Nonstationary Deformed Singular Oscillator: Quantum Invariants and the Factorization Method

K Zelaya

Centre de Recherches Mathématiques, Université de Montréal, Montréal H3C 3J7, QC, Canada

E-mail: `zelayame@crm.umontreal.ca`

Abstract. New families of time-dependent potentials related with the stationary singular oscillator are introduced. This is achieved after noticing that a nonstationary quantum invariant can be constructed for the singular oscillator. Such a quantum invariant depends on coefficients related to solutions of the Ermakov equation, where the latter guarantees the regularity of the solutions at each time. In this form, after applying the factorization method to the quantum invariant rather than to the Hamiltonian, one manages to introduce the time parameter into the transformation, leading to factorized operators that become the constants of motion for the new time-dependent Hamiltonians. At the appropriate limit, the initial quantum invariant reproduces the stationary singular oscillator Hamiltonian. Some families of stationary potentials already reported by other authors are also recovered as particular cases. A striking feature of the method is that the singular barrier of the potential can be managed to vanish, which leads to non-singular time-dependent potentials.

1. Introduction

The dynamics of non-relativistic quantum mechanical systems is determined by the Schrödinger equation. An essential part of such equation is the Hamiltonian operator, which characterizes the system under consideration. For either *stationary* or time-independent systems, the Hamiltonian plays the role of the energy observable, and the corresponding mathematical problem is reduced to solve a simple eigenvalue equation. Even in this case, only some few Hamiltonians are known to admit exact solutions. Examples include the harmonic oscillator, the hydrogen atom, and the interaction between diatomic molecules. But the set of such potentials does not include much more than a dozen of well known cases. The search of new exactly-solvable models is indeed a mathematical challenge. In this regard, the factorization method [1–5] is an outstanding technique to explore the existence and construction of new exactly-solvable stationary models. The method is intimately connected to the Darboux transformation [6] and serves as the mathematical foundation of supersymmetric quantum mechanics [1, 2]. Using the factorization method, a wide class of new exactly-solvable models has been reported for Hermitian Hamiltonians [7–9], non-Hermitian Hamiltonians with all-real spectra in both, $\mathcal{PT}$ and non-$\mathcal{PT}$ regimes [10–15], and position-dependent mass models [16–18], among others.

For *nonstationary* (time-dependent) systems the Schrödinger equation can not be reduced to an eigenvalue equation and, in most of the cases, it must be solved directly. The latter may require approximation techniques to face the related mathematical difficulties, see e.g. [19]. In this context, it is notable that despite its complexity, time-dependent phenomena find immediate applications in electromagnetic traps of charged particles [20–23], as well as in optical-analogs under the paraxial approximation [24–26]. Among the nonstationary quantum systems, the parametric oscillator [27, 28] is perhaps the most well-known model that admits a set of exact solutions. Lewis and Riesenfeld addressed the problem by noticing the existence of an *constant of motion* (*quantum invariant*) for the corresponding nonstationary eigenvalue equation [29].

On the other hand, in agreement with the conventional factorization method, some years ago Bagrov and Samsonov proposed an approach to construct new solvable time-dependent potentials [30, 31]. That is, two different Schrödinger equations are linked by the appropriate intertwining relationships and assuming that one of the equations is exactly solvable with well known solutions. The method has been successfully applied to construct nonstationary quantum potentials with exactly solvable Schrödinger equation [30–37]. However, it is necessary to emphasize that the solutions of the new equations are not necessarily orthogonal, even if the solutions of the initial equation form an orthogonal set, see e.g. [35]. In addition, the physical meaning of such solutions is unclear since they are not eigenfunctions of the corresponding Hamiltonian and the Bagrov-Samsonov method (by itself) does not provide information about the constants of motion of the system, so the latter have to be computed in independent form.

Considering the usefulness of the stationary singular oscillator [38–40] to characterize two-ion traps [41], in the present work we apply the factorization method to generate time-dependent versions of such potential. We show that the appropriate quantum invariant of the singular oscillator admits factorization in terms of time-dependent ladder operators, and that such operators are useful to intertwine the stationary singular oscillator with its time-dependent counterparts. The orthogonality and physical meaning of the solutions of the new systems are therefore associated to both invariants, the one belonging to the initial singular oscillator and that arising from the intertwining relationships.

The organization of this paper is as follows. In Sec. 2, the solutions of the stationary singular oscillator are briefly discussed. Then, an additional quantum invariant, different from the Hamiltonian, is constructed and its spectral problem is properly identified. In Sec. 3, the implementation of the factorization method on the aforementioned quantum invariant is introduced, leading to a new family of time-dependent potentials whose solutions are mapped from the initial system. Some particular cases are discussed in Sec. 4, where as a particular limit the well known stationary results are recovered. Additional details on the construction of the new Hamiltonians are presented in Appendix A. In Appendix B, the intermediate steps required in the calculation of the normalization constant associated with the added eigenvalue are presented. Final comments and perspectives of this work are presented in Sec. 5.

2. Singular Oscillator

The stationary singular oscillator is defined through the Hamiltonian

$$\hat{H}_1 = \hat{p}^2 + V_1(\hat{x}) \, , \quad V_1(\hat{x}) = \hat{x}^2 + \frac{g(g+1)}{\hat{x}^2}, \tag{1}$$

with $g \geq 0$ an arbitrary constant. Given that $\hat{H}_1$ is time-independent, the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(x,t) = \hat{H}_1\psi(x,t), \tag{2}$$

admits a set of stationary orthonormal solutions $\{\psi_n(x)\}_{n=0}^{\infty}$ computed through the time-evolution operator $\hat{U}(t) = e^{-i\hat{H}_1 t}$ and the eigenvalue equation

$$\hat{H}_1\phi_n(x) \equiv -\frac{\partial^2\phi_n(x)}{\partial x^2} + \left[x^2 + \frac{g(g+1)}{x^2}\right]\phi_n(x) = E_n\phi_n(x) \, , \quad \psi_n(x,t) = e^{-iE_n t}\phi_n(x), \tag{3}$$

Journal of Physics: Conference Series **1540** (2020) 012017 doi:10.1088/1742-6596/1540/1/012017

where the coordinate representation $\hat{x} \to x$ and $\hat{p} \to -i\partial/\partial x$ has been used. The singular oscillator is one of the few exactly-solvable models in quantum mechanics, and its eigenfunctions $\phi_n(x)$ and eigenvalues E_n have been reported in the literature [38, 40],

$$\phi_n(x) = \mathcal{N}_n e^{-\frac{x^2}{2}} x^{g+1} L_n^{(g+1/2)}\left(x^2\right), \quad \mathcal{N}_n^2 = 2\frac{\Gamma(n+1)}{\Gamma(n+g+3/2)}, \quad E_n = 4n + 2g + 3, \tag{4}$$

with $L_n^{(m)}(z)$ the *associated Laguerre polynomials* [46]. The normalization constant $\mathcal{N}_n$ was fixed from the condition $\langle \phi_n | \phi_n \rangle = 1$, where the physical inner-product is defined as

$$\langle f | g \rangle = \int_0^\infty dx f^*(x) g(x), \tag{5}$$

with z^* the complex conjugate of z and $f(x) = \langle x | f \rangle$ is the coordinate representation of the vector $|f\rangle$.

2.1. Nonstationary quantum invariant

Remarkably, even if the Hamiltonian is time-independent, there is a constant of motion, different from the Hamiltonian $\hat{H}_1$. The latter is a fact that was explored for the stationary oscillator [28], and it was used in the construction of new solvable time-dependent models [37, 42].

To illustrate the existence of such a constant of motion, consider the time-dependent operator of the form

$$\hat{I}_1(t) = C_0(t)\left(\frac{\hat{p}^2}{2m} + \frac{g(g+1)}{\hat{x}^2}\right) + C_1(t)\hat{x}^2 + C_2(t)\{\hat{x}, \hat{p}\}, \tag{6}$$

where $\{\hat{x}, \hat{p}\} = \hat{x}\hat{p} + \hat{p}\hat{x}$ is the anti-commutation relationship and the real-valued functions $C_i(t)$, for $i = 1, 2, 3$, are determined from the quantum invariant condition

$$\frac{d}{dt}\hat{I}_1(t) = i[\hat{H}_1, \hat{I}_1(t)] + \frac{\partial}{\partial t}\hat{I}_1(t) = 0. \tag{7}$$

Notice that a particular solution should be given as $C_0 = C_1 = 1$ and $C_2 = 0$, where the operator $\hat{I}_1(t)$ simply reduces to the Hamiltonian $\hat{H}_1$, which is indeed a constant of motion of the system. In the general case, with the use of the identities

$$\begin{aligned} &\left[\hat{x}^2, \hat{p}^2 + \frac{g(g+1)}{\hat{x}^2}\right] = 2i\{\hat{x}, \hat{p}\}, \quad [\hat{x}^2, \hat{p}^2] = 2i\{\hat{x}, \hat{p}\}, \\ &\left[\{\hat{x}, \hat{p}\}, \hat{p} + \frac{g(g+1)}{\hat{x}^2}\right] = 4i\left(\hat{p}^2 + \frac{g(g+1)}{\hat{x}^2}\right), \end{aligned} \tag{8}$$

the following set of coupled equations are obtained:

$$2(C_1 - C_0) + \dot{C} = 0, \quad 4C_2 + \dot{C}_0 = 0, \quad -4C_2 + \dot{C}_1 = 0, \quad \dot{f}(t) = \frac{df(t)}{dt}. \tag{9}$$

The latter can be solved with ease, but it is convenient to introduce the re-parametrization $C_0(t) = \sigma^2(t)$ such that, after some calculations, one obtains

$$\ddot{\sigma} + 4\sigma = \frac{4}{\sigma^3}, \quad C_0(t) = \sigma^2, \quad C_1(t) = \frac{\dot{\sigma}^2}{4} + \frac{1}{\sigma^2}, \quad C_2(t) = -\frac{\sigma\dot{\sigma}}{2}, \tag{10}$$

where $\sigma(t)$ solves the Ermakov equation [43]. Such an equation admits a solution through the nonlinear combination [12–14]:

$$\sigma^2(t) = aq_1^2(t) + bq_1(t)q_2(t) + cq_2^2(t), \quad b^2 - 4ac = -\frac{16}{W_0^2}, \tag{11}$$

with $W_0 = q_1\dot{q}_2 - \dot{q}_1 q_2 \neq 0$ the Wronskian of two linearly independent solutions of the classical equation of motion $\ddot{q}_{1,2} + 4q_{1,2} = 0$. These two solutions are given by

$$q_1(t) = \cos[2(t-t_0)], \quad q_2(t) = \sin[2(t-t_0)], \quad W_0 = 2, \tag{12}$$

with t_0 an arbitrary real phase-shift. After some calculations, the solution of the Ermakov equation takes the form

$$\sigma^2(t) = \frac{a+c}{2} + \frac{a-c}{2}\cos[4(t-t_0)] + \sqrt{ac-1}\,\sin[4(t-t_0)], \tag{13}$$

where the parameters $a, c > 0$, together with the constraint in (11), ensure that $\sigma(t)$ is a nodeless function for $t \in \mathbb{R}$, for details see [14].

Now, from the Lewis-Riesenfeld approach [29], it follows that the quantum invariant $\hat{I}_1(t)$ solves an eigenvalue equation of the form

$$\hat{I}_1(t)\varphi_n^{(1)}(x,t) = \lambda_n^{(1)}\varphi_n^{(1)}(x,t), \tag{14}$$

where $\lambda_n^{(1)}$ are the time-independent eigenvalues [29], and $\varphi_n^{(1)}(x,t)$ the *nonstationary eigenfunctions* that satisfy the finite-norm condition $\langle\varphi_n^{(1)}(t)|\varphi_n^{(1)}(t)\rangle < \infty$, with the inner product as defined in (5). It is worth to remark that $\varphi_n^{(1)}(x,t)$, for $n = 0, 1, \cdots$, are not solutions of the Schrödinger equation (1), but they are used to construct the solutions $\psi_n^{(1)}(x,t)$ through the addition of the appropriate time-dependent complex-phase [29]:

$$\psi_n^{(1)}(x,t) = e^{i\theta_n^{(1)}(t)}\varphi_n^{(1)}(x,t), \quad \frac{d}{dt}\theta_n^{(1)}(t) = \langle\varphi_n^{(1)}(t)|i\frac{\partial}{\partial t} - \hat{H}_1|\varphi_n^{(1)}(t)\rangle, \tag{15}$$

where $\psi_n^{(1)}(x,t)$ are indeed solutions of the Schrödinger equation. Contrary to the stationary solutions $\psi_n(x,t)$ of (3), the phase $\theta_n^{(1)}(t)$ is not related with the time evolution of the system, except for the cases in which $\hat{I}_1(t) = \hat{H}_1$.

Before proceeding, one has to solve the eigenvalue equation (14). To this end, it is convenient to introduce the coordinate representation and the reparametrization

$$\varphi_n^{(1)}(x,t) = \frac{e^{i\frac{\dot{\sigma}}{4\sigma}x^2}}{\sqrt{\sigma}}\chi_n(z(x,t)), \quad z = z(x,t) = \frac{x}{\sigma}. \tag{16}$$

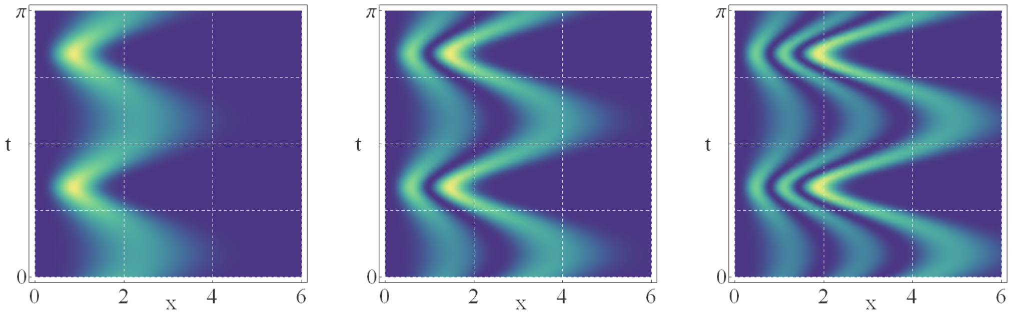
Notice that the re-parametrization $z(x,t)$ is well defined at each time, since it has been guaranteed that $\sigma(t)$ is a nodeless function at each time. After substituting (16) in (14), one recovers a differential equation for $\chi_n(z(x,t))$ of the form

$$-\frac{\partial^2\chi_n}{\partial z^2} + \left[z^2 + \frac{g(g+1)}{z^2}\right]\chi_n = \lambda_n^{(1)}\chi_n, \tag{17}$$

where it is clear that $\chi_n(z)$ solves the same eigenvalue equation (3), but in the z-parameter instead. One thus has

$$\chi_n(z) = \mathcal{N}_n e^{-z^2 2} z^{g+1} L_n^{(g+1/2)}\left(z^2\right), \quad \lambda_n^{(1)} = 4n + 2g + 3, \tag{18}$$

with $\mathcal{N}_n$ the normalization constant given in (4). Interestingly, the nonstationary eigenfunctions (16) have found applications in wave propagation in optical models, where wave-packets are described by self-focusing Laguerre-Gaussian modes [44].



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$

Figure 1: Nonstationary probability density $|\psi_n^{(1)}(x,t)|^2$ given in (16) for $g = 1$, $a = 2$, $c = 1$ and $t_0 = 0$.

The re-parametrization $z(x,t)$ also simplifies the calculation of the complex-phase $\theta_n^{(1)}(t)$ in (15), leading to

$$\theta_n^{(1)}(t) = -\lambda_n^{(1)} \int^t \frac{dt'}{\sigma^2(t')} = -\frac{\lambda_n^{(1)}}{2} \arctan\left(\sqrt{ac-1} + c\tan[2(t-t_0)]\right), \tag{19}$$

where the integral has been solved using the properties of the Ermakov equation, for details see [14]. It is worth to mention that the orthogonality of the set $\{\psi_n^{(1)}(x,t)\}_{n=0}^{\infty}$ holds provided that the solutions are evaluated at the same time, that is, $\langle\psi_m^{(1)}(t)|\psi_n^{(1)}(t)\rangle = \delta_{n,m}$. For different times though, the orthogonality does not longer hold, $\langle\psi_m^{(1)}(t')|\psi_n^{(1)}(t)\rangle \neq \delta_{n,m}$. From the completeness of the associated Laguerre polynomials, it is guaranteed that the nonstationary solutions $\psi_n^{(1)}(x,t)$ form a complete set of solutions, with a well defined number of zeros and interlacing properties at each time. The respective probability densities $|\psi_n^{(1)}(x,t)|^2$ are depicted in Figure 1 for $n = 0, 1, 2$.

3. Nonstationary deformed singular oscillator

The time-dependent quantum invariant of the previous section, along with its respective set of nonstationary eigenfunctions, provides an alternative set of solutions to the Schrödinger equation of the singular oscillator. Those results can be used further in an attempt to construct new exactly solvable model. For stationary systems, the factorization method has been applied to the singular oscillator, and used to construct new families of stationary Hamiltonians such that the spectrum is preserved, with the exception of one possible added level [45]. In this section, an alternative factorization is explored such that, even if the initial Hamiltonian is time-independent, the new resulting Hamiltonians are in general time-dependent. This is achieved by applying the factorization method to the quantum invariant $\hat{I}_1(t)$ instead of the Hamiltonian $\hat{H}_1$. The latter procedure has been proved useful while exploring the time-dependent rational extensions of the parametric oscillator [42].

Consider a set of mutually adjoint operators, defined in coordinate representation as first-order differential operators in the spatial variable of the form [42]:

$$\hat{A}(t) := \sigma\frac{\partial}{\partial x} + w(x,t), \quad \hat{A}^\dagger(t) := -\sigma\frac{\partial}{\partial x} + w^*(x,t), \tag{20}$$

where σ is the solution of the Ermakov equation given in (13), and complex-valued function $w(x,t)$ is determined from the factorization condition

$$\hat{I}_1 := \hat{A}^\dagger(t)\hat{A}(t) + \epsilon, \tag{21}$$

with ϵ a real constant and $\hat{I}_1(t)$, given in (6), is rewritten in coordinate representation as

$$\hat{I}_1(t) \equiv -\sigma^2 \frac{\partial^2}{\partial x^2} + i\sigma\dot{\sigma}x\frac{\partial}{\partial x} + R(x,t) + \sigma^2\frac{g(g+1)}{x^2}\,, \quad R(x) := i\frac{\sigma\dot{\sigma}}{2} + \left(\frac{\dot{\sigma}^2}{4} + \frac{1}{\sigma^2}\right)x^2. \tag{22}$$

After substituting (20) in (21) and comparing with (22) one obtains

$$w(x,t) = -i\frac{\dot{\sigma}}{2}x + W(x,t)\,, \quad -\sigma\frac{\partial W}{\partial x} + W^2 = \frac{x^2}{\sigma^2} + \sigma^2\frac{g(g+1)}{x^2} - \epsilon, \tag{23}$$

where $W(x,t)$ is a real-valued function. Notice that the re-parametrization $z = x/\sigma$ leads to a Riccati equation of the form

$$-\frac{\partial W}{\partial z} + W^2 = z^2 + \frac{g(g+1)}{z^2} - \epsilon, \tag{24}$$

where $W \equiv W(z(x,t))$ becomes a function of z, which is solved though the linear equation

$$-\frac{\partial^2}{\partial z^2}u(z) + \left[z^2 + \frac{g(g+1)}{z^2}\right]u(z) = \epsilon u(z)\,, \quad W(z) = -\frac{1}{u(z)}\frac{\partial u(z)}{\partial z}. \tag{25}$$

The latter coincides with the spectral problem associated with the stationary singular oscillator. But in this case, the solutions $u(z)$ are not required to have a finite-norm. Nevertheless, given the relationship between $W(z)$ and $u(z)$, it is necessary to impose $u(z(x,t))$ to be a nodeless function in $x \in \mathbb{R}^+$, such that $W(z)$ is a regular function. In general, the solutions of (25) are determined by taking (25) into the hypergeometric differential equation form [47], leading to a general solution of the form

$$u(z) = \frac{e^{-\frac{z^2}{2}}}{z^g}\left[k_a\, z^{2g+1}\,{}_1F_1\left(\frac{3+2g-\epsilon}{4}, \frac{3}{2}+g; z^2\right) + k_b\,{}_1F_1\left(\frac{1-2g-\epsilon}{4}, \frac{1}{2}-g; z^2\right)\right], \tag{26}$$

where ${}_1F_1(\cdot,\cdot;z)$ stands for the *confluent hypergeometric function* [46]. The arbitrary real constants k_a, k_b and ϵ are constrained such that $u(z)$ satisfy the nodeless condition. A first condition is given by $\epsilon < \lambda_0^{(1)}$, this guarantees that the linear combination of the confluent hypergeometric functions in (26) have at most one zero in $x \in \mathbb{R}^+$. Then, with the use of the asymptotic behavior of the confluent hypergeometric function [46], a relationship between k_a and k_b is determined such that the aforementioned zero is placed at $x \to \infty$. After some calculations one finds the conditions

$$\frac{k_a}{k_b} > -\frac{\Gamma\left(\frac{1}{2}-g\right)\Gamma\left(\frac{3+2g-\epsilon}{4}\right)}{\Gamma\left(\frac{3}{2}+g\right)\Gamma\left(\frac{1-2g-\epsilon}{4}\right)}, \quad \epsilon < 2g+3 \quad k_b \neq 0. \tag{27}$$

With (27), the solutions to the Riccati equation $W(z(x,t))$ are free of singularities, except perhaps in $x \to 0$.

Now, with the factorization operators $\hat{A}(t)$ and $\hat{A}^\dagger(t)$ already determined, it is convenient to introduce a new operator that is factorized as

$$\hat{I}_2(t) := \hat{A}(t)\hat{A}^\dagger(t) + \epsilon, \tag{28}$$

which in coordinate representation takes the form

$$\hat{I}_2 \equiv -\sigma^2\frac{\partial^2}{\partial x^2} + i\dot{\sigma}\sigma x\frac{\partial}{\partial x} + R(x) + \frac{g(g+1)}{z(x,t)^2} + F(z(x,t)), \tag{29}$$

where

$$F(z(x,t)) = 2\frac{\partial}{\partial z}W(z(x,t)) = -2\frac{\partial^2}{\partial z^2}\ln u(z(x,t)). \tag{30}$$

It is clear that $\hat{I}_2(t)$ is not a quantum invariant of the singular oscillator. However, one may determine the respective Hamiltonian $\hat{H}_2(t)$ for which $\hat{I}_2(t)$ is its quantum invariant. Such Hamiltonian is given as (see Appendix A and [42] for details):

$$\hat{H}_2(t) \equiv -\frac{\partial^2}{\partial x^2} + V_2(x,t), \quad V_2(x,t) = x^2 + \frac{g(g+1)}{x^2} + \frac{1}{\sigma^2(t)}F(z(x,t)). \tag{31}$$

Notice that, in general, the time-dependent potential $V_2(x,t)$ is not trivially separable as the sum of a spatial part plus a time-dependent part. Thus, the solutions of the Schrödinger equation may not be determined in a straightforward way if one tries to solve it directly. Nevertheless, in the sequel it is shown that the factorization operators lead to a mechanism to compute the solutions through simple mappings.

3.1. Spectral properties of $\hat{I}_2(t)$ and solutions of $\hat{H}_2(t)$

Now, with the new time-dependent Hamiltonian $\hat{H}_2(t)$ already identified, one has to address the solutions of the respective Schrödinger equation. As discussed in Sec. 2.1, it is required to solve the spectral problem associated with $\hat{I}_2(t)$, and then the appropriate time-dependent complex-phase must be added to the nonstationary eigenfunctions. Remarkably, the spectral problem

$$\hat{I}_2(t)\varphi_n^{(2)}(x,t) = \lambda_n^{(2)}\varphi_n^{(2)}(x,t), \tag{32}$$

with $\varphi_n^{(2)}(x,t)$ and the $\lambda_n^{(2)}$ the respective nonstationary eigenfunctions and eigenvalues, is determined from the *intertwining relationships* between $\hat{I}_1(t)$ and $\hat{I}_2(t)$. The latter is obtained from the factorizations defined in (21) and (28), leading to

$$\hat{I}_1(t)\hat{A}^{\dagger}(t) = \hat{A}^{\dagger}(t)\hat{I}_2(t)\,, \quad \hat{I}_2(t)\hat{A}(t) = \hat{A}(t)\hat{I}_1(t). \tag{33}$$

Eq. (33) provides a mechanism to map the eigenfunctions of $\hat{I}_1(t)$ into eigenfunctions of $\hat{I}_2(t)$, and vice versa, it also allows determining the respective eigenvalues $\lambda_n^{(2)}$ in terms of $\lambda_n^{(1)}$. In Sec. 2.1, the spectral problem related to $\hat{I}_1(t)$ was already identified. Thus, it is straightforward to obtain the spectral information for $\hat{I}_2(t)$ as

$$\varphi_{n+1}^{(2)}(x,t) = \frac{1}{\sqrt{\lambda_n^{(1)}-\epsilon}}\hat{A}(t)\varphi_n^{(1)}(x,t), \quad \lambda_{n+1}^{(2)} = \lambda_n^{(1)}, \quad n = 0,1\ldots\,, \tag{34}$$

where the orthogonality condition $\langle\varphi_m^{(2)}(t)|\varphi_n^{(2)}(t)\rangle = \delta_{n,m}$ is inherited from that of the set $\{\varphi_n^{(1)}(x,t)\}$, with respect to the physical inner-product (5). The additional factor $(\lambda_n^{(1)}-\epsilon)^{-1/2}$ has been introduced as a normalization constant. Notice that the sub-index of the mapped eigenfunctions in (34) has been fixed at $n+1$, this is because of the existence of an additional nonstationary eigenfunction $\varphi_\epsilon^{(2)}(x,t)$ that can not be constructed through the mapping provided by $\hat{A}(t)$. The existence of such an eigenfunction, henceforth called *missing state*, is well-known in the literature of the factorization method for stationary systems [1,5,8]. The missing state is determined from the orthogonality condition $\langle\varphi_{n+1}^{(2)}(t)|\varphi_\epsilon^{(0)}(t)\rangle = 0$ for all $n = 0,1,\cdots$. In this form it is guaranteed that $\varphi_\epsilon^{(2)}(x,t)$ is not a linear combination of the eigenfunctions in (34), and thus it should be added to the set of elementary solutions, provided that $\varphi_\epsilon^{(2)}(x,t)$ satisfy the finite-norm condition. Straightforward calculations show that the orthogonality condition implies $\hat{A}^{\dagger}(t)\varphi_{\epsilon}^{(2)}(x,t) = 0$, which also means that $\hat{I}_2(t)\varphi_{\epsilon}^{(2)} = \epsilon\varphi_{\epsilon}^{(2)}(x,t)$. That is, ϵ is an eigenvalue of the new quantum invariant and, from the nodeless condition $\epsilon < 2g+3 = \lambda_0^{(1)}$, it is the lowest eigenvalue in the spectrum. Thus, after some calculations, the normalized nonstationary eigenfunction $\varphi_{\epsilon}^{(2)}(x,t)$ takes the form

$$\varphi_0^{(2)}(x,t) \equiv \varphi_{\epsilon}^{(2)}(x,t) = \frac{\mathcal{N}_{\epsilon}}{\sqrt{\sigma}}\frac{e^{i\frac{\dot{\sigma}}{4\sigma}x^2}}{u(z(x,t))}\,, \quad \lambda_0^{(2)} = \epsilon\,, \tag{35}$$

where $\mathcal{N}_{\epsilon}$ stands for the normalization constant, given as (see [48] and Appendix B for details):

$$\mathcal{N}_{\epsilon}^2 = (1+2g)\left[k_a k_b + k_b^2 \frac{\Gamma\left(\frac{1-2g}{2}\right)\Gamma\left(\frac{3+2g-\epsilon}{4}\right)}{\Gamma\left(\frac{3+2g}{2}\right)\left(\frac{1-2g-\epsilon}{4}\right)}\right]. \tag{36}$$

Eq. (36) holds provided that the constraints (27) are fulfilled.

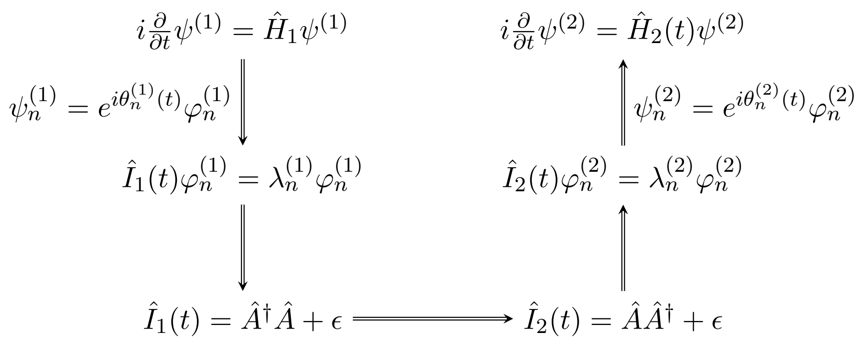


Figure 2: Scheme summarizing the construction of $\hat{H}_2(t)$, together with the respective solutions of the Schrödinger equation $\psi_n^{(2)}(x,t)$.

Now, following the discussion at the end of Sec. 2, the nonstationary eigenfunctions of the quantum invariant are mapped into solutions of the Schrödinger equation $\psi_n^{(2)}(x,t)$ through the addition of the complex-phase

$$\psi_n^{(2)}(x,t) = e^{i\theta_n^{(2)}(t)}\varphi_n^{(2)}(x,t)\,, \quad \frac{d}{dt}\theta_n^{(2)}(t) = \langle\varphi_n^{(2)}(t)|i\frac{\partial}{\partial t} - \hat{H}_2(t)|\varphi_n^{(2)}(t)\rangle, \tag{37}$$

where $n = 0, 1, \cdots$. With the use of the re-parametrization $z(x,t) = z/\sigma$, and after some calculations, one obtains (for details, see Appendix A in [42]):

$$\theta_n^{(2)}(t) = -\frac{\lambda_n^{(2)}}{2}\arctan\left(\sqrt{ac-1} + c\tan[2(t-t_0)]\right)\,. \tag{38}$$

Therefore, both the spectral information of the invariant $\hat{I}_2(t)$ and the solutions of the Schrödinger equation associated with $\hat{H}_2(t)$ have been completely determined from the information of the initial stationary singular oscillator and its quantum invariant. It is worth to remark that, contrary to the stationary case, the factorization on the quantum invariant adds an additional level which is not necessarily an energy eigenvalue, nevertheless it provides a physical solution that can not be disregarded. A summary of the factorization method implemented in this work is depicted in the scheme of Figure 2.

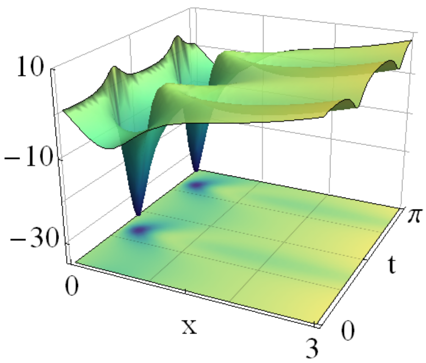
4. Particular cases

4.1. Stationary limit

The condition $a = c = 1$ leads to $\sigma(t) = 1$ and consequently to $z = x$. In this limit, it is clear that $\hat{I}_1(t) \to \hat{H}_1$. Moreover, the nonstationary eigenfunctions converge to the eigenfunctions of the singular oscillator $\phi_n(x)$. The eigenvalues of both the Hamiltonian and the quantum invariant are the same, regardless of the stationary limit. In the same limit, both the new quantum invariant $\hat{I}_2(t)$ and the Hamiltonian $\hat{H}_2(t)$ converge the stationary models obtained through the conventional factorization method, already reported in the literature [45].

4.2. Non-singular potentials $V_2(x,t)$

It is worth to discuss the special class of potentials in $V_2(x,t)$ for which the singularity at $x = 0$ is removed. A first case is obtained in the limit $g \to 0$, the initial potential $V_1(x)$ reduces to the *truncated oscillator* [49], which is a shape-invariant case of the singular oscillator $V_1(x)$ given in (1). In such a limit, the new potential $V_2(x,t)|_{g\to 0}$ is still time-dependent and non-singular at the origin. Clearly, in the stationary limit $a = c = 1$, the potential $V_2(x,t)|_{g\to 0}$ reduces to the first-step transformed potentials reported in [49].



(a)

Figure 3: Non-singular potential $V_2(x,t)$ for $g = 1$, $\epsilon = -2$, $k_a = 1$, $k_b = 1/4$, $a = 2$, $c = 1$ and $t = 0$.

The singularity at $x = 0$ is also removed with $g = 1$. In this case, the factorization method adds an additional singular term in the potential such that the singular-barrier vanishes. The behavior of $V_2(x,t)$ and the respective probability densities are depicted in Figure 3. In the latter, it can be seen that indeed the potential is finite at $x = 0$, with $V(0,t)$ a periodic function on time.

From the explicit form of the potential $V_2(x,t)$ it is easy to show that only the cases $g = 0, 1$ lead to non-singular potentials. Also, it is worth to remark that, although the singularity has been removed, the domain of definition $x \in \mathbb{R}^+$ is still preserved.

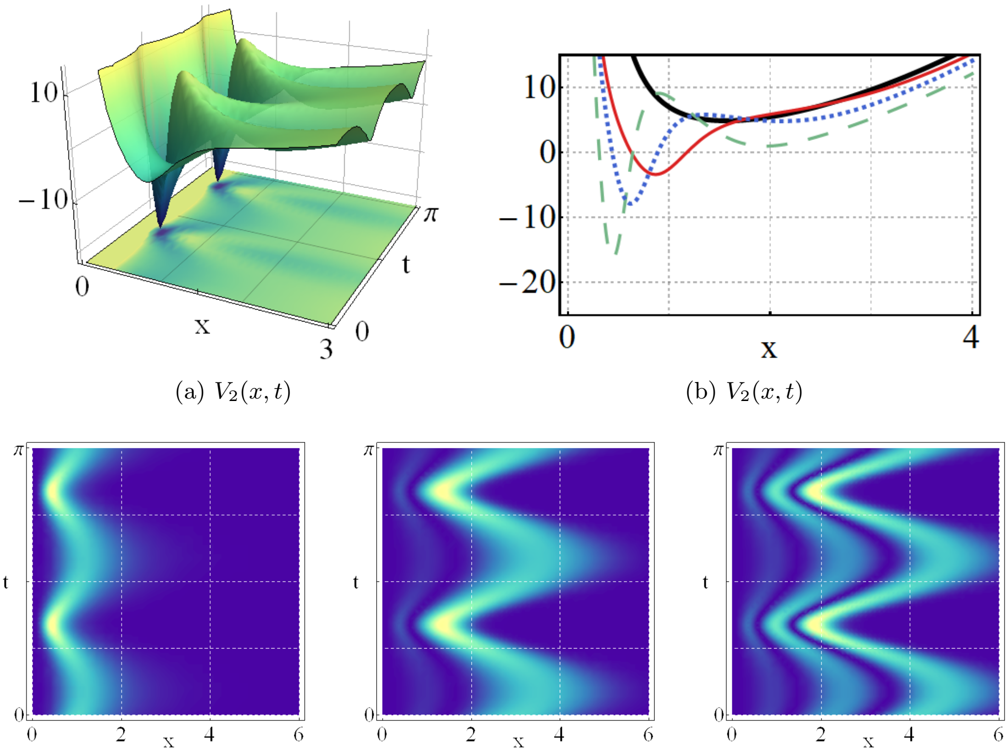
4.3. Equidistant spectrum in $\hat{I}_2(t)$

From (18), it is clear that the spectrum of the initial quantum invariant $\hat{I}_1(t)$ is equidistant, $\lambda_{n+1}^{(1)} - \lambda_n^{(1)} = 4$, for $n = 0, 1, \ldots$ The new quantum invariant $\hat{I}_2(t)$ admits equidistant spectrum if $\epsilon = 2g - 1$, which is physically admissible since it satisfies the constraint imposed in (27). The respective eigenvalues of $\hat{I}_2(t)$ are then $\lambda_n^{(2)} = 4n + 2g - 1$, $n = 0, 1, \ldots$

To illustrate the form of the new potential, the parameters are fixed as $g = 2$ and $\epsilon = 3$. Thus, one has

$$V_2(x,t) = x^2 + \frac{2}{x^2} - \frac{2}{\sigma^2}\left(1 + \frac{\partial^2}{\partial z^2}\ln\left[15\sqrt{\pi}k_a \mathrm{Erf}(z) + 8k_b - 10k_a z(3+2z^2)e^{-z^2}\right]\right), \qquad (39)$$

where $z(x,t) = x/\sigma(t)$ and $k_b > -\Gamma(7/2)k_a$. That is, the new potential and its solutions are well defined for any positive constants k_a and k_b. The behavior of the potential and the respective probability densities are depicted in Fig. 4. From Fig. 4b, the new time-dependent potential can be compared to the initial singular oscillator. Its clear that the minimum of the potential $V_2(x,t)$ is lower to that of $V_1(x)$ at any time, as expected, since the new time-dependent potential admits a new eigenvalue at $\epsilon < \lambda_0^{(1)}$. Asymptotically, at $x \to \infty$, both potentials have the same behavior, for it the deformation produced by the factorization method is localized around the origin for the parameters used in this particular case.



(a) $V_2(x,t)$ (b) $V_2(x,t)$

(c) $n = 0$ (d) $n = 1$ (e) $n = 2$

Figure 4: (Upper row) (a) Time-dependent potential $V_2(x,t)$ for $t \in [0, 2\pi]$ and (b) the respective 2-D projection for $t = 0$ (solid-red), $t = 3\pi/8$ (dashed-green) and $t = 3\pi/4$ (dotted-blue) together with the stationary singular oscillator $V_1(x)$ (thick-black). (Lower row) Probability density $|\psi_n^{(2)}(x,t)|^2$ for the $n = 0, 1, 2$. In all the cases, the parameters are fixed as $\epsilon = 3$, $g = 2$, $a = 2$, $c = 1$, $t_0 = 0$, $k_a = 1$ and $k_b = 1/4$.

5. Conclusions

It has been shown that the factorization method can be also implemented to construct exactly solvable time-dependent potentials. In particular, we have applied the method to the eigenvalue problem defined by a time-dependent quantum invariant of the singular oscillator. Then we have found that, although the initial model is stationary, the new potentials and their solutions are time-dependent. Indeed, we have shown that the method provides a constant of motion for the new singular oscillators that inherits the initial spectral problem, which is already solved. As the time-dependence of the potentials so constructed is periodic, the results reported in this paper may be useful in the trapping of particles, where such a profile is addressed to confine the particles in a given region of space [22].

To mention some possible directions in which this work can be addressed consider the rational extensions of the stationary singular oscillator reported in, e.g. [50,51]. As the solutions are given in terms of the exceptional Laguerre polynomials, one may explore such a construction in the time-dependent regime (see the development associated to the parametric oscillator in [42]). Work in this direction is in progress.

Appendix A.

The new time-dependent Hamiltonian $\hat{H}_2(t)$ related to the quantum invariant $\hat{I}_2(t)$ is determined from the following ansatz:

$$\hat{H}_2(t) := \hat{H}_1 + G(t)F(z(\hat{x},t)), \tag{A.1}$$

where $F(z(\hat{x},t))$ is defined in (30) and $G(t)$ is determined from the quantum invariant condition

$$\frac{d}{dt}\hat{I}_2(t) = i[\hat{H}_2(t), \hat{I}_2(t)] + \frac{\partial}{\partial t}\hat{I}_2(t) = 0. \tag{A.2}$$

With aid of the commutation relationships (8), together with the identity $[\{\hat{x},\hat{p}\}, T(\hat{x},t)] = 2x[\hat{p}, T(\hat{x},t)]$, valid for any smooth function $T(x,t)$ in the real x-variable, one obtains

$$-i\left(G - \frac{1}{\sigma^2}\right)\left\{\sigma^2[\hat{p}^2, F(z(\hat{x},t))] - \sigma\dot{\sigma}\hat{x}[\hat{p}, F(z(\hat{x},t))]\right\} = 0. \tag{A.3}$$

It is clear that $G(t) = \sigma^{-2}(t)$ solves (A.3). With $G(t)$ and (A.1), the time-dependent Hamiltonian $\hat{H}_2(t)$ is then given as in (31).

Appendix B.

The normalization constant $\mathcal{N}_\epsilon$ in (36) is determined from the finite-norm condition and the re-parametrization $z(x,t) = x/\sigma$, leading to the integral

$$1 = |\mathcal{N}_\epsilon|^2 \int_0^\infty \frac{dz}{(k_a u_1(z) + k_b u_2(z))^2} = |\mathcal{N}_\epsilon|^2 \int_0^\infty \frac{dz}{u_2^2(z)} \frac{1}{[k_a u_1(z)/u_2(z) + k_b]^2}, \tag{B.1}$$

where

$$u_1(z) = e^{-\frac{z^2}{2}} z^{g+1} \, {}_1F_1\left(\tfrac{3+2g-\epsilon}{4}, \tfrac{3}{2}+g; z^2\right), \quad u_2(z) = e^{-\frac{z^2}{2}} z^{-g} \, {}_1F_1\left(\tfrac{1-2g-\epsilon}{4}, \tfrac{1}{2}-g; z^2\right),$$

are the two linearly independent solutions of (25), with $\tilde{W} = u_1(z)\partial_z u_2(z) - u_2(z)\partial_z u_1(z)$ the respective Wronskian. Given that (25) is an incomplete second-order differential equation, it is straightforward to realize that $\tilde{W}$ is in general a constant [48]. Now, the change of variable $w = u_1/u_2$, together with $dz = -\tilde{W}/u_2^2(z)$, leads to

$$\frac{1}{|\mathcal{N}_\epsilon|^2} = -\frac{1}{\tilde{W}} \int_{w(z=0)}^{w(z\to\infty)} \frac{dw}{(k_a w + k_b)^2} = \frac{1}{\tilde{W} k_a}\left[\left.\frac{1}{k_a u_1(z)/u_2(z) + k_b}\right|_0^\infty\right]. \tag{B.2}$$

From (B.2) and the asymptotic behavior of the confluent hypergeometric function [46] one recovers the normalization constant in (36).

Acknowledgment
The author acknowledges the support from the *Mathematical Physics Laboratory* of the *Centre de Recherches Mathématiques*, through a postdoctoral scholarship. This research was also funded by Consejo Nacional de Ciencia y Tecnología (Mexico), grant number A1-S-24569.

References

[1] Mielnik B and Rosas-Ortiz O 2004 Factorization: Little or great algorithm? *J. Phys. A: Math. Gen.* **37** 10007.
[2] Cooper F Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore).
[3] Dong S H 2007 *Factorization Method in Quantum Mechanics* (Springer, Dordrecht).
[4] Cariñena J and Ramos A (2000) Riccati equation, Factorization Method and Shape Invariance *Rev. Math. Phys. A* **12** 1279.
[5] Khare A and Sukhatme U P 1993 New shape-invariant potentials in supersymmetric quantum mechanics, *J. Phys. A: Math. Gen.* **26** L901.
[6] Matveev V B and Salle M A 1991 *Darboux transformation and solitons* (Springer-Verlag, Berlin).
[7] Mielnik B, Nieto L M and Rosas-Ortiz O 2000 The finite difference algorithm for higher order supersymmetry *Phys. Lett. A* **269** 70.
[8] Mielnik B 1984 Factorization method and new potentials with the oscillator spectrum, *J. Math. Phys.* **25** 3387.
[9] Fernández D J and Hussin V 1999 Higher-order SUSY, linearized nonlinear Heisenberg algebras and coherent states *J. Phys. A: Math. Gen.* **32** 3603.
[10] Znojil Z, Cannata F, Bagchi B and Roychoudhury R 2000 Supersymmetry without hermiticity within PT symmetric quantum mechanics *Phys. Lett. B* **483** 284.
[11] Bagchi B, Mallik S and Quesne C 2001 Generating complex potentials with real eigenvalues in supersymmetric quantum mechanics *Int. J. Mod. Phys. A* **16** 2859.
[12] Rosas-Ortiz O, Castaños O and Schuch D 2015 New supersymmetry-generated complex potentials with real spectra *J. Phys. A: Math. Theor.* **48** 445302.
[13] Rosas-Ortiz O and Zelaya K 2018 Bi-Orthogonal Approach to Non-Hermitian Hamiltonians with the Oscillator Spectrum: Generalized Coherent States for Nonlinear Algebras *Ann. Phys.* **388** 26.
[14] Blanco-Garcia Z, Rosas-Ortiz O and Zelaya K 2019 Interplay between Riccati, Ermakov and Schrödinger equations to produce complex-valued potentials with real energy spectrum *Math. Meth. Appl. Sci.* **42** 4925.
[15] Correa F, Jakubskỳ V and Plyushchay M S 2015 PT-symmetric invisible defects and confluent Darboux-Crum transformations *Phys. Rev. A* **92** 023839.
[16] Quesne C 2006 First-order intertwining operators and position-dependent mass Schrödinger equations in d dimensions *Ann. Phys.* **321**1221.
[17] Cruz y Cruz S and Rosas-Ortiz O 2009 Position-dependent mass oscillators and coherent states, *J. Phys. A: Math. Theor.* **42** 185205.
[18] Cruz y Cruz S 2013 Factorization Method and the Position-dependent Mass Problem, in Kielanowski P, Ali S, Odzijewicz A, Schlichenmaier M and Voronov T (Eds), *Geometric Methods in Physics. Trends in Mathematics.* (Birkhäuser, Basel) p. 229.
[19] Schwabl F 2002 *Quantum Mechanics* 3rd. Edition (Springer-Verlag, Berlin).
[20] Paul W 1990 Electromagnetic traps for charged and neutral particles *Rev. Mod. Phys.* **62** 531.
[21] Pritchard D E 1983 Cooling Neutral Atoms in a Magnetic Trap for Precision Spectroscopy *Phys. Rev. Lett.* **51** 1336.
[22] Glauber R J 1992The Quantum Mechanics of Trapped Wavepackets, in Arimondo E, Philips W D and Sttrumia F (Eds.) *Laser Manipulation of Atoms and Ions: Proceedings of the International Enrico Fermi School, Course 118, Varenna, Italy, July 1-19* ()North Holland, Amsterdam) p. 643.
[23] Combescure M 1986 A quantum particle in a quadrupole radio-frequency trap *Ann. Inst. Henri Poincare A* **44** 293.
[24] Cruz y Cruz and Gress Z 2017 Group approach to the paraxial propagation of Hermite-Gaussian modes in a parabolic medium *Ann. Phys.* **383** 257.
[25] Razo R and Cruz y Cruz S 2019 New confining optical media generated by Darboux transformations *J. Phys.: Conf. Ser.* **1194** 012091.
[26] Contreras A and Jakubský V 2019 Photonic systems with two-dimensional landscapes of complex refractive index via time-dependent supersymmetry *Phys. Rev. A* **99** 053812.

[27] Dodonov V V, Manko O V and Manko V I 1995 Quantum nonstationary oscillator: Models and applications *J. Russ. Laser Res.* **16** 1.

[28] Zelaya K and Rosas-Ortiz O 2020 Quantum nonstationary oscillators: Invariants, dynamical algebras and coherent states via point transformations, *Phys. Scr.* (in press) https://doi.org/10.1088/1402-4896/ab5cbf

[29] Lewis H R and Riesenfeld W B 1969 An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field *J. Math. Phys.* **10**1458.

[30] Bagrov V G, Samsonov B F and Shekoyan L A 1995 Darboux transformation for the nonsteady Schrödinger equation *Russ. Phys. J.* **38** 706.

[31] Bagrov V G and Samsonov B F 1996 Supersymmetry of a nonstationary Schrödinger equation *Phys. Lett. A* **201** 60.

[32] Cannata F, Ioffe M, Junker G and Nishnianidze D 1999 Intertwining relations of non-stationary Schrödinger operators *J. Phys. A: Math. Gen.* **32** 3583.

[33] Suzko A A and Schulze-Halberg A (2009) Darboux transformations and supersymmetry for the generalized Schrödinger equations in (1+1) dimensions *J. Phys. A: Math. Theor.* **42** 295203.

[34] Schulze-Halberg A and Roy B (2014) Time dependent potentials associated with exceptional orthogonal polynomials *J. Math. Phys.* **55** 123506.

[35] Zelaya K and Rosas-Ortiz O 2017 Exactly Solvable Time-Dependent Oscillator-Like Potentials Generated by Darboux Transformations *J. Phys.: Conf. Ser.* **839** 012018.

[36] Contreras-Astorga A 2017 A Time-Dependent Anharmonic Oscillator *J. Phys.: Conf. Ser.* **839** 012019.

[37] Cruz y Cruz S, Razo R, Rosas-Ortiz O and Zelaya K 2020 Coherent states for exactly solvable time-dependent oscillators generated by Darboux transformations *Phys. Scr.* **95** 044009.

[38] Goldman I I and Krivchenkov V D 1961 *Problems in quantum mechanics* (Pergamon, London)

[39] Weissman Y and Jortner J 1979 The isotonic oscillator *Phys. Lett.* **70A** 177.

[40] Cruz y Cruz and Rosas-Ortiz O 2011 $SU(1,1)$ Coherent States For Position-Dependent Mass Singular Oscillators *Int. J. Theor. Phys.* **50** 2201.

[41] Dodonov V V, Man'ko V I and Rosa L (1998) Quantum singular oscillator as a model of a two-ion trap: An amplification of transition probabilities due to small-time variations of the binding potential *Phys. Rev. A* **57** 2851.

[42] Zelaya K and Hussin V 2020 Time-dependent rational extensions of the parametric oscillator: quantum invariants and the factorization method *J. Phys. A: Math. Theor.* (in press) https://doi.org/10.1088/1751-8121/ab78d1

[43] Ermakov V P 1880 Second order differential equations. Conditions of complete integrability *Kiev University Izvvestia* Series III 9 (1880) 125 (in Russian);
Harin A O 2008 *Appl. Anal. Discrete Math.* **2** 123 (English translation).

[44] Cruz y Cruz S, Gress Z, Jiménez-Macías P and Rosas-Ortiz O, Laguerre-Gaussian wave propagation in parabolic media arXiv:2001.03456.

[45] Fernández D J and Morales-Salgado V S (2013) Supersymmetric partners of the harmonic oscillator with an infinite potential barrier *J. Phys. A: Math. Theor.* **47** 035304.

[46] Olver F W J, Lozier D W, Boisvert R F and Clark C W (Eds) 2010 *NIST Handbook of Mathematical Functions* (Cambridge University Press, New York).

[47] Nikiforov A F and Uvarov V B 1998 *Special functions of Mathematical Physics: A Unified Introduction with Applications* (Birkhäuser, Boston).

[48] Zelaya K 2019 Non-Hermitian and time-dependent systems: Exact solutions, generating algebras and nonclassicality of quantum states. Ph D Thesis (Cinvestav, Mexico).

[49] Fernández D J, Hussin V and Morales-Salgado V S 2019 Coherent states for the supersymmetric partners of the truncated oscillator *Eur. Phys. J. Plus* **134** 18.

[50] Quesne C 2011 Rationally-extended radial oscillators and Laguerre exceptional orthogonal polynomials in kth-order SUSYQM *Int. J. Mod. Phys. A* **26** 5337.

[51] Marquette I and Quesne C 2013 New families of superintegrable systems from Hermite and Laguerre exceptional orthogonal polynomials *J. Math. Phys.* **54** 042102.