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Black Holes with a Cloud of Strings and Quintessence in a Non-Linear Electrodynamics Scenario

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Abstract: We obtain exact black hole solutions to Einstein gravity coupled with a nonlinear electrodynamics field, in the presence of a cloud of strings and quintessence, as sources. The solutions have four parameters, namely m, k, a, and α, corresponding to the physical mass of the black hole, the nonlinear charge of a self-gravitating magnetic field, the cloud of strings, and the intensity of the quintessential fluid. The consequences of these sources on the regularity or singularity of the solutions, on their horizons, as well as on the energy conditions, are discussed. We study some aspects concerning the thermodynamics of the black hole, by taking into account the mass, Hawking temperature, and heat capacity and show how these quantities depend on the presence of the cloud of strings and quintessence, in the scenario considered.

Keywords: black hole; cloud of strings; quintessence; non-linear electrodynamics

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1. Introduction

Black holes are sufficiently massive objects that create "a region of no escape", i.e., a region from where no object can escape, not even light [1]. The first and simplest black hole solution was obtained by Schwarzschild in 1916 by solving the Einstein fields equations for a static and spherically symmetric gravitating body [2]. Since then, black holes have attracted the attention of scientists and have been extensively studied. Recently, astrophysics experiments have confirmed the existence of black holes by observing a black hole shadow in M87 [3] or, indirectly, with the detection of a gravitational-wave signal from a binary black hole merge [4,5].

Based on the idea that the universe can be represented by a collection of one-dimensional strings rather than point particles, Letelier [6] introduced a gauge invariant model of a cloud of strings in the framework of General Relativity and obtained a generalization of the Schwarzschild solution, in the sense that it corresponds to the space-time of a static and spherically symmetric gravitating body, surrounded by a cloud of strings [6]. It is important to mention that some authors obtained space-times mathematically similar to the one obtained by Letelier, but from different matter-energy content. For example, Guendelman and Rabinowitz proposed a spherically symmetric "hedgehog" configuration in scalar field theories, which has the same space-time metric of a black hole with a cloud of strings [7]. The effect of a cloud of strings can also be obtained by a nonlinear gauge theory [8], or considering a solid angle deficit [9].

Since being introduced in the literature, the spherically symmetric cloud of strings has been considered in many black hole scenarios, not only in General Relativity [10–14], but also in Modified Gravity Theories [15–19]. It is important to call attention to the fact that some astrophysics analyses were performed from the shadow of Sgr. A, which was detected by Event Horizon Telescope, assuming that this black hole is surrounded by a cloud of strings [20].

Astrophysical observations suggest that the universe is filled by baryons, dark matter, and dark energy, with the latter component being used to understand the present stage of accelerated expansion of the universe [21–23]. Many theoretical models were developed to explain this exotic energy and, among them, the so-called quintessence, which can be described as a scalar field slowly rolling down a potential barrier [24], and which behaves like inflation. The solution corresponding to a static black hole surrounded by quintessence was obtained by Kiselev [25] and, after this, several studies were performed to know how is the role played by quintessence in the behavior of astrophysical systems [14,17,26–29].

In the mid-1930s, Born and Infeld proposed a generalization of the Maxwell's electrodynamics [30,31], in such a way to remove the energy divergence associated with a point-like charge, turning its self-energy finite. An extension of the Nonlinear Electrodynamics proposed by Born and Infeld [30,31], to the framework of the General Relativity was performed by Plebanski [32]. In this context, a static and spherically symmetric solution was obtained some decades ago, by Pellicer and Torrence [33].

Considering nonlinear electromagnetic fields as physical sources of black holes in the context of the General Relativity, it is possible to obtain regular solutions which obey the weak energy condition [34,35]. The thermodynamics properties of these gravitating bodies have been studied [36–38], as well as many other characteristics and their influence on some physical systems [36,37,39–49]. In summary, the interest in considering the coupling of nonlinear electromagnetic fields with gravity is largely motivated by the fact that it is possible to obtain various regular (as well as singular) space-times, with geometries of physical interest, with several publications about this subject; among them, we mention [34,35,50–64].

It is worth calling attention to the fact that black holes are gravitational bodies that involve very high energies and strong nonlinearities. This means that, in order to consider appropriately the behavior of charged particles placed in the space-times generated by these bodies, it is necessary to include a nonlinear electromagnetic field as a possible source of the gravitational field as described by General Relativity. Thus, it is important to think about the question concerning the coupling between gravitational and nonlinear electromagnetic fields.

This paper is organized as follows. In Section 2, we obtain the solution corresponding to the black hole with a cloud of strings and quintessence in a non-linear electrodynamics scenario. In order to obtain this solution, we firstly consider the particular case when the cloud of strings and the quintessence are absent. In the sequence, we consider firstly that only the cloud of strings is present, and then that only the quintessence is present. Finally, we consider the general case with the presence of both sources, namely the cloud of strings and quintessence, in the nonlinear electrodynamics scenario. In Section 3, we focus the discussion on the horizons and Kretschmann scalar analysis. In Sections 4 and 5, we discuss the black hole energy conditions and its thermodynamics, respectively. Finally, in Section 6, we present the discussion and conclusions.

2. Black Hole with a Cloud of Strings and Quintessence in a Non-Linear Electrodynamics Scenario

In this section, we obtain the metrics corresponding to a static black hole with a cloud of strings and quintessence in a non-linear electrodynamics scenario.

Einstein's field equations in vacuum can be obtained from the following action:

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R, \tag{1}$$

with $\kappa^2 = 8\pi G/c^4$, where G is the Newton Gravitational constant and c is the velocity of light in vacuum. The quantities $g = det(g_{\mu\nu})$ and $R = g^{\mu\nu} R_{\mu\nu}$ are the metric determinant and the Ricci scalar, respectively.

The Einstein equations with matter sources are obtained by adding a matter Lagrangian to the functional S. Thus, considering the Lagrangian of the nonlinear electromagnetic theory, $L(F)$, and coupling this quantity with the one associated with the cloud of strings, L_{CS}, we obtain the following action:

$$S_m = \int d^4x \sqrt{-g}\,(L(F) + L_{CS}), \tag{2}$$

where $F = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, with $F_{\mu\nu}$ being the Maxwell–Faraday tensor.

Therefore, the total action describing the black hole with a cloud of strings as a source in the framework of a non-linear electrodynamics scenario is given by

$$\begin{aligned} S &= S_{EH} + S_m \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R + \int d^4x\,\sqrt{-g}\,(L(F) + L_{CS}). \end{aligned} \tag{3}$$

Varying the action (3) with respect to the metric, we find

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2(T^{NED}_{\mu\nu} + T^{CS}_{\mu\nu}), \tag{4}$$

where the stress–energy tensor of the non-linear electromagnetic theory $(T^{NED}_{\mu\nu})$ and the stress–energy tensor for cloud of strings $(T^{CS}_{\mu\nu})$ are given, respectively, by [6,65]:

$$T^{NED}_{\mu\nu} = 2\left(L(F)g_{\mu\nu} - \frac{\partial L(F)}{\partial F}F_{\mu}{}^{\alpha}F_{\nu\alpha}\right), \tag{5}$$

$$T^{CS}_{\mu\nu} = \frac{\rho\Sigma_{\mu}^{\ \beta}\Sigma_{\beta\nu}}{\kappa^2(-\gamma)^{1/2}}. \tag{6}$$

Let us assume that the Lagrangian density of the non-linear electrodynamics, with a quintessential fluid as a source, can be obtained by combining the Lagrangian densities presented in [37,66]. Considering only the contribution arising from the quintessence coupled with the nonlinear electromagnetic field in the Lagrangian density presented in [66], and redefining the constants, we find that

$$L(F) = \frac{Fe^{-\frac{k\sqrt[4]{2q^2F}}{q}}}{\kappa^2} - \frac{3}{2\kappa^2}\alpha\omega_q\left(\frac{2F}{q^2}\right)^{\frac{3(\omega_q+1)}{4}}, \tag{7}$$

where $k = q^2/2m$ is the non-linear self-gravitating magnetic charge, and m is a quantity related to the mass black hole. For a spherically symmetric space-time sourced only by a magnetic charge, the only non-null component of $F_{\mu\nu}$ is $F_{23} = q\sin\theta$ and the scalar F is $F = \frac{q^2}{2r^4}$.

The action that describes the cloud of strings is given by [6,12]

$$S_{CS} = \int d^4x\,\sqrt{-g}\,L_{CS} = \int (-\gamma)^{1/2}\mathcal{M}d\lambda^0 d\lambda^1 = \int \mathcal{M}\left(-\frac{1}{2}\Sigma^{\mu\nu}\Sigma_{\mu\nu}\right)^{1/2} d\lambda^0 d\lambda^1, \tag{8}$$

with $\mathcal{M}$ being a dimensionless constant associated with each string, and γ being the determinant of the induced metric, γ_{AB}, which is given by

$$\gamma_{AB} = g_{\mu\nu}\frac{\partial x^\mu}{\partial\lambda^A}\frac{\partial x^\nu}{\partial\lambda^B}, \tag{9}$$

where $x^\mu = x^\mu(\lambda^A)$ describes the string world sheet, with λ^0 and λ^1 being time-like and space-like parameters, respectively. The bivector $\Sigma^{\mu\nu}$ can be associated with the string world sheet, and is written as

$$\Sigma^{\mu\nu} = \epsilon^{AB} \frac{\partial x^{\mu}}{\partial \lambda^{A}} \frac{\partial x^{\nu}}{\partial \lambda^{B}}, \tag{10}$$

where ϵ^{AB} is the Levi–Civita symbol, such that $\epsilon^{01} = -\epsilon^{10} = 1$. Due to the space-time symmetry, the only non-null component of the bivector $\Sigma^{\mu\nu}$ is $\Sigma^{01} = a/\rho r^2$, which depends only on the radial coordinate. In this identity, a is an integration constant related to the cloud of strings, which assumes values in the range $0 < a < 1$.

For a spherically symmetric space-time, the general metric can be written as

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{11}$$

Concerning the stress–energy tensors of the non-linear electromagnetic field $(T^{NED}_{\mu\nu})$ and of the cloud of strings $(T^{CS}_{\mu\nu})$, the non-null components are given by

$$T^{NED}_{00} = \frac{2kmf(r)e^{-\frac{k}{r}}}{\kappa^2 r^4} - \frac{3\alpha\omega_q f(r) r^{-3\omega_q - 3}}{\kappa^2}. \tag{12}$$

$$T^{NED}_{11} = -\frac{q^2(k-2r)e^{-\frac{k}{r}}}{2\kappa^2 r^3} - \frac{3}{2\kappa^2}\alpha\omega_q(3\omega_q+1)r^{-3\omega_q-1}. \tag{13}$$

$$T^{CS}_{00} = \frac{f(r)}{\kappa^2}\frac{a}{r^2}, \qquad T^{CS}_{11} = 0. \tag{14}$$

Using Equations (12)–(14) and considering the line element expressed by Equation (11), the Einstein field equations are written as

$$\begin{aligned} &1 - a - f(r) + rf'(r) + r^2 f''(r) \\ &- \frac{2kme^{-\frac{k}{r}}}{r^2} + \frac{q^2 e^{-\frac{k}{r}}(k-2r)}{r^3} \\ &+ 3\alpha\omega_q(3\omega_q+2)r^{-3\omega_q-1} = 0, \end{aligned} \tag{15}$$

whose solution is

$$f(r) = 1 - a - \frac{2me^{-\frac{k}{r}}}{r} - \alpha r^{-3\omega_q - 1} + C_2 r + \frac{C_1}{r}. \tag{16}$$

In the analysis that will be performed, only one case will be considered, namely quintessence dark energy, which corresponds to $\omega_q = -2/3$. In this case, the fourth and the fifth terms in Equation (16) can be combined. Then, also combining the third and last terms as well, we can write $f(r)$ in a simplified form. Thus, substituting this simplified form of $f(r)$ into Equation (11), we finally obtain the space-time metric corresponding to a black hole surrounded by quintessence and a cloud of strings in a non-linear electrodynamics scenario coupled to Einstein gravity, which is given by

$$\begin{aligned} ds^2 = &\left(1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)dt^2 \\ &- \left(1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)^{-1} dr^2 - r^2 d\Omega^2. \end{aligned} \tag{17}$$

Note that, in Equation (17), the constant ω_q was preserved, but we have in mind that $\omega_q = -2/3$.

The above metric can be understood as the solution of the Einstein field equation, in which the gravitational field is coupled to a nonlinear Maxwell field with two sources, namely the cloud of strings and the quintessence. The metric can also be interpreted as generated in the framework of the General Relativity, with three physical sources, namely a nonlinear electromagnetic field, a cloud of strings, and a quintessential fluid. Note that our result when both sources are included, namely the cloud of strings and the quintessence, is a generalization of a result already published [37], which investigated the role played by the presence of the cloud of strings on the thermodynamics properties, quasi-normal modes, and the shadow.

As should be expected, in the limit $k = 0$, the solution is reduced to the Letelier solution with quintessence [11], and if $k = 0$ and $a = 0$, we recover the Kiselev solution [25]. If $k = 0$ and $\alpha = 0$, we obtain the Letelier solution [6] and, finally, we obtain the Schwarzschild solution [2] for $k = 0$, $a = 0$, and $\alpha = 0$. It is worth calling attention to the fact that, for $r >> k$, the obtained solution reduces to the Reissner–Nordström with cloud of strings and quintessence as should be expected [67].

3. Black Hole Horizons, Regularity and Singularity

3.1. Black Hole Horizons

Now, let us investigate the black hole horizons, which can be determined by solving the following equation:

$$f(r) = 1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r} = 0. \tag{18}$$

In Figure 1, we represent the function $f(r)$ for different values of a, α, k and for $\omega_q = -2/3$, which is the only case to be considered. The horizons of the black hole are defined by the roots of the function $f(r)$. We can see that the black hole will have, at most, three horizons.

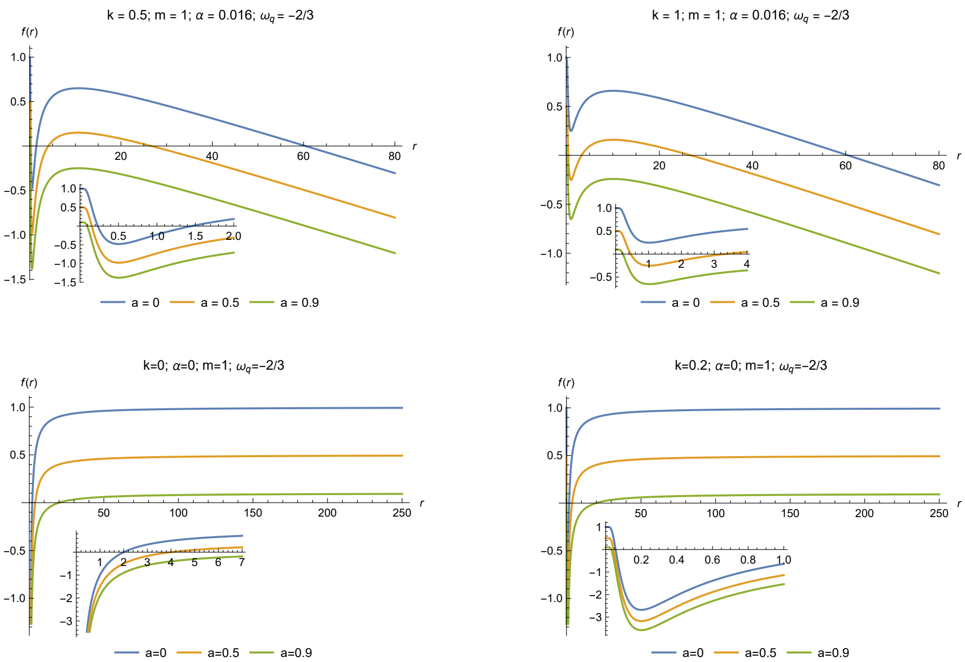


Figure 1. The function $f(r)$ for different values of k, α and a.

3.2. Regularity and Singularity

The search with the aim of knowing if a given black hole is regular or singular can be performed following different routes, such as by investigating whether singularities exist or not in some invariant constructed with the curvature tensor, or by examining whether the geodesics are extensible or not.

Among these two procedures, the analysis of the behavior of the scalar constructed with the curvature, furnish us a local and simple analysis concerning the existence of singularities or not.

We will adopt the criterion to test for the existence of singularities of a metric by examining one of the coordinate invariants quantities constructed with $R_{\mu\nu}$ or $R_{\mu\nu\sigma\rho}$. There is a finite number of possibilities to construct these invariants. We will consider one of them, named the Kretschmann scalar, which is given by $K = R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$.

In what follows, let us calculate the Kretschmann scalar, and check for singularities by examining its behavior for $r \to 0$ and $r \to \infty$. These calculations will be performed in four different cases, namely, a black hole solution coupled with nonlinear electrodynamics, which is equivalent to consider as a source, a nonlinear electromagnetic field. In the sequence, we will add a cloud of strings as a source, then substitute this source by quintessence and, finally, add the cloud of strings and quintessence as sources.

3.2.1. Black Hole Solution Coupled with Nonlinear Electrodynamics

Firstly, as stated before, let us consider the case where the cloud of strings and the quintessence are absent. Thus, $a = 0$ and $\alpha = 0$, and Equation (17) turns into

$$ds^2 = \left(1 - \frac{2m}{r}e^{-k/r}\right)dt^2 - \left(1 - \frac{2m}{r}e^{-k/r}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{19}$$

whose Kretschmann scalar is

$$K = \frac{4m^2 e^{-\frac{2k}{r}}\left(k^4 - 8k^3 r + 24k^2 r^2 - 24kr^3 + 12r^4\right)}{r^{10}}. \tag{20}$$

Calculating the limits of the Kretschmann scalar as $r \to 0$ and $r \to \infty$, we obtain

$$\lim_{r\to 0} K = 0 \text{ and } \lim_{r\to\infty} K = 0. \tag{21}$$

These results tell us that the metric describing a black hole space-time obtained in a scenario where a nonlinear electromagnetic field is coupled to Einstein's gravity is regular at the origin, $r = 0$, i.e., it has no singularity at this point, confirming the results from the literature [68].

3.2.2. Black Hole Solution Coupled with Nonlinear Electrodynamics and a Cloud of Strings as a Source

Now, we consider the presence of a cloud of strings. The corresponding solution is obtained by assuming that $\alpha = 0$ in Equation (17). Thus, the resulting metric is given by

$$ds^2 = \left(1 - a - \frac{2m}{r}e^{-k/r}\right)dt^2 - \left(1 - a - \frac{2m}{r}e^{-k/r}\right)^{-1}dr^2 - r^2 d\Omega^2. \tag{22}$$

The Kretschmann scalar is now given by

$$K = \frac{4a^2}{r^4} + \frac{16ame^{-\frac{k}{r}}}{r^5} + \frac{4m^2 e^{-\frac{2k}{r}}\left(k^4 - 8k^3 r + 24k^2 r^2 - 24kr^3 + 12r^4\right)}{r^{10}}, \tag{23}$$

whose limits, when $r \to 0$ and $r \to \infty$, are

$$\lim_{r\to 0} K = \infty \text{ and } \lim_{r\to\infty} K = 0. \tag{24}$$

Thus, we conclude that the inclusion of the cloud of strings in the non-linear electrodynamics black hole affects the regularity of the metric by introducing a true singular at the origin [68].

3.2.3. Black Hole Solution Coupled with Nonlinear Electrodynamics and Quintessence as a Source

In what follows, let us consider another particular case of the metric given by Equation (17), namely by considering $a = 0$, which means that there is no cloud of strings, but only quintessence. In this case, the metric given by Equation (17) turns into

$$\begin{aligned} ds^2 = & \left(1 - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)dt^2 \\ & - \left(1 - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)^{-1}dr^2 - r^2 d\Omega^2, \end{aligned} \tag{25}$$

whose Kretschmann scalar is now given by

$$\begin{aligned} K = & + \frac{4m^2 e^{-\frac{2k}{r}}\left(k^4 - 8k^3 r + 24k^2 r^2 - 24kr^3 + 12r^4\right)}{r^{10}} \\ & + 3\alpha^2\left(27\omega_q^4 + 54\omega_q^3 + 51\omega_q^2 + 20\omega_q + 4\right)r^{-6(\omega_q+1)} \\ & + 24\alpha m\left(3\omega_q^2 + 5\omega_q + 2\right)e^{-\frac{k}{r}}r^{-3(\omega_q+2)} \\ & - 48\alpha k m\left(3\omega_q^2 + 4\omega_q + 1\right)e^{-\frac{k}{r}}r^{-3\omega_q-7} \\ & + 4\alpha k^2 m\left(9\omega_q^2 + 9\omega_q + 2\right)e^{-\frac{k}{r}}r^{-3\omega_q-8}. \end{aligned} \tag{26}$$

Let us consider the particular case $\omega_q = -2/3$, which corresponds to the quintessence. Thus, we obtain the following results:

$$\lim_{r\to 0} K = \infty \text{ and } \lim_{r\to\infty} K = 0. \tag{27}$$

In conclusion, we can say that, similarly to the previous case, when the quintessence is present, namely the obtained metric has a true singularity at the origin, it is different from what happens when there is no quintessence.

3.2.4. Black Hole Solution Coupled with Nonlinear Electrodynamics and a Cloud of Strings and Quintessence as Sources

The metric of the non-linear electrodynamics space-time with quintessence and cloud of strings is given by

$$\begin{aligned} ds^2 = & \left(1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)dt^2 \\ & - \left(1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r}e^{-k/r}\right)^{-1}dr^2 - r^2 d\Omega^2, \end{aligned} \tag{28}$$

whose Kretschmann scalar is

$$\begin{aligned} K = & + \frac{4m^2 e^{-\frac{2k}{r}}\left(k^4 - 8k^3 r + 24k^2 r^2 - 24kr^3 + 12r^4\right)}{r^{10}} \\ & + \frac{16ame^{-\frac{k}{r}}}{r^5} + 4\alpha k^2 m\left(9\omega_q^2 + 9\omega_q + 2\right)e^{-\frac{k}{r}}r^{-3\omega_q-8} \\ & + \frac{4a^2}{r^4} + 3\alpha^2\omega_q\left(27\omega_q^3 + 54\omega_q^2 + 51\omega_q + 20\right)r^{-6(\omega_q+1)} \\ & - 48\alpha k m\left(3\omega_q^2 + 4\omega_q + 1\right)e^{-\frac{k}{r}}r^{-3\omega_q-7} \\ & + 24\alpha m\left(3\omega_q^2 + 5\omega_q + 2\right)e^{-\frac{k}{r}}r^{-3(\omega_q+2)} \\ & + 4\alpha r^{-6(\omega_q+1)}\left(2ar^{3\omega_q+1} + 3\alpha\right). \end{aligned} \tag{29}$$

We will now determine the limit of the Kretschmann scalar with $r \to 0$, which is given by

$$\lim_{r\to 0} K = \infty. \tag{30}$$

Thus, we conclude that including the cloud of strings in the non-linear electrodynamics black hole with quintessence affects the regularity of the metric by making it singular at the origin.

4. Energy Conditions

In the context of General Relativity, the energy conditions were introduced as physically reasonable restrictions on matter. Basically, they are restrictions concerning the local energy density, which should be positive, and the velocity of current of the fluid of matter, which cannot be superluminal. There are four main energy conditions: the weak energy condition (WEC), the strong energy condition (SEC), the dominant energy condition (DEC) and the null energy condition (NEC) [69].

It is known that regular black holes violate energy conditions [70]. Since introducing a cloud of strings and quintessence has transformed the black hole metric into a singular metric, we must analyze the energy conditions of the obtained black hole solution.

The stress–energy tensor can be written in the form

$$T^{\mu}_{\mu} = \mathrm{diag}[\rho, -p_r, -p_t, -p_t], \tag{31}$$

where ρ is the energy density, p_r is the radial pressure, and p_t is the tangential pressure.

Considering the Einstein equations and Equation (11), we find

$$\rho = \frac{1 - f(r) - rf'(r)}{\kappa^2 r^2}, \tag{32}$$

$$p_r = \frac{rf'(r) + f - 1}{\kappa^2 r^2}, \tag{33}$$

$$p_t = \frac{rf''(r) + 2f'(r)}{2\kappa^2 r}, \tag{34}$$

where $f(r)$ is given by

$$f(r) = 1 - a - \frac{\alpha}{r^{3\omega_q+1}} - \frac{2m}{r} e^{-k/r}, \tag{35}$$

which corresponds to Equation (16), assuming that the constants C_1 and C_2 are equal to zero, or in other words, the fifth term was put together with the third and the last term with the second. From now on, we will consider this algebraic form for the function $f(r)$. These conditions are given by the inequalities [13,69,71]

$$NEC_{1,2} = WEC_{1,2} = SEC_{1,2} \Leftrightarrow \rho + p_{r,t} \geq 0, \tag{36}$$

$$SEC_3 \Leftrightarrow \rho + p_r + 2p_t \geq 0, \tag{37}$$

$$DEC_{1,2} \Leftrightarrow \rho - |p_{r,t}| \geq 0, \tag{38}$$

$$DEC_3 = WEC_3 \Leftrightarrow \rho \geq 0, \tag{39}$$

with the indexes 1 and 2 being related to the radial and tangential components of the pressure, respectively. From the previous equations, we can observe that $DEC_{1,2} \Leftrightarrow ((NEC_{1,2})$ and $(\rho - p_{r,t} \geq 0))$. Thus, we can replace $DEC_{1,2}$ by $\rho - p_{r,t} \geq 0$.

In what follows, let us consider $f(r) > 0$, in which case t is a time-like coordinate. Thus, the energy conditions are given by the following relations:

$$NEC_1 \Leftrightarrow 0, \tag{40}$$

$$NEC_2 \Leftrightarrow \frac{a}{\kappa^2 r^2} - \frac{kme^{-\frac{k}{r}}(k-4r)}{\kappa^2 r^5} - \frac{9\alpha\omega_q(\omega_q+1)r^{-3(\omega_q+1)}}{2\kappa^2} \geq 0, \tag{41}$$

$$WEC_3 \Leftrightarrow \frac{a}{\kappa^2 r^2} + \frac{2kme^{-\frac{k}{r}}}{\kappa^2 r^4} - \frac{3\alpha\omega_q r^{-3(\omega_q+1)}}{\kappa^2} \geq 0, \tag{42}$$

$$SEC_3 \Leftrightarrow -\frac{2kme^{-\frac{k}{r}}(k-2r)}{\kappa^2 r^5} - \frac{3\alpha\omega_q(3\omega_q+1)r^{-3(\omega_q+1)}}{\kappa^2} \geq 0, \tag{43}$$

$$DEC_1 \Leftrightarrow \frac{2a}{\kappa^2 r^2} + \frac{4kme^{-\frac{k}{r}}}{\kappa^2 r^4} - \frac{6\alpha\omega_q r^{-3\omega_q-3}}{\kappa^2} \geq 0, \tag{44}$$

$$DEC_2 \Leftrightarrow \frac{a}{\kappa^2 r^2} + \frac{k^2 me^{-\frac{k}{r}}}{\kappa^2 r^5} + \frac{3\alpha\omega_q(3\omega_q-1)r^{-3(\omega_q+1)}}{2\kappa^2} \geq 0. \tag{45}$$

If we consider $-1 < \omega_q < -\frac{1}{3}$ and the limit $r \to \infty$, we obtain

$$NEC_1 \Leftrightarrow 0, \tag{46}$$
$$NEC_2 \Leftrightarrow 0, \tag{47}$$
$$WEC_3 \Leftrightarrow 0, \tag{48}$$
$$SEC_3 \Leftrightarrow 0, \tag{49}$$
$$DEC_1 \Leftrightarrow 0, \tag{50}$$
$$DEC_2 \Leftrightarrow 0. \tag{51}$$

The results concerning the energy conditions show us that, in the case considered, the energy conditions should be satisfied or violated, depending on the relation between the parameters. For $r \to \infty$, we can observe that all energy conditions are satisfied. We can also state that the conclusions about energy conditions are the same for $f(r) < 0$, in which case t is a space-like coordinate.

5. Black Hole Thermodynamics

In this section, we study some thermodynamic quantities associated with the black hole solution obtained. Specifically, we will examine the behavior of the mass, Hawking temperature and heat capacity.

5.1. Black Hole Mass

For any horizon of radius r_h, we know that $f(r_h) = 0$. This condition gives us the mass

$$m = -\frac{1}{2}e^{\frac{k}{r_h}} r_h^{-3\omega_q}\left[(a-1)r_h^{3\omega_q+1} + \alpha\right], \tag{52}$$

which is written in terms of the parameters that codify the coupling to the nonlinear electrodynamics, the presence of the cloud of strings, as well as, of the quintessence. Note that when $k = 0$, $a = 0$, and $\alpha = 0$, we recover the mass of the Schwarzschild black hole in terms of the horizon radius. Therefore, the expression given in Equation (52) represents the mass (energy) of the black hole in a nonlinear electrodynamics scenario, appropriately modified by the presence of the cloud of strings and the quintessence.

The mass given by (52) can be rewritten in terms of the entropy of the black hole whose relation to the area of the horizon, A, as in ref. [72], is given by $S = A/4 = \pi r_h^2$. Thus, in terms of the entropy, Equation (52) turns into

$$m = \frac{e^{\frac{k\sqrt{\pi}}{\sqrt{S}}} S^{-\frac{3\omega_q}{2}}\left[(1-a)S^{\frac{3\omega_q}{2}+\frac{1}{2}} - \alpha\pi^{\frac{3\omega_q}{2}+\frac{1}{2}}\right]}{2\sqrt{\pi}}, \tag{53}$$

whose behavior, is shown in Figure 2 for different values of k, a, and α.

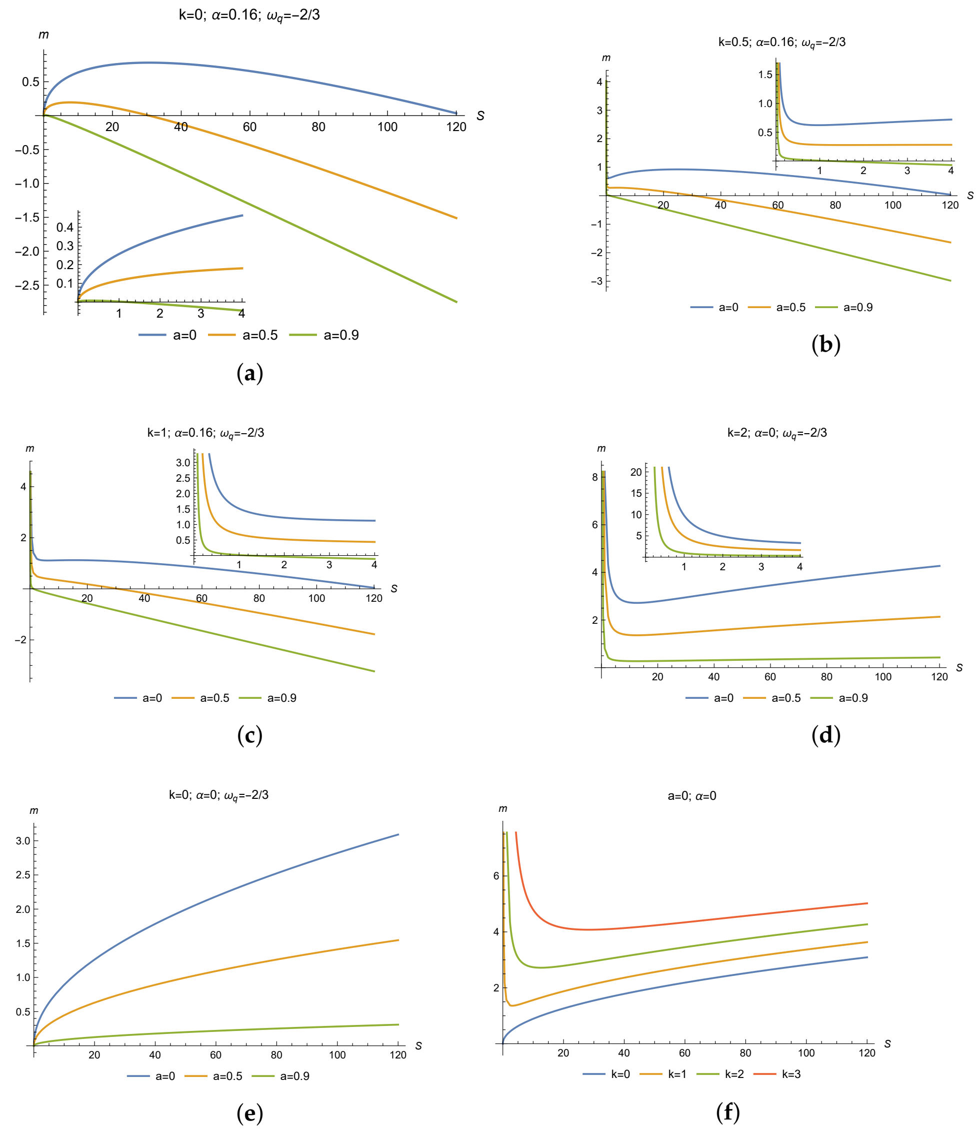


Figure 2. Black hole mass as a function of the entropy $m(S)$ for different values of k, a, and α.

In Figure 2e, the behavior of the mass parameter, m, as a function of the black hole entropy, S, in the absence of quintessence, $\alpha = 0$ and without coupling with non-linear electrodynamics, which means that $k = 0$ or, in other words, the Letelier space-time is represented. It can be seen that, in this scenario, the mass parameter has only positive values, for $0 < a < 1$, and positive values of the entropy, S. Analogous behavior occurs when we consider space-time in a non-linear electrodynamics scenario, Figure 2f. When we add the quintessence, Figure 2a–c, the mass parameter will present positive and negative values depending on the black hole parameters.

5.2. Hawking Temperature

The black hole Hawking temperature of the radiation emitted is given by [73]

$$T_\kappa = \frac{\kappa}{2\pi}, \tag{54}$$

where

$$\kappa = \left.\frac{f'(r)}{2}\right|_{r_h} \tag{55}$$

is the black hole surface gravity and $'$ denotes the derivative concerning the radial coordinate.

Using Equation (54) with κ given by (55), it is possible to calculate the Hawking temperature, $T_\kappa = T$, for the black holes with a cloud of strings, quintessence, and non-linear electrodynamics,

$$T_\kappa = \frac{S^{-\frac{3}{2}(\omega_q+1)}}{4\sqrt{\pi}}\left[k\sqrt{\pi}(a-1)S^{\frac{3\omega_q}{2}+\frac{1}{2}} \right. \\ \left. -(a-1)S^{\frac{3\omega_q}{2}+1} + k\alpha\pi^{\frac{3\omega_q}{2}+1} + 3\alpha\omega_q\sqrt{S}\pi^{\frac{3\omega_q}{2}+\frac{1}{2}}\right]. \tag{56}$$

In Figure 3, we represent the behavior of the temperature parameter, T, as a function of the entropy of the black hole, S, in different situations. Note that for the Schwarzschild space-time ($k = 0$, $a = 0$ and $\alpha = 0$), the temperature parameter presents only positive values for $S > 0$. Analogously, for the Letelier space-time, $k = 0$, $0 < a < 1$, and $\alpha = 0$, as shown in Figure 3e.

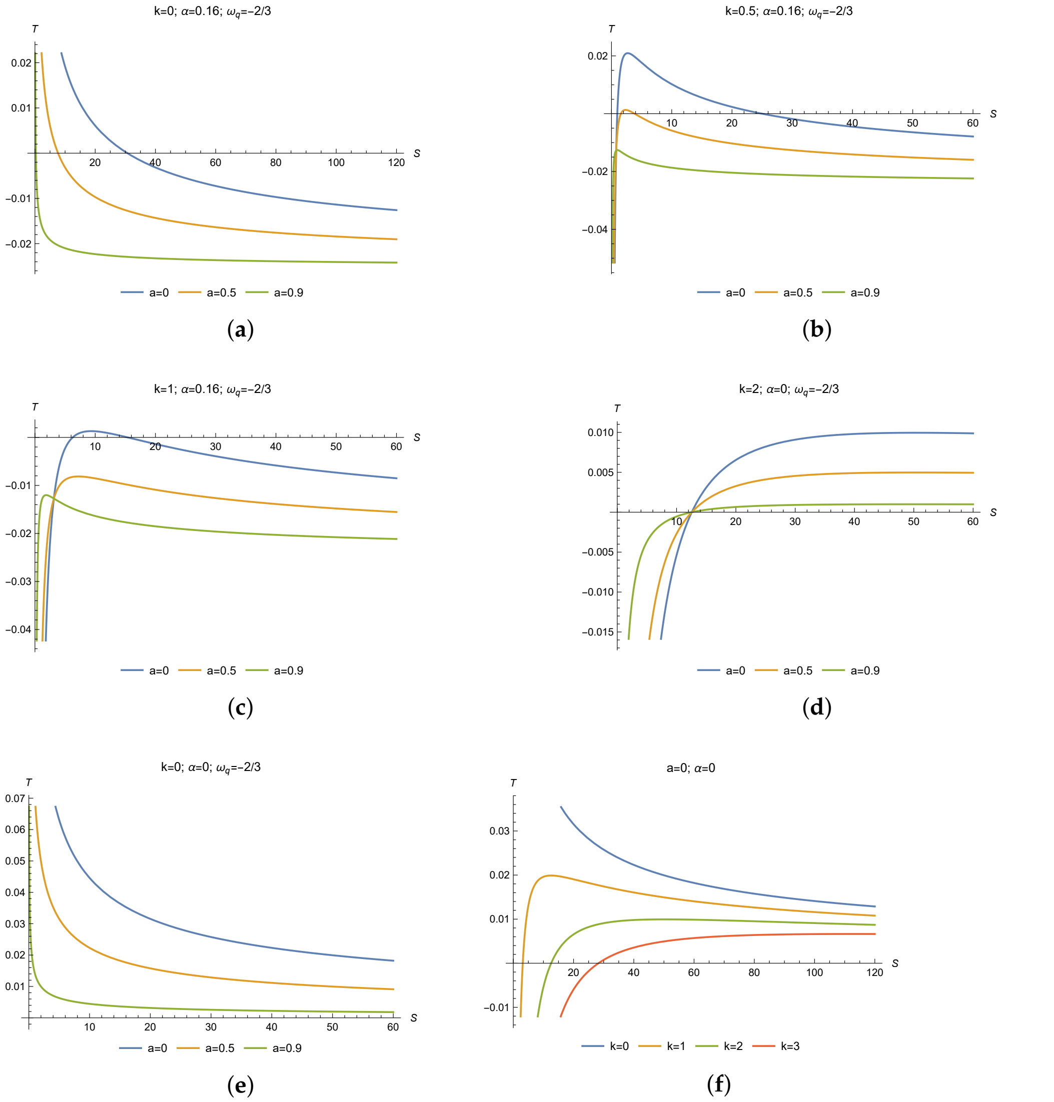


Figure 3. Black hole temperature as a function of the entropy $T(S)$ for different values of k, a and α.

When we consider the non-linear electrodynamics space-time ($a = 0$ and $\alpha = 0$), Figure 3f, it is already possible to notice that the temperature parameter will present positive and negative values for $S > 0$. This also occurs when considering the cloud of strings and quintessence in non-linear electrodynamics space-time ($\alpha = 0.16$ and $0 < a < 1$).

5.3. The First Law of Black Hole Thermodynamics

The black hole mass of Equation (53) can be written as $m = m(S, \alpha, a)$. If we consider that the parameters related to the cloud of string and the quintessence, respectively a and α, are extensive thermodynamic parameters, we can write the first law of black hole thermodynamics as [74,75]

$$dm = T_H dS + \mathcal{A}_H d\alpha + \mathcal{B}_H da. \tag{57}$$

with

$$T_H = \frac{\partial m}{\partial S}, \ \mathcal{A}_H = \frac{\partial m}{\partial \alpha} \text{ and } \mathcal{B}_H = \frac{\partial m}{\partial a}, \tag{58}$$

where $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{B}_{\mathcal{H}}$ are the intensive thermodynamic variables conjugate to the parameters α and a, respectively.

Now, let us calculate the temperature T_H using Equations (57), (58) and (53), which results in

$$\begin{aligned} T_H =& \frac{\partial m}{\partial S} = \frac{e^{\frac{k\sqrt{\pi}}{\sqrt{S}}} S^{-\frac{3}{2}(\omega_q+1)}}{4\sqrt{\pi}} \times \\ &\left[\sqrt{\pi}(a-1)kS^{\frac{3\omega_q}{2}+\frac{1}{2}} - (a-1)S^{\frac{3\omega_q}{2}+1} + \alpha k \pi^{\frac{3\omega_q}{2}+1} + 3\alpha\sqrt{S}\pi^{\frac{3\omega_q}{2}+\frac{1}{2}}\omega_q\right]. \end{aligned} \tag{59}$$

We can notice that the temperature T_κ is different from T_H. In addition to the temperature, the other quantities derived from the first law of thermodynamics can also present similar behavior. To solve this problem, the first law of black hole thermodynamics needs to be modified [76–79].

The usual first law of black hole thermodynamics arises in a context in which the Lagrangian of the theory does not explicitly depend on the mass of the black hole. When building the first law for metrics with nonlinear electrodynamics and other similar theories, derivatives of the stress–energy tensor must be taken into account and these corrections modify the first law in such a way that the old thermodynamic quantities are relate to the new ones through a correction factor [76–79]. Thus, the new first law of thermodynamics of Equation (57) can be written as [76–79]

$$d\mathcal{M} = TdS + \mathcal{A}d\alpha + \mathcal{B}da, \tag{60}$$

where

$$d\mathcal{M} = Wdm, \tag{61}$$

and W is the correction factor given by

$$W = \left(1 + 4\pi \int_{r_+}^{\infty} r^2 \frac{\partial T_0^0}{\partial m} dr\right), \tag{62}$$

and T_0^0 is a stress–energy tensor component.

Thus, we can write

$$\begin{aligned} T_\kappa &= WT_H = W\frac{\partial m}{\partial S}, \\ \mathcal{A} &= W\mathcal{A}_H = W\frac{\partial m}{\partial \alpha}, \\ \mathcal{B} &= W\mathcal{B}_H = W\frac{\partial m}{\partial a}, \end{aligned} \tag{63}$$

with the factor W being given by

$$W = e^{-k\sqrt{\frac{\pi}{S}}} = e^{-\frac{k}{r_h}}. \tag{64}$$

The intensive thermodynamic variables conjugate to the parameters α and a are, respectively, given by

$$\mathcal{A} = W\frac{\partial m}{\partial \alpha} = -\frac{1}{2}\pi^{\frac{3\omega_q}{2}} S^{-\frac{3\omega_q}{2}} = -\frac{1}{2} r_h^{-3\omega_q}, \quad \mathcal{B} = W\frac{\partial m}{\partial a} = -\frac{1}{2}\sqrt{\frac{S}{\pi}} = -\frac{r_h}{2}. \tag{65}$$

5.4. Heat Capacity

The black hole heat capacity can be calculated from

$$C = T_\kappa \frac{\partial S}{\partial T_\kappa} = T_\kappa \left(\frac{\partial T_\kappa}{\partial S}\right)^{-1}. \tag{66}$$

Substituting Equation (56) into Equation (66), we obtain the following expression for the black hole heat capacity as a function of the entropy (S):

$$C = \frac{2S\psi_1}{\psi_2 + \psi_3}, \tag{67}$$

where

$$\psi_1 = \sqrt{\pi}(a-1)kS^{\frac{3\omega_q}{2}+\frac{1}{2}} - \left[(a-1)S^{\frac{3\omega_q}{2}+1}\right] + \alpha k \pi^{\frac{3\omega_q}{2}+1} + 3\alpha\sqrt{S}\pi^{\frac{3\omega_q}{2}+\frac{1}{2}}\omega_q, \tag{68}$$

$$\psi_2 = -2\sqrt{\pi}(a-1)kS^{\frac{3\omega_q}{2}+\frac{1}{2}} + (a-1)S^{\frac{3\omega_q}{2}+1} - 3\alpha k\pi^{\frac{3\omega_q}{2}+1}, \tag{69}$$

$$\psi_3 = -3\alpha\left(\pi k + 2\sqrt{\pi}\sqrt{S}\right)\pi^{\frac{3\omega_q}{2}}\omega_q - 9\alpha\sqrt{S}\pi^{\frac{3\omega_q}{2}+\frac{1}{2}}\omega_q^2, \tag{70}$$

whose behavior is shown in Figure 4, as a function of the black hole entropy, for different values of the parameters of the cloud of strings, non-linear electrodynamics, and quintessence.

It is known that the sign of the heat capacity is related to the thermodynamic stability of a system. If $C > 0$, the system is called stable, and it is called unstable if $C < 0$.

From Figure 4, we can conclude that there are values of S for which the heat capacity is positive, as well as there are values for which the heat capacity is negative. In other words, the black hole can be thermodynamically stable or unstable and this stability is related to the values of the cloud of strings, non-linear electrodynamics, and the quintessential parameters. We can see that the transition point, in which the heat capacity diverges, changes when we vary these parameters.

In the absence of the quintessence ($\alpha = 0$) and of the non-linear electrodynamics ($k = 0$), Equation (67) reduces to $C = -2S$, as expected for the case of Schwarzschild space-time. In this case, the heat capacity is negative for $S > 0$, which indicates an unstable thermodynamic system, as shown in Figure 4e.

Otherwise, when we consider the effect of the cloud of strings, non-linear electrodynamics, and the quintessence, there are values of the entropy for which the heat capacity acquires positive values.

It is important to highlight that the cloud of strings and non-linear electrodynamics parameters plays a role in the behavior of the heat capacity, modifying the phase transition point as shown in Figure 4b–f, since for $k = 0$, $\alpha = 0.16$, and $\omega_q = -2/3$, there are no phase transition points (Figure 4a).

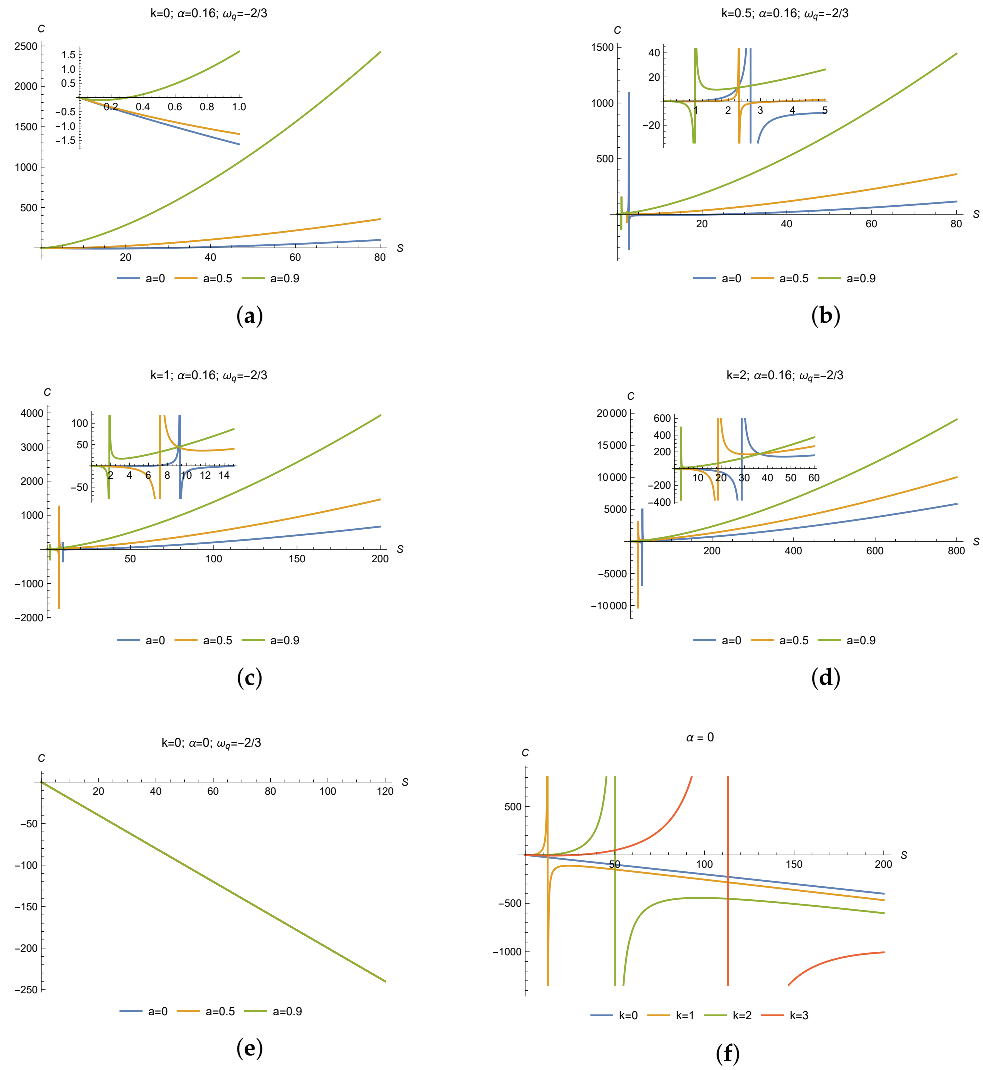


Figure 4. Black hole heat capacity as a function of the entropy $C(S)$ for different values of k, a, and α.

5.5. Gibbs Free Energy

The Gibbs free energy of a thermodynamical system can be written as (Figure 5)

$$G = M - TS. \tag{71}$$

From Equations (53) and (56), we can write

$$G = \frac{1}{4}\left(-k(a + \alpha r_h - 1) - 2r_h e^{\frac{k}{r_h}}(a + \alpha r_h - 1) + r_h(a + 2\alpha r_h - 1)\right). \tag{72}$$

The stability of a black hole can be analyzed by calculating the Hessian matrix, which is given by [80]

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{73}$$

where $H_{11} = \frac{\partial^2 G}{\partial T^2}$, $H_{22} = \frac{\partial^2 G}{\partial \mathcal{A}_H^2}$ and $H_{12} = H_{21} = \frac{\partial^2 G}{\partial \mathcal{A}_H \partial T}$. Note that we are considering only the intensive thermodynamic parameter associated with quintessence. Since $H_{12} = H_{21}$, the thermodynamical stability of the black hole can be analyzed by evaluating the trace of $\mathbf{H}$ [80],

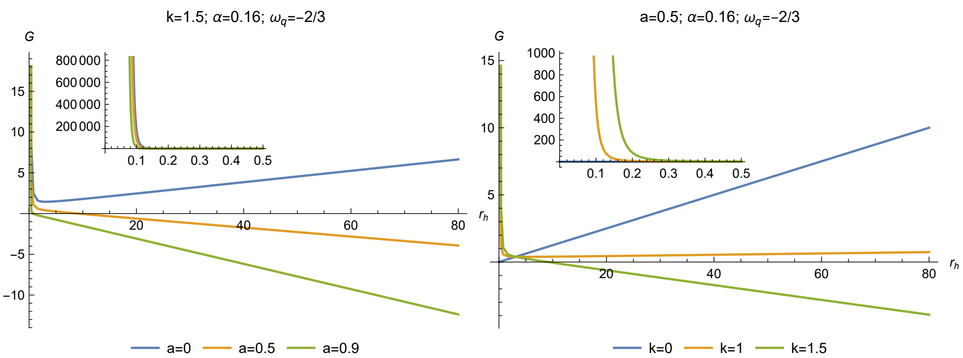
$$Tr(\mathbf{H}) = H_{11} + H_{22}. \tag{74}$$

Fixing $\omega_q = -2/3$, we obtain

$$
\begin{aligned}
H_{11} = & -\frac{8\pi^2 r_h^3}{(k(2a+\alpha r_h - 2) - a r_h + r_h)^3}[\\
& r_h^2\left(3\alpha(a-1)\left(k^2 - 6kr_h + 2r_h^2\right) + (a-1)^2(r_h - 3k) + \alpha^2 k r_h (k - 6r_h)\right) \\
& +(a-1)^2 e^{k/r_h}\left(2k^3 - 7k^2 r_h + 8kr_h^2 - 2r_h^3\right) \\
& +(a-1)\alpha r_h (3k - r_h) e^{k/r_h}\left(k^2 - 4kr_h + 6r_h^2\right) \\
& + \alpha^2 k r_h^2 e^{k/r_h}\left(k^2 - 4kr_h + 6r_h^2\right)],
\end{aligned}
\tag{75}
$$

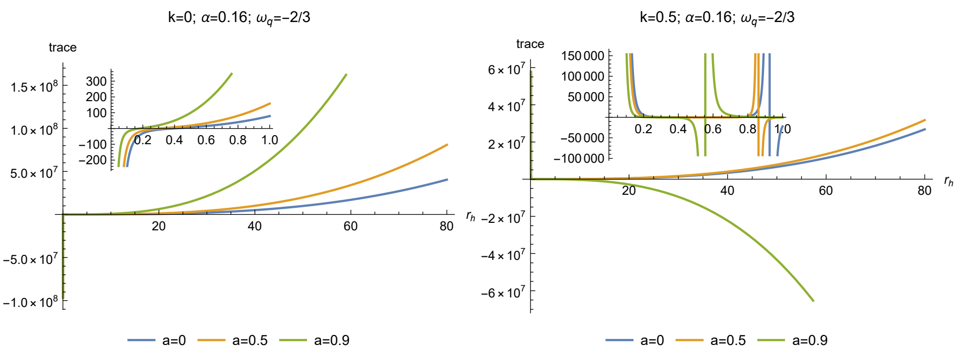
$$
H_{22} = \frac{r_h^2(-a + \alpha k + 1) - 2e^{\frac{k}{r_h}}\left(k^2(a + \alpha r_h - 1) - kr_h(-a + \alpha r_h + 1) - (a-1)r_h^2\right)}{4r_h^5}. \tag{76}
$$

In Figure 6, we represent the trace of the Hessian matrix as a function of the horizon radius. Note that the black hole shows stability for $k = 0$, $\alpha = 0.16$, $\omega_q = -2/3$, and positive values of r_h. On the other hand, for for $k = 0.5$, $\alpha = 0.16$ and $\omega_q = -2/3$, the black hole faces transitions as r_h increases.



(**a**) (**b**)

Figure 5. Plot of Gibbs free energy as a function of horizon radius.



(**a**) (**b**)

Figure 6. The trace of Hessian matrix as a function of the horizon radius.

6. Concluding Remarks

We obtain exact black hole solutions to Einstein equations coupled with nonlinear electrodynamic field, taking into account two sources, namely a cloud of strings and quintessential fluid. The solutions have four parameters, m, k, a, and α, corresponding to the physical mass of the black hole, the non-linear charge of a self-gravitating magnetic field, the cloud of strings, and the intensity of the quintessential fluid.

Concerning the horizons, Figure 1 tell us that there exist, at most, three horizons, depending on the values of the parameters that appear in the solutions which characterize the system under consideration. Examining the behavior of the parameter $f(r)$ in the metric of Equation (17), we can observe, in Figure 1, that the number and location of the black hole horizons are determined by the parameters associated with the matter contents. For $k = 0$, $m = 1$, $\alpha = 0$, and $\omega_q = -2/3$, for example, a naked singularity is formed.

The obtained results when the sources are absent confirm that the metric describing a black hole space-time obtained in a scenario where a nonlinear electromagnetic field is coupled to Einstein gravity, is regular at the origin, $r = 0$, i.e., it has no singularity at this point [68].

On the other hand, when a cloud of strings or a quintessential field is present, the metric is no more regular at the origin. In fact, there is a true singularity at the origin [63]. Curiously, when both sources are present, namely, the cloud of strings and the quintessence, the resulting metric is singular at the origin. In other words, the presence of one of these sources, or both of them, removes the regularity of the metric at $r = 0$.

The conclusion with respect to the energy conditions is that they can be satisfied or violated, depending on the relation between the parameters that appear in the metric. On the other hand, if we consider these conditions for very large values of the radial coordinate, we conclude that they are all satisfied and equal to zero. It is worth mentioning that if we consider $f(r) < 0$, in which case the coordinate t is space-like, the conclusions about the energy conditions are similar.

The behavior of the thermodynamic quantities as mass parameter and the Hawking temperature as functions of the entropy S are represented, respectively, in Figures 2 and 3, in which are pointed out the dependence of these quantities with the parameters a, k, and α.

Concerning the heat capacity, we can observe that for different values of the non-linear electrodynamics parameter, this quantity has the same qualitative dependence with the entropy, for different values of the parameter associated with the presence of the cloud of strings, as shown in Figure 4a–d.

In Figure 4e, we can see that the heat capacity varies linearly with the entropy, such that it decreases with the increasing of the parameter a. In the last figure is shown how the heat capacity depends on the entropy, when the quintessence and the cloud of strings is absent, taking into account different values of the parameter associated with the presence of the nonlinear electromagnetic field.

Finally, it is important to observe that, in the absence of the non-linear electrodynamics interaction ($k = 0$), we recover the results already obtained in the literature concerning the Letelier space-time with quintessence [11].

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References

1. Wald, R.M. *General Relativity*; University of Chicago Press: Chicago, IL, USA, 2010.
2. Schwarzschild, K. Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie. *Sitzungsberichte Königlich Preuss. Akad. Der Wiss.* **1916**, 189–196.

3. Akiyama, K.; Alberdi, A.; Alef, W.; Asada, K.; Azulay, R.; Baczko, A.K.; Ball, D.; Baloković, M.; Barrett, J.; Bintley, D.; et al. First M87 event horizon telescope results. IV. Imaging the central supermassive black hole. *Astrophys. J. Lett.* **2019**, *875*, L4.
4. Abbott, B.P. et al. [LIGO Scientific Collaboration and Virgo Collaboration] GW150914: First results from the search for binary black hole coalescence with Advanced LIGO. *Phys. Rev. D* **2016**, *93*, 122003. [CrossRef] [PubMed]
5. Sigg, D. The advanced LIGO detectors in the era of first discoveries. In Proceedings of the Interferometry XVIII, San Diego, CA, USA, 28 August–1 September 2016; Volume 9960, pp. 72–80.
6. Letelier, P.S. Clouds of strings in general relativity. *Phys. Rev. D* **1979**, *20*, 1294. [CrossRef]
7. Guendelman, E.; Rabinowitz, A. Gravitational field of a hedgehog and the evolution of vacuum bubbles. *Phys. Rev. D* **1991**, *44*, 3152. [CrossRef]
8. Övgün, A. Black hole with confining electric potential in scalar-tensor description of regularized 4-dimensional Einstein-Gauss-Bonnet gravity. *Phys. Lett. B* **2021**, *820*, 136517. [CrossRef]
9. Barriola, M.; Vilenkin, A. Gravitational field of a global monopole. *Phys. Rev. Lett.* **1989**, *63*, 341. [CrossRef]
10. Barbosa, D.; Bezerra, V.B. On the rotating Letelier spacetime. *Gen. Relativ. Gravit.* **2016**, *48*, 149. [CrossRef]
11. Dias e Costa, M.M.; Toledo, J.; Bezerra, V.B. The Letelier spacetime with quintessence: Solution, thermodynamics and Hawking radiation. *Int. J. Mod. Phys. D* **2019**, *28*, 1950074. [CrossRef]
12. Nascimento, F.; Bezerra, V.B.; Toledo, J. Some remarks on Hayward black hole surrounded by a cloud of strings. *Ann. Phys.* **2023**, *460*, 169548. [CrossRef]
13. Rodrigues, M.E.; Silva, M.V.d.S. Embedding regular black holes and black bounces in a cloud of strings. *Phys. Rev. D* **2022**, *106*, 084016. [CrossRef]
14. Toledo, J.; Bezerra, V.B. Kerr–Newman–AdS black hole with quintessence and cloud of strings. *Gen. Relativ. Gravit.* **2020**, *52*, 34. [CrossRef]
15. Ghosh, S.G.; Maharaj, S.D. Cloud of strings for radiating black holes in Lovelock gravity. *Phys. Rev. D* **2014**, *89*, 084027. [CrossRef]
16. Singh, D.V.; Ghosh, S.G.; Maharaj, S.D. Clouds of strings in 4D Einstein–Gauss–Bonnet black holes. *Phys. Dark Universe* **2020**, *30*, 100730. [CrossRef]
17. Toledo, J.d.M.; Bezerra, V.B. Black holes with cloud of strings and quintessence in Lovelock gravity. *Eur. Phys. J. C* **2018**, *78*, 534. [CrossRef]
18. Toledo, J.; Bezerra, V.B. Black holes with a cloud of strings in pure Lovelock gravity. *Eur. Phys. J. C* **2019**, *79*, 117. [CrossRef]
19. Li, Z.; Zhou, T. Kerr black hole surrounded by a cloud of strings and its weak gravitational lensing in Rastall gravity. *Phys. Rev. D* **2021**, *104*, 104044. [CrossRef]
20. Vagnozzi, S.; Roy, R.; Tsai, Y.D.; Visinelli, L.; Afrin, M.; Allahyari, A.; Bambhaniya, P.; Dey, D.; Ghosh, S.G.; Joshi, P.S.; et al. Horizon-scale tests of gravity theories and fundamental physics from the Event Horizon Telescope image of Sagittarius A. *Class. Quantum Gravity* **2022**, *40*, 165007. [CrossRef]
21. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009. [CrossRef]
22. Riess, A.G.; Kirshner, R.P.; Schmidt, B.P.; Jha, S.; Challis, P.; Garnavich, P.M.; Esin, A.A.; Carpenter, C.; Grashius, R.; Schild, R.E.; et al. BVRI light curves for 22 type Ia supernovae. *Astron. J.* **1999**, *117*, 707. [CrossRef]
23. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 high-redshift supernovae. *Astrophys. J.* **1999**, *517*, 565. [CrossRef]
24. Caldwell, R. An introduction to quintessence. *Braz. J. Phys.* **2000**, *30*, 215–229. [CrossRef]
25. Kiselev, V.V. Quintessence and black holes. *Class. Quantum Gravity* **2003**, *20*, 1187. [CrossRef]
26. Ghaderi, K.; Malakolkalami, B. Thermodynamics of the Schwarzschild and the Reissner–Nordström black holes with quintessence. *Nucl. Phys. B* **2016**, *903*, 10–18. [CrossRef]
27. Abdujabbarov, A.; Toshmatov, B.; Stuchlík, Z.; Ahmedov, B. Shadow of the rotating black hole with quintessential energy in the presence of plasma. *Int. J. Mod. Phys. D* **2017**, *26*, 1750051. [CrossRef]
28. Ghosh, S.G.; Maharaj, S.D.; Baboolal, D.; Lee, T.H. Lovelock black holes surrounded by quintessence. *Eur. Phys. J. C* **2018**, *78*, 90. [CrossRef]
29. Guo, X.Y.; Li, H.F.; Zhang, L.C.; Zhao, R. Continuous phase transition and microstructure of charged AdS black hole with quintessence. *Eur. Phys. J. C* **2020**, *80*, 168. [CrossRef]
30. Born, M.; Infeld, L. Quantum Theory of the Electromagnetic Field. *Proc. R. Soc. Lond. Ser. A* **1934**, *143*, 410.
31. Born, M.; Infeld, L. Foundations of the new field theory. *Proc. R. Soc. Lond. Ser. A* **1934**, *144*, 425–451. [CrossRef]
32. Plebański, J. *Lectures on Non-Linear Electrodynamics: An Extended Version of Lectures Given at the Niels Bohr Institute and NORDITA, Copenhagen, in October 1968*; Nordita: Stockholm, Sweden, 1970; Volume 25.
33. Pellicer, R.; Torrence, R. Nonlinear electrodynamics and general relativity. *J. Math. Phys.* **1969**, *10*, 1718–1723. [CrossRef]
34. Ayon-Beato, E.; Garcia, A. Regular black hole in general relativity coupled to nonlinear electrodynamics. *Phys. Rev. Lett.* **1998**, *80*, 5056. [CrossRef]
35. Ayon-Beato, E.; Garcıa, A. New regular black hole solution from nonlinear electrodynamics. *Phys. Lett. B* **1999**, *464*, 25–29. [CrossRef]

36. Gonzalez, H.A.; Hassaine, M.; Martinez, C. Thermodynamics of charged black holes with a nonlinear electrodynamics source. *Phys. Rev. D* **2009**, *80*, 104008. [CrossRef]
37. Singh, D.V.; Shukla, A.; Upadhyay, S. Quasinormal modes, shadow and thermodynamics of black holes coupled with nonlinear electrodynamics and cloud of strings. *Ann. Phys.* **2022**, *447*, 169157. [CrossRef]
38. Singh, D.V.; Bhardwaj, V.K.; Upadhyay, S. Thermodynamic properties, thermal image and phase transition of Einstein-Gauss-Bonnet black hole coupled with nonlinear electrodynamics. *Eur. Phys. J. Plus* **2022**, *137*, 969. [CrossRef]
39. Flachi, A.; Lemos, J.P. Quasinormal modes of regular black holes. *Phys. Rev. D* **2013**, *87*, 024034. [CrossRef]
40. de Paula, M.A.A.; dos Santos Leite, L.C.; Crispino, L.C.B. Massless scalar scattering by a charged regular black hole. *Astron. Nachrichten* **2023**, *344*, e220115. [CrossRef]
41. Nomura, K.; Yoshida, D.; Soda, J. Stability of magnetic black holes in general nonlinear electrodynamics. *Phys. Rev. D* **2020**, *101*, 124026. [CrossRef]
42. de Paula, M.A.; Junior, H.C.L.; Cunha, P.V.; Crispino, L.C. Electrically charged regular black holes in nonlinear electrodynamics: Light rings, shadows, and gravitational lensing. *Phys. Rev. D* **2023**, *108*, 084029. [CrossRef]
43. Habibina, A.; Ramadhan, H. Geodesic of nonlinear electrodynamics and stable photon orbits. *Phys. Rev. D* **2020**, *101*, 124036. [CrossRef]
44. Toshmatov, B.; Ahmedov, B.; Malafarina, D. Can a light ray distinguish the charge of a black hole in nonlinear electrodynamics? *Phys. Rev. D* **2021**, *103*, 024026. [CrossRef]
45. Kruglov, S. The shadow of M87 black hole within rational nonlinear electrodynamics. *Mod. Phys. Lett. A* **2020**, *35*, 2050291. [CrossRef]
46. Rayimbaev, J.; Figueroa, M.; Stuchlík, Z.; Juraev, B. Test particle orbits around regular black holes in general relativity combined with nonlinear electrodynamics. *Phys. Rev. D* **2020**, *101*, 104045. [CrossRef]
47. Matyjasek, J. Extremal limit of the regular charged black holes in nonlinear electrodynamics. *Phys. Rev. D* **2004**, *70*, 047504. [CrossRef]
48. Liang, J. Strong gravitational lensing by regular electrically charged black holes. *Gen. Relativ. Gravit.* **2017**, *49*, 137. [CrossRef]
49. Rodrigues, M.E.; Junior, E.L.; Silva, M.V.d.S. Using dominant and weak energy conditions for build new classe of regular black holes. *J. Cosmol. Astropart. Phys.* **2018**, *2018*, 059. [CrossRef]
50. Bronnikov, K.A. Regular magnetic black holes and monopoles from nonlinear electrodynamics. *Phys. Rev. D* **2001**, *63*, 044005. [CrossRef]
51. Dymnikova, I. Regular electrically charged vacuum structures with de Sitter centre in nonlinear electrodynamics coupled to general relativity. *Class. Quantum Gravity* **2004**, *21*, 4417. [CrossRef]
52. Balart, L.; Vagenas, E.C. Regular black holes with a nonlinear electrodynamics source. *Phys. Rev. D* **2014**, *90*, 124045. [CrossRef]
53. Culetu, H. On a regular charged black hole with a nonlinear electric source. *Int. J. Theor. Phys.* **2015**, *54*, 2855–2863. [CrossRef]
54. Kruglov, S. Asymptotic Reissner-Nordström solution within nonlinear electrodynamics. *Phys. Rev. D* **2016**, *94*, 044026. [CrossRef]
55. Salazar I, H.; García D, A.; Plebański, J. Duality rotations and type D solutions to Einstein equations with nonlinear electromagnetic sources. *J. Math. Phys.* **1987**, *28*, 2171–2181. [CrossRef]
56. García, D.A.; Salazar, I.H.; Plebański, J. Type-D solutions of the Einstein and Born-Infeld nonlinear-electrodynamics equations. *Il Nuovo C. B* **1984**, *84*, 65–90. [CrossRef]
57. Amirabi, Z.; Mazharimousavi, S.H. Black-hole solution in nonlinear electrodynamics with the maximum allowable symmetries. *Eur. Phys. J. C* **2021**, *81*, 207. [CrossRef]
58. Kruglov, S. Nonlinear electrodynamics and magnetic black holes. *Ann. Der Phys.* **2017**, *529*, 1700073. [CrossRef]
59. Sajadi, S.; Riazi, N. Nonlinear electrodynamics and regular black holes. *Gen. Relativ. Gravit.* **2017**, *49*, 45. [CrossRef]
60. Ayon-Beato, E.; Garcia, A. Non-singular charged black hole solution for non-linear source. *Gen. Relativ. Gravit.* **1999**, *31*, 629–633. [CrossRef]
61. Ayón-Beato, E.; Garcıa, A. The Bardeen model as a nonlinear magnetic monopole. *Phys. Lett. B* **2000**, *493*, 149–152. [CrossRef]
62. Ma, M.S. Magnetically charged regular black hole in a model of nonlinear electrodynamics. *Ann. Phys.* **2015**, *362*, 529–537. [CrossRef]
63. Stuchlík, Z.; Schee, J.; Ovchinnikov, D. Generic regular black holes related to nonlinear electrodynamics with Maxwellian weak-field limit: Shadows and images of Keplerian disks. *Astrophys. J.* **2019**, *887*, 145. [CrossRef]
64. Rodrigues, M.E.; Silva, M.V.d.S. Bardeen regular black hole with an electric source. *J. Cosmol. Astropart. Phys.* **2018**, *2018*, 25. [CrossRef]
65. Ghosh, S.G.; Singh, D.V.; Maharaj, S.D. Regular black holes in Einstein-Gauss-Bonnet gravity. *Phys. Rev. D* **2018**, *97*, 104050. [CrossRef]
66. Rodrigues, M.E.; Silva, M.V.d.S.; Vieira, H.A. Bardeen-Kiselev black hole with a cosmological constant. *Phys. Rev. D* **2022**, *105*, 084043. [CrossRef]
67. Toledo, J.; Bezerra, V.B. The Reissner–Nordström black hole surrounded by quintessence and a cloud of strings: Thermodynamics and quasinormal modes. *Int. J. Mod. Phys. D* **2019**, *28*, 1950023. [CrossRef]
68. Sood, A.; Kumar, A.; Singh, J.; Ghosh, S.G. Thermodynamic stability and P–V criticality of nonsingular-AdS black holes endowed with clouds of strings. *Eur. Phys. J. C* **2022**, *82*, 227. [CrossRef]

69. Kontou, E.A.; Sanders, K. Energy conditions in general relativity and quantum field theory. *Class. Quantum Gravity* **2020**, *37*, 193001. [CrossRef]
70. Zaslavskii, O. Regular black holes and energy conditions. *Phys. Lett. B* **2010**, *688*, 278–280. [CrossRef]
71. Visser, M. *Lorentzian Wormholes. From Einstein to Hawking*; American Institute of Physics: Woodbury, NY, USA, 1995.
72. Bekenstein, J.D. Black holes and entropy. *Phys. Rev. D* **1973**, *7*, 2333. [CrossRef]
73. Hawking, S.W. Particle creation by black holes. *Commun. Math. Phys.* **1975**, *43*, 199–220. [CrossRef]
74. Liu, H.; Meng, X.H. Effects of dark energy on the efficiency of charged AdS black holes as heat engines. *Eur. Phys. J. C* **2017**, *77*, 556. [CrossRef]
75. Toledo, J.; Bezerra, V. Some remarks on the thermodynamics of charged AdS black holes with cloud of strings and quintessence. *Eur. Phys. J. C* **2019**, *79*, 110. [CrossRef]
76. Ma, M.S.; Zhao, R. Corrected form of the first law of thermodynamics for regular black holes. *Class. Quantum Gravity* **2014**, *31*, 245014. [CrossRef]
77. Maluf, R.; Neves, J.C. Thermodynamics of a class of regular black holes with a generalized uncertainty principle. *Phys. Rev. D* **2018**, *97*, 104015. [CrossRef]
78. Singh, D.V.; Siwach, S. Thermodynamics and Pv criticality of Bardeen-AdS black hole in 4D Einstein-Gauss-Bonnet gravity. *Phys. Lett. B* **2020**, *808*, 135658. [CrossRef]
79. Nascimento, F.; Morais, P.H.; Toledo, J.; Bezerra, V. Some remarks on Bardeen-AdS black hole surrounded by a fluid of strings. *Gen. Relativ. Gravit.* **2024**, *56*, 86. [CrossRef]
80. Rani, S.; Jawad, A.; Raza, H.; Shaymatov, S.; Muzaffar, M.; Riaz, H. Thermodynamic properties and geometries of bardeen black hole surrounded by string clouds. *Eur. Phys. J. C* **2024**, *84*, 904. [CrossRef]