa' = \frac{16}{\kappa}.$$

Space-filling $\text{SLE}_{\kappa'}$. When $\kappa' > 4$, the paper [MS17] introduces a variant of $\text{SLE}_{\kappa'}$ [Sch00] which is space-filling. The construction in the case $\kappa' \geq 8$ is easy, since $\text{SLE}_{\kappa'}$ is in itself already space-filling. However, it is relevant when $\kappa' \in (4, 8)$, which corresponds to our setting. Indeed, in this case ordinary $\text{SLE}_{\kappa'}$ bounces off itself and the boundary, creating bubbles that will never be visited. The *space-filling* variant can roughly be obtained by iteratively filling in these bubbles with space-filling loops, see [MS17, GHS19].

Following [AG21] and the mating-of-trees theorem presented in the introduction (Theorem 5.1.1), we will use a particular instance of space-filling $\text{SLE}_{\kappa'}$ called a *counterclockwise space-filling loop*. The setting is as follows. Consider a simply connected domain D , on which one draws a space-filling $\text{SLE}_{\kappa'}$ curve η from $x \in \partial D$ to $y \in \partial D$. Then the *counterclockwise space-filling $\text{SLE}_{\kappa'}$* from x to x is defined as the limit of η when $y \rightarrow x$ in the counterclockwise direction (see [BG20]). For this process, a point z on the boundary is almost surely visited once, although some exceptional points are visited twice. An important fact is that counterclockwise space-filling $\text{SLE}_{\kappa'}$ from x to x will visit the former type of points in the counterclockwise order starting from x .

The *mating-of-trees* theorem for the γ -quantum disc gives the law of the left/right boundary lengths in the counterclockwise space-filling $\text{SLE}_{\kappa'}$ exploration of the quantum disc, as follows.

Theorem 5.2.3. *Let $\gamma \in (0, 2)$ and $(\mathbb{D}, \psi, -i)$ be a unit boundary marked quantum disc with random quantum area $\mu_\psi^\gamma(\mathbb{D})$. Consider a counterclockwise space-filling $\text{SLE}_{\kappa'}$ $\eta : [0, \mu_\psi^\gamma(\mathbb{D})] \rightarrow \overline{\mathbb{D}}$ from $-i$ to $-i$, independent of ψ , but parametrised by quantum area. Denote by L_t and R_t the change in quantum boundary lengths of the left and right sides of $\eta([0, t])$ relative to time 0 as in Figure 5.1, normalised so that $(L_0, R_0) = (0, 1)$. Then $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ is a correlated Brownian motion conditioned to remain in the positive quadrant $\mathbb{R}_+ \times \mathbb{R}_+$, started from $(0, 1)$ on the boundary, and run until it hits the origin. The covariance structure of $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ is given by*

$$\text{Var}(L_t) = \text{Var}(R_t) = \mathfrak{a}^2 t, \quad \text{Cov}(L_t, R_t) = -\cos\left(\frac{4\pi}{\kappa'}\right) \mathfrak{a}^2 t, \quad (5.2.1)$$

where $\mathfrak{a}$ is a constant depending on γ . Furthermore, the pair $(L_t, R_t)_{t \in [0, \mu_\psi^\gamma(\mathbb{D})]}$ almost surely determines $(\mathbb{D}, \psi, \eta, -i)$ modulo conformal change of coordinates (5.1.2).

Mating-of-trees theory provides a bridge between SLE on LQG and instances of (correlated) Brownian motion. See also [DMS14] and [AG21] for other types of quantum surfaces. In the case of Theorem 5.2.3, the instance of Brownian motion involved is a correlated Brownian excursion in the positive quadrant. This tells us that some quantities in the quantum disc setting can be investigated using their Brownian counterparts in the cone. We conclude this paragraph with an application taken from [AG21] of this mating-of-trees dictionary: the description of the law of the quantum area of a unit boundary quantum disc. Indeed, the parametrisation of Theorem 5.2.3 is such that the area of the unit boundary quantum disc straightforwardly corresponds to the duration of the cone excursion. By performing the calculations at the Brownian level, Ang and Gwynne determined the following law for the quantum area of a unit boundary γ -quantum disc [AG21].

Theorem 5.2.4. *The law of the area of the unit boundary length quantum disc is*

$$\mathbb{P}(\mu_\psi^\gamma(\mathbb{D}) \in dt) = \exp\left(-\frac{1}{2(\mathfrak{a} \sin(\pi\gamma^2/4))^2 t}\right) \frac{dt}{ct^{1+4/\gamma^2}},$$

where

$$c = 2^{4/\gamma^2} \Gamma(4/\gamma^2) (\mathfrak{a} \sin(\pi\gamma^2/4))^{8/\gamma^2}.$$

Chordal SLE₆ explorations of the $\sqrt{8/3}$ -quantum disc. We now turn to the description of chordal SLE₆ explorations of the $\sqrt{8/3}$ -quantum disc. For the rest of this section, we thus take

$$\gamma = \sqrt{8/3} \quad \text{and} \quad \kappa' = 16/\gamma^2 = 6.$$

Consider a doubly marked $\sqrt{8/3}$ -quantum disc $(\mathbb{D}, h, -i, i)$ with left/right boundary lengths $\ell_L, \ell_R > 0$. Let η an independent chordal SLE₆ in $\mathbb{D}$, from $-i$ to i . For $u \geq 0$, let L_u and R_u denote the left and right boundary lengths of the connected component of $\mathbb{D} \setminus \eta([0, u])$ containing i . The processes L and R are respectively started from ℓ_L and ℓ_R , are both càdlàg, and have only downward jumps, which occur when η disconnects a bubble from i (either on its left or on its right respectively). The boundary length of the bubble cut out by η is obtained as the opposite of this jump size (Figure 5.6).

We will mainly be interested in the trajectory of η targeted at i , that is without visiting the cut-out bubbles. This means that from now on, we parametrise η by so-called *quantum natural time*, which roughly corresponds to parametrise it by the local time on the set of points where η disconnects bubbles from the target point i . A more rigorous definition may be found in [DMS14, Section 6.5.3]. We write ς for the total quantum natural time of η , so that with our parametrisation η is defined on $[0, \varsigma]$. In addition, it will be convenient to extend η by setting $\eta(u) = i$ for $u > \varsigma$. The following theorem, proved in [GM18], describes the law of the remaining-to-be-explored domain and of the cut-out surfaces in the chordal exploration. Let $\mathcal{W}_u$ the remaining-to-be-explored domain at time $u \geq 0$, i.e. $\mathcal{W}_u$ is the doubly marked quantum surface obtained by restricting the field h to the connected component of $\mathbb{D} \setminus \eta([0, u])$ containing i , with mark points $\eta(u)$ and i .

Theorem 5.2.5. *Conditionally on $(L_s, R_s, s \leq u)$, $\mathcal{W}_u$ has the law of a doubly marked quantum disc with left/right boundary lengths (L_u, R_u) . Moreover, the conditional law of the surfaces obtained by restricting h to the bubbles cut out by η up to time u , each marked with the point where η finishes tracing its boundary, is that of a collection of independent singly marked quantum discs, with respective boundary lengths given by the jumps of $(L_s, R_s, s \leq u)$.*

In the first assertion of Theorem 5.2.5, we implicitly take the convention that a quantum disc with boundary length 0 (when $u > S$) is a single point. The other main result of [GM18] is the following theorem, determining the law of the left/right boundary lengths $(L_s, R_s, s \leq u)$ in terms of a Radon-Nikodym derivative.

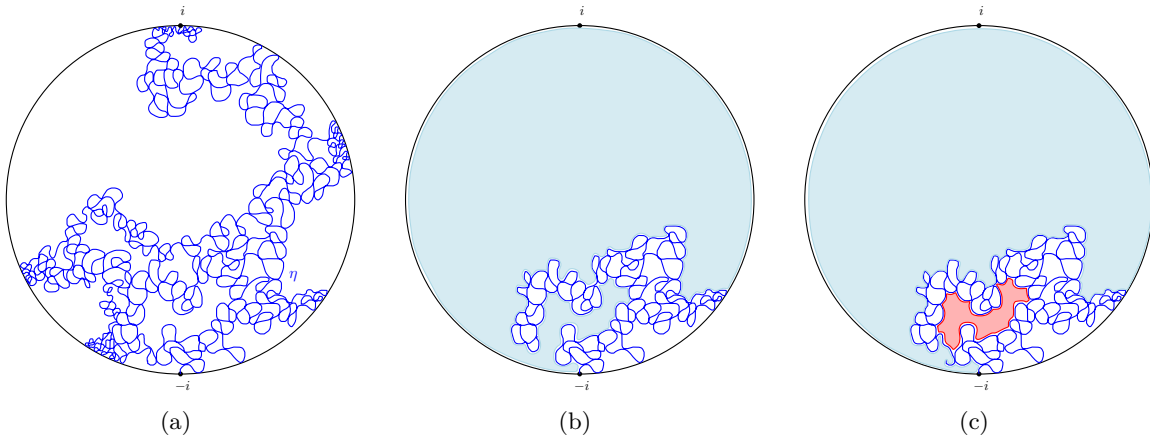


Figure 5.6 – Disconnecting a bubble from i : (a) A chordal SLE_6 η on a $\sqrt{8/3}$ -quantum disc from $-i$ to i . (b) Before disconnection: the domain containing i is in shaded blue, and its total boundary length is the ν_h -length of its boundary. (c) After disconnection: the shaded red bubble has been cut out. The total quantum boundary of the bubble corresponds to the jump of the total boundary length of the blue region (in this case, the jump occurs for the *left* boundary length).

Theorem 5.2.6. (*Law of the left/right boundary lengths in the chordal SLE_6 exploration*).

- (i) *Almost surely, ς is finite and can simultaneously be written as*

$$\varsigma = \inf\{u \geq 0, L_u = 0\} = \inf\{u \geq 0, R_u = 0\}.$$

Furthermore, $\lim_{u \rightarrow \varsigma} \eta_u = i$ and $\lim_{u \rightarrow \varsigma} (L_u, R_u) = (0, 0)$.

- (ii) *Let (L^∞, R^∞) be a pair of independent spectrally negative $3/2$ -stable Lévy processes starting from (ℓ_L, ℓ_R) . Define*

$$\varsigma^\infty := \inf\{u \geq 0, L_u^\infty \leq 0 \text{ or } R_u^\infty \leq 0\}.$$

Then, on the event $u < \varsigma$ the law of $(L_s, R_s, s < u)$ is absolutely continuous with respect to the one of $(L_s^\infty, R_s^\infty, s \leq u)$, with Radon-Nikodym derivative

$$\left(\frac{L_u^\infty + R_u^\infty}{\ell_L + \ell_R} \right)^{-5/2} \cdot \mathbf{1}_{u < \varsigma^\infty}.$$

The Lévy measures of L^∞ and R^∞ are of the form $c|y|^{-5/2} \mathbf{1}_{y < 0} dy$ for some constant $c > 0$ which is not explicit. We conclude this paragraph by emphasizing that the pair (L^∞, R^∞) corresponds to the left/right boundary length process in the chordal exploration of the $\sqrt{8/3}$ -quantum wedge, which is another type of quantum surface that can be thought of as the infinite volume and boundary length limit of the quantum disc when zooming in at a fixed boundary point.

5.2.3 The growth-fragmentation process

Recall that we have set $\gamma = \sqrt{8/3}$ and $\kappa' = 6$. Let $(\mathbb{D}, \psi, -i)$ be a singly marked unit boundary quantum disc, and consider an independent counterclockwise space-filling SLE_6 curve η , parametrised by quantum area. For $z \in \mathbb{D}$, one can consider the *branch* η^z of η targeted at z , which is obtained by reparametrising η with quantum natural time with

respect to z . This means that one explores the domain according to η , except for the components that η disconnects from z . Such a branch η^z is defined until some time t_z when it hits z (since η is space-filling). For $0 \leq u < t_z$, one can define the *total quantum boundary length* $S^z(u)$ (Figure 5.7 of the connected component containing z of $\mathbb{D} \setminus \eta([0, u])$). This defines a positive process for every $z \in \mathbb{D}$. One can go one step further and notice that, when $x, y \in \mathbb{D}$, the branches η^x and η^y (hence S^x and S^y) coincide up to some time when η disconnects the two (see Figure 5.8).

The goal of this section is to describe this branching structure for the total boundary length. More precisely, for $t \geq 0$ we set

$$\mathbf{Z}(t) := \{S^z(t), z \in \mathbb{D} \text{ and } t < t_z\}.$$

Observe that many different points $z \in \mathbb{D}$ produce the same $S^z(t)$ (in fact, this is true for any point in the remaining-to-be-explored domain at time t when exploring towards some z). The family of sizes in $\mathbf{Z}(t)$ yields a countable collection of positive real numbers as for any $\varepsilon > 0$ there can only be a finite number of such number of these sizes which are larger than ε since the branches are càdlàg. We want to see these sizes as *cells* in some branching structure, where we see the moments when η disconnects two regions as birth events for one cell, with initial size the total boundary length of the cut-out region.

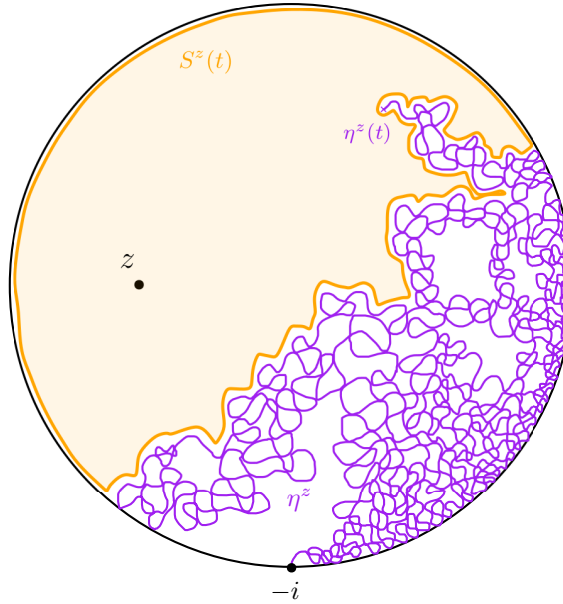


Figure 5.7 – Total quantum boundary length of the domain containing z . The picture shows the branch η^z targeted towards z until quantum natural time t (in purple). The remaining-to-be-explored domain is the orange region, and $S^z(t)$ is the ν_ψ -length of the orange boundary of this region.

Let X be the positive self-similar Markov process with index $\frac{3}{2}$ defined via (5.1.6), with Lamperti exponent as in (5.1.7). We briefly recall the definition of the *growth-fragmentation process* driven by X . At any time $t > 0$ when X has a jump $\Delta X(t) = X(t) - X(t^-)$, one creates a particle which evolves according to the same law as X , started from $-\Delta X(t)$ ¹. Conditionally on the jump times and sizes, we take all these newborn cells to be independent of each other, and of the mother cell X . Then, one repeats the procedure,

¹Note from the form of (5.1.7) that ξ is a *spectrally negative* Lévy process, so that X also has only negative jumps, and $-\Delta X(t) > 0$

thereby constructing grandchildren of X , and so on. Finally, introduce the collection $\mathbf{X}(t)$ of the sizes of cells alive at time t . We rephrase our main theorem (Theorem A).

Theorem 5.2.7. *The following equality in law holds*

$$\mathbf{Z} \stackrel{d}{=} \mathbf{X}.$$

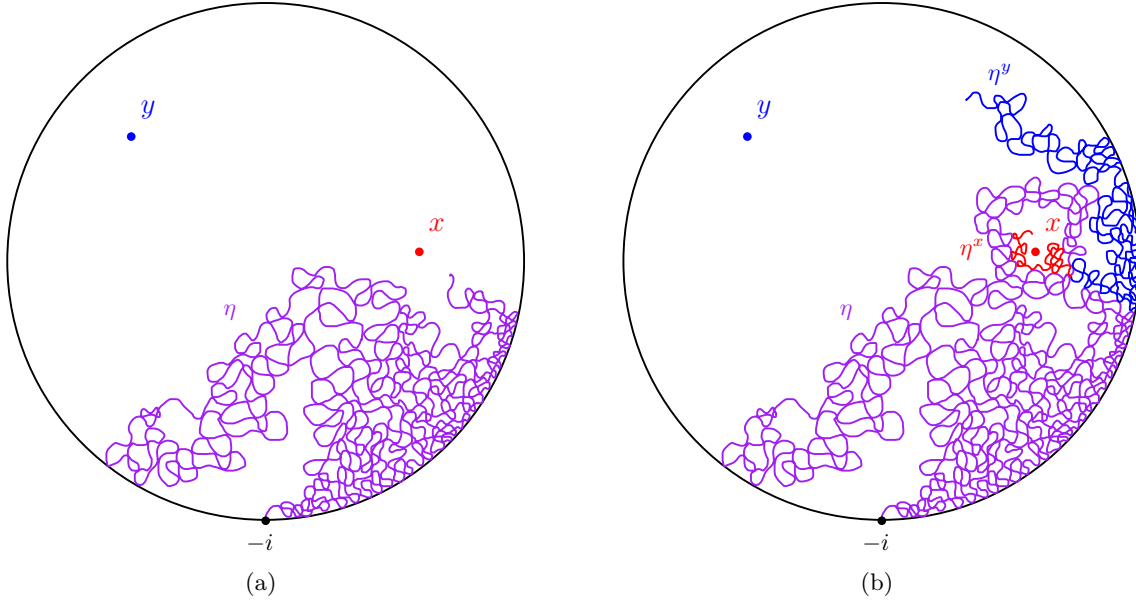


Figure 5.8 – Branches of the space-filling SLE_6 η on the $\sqrt{8/3}$ -quantum disc towards x and y . (a) The branch of η towards x and y (purple) is the same. (b) The two branches get disconnected: a loop has been cut out, surrounding x . The branch η^x targeted at x is shown in (purple and then) red, and the branch η^y targeted at y is in (purple and then) blue.

It is perhaps useful to restate the theorem as follows. In the branching structure $\mathbf{Z}$, one may consider the so-called *locally largest fragment*, meaning that one follows the branch with maximal size at each splitting (beware that this may not be the global maximum at some fixed time t). The core message of Theorem 5.2.7 is the following. First, the law of the total quantum boundary length $(S^*(t), t \geq 0)$ of this locally largest branch is as described by (5.1.7) after Lamperti transformation. Then, the quantum surfaces that are cut out by this branch (defined by restriction of the free field ψ), each marked by the point where η finishes tracing the boundary, builds up an independent collection of singly marked quantum discs, with total boundary lengths given by the magnitude of the jumps of S^* . Let us mention that Theorem 5.2.7 can be used to find martingales with the growth-fragmentation machinery. Denote $\mathbf{Z}(t) := \{Z_i(t), i \geq 1\}$ for simplicity, with the convention that $Z_i(t) = 0$ for all $i \geq 1$ when $\mathbf{Z}(t)$ is empty.

Corollary 5.2.8. *In the setting described above, the process*

$$\mathcal{M}(t) := \sum_{i \geq 1} Z_i(t)^3,$$

is a martingale for the natural filtration of $\mathbf{Z}$.

Corollary 5.2.8 is a straightforward consequence of [BBCK18, Section 3.3], and of the expression of the cumulant, a formula for which may be found in their Proposition 5.2. In particular, note that this provides another martingale exponent $\omega_- = 2$ associated with a genealogical martingale for the growth-fragmentation.

Proof of Theorem 5.2.7. The idea of the proof is the following adaptation of [MSW20, Section 5.3]. We first relate the locally largest evolution to a chordal exploration in the quantum disc. Loosely speaking, we give the Radon-Nikodym derivative between the two exploration mechanisms. Then, we make use of Theorem 5.2.6, which relates the left/right boundary lengths in the chordal exploration to a pair of independent stable spectrally negative Lévy processes (L^∞, R^∞) . Finally, we combine these two results to show an absolute continuity relationship between the total boundary length in the locally largest exploration and the sum $L^\infty + R^\infty$. We prove that the Lamperti exponent for the locally largest fragment roughly appears as that of a $\frac{3}{2}$ -stable Lévy process with no upward jumps, biased by some explicit martingale, which eventually enables us to compute its Laplace exponent.

As a first step, we relate the locally largest fragment S^* to the chordal exploration in the quantum disc. Let t^* denote the extinction time of the locally largest fragment, and fix some time $t < t^*$. Consider first the case when the locally largest branch has not yet been disconnected i at time t . This means that, up to time t , S^* corresponds to the total boundary length in a chordal exploration of the quantum disc. Such an exploration starts from $-i$, and is targeted at i on the boundary $\partial\mathbb{D}$, subject to the condition that S^i is the *locally largest* at each jump time occurring before t , *viz.*

$$\forall s \leq t, \quad S^i(s) > -\Delta S^i(s).$$

Now it may happen that the locally largest branch gets disconnected from i before time t . This means that, at some time $s < t$, the branch η^* corresponding to the locally largest fragment cuts out a domain whose total boundary length is larger than the total boundary length of the other domain containing i . The target-invariance property of the SLE_6 η [She09] entails that the conditional law of the branch after time s only depends on the *total* boundary of the cut-out bubble. More precisely, it yields that, conditionally on the past up to s , the bubble is a quantum disc (marked with the point where η finishes tracing the contour of the bubble) with total boundary length given by $S^*(s)$, and that $(S^*(u+s), u \leq t-s)$ evolves as the locally largest evolution in this quantum disc started from $S^*(s)$. In this quantum disc, one can again consider a chordal exploration, until the locally largest fragment gets disconnected, and so on.

In short, one can introduce the successive times when the locally largest branch is disconnected from the marked point on the boundary of the current domain (call these *disconnection* times). The previous argument based on target invariance of SLE_6 gives the following key identity:

$$\mathbb{E}(F(S^*(s), s \leq t) \mathbf{1}_{t < t^*}) = \mathbb{E}\left(F(L_s + R_s, s \leq t) \mathbf{1}_{\forall s \leq t, (L+R)(s) > -\Delta(L+R)(s)}\right), \quad (5.2.2)$$

where L and R describe the left and right boundary lengths in a chordal exploration as in Theorem 5.2.6. The rigorous derivation of equation (5.2.2) comes from summing over all disconnection times, and applying the target-invariance property at each step. Note that there is a subtlety hidden in (5.2.2). Namely, the indicator function on the right-hand side of the identity forces both L and R to stay positive until time t (and in particular not to vanish). Indeed, it clearly implies that the sum $L + R$ be positive, and by the first item of Theorem 5.2.6, one sees that L and R must both stay positive.

Now, we use the absolute continuity relationship in Theorem 5.2.6. Write

$$\mathcal{E}_t^\infty := \{\forall s \leq t, (L^\infty + R^\infty)(s) > -\Delta(L^\infty + R^\infty)(s)\},$$

where L^∞ and R^∞ are as in Theorem 5.2.6. We come to the conclusion that

$$\mathbb{E}(F(S^*(s), s \leq t) \mathbf{1}_{t < t^*}) = \mathbb{E}\left((L_t^\infty + R_t^\infty)^{-5/2} F(L_s^\infty + R_s^\infty, s \leq t) \cdot \mathbf{1}_{t < \zeta^\infty} \cdot \mathbf{1}_{\mathcal{E}_t^\infty}\right). \quad (5.2.3)$$

Note that in our case $\ell_L + \ell_R = 1$, since we started with a unit boundary quantum disc. The expression on the right-hand side of (5.2.3) only depends on the sum $S^\infty = L^\infty + R^\infty$, except *a priori* for the indicator that $(t < \varsigma^\infty)$. However, the formula involves an h -transform which forces L^∞ and R^∞ to be killed at 0 exactly at the same time (this can be seen as a consequence of the first assertion of Theorem 5.2.6). This means that we can rewrite (5.2.3) as

$$\mathbb{E}(F(S^*(s), s \leq t) \mathbf{1}_{t < t^*}) = E\left(F(Y_s, s \leq t) Y_t^{-5/2} \cdot \mathbf{1}_{Y_s \leq t, Y_s > -\Delta Y_s}\right), \quad (5.2.4)$$

where under P , Y is a spectrally negative $\frac{3}{2}$ -stable Lévy process starting from 1, killed when entering the negative half-line (again, the killing is hidden in the indicator). Results from [CC06] show that Y can be written $Y_s = \exp(\xi^0(\tau^0(s)))$, where

$$\tau^0(s) := \inf\left\{u \geq 0, \int_0^u e^{\frac{3}{2}\xi^0(u)} du \geq s\right\},$$

and ξ^0 is the Lévy process starting at 0, with Laplace exponent

$$\psi^0(q) = 4cq + 2c \int_{-\infty}^0 (e^{qy} - 1 - q(e^y - 1) \mathbf{1}_{|e^y - 1| < 1}) \frac{e^y dy}{(e^y - 1)^{5/2}}, \quad -1 < q < \frac{3}{2}.$$

Here c is the same as the positive jump rate of L^∞ and R^∞ (see the discussion following Theorem 5.2.6). This essentially concludes the derivation of the law of the locally largest fragment. It only remains to prove that the expression of the Laplace exponent ξ^0 biased by the h -transform in (5.2.4) matches the expression (5.1.7). Actually, [LGR20] spares us some work here, since the very same computation already appears in their description of the locally largest fragment. The following lemma is a simple rephrasing of [LGR20, Lemma 17]². A analogous statement also appears in [AS20, Sil21] for other growth-fragmentation processes related to excursions in the half-plane.

Lemma 5.2.9. *For $t \geq 0$, introduce*

$$M_t^0 := e^{-\frac{5}{2}\xi^0(t)} \cdot \mathbf{1}_{\forall 0 \leq s \leq t, \Delta\xi^0(s) > -\log(2)}.$$

Then $(M_t^0, t \geq 0)$ is a martingale with respect to the canonical filtration of ξ^0 . Moreover, under the tilted probability measure $M_t^0 \cdot P$, $(\xi^0(s), s \leq t)$ evolves as the Lévy process ξ with Laplace exponent given by (5.1.7).

We finish the proof of Theorem 5.2.7 by appealing to Lemma 5.2.9, together with the arguments of [LGR20] that we reproduce here to be self-contained. With this notation at hand, identity (5.2.4) now reads

$$\mathbb{E}(F(S^*(s), s \leq t) \mathbf{1}_{t < t^*}) = E(M_{\tau^0(t)}^0 \cdot F(\exp(\xi^0(\tau^0(s))), s \in [0, t])).$$

The optional stopping theorem implies that for any $A > 0$,

$$\begin{aligned} & E\left[M_{\tau^0(t)}^0 \cdot F(\exp(\xi^0(\tau^0(s))), s \in [0, t]) \mathbf{1}_{\{A > \tau^0(t)\}}\right] \\ &= E\left[M_A^0 \cdot F(\exp(\xi^0(\tau^0(s))), s \in [0, t]) \mathbf{1}_{\{A > \tau^0(t)\}}\right]. \end{aligned}$$

²Note that [LGR20, Lemma 17] expresses the result in terms of (the Lamperti exponent of) the spectrally negative $\frac{3}{2}$ -stable process *conditioned to remain positive*. To transfer the results to the killed process ξ^0 , one needs to recall from [CC06] that this conditioning is obtained as an h -transform of the killed process Y , with harmonic function $h(x) = x^{-1/2}$.

By Lemma 5.2.9, the right-hand side is

$$E\left[F(\exp(\xi(\tau(s))), s \in [0, t])\mathbb{1}_{\{A > \tau(t)\}}\right],$$

where

$$\tau(s) := \inf\left\{u \geq 0, \int_0^u e^{\frac{3}{2}\xi(u)} du \geq s\right\}.$$

Taking $c \rightarrow \infty$ and using dominated convergence completes the proof for the law of the locally largest fragment.

One should then prove the statements about the conditional independence of the offspring. This can be derived using the conditional independence statement Theorem 5.2.5. We omit the proof here, as the arguments of [MSW20] work in our setting. $\square$

5.3 A decomposition of the Brownian cone excursion

The processes involved in Section 5.2.3 have a natural translation in terms of mating-of-trees (Theorem 5.1.1). We present some applications of Theorem A showing that the growth-fragmentation process $\mathbf{Z}$ in Section 5.2.3 can be constructed naturally using Brownian motion. We establish a general Bismut-type description for cone excursions, which could be of independent interest, and state a conjecture on the spectrally positive $\frac{3}{2}$ -stable process conditioned to remain positive.

5.3.1 Cone points of Brownian motion

Two different types of cone points will naturally come into play in our setting: the *forward* ones or the *backward* ones. We will define and review both of them here. Then, we introduce Brownian cone excursions in the forward and backward setting. In particular, we shall be interested in the density of the endpoint for these cone excursions. We stress that, although we are interested in the case when $\theta = \frac{2\pi}{3}$, we describe the results in full generality. Recall that for $\theta \in (0, 2\pi]$, $\mathcal{C}_\theta = \{z \in \mathbb{C}, \arg(z) \in (0, \theta)\}$ is the cone with apex angle θ . We will use the following notation:

- W denotes planar Brownian motion started at 0, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$. We extend the definition of $\mathcal{F}_T$ to stopping times T in the usual way;
- Γ is the set of functions γ defined on a finite interval $[0, \zeta(\gamma)]$, with values in $\mathbb{C}$ and vanishing at $\zeta(\gamma)$ (the γ 's we will be interested in will actually remain in a cone, and start somewhere inside it or on the boundary). By convention, we add a cemetery function $\diamond$ to Γ . We endow Γ with the σ -field $\mathcal{G}$ generated by the coordinate mappings.

Some results will be better expressed in terms of the correlated Brownian motion $\widetilde{W}$ obtained by applying the transformation Λ^{-1} from (5.1.8). For ease of notation, we will mix up the quantities associated with $\widetilde{W}$ (such as local times, inverse local times, etc.) with the ones defined in terms of W , noting that there is always a bijective path from one to the other, which transforms cones into quadrants.

Forward cone points of Brownian motion. The first instance of cone points that we will be interested in is already of particular importance in [DMS14]. We say that $t \in \mathbb{R}_+$ is a *forward cone time* if there exists $\varepsilon > 0$ such that $W_s \in W_t + \mathcal{C}_\theta$ for $s \in (t, t + \varepsilon)$. It has been proved [Bur85, Shi85] that such times exist if, and only if, $\theta > \frac{\pi}{2}$. By an approximation procedure, Shimura constructed the law of Brownian motion starting from 0 and conditioned

to remain in $\mathcal{C}_\theta$ at least for one unit of time. If $u > 0$ is straddled by a forward cone excursion, meaning that there exists $t \leq u$ such that $W_s \in W_t + \mathcal{C}_\theta$ for all $s \in (t, u]$, then (following [DMS14, Section 10.2]) we say that u is a *pinched* time. We remark that the set of pinched times almost surely forms an open subset of $[0, \infty)$, so that we can express it as a countable disjoint union of open intervals. These intervals will be referred to as *forward cone excursions*. See Figure 5.9.

If u is *not* a pinched time, we say that u is *cone-free* (ancestor-free in [DMS14]). One can see that cone-free times form a *regenerative* set in the sense of [Mai71], so that one can define a local time $(\ell_\theta(t), t \geq 0)$ supported on the set of cone-free times. The inverse τ_θ of this local time,

$$\tau_\theta(t) := \inf\{s \geq 0, \ell_\theta(s) > t\}, \quad t > 0,$$

gives a way to define the forward cone excursions $(e_\theta(s), s > 0)$. The following proposition describes the structure of these forward cone excursions as a Poisson point process. We will not give the details, as this is essentially done in [DMS14, Section 10.2], and since we will further discuss the analogue for *backward* cone excursions.

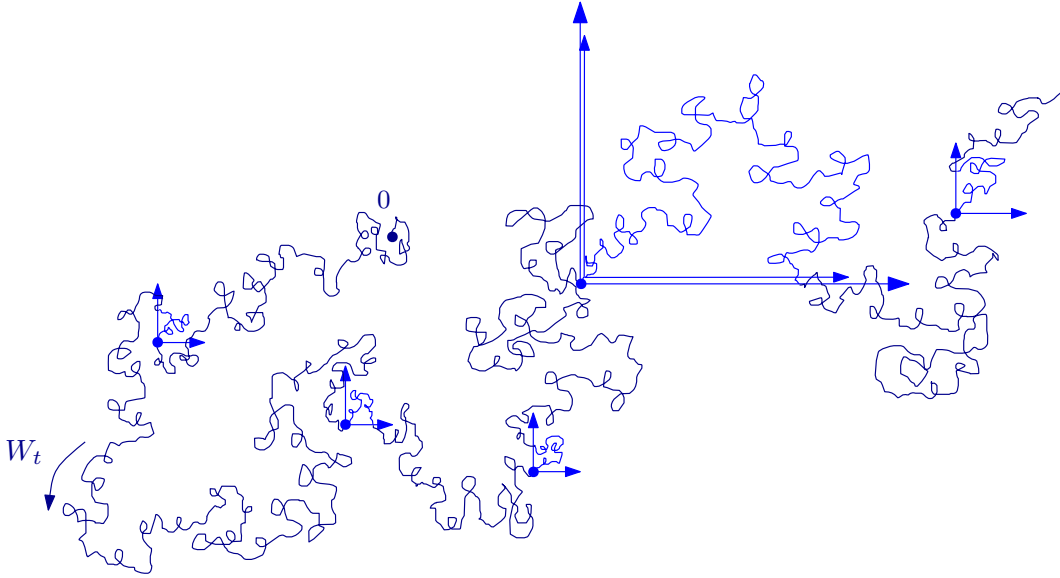


Figure 5.9 – Forward cone times of planar Brownian motion. The picture is drawn in the correlated setting, so that we look at excursions in a quadrant made by a *correlated* planar Brownian motion W . The reader should imagine accumulation of cone times, as emphasized on the large excursion in the middle. The forward cone excursions are shown in blue.

Proposition 5.3.1. *The forward cone excursions $(e_\theta(s), s > 0)$ form a $(\mathcal{F}_{\tau_\theta(s)}, s > 0)$ -Poisson point process in $(\Gamma, \mathcal{G})$. We denote its intensity measure by $\mathfrak{m}_\theta$.*

Notice that excursions under the measure $\mathfrak{m}_\theta$ remain in $\mathcal{C}_\theta$, starting at the apex, and ending on the boundary of $\mathcal{C}_\theta$. It will be important for our purposes to establish the density of the endpoint under $\mathfrak{m}_\theta$. Although not explicitly stated, it appears that this density was already described in the proof of [DMS14, Proposition 10.3]. Fix

$$\lambda = \frac{\pi}{\theta}.$$

Proposition 5.3.2. *We have the following disintegration formula for $\mathfrak{n}_\theta$:*

$$\mathfrak{n}_\theta = \int_{\partial\mathcal{C}_\theta} \frac{dz}{\theta|z|^{1+\lambda}} P_\theta^z, \quad (5.3.1)$$

where the P_θ^z are probability measures supported on cone excursions ending at $z \in \partial\mathcal{C}_\theta$.

The interpretation of P_θ^z is that it corresponds (via (5.3.1)) to the measure n_θ conditional on the endpoint being z . Finally, it is natural to wonder what is the law of W time-changed by the inverse local time. In the SLE on quantum cone picture, this roughly amounts to recording the left/right boundary lengths of the branch towards infinity, as we discussed in Section 5.2.2, in the sense that we do not explore the bubbles disconnected by the space-filling SLE along the way. It is actually easier to phrase it in the quadrant, in terms of *correlated* Brownian motion. Write $\tilde{W} = (\tilde{X}, \tilde{Y})$. The following proposition is [DMS14, Proposition 1.13].

Theorem 5.3.3. *The time-changed process $(\tilde{X}_{\tau_\theta(t)}, \tilde{Y}_{\tau_\theta(t)})$ evolves as a pair of independent spectrally positive λ -stable Lévy processes.*

We stress once more that τ_θ was rather defined in terms of W , but it is not hard to translate it in the correlated framework by looking at sub-excursions in the quadrant rather than the cone. Besides, note that the reason why this setting is more convenient here is because $\tilde{X}, \tilde{Y}$ never jump at the same time, hence the independence statement. Theorem 5.3.3 is also related to Proposition 5.3.2, since the density of the endpoint corresponds to the Lévy measure of the time-changed process.

Backward cone times of Brownian motion. The other type of cone points we want to discuss is the one introduced by Le Gall [LG87] (actually, the cone points we describe here are obtained after rotation of the plane). Call $t \in \mathbb{R}_+$ a *backward cone time* if $W_s \in W_t + \mathcal{C}_\theta$ for all $s \in [0, t)$ (Figure 5.10). We focus on the case $\theta \in (\frac{\pi}{2}, \pi]$, since the results of [Bur85, Shi85] prove that such times exist if, and only if, $\theta > \frac{\pi}{2}$. Moreover, the set H_θ of backward cone times is also regenerative, so that one can again define a local time $(\mathfrak{l}_\theta(s), s \geq 0)$ supported on H_θ [Mai71]. Le Gall also constructs an approximation of the local time, see [LG87, Section 3]. Let $\mathfrak{t}_\theta$ be the inverse local time

$$\mathfrak{t}_\theta(t) := \inf\{s \geq 0, \mathfrak{l}_\theta(s) > t\}.$$

Then [LG87, Theorem 5.2] determines in particular the law of W time-changed by $\mathfrak{t}_\theta$ as follows (note that the case $\theta = \pi$ gives back Spitzer's construction of the Cauchy process [Spi58]). Recall again that $\lambda = \frac{\pi}{\theta}$.

Theorem 5.3.4. *The process $(W_{\mathfrak{t}_\theta(t)}, t \geq 0)$ is a stable Lévy process in the plane, with index $2 - \lambda$.*

At this point, we should emphasize that the structure of $W \circ \mathfrak{t}_\theta$ is much more involved than the one of $\tilde{W} \circ \tau_\theta$ appearing in Theorem 5.3.3. The issue is that there is *no independence* in the backward cone times framework. Indeed, the backward cone excursions that are cut out in W go from the *interior* of the cone to the apex, and therefore at a jump time of $W \circ \mathfrak{t}_\theta$, both components jump simultaneously. In particular, the Lévy measure of $W \circ \mathfrak{t}_\theta$ can be written in polar coordinates

$$\mathfrak{L}_\theta(dr, d\phi) = \frac{dr}{r^{3-\lambda}} \cdot \mathfrak{m}(d\phi), \quad (5.3.2)$$

but the angular part $\mathfrak{m}$ of the measure does not seem to be known in general. Although $\mathfrak{m}$ is not explicit, it is characterised by formula (5.j) in [LG87]. However, in the case $\theta = \frac{2\pi}{3}$,

a closed formula has been found in the Liouville quantum gravity framework by [AHS21], namely

$$\mathfrak{L}_{\frac{2\pi}{3}}(dx, dy) = \frac{C dx dy}{(x+y)^{5/2}} \mathbb{1}_{x,y \geq 0}, \quad (5.3.3)$$

where C is an explicit constant. See Proposition 5.2 there. Note also that [LG87, Theorem 5.2] gives the law of $\mathfrak{t}_\theta$ as a stable subordinator with index $1 - \frac{\lambda}{2}$.

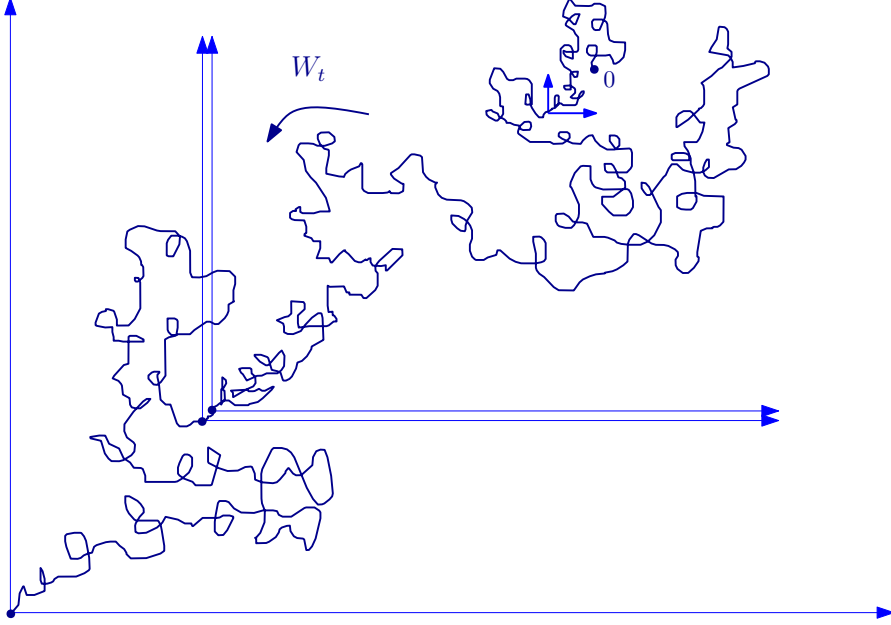


Figure 5.10 – Backward cone times of planar Brownian motion. Again, the picture is drawn in the correlated setting, so that we look at quadrants containing the past trajectory of a *correlated* planar Brownian motion W . The reader should imagine accumulation of cone times, as suggested by the two quadrants in the middle of the picture.

In the same vein as for forward cone points, we can describe the *backward cone excursions* process. Recall that Γ is the set of functions γ defined on a finite interval $[0, \zeta]$, with values in $\mathbb{C}$ and vanishing at ζ (with a cemetery function denoted by $\diamond$).

Definition 5.3.5. *The backward cone excursion process is the process $\mathfrak{e}_\theta = (\mathfrak{e}_\theta(s), s > 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(\Gamma, \mathcal{G})$, defined as follows:*

(i) if $\mathfrak{t}_\theta(s) > \mathfrak{t}_\theta(s^-)$, then

$$\mathfrak{e}_\theta(s) : r \mapsto W(\mathfrak{t}_\theta(s^-) + r) - W(\mathfrak{t}_\theta(s)), \quad 0 \leq r \leq \mathfrak{t}_\theta(s) - \mathfrak{t}_\theta(s^-),$$

(ii) if $\mathfrak{t}_\theta(s) = \mathfrak{t}_\theta(s^-)$ then $\mathfrak{e}_\theta(s) := \diamond$.

This definition is made so that the $\mathfrak{e}_\theta(s)$ are cone excursions ending at the apex, and there is a non-degenerate cone excursion $\mathfrak{e}_\theta(s)$ whenever $\mathfrak{t}_\theta$ has a constant stretch at time s . We claim that this process defines a Poisson point process.

Proposition 5.3.6. *The process $(\mathfrak{e}_\theta(s), s > 0)$ is a $(\mathcal{F}_{\mathfrak{t}_\theta(s)}, s > 0)$ -Poisson point process with some intensity measure denoted by $\mathfrak{n}_\theta$.*

Proof. We check the properties listed in [RY99, Definition XII.1.8].

- (i) Plainly, $(\mathfrak{e}_\theta(s), s > 0)$ is a point process in the sense of [RY99, Definition XII.1.1] (the fact that there are at most countably many non-degenerate excursions can be seen as a consequence of the fact that the jump times of $\mathfrak{t}_\theta$ are at most countable).
- (ii) We check that $(\mathfrak{e}_\theta(s), s > 0)$ is σ -discrete [RY99, Definition XII.1.2]. Let $\Gamma_n := \{\gamma \in \Gamma, R(\gamma) > 1/n\}$, $n \geq 1$. Then $\Gamma = \bigcup_{n \geq 1} \Gamma_n$, and the Γ_n are measurable. The counting functions

$$N_t^{\Gamma_n} := \sum_{s \leq t} \mathbf{1}_{\mathfrak{e}_\theta(s) \in \Gamma_n}, \quad t > 0,$$

are *a.s.* finite random variables. Indeed, set $T_0 := 0$ and $T_{k+1} := \inf\{s > T_k, \mathfrak{t}_\theta(s) - \mathfrak{t}_\theta(s^-) > 1/n\}$, $k \geq 0$. By definition,

$$N_t^{\Gamma_n} := \sum_{k \geq 1} \mathbf{1}_{T_k \leq t}, \quad t > 0,$$

and $N_t^{\Gamma_n} \leq n \mathfrak{t}_\theta(t)$.

- (iii) The process $\mathfrak{e}_\theta$ is clearly $(\mathcal{F}_{\mathfrak{t}_\theta(s)})$ -adapted.
- (iv) Finally, for any subset X of Γ , and $t, r > 0$, write

$$N_{(t, t+r]}^X := \sum_{t < s \leq t+r} \mathbf{1}_{\mathfrak{e}_\theta(s) \in X}.$$

Since $\mathfrak{e}_\theta(s) \circ \theta_r = \mathfrak{e}_\theta(s+r)$ (see [RY99, Section XII.1], and in particular the remark following Proposition XII.1.3 on finite continuous additive functionals), by shifting the excursion process we get

$$\mathbb{P}\left(N_{(t, t+r]}^X \in A \mid \mathcal{F}_{\mathfrak{t}_\theta(t)}\right) = \mathbb{P}\left(N_r^X \circ \Theta_{\mathfrak{t}_\theta(t)} \in A \mid \mathcal{F}_{\mathfrak{t}_\theta(t)}\right),$$

where Θ denotes the shift operator. Now by the strong Markov property of W , this is

$$\mathbb{P}\left(N_{(t, t+r]}^X \in A \mid \mathcal{F}_{\mathfrak{t}_\theta(t)}\right) = \mathbb{P}_{W(\mathfrak{t}_\theta(t))}\left(N_r^X \in A\right).$$

But the excursion process $\mathfrak{e}_\theta$ is by definition independent of the starting point of W , hence finally

$$\mathbb{P}\left(N_{(t, t+r]}^X \in A \mid \mathcal{F}_{\mathfrak{t}_\theta(t)}\right) = \mathbb{P}\left(N_r^X \in A\right).$$

This concludes the proof of the Poisson point process property. □

The next section is devoted to deriving basic features of the excursion measure $\mathfrak{n}_\theta$. In particular, we will make explicit the density of the endpoint (or rather the starting point, given our definition) under $\mathfrak{n}$. We postpone this preliminary study to cast a new light on the growth-fragmentation presented in Section 5.2.3 in the LQG setting.

The growth-fragmentation embedded in Brownian $\frac{2\pi}{3}$ -cone excursions. We now focus on the case when $\theta = \frac{2\pi}{3}$. We remind the reader that this is the case connected to the space-filling SLE₆ explorations of the $\sqrt{8/3}$ -quantum disc Theorem 5.1.1. The mating-of-trees naturally translates cutpoints of the SLE₆ η into *subcone excursions* of the previous types. We can thus rephrase our findings in Section 5.2.3 in the setting of cone excursions. We take the viewpoint of *correlated* Brownian excursions to stick to the mating-of-trees statement. For clarity, we will also drop the subscript θ in this case. In particular, the

normalised excursion measure P_θ^z from the disintegration property 5.3.2 will be denoted by P^z when $\theta = \frac{2\pi}{3}$.

Let $z \in \partial\mathbb{R}_+^2 \setminus \{0\}$. Under measure P^z , let $((A_t, B_t) : 0 \leq t \leq \zeta)$ be a correlated Brownian excursion in the positive quadrant, with correlation $-\cos(\frac{2\pi}{3}) = -\frac{1}{2}$, starting from z and with duration ζ . Note that – in addition to the correlation – under P^z we are now considering the time-reversal of a forward cone excursion in the sense of Section 5.3.1, *i.e.* $((A_t, B_t) : 0 \leq t \leq \zeta)$ starts at z on the boundary and ends at the apex. Fix $0 \leq t \leq \zeta$. In analogy to the forward cone times setting, we say that an interval (g, d) is a t -cone excursion if

- (i) $d < t$
- (ii) for all $s \in (g, d)$, $A_s > A_d$ and $B_s > B_d$,
- (iii) either $A_g = A_d$ or $B_g = B_d$
- (iv) (g, d) is the maximal interval for which these hold (with respect to inclusion).

Let $\mathcal{E}_t = \{(g, d) \text{ an } s\text{-cone excursion, } s \leq t\}$. Note that excursions in $\mathcal{E}_t$ may be nested but never overlap. The set $\bigcup \mathcal{E}_t$, namely the union of the sets in $\mathcal{E}_t$, contains only the t -cone excursions themselves. The set $\mathcal{K}_t = [0, t] \setminus \bigcup \mathcal{E}_t$ is again a regenerative set, and hence one can define a local time $\mathcal{L}^t = (\mathcal{L}_s^t, 0 \leq s \leq t)$. Let $\tau_u^t = \inf\{s \geq 0 : \mathcal{L}_s^t \geq u\}$, $0 \leq u \leq \mathcal{L}_t^t$, be the inverse local time on $\mathcal{K}_t$. In the SLE on LQG picture, this inverse local time corresponds to the quantum natural time targeted at t . In other words, the times when (A, B) starts cutting out a subcone excursion along the way to time t correspond to the times when η disconnects a bubble from the target point $\eta(t)$. Moreover, the total quantum boundary length of the bubble is the *size* of the cut-out excursion (*i.e.* the difference between starting point and endpoint).

Along its way, the space-filling SLE₆ η draws a collection of loops which forms a CLE₆. We now remark that one can also see the CLE₆ loops that are cut out by η in this cone picture. Indeed, there is a distinguished countable subset $E \subset \mathcal{E}_\zeta$ of cone excursions (g, d) such that for some $\varepsilon > 0$ there is no cone excursion (g', d') with $(g, d) \supset (g', d') \supset (g + \varepsilon, d - \varepsilon)$. Each of these $(g, d) \in E$ corresponds to a distinct (nested) CLE₆ loop, where d is the last time that the space-filling SLE₆ intersects the loop. Note that intervals of E are disjoint or nested, and appear in [BHS21] under the name of *envelope intervals*.

Now, set $\mathcal{Y}_t = \mathcal{L}_t^t$. For clarity, assume that we are speaking about a left-continuous version of the process. Fix $t \in (0, \zeta)$. For each $a > 0$, define $(g_t(a), d_t(a))$ to be the unique interval (not necessarily an excursion interval) containing t such that $\mathcal{Y}_{g_t(a)} = a \leq \mathcal{Y}_{d_t(a)}$, and for $u \in (g_t(a), d_t(a))$, $\mathcal{Y}_u > a$. Note that when $\mathcal{Y}$ has jumps, $d_t(a)$ may remain constant for some period as a increases. Let $L_a^t := A_{g_t(a)} - A_{d_t(a)}$, and likewise $R_a^t := B_{g_t(a)} - B_{d_t(a)}$. Then the interpretation of (L_a^t, R_a^t) is that they describe the left/right quantum boundary length of the remaining-to-be-explored domain when targeting $\eta(t)$, at quantum natural time a . For $a > 0$, say that $(s, t) \in \mathcal{I}(a)$ if and only if for all $u \in (s, t)$, $\mathcal{Y}_u > a$ and (s, t) is maximal among such intervals (that is, elements of $\mathcal{I}(a)$ are *superlevel sets* of $\mathcal{Y}$ at level a). Then the previous discussion boils down to

$$\mathbf{Z}(a) = \sum_{(s,t) \in \mathcal{I}(a)} \delta_{A_s + B_s - A_t - B_t}, \quad (5.3.4)$$

where $\mathbf{Z}$ is the branching process defined in Section 5.2.3. In particular, Theorem 5.2.7 shows that there is a growth-fragmentation embedded in $\frac{2\pi}{3}$ -subcone excursions, and that, moreover, this process is explicitly described *via* the positive self-similar Markov process with index $\frac{3}{2}$ and Lamperti exponent as in (5.1.7).

The remainder of this text is devoted to make use of this characterisation to deduce some fine properties of the $\frac{2\pi}{3}$ -Brownian cone excursions.

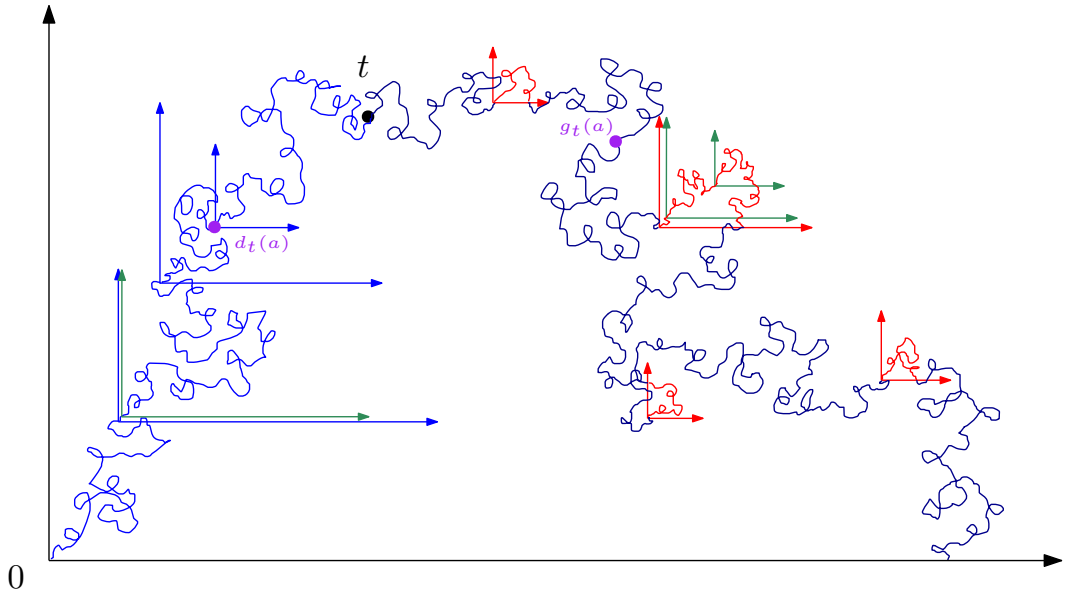


Figure 5.11 – The growth-fragmentation embedded in $\frac{2\pi}{3}$ -cone excursions. If t is a time on the excursion, we record the t -cone excursions $\bigcup \mathcal{E}_t$ (red). One should think that cone excursions can be nested, typically with an accumulation of cone points (green). The complement of this set $\bigcup \mathcal{E}_t$ gives rise to a local time $\mathcal{L}^t$. For $0 \leq a < t$, we construct the interval $(g_t(a), d_t(a))$ straddling t corresponding to the component targeted at $\eta(t)$ after quantum natural time a in the LQG picture. These times satisfy in particular that $\mathcal{L}_{g_t(a)}^{g_t(a)} = a$ and $d_t(a)$ is the first *backward* cone time of the trajectory after t that falls below the whole trajectory from $g_t(a)$ to t .

5.3.2 Basic properties of the backward cone excursion measure $\mathbf{n}_\theta$

We resume the study of $\mathbf{n}_\theta$, in the general case when $\theta \in (\frac{\pi}{2}, \pi]$, by establishing a kind of Markov property under $\mathbf{n}_\theta$, deriving the density of the endpoint, and proving the convergence of the normalised backward cone excursion measure to the normalised forward one when the point is sent to the boundary. We will denote by γ a generic cone excursion, and ζ its duration.

The Markov property of $\mathbf{n}_\theta$. One of the core properties of the classical Itô measure of Brownian motion is its Markov property, see [RY99, Theorem XII.4.1]. In this case, it roughly states that for $t > 0$, on the event $t < \zeta$, then conditioned on the past, the trajectory of γ from time t onwards is an independent Brownian motion starting at $\gamma(t)$, and killed upon reaching 0. Of course, under $\mathbf{n}_\theta$, the statement is less straightforward, as there is some dependence on the past. Indeed, backward cone times are defined so that the *whole* past trajectory is contained in a cone, hence ending the excursion should depend on the past even before t . The next result states that, loosely speaking, this happens to be the only dependence.

Proposition 5.3.7. *(The Markov property under $\mathbf{n}_\theta$.)*

Let $t > 0$. On the event that $t < \zeta$, and conditioned on $(\gamma(s), 0 \leq s \leq t)$, the law of $(\gamma(t+s), 0 \leq s \leq \zeta)$ is that of an independent planar Brownian motion W started at $\gamma(t)$, and stopped at its first backward cone time I for which the cone also contains the path $(\gamma(s), 0 \leq s \leq t)$, and shifted by $-W_I$.

Remark 5.3.8. We could have stated Proposition 5.3.7 in the positive quadrant instead of the cone (using again the mapping Λ^{-1}). In this setup, the statement reads a bit more

nicely, because backward cone times correspond to *simultaneous running infima* for planar Brownian motion, meaning that both components are running infima.

Proof of Proposition 5.3.7. The proof follows the lines of [RY99, Theorem XII.4.1]. Denote by $\Theta = (\Theta_t, t \geq 0)$ the shift operator. We want to prove that for all measurable sets $A \subset \mathbb{C}$ and $\tilde{\Gamma} \subset \Gamma$,

$$\mathbf{n}_\theta(\{\gamma(t) \in A\} \cap \{\gamma \circ \Theta_t \in \tilde{\Gamma}\}) = \mathbf{n}_\theta(\mathbf{1}_{\gamma(t) \in A} \cdot \mathbb{P}_{\gamma(t)}((W_s - W_I, s \leq I) \in \tilde{\Gamma})), \quad (5.3.5)$$

where I is the first backward cone time for which the cone contains $(\gamma(s), 0 \leq s \leq t)$ (hence I is also averaged under $\mathbf{n}_\theta$).

We will derive identity (5.3.5) from the Poisson point process structure of the excursion process in Proposition 5.3.6. Denote by $\mathfrak{e}_\theta^{\{\gamma(t) \in A\}}$ the Poisson point process obtained by restriction of $\mathfrak{e}_\theta$ to those excursions which satisfy $\gamma(t) \in A$. Note that the intensity measure of this point process is finite, and therefore we can consider its first jump time S_1 . Now recall (for instance from [RY99, Lemma XII.1.13]) the following classical identity:

$$\frac{\mathbf{n}_\theta(\{\gamma(t) \in A\} \cap \{\gamma \circ \Theta_t \in \tilde{\Gamma}\})}{\mathbf{n}_\theta(\gamma(t) \in A)} = \mathbb{P}(\mathfrak{e}_\theta^{\{\gamma(t) \in A\}}(S_1) \in \Theta_t^{-1}(\tilde{\Gamma})). \quad (5.3.6)$$

We can write

$$\mathfrak{e}_\theta^{\{\gamma(t) \in A\}}(S_1) = (W_{\mathfrak{t}_\theta(S_1^-)+r} - W_{\mathfrak{t}_\theta(S_1)}, 0 \leq r \leq \mathfrak{t}_\theta(S_1) - \mathfrak{t}_\theta(S_1^-)).$$

Since S_1 is a $(\mathcal{F}_{\mathfrak{t}_\theta(s)}, s \geq 0)$ -stopping time, $\mathfrak{t}_\theta(S_1^-)$ and $\mathfrak{t}_\theta(S_1)$ are $(\mathcal{F}_s, s \geq 0)$ -stopping times. For clarity, write $T = \mathfrak{t}_\theta(S_1^-) + t$. Then

$$\mathbb{P}(\mathfrak{e}_\theta^{\{\gamma(t) \in A\}}(S_1) \in \Theta_t^{-1}(\tilde{\Gamma})) = \mathbb{P}((W_{T+r} - W_J, r \leq J - T) \in \tilde{\Gamma} | W_T \in A),$$

where J is the first backward cone time of W after T such that the cone also contains $(W_s, \mathfrak{t}_\theta(S_1^-) \leq s \leq T)$. Applying the strong Markov property at time T yields

$$\mathbb{P}(\mathfrak{e}_\theta^{\{\gamma(t) \in A\}}(S_1) \in \Theta_t^{-1}(\tilde{\Gamma})) = \mathbb{P}(\mathbb{P}_{W_T}((\tilde{W}_r - \tilde{W}_I, r \leq I) \in \tilde{\Gamma}) | W_T \in A),$$

where I is the first backward cone time of $\tilde{W}$ such that the cone also contains $(W_s, s \leq T)$. Coming back to (5.3.6), we proved that

$$\frac{\mathbf{n}_\theta(\{\gamma(t) \in A\} \cap \{\gamma \circ \Theta_t \in \tilde{\Gamma}\})}{\mathbf{n}_\theta(\gamma(t) \in A)} = \mathbb{P}(\mathbb{P}_{W_T}((\tilde{W}_r - \tilde{W}_I, r \leq I) \in \tilde{\Gamma}) | W_T \in A).$$

The same argument entails that the law of $\gamma(t)$ under $\mathbf{n}_\theta(\cdot | \gamma(t) \in A)$ is that of W_T under $\mathbb{P}(\cdot | W_T \in A)$. Therefore, we conclude that

$$\mathbf{n}_\theta(\{\gamma(t) \in A\} \cap \{\gamma \circ \Theta_t \in \tilde{\Gamma}\}) = \mathbf{n}_\theta(\mathbf{1}_{\gamma(t) \in A} \cdot \mathbb{P}_{\gamma(t)}((W_s - W_I, s \leq I) \in \tilde{\Gamma})),$$

which is the desired Markov property (5.3.5). $\square$

Density of the endpoint under $\mathbf{n}_\theta$, and the normalised backward cone excursion measure. We will want to relate the two types of cone excursions in the limit when the starting point is taken to the boundary. A straightforward consequence of the results of [LG87] concerning the $(2 - \lambda)$ -stable process is that we can disintegrate the backward measure $\mathbf{n}_\theta$ over the starting point.

Proposition 5.3.9. *We have the following disintegration formula for $\mathbf{n}_\theta$ in polar coordinates:*

$$\mathbf{n}_\theta = \int_{\mathcal{C}_\theta} \frac{dr}{r^{3-\lambda}} \mathbf{m}(d\phi) Q_\theta^{re^{i\phi}}, \quad (5.3.7)$$

where $\mathbf{m}$ is the finite positive measure on $[0, \theta]$ which appears in (5.3.2) and the $Q_\theta^{re^{i\phi}}$ are probability measures supported on cone excursions from $re^{i\phi} \in \mathcal{C}_\theta$ to the apex 0.

Remark 5.3.10. Recall from equation (5.3.3) that in the case when $\theta = \frac{2\pi}{3}$, a formula for μ is known. In this case, the disintegration reads more conveniently if we switch away from polar coordinates:

$$\mathbf{n}_\theta = \int_{x,y \geq 0} \frac{C dx dy}{(x+y)^{5/2}} Q^{x+iy}. \quad (5.3.8)$$

Proof of Proposition 5.3.9. By definition of $\mathbf{n}_\theta$ as the intensity measure of the Poisson point process of excursions, we know that for any measurable $A \in \mathbb{C}$,

$$\mathbf{n}_\theta(\mathbf{1}_{\gamma(0) \in A} \cdot f(\gamma)) = \mathbb{E} \left(\sum_{0 < s \leq 1} \mathbf{1}_{\Delta \mathbf{e}_\theta(s) \in A} \cdot f(\mathbf{e}_\theta(s)) \right),$$

where $\Delta \mathbf{e}_\theta(s)$ consists in taking the difference between the starting point and endpoint of the excursion $\mathbf{e}_\theta(s)$ (with the convention that the indicator is zero when $\mathbf{e}_\theta(s) = \diamond$). It remains to notice that the quantities $\Delta \mathbf{e}_\theta(s)$ correspond exactly to the jumps of the process $W \circ \mathbf{t}_\theta$. We now use Theorem 5.3.4 borrowed from [LG87] to express the above expectation. Indeed, we know that $W \circ \mathbf{t}_\theta$ is a $(2 - \lambda)$ -stable Lévy process in the plane, hence its Lévy measure has the form given by (5.3.2). An application of the compensation formula then yields

$$\mathbf{n}_\theta(\mathbf{1}_{\gamma(0) \in A} \cdot f(\gamma)) = \int_{\mathcal{C}_\theta} \mathfrak{L}_\theta(dr, d\phi) Q_\theta^{re^{i\phi}}(f). \quad (5.3.9)$$

Plugging (5.3.2) into (5.3.9), we obtain the desired identity. $\square$

We now want to relate the backward excursion measure $\mathbf{n}_\theta$ and the forward one $\mathbf{m}_\theta$ when the starting point under $\mathbf{n}_\theta$ is taken to the boundary. For practical purposes, we will take the viewpoint of correlated Brownian motions (*i.e.* now we consider that P_θ^z and $Q_\theta^{re^{i\phi}}$ are measures on excursions in the positive quadrant $\mathbb{R}_+^2$). For $\varepsilon > 0$, let $B_+(z, \varepsilon)$ be the intersection of the ball with radius ε around z and the quadrant $\mathbb{R}_+^2$.

Proposition 5.3.11. *Let $z \in \partial \mathbb{R}_+^2$ such that $\Im(z) = 0$ and $\Re(z) > 0$. Then the law of γ under $\mathbf{n}_\theta(\cdot | \gamma(0) \in B_+(z, \varepsilon))$ converges weakly as $\varepsilon \rightarrow 0$ to P_θ^z .*

Proof. We use [MS19, Theorem 3.1], which gives a uniqueness result for the excursion law P_θ^z . It characterises a correlated Brownian excursion of duration 1 in the quadrant, from 0 to $z \in \partial \mathbb{R}_+^2$, as the unique process such that conditioned on any initial and final segment of the process, the law of the middle part is a correlated Brownian bridge conditioned to stay in the quadrant. Fix $z \in \partial \mathbb{R}_+^2$.

Write n_1 for the law of a backward cone excursion under $\mathbf{n}_\theta$ conditioned to have lifetime 1. Consider $0 < s < t < 1$. Let A be the event that for all $u \in [s, t]$, $\gamma(u)$ stays in the quadrant with apex at the origin. Let B be the event that $\gamma|_{[s, t]}$ has no backward cone point below the whole path $\gamma([0, s])$. Then we claim that given $\gamma|_{[0, s]}$ and $\gamma|_{[t, 1]}$, the law of $\gamma|_{[s, t]}$ under n_1 is that of a Brownian bridge from $\gamma(s)$ to $\gamma(t)$, conditioned on the event $A \cap B$. Indeed, by the Markov property under $\mathbf{n}_\theta$ (Proposition 5.3.7), we know that conditioned on the past until time s , $\gamma|_{[s, 1]}$ is a planar Brownian motion such that the first cone point falling below $(\gamma(u), u \leq s)$ occurs at time 1. Conditioning on $\gamma|_{[t, 1]}$ then yields the claim.

Thus, we get that for all nonnegative bounded measurable functions F, g_1 and g_2 ,

$$\begin{aligned} n_1 & \left(F(\gamma(u), s \leq u \leq t) g_1(\gamma(u), u \leq s) g_2(\gamma(u), t \leq u \leq 1) \mathbf{1}_{\gamma(0) \in B_+(z, \varepsilon)} \right) \\ &= n_1 \left(g_1(\gamma(u), u \leq s) g_2(\gamma(u), t \leq u \leq 1) \mathbf{1}_{\gamma(0) \in B_+(z, \varepsilon)} \mathbb{E}_{t-s}^{\gamma(s) \rightarrow \gamma(t)} (F(w) \cdot \mathbf{1}_{w \in A \cap B}) \right), \end{aligned} \quad (5.3.10)$$

where $\mathbb{E}_u^{a \rightarrow b}$ denotes the law of a Brownian bridge w between a and b with duration u . Now, note that, on the event that $\gamma(0) \in B_+(z, \varepsilon)$,

$$F(w) \cdot \mathbf{1}_{w \in A \cap B} \xrightarrow{\varepsilon \rightarrow 0} F(w) \cdot \mathbf{1}_{w \in A},$$

uniformly over all paths w . Hence taking a scaling limit in (5.3.10) entails that $\mathbf{n}_\theta(\cdot | \gamma(0) \in B_+(z, \varepsilon))$ converges weakly to a probability measure satisfying the criterion of [MS19, Theorem 3.1]. This gives the characterisation statement for a forward cone excursion of duration 1, which is then easily extended to any duration. $\square$

5.3.3 A Bismut description of the backward cone excursion measure $\mathbf{n}_\theta$

The classical Bismut description deals with the one-dimensional Itô measure and roughly describes the infinite excursion measure seen from a time t chosen according to the Lebesgue measure on the excursion lifespan (see [RY99, Theorem XII.4.7]). Although it is a straightforward consequence of the one-dimensional case, the Bismut description of Brownian excursions in the half-plane [AS20, Proposition 2.6] is perhaps more relevant to our case, since it corresponds intuitively to taking $\theta = \pi$. In this case, it first states that the height at time t is distributed according to the Lebesgue measure da . Secondly, it describes the left and right parts of the trajectory from time t onwards as two independent Brownian motion stopped when reaching the horizontal line $\{z \in \mathbb{C}, \Im(z) = -a\}$. The nature of the cone excursions make the Bismut description of $\mathbf{n}_\theta$ more involved, although it remains similar in spirit. Recall the definition of $\mathcal{L}_t^t$ from the last paragraph of Section 5.3.1 (the construction there is done for excursions starting from the boundary, but one can also consider excursions starting from an interior point). See Figure 5.12 below.

Theorem 5.3.12. (*Bismut description of $\mathbf{n}_\theta$*)

Let $\bar{\mathbf{n}}_\theta$ be the measure on $\mathbb{R}_+ \times \Gamma$ defined by

$$\bar{\mathbf{n}}_\theta(dt, d\gamma) = \mathbf{1}_{0 \leq t \leq \zeta} dt \cdot \mathbf{n}_\theta(d\gamma).$$

Then under $\bar{\mathbf{n}}_\theta$, $\mathcal{L}_t^t$ is distributed according to the Lebesgue measure $c'da$, where c' is some (explicit) constant. Moreover, conditionally on $\mathcal{L}_t^t = a$, the law of $\gamma^{t,-} := (\gamma(t-s) - \gamma(t), 0 \leq s \leq t)$ and $\gamma^{t,+} := (\gamma(t+s) - \gamma(t), 0 \leq s \leq \zeta - t)$ is described as follows. First, $\gamma^{t,-}$ is a planar Brownian motion run until time $\tau_\theta(a)$ when its cone free local time equals a . Then, $\gamma^{t,+}$ has the law of an independent planar Brownian motion, stopped at its first backward cone point for which the cone also contains the whole path $\gamma^{t,-}$.

Remark 5.3.13. It is important to point out that, unlike in the one-dimensional or in the half-plane case, the paths $\gamma^{t,-}$ and $\gamma^{t,+}$ are no longer independent conditionally on $\mathcal{L}_t^t$. This dependence makes the Bismut description of $\mathbf{n}_\theta$ much more involved, although we stress that the only dependence concerns the stopping time for $\gamma^{t,+}$.

Proof of Theorem 5.3.12. Our proof relies on size-biased sampling of $\mathbf{n}_\theta$ related to the famous *waiting time paradox* of Kingman [Kin92]. Indeed, it is a general fact (see for

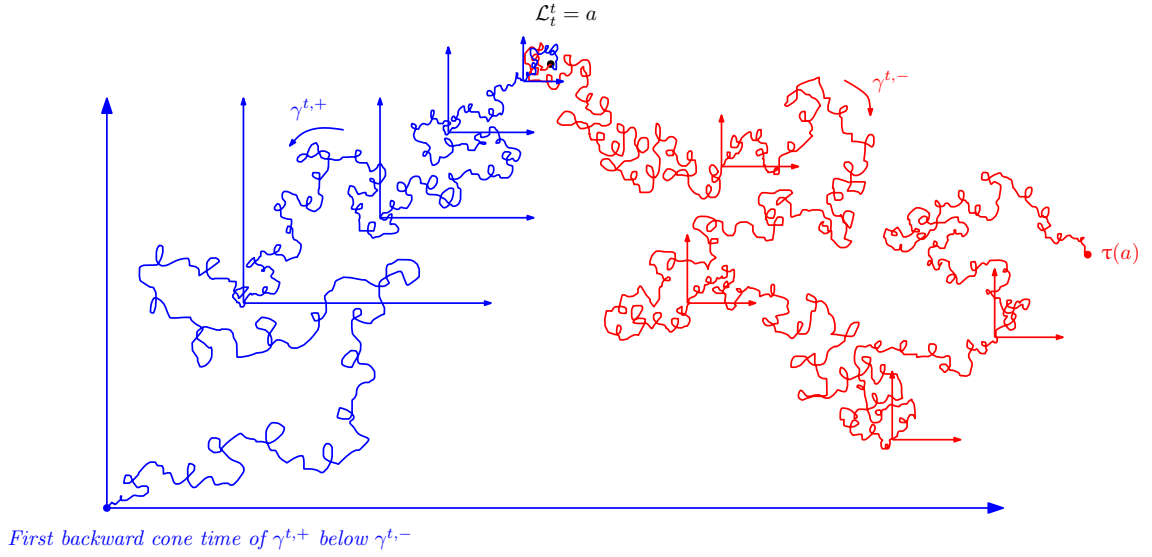


Figure 5.12 – The Bismut description of $\mathfrak{n}_\theta$.

example [BVHS99]) that the size-biased measure $\bar{\mathfrak{n}}_\theta$ can be obtained in the limit when considering the backward cone excursion straddling time $t > 0$ in a planar Brownian motion and taking $t \rightarrow \infty$. The intuition is that such an excursion should be biased because it straddles this fixed time t .

Under law $\mathbb{P}$, let W be a planar Brownian motion started at the origin. Fix $t > 0$. As in the last paragraph of Section 5.3.1, we can define $\mathcal{L}_t^t$ by recording all the forward cone excursions between times 0 and t . Let us prove the first claim concerning the ‘law’ of $\mathcal{L}_t^t$. Take any nonnegative measurable function f . By definition, $\mathcal{L}_t^t$ is the *cone-free local time* of the time-reversed trajectory from time t to 0. We shall therefore consider the time-reversal $\widehat{W}$ of W from t to 0, shifted by $-W_t$ so that it starts at 0. Then $\widehat{W}$ is a Brownian motion stopped at time t . In particular, we denote by $(\tau_\theta(s), s > 0)$ the inverse of the local time, and $(\mathfrak{e}_\theta(s), s > 0)$ the forward cone excursion process associated with $\widehat{W}$. We want to decompose $\mathcal{L}_t^t$ over the excursion process $(\mathfrak{e}_\theta(s), s > 0)$. This provides the simple identity:

$$\mathbb{E}(f(\mathcal{L}_t^t)) = \mathbb{E}\left(\sum_{s>0} f(s) \mathbb{1}_{\zeta(\mathfrak{e}_\theta(s)) > t - \tau_\theta(s^-)} \mathbb{1}_{t > \tau_\theta(s^-)}\right). \quad (5.3.11)$$

We stress that the reason for the indicator in (5.3.11) is because we want to count the cone-free local time until time t ; hence we stop the excursion process $(\mathfrak{e}_s, s > 0)$ when we start tracing a forward cone excursion straddling 0 in the original picture (see Figure 5.13). Now we can use the compensation formula for the Poisson point process in (5.3.11). We get

$$\mathbb{E}(f(\mathcal{L}_t^t)) = \mathbb{E}\left(\int_0^\infty da f(a) \cdot \mathfrak{n}_\theta(\zeta > t - T_a^{\widehat{W}}) \mathbb{1}_{t > T_a^{\widehat{W}}}\right), \quad (5.3.12)$$

where $T_a^{\widehat{W}}$ is the inverse cone-free local time at time a for $\widehat{W}$ and is averaged under the first expectation. We are therefore left with the tail distribution of the duration under the forward excursion measure $\mathfrak{n}_\theta$.

We take a shortcut here and use the results of [AG21] determining the law of the duration under $\mathfrak{n}_\theta$. Actually, their results are much finer since they derive the law of the duration under the *normalised* excursion P_θ^z , but we will only need the result under $\mathfrak{n}_\theta$. Recall that

we have set $\lambda = \pi/\theta = \kappa'/4$. We know from [AG21, Theorem 1.2] that

$$P_\theta^1(\zeta \in dt) = \frac{1}{ct^{1+\lambda}} \exp\left(-\frac{1}{2(\mathfrak{a} \sin(\theta))^2 t}\right) dt, \quad (5.3.13)$$

where

$$c = 2^\lambda \Gamma(\lambda) (\mathfrak{a} \sin(\theta))^{2\lambda}. \quad (5.3.14)$$

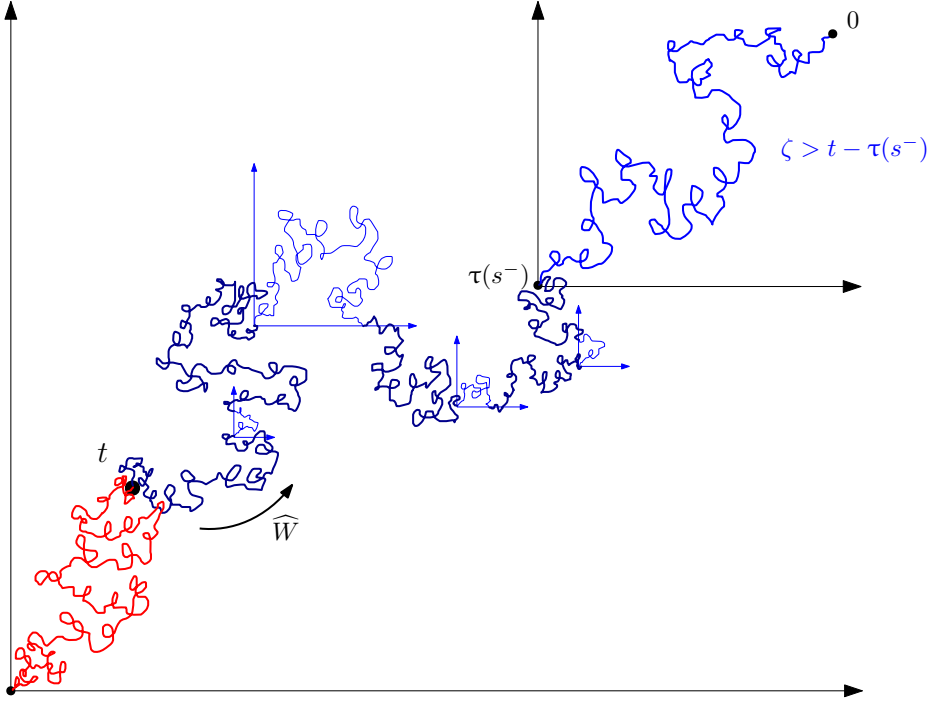


Figure 5.13 – The backward cone excursion straddling t viewed in the correlated framework. We start a Brownian motion from 0, and look at the backward cone excursion straddling large time t (labelled by the two black cones). Looking back from time t (blue trajectory), we record all the forward cone excursions (bright blue). We define $\mathcal{L}_t^i$ as the local time on the complement on these excursions. The excursion process is stopped at time s when reaching an excursion such that $\zeta(\mathfrak{e}_\theta(s)) > t - \tau(s^-)$ (the last excursion in bold blue).

Using Proposition 5.3.2 and (5.3.13) with a scaling argument, we have that for any nonnegative measurable function f ,

$$\begin{aligned} \mathfrak{m}_\theta(f(\zeta)) &= \int_{\partial\mathcal{C}_\theta} \frac{dz}{\theta|z|^{1+\lambda}} \int_0^\infty \frac{1}{ct^{1+\lambda}} \exp\left(-\frac{1}{2(\mathfrak{a} \sin(\theta))^2 t}\right) f(|z|^2 t) dt \\ &= \int_0^\infty ds f(s) \int_{\partial\mathcal{C}_\theta} \frac{dz}{\theta|z|^{1+\lambda}} \frac{|z|^{2\lambda}}{cs^{1+\lambda}} \exp\left(-\frac{|z|^2}{2(\mathfrak{a} \sin(\theta))^2 s}\right) \\ &= \int_0^\infty ds f(s) \int_{\partial\mathcal{C}_\theta} dz \frac{|z|^{\lambda-1}}{c\theta s^{1+\lambda}} \exp\left(-\frac{|z|^2}{2(\mathfrak{a} \sin(\theta))^2 s}\right). \end{aligned}$$

The change of variables $u := z/\sqrt{s}$ triggers

$$\begin{aligned} \mathfrak{m}_\theta(f(\zeta)) &= \int_0^\infty ds f(s) \int_{\partial\mathcal{C}_\theta} du \frac{\sqrt{s}}{(\sqrt{s})^{1-\lambda}} \frac{|u|^{\lambda-1}}{c\theta s^{1+\lambda}} \exp\left(-\frac{|u|^2}{2(\mathfrak{a} \sin(\theta))^2}\right) \\ &= \int_0^\infty \frac{ds}{c\theta s^{1+\frac{\lambda}{2}}} f(s) \int_{\partial\mathcal{C}_\theta} du |u|^{\lambda-1} \exp\left(-\frac{|u|^2}{2(\mathfrak{a} \sin(\theta))^2}\right). \end{aligned}$$

A back-of-the-envelope calculation finally yields

$$\mathfrak{n}_\theta(f(\zeta)) = \int_0^\infty \frac{2^{\lambda/2} (a \sin(\theta))^\lambda \Gamma(\frac{\lambda}{2}) ds}{c \theta s^{1+\frac{\lambda}{2}}} f(s), \quad (5.3.15)$$

which may further be simplified using equation (5.3.14). In any case, we get that the tail distribution of ζ under $\mathfrak{n}_\theta$ is

$$\mathfrak{n}_\theta(\zeta > T) = \frac{c'}{T^{\lambda/2}}, \quad (5.3.16)$$

for some (explicit) constant c' . Coming back to (5.3.12), we get

$$\frac{1}{t^{\lambda/2}} \mathbb{E}(f(\mathcal{L}_t^t)) \xrightarrow{t \rightarrow \infty} c' \int_0^\infty da f(a).$$

This proves the first claim of Theorem 5.3.12. The second claim on the distribution of $\gamma^{t,-}$ and $\gamma^{t,+}$ follows the same ideas, but involves cumbersome notation, so that we only mention the main ideas. Essentially, one takes two other functions g and h of the past and future trajectories $\gamma^{t,-}$ and $\gamma^{t,+}$, uses the Markov property at time t and then decomposes the past trajectory over the excursion process as in (5.3.11). Then, one is basically left with the same tail distribution under $\mathfrak{n}_\theta$ as in (5.3.12), together with a function g of the past trajectory run until $T_a^{\widehat{W}}$, and a function h of the future trajectory run until its first cone time for which the cone also contains the past trajectory. The estimate (5.3.16) again yields the result when taking the scaling limit. $\square$

5.3.4 A Brownian motion interpretation of the spectrally positive $3/2$ -stable process conditioned to remain positive

It is known that the growth-fragmentation processes $\mathbf{X}^\theta$ corresponding to (5.1.6) are closely related to θ -stable processes conditioned to remain positive or to be absorbed continuously at 0, which appear as a spine in the growth-fragmentation structure (see [BBCK18]). Moreover, these processes can also be seen in the scaling limit from the planar map perspective [BBCK18, Proposition 6.6]. For example, the stable process conditioned to be absorbed continuously shows up when considering *pointed* planar maps, which are a size-biased version of the planar maps considered in [BBCK18], very similar in spirit to $\mathfrak{n}_\theta$. We state a conjecture in this vein, and prove that this conjecture yields another proof of Theorem A only using Brownian motion arguments without reference to LQG (other than the mating-of-trees dictionary).

Conjecture 5.3.14. *Let W and W' be two independent planar Brownian motions starting from the origin. For $a \geq 0$, consider the $\frac{3}{2}$ -stable process $(W'(\tau(t)), t \leq a)$ corresponding to the forward $\frac{2\pi}{3}$ -cone times up to time a . Introduce the first passage time*

$$\mathfrak{s}(a) := \inf\{s \geq 0, W'(\tau(t)) \in W(\mathfrak{t}(s)) + \mathcal{C}_\theta \text{ for all } t \leq a\},$$

of the $\frac{1}{2}$ -stable process $(W(\mathfrak{t}(t)), t \geq 0)$ below the path $(W'(\tau(t)), t \leq a)$. Let $S(a)$ be the sum of the coordinates of $W' \circ \tau(a) - W \circ \mathfrak{t}(\mathfrak{s}(a))$. Then S evolves as a spectrally positive $\frac{3}{2}$ -stable Lévy process conditioned to remain positive.

Let us conclude this work by a tentative proof of Theorem A directly in the Brownian framework, subject to Conjecture 5.3.14. More precisely, we will show how Theorem 5.3.12 and Conjecture 5.3.14 together determine the law of the locally largest fragment from the cone excursion viewpoint. The argument is similar to that of [AS20, Theorem 3.3], which roughly corresponds to the case $\theta = \pi$ (see also [LGR20, Sil21]). Note also that, in order

to prove Theorem A (and not only determine the law of the locally largest fragment), one would also need to establish the conditional independence of the offspring, which corresponds in our setting to the conditional independence of the cut-out cone excursions (either to the left or to the right) in the locally largest exploration, as in [AS20, Theorem 3.6].

We now briefly mention the main arguments towards proving Theorem A. From now on, we fix $\theta = \frac{2\pi}{3}$, and drop the subscript θ for ease of notation (in what follows, we view cone excursions as their correlated version in the positive quadrant). Let H be a bounded continuous nonnegative function defined on the space of finite càdlàg paths, and $a \geq 0$. We denote by t^* the random time corresponding to the locally largest evolution in the cone excursion under $\mathbf{n}$, and Z^* the size of the fragment targeted at t^* in the branching process $\mathbf{Z}$ of (5.3.4). Recall the notation S from Conjecture 5.3.14. The first step is to prove that

$$\mathbf{n}(H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}}) = \int_{x, y \geq 0} \frac{C dx dy}{(x + y)^{5/2}} h(-a, x + y), \quad (5.3.17)$$

where C is the constant appearing in (5.3.3), and h is

$$h(-a, z) := \mathbb{E}\left(H(z + S(a - b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S(a-b) > -\Delta S(a-b)}\right). \quad (5.3.18)$$

Indeed, integrating over all points t above local time a and denoting by Z^t the size of the fragment targeted at t , we see that for $\mathbf{n}$ -almost every excursion γ ,

$$H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}} = \int_0^{\zeta(\gamma)} H(Z^t(b), b \in [0, a]) \mathbb{1}_{\{\mathcal{L}_t^t > a\} \cap \mathcal{E}_a^t} \frac{dt}{\zeta(\gamma_a^{(t)})}, \quad (5.3.19)$$

where $\gamma_a^{(t)}$ is the subpath of γ between $g_t(a)$ and $d_t(a)$, and

$$\mathcal{E}_a^t = \left\{ \forall b \in [0, a], Z^t(b) > -\Delta Z^t(b) \right\},$$

is the event that the fragment targeted at t follows the locally largest evolution up to (local) time a . Now observe that, taking the $\mathbf{n}$ -expectation of (5.3.19) and applying the Bismut description of $\mathbf{n}$ (Theorem 5.3.12), we obtain with the notation of Conjecture 5.3.14,

$$\mathbf{n}(H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}}) = c' \int_a^\infty dA \cdot \mathbb{E}\left(\frac{H(S(A - b), b \in [0, a])}{\tau(A - a) + \mathbf{t}(\mathbf{s}(A - a))} \mathbb{1}_{\forall b \in [0, a], S(A-b) > -\Delta S(A-b)}\right).$$

Then we use Conjecture 5.3.14. By the strong Markov property of Brownian motion at times $\tau(A - a)$ and $\mathbf{t}(\mathbf{s}(A - a))$,

$$\mathbf{n}(H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}}) = c' \int_a^\infty dA \cdot \mathbb{E}\left(\frac{h(-a, S(A - a))}{\tau(A - a) + \mathbf{t}(\mathbf{s}(A - a))}\right),$$

with h defined as in (5.3.18). A simple change of variables yields

$$\mathbf{n}(H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}}) = c' \int_0^\infty dA \cdot \mathbb{E}\left(\frac{h(-a, S(A))}{\tau(A) + \mathbf{t}(\mathbf{s}(A))}\right). \quad (5.3.20)$$

On the other hand, using the Bismut description (Theorem 5.3.12) backwards, the right-hand side of (5.3.20) is

$$c' \int_0^\infty dA \cdot \mathbb{E}\left(\frac{h(-a, S(A))}{\tau(A) + \mathbf{t}(\mathbf{s}(A))}\right) = \mathbf{n}(h(-a, \gamma(0))),$$

where $\gamma(0)$ is here implicitly understood as the sum of its coordinates. Thus, (5.3.20) boils down to

$$\mathbf{n}(H(Z^*(b), b \in [0, a]) \mathbb{1}_{\{a < \mathcal{L}_{t^*}^*\}}) = \mathbf{n}(h(-a, \gamma(0))).$$

Identity (5.3.17) is retrieved by using the density of the starting point under $\mathbf{n}$ (see Proposition 5.3.9 and formula (5.3.8) in the remark following it).

By a simple change of variables, formula (5.3.17) rewrites as

$$\mathbf{n}(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) = \int_0^\infty \frac{C d\ell}{\ell^{3/2}} h(-a, \ell).$$

The function h defined in (5.3.18) involves the time-reversal of S . According to Conjecture 5.3.14, S itself is a stable process conditioned to stay positive. Now recall for example from [CC06] that the spectrally positive $\frac{3}{2}$ -stable process conditioned to remain positive can be written as a Doob h -transform of the spectrally positive $\frac{3}{2}$ -stable process S_+^0 killed when entering the negative half-line, with harmonic function $g(x) = x$. As a consequence,

$$\begin{aligned} \mathbf{n}(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) \\ = \int_0^\infty C d\ell E_\ell \left(\frac{S_+^0(a)}{\ell^{5/2}} H(S_+^0(a-b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_+^0(a-b) > -\Delta S_+^0(a-b)} \right), \end{aligned}$$

where under P_ℓ , $S_+^0(0) = \ell$. Now, we use duality with respect to the Lebesgue measure of the killed Lévy process S_+^0 (see [Ber96, Section II.1]). Let $S_-^0 := -S_+^0$ be the spectrally negative $\frac{3}{2}$ -stable process killed above 0. This provides

$$\begin{aligned} \mathbf{n}(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) \\ = \int_0^\infty C d\ell E_\ell \left(\frac{\ell}{S_-^0(a)^{5/2}} H(S_-^0(b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_-^0(b) > -\Delta S_-^0(b)} \right). \end{aligned}$$

Disintegrating $\mathbf{n}$ over the endpoint as in Remark 5.3.10, we get that

$$\begin{aligned} \int_{x, y \geq 0} \frac{dx dy}{(x+y)^{5/2}} Q^{x+iy}(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) \\ = \int_0^\infty d\ell E_\ell \left(\frac{\ell}{S_-^0(a)^{5/2}} H(S_-^0(b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_-^0(b) > -\Delta S_-^0(b)} \right). \end{aligned}$$

The law of Z^* under Q^{x+iy} only depends on the sum $x+y$; when $x+y = \ell$, we use the slight abuse of notation Q^ℓ to denote Q^{x+iy} . Then by a change of variables,

$$\begin{aligned} \int_0^\infty \frac{d\ell}{\ell^{3/2}} Q^\ell(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) \\ = \int_0^\infty d\ell E_\ell \left(\frac{\ell}{S_-^0(a)^{5/2}} H(S_-^0(b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_-^0(b) > -\Delta S_-^0(b)} \right). \end{aligned}$$

Now, take another function f of $Z^*(0)$ in H . The previous identity becomes

$$\begin{aligned} \int_0^\infty \frac{d\ell}{\ell^{3/2}} f(\ell) Q^\ell(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) \\ = \int_0^\infty d\ell f(\ell) E_\ell \left(\frac{\ell}{S_-^0(a)^{5/2}} H(S_-^0(b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_-^0(b) > -\Delta S_-^0(b)} \right). \end{aligned}$$

This equality holds for all nonnegative measurable function f , and a continuity argument brings

$$Q^\ell(H(Z^*(b), b \in [0, a])\mathbb{1}_{\{a < \mathcal{L}_{t^*}^{t^*}\}}) = E_\ell \left(\frac{\ell^{5/2}}{S_-^0(a)^{5/2}} H(S_-^0(b), b \in [0, a]) \mathbb{1}_{\forall b \in [0, a], S_-^0(b) > -\Delta S_-^0(b)} \right).$$

This essentially establishes the law of Z^* as a Doob h -transform of S_-^0 . In particular, it gives that Z^* is a positive self-similar Markov process with index $\frac{3}{2}$. We then short-circuit the derivation of the Lamperti exponent of Z^* using Lemma 5.2.9 as an input, which enables to recover the Laplace exponent (5.1.7). Finally, we leverage Proposition 5.3.11 to establish the result under P^z .

As a side remark, note that we argued under $\mathfrak{n}$ rather than $\mathfrak{n}$ all the way. This is surprising in view of the Liouville quantum gravity approach (Section 5.2.3), where this problem did not appear. In fact, the measure $\mathfrak{n}$ describes another type of quantum disc (with a different *weight* $W = \frac{3\gamma^2}{2} - 2$ related to the angle of the cone). We believe that the results obtained under $\mathfrak{n}$ in this work may be useful to derive properties for both these weight- W quantum discs and the usual (weight $W = 2$) quantum discs which were our original interest.

Bibliography

- [ACGŻ17] L. Alili, L. Chaumont, P. Graczyk, and T. Żak. Inversion, duality and Doob h -transforms for self-similar Markov processes. *Electronic Journal of Probability*, **22**:1–18, 2017.
- [AG21] M. Ang and E. Gwynne. Liouville quantum gravity surfaces with boundary as matings of trees. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, **57**(1):1–53, 2021.
- [AHPS21] J. Aru, N. Holden, E. Powell, and X. Sun. Mating of trees for critical Liouville quantum gravity, 2021. <https://arxiv.org/abs/2109.00275>.
- [AHS20] E. Aïdékon, Y. Hu, and Z. Shi. Points of infinite multiplicity of planar Brownian motion: measures and local times. *The Annals of Probability*, **48**(4):1785–1825, 2020.
- [AHS21] M. Ang, N. Holden, and X. Sun. Integrability of SLE via conformal welding of random surfaces. *arXiv:2104.09477*, 2021. <https://arxiv.org/abs/2104.09477>.
- [Aïd13] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *The Annals of Probability*, **41**(3A):1362–1426, 2013.
- [AP98] D. Aldous and J. Pitman. The standard additive coalescent. *The Annals of Probability*, **26**(4):1703–1726, 1998.
- [ARS21] M. Ang, G. Remy, and X. Sun. FZZ formula of boundary Liouville CFT via conformal welding. *arXiv:2104.09478*, 2021. <https://arxiv.org/abs/2104.09478>.
- [AS20] E. Aïdékon and W. Da Silva. Growth-fragmentation process embedded in a planar Brownian excursion. *arXiv:2005.06372*, 2020. <https://arxiv.org/abs/2005.06372>.
- [Asm08] S. Asmussen. *Applied probability and queues*, volume **51**. Springer Science & Business Media, 2008.
- [BBCK18] J. Bertoin, T. Budd, N. Curien, and I. Kortchemski. Martingales in self-similar growth-fragmentations and their connections with random planar maps. *Probab. Th. Rel. Fields*, **172**:663–724, 2018.
- [BBG11] G. Borot, J. Bouttier, and E. Guitter. A recursive approach to the $O(n)$ model on random maps via nested loops. *Journal of Physics A: Mathematical and Theoretical*, **45**(4), 2011.

- [BBG12] G. Borot, J. Bouttier, and E. Guitter. Loop models on random maps via nested loops: the case of domain symmetry breaking and application to the Potts model. *Journal of Physics A: Mathematical and Theoretical*, **45**(49), 2012.
- [BC17] T. Budd and N. Curien. Geometry of infinite planar maps with high degrees. *Electronic Journal of Probability*, **22**:1–37, 2017.
- [BCK18] J. Bertoin, N. Curien, and I. Kortchemski. Random planar maps and growth-fragmentations. *The Annals of Probability*, **46**(1):207–260, 2018.
- [Ber96] J. Bertoin. *Lévy processes*. Cambridge Tracts in Mathematics, **121**, Cambridge University Press, Cambridge, 1996.
- [Ber00] J. Bertoin. A fragmentation process connected to Brownian motion. *Probability Theory and Related Fields*, **117**(2):289–301, 2000.
- [Ber02] J. Bertoin. Self-similar fragmentations. *Annales de l’I.H.P. Probabilités et statistiques*, **38**(3):319–340, 2002.
- [Ber07] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electronic Journal of Combinatorics*, **14**, 2007.
- [Ber17a] N. Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electronic communications in Probability*, **22**:1–12, 2017.
- [Ber17b] J. Bertoin. Markovian growth-fragmentation processes. *Bernoulli*, **23**:1082–1101, 2017.
- [Bet15] J. Bettinelli. Scaling limit of random planar quadrangulations with a boundary. **51**(2):432–477, 2015.
- [BG20] N. Berestycki and E. Gwynne. Random walks on mated-CRT planar maps and Liouville Brownian motion. *arXiv preprint:2003.10320*, 2020. <https://arxiv.org/abs/2003.10320>.
- [BGR61] R. M. Blumenthal, R. K. Gettoor, and D.B. Ray. On the distribution of first hits for the symmetric stable processes. *Transactions of the American Mathematical Society*, **99**(3):540–554, 1961.
- [BHS21] O. Bernardi, N. Holden, and X. Sun. Percolation on triangulations: a bijective path to Liouville quantum gravity. *arXiv preprint:1807.01684*, 2021. <https://arxiv.org/abs/1807.01684>.
- [Big77] J.D. Biggins. Martingale convergence in the branching random walk. *Journal of Applied Probability*, **14**(1):25–37, 1977.
- [BK04] J. D. Biggins and A. E. Kyprianou. Measure change in multitype branching. *Advances in Applied Probability*, **36**(2):544–581, 2004.
- [BM17] J. Bettinelli and G. Miermont. Compact brownian surfaces I: Brownian disks. *Probability Theory and Related Fields*, **167**(3-4):555–614, 2017.
- [BMR19] E. Baur, G. Miermont, and G. Ray. Classification of scaling limits of uniform quadrangulations with a boundary. *The Annals of Probability*, **47**(6):3397–3477, 2019.
- [BP21] N. Berestycki and E. Powell. Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos. *Lecture notes*, 2021.
- [BPR20] N. Berestycki, E. Powell, and G. Ray. A characterisation of the Gaussian free field. *Probability Theory and Related Fields*, **176**(3):1259–1301, 2020.
- [Bud15] T. Budd. The peeling process of infinite Boltzmann planar maps. *The Electronic Journal of Combinatorics*, (**23**):1–28, 2015.

- [Bud18] T. Budd. The peeling process on random planar maps coupled to an $O(n)$ -loop model (with an appendix by Linxiao Chen). *arXiv preprint arXiv:1809.02012*, 2018. <https://arxiv.org/abs/1809.02012>.
- [Bur85] K. Burdzy. Brownian paths and cones. *The Annals of Probability*, **13**(3):1006–1010, 1985.
- [Bur86] K. Burdzy. Brownian excursions from hyperplanes and smooth surfaces. *Transactions of the American mathematical society*, 1986.
- [BVHS99] J. Bertoin, K. Van Harn, and F. W. Steutel. Renewal theory and level passage by subordinators. *Statistics & probability letters*, **45**(1):65–69, 1999.
- [BW96] Jean Bertoin and Wendelin Werner. Stable windings. *The Annals of Probability*, pages 1269–1279, 1996.
- [BY05] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probability Surveys*, **2**, 2005.
- [CC06] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *Journal of Applied Probability*, **43**(4):967–983, 2006.
- [CCM20] L. Chen, N. Curien, and P. Maillard. The perimeter cascade in critical Boltzmann quadrangulations decorated by an $o(n)$ loop model. *Annales de l’Institut Henri Poincaré D.*, **7**(4):535–584, 2020.
- [Cer19] B. Cerclé. Unit boundary length quantum disk: a study of two different perspectives and their equivalence. *arXiv preprint:1912.08012*, 2019. <https://arxiv.org/abs/1912.08012>.
- [Che17] L. Chen. Basic properties of the infinite critical-FK random map. *Annales de l’Institut Henri Poincaré D*, **4**(3):245–271, 2017.
- [Che18] L. Chen. *Random Planar Maps coupled to Spin Systems*. PhD thesis, Université Paris-Saclay, 2018.
- [Chy06] O. Chybiryakov. The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stochastic Processes and their Applications*, **116**(5):857–872, 2006.
- [Çin75] E. Çinlar. Lévy systems of Markov additive processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **31**(3):175–185, 1975.
- [CPR13] L. Chaumont, H. Pantí, and V. Rivero. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, **19**:2494–2523, 2013.
- [CPY97] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. *Exponential functionals and principal values related to Brownian motion*, pages 73–121, 1997.
- [Cur19] N. Curien. Peeling random planar maps, 2019. <https://www.imo.universite-paris-saclay.fr/~curien/cours/peccot.pdf>.
- [DC13] H. Duminil-Copin. Parafermionic observables and their applications to planar statistical physics models. *Ensaio Matemáticos*, **25**:1–371, 2013.
- [DDK17] S. Dereich, L. Döring, and A. E. Kyprianou. Real self-similar processes started from the origin. *The Annals of Probability*, **45**(3):1952–2003, 2017.
- [Dei10] K. Deimling. *Nonlinear functional analysis*. Courier Corporation, 2010.
- [DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *arXiv preprint arXiv:1409.7055*, 2014. <https://arxiv.org/abs/1409.7055>.
- [DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Inventiones mathematicae*, **185**(2):333–393, 2011.

- [Eva85] S. N. Evans. On the Hausdorff dimension of Brownian cone points. *Mathematical Proceedings of the Cambridge Philosophical Society*, **98**(2):343–353, 1985.
- [GHMS17] E. Gwynne, N. Holden, J. Miller, and X. Sun. Brownian motion correlation in the peanosphere for $\kappa > 8$. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, **53**(4):1866–1889, 2017.
- [GHS19] E. Gwynne, N. Holden, and X. Sun. Mating of trees for random planar maps and Liouville quantum gravity: a survey. *arXiv preprint:1910.04713*, 2019. <https://arxiv.org/abs/1910.04713>.
- [GM17] E. Gwynne and J. Miller. Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov-Hausdorff-Prokhorov-uniform topology. *Electronic Journal of Probability*, **22**:1–47, 2017.
- [GM18] E. Gwynne and J. Miller. Chordal SLE₆ explorations of a quantum disk. *Electronic Journal of Probability*, **23**:1–24, 2018.
- [GM19] E. Gwynne and J. Miller. Convergence of the free Boltzmann quadrangulation with simple boundary to the Brownian disk. **55**(1):551–589, 2019.
- [GMS17] E. Gwynne, J. Miller, and S. Sheffield. The Tutte embedding of the mated-CRT map converges to Liouville quantum gravity. *arXiv preprint arXiv:1705.11161*, 2017. <https://arxiv.org/abs/1705.11161>.
- [GMS19] E. Gwynne, C. Mao, and X. Sun. Scaling limits for the critical Fortuin–Kasteleyn model on a random planar map i: cone times. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, **55**(1):1–60, 2019.
- [HK71] R. Høegh-Krohn. A general class of quantum fields without cut-offs in two space-time dimensions. *Communications in Mathematical Physics*, **21**(3):244–255, 1971.
- [HRV18] Y. Huang, R. Rhodes, and V. Vargas. Liouville quantum gravity on the unit disk. **54**(3):1694–1730, 2018.
- [HS19] N. Holden and X. Sun. Convergence of uniform triangulations under the Cardy embedding. *arXiv preprint arXiv:1905.13207*, 2019. <https://arxiv.org/abs/1905.13207>.
- [Itô72] K. Itô. Poisson point processes attached to Markov processes. *Proceedings Sixth Berkeley Symposium on Mathematical Statistics and Probability*, III:225–239, 1972.
- [Iva11] Ivanovs. *One-sided Markov additive processes and related exit problems*. BOX-Press, 2011.
- [Kah85] J.-P. Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, **9**(2):105–150, 1985.
- [Ken01] R. Kenyon. Dominos and the Gaussian free field. *Annals of probability*, pages 1128–1137, 2001.
- [Kin75] J.F.C. Kingman. The first birth problem for an age-dependent branching process. *The Annals of Probability*, **3**(5):790–801, 1975.
- [Kin92] J. F. C. Kingman. *Poisson processes*, volume **3**. Oxford University Press, 1992.
- [Kiu80] S. Kiu. Semi-stable Markov processes in $\mathbb{R}^n$. *Stochastic Processes and their Applications*, **10**(2):183–191, 1980.
- [KKPW14] A. Kuznetsov, A.E. Kyprianou, J.C. Pardo, and A. Watson. The hitting time of zero for a stable process. *Electronic Journal of Probability*, **19**:1–26, 2014.
- [KN04] W. Kager and B. Nienhuis. A guide to stochastic Löwner evolution and its applications. *Journal of statistical physics*, **115**(5):1149–1229, 2004.

- [KP76] J.-P. Kahane and J. Peyriere. Sur certaines martingales de Benoit Mandelbrot. *Advances in mathematics*, **22**(2):131–145, 1976.
- [KP21] A.E. Kyprianou and J.C. Pardo. *Stable Lévy processes via Lamperti-type representations*. Cambridge University Press, Cambridge, 2021.
- [KRS18] A. E. Kyprianou, V. Rivero, and W. Satitkanitkul. Stable Lévy processes in a cone. *arXiv preprint arXiv:1804.08393*, 2018. <https://arxiv.org/pdf/1804.08393.pdf>.
- [KŞY20] V. Kyprianou, A.E. and Rivero, B. Şengül, and T. Yang. Entrance laws at the origin of self-similar Markov processes in high dimensions. *Transactions of the American Mathematical Society*, **373**(9):6227–6299, 2020.
- [Kyp98] A. E. Kyprianou. Slow variation and uniqueness of solutions to the functional equation in the branching random walk. *Journal of applied probability*, **35**(4):795–801, 1998.
- [Lam72] J. Lamperti. Semi-stable Markov processes I. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **22**(3):205–225, 1972.
- [Law08] G. Lawler. *Conformally invariant processes in the plane*. Number 114. American Mathematical Soc., 2008.
- [LG87] J.-F. Le Gall. Mouvement brownien, cônes et processus stables. *Probability Theory and Related Fields*, **76**:587–627, 1987.
- [LG13] J.-F. Le Gall. Uniqueness and universality of the brownian map. *The Annals of Probability*, **41**(4):2880–2960, 2013.
- [LGM11] J.-F. Le Gall and G. Miermont. Scaling limits of random planar maps with large faces. *The Annals of Probability*, **39**(1):1–69, 2011.
- [LGR20] J.-F. Le Gall and A. Riera. Growth-fragmentation processes in Brownian motion indexed by the Brownian tree. *The Annals of Probability*, **48**(4):1742–1784, 2020.
- [LPP95] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *The Annals of Probability*, **23**(3):1125–1138, 1995.
- [LS87] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *The Annals of Probability*, **15**(3):1052–1061, 1987.
- [Lyo97] R. Lyons. A simple path to Biggins’ martingale convergence for branching random walk. pages 217–221. Springer, 1997.
- [Mai71] B. Maisonneuve. Ensembles régénératifs, temps locaux et subordinateurs. *Séminaire de Probabilités Université de Strasbourg*, pages 147–169, 1971.
- [Man74] B. Mandelbrot. Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire. *CR Acad. Sci. Paris*, **278**(289-292):355–358, 1974.
- [Mie13] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta mathematica*, **210**(2):319–401, 2013.
- [MS16] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *arXiv:1605.03563*, 2016. <https://arxiv.org/pdf/1605.03563.pdf>.
- [MS17] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probability Theory and Related Fields*, **169**(3):729–869, 2017.

- [MS19] J. Miller and S. Sheffield. Liouville quantum gravity spheres as matings of finite-diameter trees. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, **55**(3):1712–1750, 2019.
- [MS20] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: the QLE(8/3, 0) metric. *Inventiones mathematicae*, **219**(1):75–152, 2020.
- [MS21] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *Probability Theory and Related Fields*, **179**(3):1183–1211, 2021.
- [MSW17] J. Miller, S. Sheffield, and W. Werner. CLE percolations. 5, 2017.
- [MSW20] Jason Miller, Scott Sheffield, and Wendelin Werner. Simple Conformal Loop Ensembles on Liouville Quantum Gravity. *arXiv preprint arXiv:2002.05698*, 2020.
- [Mul67] R. C. Mullin. On the enumeration of tree-rooted maps. *Canadian Journal of Mathematics*, **19**:174–183, 1967.
- [PR13] Juan Carlos Pardo and Víctor Rivero. Self-similar Markov processes. *Bol. Soc. Mat. Mexicana (3)*, 19(2):201–235, 2013.
- [PW20] E. Powell and W. Werner. Lecture notes on the Gaussian free field. *arXiv:2004.04720*, 2020. <https://arxiv.org/pdf/2004.04720>.
- [Riv12] V. Rivero. Tail asymptotics for exponential functionals of Lévy processes: the convolution equivalent case. **48**(4):1081–1102, 2012.
- [RS05] S. Rohde and O. Schramm. Basic properties of SLE. *Annals of Mathematics*, pages 883–924, 2005.
- [RV07] B. Rider and B. Virág. The noise in the circular law and the Gaussian free field. *International Mathematics Research Notices*, **2007**, 2007.
- [RV14] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: a review. *Probability Surveys*, **11**:315–392, 2014.
- [RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer-Verlag, Berlin, Third Edition, 1999.
- [Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, **118**(1):221–288, 2000.
- [She07] S. Sheffield. Gaussian free fields for mathematicians. *Probability theory and related fields*, **139**(3-4):521–541, 2007.
- [She09] S. Sheffield. Exploration trees and conformal loop ensembles. *Duke Mathematical Journal*, **147**(1):79–129, 2009.
- [She16a] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *The Annals of Probability*, **44**(5):3474–3545, 2016.
- [She16b] S. Sheffield. Quantum gravity and inventory accumulation. *The Annals of Probability*, **44**(6):3804–3848, 2016.
- [Shi85] M. Shimura. Excursions in a cone for two-dimensional Brownian motion. *Journal of Mathematics of Kyoto University*, **25**(3):433–443, 1985.
- [Shi15] Z. Shi. *Branching random walks, Lecture notes from the 42nd Probability Summer School held in Saint-Flour*, volume **2151**. Lecture Notes in Mathematics, Springer, Cham., 2015.
- [Shi17] Q. Shi. Growth-fragmentation processes and bifurcators. *Electronic Journal of Probability*, Vol. 22, **15**:1–25, 2017.
- [Sil21] W. Da Silva. Self-similar signed growth-fragmentations. *arXiv:2101.02582*, 2021. <https://arxiv.org/abs/2101.02582>.

- [Spi58] F. Spitzer. Some theorems concerning two-dimensional Brownian motion. *Transactions of the American Mathematical Society*, **87**:187–197, 1958.
- [Ste18a] R. Stephenson. Local convergence of large critical multi-type Galton–Watson trees and applications to random maps. *Journal of Theoretical Probability*, **31**(1):159–205, 2018.
- [Ste18b] R. Stephenson. On the exponential functional of Markov Additive Processes, and applications to multi-type self-similar fragmentation processes and trees. *ALEA*, **15**:1257–1292, 2018.
- [SW12] S. Sheffield and W. Werner. Conformal loop ensembles: the markovian characterization and the loop-soup construction. *Annals of Mathematics*, pages 1827–1917, 2012.
- [SW16] S. Sheffield and M. Wang. Field-measure correspondence in Liouville quantum gravity almost surely commutes with all conformal maps simultaneously. *arXiv:1605.06171*, 2016. <https://arxiv.org/abs/1605.06171>.
- [Wat21] A. Watson. A growth-fragmentation connected to the ricocheted stable process. *arXiv preprint arXiv:2101.05658*, 2021. <https://arxiv.org/abs/2101.05658>.