Perturbative and nonperturbative aspects of heterotic sigma models

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Dedication

To those who held me up over the years.

Abstract

Supersymmetric nonlinear sigma models are interesting from various perspectives. They are useful for understanding the most fundamental theory of our world, and for low-energy effective model-building. Mathematically, they make surprising connections between different exciting areas such as complex geometry, deformation theory, quantum algebra and topology.

In this thesis, we study perturbative and nonperturbative aspects of sigma models with $\mathcal{N} = (0, 2)$ supersymmetry, with an emphasize on a possible version of extended 4d/2d correspondence.

We showed that in some $\mathcal{N} = (0, 2)$ models, β functions calculated through Feynman graphs can be reproduced by nonrenormalization theorems. And the result can further be compared with the supercurrent analysis. These cases including linear models, minimal $CP(1)$ model (other $CP(N)$ models are obstructed by global anomaly) together with its extended cousins, and heterotic $CP(N)$ models.

Nonperturbatively we built the instanton measure for minimal $CP(1)$ model and its $(0, 2)$-extended cousins. The instanton measure bears similarity to the instanton measure for 4d super-Yang-Mills theories. Through this analogy, there seems to be a correspondence between $\mathcal{N} = 1$ theories in 4d and $\mathcal{N} = (0, 2)$ theories in 2d, which extends previous results initiated by Edalati-Tong and Shifman-Yung.

An interesting by-product is also obtained during the procedure, which shows that for non-minimal (globally anomaly-free) $\mathcal{N} = (0, 2)$ models with $CP(1)$ as target spaces, there seems always exist certain infrared fixed points, induced by the behavior of chiral fermions.

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Chapter 1

Introduction

1.1 Background

Two dimensional (2d) sigma models are theories of fields that map from two-dimensional space-time manifold to certain target manifold. This type of field theories are also important in understanding various realistic problems in physics. In particular, they bear a similarity to four dimensional (4d) nonabelian gauge theories — modeling asymptotic freedom, soliton and instanton solutions, gauge symmetry, large-N limits, etc etc [1]. Being in 2d makes things more computable (in some cases even exactly solvable), hence 2d sigma models could help us to better understand 4d theories. 2d nonlinear sigma models are also related to string theory, where their target spaces represent higher dimensional space-time.

Moreover, 2d sigma models are appealing due to their explicit geometric meaning. The world-sheets are commonly taken to be Riemann surfaces, instead of just merely spaces. And the target spaces are manifolds with some extra structures. So the fields can be viewed as maps from a Riemann surface Σ to a given manifold M. Naturally the geometric information on Σ and M will impact the theory. Inspecting such theories may tell us various structures on the target manifold and on the mapping space, i.e., topological invariants.

The development of supersymmetry makes these models even more important. The superfield formulation of sigma models is both elegant and geometric, and most importantly, it gives the most general form of the Lagrangian for supersymmetric field theories. One could also gauge supersymmetry to obtain a supergravity theory [2, 3], which is, of course, exciting.

Besides the phenomenological importance, supersymmetry makes possible cancellation of contributions from bosons and fermions, which sets up strong constraints on their quantum behavior. Due to such cancellation, some observables can be calculated exactly [4, 5, 6, 7, 8, 9, 10, 11, 12]. Various dualities hold, which can be used to probe strong interacting systems where perturbation theory is not applicable[13, 14, 15, 16, 17]. It is thus important to understand when and how these simplification could happen, with the hope that our knowledge on supersymmetry could shed light on calculation of lower-level supersymmetric, or nonsupersymmetric theories as well.

Finally, supersymmetry makes the correspondence between 4d super-Yang-Mills theories and 2d supersymmetric sigma models more transparent. Explicit formulation of such correspondence has been established in many cases, see [18, 19, 20, 21, 22, 23, 24, 25]. These are different versions of 4d/2d correspondence from various points of view, and the relation among them is not entirely clear. But for some reason they are all related to the Bogomol'nyi-Prasad-Sommerfield (BPS) sector of 4d theories, which can not be formulated without supersymmetry. From this point of view, we can say that supersymmetry is crucial in establishing 4d/2d correspondences.

In this thesis we will mainly discuss perturbative and non-perturbative aspects of 2d $\mathcal{N} = (0,2)$ sigma models, with an emphasis on their renormalization properties and the relation to 4d $\mathcal{N} = 1$ super-Yang-Mills theories. Similar results in case of 2d $\mathcal{N} = (2,2)$ sigma models are well-known. We will briefly review them in the remainder section, and then try to advance our understanding to the case of $\mathcal{N} = (0,2)$ theories in the subsequent chapters.

1.1.1 $\mathcal{N} = (2,2)$ nonlinear sigma models and topological theories

A bosonic sigma model can be viewed as a theory of maps from our spacetime manifold Σ to the target space M:

$$\mathcal{C} = \{\phi : \Sigma \to M\} \tag{1.1}$$

. In particular, such model preserves the diffeomorphisms on M and Σ, i.e., the action

$$S = \int_\Sigma d^2x\sqrt{h}\frac{1}{2}g_{i,j}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \tag{1.2}$$

is invariant under diffeomorphic transformation, where h is the metric on the Rieman surface Σ, and g is the metric on the target manifold M.

Having $\mathcal{N} = (1,1)$ supersymmetry means that we have, for each bosonic field ϕ^i, a Majorana Dirac fermion ψ^i living in the tangent space of M. In particular, fermions can be viewed as sections of the Spin bundle on Σ tensored with the pullback of TM along ϕ,

$$\psi_+ \in \Gamma(\Sigma, S_+ \otimes \phi^* TM)\,, \quad \psi_- \in \Gamma(\Sigma, S_- \otimes \phi^* TM)\,, \tag{1.3}$$

where we used S_+ and S_- to denote the square-root of the canonical line bundle over Σ with the positive and negative eigenvalues respectively. The action is now

$$S = \int_\Sigma d^2x \sqrt{h} \frac{1}{2} \{ g_{i,j}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + i g_{i,j} \bar{\psi}^j \not{D} \psi^i - R_{i,j,k,l} \bar{\psi}^i \psi^j \bar{\psi}^k \psi^l \}\,. \tag{1.4}$$

The structure is uniquely determined by the geometry information and supersymmetry.

Since we will also be interested in exploring holomorphic information, we will focus on the Kähler manifolds. In this case, the $\mathcal{N} = (1,1)$ supersymmetry gets automatically enhanced to $\mathcal{N} = (2,2)$ [26]. The Dirac fermions now can carry holomorphic data, as they can be viewed as pullback of $T^{0,1}M$, or $T^{1,0}M$ via ϕ.

The renormalization of the theory is characterized purely by the metric g. Perturbatively the calculation can be carried out using the background field method. The method can be useful for more generic models too, as long as some technical issues being taken care of carefully. In Chap. 2 we present the plane-wave background field method, and use that to illustrate the generic procedure of calculation.

In addition to the direct loop calculation, in case of M being flag manifolds, we could also show that the theory is renormalized exactly at one loop level. The main idea is some version of nonrenormalization theorem. We will make use of instanton solutions in these models. In such a background the nonrenormalization theorem makes the cancellation between bosonic and fermionic loops explicit. Given an instanton map $\phi : \Sigma \to M$ of degree k, we can consider the quantum correction to the vacuum in this background. It turns out, the bosonic and fermionic nonzero modes exactly cancel each other at one loop level due to supersymmetry, and hence the instanton measure is determined only by the zero modes. Nonrenormalization theorem further says that there is no higher loop correction to the vacuum transition rate in such background, and hence one can build all loop exact instanton measure. Now from this we can deduce the exact beta function of the theory.

Not only is the perturbative calculation in $\mathcal{N} = (2,2)$ models simpler, so is the non-perturbative sector. Applying an A-twist to the given theory, we arrive at topological field theories (see [27, 28]). The idea is that, by a topological twist, we now consider the fermions being space-time scalars or vectors, instead of space-time spinors:

$$\psi_+ \in \Gamma(\Sigma, \phi^* T^{1,0} M)\,, \quad \psi_- \in \Gamma(\Sigma, \bar{K} \otimes \phi^{0,1} TM^\vee)\,. \tag{1.5}$$

Hence, half of the supersymmetric transformations, which previously couple to sections of the square root of the canonical line bundle K, now are independent of these sections. Hence, we obtain a global BRST symmetry, whose symmetric charge we denote by Q. When our space-time manifold Σ is trivial, global sections of S_+ and S_- can always be found, and then the Lagrangian of the twisted model looks the same as the untwisted one.

The real difference between the twisted and untwisted model lies in the identification of physical observables. The twisted models have as all their physical observables only Q-closed states/operators. A priori, any Q-exact operators will have vanishing quantum correction. Now the operators that we are interested in all live in the Q-cohomology, or, quantum cohomology ring of the target space. These are called chiral rings.

The calculation of the chiral rings is much easier than the calculation of the operator product expansions of the full-fledged quantum field theory. Due to the localization of the path integral, two-loop and higher loop perturbative corrections are absent in these cases. At the end of the day, one only needs to think about contributions from the zero modes of the given theory. Namely, one needs to integrate over the moduli space of these bosonic and fermionic zero modes. Conceptually to make sense of the functional integration, we will need the moduli space to be compact; otherwise a reasonable compactification procedure is required. Barring the technical details of the compactification, the calculation is well-defined [29].

The method of localization became a very important and so far the only way to carry out exact calculations of the correlations functions. Although the topological sector does not control the full quantum field theory, it is, nonetheless, both useful and computable. There are also works trying to start from a TFT obtained in the large-volume limit, and approximate full quantum field theories by going away from the large-volume limit, see [30, 31, 32].

1.1.2 4d/2d correspondence

Previously we mentioned that there are different versions of the correspondence between 4d super-Yang-Mills theories and 2d supersymmetric nonlinear sigma models. Here we will be specific and focus on the version first explored by Hanany and Tong [18, 19], which was invented to understand the BPS spectra of SQCD.

The simplest case is to consider an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $U(N)$ and flavor group $SU(N)$. In this case, the superpotential is uniquely determined as

$$\mathcal{W} = q\,A\,\tilde{q} + \sum_a q\,A^a t^a \tilde{q}\,. \tag{1.6}$$

Here we denote the gauge vector multiplet that corresponds to the $U(1)$ part by A, and those to the $SU(N)$ part by A^a. The chiral superfields q and $\tilde{q}$ makes a hypermultiplet, which is in the fundamental representation of the gauge group. The gauge multiplet is made up of a vector supermultiplet A and a chiral scalar ϕ, the latter in the adjoint representation. We will also have Fayet-Iliopoulos term in our theory denoted by v. To see what this leads to, we have to inspect the scalar potential. (Here we make no difference between the superfield and its lowest components. Their meaning should be clear from the context.)

$$V \propto \frac{e^2}{2}\mathrm{Tr}(\sum_{i=1}^{N}|q_i|^2 - |\tilde{q}_i|^2 - v^2)^2 + e^2\mathrm{Tr}|\sum_{i=1}^{N}\tilde{q}_i q_i|^2 + \frac{1}{2e^2}\mathrm{Tr}|[A, A^\dagger]| + \cdots\,. \tag{1.7}$$

To minimize the potential, we will need the vacuum expectation value of q to be diagonal with respect to color and flavor indices. The fields $\tilde{q}$ and A will keep their vev vanishing. Hence, the gauge and flavor symmetries are broken to the diagonal combination of these two:

$$U(N) \times SU(N) \to SU(N)_{\mathrm{diag}}\,. \tag{1.8}$$

Around this supersymmetric vacuum one can study the small fluctuations that preserve the ground state. When $N = 1$, it was known that there are Abelian vortex strings. When $N > 1$, we could embed the Abelian vortex string solution into the field configuration of q to get the non-Abelian vortex string.

Vortex strings are a class of topological, string-like solutions carrying magnetic flux threaded through their core. Like typical topological solution, their equation is first-order, and preserves half of supersymmetry (the so-called BPS solution).

Suppose we have the vortex string oriented along the third spacial dimension z, then we will have to have q winding in the $x - y$ plane at special infinity. The topological quantity that describes the winding is

$$\pi_1(U(N) \times SU(N)/SU(N)_{\mathrm{diag}}) \cong \mathbb{Z}\,. \tag{1.9}$$

So the solution is classified by the winding number. For a fixed winding number k, there are $2Nk$ bosonic collective coordinates that parameterizes the solution (i.e., moduli of the solution). In the case we are interested in, we take $k = 1$. The way of obtaining non-Abelian vortex string solution from known Abelian one tells us that the embedding of abelian vortex string is invariant under the $SU(N-1) \times U(1)$ action. Hence, the true moduli for the bosonic orientation zero modes, is given by

$$SU(N)/SU(N-1) \times U(1) \cong CP(N-1)\,. \tag{1.10}$$

We can further obtain the fermionic collective coordinates by directly solving the instanton equation or applying supersymmetry on the bosonic ones. At the end of the day, by promoting these collective coordinates to quantum fields along t and z direction we are led to the 2d $\mathcal{N} = 2$ sigma model with target space $CP(N-1)$.

The vortex string solution of the bulk theory saturates its bounds, and is thus 1/2-BPS. It is, hence, possible to understand the correspondence from the perspective of topological field theories. The author is not aware of any explicit construction though.

1.2 Main results

The difficulty of calculating nonlinear field theories firstly lies in the nonlinearity, which means that we have infinitely many interaction terms guaranteed by some geometry. To simplify the calculation, we need to expand the Lagrangian in power series and at the same time keep track of the symmetry. For general Riemannian manifold it is possible to use geodesic background method restricting oneself to Riemann normal coordinate systems. In that way the geometric structure is explicit. Unfortunately, in case of $\mathcal{N} = (2, 2)$ theory it is hard to keep track of the Kahler structure explicitly, and the extension to superfield calculation seems unlikely. (For some work along this direction, the reader may refer to [33, 34, 35].)

Here we will describe the renormalization of $N = (2,2)$ sigma models using plane-wave background, taking as an example the $CP(1)$ model. Unlike geodesic background field method, the linear background field method can be extended to superfield formulation without any difficulty. The main aim is to set up a universal frame work for later discussion on $N = (0,2)$ theories. We believe that the way to carry out plane-wave background calculation at two-loop order is well-known to experts, but it has not been written up anywhere. So we include a calculation here.

Unlike the $\mathcal{N} = (2,2)$ models, which has only Dirac-type fermions, chiral or heterotic sigma models are characterized by the existence of chiral fermions. Generally these theories are trickier, since the massless chiral fermions may render the renormalization procedure difficult, causing various IR issues. The $\mathcal{N} = (0,2)$ models belong to this class.

We first construct a $\mathcal{N} = (0,2)$ linear sigma models and its renormalization. We derive the supergraph Feynman rule to be used for the calculation. Fixing one such model, we show, by explicit calculations up to two-loop order, that the Wilsonian β function is one-loop-exact. We derive a nonrenormalization theorem which is valid to all orders. This nonrenormalization theorem is rather unusual since it refers to (formally) D terms. It is based on the fact that supersymmetry combined with target space symmetries and "flavor" symmetries is sufficient to guarantee the absence of loop corrections. We analyze the supercurrent supermultiplet (i.e., the hypercurrent) providing further evidence in favor of the absence of higher loops in the Wilsonian β function. Similar argument should be easily extended to other linear models as well.

Heading toward the nonlinear world, we consider two-dimensional $\mathcal{N} = (0,2)$ sigma models with the target space $CP(1)$. A minimal model of this type has one left-handed fermion. Nonminimal extensions contain, in addition, N_f right-handed fermions. We want to derive expressions for the β functions valid to all orders. To this end we use a variety of methods: (i) perturbative analysis; (ii) instanton calculus; (iii) analysis of the supercurrent supermultiplet (the so-called hypercurrent) and its anomalies, and some other arguments. All these arguments, combined, indicate a direct parallel between the heterotic $\mathcal{N} = (0,2)$ $CP(1)$ models and four-dimensional super-Yang-Mills theories. In particular, the minimal $\mathcal{N} = (0,2)$ $CP(1)$ model is similar to $\mathcal{N} = 1$ supersymmetric gluodynamics. Its exact β function can be found; it has the structure of the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function of supersymmetric gluodynamics. The passage to nonminimal $\mathcal{N} = (0,2)$ sigma models is equivalent to adding matter. In this case an NSVZ-type exact relation between the β function and the anomalous dimensions γ of the "matter" fields is established. We derive an analog of the Konishi anomaly. At large N_f our β function develops an infrared fixed point at small values of the coupling constant (analogous to the Banks-Zaks fixed point). Thus, we reliably predict the existence of a conformal window. At $N_f = 1$ the model under consideration reduces to the well-known $\mathcal{N} = (2,2)$ $CP(1)$ model.

Finally we study the renormalization of the heterotically deformed $\mathcal{N} = (0,2)$ $CP(N-1)$ sigma models. In addition to the coupling constant g^2 of the undeformed N=(2,2) model, there is the second coupling constant γ describing the strength of the heterotic deformation. We calculate both β functions, β_g and β_γ at one loop, determining the flow of g^2 and γ. Under a certain choice of the initial conditions, the theory is asymptotically free. The β function for the ratio $\rho = \gamma^2/g^2$ exhibits an infrared fixed point at $\rho = \frac{1}{2}$. Formally this fixed point lies outside the validity of the one-loop approximation. We argue, however, that the fixed point at $\rho = \frac{1}{2}$ may survive to all orders. The reason is the enhancement of symmetry - emergence of a chiral fermion flavor symmetry in the heterotically deformed Lagrangian - at $\rho = \frac{1}{2}$. Next we argue that β_ρ formally obtained at one loop, is exact to all orders in the large-N (planar) approximation. Thus, the fixed point at $\rho = \frac{1}{2}$ is definitely the feature of the model in the large-N limit. It is expected that similar nonrenormalization theorem should be derived also in this case, and this is part of the ongoing work.

1.3 Relation to other works

Here we comment on the relevance between our work and that of others, with the hope that the understanding on both sides could benefit each other.

Starting with $N = (0,2)$ theories one could do topological twist using one of the supercharges. The twist is similar to the one with $\mathcal{N} = (2,2)$ models, but the theories we obtain after the twist are generally not topological field theories. However, they also have nicer property than the untwisted theories. For example, the class of $\mathcal{N} = (0,2)$ $CP(1)$

theories, that we will consider in Chap. 4, after topological twist, become so-called "holomorphic Chern-Simons theories", and it was showed in [36] that the obstruction to quantize these theories around constant holomorphic background is the second Chern character of the target space. This criterion makes use of the BV formalism established in [37], and it remains to see whether their formulation can further be generalized from holomorphic Chern-Simons theories to full-fledged quantum field theories.

Going further along the direction of nonperturbative aspects, one can probe the nature of chiral rings of these models. In [38], it has been shown that in the case of the minimal model, the chiral algebra will be trivialized at quantum level whenever the first chern class of the target space is nontrivial. However, in Chap. 4 we will study the extended chiral models with target space $CP(1)$. We will build up the instanton measure, providing a starting point to do the calculation for the chiral rings for these types of generalizations to the minimal models. It will be interesting to see whether the trivialization of chiral algebras is generic in all models, or it will change when we add in more chiral fermions. At least in the case of $CP(1)$ models, the minimal model and the extended models are drastically different. For example, the extended models may have bi-quark condensate, while the very phenomenon is absent in the minimal models.

Other than direct calculation (which is often quite cumbersome), sheaf cohomology is a way to calculate the chiral rings which has been developed in the context of $(0,2)$ mirror symmetry [39]. However, there is no comparison between direct calculations and the sheaf cohomology method. We hope that building up the instanton measure could lead us one step closer to the direct calculation, and finally help to compare these aspects.

Going beyond topological sector there has not been many discussion on the chiral theories in the literature. One works goes to Frenkel-Losev-Nekrasov [30, 31, 32], where these authors explored correlation functions from non-topological sectors and are hoping to get exact calculations away from the large volume limit. We hope that our analysis on nonrenormalization theorems in $\mathcal{N}=(0,2)$ theories should have some incarnation in their context.

In light of 4d/2d correspondences, we hope to see a more general version of such correspondence between $\mathcal{N}=1$ 4d gauge theories and $\mathcal{N}=(0,2)$ 2d nonlinear sigma models. This type of correspondence has never been directly probed before. But we can see one example of it through a $\mathcal{N} = 1$-preserving deformation of the $\mathcal{N} = 2$ 4d SQCD, which, in its turn, leads to heterotic $\mathcal{N} = (0, 2)$ nonlinear models in 2d in a quite explicit way. There is also one more way to understand the correspondence from the point view of chiral rings. It was pointed out by Dorey-Lee-Hollowood that the above version of 4d/2d correspondence can be viewed from the following aspect: an isomorphism between chiral rings of the corresponding theories. [1] What they have actually established is that the F-terms on both sides are the same, and hence the supersymmetric vacua. Since in $N = 2$ theory we do have the operator-state correspondence, their argument is enough to show that the chiral rings are isomorphic.

1.4 Outline of the thesis

The layout of the thesis is as follows.

In Chap. 2 we show linear background field method for perturbative calculations of $N = (2, 2)$ sigma models.

In Chap. 3 we define a class of $\mathcal{N} = (0, 2)$ linear sigma models, calculate their β functions and discuss nonrenormalization theorem from various prospects.

In Chap. 4 we consider two-dimensional $\mathcal{N} = (0, 2)$ sigma models with $CP(1)$ target space. In addition to perturbative calculation and nonrenormalization theorems, we also build instanton measure for these models, and point out an analogy between these models and 4d $\mathcal{N} = 1$ super-Yang-Mills theories.

In Chap. 5 we study renormalizations of a class of the heterotically deformed $\mathcal{N} = (0, 2)$ $CP(N - 1)$ sigma models. An infrared fixed point is discovered, and a possible chiral fermion flavor symmetry enhancement is discussed. The result is compatible with large-N calculation.

[1] The interpretation in our context is slightly tricky, as mentioned in [24].

Chapter 2

Plane-wave background field method for $\mathcal{N} = (2,2)$ sigma models

2.1 Introduction

It has been well-accepted, that all the fundamental quantum field theories should be renormalizable. One way to do a quick estimation is to count the superficial degrees of divergence of Feynman diagrams, which is given by considering the dimension of each quantum field. In 4-dimensional space-time, there are only certain kinds of interactions that are allowed. But in 2-dimensional space-time people could have more choices. In that case, the bosonic field is dimensionless, thus the interaction could invlove arbitrary functions of bosonic fields, without changing the superficial degrees of divergence.

Nowadays the renormalizability is no longer a serious issue. Basically because people can find a UV completion for such theories. We could take the non-renormalizable theory as the low energy effective theory of some more fundamental theory, and the latter is renormalizable. But it still makes sense to talk about this issue. More specifically, it is interesting to understand how many counter can terms we get from loop calculation. The non-renormalizable theories correspond here to theories with infinity many terms due to quantum loop effects.

It is interesting to talk about this issue for non-linear field theories, which are hard to calculate due to the non-linearity. However, such theories, if interesting, always contains certain symmetry. So if there is no quantum anomaly associated to the symmetry (and often, such geometric “anomalies” are absent), after renormalization the symmetry should be restored and only a finite number of counter terms are allowed to arise. One can consider the examples of non-abelian gauge theories and non-linear sigma models. The former have non-abelian gauge symmetry, and the loop calculation results in a redefinition of gauge coupling constant. The latter have reparameterization symmetry on the target space, and the loop calculation result is strongly constrained by the symmetry.

The situation of non-linear sigma model is more complicated. Generally the Lagrangian is given by the following form $\mathcal{L} = g(\phi)_{ij}\partial^{\mu}\phi^{i}\partial_{\mu}\phi^{j}$. So the kinetic term is non-canonical, and contains interactions. Here our difficulty is two-folded. Firstly, the quantum degrees of freedom manifest themselves often in a non-polynomial interaction, where field quantization becomes difficult. Secondly, even if we forget about quantization and just think about equation of motion at zero-th order, it is usually hard to solve.

The solution to the first difficulty is to consider background field method. We consider each field has a background part and a quantum part: $\phi(x) = \phi_{bk}(x) + q(x)$ and assume that the quantum fluctuation is small. So we could expand the Lagrangian as a power series of quantum fields. Then the quantum degrees of freedom do have a polynomial-typed interaction. The second problem now also get simplified. We should think about the equations of motion for the background fields and that of quantum fields separately. For background fields, the equation of motion at zero-th order is what we had before, which generally is again hard to be solved. But we almost never need to solve it. For quantum fields the equation of motion at zero-th order is usually good enough for us to solve for the propagators of quantum fields.

Different approaches could be taken here. One way is to assume that the background field is the classical equation of motion, so it also has the target space symmetry. Due to this symmetry, the kinetic term of quantum field can be simplified by replacing the ordinary partial derivatives with the properly defined covariant derivatives. Now we could choose a convenient gauge (i.e., a choice of local coordinates), which guarantees that each interaction term has a tensor ”coefficient”. So during the loop calculation the symmetry is maintained explicitly, and the quantum fields look locally like the quantum fields with flat background (which is the tangent space of the target manifold). For example, if the target space is a Riemann manifold, we could choose the local coordinate to be the Riemann normal coordinate, and the calculation can be greatly simplified by such choice, as is shown in Ref. [33]. However, we are not always lucky to find a good choice of local coordinate system which makes the loop calculation straightforward.

Another way is to take a background that could best simplify the equation of motion of quantum field. In this case the background field may not satisfy its equation of motion, hence a source term is needed, which adds no complexity to loop calculation. Typically doing this could explicitly break the target space symmetry. However, by specifying the form of the background we keep track of such breaking, and finally the renormalized Lagrangian will restore it. One loop calculation can always be done without any ambiguity, and it does not strongly depend on the choice of background fields. However, the calculation at two loop level is non-trivial, with some points to be paid attention to, as will be shown here. The calculation will be made explicit for $CP(1)$ model only. However, we note that this method can be unambiguously applied to other models with Kahler manifolds as targets.

2.2 Loop calculations in the $\mathcal{N} = (2,2)$ CP(1) sigma model

In this section we describe the background field technique in application to two-loop calculations in the CP(1) model. We exploit a special choice of the backround field [1] — a slowly-varing plane-wave background — which is convenient in this problem. Our result is to be compared with the standard result to show the validity of our method. Within this section we also introduce some relevant notations and conventions.

2.2.1 The model, notation and one-loop result

The Lagrangian of the $\mathcal{N} = (2,2)$ CP(1) sigma model is as follows:

$$\mathcal{L} = G\left\{\partial^\mu\phi\partial_\mu\bar{\phi} + i\bar{\psi}\partial\!\!\!/\psi - 2i\,\frac{1}{\chi}\,\bar{\psi}\gamma^\mu\psi\,\bar{\phi}\partial_\mu\phi + \frac{1}{\chi^2}(\bar{\psi}\psi)^2\right\}\,. \tag{2.1}$$

Here

$$\chi = 1 + \bar{\phi}\phi\,, \qquad G = \frac{2}{g^2\chi^2}\,, \tag{2.2}$$

ϕ is a complex scalar field, G is the target-space metric in the Fubini-Study form, and ψ is a two-component Dirac spinor.

To get started it is instructive to calculate one-loop renormalization for the coupling constant g^{-2}. The result is well-known, of course [40]; this exercise must be viewed merely as an opportunity to describe our computational machinery.

As usual in the background field method we split the fields in two parts: background and quantum,

$$\phi \to \phi_0 + q \tag{2.3}$$

where q represents the quantum part which determines loops. For the background part we choose

$$\phi_0 = f e^{-ikx}\,, \qquad \bar\phi_0 = \bar f e^{ikx}\,, \qquad \bar\phi_0\phi_0 = \bar f f\,, \tag{2.4}$$

where f is a constant and at the very end (after separating terms quadratic in k) we can tend $k \to 0$. The fermion field is quantum, by definition. The background field expression for the Lagrangian is

$$\mathcal{L}_0 = \frac{2}{g_0^2}\,\frac{k^2\,\bar f f}{(1+\bar f f)^2}\,. \tag{2.5}$$

We will concentrate on the boson loops because the fermion loop at this level is finite and, thus, does not contribute to the β function. For further discussion see App. E.

Up to the second order in the quantum field, we have

$$\begin{aligned}
\mathcal{L}_{\phi(1)} &= (G_0^{1,0} q + G_0^{0,1}\bar q)\partial_\mu\phi_0\partial^\mu\bar\phi_0 + G_0(\partial_\mu q\partial^\mu\bar\phi_0 + \partial_\mu\phi_0\partial^\mu\bar q)\,,\\
\mathcal{L}_{\phi(2)} &= (G_0^{2,0} q^2 + G_0^{0,2}\bar q^2 + G_0^{1,1} q\bar q)\partial_\mu\phi_0\partial^\mu\bar\phi_0\\
&+ (G_0^{1,0} q + G_0^{0,1}\bar q)(\partial_\mu q\partial^\mu\bar\phi_0 + \partial_\mu\phi_0\partial^\mu\bar q) + G_0\partial_\mu q\partial^\mu\bar q
\end{aligned} \tag{2.6}$$

where

$$G_0^{i,j} = \frac{1}{i!j!}\,\frac{\partial^i}{\partial\phi^i}\,\frac{\partial^j}{\partial\bar\phi^j}\,\frac{2}{g_0^2\,(1+\phi\bar\phi)^2}\bigg|_{\phi_0,\bar\phi_0}\,, \tag{2.7}$$

and the metric G_0 is the quantity defined in Eq. (2.2) restricted to the background fields. In what follows we may drop the subscript 0 on some quantities, whose meaning is obvious from the context. In the first line in Eq. 2.6 addition of source terms (irrelevant for what follows) may be necessary in order to be able to neglect $\mathcal{L}_{\phi(1)}$ because of its proportionality to the equation of motion for the given background field.[1]

[1] In two-loop calculations one should also add to (2.6) the fermion term $\mathcal{L}_\psi$ which can be read off from Eq. (2.1).

In calculating the graphs which determine the β function one should keep in mind that they are divergent both in the ultraviolet (UV) and infrared (IR). To regularize the UV divergence we will use dimensional reduction working in $D = 2 - \epsilon$ dimensions. To regularize the IR divergence we introduce a small mass term for the quantum field (see [?]),

$$\mathcal{L}_m = G_0 \left(-m_0^2 q\bar{q} - m_0 \bar{\psi}\psi \right) . \tag{2.8}$$

At the very end both regularizing parameters, m_0 and ϵ are supposed to tend to zero. In logarithmically divergent graphs the following relation takes place:

$$\frac{1}{\epsilon} \leftrightarrow \ln \frac{M_{\text{uv}}}{m_0} , \tag{2.9}$$

where M_{uv} is the mass of the Pauli–Villars regulator.

Now we can pass to one-loop calculation. As was mentioned, the fermion loop is irrelevant. The relevant boson diagrams are depicted in Fig. 2.1. For dimensional

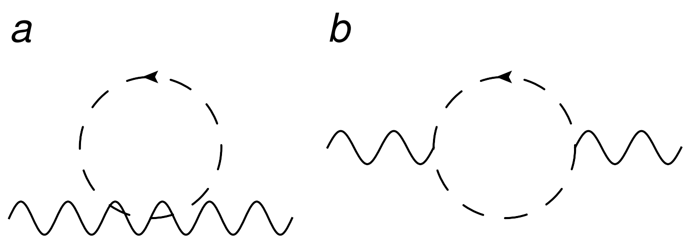

Figure 2.1: Boson loops contributing to the one-loop β function. The wavy lines stand for the background field $\partial_\mu \phi_0$ while the dashed lines for q.

reasons we need each diagram to be at most quadratic in $\partial^\mu \phi_0$. Higher orders in this parameter will not contribute to the effective action (see Eq. 2.5). As a result we have the following contributions to the effective Lagrangian:

$$(1) \quad = \quad G^{1,1}\,(\partial^\mu\phi_0\partial_\mu\phi_0)\int\frac{d^dp}{(2\pi)^d}\frac{i}{p^2-m^2} = iG^{1,1}\partial^\mu\phi_0\partial_\mu\phi_0 IG^{-1}\,, \tag{2.10}$$

$$\begin{aligned}(2a) \quad &= \quad \frac{i}{2!}\{G^{1,0^2}\partial_\mu\phi_0\partial_\nu\phi_0 + G^{0,1^2}\partial_\mu\bar{\phi}_0\partial_\nu\bar{\phi}_0\}\int\frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{(p^2-m^2)^2} \\ &= \quad \frac{i}{2d}\{G^{1,0^2}\partial_\mu\phi\partial^\mu\phi_0 + G^{0,1^2}\partial_\mu\bar{\phi}_0\partial^\mu\bar{\phi}_0\}IG^{-2}\,,\end{aligned} \tag{2.11}$$

$$\begin{aligned}(2b) \quad &= \quad iG^{1,0}G^{0,1}\partial_\mu\phi_0\partial_\nu\bar{\phi}_0\int\frac{d^dp}{(2\pi)^d}\frac{i^2p^\mu p^\nu}{(p^2-m^2)^2} \\ &= \quad -\frac{i}{2}G^{1,0}G^{0,1}\left(\partial_\mu\phi_0\partial^\mu\bar{\phi}_0\right)I\,G^{-2}\,,\end{aligned} \tag{2.12}$$

where we define the integral ($d = 2-\epsilon$)

$$I = \int\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2-m^2} = -\frac{i}{4\pi}\Gamma(\frac{\epsilon}{2})\,(\frac{\sqrt{4\pi}}{m})^\epsilon\,. \tag{2.13}$$

It seems that there are more manipulations needed in order to drop the unwanted terms that contains $\partial_\mu\phi\partial^\mu\phi$ and its hermitian conjugate. But this indeed is not necessary for a slow varying background. The reason is as follows. We know from physics argument that if we have a complex valued function ϕ on x, it must be a function on $ik\cdot x$ for certain k's to balance the dimension. Then the imaginary part of ϕ is totally determined by $ik\cdot x$. Now we consider $\bar{\phi}\phi$, which is a function on $ik\cdot x$, its taylor expansion must be of the form $A + B(k\cdot x)^2$. But for any place where we have such combination, we also have $\partial_\mu\phi\partial^\mu\phi$, etc. The later again is of quadratic order in k. Hence we could safely neglect either $\partial_\mu\phi\partial^\mu\phi$ or $B(k\cdot x)^2$. The first choice is stupid because that would vanish the whole result. The second choice is equivalent to the condition that $1+\bar{\phi}\phi$ being constant, which leads to the final result.

2.2.2 One-loop β function

Using ϕ_0 from Eq. (2.4) and assembling (2.10) – (2.12) we arrive at the following effective one-loop Lagrangian

$$\begin{aligned}\Delta\mathcal{L}_1 &= ik^2\, I\left(G^{1,1}Gf\bar{f} - \frac{1}{4}G^{1,0^2}f^2 - \frac{1}{4}G^{0,1^2}\bar{f}^2 - \frac{1}{2}G^{1,0}G^{0,1}f\bar{f}\right)G^{-2} \\ &= -2ik^2 f\bar{f}I\frac{1}{(1+f\bar{f})^2}\,, \end{aligned} \tag{2.14}$$

to be compared with (2.5). This one-loop result obviously maintans the target-space covariance. The integral I can be readily calculated upon transition to the Euclidean space,

$$I = \frac{-i}{(4\pi)^{d/2}}\Gamma(1-\frac{d}{2})(m^2)^{-1+\frac{d}{2}} \tag{2.15}$$

implying

$$\Delta\mathcal{L}_1 = -\frac{2}{(1+\bar{\phi}\phi)^2}\partial^\mu\bar{\phi}\partial_\mu\phi\cdot\frac{1}{4\pi m^\epsilon}\frac{2}{\epsilon} \tag{2.16}$$

Comparing with Eq. (2.5) it is easy to see that

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{1}{2\pi}\,\ln\frac{M}{m} \tag{2.17}$$

Now, the general definition of the β function is

$$\beta = (\partial/\partial\ln(m))\,g^2(m)\,. \tag{2.18}$$

Having this definition in mind we obtain the following one-loop β function,

$$\beta(g^2) = -\frac{g^4}{2\pi}\,. \tag{2.19}$$

If we rewrite the right-hand side as $-2g^2/4\pi$ and then replace 2 by N we will get the β function for the CP$(N-1)$ model [41],

$$\beta_{\mathrm{CP}(N-1)} = -\frac{N\,g^4}{4\pi}\,. \tag{2.20}$$

At one loop level choosing a particular (plane-wave) background neither simplifies nor complicates our computational routine compared to the case of an arbitrary background field representing a generic solution of the equation of motion. Subtle differences show up at two loops. See Sect. 2.3.

2.3 Two loops

Choosing the plane-wave bosonic background field and treating the fermion field ψ as purely quantum (for the time being) we expand the Lagrangian to the forth order in the quantum fields,

$$
\begin{aligned}
\mathcal{L}_{\phi(1)} &= (G^{1,0}q + G^{0,1}\bar{q})\partial_\mu\phi\partial^\mu\bar{\phi} + G(\partial_\mu q\partial^\mu\bar{\phi} + \partial_\mu\phi\partial^\mu\bar{q})\,, \\
\mathcal{L}_{\phi(2)} &= (G^{2,0}q^2 + G^{0,2}\bar{q}^2 + G^{1,1}q\bar{q})\partial_\mu\phi\partial^\mu\bar{\phi} \\
&+ (G^{1,0}q + G^{0,1}\bar{q})(\partial_\mu q\partial^\mu\bar{\phi} + \partial_\mu\phi\partial^\mu\bar{q}) + G\partial_\mu q\partial^\mu\bar{q}\,, \\
\mathcal{L}_{\phi(3)} &= (G^{2,1}q^2\bar{q} + G^{1,2}q\bar{q}^2)\partial_\mu\phi\partial^\mu\bar{\phi} + (G^{2,0}q^2 + G^{0,2}\bar{q}^2 + G^{1,1}q\bar{q})(\partial_\mu q\partial^\mu\bar{\phi} + \partial_\mu\phi\partial^\mu\bar{q}) \\
&+ (G^{1,0}q + G^{0,1}\bar{q})\partial_\mu q\partial^\mu\bar{q}\,, \\
\mathcal{L}_{\phi(4)} &= G^{2,2}q^2\bar{q}^2\partial_\mu\phi\partial^\mu\bar{\phi} + (G^{2,1}q^2\bar{q} + G^{1,2}q\bar{q}^2)(\partial_\mu q\partial^\mu\bar{\phi} + \partial_\mu\phi\partial^\mu\bar{q}) \\
&+ (G^{2,0}q^2 + G^{0,2}\bar{q}^2 + G^{1,1}q\bar{q})\partial_\mu q\partial^\mu\bar{q}\,, \\
\mathcal{L}_{\psi(2)} &= Gi\bar{\psi}\partial\!\!\!/\psi + \Gamma i\bar{\psi}\gamma^\mu\psi\partial_\mu\phi\,, \\
\mathcal{L}_{\psi(3)} &= (G^{1,0}q + G^{0,1}\bar{q})i\bar{\psi}\partial\!\!\!/\psi + (\Gamma^{1,0}q + \Gamma^{0,1}\bar{q})i\bar{\psi}\gamma^\mu\psi\partial_\mu\phi + \Gamma i\bar{\psi}\gamma^\mu\psi\partial_\mu q\,, \\
\mathcal{L}_{\psi(4)} &= (G^{2,0}q^2 + G^{0,2}\bar{q}^2 + G^{1,1}q\bar{q})i\bar{\psi}\partial\!\!\!/\psi + (\Gamma^{2,0}q^2 + \Gamma^{0,2}\bar{q}^2 + \Gamma^{1,1}q\bar{q})i\bar{\psi}\gamma^\mu\psi\partial_\mu\phi \\
&+ (\Gamma^{1,0}q + \Gamma^{0,1}\bar{q})i\bar{\psi}\gamma^\mu\psi\partial_\mu q + \frac{G}{(1+\bar{\phi}\phi)^2}(\bar{\psi}\psi)^2\,.
\end{aligned} \tag{2.21}
$$

We shall give the definition of Γ being $G_0^{1,0} = -\frac{4\bar{\phi}_0}{(1+\bar{\phi}_0\phi_0)^3}$ and $\Gamma^{i,j} = \frac{1}{i!\cdot j!}\frac{\partial^i}{\partial^i\phi}\frac{\partial^j}{\partial^j\bar{\phi}}\Gamma|_{\phi=\phi_0,\bar{\phi}=\bar{\phi}_0}$. In what follows we will neglect $\mathcal{L}_{\phi(1)}$.

Before analyzing all possible diagrams it is useful to have a look at the vertices. Each of them can be divided into background part (the part only depend on the background field) and quantum part (that on quantum fields). If the background part is hermitian, it is a real number, hence the momentum that are flowing through quantum part is conserved, which leads to the ordinary Feymann rule that we are familiar with. If not then there is possible momentum non-conservation due to the non-trivial background. Let us look at the term $G^{1,0}q\partial^\mu q\partial_\mu\bar{q}$ for example.

$$\begin{aligned}\langle p_f, l_f|e^{-ik\cdot x}q\partial^\mu q\partial_\mu\bar{q}|p_i\rangle &= \int d^2x e^{-ip_i\cdot x+i(p_f+l_f)\cdot x-ik\cdot x}p_i\cdot(p_f+l_f) \\ &= p_i\cdot(p_f+l_f)(2\pi)^2\delta(p_i+k-p_f-l_f) \qquad (2.22)\end{aligned}$$

The outgoing momentum $p_f + l_f$ does not equal to the incoming momentum p_i. This tells us that some of the diagrams should have momentum insertion. Especially, this means that the result of loop calculation with momentum non-conservation should depend on k. We shall see that having this in mind is crucial for us to get the correct answer.

We still need to regulate the IR behavior of q and ψ. In the sprit of supersymmetric dimensional regularization, we explicitly introduce the mass term $-G_0(m^2q\bar{q} + m\bar{\psi}\psi)$. In general the mass term will receive one loop correction, which is going to affect the two loop result. So one should be careful about it. In this section we will only concentrate on the genuine single pole (free of sub-diagram divergence) of fermion loops, which is independent of the structure of mass terms. We will return to this issue in Sec. 2.3.2.

Basically there are 2 types of diagrams — the tadpole diagram and the sunset. We further classify them according to the number and positions of external lines $\partial_\mu\phi$ and $\partial^\mu\bar{\phi}$. The quantum loop is expected to have dimension no less than zero because otherwise the superficial divergence would be negative and hence no genuine UV poles could generate. Hence the number of external lines of each diagram should be at most 2.

2.3.1 Calculation of the “vacuum” diagrams

At two-loop level it is convenient to start our analysis from a subset of diagrams containing the fermion loop. The reason is that this subset is easily identifiable, has fewer graphs, and, generally speaking, behaves better in the UV and IR regions compared to purely bosonic diagrams. In fact, if for orientation one ignores verification of the target-spave invariance of the result, the above subset reduces just to one diagram which is trivially calculable.

Based on this, we have a complete list of possible loops shown in Fig.2.2. In the case of diagrams with two external lines, the leading contribution, being logarithmic, can be obtained by turning off the external momentum, i.e., we just calculate vacuum

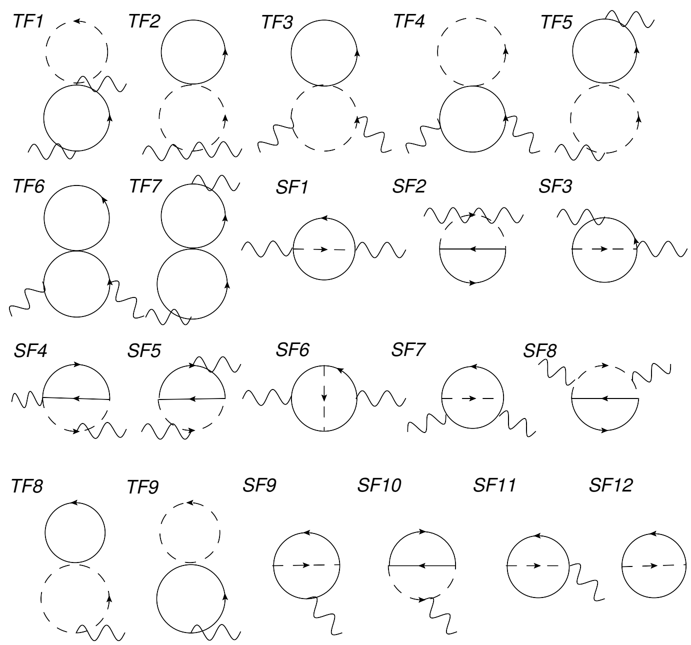

Figure 2.2: Two-loop diagrams with one or two fermion loops contributing to the two-loop β function. The wavy lines stand for the background field ϕ_0, the solid lines for fermions while the dashed lines for q.

diagrams. This can be verified by expanding the loop integrand with respect to small external momentum k (actually one need to consider $\frac{k^2}{m^2}$ to be a small quantity for Taylor expansion). All the sub-leading terms would lead to a term with higher order in k than we actually need.

Have all the analysis in mind, we could start the evaluation of all vacuum diagrams. Our calculation is based on the following algebraic identities:

$$\begin{aligned} p \cdot r &= \frac{1}{2}(D1 + D2 - D3 + m^2) \\ \frac{p^2}{D1} &= 1 + \frac{m^2}{D1}, \end{aligned}$$

where we define $D1 = p^2 - m^2$, $D2 = r^2 - m^2$ and $D3 = (p-r)^2 - m^2$, p and r being the momentum flowing in the quantum loops. Our integration thus is with respect to p and r. After integration, there will be no external momentum in our expression, so all the quantum loops will give the contribution either being a constant scalar or proportional to $g^{\mu\nu}$, depending on whether or not they have the two external lines on the same vertex. For the scalar case all the loop integrands can be expressed by $D1$, $D2$, $D3$ and mass term m^2. Hence if we define $I = \int \frac{d^d p^2}{(2\pi)^d} \frac{1}{D1}$ and $J = \int \frac{d^d p^2}{(2\pi)^d} \frac{m^2}{D1^2}$, the final result would be a polynomial depending on I, J and space-time dimension d. From the definition we can see that I represents the UV divergence of a single loop while J is UV finite and is IR regularized by the mass terms. This helps us to distinguish without ambiguity among genuine single poles and sub-diagram divergences. The validity of this approach will be talked about in App. B.1. If our integration is proportional to $g^{\mu\nu}$, we can simply replace the structure in numerator of the integrands such as $p^\mu p^\nu$ and $r^\mu p^\nu$ by $\frac{1}{d} g^{\mu\nu} p^2$ and $\frac{1}{d} g^{\mu\nu} p \cdot r$.

We calculated all the vacuum diagrams in the way mentioned above. The reason for stressing this method is that we found that we want to retrieve genuine single poles from diagrams. Actually one can see that the only way to generate IJ term is to have at least two propagator with the same momentum flowing into them, and thus make the diagram factorizable. All the vertices that have such consequence contain exactly two quantum field, i.e., the terms such as $q\bar{q}$, $q\partial^\mu \bar{q}$, etc.. Hence they should be absent due to the mass and wave-function renormalizations by Hepp's theorem, as we will show in Sect. 2.3.4.

2.3.2 Diagrams with external momenta

The diagrams with one external line (resp., no external lines) have dimension of mass (resp., mass square). But linear or quadratic UV-divergence is unphysical, we need to collect the terms linear (resp. quadratic) in external momentum to give us the correct dimension. The external momentum is totally fixed by our choice of background field. This can be easily understood by an examples of the following two-point quantum Green function.

$$\begin{aligned} G_{\text{qu}}(x-y) &= \langle T(\text{vertex with some } q\text{'s and } \bar{q}\text{'s})_x(\text{vertex with some } q\text{'s and } \bar{q}\text{'s})_y\rangle \\ &= \int \frac{d^2p}{(2\pi)^2}\hat{f}(p)e^{-ip\cdot(x-y)} = \hat{f}(i\partial_{(x-y)})\delta(x-y). \end{aligned} \tag{2.23}$$

Following the same notation of [42], a consistent choice of p flows from y to x. If we further suppose the space-time dependence of background field gives $e^{-ik\cdot(x-y)}$, the resultant two loop correction gives

$$\delta\mathcal{L} \;\propto\; \int d^2y\hat{f}(i\partial_{(x-y)})\delta(x-y)e^{-ik\cdot(x-y)} = \int d^2y\hat{f}(i^2k)e^{-ik\cdot(x-y)}\delta(x-y) = \hat{f}(i^2k).$$

This can be easily generalized to n-point functions. Note that our chosen background field only manifest itself in form of momentum insertion, which make the calculation straightforward.

If one would like to use the method we mentioned in last section to evaluate these diagrams, one has to be more careful in recognizing genuine single poles. Suppose we have the following integration, $\int \frac{d^dr}{(2\pi)^2}\frac{d^dp}{(2\pi)^2}\frac{1}{r^2-m^2}\frac{p^\mu+l^\mu}{(p+l)-m^2}$, one could easily say it vanishes by shifting p to $p+l$. On the other hand if we expand the integrand for small p, we get

$$\begin{aligned} &\int \frac{d^dr}{(2\pi)^2}\frac{d^dp}{(2\pi)^2}\frac{1}{r^2-m^2}\frac{l^\mu+p^\mu}{(p+l)-m^2} = I\int\frac{d^dp}{(2\pi)^2}\left(\frac{l^\mu}{p^2-m^2}-\frac{2p\cdot lp^\mu}{(p^2-m^2)^2}\right) \\ &= l^\mu(1-\frac{2}{d})I^2 - l^\mu IJ \end{aligned} \tag{2.24}$$

Quantitively this answer is again vanishing. But if one follows the argument in previous session about genuine single poles, one would have identified the first term as the contribution in β function and the second term to be sub-diagram divergence. In fact it is not the case. None of them contributes to β function. The reason is that in the loop where p is running over, it is IR safe not because of m^2 but because of the external momentum

l. Graphically there is no vertices that could lead to factorizable diagrams inserted here. Hence no sub-diagram divergence should be expected.

The philosophy here is that expansion with respect to small l may mix up the genuine single poles with sub-diagram divergences. So we need to consider some other way to identify the genuine single poles. The methods vary, but they always exist. We show a very simple one here, and more detailed analysis on this issue can be found in App. B.2. First observe that for vacuum diagrams, all the IJ terms come from diagrams with vertices quadratic in quantum field. This type of vertices acts like a mass insertion. So the appearance of IJ term is due to certain mechanism of mass insertion. Then think, for example, about a bosonic propagator with momentum $p+k$, where k is small. And imagine that we have an integration over p, where the integrand is regularized at the IR region by our propagator. Without the small k expansion the integration is regularized by the momentum k instead of IR cutoff, thus could possibly be free of sub-diagram divergence. But if we do the expansion, it is regularized by IR cutoff m^2, which will lead to sub-diagram divergences. This gives us the following judgement. If a momentum insertion is between two 3-vertices, and can be assigned to the propagator without any mass insertion, there should be genuine single poles. If there is a way to make the momentum insertion overlaps the mass insertion, this is just sub-diagram divergence. There are other ways to identify sub-diagram divergence for diagrams with momentum insertions. But the result is the same and our method is elegant enough to make a judgement without doing loop calculations.

Up to now we have exhausted all techniques that could be taken in calculating the relevant fermion diagrams. The result is listed in the following table, only non-trivial contributions are included.

We will neglect the IJ terms within this section and come back to them later. One can see explicitly that the calculation gives the covariant structure $\frac{g^2}{(1+\phi\bar{\phi})}\partial^\mu\phi\partial_\mu\bar{\phi}(d-2)I^2$.

2.3.3 Calculation of the boson loops

The calculation for Boson loops is largely the same. But one needs to pay attention to two more point. The first one is that we will encounter I^2 terms which generally is not suppressed by $(d-2)$, as we saw in the case of fermion loops. We are going to see the canceling of such terms, which is independent of the choice of IR cutoff. The second issue is that we will again have IJ terms, which are supposed to be distinguished from the genuine single poles. The latter solely come from the sub-leading contribution of I^2 terms.

Since the calculation for diagrams with 2 external line is trivial, we will simply give the result without mentioning the calculation details. That of diagrams with momentum insertion is tricky. But the method mentioned by the end of last section can still be used. A more detailed analysis will be provided in App. B.2.

Now we are ready to write down all we get from boson loops: The calculation

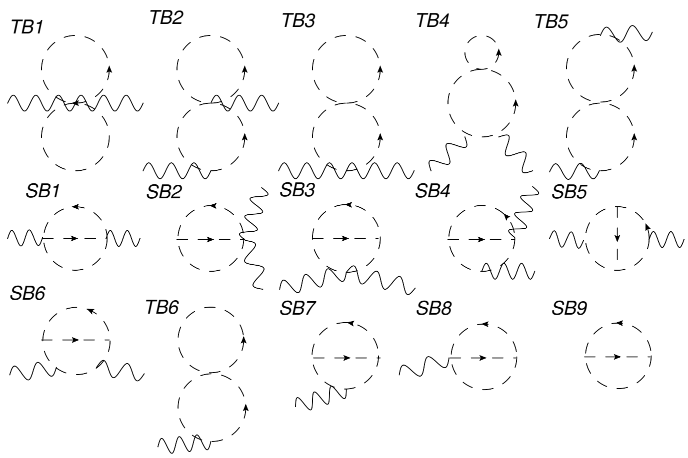

Figure 2.3: Two-loop diagrams with bosonic loops contributing to the two-loop β function. The wavy lines stand for the background field ϕ_0 and $\bar{\phi}_0$, the dashed lines for quantum field q and $\bar{q}$.

is given in Tab. 2.2.

Since now we are only interested in genuine single poles, we could neglect IJ terms temporarily. Then the result can be concluded as:

$$\text{Tadpoles with two external lines} = -f\bar{f}k^2g^2I^2 + f\bar{f}k^2g^2I^2(d-2)\frac{4f^2\bar{f}^2+4f\bar{f}}{(1+f\bar{f})^2}$$

$$\text{Sunsets with two external lines} = f\bar{f}k^2g^2I^2\frac{1}{2} - f\bar{f}k^2g^2I^2(d-2)\frac{17f^2\bar{f}^2+14f\bar{f}+1}{4(1+f\bar{f})^2}$$

$$\text{Diagrams with one external lines} = f\bar{f}k^2g^2I^2\frac{1}{2} + f\bar{f}k^2g^2I^2(d-2)\frac{f^2\bar{f}^2-2f\bar{f}-3}{4(1+f\bar{f})}.$$

It is clear that I^2 structure is vanishing and the genuine single poles is canceled by that from fermion loops. Indeed supersymmetry is functioning in such a way that it actually can guarantee the canceling of boson and fermion genuine single poles among each type of diagrams. But in our results this point becomes be vague, since our choice of Lagrangian for fermion field is not explicitly symmetric with respect to ψ_L and ψ_R. Nevertheless, the readers will be able to see this by using the symmetric Lagrangian.

2.3.4 Canceling sub-diagram divergence

According to the symmetry, everything after the loop calculation should be invariant with respect to symmetry transformations, the IJ terms included. And by Hepp's theorem, all the sub-diagram divergence should be identified as one-loop effects. But if we collect, for example, the IJ terms from bosonic loops, we will see that these terms are present in $\Delta\mathcal{L}$, and moreover, in a non-invariant form! We should try harder to get rid of these annoying stuffs. Our task here is two folded: firstly we would like to show that the requirement of target space covariance leads to the addition of more terms to give mass to the quantum field; secondly we need to show that altogether the IJ terms do not contribute to β function.

We consider pure bosonic model first. In our previous calculation we used the mass term $m^2 Gq\bar{q}$. In principle we should make sure that the mass term does not spoil the following symmetry of our theory.

$$\begin{aligned} &\phi \to \phi + \epsilon + \bar{\epsilon}\phi^2, \qquad \bar{\phi} \to \bar{\phi} + \bar{\epsilon} + \epsilon\bar{\phi}^2, \\ &q \to q + 2\bar{\epsilon}\phi q + \bar{\epsilon} q^2, \qquad \bar{q} \to \bar{q} + 2\epsilon\bar{\phi}\bar{q} + \epsilon\bar{q}^2. \end{aligned} \tag{2.25}$$

At one loop level, where the transformation of quantum field is linear, it is invariant. At two loop level, however, it is not. So we need to add some more 'mass' terms to maintain the symmetry. These terms are proportional to m^2, but with higher powers in quantum fields in order to maintain a canonical form of quantum field propagators.

The structure of mass term is almost totally determined by the symmetry, at two-loop level. So we have

$$\begin{aligned}\mathcal{L}_{\phi,m} = & - m^2 G q\bar{q} - m^2\frac{G^{1,0}}{2!}q^2\bar{q} - m^2\frac{G^{0,1}}{2!}q\bar{q}^2 - m^2\frac{G^{1,1}}{2!2!}q^2\bar{q}^2 - m^2 2!\frac{G^{2,0}}{3!}q^3\bar{q} \\ & - m^2 2!\frac{G^{0,2}}{3!}q\bar{q}^3 - m^2 c\frac{R}{2!2!}q^2\bar{q}^2, \end{aligned} \tag{2.26}$$

where $R = -\frac{4}{g^2(1+\phi\bar{\phi})^4}$ and c is an arbitrary constant.

New diagrams out of these terms are superficially convergent, but may contain subdiagram divergences. The calculation result is listed in Tab. 2.3.

Together with the calculation in previous section, the IJ term is now covariant:

$$\Delta\mathcal{L}_\phi = -\frac{g^2\partial_\mu\phi\partial^\mu\bar{\phi}}{(1+\phi\bar{\phi})^2}(d-2)I^2 - \frac{2g^2\partial_\mu\phi\partial^\mu\bar{\phi}}{(1+\phi\bar{\phi})^2}(c-1)IJ. \tag{2.27}$$

But it could take arbitrary value because of the constant c. We will see this does not affect the β function for g. Indeed, all these IJ terms are certain kinds of manifestation of one loop renormalization.

Let us take again the full Lagrangian for bosonic theory and pretend that we only integrate over the first loop. The kinetic term for quantum field q get renormalized in the same way as that of background field, hence $\delta(G\partial_\mu q\partial^\mu\bar{q}) = -iIg^2(G\partial_\mu q\partial^\mu\bar{q})$. The mass term for q is renormalized by tadpoles attached to $m^2\frac{G^{1,1}}{2!2!}q^2\bar{q}^2 + m^2 c\frac{R}{2!2!}q^2\bar{q}^2$ and $G^{1,1}q\bar{q}\partial_\mu q\partial^\mu\bar{q}$. To shorten the list here only the covariant contribution is taken into account. So we have $\delta(m^2 Gq\bar{q}) = -ciIg^2(m^2 Gq\bar{q})$. For mass renormalization we get $m_R^2 = m_0^2(1-(c-1)iIg^2)$. Now consider one loop result, we could express m_R by bare quantity m_0 since the later is independent of the renormalization scale, hence

$$\begin{aligned}\frac{2}{g_R^2} &= \frac{2}{g_0^2}(1 - iIg^2(m_0^2)^{-\frac{\epsilon}{2}}) = \frac{2}{g_0^2}\{1 - iIg^2(m_R^2)^{-\frac{\epsilon}{2}}(1+(c-1)iIg^2)^{-\frac{\epsilon}{2}}\} \\ &= \frac{2}{g_0^2}\{1 - iI(m_R^2)^{-\frac{\epsilon}{2}}g_0^2 - (1-c)IJ(m_R^2)^{-\frac{\epsilon}{2}}g_0^4\}. \end{aligned} \tag{2.28}$$

The second term exactly cancels with the IJ term in $\Delta\mathcal{L}_\phi$.

For $\mathcal{N} = (2,2)$ supersymmetric model the situation is similar. The mass term from ψ naturally involves another constant c'. And again one should be able to see that this arbitrariness does not affect the β function. However, if one wants to maintain supersymmetry one needs to fine tune the constant c or c'. If we choose $c' = 0$ and assume that c

takes proper value such that all symmetry is maintained, the fermionic loop diagrams that involve new mass terms give contributions as shown in Tab. 2.4.

Hence fermionic loops give,

$$\Delta\mathcal{L}_{\psi} = \frac{g^2 \partial_\mu \phi \partial^\mu \bar{\phi}}{(1+\phi\bar{\phi})^2}(d-2)I^2 \,. \tag{2.29}$$

We will discuss this issue of fine tuning in App. C.

Diagram	I^2 term	IJ term
TF1	$\frac{12f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{24f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF2	0	$-\frac{4f\bar{f}(13f^2\bar{f}^2-4f\bar{f}+1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF3	0	$\frac{8f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF4	$-\frac{4f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{8f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF5	$-\frac{4f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{8f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF8	0	$\frac{24f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SF1	$\frac{3f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	0
SF2	0	$\frac{24f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SF3	$-\frac{6f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{8f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SF5	$\frac{8f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{16f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SF6&7	$\frac{20f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{32f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SF8	0	$-\frac{48f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SF9	$\frac{2f^2\bar{f}^2g^2k^2}{1+f\bar{f}}(d-2)I^2$	0
SF10	0	$-\frac{48f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SF11	$-\frac{f\bar{f}(2f\bar{f}-1)g2k^2}{1+f\bar{f}}(d-2)I^2$	0

Table 2.1: Two-loop calculation with fermionic loops contributing to the two-loop β function

Diagram	I^2 term	IJ term
TB1	$-\frac{3f\bar{f}(3f^2\bar{f}^2-6f\bar{f}+1)g^2k^2}{(1+f\bar{f})^2}I^2$	0
TB2	$\frac{24f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2 - \frac{12f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{24f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TB3	$\frac{2f\bar{f}(1-2f\bar{f})^2g^2k^2}{(1+f\bar{f})^2}I^2$	$\frac{2f\bar{f}(17f^2\bar{f}^2-8f\bar{f}+2)g^2k^2}{(1+f\bar{f})^2}IJ$
TB4	$-\frac{8f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2 + \frac{4f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{16f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TB5	$-\frac{4f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2 + \frac{4f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{8f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TB6	0	$-\frac{12f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SB1	$\frac{f\bar{f}(1-5f\bar{f})^2g^2k^2}{2(1+f\bar{f})^2}I^2 - \frac{f\bar{f}(1-5f\bar{f})^2g^2k^2}{4(1+f\bar{f})^2}(d-2)I^2$	0
SB2	$\frac{12f^2\bar{f}^2(f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2$	0
SB3	$-\frac{6f^2\bar{f}^2(2f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2$	$-\frac{12f^2\bar{f}^2(4f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SB4	$-\frac{12f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}I^2 + \frac{6f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$-\frac{8f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SB5	$\frac{24f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}I^2 - \frac{16f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{32f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SB6	$\frac{24f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}I^2 - \frac{12f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}(d-2)I^2$	$\frac{40f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SB7	$-\frac{2f^2\bar{f}^2g^2k^2}{1+f\bar{f}}(d-2)I^2$	$\frac{16f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SB8	$f\bar{f}g^2k^2I^2 + \frac{f\bar{f}(5f\bar{f}-1)g^2k^2}{2(1+f\bar{f})}(d-2)I^2$	0
SB9	$-\frac{1}{2}f\bar{f}g^2k^2I^2 - \frac{1}{4}f\bar{f}g^2k^2(d-2)I^2$	0

Table 2.2: Two-loop calculation with only bosonic loops contributing to the two-loop β function.

Diagram	IJ term
TB3	$-\frac{2f\bar{f}(13f^2\bar{f}^2-2(c+2)f\bar{f}+c+1)g^2k^2}{(1+f\bar{f})^2}IJ$
TB4	$\frac{4f^2\bar{f}^2(2f\bar{f}-c-1)g^2k^2}{(1+f\bar{f})^2}IJ$
TB6	$\frac{12f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SB3	$\frac{8f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SB6	$-\frac{16f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SB7	$-\frac{16f^2\bar{f}^2g^2k^2}{1+f\bar{f}2}IJ$

Table 2.3: Two-loop calculation with bosonic loops which involve new mass vertices. The labeling of the diagrams still follows that of Figure 2.3.

Diagram	IJ term
TF2	$\frac{4f\bar{f}(13f^2\bar{f}^2-4f\bar{f}+1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF3	$\frac{8f^2\bar{f}^2(-2f\bar{f}+1)g^2k^2}{(1+f\bar{f})^2}IJ$
TF8	$-\frac{24f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$
SF2	$-\frac{24f^2\bar{f}^2(5f\bar{f}-1)g^2k^2}{(1+f\bar{f})^2}IJ$
SF8	$\frac{48f^3\bar{f}^3g^2k^2}{(1+f\bar{f})^2}IJ$
SF10	$\frac{48f^2\bar{f}^2g^2k^2}{1+f\bar{f}}IJ$

Table 2.4: Two-loop calculations for diagrams with one or two fermionic loops which involve new mass vertices. The labeling of the diagrams still follows that of Figure 2.2.

Chapter 3

Linear $(0, 2)$ sigma model and nonrenormalization theorem

3.1 Introduction

In this section we discuss multiloop calculations in a specific $\mathcal{N} = (0, 2)$ linear sigma model. The motivation is two-folded. On the one hand, this is a preparation for our study Chap. **??** of a class of two-dimensional $\mathcal{N} = (0, 2)$ $CP(N-1)$ nonlinear sigma models (heterotic $CP(N-1)$ models for short). On the other hand, the linear model we suggest has its own field-theoretical significances, among which the most interesting are a peculiar supergraph technique and a version of nonrenormalization theorem. Surprisingly, it is a renoramlization theorem for D terms!

So we will focus on a simple, linear version of $\mathcal{N} = (0, 2)$ sigma models, which serves our purposes at this stage.

We start from developing an appropriate $\mathcal{N} = (0, 2)$ supergraph technique to carry out an explicit two-loop calculation. The result is as follows: the interaction term proportional to γ is not renormalized, and neither are the Z factors of the superfield A in the Wilsonian sense (see Eq. (3.8)). The Z factors of the superfields $\mathcal{B}$ and B are renormalized, but this is just an iteration of the one-loop contribution. Then we prove the nonrenormalization theorem, which extends the first result to all orders. What is remarkable is the fact that the nonrenormalization theorem emerges for a D term provided there are certain target space conditions. Thus, up to two-loop order, the β function in the heterotic model at hand is

$$\beta(\gamma) = \frac{\gamma^3}{2\pi} \,. \tag{3.1}$$

Due to the fact that the nonrenormalization theorem generally fails to detect the geometric progression in the Z factors of $\mathcal{B}$ and B, at the moment we can not directly extend this result to three loops and higher in $\beta(\gamma)$. But it is reasonable to conjecture that this is the case. An argument substantiating this statement is presented in Sec. 3.7.

This chapter is organized as follow. In Sec. 3.2 we introduce the simplified heterotic $\mathcal{N} = (0, 2)$ linear model, which captures in full the quantum behavior of the deformation strength γ. In Sec. 3.3 we give the Feynman rules for supergraph calculations in $\mathcal{N} = (0, 2)$ theories. In Sec. 3.4, we calculate the two-loop contribution to $\beta(\gamma)$. Vanishing of certain diagrams provides us with an indication of the nonrenormalization theorem. In Sec. 3.5, we give the D term nonrenormalization theorem, which is valid perturbatively. In Sec. 3.6 we extend this statement beyond perturbation theory. In Sec. 3.7 we analyze the supercurrent supermultiplet of this model (the so-called hypercurrent), following the line of reasoning of [43].

3.2 An $\mathcal{N} = (0, 2)$ linear model

In Chap. 5 we will show that in the heterotic $CP(1)$ model, there is a fermionic SU(2) flavor symmetry, which mixes the chiral fields $\mathcal{B}$ and B (see Eq. (3.11)). To mimic this phenomenon, we introduce a simplified $\mathcal{N} = (0, 2)$ linear model, which emphasizes the mechanism of the $\mathcal{N} = (2, 2)$ deformation and retains the fermion flavor symmetry.

We begin by briefly reviewing $\mathcal{N} = (0, 2)$ supersmmetry and some notations. We define the left moving and right moving derivatives as

$$\partial_L = \partial_t + \partial_z \,, \qquad \partial_R = \partial_t - \partial_z \,, \tag{3.2}$$

and use the following definition for the superderivatives:

$$D_R = \frac{\partial}{\partial\theta_R} - i\theta_R^\dagger \partial_L \,, \qquad \bar{D}_R = -\frac{\partial}{\partial\theta_R^\dagger} + i\theta_R \partial_L \,. \tag{3.3}$$

Their commutator gives $\{D_R, \bar{D}_R\} = 2i\partial_L$, as it should. All integrations and differentiations are understood as acting from the left, if not stated to the contrary. The shifted space-time coordinates that satisfy the chiral condition are

$$y^0 = t + i\theta_R^\dagger \theta_R\,, \qquad y^1 = z + i\theta_R^\dagger \theta_R\,. \tag{3.4}$$

The antichiral counterparts are

$$\tilde{y}^0 = t - i\theta_R^\dagger \theta_R\,, \qquad \tilde{y}^1 = z - i\theta_R^\dagger \theta_R\,. \tag{3.5}$$

Under supersymmetric transformation $\delta_\epsilon + \delta_{\bar{\epsilon}}$

$$\begin{aligned} \theta_R &\to \theta_R + \epsilon\,, & \theta_R^\dagger &\to \theta_R^\dagger + \bar{\epsilon}\,, \\ y^\mu &\to y^\mu + 2i\bar{\epsilon}\theta_R\,, & \tilde{y}^\mu &\to \tilde{y}^\mu - 2i\theta_R^\dagger \epsilon\,, \end{aligned} \tag{3.6}$$

where $\mu = 0, 1$.

We can now define the chiral $\mathcal{N} = (0, 2)$ superfields in our model,

$$\begin{aligned} A(y^\mu, \theta_R) &= \phi(y^\mu) + \sqrt{2}\theta_R \psi_L(y^\mu)\,, \\ B(y^\mu, \theta_R) &= \psi_R(y^\mu) + \sqrt{2}\theta_R F(y^\mu)\,, \\ \mathcal{B}(y^\mu, \theta_R) &= \zeta_R(y^\mu) + \sqrt{2}\theta_R \mathcal{F}(y^\mu)\,. \end{aligned} \tag{3.7}$$

Here ϕ, ψ_L, ψ_R and ζ_R describe physical degrees of freedom, while F and $\mathcal{F}$ will enter without derivatives and, thus, can be eliminated by virtue of equations of motion.

In the $\mathcal{N} = (0, 2)$ superfield formalism the Lagrangian of the simplified model is as follow:

$$\mathcal{L} = \frac{1}{2}\int d^2\theta_R \left[\frac{1}{2}\left(iA^\dagger \partial_R A - iA\partial_R A^\dagger\right) + B^\dagger B + \mathcal{B}^\dagger \mathcal{B} - \left(\gamma \mathcal{B} B A^\dagger + \text{H.c.}\right)\right]. \tag{3.8}$$

In the component language, after eliminating F and $\mathcal{F}$, we have

$$\begin{aligned} \mathcal{L} &= \partial^\mu \phi^\dagger \partial_\mu \phi + i\bar{\psi}\not{\partial}\psi + i\zeta_R^\dagger \partial_L \zeta_R + \left[\gamma \zeta_R \psi_R \partial_L \phi^\dagger + \text{H.c.}\right] \\ &\quad + \gamma^2 \left(\zeta_R^\dagger \zeta_R\right)\left(\psi_L^\dagger \psi_L\right) + \gamma^2 \left(\psi_R^\dagger \psi_R\right)\left(\psi_L^\dagger \psi_L\right)\,. \end{aligned} \tag{3.9}$$

Note that $\mathcal{N}=(0,2)$ supersymmetry completely fixes the second line in terms of the first line.

The Lagrangian is invariant under SU(2) rotations of B and $\mathcal{B}$. Actually, if we define an SU(2) superfield doublet

$$\Psi = \left(\begin{smallmatrix} B \\ \mathcal{B} \end{smallmatrix}\right) , \tag{3.10}$$

the part of the Lagrangian that involves all right-handed fermions can be rewritten as

$$\frac{1}{2}\int d^2\theta_R\, \Psi_a^\dagger \Psi_a + \left[\frac{\gamma}{2} A^\dagger \varepsilon^{ab} \Psi_a \Psi_b + \text{H.c.}\right] , \tag{3.11}$$

which is obviously SU(2) invariant.

To see the importance of this model in helping us understanding the nonlinear models, one can compare this with Eq. (5.2), we indeed see that Eq. (3.9) is the limiting case of the former with $\gamma^2/g^2 \to \infty$. The opposite limiting case, $\gamma^2/g^2 \to 0$, is well-understood; it is just the undeformed $\mathcal{N}=(2,2)$ model in Eq. (5.1). The model in Eq. (3.9) can be viewed as a preparatory step to developing perturbation theory in the $\mathcal{N}=(0,2)$ heterotic $CP(N-1)$ models. We will show that this model exhibits a nonrenormalization theorem. The proof of the latter strengthens our understanding of heterotic supersymmetry.

3.3 Supergraph method

In this section we explicitly formulate superfield/supergraph calculus for the given model. Calculations in the $\mathcal{N}=(0,1)$ language were previously discussed in the literature, see e.g. [44, 45]. We feel that it is worth developing a similar formalism for $\mathcal{N}=(0,2)$ theories, for the following reasons. First, most $\mathcal{N}=(0,2)$ models can be obtained as deformations from $\mathcal{N}=(2,2)$, where holomorphic structures are crucial. It would be best if we preserve them explicitly. Second, this language is useful in deriving the nonrenormalization theorem of Sec. 3.5, a phenomenon not so easy to see when manipulating with $\mathcal{N}=(0,1)$ superalgebras. Third, so far no calculations were performed at two-loop level. The tools we develop here are expected to be helpful in the heterotic $CP(N-1)$ models too.

To derive the superpropagator, we define the functional variation for a bosonic chiral and antichiral superfields,

$$\begin{aligned}
\frac{\delta}{\delta A(y,\theta_R)} A'(y',\theta'_R) &= \delta(y-y')\delta(\theta_R-\theta'_R)\,, \\
\frac{\delta}{\delta A^\dagger(\tilde{y},\theta^\dagger_R)} A'^\dagger(\tilde{y}',\theta'^\dagger_R) &= \delta(\tilde{y}-\tilde{y}')\delta(\theta^\dagger_R-\theta'^\dagger_R)\,,
\end{aligned} \tag{3.12}$$

where y and $\tilde{y}$ are defined in Eq. (D.10) and (D.11). For a generic function $F(x,\theta_R,\theta^\dagger_R)$, we have

$$\begin{aligned}
&\frac{\delta}{\delta A(y,\theta_R)} \int d^2x' d\theta'_R d\theta'^\dagger_R\, A(y',\theta'_R) F(x',\theta'_R,\theta'^\dagger_R) \\
= &\int d^2y' d\theta'_R d\theta'^\dagger_R\, \delta(y-y')\delta(\theta_R-\theta'_R) F(y'-i\theta'^\dagger_R\theta'_R,\theta'_R,\theta'^\dagger_R) \\
= &-\int d\theta^\dagger_R\, F(y-i\theta^\dagger_R\theta_R,\theta_R,\theta^\dagger_R) = \bar{D}_R F(x,\theta_R,\theta^\dagger_R)\,.
\end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned}
&\frac{\delta}{\delta A^\dagger(\tilde{y},\theta^\dagger_R)} \int d^2x' d\theta'_R d\theta'^\dagger_R\, F(x',\theta'_R,\theta'^\dagger_R) A^\dagger(\tilde{y}',\theta'^\dagger_R) \\
= &\int d^2y' d\theta'_R d\theta'^\dagger_R\, F(\tilde{y}'+i\theta'^\dagger_R\theta'_R,\theta'_R,\theta'^\dagger_R)\delta(\tilde{y}-\tilde{y}')\delta(\theta^\dagger_R-\theta'^\dagger_R) \\
= &\int F(\tilde{y}+i\theta^\dagger_R\theta_R,\theta_R,\theta^\dagger_R)\, d\theta_R = F(x,\theta_R,\theta^\dagger_R)\overleftarrow{D}_R\,.
\end{aligned} \tag{3.14}$$

Note that we intentionally write D_R acting from the right, because we want our expression to be explicitly Hermitean-conjugate to the previous result.

On the other hand, we compare the result with

$$\begin{aligned}
&\int d^2x' d\theta'_R d\theta'^\dagger_R\, F(x',\theta'_R,\theta'^\dagger_R)\bar{D}_R\delta(x-x')\delta(\theta^\dagger_R-\theta'^\dagger_R)\delta(\theta_R-\theta'_R) \\
= &-\int d^2x' d\theta'_R d\theta'^\dagger_R\, \delta(x-x')\delta(\theta^\dagger_R-\theta'^\dagger_R)\delta(\theta_R-\theta'_R)\bar{D}_R F(x',\theta'_R,\theta'^\dagger_R) \\
= &-\bar{D}_R F(x,\theta_R,\theta^\dagger_R)\,,
\end{aligned} \tag{3.15}$$

which implies that, upon integration,

$$\frac{\delta}{\delta A(x,\theta_R,\theta_R^\dagger)} A'(x',\theta'_R,\theta'^\dagger_R) = -\bar{D}_R\delta(x-x')\delta(\theta_R^\dagger-\theta'^\dagger_R)\delta(\theta_R-\theta'_R)\,,$$

$$\frac{\delta}{\delta A^\dagger(x,\theta_R,\theta_R^\dagger)} A'^\dagger(x',\theta'_R,\theta'^\dagger_R) = -\delta(x-x')\delta(\theta_R^\dagger-\theta'^\dagger_R)\delta(\theta_R-\theta'_R)\overleftarrow{D}_R\,. \quad (3.16)$$

For a chiral field J_A, we have the projection

$$\frac{\bar{D}_R D_R}{2i\partial_L} J_A = \frac{\{\bar{D}_R, D_R\}}{2i\partial_L} J_A = J_A\,. \quad (3.17)$$

Using this we can conveniently pass from the F term to the integration over the full superspace, namely

$$\int d^2x d\theta_R\, A J_A = \int d^3z\, A\frac{-D_R}{2i\partial_L} J_A = \int d^3z\, J_A\frac{D_R}{2i\partial_L} A\,,$$

$$\int d^2x\, J_A^\dagger A^\dagger d\theta_R^\dagger = \int d^3z\, A^\dagger\frac{\bar{D}_R}{2i\partial_L} J_A^\dagger = \int d^3z\, J_A^\dagger\frac{-\bar{D}_R}{2i\partial_L} A^\dagger\,. \quad (3.18)$$

Here and in what follows in this section we use z to denote the triplet of (super)coordinates $(x^\mu,\theta_R,\theta_R^\dagger)$. Note that the currents J_A and $J_A^\dagger$ are Grassmannian. We can write the partition function as

$$Z[J_A,J_A^\dagger] = \int \mathcal{D}A\mathcal{D}A^\dagger \exp\left(i\int d^3z\,\frac{i}{2}A^\dagger\overleftrightarrow{\partial_R}A + A\frac{-D_R}{2i\partial_L}J_A + A^\dagger\frac{\bar{D}_R}{2i\partial_L}J_A^\dagger\right)\,, \quad (3.19)$$

and, by virtue of the functional integration, we get

$$\exp\left[-\frac{i}{2}\int d^3z \begin{pmatrix} J_A & J_A^\dagger \end{pmatrix} \begin{pmatrix} \frac{D_R}{2i\partial_L} & 0 \\ 0 & \frac{-\bar{D}_R}{2i\partial_L} \end{pmatrix} \begin{pmatrix} 0 & -\frac{2i}{\partial_R} \\ \frac{2i}{\partial_R} & 0 \end{pmatrix} \begin{pmatrix} \frac{-D_R}{2i\partial_L}J_A \\ \frac{\bar{D}_R}{2i\partial_L}J_A^\dagger \end{pmatrix}\right]$$

$$= \exp\int d^3z - \frac{i}{2}\left(J_A^\dagger\frac{1}{\Box}J_A + J_A\frac{-1}{\Box}J_A^\dagger\right)\,. \quad (3.20)$$

As a result we get the Feynman propagator for the chiral field A in the form

$$\langle 0|T\{A(x,\theta_R,\theta_R^\dagger)\,,\ A^\dagger(y,\eta_R,\eta_R^\dagger)\}|0\rangle = \frac{i}{\Box}\delta(x-y)\delta(\theta_R^\dagger-\eta_R^\dagger)\delta(\theta_R-\eta_R)\,. \quad (3.21)$$

Using the same line of reasoning now we will determine the propagators for the superfields B and $\mathcal{B}$. Note that due to the fermionic symmetry (see Eq. (3.11)), they are exactly the same. Take B for example; the partition function is

$$Z[J_B,J_B^\dagger] = \int \mathcal{D}B\mathcal{D}B^\dagger \exp\left(i\int d^3z\frac{1}{2}B^\dagger B + B\frac{D_R}{2i\partial_L}J_B + B^\dagger\frac{-\bar{D}_R}{2i\partial_L}J_B^\dagger\right)\,. \quad (3.22)$$

By virtue of the functional integration, we arrive at

$$\exp\left[-\frac{i}{2}\int dz\,(\,J_B\;J_B^\dagger\,)\begin{pmatrix}\frac{-D_R}{2i\partial_L} & 0\\ 0 & \frac{\bar{D}_R}{2i\partial_L}\end{pmatrix}\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}\begin{pmatrix}\frac{D_R}{2i\partial_L}J_B\\ \frac{-\bar{D}_R}{2i\partial_L}J_B^\dagger\end{pmatrix}\right]$$

$$= \quad \exp\int d^3z - \frac{i}{2}\left(J_B^\dagger\frac{1}{i\partial_L}J_B + J_B\frac{-1}{i\partial_L}J_B^\dagger\right)\,. \tag{3.23}$$

As a result,

$$\langle 0|T\{B(x,\theta_R,\theta_R^\dagger)\,,\,B^\dagger(y,\eta_R,\eta_R^\dagger)\}|0\rangle = \frac{-1}{\partial_L}\delta(x-y)\delta(\theta_R^\dagger-\eta_R^\dagger)\delta(\theta_R-\eta_R)\,. \tag{3.24}$$

The same applies to $\mathcal{B}$.

Now, let us pass to the interaction vertices. They can be obtained by considering

$$S_{\rm int}\left[\frac{\delta}{\delta J}\right]J_{\mathcal{B}}J_BJ_A^\dagger = -\frac{\gamma}{2}\int d^3z\left(\frac{\delta}{\delta J_{\mathcal{B}}}\frac{\delta}{\delta J_B}\frac{\delta}{\delta J_A^\dagger}\right)J_{\mathcal{B}}(z_1)J_B(z_2)J_A^\dagger(z_3)$$

$$= \quad \frac{\gamma}{2}\int d^3z\bar{D}_R\delta(z-z_1)\,\bar{D}_R(z-z_2)\,\delta(z-z_3)\overleftarrow{D}_R\,. \tag{3.25}$$

We can summarize the Feynman rules for the model at hand in the momentum space:

- For each propagator $\langle 0|T\{A_1\,,\,A_2^\dagger\}|0\rangle$, write

  $$-\frac{i}{p^2}\delta(\theta_{12})$$

  where $\delta(\theta_{12}) = \delta(\theta_1^\dagger-\theta_2^\dagger)\delta(\theta_1-\theta_2)$, with the momentum p flowing from 2 to 1.

- for each propagator $\langle 0|T\{B_1\,,\,B_2^\dagger\}|0\rangle$ or $\langle 0|T\{\mathcal{B}_1\,,\,\mathcal{B}_2^\dagger\}|0\rangle$, write

  $$-\frac{i}{p_L}\delta(\theta_{12})\,,$$

  with the momentum p flowing from 2 to 1.

- For each vertex, write $i\frac{\gamma}{2}$.

- For each propagator that connects a chiral field to the vertex, put $\bar{D}_R(p,\theta_R^\dagger,\theta_R)$ acting on it; for that connecting an antichiral field, put $\overleftarrow{D}_R(p,\theta_R^\dagger,\theta_R)$ acting on it, where p is the momentum that flows into the vertex through the propagator.

- Integrate over $\int d^2\theta_R$ and impose momentum conservation at each vertex, integrate over the momentum $\int \frac{d^2p}{(2\pi)^2}$ for each loop.

- For each external chiral or antichiral line, we have a factor for the field, but no D_R or $\bar{D}_R$ factors.

This set of the Feynman rules is displayed in Fig. 3.1.

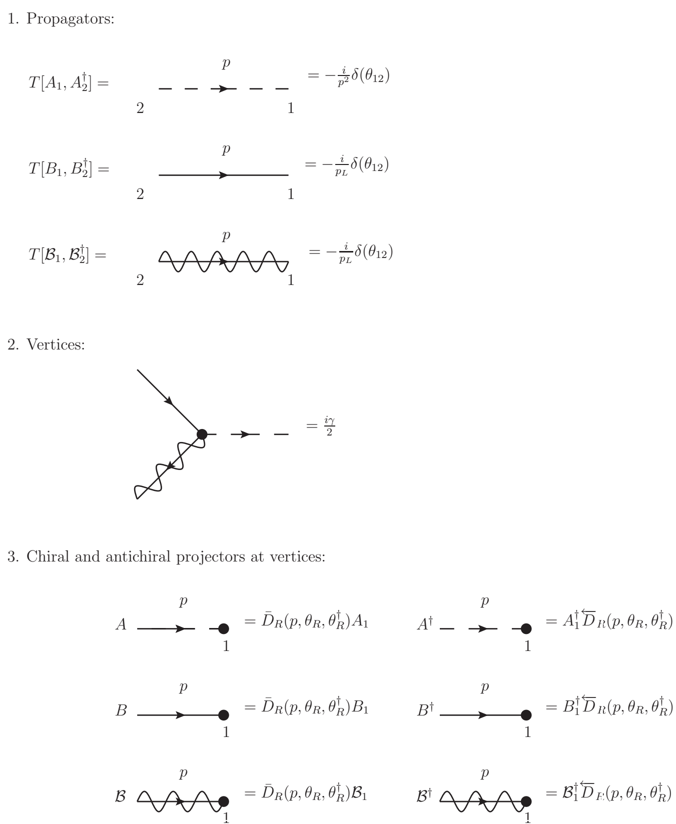

Figure 3.1: Feynman rules for the linear $\mathcal{N} = (0, 2)$ sigma model.

To facilitate our calculation, let us present here some useful identities. Verification of these identities is straightforward and is left as an exercise for the reader. In what follows, we will omit the subscript R in θ_R and D_R,

$$\begin{aligned} \delta(\theta_1-\theta_2)\overleftarrow{D}_2(\theta_2,p) &= -D_1(\theta_1,-p)\delta(\theta_1-\theta_2)\,, \\ \bar{D}_1 D_1(\theta_1,p)\delta(\theta_{12})|_{\theta_1=\theta_2} &= -D_2\bar{D}_2(\theta_2,-p)\delta(\theta_{12})|_{\theta_1=\theta_2}=1\,. \end{aligned} \tag{3.26}$$

3.4 One and two-loop results

Now we are ready to undertake the loop calculations using the superfield technique. We start from the Lagrangian with the bare coupling in UV, and evolve it down, where we have

$$\mathcal{L}=\frac{1}{2}\int d^2\theta_R\,\frac{1}{2}Z_A\left(iA^\dagger\partial_R A-iA\partial_R A^\dagger\right)+Z_B B^\dagger B+Z_{\mathcal{B}}\mathcal{B}^\dagger\mathcal{B}-Z_\gamma\left(\gamma_0\mathcal{B}BA^\dagger+\text{H.c.}\right)\,. \tag{3.27}$$

First, we would like to calculate the one-loop correction to the Z factors. The diagrams to be considered are collected in Fig 3.2.

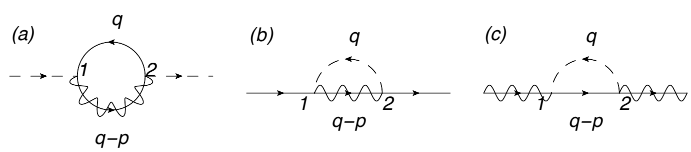

Figure 3.2: We use dashed line for the field A, straight arrowed line for the field B, and straight with wavy lines superimposed for the field $\mathcal{B}$.

For diagram (a), we get

$$\begin{aligned} &\int d\theta_1 d\theta_2\frac{d^2q}{(2\pi)^2}\,A_1^\dagger A_2\frac{-i}{q_L}\frac{-i}{p_L-q_L}\bar{D}_1\delta(\theta_{12})\overleftarrow{D}_2\,\bar{D}_1\delta(\theta_{12})\overleftarrow{D}_2 \\ = &\int d\theta_1\frac{d^2q}{(2\pi)^2}\,\{D_1,\bar{D}_1\}A_1^\dagger\,A_1\frac{1}{p_L(p_L-q_L)}\,. \end{aligned} \tag{3.28}$$

In the above calculation we used integration by parts to move all D's on one delta-function, and then do the integration over θ_2. One can show that the integration over the momentum is finite, and, hence, this graph does not contribute to Z_A.

As for diagram (b), we obtain

$$\begin{aligned}
&\left(i\frac{\gamma}{2}\right)^2 \int d\theta_1 d\theta_2 \frac{d^2q}{(2\pi)^2}\, B_1 B_2^\dagger \frac{-i}{p_L - q_L}\frac{-i}{q^2}\bar{D}_1\delta(\theta_{12})\overleftarrow{D}_2\, \bar{D}_2\delta(\theta_{12})\overleftarrow{D}_1 \\
= &-\left(i\frac{\gamma}{2}\right)^2 \int d\theta_1 d\theta_2 \frac{d^2q}{(2\pi)^2}\, B_2^\dagger\, B_1 \frac{1}{q^2(q_L - p_L)}\left(\bar{D}_1 D_1 \bar{D}_1 D_1 \delta(\theta_{12})\right)\delta(\theta_{12}) \\
= &\left(i\frac{\gamma}{2}\right)^2 \int d\theta_1 \frac{d^2q}{(2\pi)^2}\, B_1^\dagger B_1 \frac{2(q_L - p_L)}{(q_L - p_L)q^2}\,. \qquad (3.29)
\end{aligned}$$

Finally, it is not difficult to see that

$$Z_B = 1 + i\gamma^2 I\,, \qquad (3.30)$$

where the integral I is defined as

$$I \equiv \int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2}\,, \qquad (3.31)$$

which gives a single pole in the UV.

Due to the fermion flavor symmetry $Z_{\mathcal{B}} = Z_B$, we do not need a separate calculation here. Also, at one-loop level there is no diagram contributing to γ, hence $\beta(\gamma)$ is totally determined by the Z factors. In this way, we recover the result of [46],

$$\beta_{\text{one-loop}}(\gamma) = \frac{\gamma^3}{2\pi}\,. \qquad (3.32)$$

Now we are ready to move on to the two-loop calculation. We would like to prove a version of nonrenormalization theorem, stating that the interaction term $\frac{\gamma}{2}\mathcal{B}BA^\dagger$ is not renormalized. First, we will verify it at the two-loop level, by considering the diagram depicted in Fig 3.3. To this end it is sufficient to manipulate a little bit with the D-algebras,

$$\begin{aligned}
&(\bar{D}_1\delta(\theta_{12})\overleftarrow{D}_2)\,(\bar{D}_3\delta(\theta_{23})\overleftarrow{D}_2)\,(\bar{D}_3\delta(\theta_{34})\overleftarrow{D}_4)\,(\bar{D}_5\delta(\theta_{45})\overleftarrow{D}_4)\,(\bar{D}_4\delta(\theta_{14})\overleftarrow{D}_1)\,(\bar{D}_2\delta(\theta_{25})\overleftarrow{D}_5) \\
= \;& \bar{D}_1 D_1\delta(\theta_{12})\; \bar{D}_3 D_3\delta(\theta_{23})\; \bar{D}_3 D_3\delta(\theta_{34})\; \bar{D}_5 D_5\delta(\theta_{45})\; \bar{D}_4 D_4\delta(\theta_{14})\; \bar{D}_2 D_2\delta(\theta_{25}) \\
= \;& -\bar{D}_1 D_1\delta(\theta_{12})\; \bar{D}_3\bar{D}_3 D_3\delta(\theta_{23})\; D_3\delta(\theta_{34})\; \bar{D}_5 D_5\delta(\theta_{45})\; \bar{D}_4 D_4\delta(\theta_{14})\; \bar{D}_2 D_2\delta(\theta_{25}) \\
= \;& 0\,, \qquad (3.33)
\end{aligned}$$

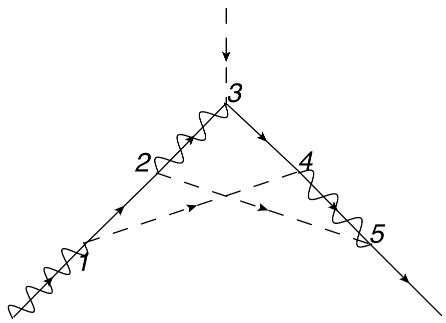

Figure 3.3: Two-loop correction to the vertex.

Here we need to emphasize that the canceling is independent of the ways of regularization one takes, as we have not come to the stage of doing actual momentum integration. One can also see this explicitly from component field calculation. Q.E.D.

One can show that the Wilsonian two-loop correction to Z_A, as shown in Fig 3.4, vanishes.

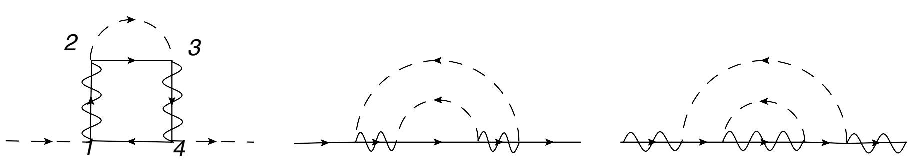

Figure 3.4: Two-loop wave-function renormalization for A, B and $\mathcal{B}$, respectively. Note that there is another diagram contributing to Z_A. It gives the identical contribution to the one presented here.

The one-particle irreducible correction of Z_A, however, is not trivial. With careful calculation, one can show that

$$Z_A = 1 + \frac{i}{2\pi}|\gamma|^4 I \,, \tag{3.34}$$

There are, also corrections to Z_B and $Z_{\mathcal{B}}$ (see Fig. 3.4). After a straight-forward calculation we get (the subscript 0 labels the bare coupling)

$$Z_B = Z_{\mathcal{B}} = 1 + i\gamma_0^2\, I + \frac{1}{2}\gamma_0^4\, I^2 \,. \tag{3.35}$$

The two-loop γ^4 term is an iteration of the one-loop γ^2 term and has no impact on the $\beta(\gamma)$ at the two-loop level. Indeed,

$$\gamma^2 = \gamma_0^2 / Z_B^2 \,, \tag{3.36}$$

and

$$\frac{1}{\gamma^2} = \frac{1}{\gamma_0^2} + 2i\, I + \text{(possibly) } O(\gamma^4) \,, \tag{3.37}$$

with no terms $O(\gamma^2)$. The right-hand side leads us back to $\beta(\gamma)$ as in Eq. (3.1), with no two-loop contribution in the Wilsonian sense.

3.5 Nonrenormalization theorem in full

Since we have both Z_A (in the Wilsonian sense) and Z_γ not corrected up to two-loop order, one can expect that they do not receive higher loop corrections at all. We will show that this is guaranteed by a nonrenormalization theorem, based on supersymmetry in conjunction with the target space symmetry of this model. Moreover, the nonrenormalization theorem is about a D term rather than an F term!

Generally speaking, each D term in the Lagrangian can be treated as an F term, by replacing the integration over θs by Ds acting on the integrand. Then, following the argument of the F term nonrenormalization, one could ask: is it possible to find some background that preserves a half of supersymmetry on which the given F term does not vanish? Can one deduce, on these grounds, that a nonrenormalization appears? The answer is negative.

Let us first understand why nonrenormalization theorems lose their validity for D terms. Assume we want to choose a background, preserved by the supertransformation $\delta_{\bar\epsilon}$. Then, for a chiral superfield ϕ and its antichiral counterpart, we have

$$\bar{D}_R \phi = 0 \,, \qquad D_R \phi^\dagger = 0 \,, \tag{3.38}$$

and

$$\delta_{\bar\epsilon} \phi = 0 \,, \qquad \delta_{\bar\epsilon} \phi^\dagger = 0 \,, \tag{3.39}$$

where we define the supertransformation to be

$$\delta_{\bar\epsilon} = \bar\epsilon \frac{\partial}{\partial \theta_R^\dagger} + i \bar\epsilon \theta_R \partial_L \,. \tag{3.40}$$

This implies strong constraints on the background field that we could choose. Indeed, ϕ has to satisfy

$$\frac{\partial}{\partial\theta_R^\dagger}\phi = 0\,, \qquad \theta_R\partial_L\phi = 0\,. \tag{3.41}$$

Equation (3.41) implies, in turn a general solution of the following form:

$$\phi = f(t-z, \theta_R) + g(t,z)\theta_R\,, \tag{3.42}$$

where f and g could be arbitrary functions. Similarly, for $\phi^\dagger$, we have

$$\phi^\dagger = h(t - i\theta_R^\dagger\theta_R, z - i\theta_R^\dagger\theta_R)\,. \tag{3.43}$$

Now, both ϕ and $\phi^\dagger$ satisfy the chiral condition. Therefore, if we have a combination of ϕ and $\phi^\dagger$ and take the integral over $\int d^2x d\theta_R d\theta_R^\dagger$, it vanishes!

Needless to say, if one first integrates over, say, $d\theta_R^\dagger$, and is left with the "fake" F term, the proof of the nonrenormalization theorem also fails. The above F term,

$$\bar{D}_R A^\dagger \mathcal{B} B\,,$$

will be a total derivative, of necessity, and, hence, the integral over d^2x will vanish (assuming the background to decay at infinity).

This is merely a recap of what we knew before, in a little bit fancy language. We can generalize the logic of the proof, however. In our problem the target space symmetry reveals itself in the invariance of the action under the shift of A,

$$A \to A + a(t-z)\,, \qquad A^\dagger \to A^\dagger + a^\dagger(t-z)\,, \tag{3.44}$$

where a and $a^\dagger$ are generic functions of $t-z$. (Note that they do not need to be Hermitean-conjugate to each other.) The reason is that the function $f(t-z)$ can be understood as being both chiral and antichiral, since both D_R and $\bar{D}_R$ vanish when acting on it. This makes it possible to combine the target space symmetry with the requirement of the supertransformation symmetry. Namely, we will require the background field to be invariant under the shift by $\delta_{\bar{\epsilon}}$ *supplemented* by the target space symmetry.

We can say that what enters in the kinetic term for A and the interaction term, is in fact not the field A itself, but, rather, its equivalence classes under the aforementioned transformation (3.44). Let us denote by $[A]$ the equivalence class to which A belongs. The key idea is that by claiming so, our constraints for the background field get weaker, and we have a "thickening" of our domain of possible solutions to Eqs. (3.38) Eq. (3.39). At the end of the day, a nontrivial background is possible.

In fact, since both the supertransformation symmetry and that of Eq. (3.44) are valid symmetries of $[A]$, if we pick one element in $[A]$, say, A, and apply $\delta_{\bar{\epsilon}}$, it may end up being another element in $[A]$, without changing the whole equivalence class it belongs to. Thus we can relaxe our condition (3.39),

$$\delta_{\bar{\epsilon}} A = \bar{\epsilon} a(t-z)\,, \qquad \delta_{\bar{\epsilon}} A^\dagger = \bar{\epsilon} a^\dagger(t-z)\,, \tag{3.45}$$

where a and $a^\dagger$ are functions of $t-z$, and $\bar{\epsilon}$ is a small supertransformation parameter.

Now, this will lead us to a more general solution for the background field $A^\dagger$,

$$A^\dagger = \theta_R^\dagger a^\dagger(t-z) + h(t - i\theta_R^\dagger\theta_R, z - i\theta_R^\dagger\theta_R)\,. \tag{3.46}$$

Furthermore, one can also show that the allowed background for the field A is not "thickened." It is straightforward to verify that by taking, for example, the following background fields:

$$\begin{aligned} A &= g(t,z)\theta_R\,, \qquad & A^\dagger &= \theta_R^\dagger a^\dagger(t-z)\,, \\ B &= 1\,, \qquad & B^\dagger &= 0\,, \\ \mathcal{B} &= f(t-z,\theta_R)\,, \qquad & \mathcal{B}^\dagger &= 0\,, \end{aligned} \tag{3.47}$$

we indeed have a desirable nontrivial background for both the A kinetic term and the interaction term.

We can then apply the argumentation which leads us to the nonrenormalization theorem. To calculate effective action we decompose the superfields into the background and the quantum parts. Due to the linearity of the target space symmetry, the symmetry transformation can be assigned only to the background part of the A field (and, of course that of $A^\dagger$, too), leaving the quantum part intact. The chosen background fields are invariant under the transformation of $\theta_R^\dagger$ supplemented by the target space shift. The symmetry is exact, it translates to the quantum level in form of a supersymmetry shift of $\theta_R^\dagger$. Therefore, the integrand in the loop calculations is homogeneous in $\theta_R^\dagger$, and, hence, is independent of $\theta_R^\dagger$.

On the other hand, we learn from the Feynman rules listed in Sec. 3.3 that all loop calculations [1] should involve the integration over $\theta_R^\dagger$. Thus, finally we have to obtain zero in two and higher-loop perturbative calculation.

At the moment we are aware of no way to predict quantum corrections for $\mathcal{B}$ and B without explicit calculations, since the constraints (3.38) and (3.39) hold "as is", leaving us with no nontrivial background for their kinetic terms.

Indeed, from the loop calculation in Sec. 3.4 we can see that they get renormalized at two-loop order. Strictly speaking, the two-loop effects contain only double poles, and are merely manifestations of the one-loop terms. However, the background field method can not distinguish between a geometric progression and genuine two-loop effects.

3.6 Generalization to nonperturbative regime a lá Seiberg

In this section we will extend the nonrenormalization theorem of Sec. 3.5 beyond perturbatiion theory. We show that Z_A and Z_γ do not receive nonperturbative corrections either.

Following arguments similar to that in [12], we promote γ to a chiral superfield. It is important to note that the chirality of γ is protected by the target space symmetry.

Indeed, let us inspect the term $\int d^2\theta_R\, \gamma \mathcal{B} B A^\dagger$. It must be invariant under the shift $A^\dagger \to A^\dagger + a^\dagger$. Then $\int d^2\theta_R\, \gamma \mathcal{B} B a^\dagger$ must vanish. This is impossible unless γ is a chiral superfield.

Now, we can assign appropriate R-charges to all fields. They are collected in Table 5.1. Using these charge assignments one can show that independent R-neutral combinations of γ, $A^\dagger$, B and $\mathcal{B}$ are

$$\gamma \mathcal{B} B A^\dagger\,, \quad |\mathcal{B}|^2\,, \quad |B|^2\,, \quad |A|^2\,, \text{ and } |\gamma|^2\,. \tag{3.48}$$

Therefore, we could the renormalized interaction term in the effective Lagrangian in the

[1] Strictly speaking, this does not include the one-loop correction, since the ultraviolet contribution does not involve integrations over $\theta_R^\dagger$. However, the one-loop calculation is easy to carry out explicitly in the way we did it.

Fields	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$
γ	1	0	$-\frac{1}{2}$
A	0	1	$-\frac{1}{2}$
B	0	1	$\frac{1}{2}$
$\mathcal{B}$	-1	0	$-\frac{1}{2}$

Table 3.1: U(1) symmetries of the linear sigma model.

most general case takes the form

$$\int d^2\theta_R\, f\left(\gamma\mathcal{B}BA^\dagger,\, |A|^2,\, |\mathcal{B}|^2,\, |B|^2,\, |\gamma|^2\right) + \text{H.c.}\,. \tag{3.49}$$

Let us suppress the dependence of f on $|\mathcal{B}|^2$, $|B|^2$ and $|\gamma|^2$ for a short while. For a generic function of $\gamma\mathcal{B}BA^\dagger$ and $|A|^2$, it does no harm to express its dependence on these variables as

$$f\left(\gamma\mathcal{B}BA^\dagger,\, \frac{\gamma\mathcal{B}B}{A}\right)\,. \tag{3.50}$$

Now let us check the symmetry: under the shift symmetry $A^\dagger \to A^\dagger + a^\dagger$ for a constant $a^\dagger$, we have

$$\begin{aligned} &\delta_c \int d^2\theta_R\, f\left(\gamma\mathcal{B}BA^\dagger, \frac{\gamma\mathcal{B}B}{A}\right) + \text{H.c.} \\ = &\int d^2\theta_R \Bigg\{ \left[f\left(\gamma\mathcal{B}B(A^\dagger + a^\dagger), \frac{\gamma\mathcal{B}B}{A}\right) - f\left(\gamma\mathcal{B}BA^\dagger, \frac{\gamma\mathcal{B}B}{A}\right)\right] \\ &+ \left[f^\dagger\left(\gamma^\dagger B^\dagger\mathcal{B}^\dagger A, \frac{\gamma^\dagger B^\dagger\mathcal{B}^\dagger}{A^\dagger + a^\dagger}\right) - f^\dagger\left(\gamma^\dagger B^\dagger\mathcal{B}^\dagger A, \frac{\gamma^\dagger B^\dagger\mathcal{B}^\dagger}{A^\dagger}\right)\right]\Bigg\}\,. \end{aligned} \tag{3.51}$$

The whole expression must vanish. Hence, we need the integrand to be a linear combination of a holomorphic and antiholomorphic functions. This tells us that the first line must be a holomorphic function, and the second line antiholomorphic. It is straightforward to see that the former constraint requires

$$f = f_0\left(\frac{\gamma\mathcal{B}B}{A}\right) + f_1\left(\frac{\gamma\mathcal{B}B}{A}\right)\gamma\mathcal{B}BA^\dagger\,, \tag{3.52}$$

where $f_{0,1}$ are some functions, generally speaking. In fact, f_1 must reduce to a constant. Otherwise, upon the shift of A in its argument, we do not get a holomorphic function. The second D term in the braces in Eq. (3.51) leads us to the same conclusion.

Now, let us stitch on possible dependences of f_0 and f_1 on $|\mathcal{B}|^2$, $|B|^2$ and $|\gamma|^2$. We immediately see that they must be free of these structures.

Finally, note that the function f_0 will vanish under integration over $d^2\theta_R$. Hence the only term that can appear in the effective Lagrangian is $\int d^2\theta_R\, f_1\gamma\mathcal{B}BA^\dagger$. Now, since f_1 is independent of γ, f_1 has to be the canonical coefficient from the classical Lagrangian. Q.E.D.

For Z_A the argument is similar. Let us assume the renormalized kinetic term to be

$$\int d^2\theta_R\, \frac{1}{2}\left(f A^\dagger \partial_R A + f^\dagger A \partial_R A^\dagger\right)\,, \tag{3.53}$$

with f and $f^\dagger$ generic functions of the superfields. They must be U(1) neutral under the R rotation, according to Table 5.1. One can show, by applying the stronger symmetry,

$$A \to A + \epsilon_1(t-z)\,, \qquad A^\dagger \to A^\dagger + \epsilon_2(t-z)\,, \tag{3.54}$$

that the functions f and $f^\dagger$ are trivial, with necessity. This completes the proof.

3.7 Supercurrent analysis

Here we present an alternative argument in favor of the absence of higher loops in the β function.

The hypercurrent we need has the form

$$\mathcal{J}_{LL} = \frac{1}{2}\bar{D}_R A^\dagger D_R A\,. \tag{3.55}$$

In components

$$\mathcal{J}_{LL} = j_{LL} + i\theta_R S_{LLL} + i\theta_R^\dagger S_{LLL}^\dagger - \theta_R\theta_R^\dagger T_{LLLL}\,. \tag{3.56}$$

Classically, the U$(1)_A$ current for the rotation of the chiral fermions is conserved,

$$j_{LL} = \psi_L^\dagger \psi_L\,, \quad \partial_R j_{LL} = 0\,. \tag{3.57}$$

The supercurrents are

$$S_{LLL} = i\sqrt{2}\partial_L \phi^\dagger \psi_L \tag{3.58}$$

and $S_{LRR} = 0$ (classically). The supercurrent concervation implies

$$\partial_R S_{LRR} = 0\,. \tag{3.59}$$

The energy momentum tensor has the components :

$$\begin{aligned} T_{LLLL} &= -2\partial_L\phi^\dagger\partial_L\phi - i\psi_L^\dagger\partial_L\psi_L + i\partial_L\psi_L^\dagger\psi_L\,, \\ T_{RRRR} &= -2\partial_R\phi^\dagger\partial_R\phi - i\psi_R^\dagger\partial_R\psi_R + i\partial_R\psi_R^\dagger\psi_R - i\zeta_R^\dagger\partial_R\zeta_R \\ &\quad +i\partial_R\zeta_R^\dagger\zeta_R - 2[i\gamma\zeta_R\psi_R\partial_R\phi^\dagger + \text{H.c.}]\,, \\ T_{LLRR} &= 0 \;\; \text{(classically)}\,. \end{aligned} \tag{3.60}$$

It is easy to see that the three currents j_{LL}, S_{LLL} and T_{LLLL} form a $\mathcal{N} = (0, 2)$ (nonchiral) supermultiplet, which we denote by $\mathcal{J}_{LL}$ and refer to as the hypercurrrent. In superfields we can write $\partial_R \mathcal{J}_{LL} = 0$.

Quantum mechanically j_{LL} is no longer conserved, due to the chiral fermion anomaly, and hence the conservation laws are adjusted in terms of superfields

$$\mathcal{W}_R = -\frac{i\gamma^2}{4\pi}\bar{D}_R(B^\dagger B + \mathcal{B}^\dagger\mathcal{B})\,, \tag{3.61}$$

which, in component, is

$$\mathcal{W}_R = -S_{LRR}^\dagger + i\theta_R(T_{LLRR} + i\partial_R j_{LL}) + i\theta_R\theta_R^\dagger\partial_L S_{LRR}^\dagger \tag{3.62}$$

In particular, there will be a nontrivial contribution to S_{LRR} and T_{LLRR}:

$$\begin{aligned} S_{LRR} &= -\frac{i}{\sqrt{2}\pi}\gamma^3\psi_L\psi_R^\dagger\zeta_R^\dagger\,, \\ T_{LLRR} &= -\frac{\gamma^2}{2\pi}\left[\gamma^2\psi_L^\dagger\psi_L(\psi_R^\dagger\psi_R + \zeta_R^\dagger\zeta_R) - i\psi_R^\dagger\overleftrightarrow{\partial}_L\psi_R - i\zeta_R^\dagger\overleftrightarrow{\partial}_L\zeta_R\right]\,. \end{aligned} \tag{3.63}$$

Thus the chiral anomaly (see Fig. 3.5) and supersymmetry fix the trace of the energy momentum T^μ_μ, which is proportional to the β function. Moreover, we could absorb the power of γ into the definition of the fields, which means that

$$T_{LLRR} = \frac{2}{\gamma}\beta(\gamma)\mathcal{L}\,. \tag{3.64}$$

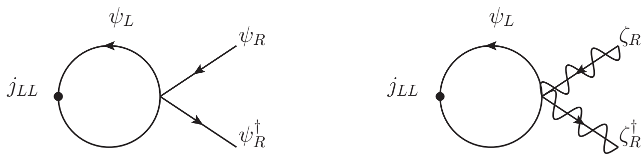

Figure 3.5: One-loop diagram for j_{LL} anomaly.

From this we can see that the $\partial_R j_{LL}$ anomaly actually controls the running of the coupling of the theory. Since the chiral fermion anomaly is a one-loop effect, there is no higher loop contribution to $\partial_R j_{LL}$, which also implies that the Wilsonian β function of γ is one-loop exact. Recall that β function also encodes the information of wave-function renormalization of ζ_R and ψ_R, we could indirectly show that their anomalous dimensions are also one loop exact.

3.8 Conclusion

In this chapter, we introduce a simplified but instructive model that illustrates the nature of the heterotic deformation of $\mathcal{N} = (2,2)$ to $\mathcal{N} = (0,2)$ theories. It was hoped that the theory should have some conformal properties, see e.g. [21]. We showed that this is partially true, due to the nonrenormalization of the interaction term and the target field A. The supergraph method for the $\mathcal{N} = (0,2)$ case that we worked out prompted us that we should expect some nonrenormalization theorems. This is due to the fact that relevant diagrams vanish at the level of the D-algebra — before the momentum integration. And indeed, the nonrenormalization theorems did materialize!

The most interesting result is the proof of D term nonrenormalization for the A kinetic and interaction terms. We generalized the conventional procedure and demonstrated that invoking the target space symmetries we can in a sense expand in realm of F terms. The key fact is that the target space symmetry "thickens" the solution for the nontrivial background field.

Usually for $\mathcal{N} = (0,2)$ nonlinear sigma models, ie, when the superfield A does not live in the flat space, the symmetry group of the theory will be the Diff(M), where M is the target manifold. In most of the cases, the symmetry on M can only realize in terms of $\phi \to f(\phi)$, which should have no direct dependence on space-time. On flat space-time, however, we do have that the transformation function could depend on space-time coordinates. This kind of symmetry has not been explored thoroughly before. On the other hand, the presence of one-particle irreducible two-loop beta function means that this kind of symmetry is somehow anomalous at the infrared, which is nonetheless interesting.

Chapter 4

$\mathcal{N} = (0,2)$ $CP(1)$ models and 2d analogs of 4d SYM

4.1 Introduction

This chapter could have been called "Perturbative and nonperturbative aspects of $\mathcal{N} = (0,2)$ sigma models: the β function, Konishi anomaly, conformal window and all that in $CP(1)$." 4d/2d correspondence is a popular topic in the current literature. Its discussion has a long history. Most theoretical efforts were focused on a relation between four-dimensional $\mathcal{N} = 2$ SQCD and two-dimensional $\mathcal{N} = (2,2)$ sigma models. The former support non-Abelian strings [19, 47]. The latter appear as low-energy effective theories on the non-Abelian string world sheet. It is not surprising then that the BPS-protected sectors of the 4d parents and 2d daughter theories are related.

Later on, the bulk theories supporting non-Abelian strings were deformed to break $\mathcal{N} = 2$ in 4D down to $\mathcal{N} = 1$. It was found [20, 21] that the low-energy theories on the string world sheet are no longer $\mathcal{N} = (2,2)$ supersymmetric. Instead, one gets $\mathcal{N} = (0,2)$ heterotic sigma models with the $CP(N-1)$ target space. This finding gave a strong impetus to explorations of these heterotic models which had been previously discussed only in general terms [39, 48, 49, 38, 32].

In this chapter we will study two-dimensional $\mathcal{N} = (0,2)$ sigma models with the $CP(1)$ target space. A minimal model of this type has one left-handed fermion which, together with a complex scalar field, enters an $\mathcal{N} = (0,2)$ chiral superfield. This minimal model can be readily extended. Nonminimal extensions contain, in addition, N_f right-handed fermions. In particular, if $N_f = 1$, the nonminimal model under consideration reduces to the conventional $\mathcal{N} = (2,2)$ $CP(1)$ model.

We will focus on various derivations of exact expressions for the β functions (valid to all orders in the $CP(1)$ coupling). Remarkably, our results will exhibit a direct parallel between the heterotic $\mathcal{N} = (0,2)$ $CP(1)$ models and four-dimensional super-Yang–Mills theories. In particular, the minimal $\mathcal{N} = (0,2)$ $CP(1)$ model is similar to $\mathcal{N} = 1$ supersymmetric gluodynamics. Its exact β function can be found; it has the structure of the Novikov–Shifman–Vainshtein–Zakharov (NSVZ) β function [10, 50] in supersymmetric Yang–Mills theory without matter. Then we pass to nonminimal $\mathcal{N} = (0,2)$ sigma models. It turns out that this passage corresponds to adding (adjoint) matter in four-dimensional super-Yang–Mills theory. Thus, in the nonminimal $\mathcal{N} = (0,2)$ $CP(1)$ models we will obtain an NSVZ-type exact relation between the β function and the anomalous dimensions γ of the "matter" fields.

Our arguments will be based on a number of methods. First, we will carry out a perturbative (super)graph analysis. This will allow us to obtain the β functions at the two-loop level. Comparison with the $N_f = 1$ case which is in fact $\mathcal{N} = (2,2)$ will give us the first indication on the emergence of the NSVZ-type β function.

Then we will study the instanton measure, using parallels with the analogous NSVZ derivation. We will obtain a version of the nonrenormalization theorem in the instanton background. Essentially we will demonstrate that the instanton measure is exhausted by a one-loop calculation, in much the same way as was the case in 4D super-Yang–Mills theories [50] and in 2D $\mathcal{N} = (2,2)$ sigma models [41]. From this result one can readily deduce a β function of the NSVZ type.

Our third argument is based on the analysis of the supercurrent supermultiplet (the so-called hypercurrent) and its anomalies. Not only will the NSVZ β function be confirmed, but, in addition we will understand the difference between the holomorphic and canonic couplings, which is exactly the same as in the 4D super-Yang–Mills [51]. *En route* we will derive a 2D analog of the Konishi anomaly. This is a necessary element of the β function derivation through the hypercurrent anomaly. The exact formula that we obtain relates the β function of the nonminimal models with the anomalous dimension of the "matter fields." The latter is known as an expansion in perturbation theory.

At large N_f our β function develops an infrared fixed point at small values of the coupling constant (analogous to the Banks–Zaks fixed point [52] [1]). Since the position of this fixed point is at $g^2 \sim 1/N_f$, we can use the leading-order result for the anomalous dimension to *prove* the existence of the fixed point. In other words, in the nonminimal models a conformal window exists starting from some critical value N_f^*. Near the lower edge of the conformal window the theory is presumably strongly coupled.

One can ask a natural question: Why do we consider only the $CP(1)$ model and do not generalize to CP$(N-1)$ with arbitrary N? This is due to an anomaly in heterotic models pointed out in [53, 48]. This anomaly prevents us from considering the models we study here for arbitrary N. However, some other nonminimal generalization of the $\mathcal{N}=(0,2)$ CP$(N-1)$ models will be studied in the next chapter.

The structure of is as follows. We formulate the minimal $\mathcal{N}=(0,2)$ models in Sec. 4.2. In Sec. 4.3 we carry out perturbative calculations of the β function up to two-loop order, in superfield formalism, as outlined in Chap. 2. In Sec. 4.4 we start studying nonperturbative effects in the minimal model (instanton and its measure). We construct exact instanton measure. In this construction we take into account zero modes, one-loop effects in the instanton background, and then, following NSVZ [1], use a nonrenormalization theorem for two and more loops. The instanton background gives us a particularly clear way to see the cancellation of higher loops. The all-loop exact β function is presented in Sec. 4.4.1. In Sec. 4.5 we calculate explicitly the supercurrent supermultiplet for this model. In Sec. 4.6 we extend the minimal model by adding "matter", i.e. the right-handed fermion fields. Following the same road as in the minimal model, we calculate the two-loop β function perturbatively in the nonminimal model. Then we exploit the instanton analysis to obtain an exact relation between the β function and the anomalous dimension γ of the "matter" fields. In Sec. 4.7 we calculate the supercurrent supermultiplets for the extended (nonminimal) models. Section 4.8 is devoted to a 2D analog of the Konishi anomaly in the extended models. Finally, Sec. 4.9 demonstrates the appearance of a conformal window. Main conclusions and prospects for future explorations are summarized in Sec. 5.8.

[1] More exactly, it should have been referred to as the Belavin–Migdal–Banks–Zaks.

4.2 Formulation of the minimal heterotic $CP(1)$ model

In this section we will formulate the minimal $\mathcal{N} = (0, 2)$ $CP(1)$ sigma model (previously it was studied e.g. in [48, 38]). We will use $\mathcal{N} = (0, 2)$ superfield formalism. Note that due to the anomaly in [53, 48] it is impossible to generalize this model to $CP(N-1)$.

The Lagrangian of the model under consideration is

$$\mathcal{L}_A = \frac{1}{g^2} \int d^2\theta_R \frac{A^\dagger i \overleftrightarrow{\partial}_{RR} A}{1 + A^\dagger A}\,, \tag{4.1}$$

where A is a bosonic chiral superfield:

$$A(x, \theta_R^\dagger, \theta_R) = \phi(x) + \sqrt{2}\theta_R \psi_L(x) + i\theta_R^\dagger \theta_R \partial_{LL}\phi\,, \tag{4.2}$$

ϕ is a complex scalar, and ψ_L is a left-handed Weyl fermion. We define $\overleftrightarrow{\partial}_{RR}$ to be $\overrightarrow{\partial}_{RR}/2 - \overleftarrow{\partial}_{RR}/2$. The superfield A can be understood as taking values on the $CP(1)$ manifold, and, thus, can be endowed with the following nonlinear transformations:

$$A \to A + \epsilon + \bar{\epsilon} A^2\,, \quad A^\dagger \to A^\dagger + \bar{\epsilon} + \epsilon (A^\dagger)^2\,, \tag{4.3}$$

plus a U(1) rotation.

In components, we can write the Lagrangian as

$$G \left\{ \partial^\mu \phi \partial_\mu \phi^\dagger + i\psi_L^\dagger \overleftrightarrow{\partial}_{RR} \psi_L - 2i\, \frac{1}{\chi}\, \psi_L^\dagger \psi_L\, \phi^\dagger \overleftrightarrow{\partial}_{RR} \phi \right\}\,. \tag{4.4}$$

The derivatives ∂_{RR} and ∂_{LL} are defined in App. D.1, see Eq. (D.3). Here we denote by G the Kähler metric on the target space (S^2 in the case at hand), in the Fubini–Study form,

$$G = \frac{2}{g^2\, \chi^2}\,, \tag{4.5}$$

where

$$\chi \equiv 1 + \phi\, \phi^\dagger\,. \tag{4.6}$$

Moreover, R is the Ricci tensor,

$$R = \frac{2}{\chi^2}\,, \tag{4.7}$$

while g^2 is the coupling constant.

The coupling constant g can be complexified. In what follows we will deal with the holomorphic coupling g_h defined as

$$\frac{2}{g_h^2} = \frac{2}{g^2} + i\frac{\omega}{2\pi}\,. \tag{4.8}$$

In terms of the holomorphic coupling the Lagrangian of the minimal model has the form

$$\begin{aligned} \mathcal{L}_A &= \int d^2\theta_R \frac{i}{2g_h^2}\frac{A^\dagger \partial_{RR} A}{1 + A^\dagger A} + \text{H.c.} \\ &= -\frac{i}{2g_h^2}\int d\theta_R \frac{\bar{D}_L A^\dagger \partial_{RR} A}{(1 + A^\dagger A)^2} + \frac{i}{2\bar{g}_h^2}\int d\theta_R^\dagger \frac{D_L A \partial_{RR} A^\dagger}{(1 + A^\dagger A)^2}\,. \end{aligned} \tag{4.9}$$

The target space invariance of the integrand is maintained in the second line. In perturbative loop calculations and in instanton analysis we will use the canonical coupling g. To differentiate between the bare and renormalized couplings we will use subscripts 0 and r where appropriate.

In Sect. 4.6 we will extend this minimal model by adding N_f "matter" fields.

4.3 Perturbative superfield calculation of the β function

Fermions do not contribute to the β function at one loop (see e.g. [1]). Therefore, the first coefficient of the β function in the minimal heterotic model is the same as in the nonsupersymmetric $CP(1)$ model (see [1, 46]). The first nontrivial task to address is the calculation of the second coefficient.

In this section we will use the superfield method to calculate the two-loop β function in the minimal model. We will use a linear background field method, setting the background field

$$A_{bk} = f e^{-ix\cdot k}\,.$$

The basic method is roughly the same as that in [1]. The superfield calculation was outlined in [54]. We expand the action around the chosen background, splitting the superfield A into two parts, classical (background) and quantum. Then we calculate relevant diagrams with quantum fields in loops.

If we limit ourselves to the origin in the target space (i.e. $\phi = 0$) and forgo the check of the target space invariance , at two-loop order the β function is determined by the diagrams in Fig. 4.1. As previously we use the ϵ regularization, where

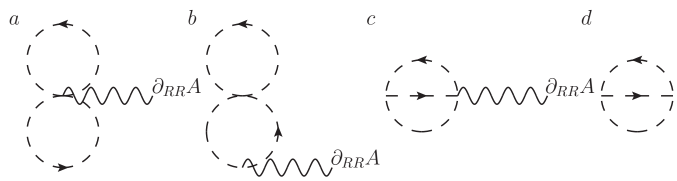

Figure 4.1: Two-loop correction to the coupling g by A-loops. The dashed lines denote the propagator of the quantum part of A in the chosen background, while the wavy lines denote the background field.

$$\epsilon = 2 - d\,.$$

The last diagram does not explicitly exhibit $\partial_{RR}A$ as the external line, but it does produce a contribution due to momentum insertion. A quick evaluation tells us that the first three diagrams contribute only double poles, and they cancel among themselves, as they should. The graph-by-graph results are listed in Table 4.1, where the following notation is used

$$I \equiv \int \frac{d^{2-\epsilon}p}{(2\pi)^d}\,\frac{1}{p^2 - m^2}\,. \tag{4.10}$$

In logarithmically divergent graphs the following correspondence takes place at one loop:

$$\frac{1}{\epsilon} \leftrightarrow \ln\frac{M}{m}\,, \tag{4.11}$$

where the left-hand side represents dimensional regularization, while the right-hand side the Pauli–Villars regularization; M is the mass of the Pauli–Villars regulator. The remaining contribution due to the last diagram results in the following two-loop β function:

$$\frac{1}{g_r^2} = \frac{1}{g_0^2}\left(1 - i g_0^2 I - \frac{i}{4\pi} g_0^4 I\right), \tag{4.12}$$

or

$$\beta(g^2) = -\frac{g^4}{2\pi}\left(1 + \frac{g^2}{4\pi}\right)\,. \tag{4.13}$$

Diagram	Double pole	Single pole
a	$-\frac{3}{2}\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
b	$\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
c	$\frac{1}{2}\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
d	0	$-\frac{g^2}{2}\frac{i}{4\pi}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I$+H.c.

Table 4.1: Two-loop calculation of g^{-2} renormalization in the ϵ expansion follows that in Figure 4.1. $I = \int \frac{d^{2-\epsilon}p}{(2\pi)^d}\frac{1}{p^2-m^2}$, see [46].

Below we will argue that higher loops iterate the two-loop expression in a geometrical progression, so that the full result for the β function in the minimal heterotic model is

$$\beta(g^2) = -\frac{g^4}{2\pi}\left(1-\frac{g^2}{4\pi}\right)^{-1}. \tag{4.14}$$

A parallel with the NSVZ β function in supersymmetric gluodynamics [50] is evident.

4.4 Non-perturbative calculation through the instanton measure

Bosonic $CP(N-1)$ models exhibit instanton solutions [55, 56]. Hence, this is also the case for the $\mathcal{N}=(0,2)$ models. For $CP(1)$, the bosonic (anti-)instanton solution with the unit topological charge is

$$\phi = \frac{y}{z-z_0}\,, \quad \phi^\dagger = \frac{\bar{y}}{\bar{z}-\bar{z}_0}\,, \tag{4.15}$$

where y and z_0 are the collective coordinates: z_0 is the instanton center while a complex number y parametrizes its size and a U(1) phase. Our notation in Euclidean space-time is explained in App. D.2 to which the reader is referred for further details. We easily get the bosonic zero modes, by taking derivatives of the instanton solution with respect to the above collective coordinates. There are four (real) zero modes, or, two complex [1].

The fermion zero modes can be obtained by applying supersymmetry and superconformal symmetry. From the supersymmetry transformation induced by $Q^\dagger$, one obtains the following fermion zero mode:

$$\psi^\dagger_{\bar z} = \frac{\bar y \alpha}{(\bar z - \bar z_0)^2}\,. \tag{4.16}$$

From the superconformal transformation, we get another zero mode,

$$\psi^\dagger_{\bar z} = \frac{\bar y \beta^\dagger}{\bar z - \bar z_0}\,. \tag{4.17}$$

Note that in the $\mathcal{N} = (0,2)$ theory we deal with *two* fermion zero modes rather than four, which appear in the $\mathcal{N} = (2,2)$ $CP(1)$ model. The reason is that, involution is lost upon transition to Euclidean space. No zero mode arises from the background $\phi = \frac{y}{z - z_0}$ (see also [38]). This means that the superinstanton under consideration has no collective coordinates $\alpha^\dagger$ and β. The fact that we deal with two rather than four fermion zero modes agrees with the coefficient in the chiral anomaly (see Sec. 4.5) which is twice smaller in $\mathcal{N} = (0,2)$ compared to $\mathcal{N} = (2,2)$.

Assembling everything together, we obtain the instanton superfield in the form

$$A_{\rm inst} = \frac{y}{z - z_0}\,, \qquad A^\dagger_{\rm inst} = \frac{\bar y (1 + 4 i \theta^\dagger \beta^\dagger)}{\bar z_{\rm ch} - \bar z_0 - 4 i \theta^\dagger \alpha}\,, \tag{4.18}$$

where [2]

$$\bar z_{\rm ch} = \bar z - 2 i \theta^\dagger \theta\,.$$

To derive the instanton measure, we need to define the integral over the collective coordinates. To this end, as usual, we proceed from the mode expansion to the collective coordinates of the zero modes (moduli). In particular, we need to calculate the normalization of the zero modes given by

$$\int dz d\bar z\; G_{ij} \delta\phi_i \delta\phi^\dagger_i\,. \tag{4.19}$$

As a technical point, we note that two of the bosonic (real) modes (conformal) and the fermionic superconformal mode are actually logarithmically divergent in the infrared under the normalization. However, these divergences are canceled by similar divergences coming

[2] Note that in Sect. 4.4 we will use θ and $\theta^\dagger$ to denote the Grassmannian variables in Euclidean superspace. We intentionally drop the subscript "R" to distinguish from those in Minkowski superspace.

from the one-loop contribution due to the nonzero modes. This was explicitly verified in the case of nonsupersymmetric $CP(1)$ models in [57]; the argument readily extends to the supersymmetric case too.

As it follows from the norm of the modes, each (complex) boson zero mode is accompanied by the factor $2/g^2$ and each (complex) fermion zero mode is accompanied by the factor $g^2/2$. The dependence on the instanton size $|y|$ will be omitted temporarily and recovered at a later stage on the basis of dimension arguments. Hereafter, we will drop the constant numerical factors, since they contribute only to an overall constant. As a result, at this stage we arrive at the following instanton measure

$$d\mu = \text{ const. } \left(\frac{1}{g^2}\right)^{n_b} \left(g^2\right)^{n_f} e^{-\frac{4\pi}{g^2}}\ dyd\bar{y}\ \ dz_0 d\bar{z}_0\, d\alpha d\beta^\dagger\,, \tag{4.20}$$

where $n_b = 2$ and $n_f = 1$. (We hasten to add that this is *not* the final result.)

So far quantum corrections have not yet been discussed. In the $\mathcal{N} = (2,2)$ model, the one-loop corrections due to the nonzero modes in the instanton background cancel each other completely [1, 41]. In the $\mathcal{N} = (0,2)$ model this is not quite the case. Let us consider the one-loop effects in more detail. For the nonzero bosonic modes, we will expand the field ϕ as

$$\phi = \phi_{\text{inst}} + \frac{g}{\sqrt{2}}\delta\phi = \phi_{\text{inst}} + \frac{g}{\sqrt{2}}\sum_n \phi_n a_n\,. \tag{4.21}$$

Note that the part ϕ_{inst} contains the boson zero modes. The functions ϕ_n in the expansion (4.21) are the eigenfunctions of the operator

$$-\frac{\partial}{\partial z}\frac{1}{\chi^2_{\text{inst}}}\frac{\partial}{\partial \bar{z}}\phi_n = E_n^2\ \frac{\phi_n}{\chi^2_{inst}} \tag{4.22}$$

normalized by the condition

$$\int \frac{\phi_n^\dagger \phi_n}{\chi^2_{inst}} d^2x = 1\,, \tag{4.23}$$

where $\chi_{\text{inst}} = 1 + \phi^\dagger_{\text{inst}}\phi_{\text{inst}}$, and E_n^2 is the n-th eigenvalue. At the one-loop level we can rewrite the action as

$$-\frac{4\pi}{g^2} - \int d^2x \sum_n E_n^2 a_n^\dagger a_n \frac{\phi_n^\dagger \phi_n}{\chi^2_{\text{inst}}}\,, \tag{4.24}$$

which, according to the standard rule of functional integration, gives

$$\int [\mathscr{D}\delta\phi][\mathscr{D}\delta\phi^\dagger] \to \left\{\det\left[-\frac{\partial}{\partial z}\frac{1}{\chi^2_{inst}}\frac{\partial}{\partial \bar{z}}\right]\right\}^{-1} = \prod_n \frac{1}{E_n^2}\,. \tag{4.25}$$

As for the fermion nonzero modes, we perform a similar expansion. Note that after the Wick rotation, the left-handed fermion ψ_L is no longer related to $\psi_L^\dagger$ by Hermitian conjugation. Therefore in this section we will use $\psi_{\bar{z}}$ and $\psi_z^\dagger$, respectively, to denote them.

Consider the expansion for the fermion fields

$$\psi_{\bar{z}} = \psi_{\bar{z},\text{inst}} + \frac{g}{\sqrt{2}}\sum_n b_n u_n\,, \quad \psi_z^\dagger = \psi_{z,\text{inst}}^\dagger + \frac{g}{\sqrt{2}}\sum_n c_n \bar{v}_n\,, \tag{4.26}$$

where b_n and c_n are complex Grassmannian parameters. The functions u_n and v_n are defined via

$$i\partial_z \frac{1}{\chi_{\text{inst}}^2} u_n = \frac{\mathcal{E}_n}{\chi_{\text{inst}}^2} v_n\,, \quad i\partial_{\bar{z}} v_n = \mathcal{E}_n u_n\,, \tag{4.27}$$

subject to the normalization conditions similar to that of ϕ_n. The part of the action that contains ψ_L now becomes

$$\int d^2x\ i\psi_z^\dagger \partial_z \frac{2}{g^2\chi_{\text{inst}}^2}\psi_{\bar{z}} = \int d^2x\ i\sum_n \mathcal{E}_n c_n b_n \frac{\bar{v}_n v_n}{\chi_{\text{inst}}^2}\,. \tag{4.28}$$

Therefore, integration over the Grassmannian parameters yields $\prod_n \mathcal{E}_n$. Note that in solving Eq. (4.27) we obtain

$$-\partial_z \frac{1}{\chi_{\text{inst}}^2}\partial_{\bar{z}} v_n = \frac{\mathcal{E}_n^2}{\chi_{\text{inst}}^2} v_n\,, \tag{4.29}$$

which is exactly the same as the equation that defines ϕ_n. Hence the boson-fermion degeneracy follows,

$$\mathcal{E}_n^2 = E_n^2\,.$$

In principle the eigenvalue $\mathcal{E}_n$ could be both positive and negative. Let us elucidate this subtle point. In fact, here we are double counting the eigenstates in calculating the fermion determinant. It is easy to see that this is the case if we turn first to the $\mathcal{N} = (2,2)$ theory. There, the relevant action is given by

$$\int d^2x \left(i\sum_n \mathcal{E}_n c_n b_n \frac{\bar{v}_n v_n}{\chi_{\text{inst}}^2} + i\sum_n \mathcal{E}_n \bar{b}_n \bar{c}_n \frac{\bar{u}_n u_n}{\chi_{\text{inst}}^2} \right). \tag{4.30}$$

Integration runs over four Grassmann parameters (at each level),

$$\prod_n db_n d\bar{b}_n dc_n d\bar{c}_n\,, \tag{4.31}$$

In the $\mathcal{N}=(0,2)$ case we have to identify

$$b_n \leftrightarrow \bar{c}_n\,, \quad c_n \leftrightarrow \bar{b}_n\,. \tag{4.32}$$

In other words, we should count only the field configurations that correspond either to $\mathcal{E}_n$ or to $-\mathcal{E}_n$ (assuming $E_n > 0$).

As a result, in our $\mathcal{N}=(0,2)$ theory, $\prod \mathcal{E}_n$ should be understood as $(\prod E_n^2)^{\frac{1}{2}}$, and, hence, symbolically we can write

$$\int [\mathscr{D}\delta\psi_{\bar z}][\mathscr{D}\delta\psi_z^\dagger] \to \left\{ \det \left[\begin{array}{cc} 0 & i\partial_z \frac{1}{\chi^2_{\rm inst}} \\ i\frac{1}{\chi^2_{\rm inst}}\partial_{\bar z} & 0 \end{array} \right] \right\}^{\frac{1}{2}} = \left(\prod_n E_n^2 \right)^{\frac{1}{2}}\,. \tag{4.33}$$

The product runs over the nonzero modes.

As a result, due to the lack of balance between the numbers of the modes (bosonic versus fermionic), we do not have complete cancellation of the one-loop correction coming from the boson nonzero modes by that coming from the fermion nonzero modes. This is in contradistinction with the situation in the $\mathcal{N}=(2,2)$ theory.

We have to evaluate the one-loop contribution from the nonzero modes in the instanton background. In four dimensions this kind of calculation was carried out in [58], and in the pure bosonic $CP(1)$ model it has been done in [57]. All we need to know is the general form of the one-loop correction due to nonzero modes in the instanton measure, $\exp\left(\text{const.}\log\frac{M}{|y|}\right)$, with no explicit g^2 dependence.[3] Here M is mass of the ultraviolet (UV) regulator of the theory. Thus, the one-loop effect will bring us a prefactor M^κ. We will determine it using our knowledge of the bosonic $CP(1)$ model.

Explicitly, we can write down the instanton measure for the bosonic $CP(1)$ model to one-loop order, which is given in [57] and also entirely fixed by the β function at the two-loop level. We will postpone the second derivation till Sec. 4.4.1, and just show the final result. The measure is

$$d\mu \sim \left(\frac{M^2}{g^2}\right)^{n_b} M^{-2}\; dy d\bar{y}\; dz_0 d\bar{z}_0\,, \qquad n_b = 2\,, \tag{4.34}$$

where the factor M^{-2} comes from the one-loop correction due to the nonzero modes, and, hence, $\prod_n E_n^{-2} = M^{-2}$. Given Eq. (4.33), we immediately conclude that the one-loop correction to the instanton measure in our $\mathcal{N}=(0,2)$ model is M^{-1}.

[3] See, however, a remark after Eq. (4.19).

With this knowledge in hand we can return to Eq. (4.20) which contains only zero modes. After inserting nonzero mode one-loop effects we find the instanton measure in the form

$$d\mu \sim \left(\frac{M^2}{g^2}\right)^{n_b} \left(\frac{g^2}{M}\right)^{n_f} M^{-1} e^{-\frac{4\pi}{g^2}}\; dy d\bar{y}\; dz_0 d\bar{z}_0 d\;\alpha d\beta^\dagger\,, \tag{4.35}$$

with $n_b = 2$ and $n_f = 1$. As we will argue in Sect. 4.4.1, this is the exact formula.

Finally, note that the instanton measure is dimensionless. Therefore, we need to reinstate an appropriate dimensional parameter. There is a unique choice, the instanton size, which is, simultaneously, the infrared cutoff in the instanton calculation. It is given by $|y|$.

This leaves us with the following master formula for the measure in the $\mathcal{N} = (0, 2)$ $CP(1)$ model:

$$\begin{aligned} d\mu &= \left(\frac{M^2}{g^2}\right)^{n_b} \left(\frac{g^2}{M}\right)^{n_f} (M)^{-1}\, e^{-\frac{4\pi}{g^2}}\; d\log(y) d\log(\bar{y})\; dz_0 d\bar{z}_0\; d\alpha d\beta^\dagger\,, \\ n_b &= 2\,, \quad n_f = 1\,. \end{aligned} \tag{4.36}$$

4.4.1 A nonrenormalization theorem

This is not the end of the story, however. We have to address the question of two- and higher-loop corrections in the instanton background. In this subsection we will argue that they vanish. Our arguments are intended to show that the instanton measure in (4.36) is all loop exact, i.e., it does not receive higher-loop corrections. The proof is a version of the nonrenormalization theorem [1].

Let us recall that in the instanton background superfield $A_{\rm inst}$ and $A^\dagger_{\rm inst}$, we can apply supersymmetry transformation given by

$$\theta \to \theta + \epsilon\,, \quad \theta^\dagger \to \theta^\dagger + \epsilon^\dagger\,, \quad \bar{z}_{\rm ch} \to \bar{z}_{\rm ch} + 4i\epsilon\theta^\dagger\,. \tag{4.37}$$

Under such a transformation the superinstanton transforms as

$$A_{\rm inst} = \frac{y}{z - z_0} \to \frac{y}{z - z_0}\,, \tag{4.38}$$

$$A^\dagger_{\rm inst} = \frac{\bar{y}(1 + 4i\theta^\dagger\beta^\dagger)}{\bar{z}_{\rm ch} - \bar{z}_0 - 4i\theta^\dagger\alpha} \to \frac{\bar{y}(1 + 4i\theta^\dagger\beta^\dagger + 4i\epsilon^\dagger\beta^\dagger)}{\bar{z}_{\rm ch} - \bar{z}_0 + 4i\epsilon\theta^\dagger - 4i\theta^\dagger\alpha - 4i\epsilon^\dagger\alpha}\,. \tag{4.39}$$

To make the background invariant under such transformations, one can assign appropriate transformation laws to the collective coordinates (moduli), namely,

$$\bar{y} \to \bar{y}(1 + 4i\epsilon^\dagger \beta^\dagger)\,, \quad \bar{z}_0 \to \bar{z}_0 + 4i\epsilon^\dagger \alpha\,, \quad \alpha \to \alpha + \epsilon\,, \quad \beta^\dagger \to \beta^\dagger\,. \tag{4.40}$$

Combining (4.37) and (4.40) it is not difficult to see that the instanton field configuration remains intact when we apply the supersymmetry transformations. Moreover, our expression for the integration measure over the collective coordinates, (4.36), is invariant too. This implies the following. Supersymmetry understood as a combined action of (4.37) and (4.40) is preserved classically by our chosen instanton background, and, hence, it will be preserved in loops. In particular, this forbids any correction to (4.36) of the form [4] $1 + cg^2 \log(M^2|y|^2)$, since such a term would change the power of $\bar{y}$, that would be in contradiction with (4.40).

Moreover, nonlogarithmic corrections of the type $1 + cg^2 + c'g^4 + \cdots$ also do not show up in multiloop calculations. This follows from a nonrenormalization argument similar to the one in Chap. 3. Consider a correction that could possibly come from two or more loops. In this case we can always write the loop integration in the form

$$\int d^2 z d\theta d\theta^\dagger f(x, \theta, \theta^\dagger, z_0, \bar{z}_0, y, \bar{y}, \alpha, \beta^\dagger)\,, \tag{4.41}$$

where the function f must be invariant under supersymmetry transformations supplemented by (4.40). This tells us that f can only be a function of the following (invariant) arguments:

$$y\,, \quad \bar{y}(1 + 4i\beta^\dagger \theta^\dagger)\,, \quad z - z_0\,, \quad \bar{z}_{\rm ch} - \bar{z}_0 + 4i\theta^\dagger \alpha\,, \quad \theta - \alpha\,, \quad \beta^\dagger\,. \tag{4.42}$$

In the subsequent analysis we will only indicate the explicit dependence of f on $\bar{y}(1 + 4i\beta^\dagger\theta^\dagger)$, $\bar{z}_{\rm ch} - \bar{z}_0 + 4i\theta^\dagger\alpha$, $\theta - \alpha$ and $\beta^\dagger$. Only these variables will be of importance. Due to the Grassmannian nature of $\theta - \alpha$ and $\beta^\dagger$, the function f can be represented as a sum

[4] The scale $|y|$ serves as a natural infrared cutoff. Note that the infrared cutoff is provided by the absolute value of y, while the dependence on the phase angle is trivial.

of two terms,

$$
\begin{aligned}
& f\left(\bar{y}(1+4i\beta^{\dagger}\theta^{\dagger}), \bar{z}_{ch}-\bar{z}_0+4i\theta^{\dagger}\alpha, \theta-\alpha, \beta^{\dagger}\right) \\
= \; & f_0\left(\bar{y}(1+4i\beta^{\dagger}\theta^{\dagger}), \bar{z}_{ch}-\bar{z}_0+4i\theta^{\dagger}\alpha\right) \\
& +(\theta-\alpha)\beta^{\dagger} f_1\left(\bar{y}(1+4i\beta^{\dagger}\theta^{\dagger}), \bar{z}_{ch}-\bar{z}_0+4i\theta^{\dagger}\alpha\right),
\end{aligned}
\tag{4.43}
$$

where $f_{0,1}$ are some other functions. It is obvious that upon integration over θ, only f_1 can survive, and the integration takes the form

$$
\int d^2z \; d\theta^{\dagger} \; \beta^{\dagger} \; f_1\left(\bar{y}, \bar{z}-\bar{z}_0+4i\theta^{\dagger}\alpha\right). \tag{4.44}
$$

Next, we shift $\bar{z}$, and then the remaining integral has to vanish. It vanishes, indeed! Note that the integration is finite and local, hence the shift in $\bar{z}$ must be valid.

4.4.2 The full β function

Now we know that our expression for the instanton measure is all-loop exact. It depends on the Pauli–Villars regulator mass M explicitly, through M^2, and implicitly, through $g^2(M)$. The overall dependence on M must cancel, i.e,

$$
\frac{d}{d\log(M)}\left(-\frac{4\pi}{g^2}-\log g^2+\log M^2\right)=0\,. \tag{4.45}
$$

This gives us the all-loop exact β function for the coupling constant g,

$$
\beta(g^2)=-\frac{g^4}{2\pi}\frac{1}{1-\frac{g^2}{4\pi}}\,. \tag{4.46}
$$

The two-loop coefficient is in agreement with (4.13) determined by a direct perturbation calculation.

4.5 Supercurrent supermultiplet (hypercurrent)

In this section we will analyze the hypercurrent (see [59, 43]) of the minimal model. This will set the stage for an alternative derivation of the β function which will be completed in Sect. 4.7. Our consideration will run parallel to that of [60].

Classically, the model under consideration has a conserved U(1) current corresponding to rotations of the chiral fermion ψ_L,

$$j_{LL} = G\psi_L^\dagger \psi_L \,. \tag{4.47}$$

The supercurrent of $\mathcal{N} = (0, 2)$ supersymmetry is

$$S_{LLL} = i\sqrt{2}G\partial_{LL}\phi^\dagger \psi_L \,, \qquad S_{LRR} = 0 \,. \tag{4.48}$$

Finally, the energy momentum tensor of our model has the form

$$\begin{aligned} T_{LLLL} &= -2G\partial_{LL}\phi^\dagger \partial_{LL}\phi - iG\psi_L^\dagger \mathcal{D}_{LL}\psi_L + iG(\mathcal{D}_{LL}\psi_L^\dagger)\psi_L \,, \\ T_{RRRR} &= -2G\partial_{RR}\phi^\dagger \partial_{RR}\phi \,, \\ T_{LLRR} &= 0 \,. \end{aligned} \tag{4.49}$$

It is easy to see that the three currents, j_{LL}, S_{LLL} and T_{LLLL} form an $\mathcal{N} = (0, 2)$ (non-chiral) supermultiplet, which we will denote by $\mathcal{J}_{LL}$ and refer to it as the hypercurrent,

$$\mathcal{J}_{LL} = j_{LL}(x) + i\theta_R S_{LLL}(x) + i\theta_R^\dagger S_{LLL}^\dagger(x) - \theta_R\theta_R^\dagger T_{LLLL}(x) \,. \tag{4.50}$$

In fact, the above multiplet has a concise superfield expression, namely,

$$\mathcal{J}_{LL} = \frac{1}{2}G(\bar{D}_L A^\dagger)D_L A \,, \tag{4.51}$$

with the left-handed fermion current as its lowest component.

As was mentioned, the Lagrangian (4.4) is invariant under U(1) chiral rotations. Therefore, the current j_{LL} is conserved classically, $\partial_{RR}j_{LL} = 0$. This also tells us that $j_{RR} = 0$. Both relations are certainly true at the classical level. In fact, it is obvious that at the classical level the hypercurrent $\mathcal{J}_{LL}$ is conserved as a whole, $\partial_{RR}\mathcal{J}_{LL} = 0$.

The supercurrent conservation is

$$\partial_{LL}S_{RRL} + \partial_{RR}S_{LLL} = 0 \,. \tag{4.52}$$

Classically we have $S_{RRL} = 0$, and, therefore, the conservation law simplifies, $\partial_{RR}S_{LLL} = 0$. As for the energy-momentum tensor, its conservation tells us that

$$\begin{aligned} \partial_{LL}T_{RRRR} + \partial_{RR}T_{LLRR} &= 0 \,, \\ \partial_{LL}T_{RRLL} + \partial_{RR}T_{LLLL} &= 0 \,. \end{aligned} \tag{4.53}$$

The condition of tracelessness is

$$T_{LLRR} + T_{RRLL} = 0\,. \tag{4.54}$$

Moreover, imposing the "symmetrycity" condition on T,

$$T_{LLRR} = T_{RRLL}\,, \tag{4.55}$$

we obtain that $T_{LLRR} = 0$.

Quantum mechanically (i.e. with loops included) the current j_{LL} is anomalous. It is easy to see that the diagram in Fig. 4.2 does not vanish,

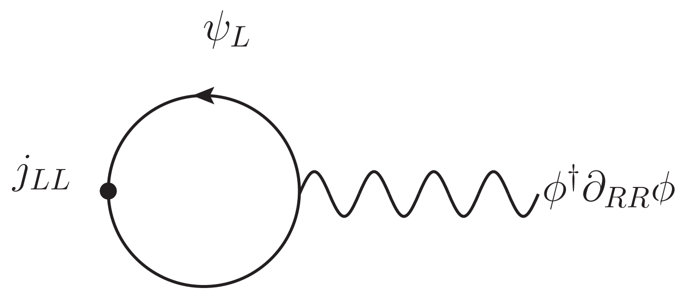

Figure 4.2: One-loop diagram for the j_{LL} anomaly.

$$\partial_{RR} j_{LL} = \frac{i}{2\pi}\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi - \partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2}\,. \tag{4.56}$$

Much in the same way as in the Adler–Bell–Jackiw anomaly [61, 62] and in [60], the chiral current nonconservation is exhausted by one loop in the *Wilsonian* sense. Invoking the superfield formalism, we can translate (4.56) in the superfield language,

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_L A^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger\bar{D}_L A}{(1+A^\dagger A)^2}\right]\,, \tag{4.57}$$

where the right-hand side is *exact* in the Wilsonian sense.

Following the general arguments of [59, 43] it is convenient to rewrite Eq. (4.57) in a general form

$$\partial_{RR}\mathcal{J}_{LL} = -\frac{1}{2}D_L\mathcal{W}_R + \frac{1}{2}\bar{D}_L\bar{\mathcal{W}}_R\,, \tag{4.58}$$

where

$$\mathcal{W}_R = -\frac{1}{2\pi}\frac{\partial_{RR}A\bar{D}_L A^\dagger}{(1+A^\dagger A)^2}\,. \tag{4.59}$$

The superfield $\mathcal{W}_R$ on the right-hand side was absent at the classical level. The expression for $\mathcal{W}_R$ contains S_{LRR} as its lowest component, namely,

$$\mathcal{W}_R = -S^\dagger_{LRR} + i\theta_R(T_{LLRR} + i\partial_{RR}j_{5,LL}) + i\theta_R\theta^\dagger_R\partial_{LL}S^\dagger_{LRR}\,. \tag{4.60}$$

Note that the coefficient in front of $i\theta_R$ contains the real and imaginary parts. The former is the trace of the energy-momentum tensor, the latter is the divergence of the U(1) current.

With this information in hand we finally arrive at

$$\begin{aligned}
S_{LRR} &= \frac{1}{\sqrt{2}\,\pi}\frac{\partial_{RR}\phi^\dagger\psi_L}{(1+\phi^\dagger\phi)^2}\,,\\
T_{LLRR} &= -\frac{1}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi + \partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2} + \frac{2\psi^\dagger_L i\mathcal{D}_R\psi_L}{(1+\phi^\dagger\phi)^2}\right]\,.
\end{aligned} \tag{4.61}$$

The first line in (5.40) presents the superconformal anomaly while the second line presents the scale anomaly. We see that in the Wilsonian sense these anomalies are exhausted by one loop. In particular, for the trace of the energy-momentum tensor we obtain

$$T_{LLRR} = -T^\mu_\mu = \frac{\beta(g^2)}{g^2}\mathcal{L}_A\,. \tag{4.62}$$

Comparing the right-hand sides of (5.40) and (5.43) we conclude that

$$\beta_{\text{Wilsonian}} = -\frac{g^4}{2\pi}\,. \tag{4.63}$$

The denominator in Eq. (4.46) is of an infrared origin. One can say that it comes at the stage of taking the matrix element of the right-hand side of (5.40). Alternatively, one can say that it appears in passing from the holomorphic coupling to the canonically normalized coupling [63]. One should compare this with exactly the same situation in four-dimensional supersymmetric gluodynamics.

4.6 Adding fermions

In this section we consider a more general (nonminimal) version of the $\mathcal{N}=(0,2)$ $CP(1)$ nonlinear model, which is analogous to four-dimensional $\mathcal{N}=1$ super-Yang–Mills theory with adjoint matter. This will turn out beneficial for two reasons: first, we will strengthen the case for our all-loop exact β function. Second, we will find a conformal window in multiflavor heterotic $CP(1)$ models.

To add "matter" we have to use the $\mathcal{N} = (0,2)$ superfield B_i $(i = 1, 2, ..., N_f)$ with the following structure

$$B_i(x, \theta_R, \theta_R^\dagger) = \psi_{R,i}(x) + \sqrt{2}\theta_R F_i(x) + i\theta_R^\dagger \theta_R \partial_{LL} \psi_{R,i}(x)\,. \tag{4.64}$$

As usual, the F terms are auxiliary. Thus, the superfield B contains only a single right-handed fermion degree of freedom (per flavor). The latter has no bosonic counterpart. Then the heterotic $CP(1)$ model with matter acquires the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B = \int d^2\theta_R \left[\frac{1}{g^2} \frac{A^\dagger i \overleftrightarrow{\partial}_{RR} A}{1 + A^\dagger A} + \frac{1}{2} \frac{\vec{B}^\dagger \cdot \vec{B}}{(1 + A^\dagger A)^2} \right] , \tag{4.65}$$

where $\vec{B}$ is the vector made of fermionic chiral superfields,

$$\vec{B} = \left\{ B_i(x, \theta_R, \theta_R^\dagger) \right\} . \tag{4.66}$$

It is easy to see that if $N_f = 1$, the model (5.44) reduces to the $\mathcal{N} = (2,2)$ $CP(1)$ model. This circumstance will be used below. The B_i fields live on the tangent space of $CP(1)$, and, hence, are endowed with the following target space symmetry transformation:

$$\vec{B} \to \vec{B} + 2\bar{\epsilon} A \vec{B}\,, \quad \vec{B}^\dagger \to \vec{B}^\dagger + 2\epsilon A^\dagger \vec{B}^\dagger\,. \tag{4.67}$$

We will find an analog of the NSVZ β function which replaces that of the minimal model (see Sec. 4.2). To this end we will exploit (a) instanton calculus and (b) hypercurrent analysis. We recall that adding fermions in the way described above is only possible for $CP(1)$. The only exception is the case $N_f = 1$. In this case we deal with the nonchiral $CP(1)$ model which can be readily generalized to $CP(N-1)$.

4.6.1 Two-loop result: direct calculation

First we will show that at the perturbative calculation at two-loop level gives exactly the answer we expect. We will collect two-loop corrections to the renormalization of g^{-2} by considering the quantum correction to the kinetic term of the superfield A. As compared with the previous case, all we have to change is that now we need to take the B_i loop into account, see Fig. 4.3.

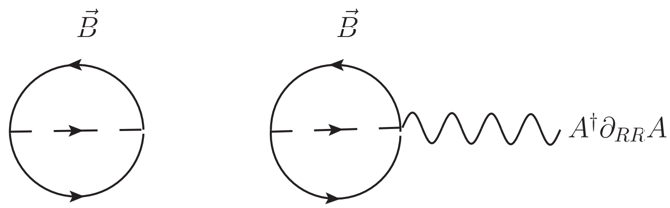

Figure 4.3: Two-loop correction to $\beta(g^2)$ due to the B_i loops.

For each flavor there will be a single diagram that contributes. At the two-loop level distinct flavors do not interfere with each other, they enter additively. Hence, we can guess the result from what happens in the $\mathcal{N} = (2,2)$ $CP(1)$ model, in which case the single-pole contribution due to B cancels that due to the A loop. Indeed, as was mentioned above, if $N_f = 1$ we deal with the $\mathcal{N} = (2,2)$ model, in which all loops in the charge renormalization higher than the first loop must cancel.

So without actually having to do the calculation (which is not difficult, though), we can write down the final result

$$\delta\mathcal{L}_{B\,\text{two loop}} = \int d^2\theta_R N_f \frac{g^2}{2}\frac{i}{4\pi}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A} I + \text{H.c.}\,, \tag{4.68}$$

where I is defined in Eq. (5.8). As a result we have, at the two-loop level

$$\beta(g^2) \equiv \frac{\partial}{\partial \log M} g^2 = -\frac{g^4}{2\pi}\left(1 - \frac{N_f - 1}{4\pi} g^2\right)\,, \tag{4.69}$$

where M is the mass of the ultraviolet regulator (e.g. the Pauli–Villars mass). Shortly we will see that Eq. (5.49) can be rewritten as

$$\begin{aligned}\beta(g^2) &= -\frac{g^4}{2\pi}\frac{1-\frac{N_f\, g^2}{4\pi}}{1-\frac{g^2}{4\pi}} \\ &= -\frac{g^4}{2\pi}\frac{1+\frac{N_f\gamma}{2}}{1-\frac{g^2}{4\pi}}\,,\end{aligned} \tag{4.70}$$

where γ is the anomalous dimension [5] of the B fields (which is one and the same for all matter fields due to the flavor symmetry of the model (5.44)). Needless to say, at $N_f = 1$ the β function degenerates into a one-loop expression. This is welcome since at $N_f = 1$ we, in fact, deal with the $\mathcal{N} = (2, 2)$ model, whose β function is exhausted by one loop [41]. At the two-loop level Eqs. (5.49) and (5.50) are identical. In higher orders (5.50) is exact, as we will argue below.

How does the anomalous dimension γ appear? In the Lagrangian (5.44), as we evolve it from M down to a current normalization point μ, we should take care of the wave-function renormalization of the B_i fields, in addition to the g^2 renormalization. The fields B_i live on the tangent space of the target manifold, and the covariant structure is uniquely fixed.

$$\mathcal{L}_{B,UV} = \int d^2\theta_R \frac{1}{2}\frac{\vec{B}_0^\dagger \vec{B}_0}{(1+A^\dagger A)^2}\,, \quad \mathcal{L}_{B,IR} = \int d^2\theta_R \frac{Z}{2}\,\frac{\vec{B}_0^\dagger \vec{B}_0}{(1+A^\dagger A)^2} \equiv \int d^2\theta_R \frac{1}{2}\frac{\vec{B}_r^\dagger \vec{B}_r}{(1+A^\dagger A)^2}\,. \tag{4.71}$$

The Z factor can be absorbed into the redefinition of B_i. It leaves a remnant, however, in the form of the Konishi anomaly, in much the same way as in four-dimensional super-Yang–Mills [60].

Alternatively, we could introduce a Z-factor in the ultraviolet as follows. We denote it by Z_0,

$$\mathcal{L}_{B,UV} = \int d^2\theta_R \frac{1}{2} Z_0 \frac{\vec{B}^\dagger \vec{B}}{(1+A^\dagger A)^2}\,, \quad \mathcal{L}_{B,IR} = \int d^2\theta_R \frac{1}{2}\frac{\vec{B}^\dagger \vec{B}}{(1+A^\dagger A)^2}\,. \tag{4.72}$$

Note that $Z_0 = Z^{-1}$. These two renormalization schemes are consistent. The ultraviolet factor Z_0 is used in instanton calculus, see Sec. 4.6.2. We introduce it here to make easier the comparison with the previous instanton calculations, for example [1, 41]

For each flavor, we have one and the same diagram for the Z factor which, after a simple and straightforward calculation, yields

$$Z = \frac{1}{Z_0} \equiv \left(\frac{B_{i,r}}{B_{i,0}}\right)^2 = 1 - ig^2 I\,, \tag{4.73}$$

and

$$\gamma(\vec{B}) \equiv -\frac{\partial}{\partial \log \mu} \log Z \equiv -\frac{\partial}{\partial \log M} \log Z_0 = -\frac{g^2}{2\pi} + O(g^6)\,. \tag{4.74}$$

[5] The anomalous dimension is defined below in (4.74).

4.6.2 Instanton calculus

In this section we will apply the instanton analysis in the multiflavor model to substantiate Eq. (5.50). In fact, the difference between the N_f-flavor model and the minimal model in essence reduces to a different number of the fermion zero modes in the given one-instanton background. The target space symmetry ensures that there are zero modes associated with the right-handed fermions $\vec{\psi}_z$. There are two of those for each "matter" field,

$$\psi_{z,i} = \frac{y\alpha_i^\dagger}{(z-z_0)^2}\,, \quad \psi_{z,i} = \frac{y\beta_i}{z-z_0}\,. \tag{4.75}$$

Note that in the instanton measure we no longer have the factor g^2 for each pair of fermion zero modes of $\vec{\psi}_z$. This is in contradistinction with what we had for the modes of the fermion component of the superfield A. The normalization of the B terms in (5.44) is canonical.

At the loop level we get corrections to the collective coordinates $\alpha_i^\dagger$ and β_i due to the corresponding wave-function renormalization. Each fermion superfield acquires $\sqrt{Z}$, and so do $\alpha_i^\dagger$ and β_i. Correspondingly, each $d\alpha_i^\dagger$ and $d\beta_i$ will introduce a factor $Z^{-\frac{1}{2}}$, where Z is defined as in (4.73) and (4.74). Therefore, summarizing, for each "matter" fermion field, we have the accompanying factor $(ZM)^{-1}$.

The second question we must address is the one-loop correction due to nonzero modes. Following the same road as in Sec. 4.4, we can build the expansion in the nonzero modes using the eigenfunctions u_n and v_n defined in (4.27). Indeed, each flavor will give

$$\int [\mathscr{D}\delta\psi_{z,i}][\mathscr{D}\delta\psi^\dagger_{\bar{z},i}] \to \left\{ \det \begin{bmatrix} 0 & i\partial_z \frac{1}{\chi^2_{\rm inst}} \\ i\frac{1}{\chi^2_{\rm inst}}\partial_{\bar{z}} & 0 \end{bmatrix} \right\}^{\frac{1}{2}} = \left(\prod_n E_n^2 \right)^{\frac{1}{2}}, \tag{4.76}$$

and, hence, each extra flavor will contribute M in the instanton measure after an appropriate regularization of the infinite product. One can easily see that when $N_f = 1$, we recover the fact that the one-loop determinant from the boson and fermion nonzero modes, respectively, cancel each other. This is certainly what we expect. As a result, at the end of the day, we get the following expression for the measure:

$$d\mu \;=\; \left(\frac{M^2}{g_0^2}\right)^2 \left(\frac{g_0^2}{M}\right)^1 \left(\frac{1}{Z_0 M}\right)^{N_f} M^{-1+N_f} e^{-\frac{4\pi}{g_0^2}}$$
$$\times \;\; d\log(y) d\log(\bar{y}) \; dz_0 d\bar{z}_0 \; d\alpha d\beta^\dagger \prod_{i=1}^{N_f} d\alpha_i^\dagger d\beta_i \,. \tag{4.77}$$

Next, we note that our nonrenormalization theorem in the instanton background derived in the minimal model (Sect. 4.4.1) holds in the nonminimal model too. The general argument telling us that in the instanton background, all one-particle irreducible diagrams with two loops or more do not contribute is essentially the same as in Sect. 4.4.1. We can illustrate how it happens in the component language for three-loop graphs shown in Fig. 4.4. The diagrams displayed on the left and on the right cancel each other.

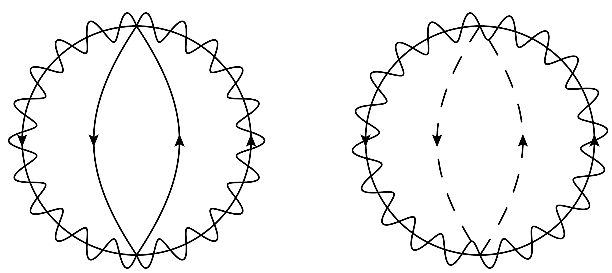

Figure 4.4: An illustration of how the cancellation at higher-loop level happens. The dashed lines are the ϕ propagators, the solid lines are those of $\psi_{\bar{z}}$, and the solid lines with the wavy lines superimposed denote the propagators of $\psi_{z,i}$.

Recall that the Z factors of the B_i fields get renormalized. These are one-particle reducible graphs in the instanton background not seen in the above consideration (in the instanton background the $\psi_{z,i}$ kinetic terms vanish due to equations of motion). They have to be included in the measure additionally, as was done in (4.77).

Asserting that the overall dependence of the instanton measure $d\mu$ on the ultraviolet cut-off M should cancel, we arrive at the exact relation between the β function and the anomalous dimension $\gamma(B_i)$,

$$\beta(g^2) = -\frac{g^4}{2\pi} \frac{1 + \frac{N_f}{2}\gamma(B_i)}{1 - \frac{1}{4\pi} g^2} \,, \tag{4.78}$$

exactly as in (5.50).

In the multiflavor model neither $\beta(g^2)$ nor $\gamma(B_i)$ are all-loop exact. But the relation between them is exact. This is similar to the situation in $\mathcal{N}=1$ super-Yang–Mills theory with matter in four dimensions. As in the NSVZ β function, the knowledge of Z's at one-loop order gives $\beta(g^2)$ at two-loop order, and so on.

4.7 Hypercurrent for N_f flavors

Now we can generalize the hypercurrent, passing from the minimal model (Sect. 4.5) to the multiflavor model. At the classical level the operator $\mathcal{J}_{LL}$ is defined exactly in the same way as in the minimal $\mathcal{N}=(0,2)$ model.

The current j_{LL} is corrected at the quantum level through the anomalous diagram depicted in Fig. 4.2 and, in addition, through a new diagram shown in Fig. 4.5. Now

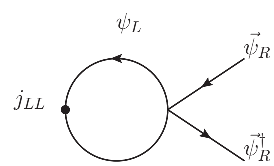

Figure 4.5: One-loop diagram for the j_{LL} anomaly in the $\mathcal{N}=(0,2)$ $CP(1)$ models with matter.

the anomaly can be expressed in superfields as

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2} + \frac{g^2}{2}\frac{\{D_L,\bar{D}_L\}}{2i}\frac{2\vec{B}^\dagger\vec{B}}{(1+A^\dagger A)^2}\right]. \tag{4.79}$$

It is not difficult to understand that the last term on the right-hand side is just the leading term of expansion of the exact (Wilsonian) formula,

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2}\right] - \frac{1}{4}\gamma\,\frac{\{D_L,\bar{D}_L\}}{2i}\frac{2\vec{B}^\dagger\vec{B}}{(1+A^\dagger A)^2}\,. \tag{4.80}$$

To substantiate this point let us consider the renormalization-group evolution of the bare Lagrangian (5.44). The exact Wilsonian effective Lagrangian has the form

$$\mathcal{L}_{\text{Wilsonian}} = \int d^2\theta_R\left[\frac{i}{2g_h^2}\frac{A^\dagger\partial_{RR}A}{1+A^\dagger A} - \frac{i}{2\bar{g}_h^2}\frac{A\partial_{RR}A^\dagger}{1+A^\dagger A} + \frac{1}{2}\frac{\vec{B}_r^\dagger\cdot\vec{B}_r}{(1+A^\dagger A)^2}\right], \tag{4.81}$$

where g_h^2 stands for the holomorphic running coupling whose renormalization is exhausted by one loop. The response of this Lagrangian to scale transformations reduces to

$$\delta\mathcal{L}_{\text{Wilsonian}} \propto \int d^2\theta_R \left[-\frac{i\beta_{\text{Wilsonian}}}{2g_h^4}\,\frac{A^\dagger\partial_{RR}A}{1+A^\dagger A} + \frac{i\bar\beta_{\text{Wilsonian}}}{2\bar g_h^4}\,\frac{A\partial_{RR}A^\dagger}{1+A^\dagger A} - \frac{\gamma}{2}\,\frac{\vec{B}_r^\dagger\cdot\vec{B}_r}{(1+A^\dagger A)^2}\right]. \tag{4.82}$$

The expression inside the square brackets in (4.82), up to a minus sign, is just another component of (5.36). Thus, Eq. (4.82) confirms (4.80). In components Eq. (4.80) is equivalent to

$$\begin{aligned} S_{LRR} &= \frac{1}{\sqrt{2}\pi}\frac{\partial_{RR}\phi^\dagger\psi_L}{(1+\phi^\dagger\phi)^2}\,,\\ T_{LLRR} &= -\frac{1}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi+\partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2}+\frac{2\psi_L^\dagger i\mathcal{D}_{RR}\psi_L}{(1+\phi^\dagger\phi)^2}\right]\\ &\quad +\gamma\left[\frac{\vec{\psi}_R^\dagger i\overleftrightarrow{\mathcal{D}}_{LL}\vec{\psi}_R}{(1+\phi^\dagger\phi)^2}-\frac{2\psi_L^\dagger\psi_L\vec{\psi}_R^\dagger\vec{\psi}_R}{(1+\phi^\dagger\phi)^4}\right]. \end{aligned} \tag{4.83}$$

Returning to Eq. (4.80) we observe that the last term on the right-hand side will convert itself into the first term through the Konishi anomaly (Sect. 4.8). This will produce the numerator of the β function, cf. Eq. (5.50). The denominator will appear upon taking the matrix element of the operator $A^\dagger\partial A/(1+A^\dagger A)$, or, alternatively, upon the transition from the holomorphic coupling to the canonical coupling.

4.8 "Konishi" anomaly

As was mentioned in the Introduction, adding "flavor" fields B_i in the tangent space is similar to adding adjoint matter in the $\mathcal{N}=1$ four-dimensional super-Yang–Mills theory. This similarity extends rather far. In particular, in this section we will derive an analog of the Konishi anomaly [64, 65].

For each matter field that we introduced, we have an extra (classical) U(1) symmetry, corresponding to individual rotations of the B_i fields, see Eq. (5.44). It is obvious that the corresponding classically conserved U(1) currents are

$$j_{RR,i} = \frac{g^2}{2}G\psi_{R,i}^\dagger\psi_{R,i}\,. \tag{4.84}$$

These currents are the lowest components of the superfield operators

$$\mathcal{J}_{RR,i} \equiv \frac{1}{(1+A^\dagger A)^2} B_i^\dagger B_i\,, \qquad i = 1,2,...,N_f\,. \tag{4.85}$$

At the quantum level, due to the anomaly, these matter U(1) currents cease to be conserved. Instead, evaluating the diagrams in Fig. 4.6, we find

$$\partial_{LL}\mathcal{J}_{RR,i} = -\frac{1}{4\pi}\left[D_L \frac{\partial_{RR} A \bar{D}_L A^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L \frac{\partial_{RR} A^\dagger D_L A}{(1+A^\dagger A)^2}\right]\,, \tag{4.86}$$

for each (fixed) value of i. Comparing this expression with the last term in Eq. (4.80) (in its imaginary part) we see that Eq. (4.80) can be rewritten as follows:

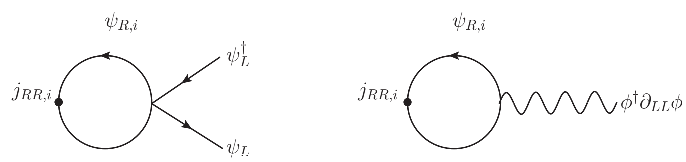

Figure 4.6: One-loop correction to the $U(1)$ current $j_{RR,i}$.

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1+(N_f\gamma/2)}{4\pi}\left[D_L \frac{\partial_{RR} A \bar{D}_L A^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L \frac{\partial_{RR} A^\dagger D_L A}{(1+A^\dagger A)^2}\right]\,. \tag{4.87}$$

The real part of the superfield relation takes the form

$$T_{LLRR} = -\frac{1+(N_f\gamma/2)}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi + \partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2} + \frac{2\psi_L^\dagger i\mathcal{D}_{RR}\psi_L}{(1+\phi^\dagger\phi)^2}\right]. \tag{4.88}$$

These are still Wilsonian operator formulas which present a direct parallel with, say, Eq. (2.111) in [66].

It is clear that the passage to the generator of the 1-particle irreducible vertices (or, alternatively, from the holomorphic coupling to the canonic coupling [63]) proceeds exactly in the same manner as in the minimal model, resulting in the replacement

$$1+(N_f\gamma/2) \to \frac{1+(N_f\gamma/2)}{1-g^2/4\pi}\,. \tag{4.89}$$

4.9 Conformal window

The β function that we have just derived, see Eq. (5.50), has a remarkable property. Assume that $N_f \gg 1$. Then it develops an infrared fixed point at parametrically small values of g^2, such that one can still trust the one-loop result for the anomalous dimension of the matter fields,

$$\frac{g_*^2}{2\pi} = \frac{2}{N_f} \ll 1\,, \tag{4.90}$$

where the asterisk labels the fixed point. If we choose the bare coupling constant in the interval $(0, 2/N_f)$ then the theory under consideration is asymptotically free in the ultraviolet and conformal in the infrared. At large N_f it is weakly coupled at all distances. Perturbative calculations of the anomalous dimensions of various operators make sense.

It is clear that on the side of small N_f the conformal window should extend to some $N_f^* > 1$. At $N_f = 1$ supersymmetry of the model under consideration is enhanced up to (2,2), and this model is certainly nonconformal. Rather, it develops a mass gap.

4.10 Conclusion

In this chapter we studied a class of two-dimensional $\mathcal{N} = (0,2)$ nonlinear sigma models with $CP(1)$ as the target space. We presented a number of arguments indicating the similarity of these models with four-dimensional super-Yang–Mills with adjoint matter. Two further questions could be asked here.

- Can we generalize our nonrenormalization theorem to other models?
- Can we see further implications of the 2D/4D correspondence?

These two questions are interrelated. As for the first one, the answer is positive, at least in part. The original NSVZ argument can be applied to a large class of models with flag manifolds as the target manifolds. We do expect our analysis to go through.

The fermion anomaly [53] does not allow us to extend our multiflavor models in the direction of $CP(N-1)$. This is due to the fact that the anomaly free condition $p_1 = 0$ (p_1 is the first Pontryagin class) rules out $CP(N-1)$ target spaces except for $N_f = 1$. If $N_f = 1$, the model becomes nonchiral.

There are two obvious "technical" questions to be explored. First, the (bi)fermion condensates. It is well-known that such a condensate develops in the $\mathcal{N} = (2,2)$ $CP(1)$ model. Moreover, it plays the role of the order parameter distinguishing between two distinct vacua of this theory. In the minimal $\mathcal{N} = (0,2)$ $CP(1)$ model such a phenomenon seems to be impossible since it is impossible to build a Lorentz scalar from the ψ_L field alone. We checked that one instanton in the minimal model does not give rise to the fermion condensate. Whether or not they develop at $N_f > 2$ remains to be seen.

Next, it is interesting to know whether the NSVZ-like β functions can be derived for heterotic models, such as the one in [21]. At the moment we can formulate a conjecture that the relation (4.78) will stay intact; all dependence on the additional coupling constants will be hidden in the anomalous dimensions.

Another issue of interest is the occurrence of conformality. The nonsupersymmetric (bosonic) $CP(1)$ model is known to be conformal when the vacuum angle ω (see Eq. (4.8)) equals to π. Is it a hint that the conformal window of the $\mathcal{N} = (0,2)$ models extends all the way down to $N_f = 2$?

Chapter 5

Heterotic $CP(N)$ sigma models

Starting from the version of 4d/2d correspondence which we illustrated in Chap. 1, we can think about various deformations. For example, if one considers $\mathcal{N}=1$ preserving deformation for the 4d $\mathcal{N}=2$, $U(N)$ gauge theory with N fundamental hypermultiplets, one arrives at heterotically deformed $CP(N)$ models (heterotic $CP(N)$ models for short).

Here we will consider a class of heterotic model obtained in this way [20, 21], which are characterized by two coupling constants: the original asymptotically free coupling and an extra one describing the strength of the heterotic deformation. These two-dimensional models are of importance on their own, since they exhibit highly nontrivial dynamics, with a number of phase transitions. This fact was recently revealed [67] in the large-N solution of the model.

Our current task is to analyze perturbative aspects of the heterotic CP(N) models. General aspects of perturbation theory in the $\mathcal{N}=(0,2)$ models were discussed by Witten [48]. We will study particular renormalization properties and calculate the β functions in the $CP(N-1)$ models heterotically deformed in a special way. In this chapter of a series we will focus on one-loop effects and demonstrate that both couplings of the model enjoy asymptotic freedom (AF). Moreover, we observe a special fixed-point regime in the infrared (IR) domain and argue that it holds beyond one loop.

Written in components, the Lagrangian of the heterotic $CP(1)$ model takes the form [21]

$$\mathcal{L}_{(2,2)} = G\left\{\partial^\mu\phi\partial_\mu\phi^\dagger + i\bar{\psi}\partial\!\!\!/\psi - 2i\,\frac{1}{\chi}\,\bar{\psi}\gamma^\mu\psi\,\phi^\dagger\partial_\mu\phi - \frac{2}{\chi^2}\,\psi_L^\dagger\psi_L\,\psi_R^\dagger\psi_R\right\}\,, \tag{5.1}$$

and [1]

$$\mathcal{L}_{(0,2)} = \zeta_R^\dagger \, i\partial_L \, \zeta_R + \left[\frac{\gamma}{g^2}\, \zeta_R \, R \left(i\, \partial_L \phi^\dagger\right) \psi_R + \text{H.c.}\right]$$

$$+\frac{|\gamma|^2}{g^2}\left(\zeta_R^\dagger \, \zeta_R\right)\left(R\, \psi_L^\dagger \psi_L\right) + G\left\{\frac{2|\gamma|^2}{g^2\chi^2}\, \psi_L^\dagger \, \psi_L \, \psi_R^\dagger \, \psi_R\right\}\,, \tag{5.2}$$

where G is the Kähler metric on the target space,

$$G = \frac{2}{g^2\, \chi^2}\,, \tag{5.3}$$

R is the Ricci tensor,

$$R = \frac{2}{\chi^2}\,, \tag{5.4}$$

and we use the notation

$$\chi \equiv 1 + \phi\, \phi^\dagger\,. \tag{5.5}$$

The coupling g^2 enters through the metric, while the deformation coupling γ appears in Eq. (5.2). Both couplings can be chosen to be real.[2] Setting g^2 to be real means we will not consider the topological term $\varepsilon_{\mu\nu}\partial^\mu\phi\partial^\nu\phi^\dagger$ now. To make γ real, one should perform a phase rotation of the bosonic field ϕ absorbing the phase of γ (see the first line in Eq. (5.2)). In fact, this corresponds to a kind of R-symmetry, as the reader will see from the $\mathcal{N} = (0,2)$ superfield formalism In Sec. 5.1. Our main results are presented by the following expressions for the one-loop β functions:

$$\beta_g \equiv \frac{\partial}{\partial \ln\mu} g^2(\mu) = -\frac{g^4}{2\pi} + \ldots \tag{5.6}$$

$$\beta_\gamma \equiv \frac{\partial}{\partial \ln\mu} \gamma(\mu) = \frac{\gamma}{2\pi}\left(\gamma^2 - g^2\right) + \ldots \tag{5.7}$$

where ellipses stand for two-loop and higher-order terms. The heterotic deformation does not affect β_g which stays the same as in the $\mathcal{N} = (2,2)$ $CP(1)$ model. Among other results, we calculate the law of running of the ratio

$$\rho = \frac{\gamma^2}{g^2}$$

[1] The sign in front of the term $\zeta_R^\dagger \, \zeta_R \psi_L^\dagger \psi_L$ in (5.2) is opposite to that in [21] due to a typo in [21]. Also notice that the definition of γ here corresponds to γg^2 in [21]. The reason for this rescaling of the deformation parameter compared to [21] is that g^2 and γ^2 as defined here are the genuine loop expansion parameters.

[2] However, in Secs. 5.1 and 5.2 we will treat γ as a complex coupling in analyzing generalized $U(1)$ symmetries and for similar purposes.

see Eq. (5.35). If at any renormalization point, in particular, in the ultraviolet (UV) limit, ρ is chosen to be smaller than $1/2$, in the IR limit it runs to $\rho \to 1/2$, which is the fixed point for this parameter. With $\rho \leq 1/2$, the theory is asymptotically free. Analogs of Eqs. (5.6) and (5.7) in the heterotic $CP(N-1)$ model with arbitrary N are presented in (5.41) and (5.43).

The chapter is organized as follows. In Sec. 5.1, we describe the model using both superfield language and component field language. We show that the $\mathcal{N} = (0,2)$ deformation structure is quite unique in certain sense. In Sec. 5.2, we show that by symmetry analysis, the renormalization structure of the deformation term is constrained. Conceptually it gives partially the answer for why the deformation term does not get renormalized at one-loop level. In Sec. 5.3 we describe the linear background field method and calculate the one-loop β function for g^2 as an example. In Sec. 5.4 we calculate the Z-factor wave-function renormalizations for the field ζ_R and ψ_R. In Sec. 5.5 we calculate the one-loop β function for γ, and thus verify our expectation in Sec. 5.2. In Sec. 5.6 we discuss the running of the two couplings. We find the region that is good for perturbative calculation, and we find an IR fixed point of ρ that exists universally at one-loop order. In Sec. 5.7 we generalize the result to $CP(N-1)$ model. We show that a factorization of β_ρ survives in all loop orders.

5.1 $\mathcal{N} = (0,2)$ $CP(1)$ sigma model

Here we will briefly review basics of the heterotic $CP(1)$ model. The metric (5.3) is the Fubini–Study metric of the two-dimensional sphere S^2. It is not difficult to see that the Lagrangian $\mathcal{L}_{(2,2)} + \mathcal{L}_{(0,2)}$ is invariant with respect to the joint U(1) rotations of the fields ϕ and ψ, and, in addition, invariant under the following nonlinear transformations:

$$\begin{aligned} &\phi \to \phi + \alpha + \alpha^\dagger \phi^2\,, \qquad \phi^\dagger \to \phi^\dagger + \alpha^\dagger + \alpha\phi^{\dagger 2}, \\ &\psi \to \psi + 2\alpha^\dagger \phi\psi\,, \qquad \bar{\psi} \to \bar{\psi} + 2\alpha\phi^\dagger\bar{\psi}. \end{aligned} \tag{5.8}$$

Here α is a complex constant. The above transformations tell us that ϕ transforms as the coordinate of the target manifold (a 2-sphere), and the fields ψ and $\partial^\mu\phi$ transform as the tangent vectors.

Supersymmetry of this model is best understood via its superfield description. The

$\mathcal{N}=(0,2)$ model description can be obtained by either integrating out the Grassmannian variables θ_L and $\theta_L^\dagger$ [21], or by constructing the model from $\mathcal{N}=(0,2)$ superfields. We will follow the second way, for reasons which will become clear shortly. As before, we define the left and right derivatives as follows.

$$\partial_L=\partial_t+\partial_z\,,\qquad \partial_R=\partial_t-\partial_z\,. \tag{5.9}$$

The shifted space-time coordinates that satisfy the chiral condition are defined as

$$x_L^0=t+i\theta_R^\dagger\theta_R\,,\qquad x_L^1=z+i\theta_R^\dagger\theta_R\,. \tag{5.10}$$

We start from two chiral $\mathcal{N}=(0,2)$ superfields A and B,

$$\begin{aligned} A(x_L^\mu,\theta_R)&=\phi(x_L^\mu)+\sqrt{2}\theta_R\psi_L(x_L^\mu)\,,\\ B(x_L^\mu,\theta_R)&=\psi_R(x_L^\mu)+\sqrt{2}\theta_R F(x_L^\mu)\,. \end{aligned} \tag{5.11}$$

The supersymmetry transformations are

$$\begin{aligned} \delta_R\phi&=\sqrt{2}\epsilon_R\psi_L\,, &\qquad \delta_R\psi_L&=-\sqrt{2}i\epsilon_R^\dagger\partial_L\phi\,,\\ \delta_R\psi_R&=\sqrt{2}\epsilon_R F\,, &\qquad \delta_R F&=-\sqrt{2}i\epsilon_R^\dagger\partial_L\psi_R\,. \end{aligned} \tag{5.12}$$

In terms of these $\mathcal{N}=(0,2)$ superfields, it is not difficult to show that the Lagrangian

$$\mathcal{L}_{(2,2)}=\frac{2}{g^2}\int d^2\theta_R\left\{\frac{iA^\dagger\partial_R A-iA\partial_R A^\dagger}{1+A^\dagger A}+\frac{2B^\dagger B}{(1+A^\dagger A)^2}\right\} \tag{5.13}$$

identically reproduces Eq. (5.1).

Needless to say, this Lagrangian is $\mathcal{N}=(0,2)$ invariant, by construction. Next, we must show that it is target-space invariant. The global U(1) symmetry of Eq. (5.13) is obvious. As far as the nonlinear transformations Eq. (5.8) are concerned, for the first term we have

$$\delta_\alpha\,\frac{A\partial_R A^\dagger+A^\dagger\partial_R A}{1+A^\dagger A}\propto\left(\alpha\partial_R A^\dagger+\alpha^\dagger\partial_R A\right), \tag{5.14}$$

which is a combination of holomorphic and antiholomorphic functions, and thus should vanish in the action. Moreover, one can show that the second term is invariant by itself, i.e.,

$$\delta_\alpha\,\frac{B^\dagger B}{(1+A^\dagger A)^2}=0\,. \tag{5.15}$$

The relative coefficient between the first and the second terms in Eq. (5.13) is unity because of the $\mathcal{N} = (2,2)$ symmetry which is implicit in Eq. (5.13). One can, of course, introduce another coupling constant in front of the term quadratic in the B field, but then one can always absorb such constant in the normalization of B.

Thus, with the given set of fields, starting from $\mathcal{N} = (0,2)$ supersymmetry, we get an enhanced $\mathcal{N} = (2,2)$ supersymmetry "for free." This phenomenon is similar to the $\mathcal{N} = (1,1)$ case, as shown in [26].

This gives us a hint on the way of converting the $\mathcal{N} = (2,2)$ model into its heterotic $\mathcal{N} = (0,2)$ extension. Namely, one should add an interaction of the field B with A and $A^\dagger$ without involving the field $B^\dagger$. Taking into consideration the target space symmetry, one can come up with the following deformation Lagrangian:

$$\Delta\mathcal{L} = -\int d^2\theta_R \, \frac{4\gamma}{g^2} \, \frac{BA^\dagger}{1 + A^\dagger A} + \text{H.c.}\,. \tag{5.16}$$

The coupling γ has dimension $m^{1/2}$, and must be viewed as a complex Grassmann number. Now, in components, the deformation Lagrangian takes the form

$$\Delta\mathcal{L} = \Big[\gamma\, G(i\partial_L\phi^\dagger)\psi_R + \text{H.c.}\Big] + |\gamma|^2 (G\psi_R^\dagger\psi_R)\,. \tag{5.17}$$

The above construction, however, suffers from the fermion number nonconservation. The very last step which fixes this problem is promoting γ to a dynamic superfield. To this end we introduce a chiral superfield [3]

$$\mathcal{B} = \zeta_R + \sqrt{2}\theta_R\mathcal{F}\,, \tag{5.18}$$

and then replace γ by

$$\gamma \to \gamma\,\mathcal{B}\,. \tag{5.19}$$

As a result, the deformation Lagrangian takes the form

$$\begin{aligned} \Delta\mathcal{L} &= \int d^2\theta_R \left\{ \left[-\frac{4\gamma}{g^2} \, \frac{\mathcal{B}BA^\dagger}{1 + A^\dagger A} + \text{H.c.} \right] + 2\mathcal{B}^\dagger\mathcal{B} \right\} \\ &= i\zeta_R^\dagger\partial_L\zeta_R + \Big[\gamma\,\zeta_R\, G(i\partial_L\phi^\dagger)\psi_R + \text{H.c.}\Big] \\ &\quad + |\gamma|^2(\zeta_R^\dagger\zeta_R)(G\psi_R^\dagger\psi_R) + |\gamma|^2 G^2 \psi_L^\dagger\psi_L\psi_R^\dagger\psi_R\,. \end{aligned} \tag{5.20}$$

[3] Warning: the definition of the superfield $\mathcal{B}$ here is slightly different from that in [21].

The target space invariance can be verified at the level of superfields,

$$\delta_\alpha \frac{\mathcal{B}BA^\dagger}{1+A^\dagger A} \propto \alpha^\dagger \mathcal{B}B\,, \tag{5.21}$$

which vanishes in the action. Needless to say, the S^2 target space transformations of $\mathcal{B}$ are trivial, $\delta\mathcal{B}=0$.

Finally, we comment on the absorption of the U(1) phase of the coupling γ in this context. This means that whenever $\gamma = |\gamma|e^{i\omega}$, we can always redefine $\gamma \to \gamma e^{-i\omega}$ and $A^\dagger \to A^\dagger e^{i\omega}$ supplemented by a θ_R rotation. So γ can always be kept real. Notably, this rotation only involves bosons. This is because, we can assign a proper U(1) charge to θ_R to keep ψ_L U(1)-neutral. Thus the rotation actually corresponds to a kind of R-symmetry. It is anomaly-free.

5.2 Global symmetries and renormalization structure

In this section we analyze the renormalization structure of the heterotic $CP(1)$ model. In the deformed model we have two dimensionless couplings, g and γ. The first one is related to geometry of the target space, and the second one parametrizes the strength of the heterotic deformation. The first question to ask at one-loop level is that whether or not there is mixing between these two couplings. In addition, we should verify that other structures, which are absent in Eq. (5.13) and Eq. (5.20), do not show up as a result of loop corrections. What can be said from analyzing the symmetries of the model at hand? We will follow the line of reasoning similar to the proof [12] of nonrenormalization theorems for superpotentials in four dimensions. We start from the U(1) symmetries.[4] There are three generalized U(1) symmetries listed in Table 5.1, with the corresponding charge assignments. Using these charge assignments one can show that the only nontrivial combination of γ, $A^\dagger$, B, and $\mathcal{B}$ invariant under all U(1) symmetries is given by $\gamma\mathcal{B}BA^\dagger$.

Since $\Delta\mathcal{L}$ presents the integral over $d\theta_R$ and $d\theta_R^\dagger$, (unfortunately), it is not analogous to F terms in conventional supersymmetric Lagrangians. Hence, with radiative corrections included, $\Delta\mathcal{L}$ can possibly contain U(1)-neutral pairs such as $A^\dagger A$, $B^\dagger B$, $\mathcal{B}^\dagger\mathcal{B}$ and the modulus of the coupling $|\gamma|^2$. Due to dimensional reasons, $B^\dagger B$ could only show up in the

[4] The coupling constant γ can and will be ascribed U(1) charges, as in [12].

Fields	$U(1)_1$	$U(1)_2$	$U(1)_3$
γ	1	0	$-\frac{1}{2}$
A	0	1	$-\frac{1}{2}$
B	0	1	$\frac{1}{2}$
$\mathcal{B}$	-1	0	$-\frac{1}{2}$

Table 5.1: Generalized U(1) symmetries of the heterotically deformed $CP(1)$ model.

combination of the type $B^\dagger B/\mathcal{B}^\dagger\mathcal{B}$, which, obviously, cannot happen in loops provided our theory is properly regularized in the infrared domain (and, of course, it will be regularized both in the IR and UV domains). Its absence can also be verified by analyzing a fermionic SU(2) symmetry discussed in Sec. 5.4.

As for $A^\dagger A$, this term is constrained by the non-linear target space symmetry. The only allowed form is the combination

$$\frac{BA^\dagger}{1+A^\dagger A}$$

and its Hermitian conjugate. This shows that the structure of the deformation is quite unique. Corrections in g^2 are strongly constrained too, as we will see momentarily. Thus, the corrections that can show up in loops are $O(\gamma^2)$, $O(\gamma^2 g^2)$, $O(\gamma^4)$, and so on.

The latter circumstance becomes clear if we take into consideration that the operator $R\partial_L\phi^\dagger\psi_R$ is, in fact, the superconformal anomaly of the undeformed theory,

$$J_{sc,L} = \frac{-i\sqrt{2}}{2\pi}\frac{\partial_L\phi^\dagger\psi_R}{\chi^2}\,. \tag{5.22}$$

Since the supercurrent is conserved, $R\partial_L\phi^\dagger\psi_R$ receives no corrections in g^2, g^4, etc. Certainly, on general grounds we can not exclude corrections of the type $|\gamma|^2 g^2$, but these can show up only in the second and higher loops. Therefore, at one loop the corrections to γ come from $O(|\gamma|^2)$ one-particle irreducible correction to the vertex, and the Z factors of the fields of which the deformation term is built. Calculation of the Z factors is carried out in Sect. 5.4. We check all assertions made above on general grounds by explicit calculation of relevant diagrams in Sect. 5.5.

5.3 One-loop β function for g^2

We will see that β_g is unaffected by the heterotic deformation at one loop. The calculation is exactly the same as in the $\mathcal{N}=(2,2)$ case, hence we again have the following one-loop β function:

$$\beta_g = -\frac{g^4}{2\pi}\,. \tag{5.23}$$

We end this section by commenting on the fermion diagrams. In the undeformed $\mathcal{N}=(2,2)$ model, the relevant diagram is the T product of two fermion U(1) currents $\bar{\psi}\gamma^\mu\psi$ (Fig. E.1), which is finite due to transversality. Moreover, this remains to be the case in the deformed $\mathcal{N}=(0,2)$ model too — all relevant fermion diagrams have no $\ln M_{\rm uv}/\mu$ structure. A more detailed argumentation can be found in App. E. Thus, Eq. (5.23) presents the full contribution at one loop.

5.4 Z-factors

The renormalization of the term $\mathcal{B}BA^\dagger(1+A^\dagger A)^{-1}$ is determined by one-particle irreducible corrections to the vertex and the wave-function renormalization for ψ_R and ζ_R. Here we will deal with the corresponding Z factors, Z_ζ and Z_{ψ_R} (for brevity the latter will be denoted as Z_ψ).

A closer look at the Lagrangian given in Eqs. (5.13) and (5.20), tells us that this Lagrangian is invariant under a SU(2) rotation of $\mathcal{B}$ and $\frac{B}{1+A^\dagger A}$. If we define a SU(2) superfield doublet

$$\Psi = \begin{pmatrix} \frac{\sqrt{2}B}{g(1+A^\dagger A)} \\ \mathcal{B} \end{pmatrix}\,, \tag{5.24}$$

the part that involves all right-handed fermions (i.e., all but the first terms in Eq. (5.26)) can be rewritten as

$$2\Psi_a^\dagger\Psi_a + \sqrt{2}\left[\frac{\gamma}{g}A^\dagger\varepsilon^{ab}\Psi_a\Psi_b + \text{H.c.}\right]\,, \tag{5.25}$$

which is obviously SU(2) invariant.[5] This symmetry has a discrete symmetry inside,

[5] We would like to comment here that this symmetry does not commute with the target space symmetry, as the first component of the doublet is not invariant while the second component is.

which is not affected by fermion anomalies. So we conclude that the wave-function renormalization for $\psi_R \sqrt{\frac{2}{g^2\chi^2}}$ on the one hand, and for ζ_R on the other, must be the same,

$$Z_\zeta = Z_\psi\,.$$

In addition to the perturbative calculation presented below, this can also be indirectly deduced from Chen's analysis [68] of the fermion zero modes in the instanton background.

Now we are ready to carry out the actual calculation to verify the above expectation, and calculate the wave-function renormalizations for ψ_R and ζ_R. We start from the Lagrangian with the bare couplings in UV,

$$\begin{aligned}
\mathcal{L}_0 &= \int d^2\theta_R \frac{2}{g_0^2}\left\{\frac{A\partial_R A^\dagger + A^\dagger\partial_R A}{1+A^\dagger A} + \frac{2B^\dagger B}{(1+A^\dagger A)^2}\right\} + 2\mathcal{B}^\dagger\mathcal{B} \\
&\quad -\frac{4}{g_0^2}\left[\frac{\gamma_0\,\mathcal{B}BA^\dagger}{1+A^\dagger A} + \text{H.c.}\right]\,, \qquad (5.26)
\end{aligned}$$

and evolve it down,[6] where we have

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \int d^2\theta_R \frac{2}{g^2}\left\{\frac{A\partial_R A^\dagger + A^\dagger\partial_R A}{1+A^\dagger A} + Z_\psi\frac{2B^\dagger B}{(1+A^\dagger A)^2}\right\} + 2Z_\zeta\mathcal{B}^\dagger\mathcal{B} \\
&\quad -\frac{4}{g_0^2}Z_\gamma\left[\frac{\gamma_0\,\mathcal{B}BA^\dagger}{1+A^\dagger A} + \text{H.c.}\right]\,. \qquad (5.27)
\end{aligned}$$

Finally, we redefine the fields $\mathcal{B}$ and B to absorb the Z_ζ and Z_ψ factors. With this absorption done, we get

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \int d^2\theta_R \frac{2}{g^2}\left\{\frac{A\partial_R A^\dagger + A^\dagger\partial_R A}{1+A^\dagger A} + \frac{2B^\dagger B}{(1+A^\dagger A)^2}\right\} + 2\mathcal{B}^\dagger\mathcal{B} \\
&\quad -\frac{4}{g_0^2}\frac{Z_\gamma}{\sqrt{Z_\psi Z_\zeta}}\left[\frac{\gamma_0\,\mathcal{B}BA^\dagger}{1+A^\dagger A} + \text{H.c.}\right]\,. \qquad (5.28)
\end{aligned}$$

The one-particle irreducible diagrams for Z_ζ and Z_ψ are shown in Fig. 5.1. The calculation can be done by applying the linear background field method to ψ_R and ζ_R separately, similarly to our description in Sec. 5.3. It is straightforward to check that

[6] At one loop there is no need distinguish between the Wilsonian action and the generator of the one-particle irreducible vertices, as they are the same.

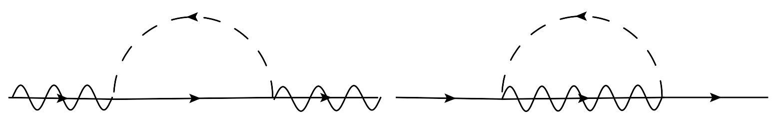

Figure 5.1: One-loop wave-function renormalization of ζ_R and ψ_R. The solid line denotes the fermion field ψ_R. The solid line with a wavy line superimposed corresponds to the field ζ_R. The fermion fields have their quantum parts and background parts marked by the same lines.

$$Z_\psi = Z_\zeta = 1 + i\gamma^2 I\,, \tag{5.29}$$

or, in other words,

$$\zeta_R \to \zeta_R \left(1 + \frac{i}{2}\gamma^2 I\right)\,, \qquad \psi_R \to \psi_R \left(1 + \frac{i}{2}\gamma^2 I\right)\,. \tag{5.30}$$

We collect only divergent terms, where I is given in Eq. (2.13). One can see that ζ_R and ψ_R are corrected by γ^2 in the same way. In principle we still need to collect the g^2 correction to ψ_R. With the canonically normalized kinetic term such a correction would be absorbed in g. With our current normalization (see Eq. (5.13)), it is simply absent. Thus, Eq. (5.29) is the full result for the wave-function renormalization.

5.5 One-loop β function for γ

From Eq. (5.28), we see that

$$\frac{\gamma}{g^2} = \frac{Z_\gamma}{\sqrt{Z_\zeta Z_\psi}}\,\frac{\gamma_0}{g_0^2}\,, \tag{5.31}$$

where Z_γ collects all one-particle irreducible diagrams that correct the $\mathcal{B}B\bar{A}$ vertex. In Sec. 5.2 we used a general argument to show that the $O(g^2)$ correction to Z_γ should vanish. Here we verify this statement, by explicit one-loop calculations. The relevant diagrams are shown in Fig. 5.2, and their contributions are listed in Table. 5.2. It is seen that the total sum of all diagrams vanishes.

As for $O(\gamma^2)$ correction to the $\mathcal{B}B\bar{A}$ vertex, at one loop the would-be relevant diagrams either contain three vertices linear in γ, or involve one four-fermion interaction which contributes $O(\gamma^2)$. In either case, these diagram are one-particle reducible, as shown in Fig. 5.3. So, now it is verified that $Z_\gamma = 1$, and, hence,

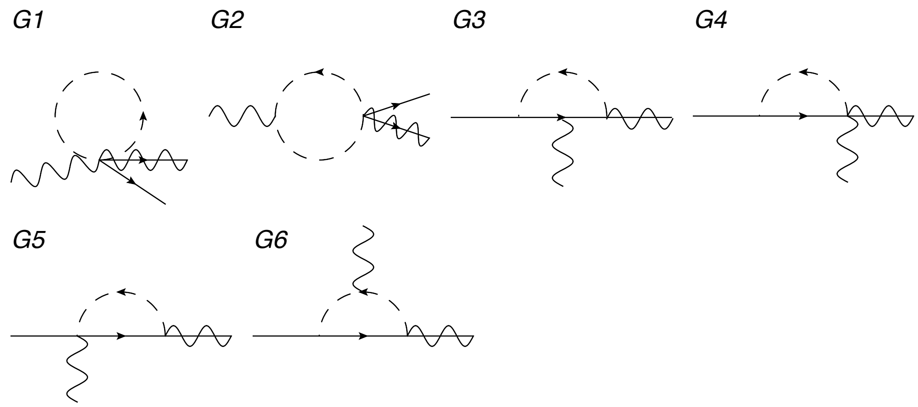

Figure 5.2: One-loop 1PI diagrams contributing to the renormalization of $\zeta_R R(i\partial_L \phi^\dagger)\psi_R$. Their overall sum vanishes, see Table 5.2.

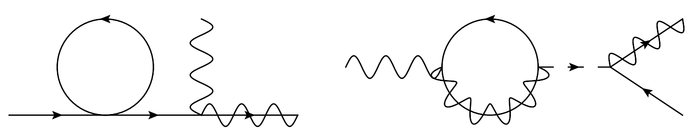

Figure 5.3: Examples of one-loop diagrams $O(\gamma^2)$, which do not contribute to the low-energy effective action since they are one-particle reducible.

Diagram	Result
G1	$-\frac{2(2f^\dagger f-1)}{(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$
G2	$\frac{8f^\dagger f}{d(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$
G3	$-\frac{4f^\dagger f}{(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$
G4	$\frac{4f^\dagger f}{(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$
G5	$\frac{2(2f^\dagger f-1)}{(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$
G6	$-\frac{8f^\dagger f}{d(1+f^\dagger f)^2}\gamma\zeta_R\partial_L\phi^\dagger\psi_R I$

Table 5.2: One-loop results for $R\zeta_R i\partial_L\phi^\dagger\psi_R$ ($1/\epsilon$ terms coming from the integral I).

$$\gamma = \left(1 - i\gamma^2 I + ig^2 I\right)\gamma_0\,, \tag{5.32}$$

implying in turn

$$\beta_\gamma = \frac{\gamma}{2\pi}(\gamma^2 - g^2)\,. \tag{5.33}$$

From symmetry arguments one should be convinced that the correction to the four-fermion interaction (the second line in Eq. (5.2)) must be totally determined by the wave-function renormalization of ζ_R and ψ_R and by renormalization of g^2 and γ. As a consistency check we present the calculation of these terms in App F, to show that it is precisely the case.

5.6 Discussion on the running of the couplings

Now that we know how the couplings g^2 and γ run in the leading order we can discuss the evolution of the two-coupling theory at hand from the UV to IR or vice versa.

The running of g^2 does not change (see Eq. (5.6)), it is still in the AF regime, much in the same way as in the undeformed $\mathcal{N} = (2,2)$ model. Given the definition of the deformation constant in Eq. (5.2), we see that of interest is the evolution law of the ratio

$$\rho(\mu) = \left(\frac{\gamma(\mu)}{g(\mu)}\right)^2\,. \tag{5.34}$$

This is the coupling constant in front of the four-fermion term. As we will see shortly, to avoid the Landau pole in the UV we must choose

$$\rho_0 = \rho(M_{\rm uv}) \le \rho_* \equiv \frac{1}{2}\,.$$

Indeed, assembling together our results for the $\beta_{g,\gamma}$ functions we find

$$\beta_\rho = \frac{g^2}{2\pi}\rho(2\rho-1)\,. \tag{5.35}$$

If $\rho < \rho_*$, the β function is negative implying the AF regime. If, on the other hand, $\rho > \rho_*$, the β function is positive, implying the existence of the Landau pole at a large value of the normalization point. The boundary value $\rho_* = 1/2$ is a fixed point. If at an intermediate normalization point [7] μ we choose $\rho < \rho_*$, it will run according to the AF law in UV and will tend to $1/2$ in IR.

Actually, it is simple to find an analytic solution for ρ as a function of g^2, by rewriting Eq. (5.35) as

$$\frac{d\rho}{\rho(2\rho-1)} = -d\left(\ln(g^2)\right), \tag{5.36}$$

where on the right-hand side we used Eq. (5.23). Eq. (5.36) implies

$$\rho(g^2) = \frac{1}{2+\frac{c}{g^2}}\,, \tag{5.37}$$

where the constant c is fixed by the boundary conditions. In Fig. 5.4 we observe a universal IR behavior of ρ approaching $1/2$ from both sides. Of course, so far our derivation was purely perturbative, and our result was obtained at one loop, which formally precludes us from penetrating too far in the infrared, where the coupling g^2 explodes. However, later we will argue that IR the fixed point at $\rho = 1/2$ survives beyond this approximation.

Figure 5.4 also exhibits the pattern of the UV behavior. If c is positive, ρ is asymptotically free. For negative c one hits the Landau pole at a large (but finite) value of the normalization point. Below we will not consider this regime because of its inconsistency. If we take $c = 0$ the value for ρ freezes at $\rho_* = 1/2$ in the weak coupling domain, and, formally, remains at $1/2$ in the strong coupling domain too. As we will see later, in the heterotic $CP(N-1)$ models with $N > 2$, the boundary value $\rho = 1/2$ is a special point where a certain chiral flavor symmetry is restored.

[7] μ cannot be too small. It must be large enough to guarantee that $g^2(\mu) < 1$. Otherwise we are outside the domain of perturbation theory.

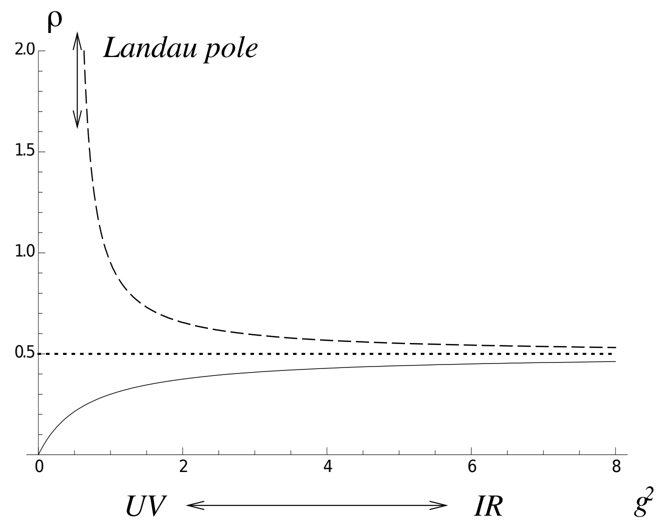

Figure 5.4: ρ versus g^2. The dashed, dotted and solid lines correspond to the cases $c < 0$, $c = 0$, $c > 0$ respectively.

5.7 Extension to $CP(N-1)$ with $N > 2$

In this section, we will generalize our analysis to the heterotic $CP(N-1)$ sigma model. We will show that the parameter ρ does not scale with N, and for any N the function β_ρ is the the same as in Eq. (5.35) at one loop. Then we will discuss a chiral fermion symmetry of the ψ_L's sector at $\rho = 1/2$, which will give us an argument that β_ρ is proportional to $2\rho - 1$ in all loops.

The heterotic $CP(N-1)$ Lagrangian in the geometric formulation can be borrowed from [21],

$$
\begin{aligned}
\mathcal{L}_{\mathrm{CP}(N-1)} \quad &= G_{i\bar{j}}\left[\partial^\mu\phi^i\partial_\mu\phi^{\dagger\bar{j}} + i\bar{\psi}^{\bar{j}}\not{D}\psi^i\right] \\
&+ i\zeta_R^\dagger\partial_L\zeta_R + \left[\gamma\zeta_R G_{i\bar{j}}\left(i\partial_L\phi^{\dagger\bar{j}}\right)\psi_R^i + \mathrm{H.c.}\right] + \gamma^2\left(\zeta_R^\dagger\zeta_R\right)\left(G_{i\bar{j}}\psi_L^{\dagger\bar{j}}\psi_L^i\right) \\
&- \frac{g^2}{2}\left(G_{i\bar{j}}\psi_R^{\dagger\bar{j}}\psi_R^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\bar{m}}\psi_L^k\right) \\
&+ \frac{g^2}{2}\left(1 - 2\frac{\gamma^2}{g^2}\right)\left(G_{i\bar{j}}\psi_R^{\dagger\bar{j}}\psi_L^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\bar{m}}\psi_R^k\right), \qquad (5.38)
\end{aligned}
$$

where

$$
\not{D}\psi^i = \not{\partial}\psi^i + \Gamma^i_{lk}\not{\partial}\phi^l\psi^k\,, \qquad \Gamma^i_{lk} = G^{i\bar{j}}\partial(G_{k\bar{j}})/\partial(\phi^l)\,. \qquad (5.39)
$$

The inverse metric $G^{i\bar{j}}$ is defined from the condition $G^{i\bar{j}}G_{k\bar{j}} = \delta^i_k$ (see [35] and the references therein). We again apply the phase rotation of γ to make it real. Moreover, $G_{i\bar{j}}$ in Eq. (5.38) is the standard Käler metric in the Fubini–Study form. It is seen that $\rho = 1/2$ is a special value nullifying the last line in Eq. (5.38). The scaling of the coupling constants with N is as follows [67]:

$$
g^2 \sim N^{-1}\,, \quad \gamma^2 \sim N^{-1}\,, \quad \rho \sim N^0\,. \qquad (5.40)
$$

Now we will establish that the β functions $\beta_{g,\gamma}$ are compatible with the above scaling.

Our background field strategy can still be applied here much in the same way as in CP(1). As well known, in the undeformed model one has (e.g. [41])

$$
\beta_g = -\frac{Ng^4}{4\pi}\,. \qquad (5.41)
$$

As was demonstrated in Sec. 5.3 (see also App. E), β_g remains intact at one loop in the deformed model. Parallelizing our previous analysis it is not difficult to get that

$$
Z_\zeta = 1 + i(N-1)\gamma^2 I\,, \qquad Z_\psi = 1 + i\gamma^2 I\,, \qquad (5.42)
$$

implying

$$
\beta_\gamma = \frac{N\gamma}{4\pi}\left(\gamma^2 - g^2\right)\,. \qquad (5.43)
$$

Finally, combining Eqs. (5.41) and (5.43) we conclude that in the heterotic $CP(N-1)$ model

$$
\beta_\rho = \frac{Ng^2}{4\pi}\rho(2\rho - 1)\,. \qquad (5.44)
$$

Thus, the scaling laws Eq. (5.40) indeed go through Eqs. (5.41), (5.43), and (5.44). Moreover, one can introduce the 't Hooft couplings

$$\lambda = Ng^2\,, \qquad \gamma' = \sqrt{N}\gamma\,, \tag{5.45}$$

in terms of which there are no explicit N factors in the β functions. In particular,

$$\beta_\rho = \frac{\lambda}{4\pi}\rho(2\rho - 1)\,, \qquad \rho = \frac{\lambda}{(\gamma')^2}\,. \tag{5.46}$$

All UV and IR regimes observed in $CP(1)$ are maintained in the heterotic CP$(N-1)$, in particular, the fixed point $\rho = 1/2$. Note that the constant c expressing the boundary condition in the solution Eq. (5.37) must be rescaled in $CP(N-1)$, namely,

$$\rho(g^2) = \frac{1}{2 + \frac{c}{Ng^2}}\,, \tag{5.47}$$

Now, after we dealt with arbitrary values of N, it is time to turn to the issue of the chiral fermion symmetry. A quick inspection of the third and fourth lines in Eq. (5.38) prompts us that their chiral structure is different. The third line is invariant under *independent* SU$(N-1)$ rotations of ψ_L^i and ψ_R^j, while the fourth line is not.[8] A class of rotations we keep in mind is the SU$(N-1)$ rotations of ψ_L's, with ψ_R's, ζ_R, and bosonic fields intact. In the geometric formulation one introduces vielbeins E_i^a and rotates

$$E_i^a\,\psi_L^i \to U_b^a\,E_i^b\,\psi_L^i \qquad E_i^a\,\psi_R^i \to E_i^a\,\psi_R^i\,, \tag{5.48}$$

where U_b^a is a matrix from SU$(N-1)$. A similar transformation law can be written in the gauge formulation [67] too.[9] Note that in the $CP(1)$ model the only possible chiral transformation of fermions is U(1), and it is anomalous. Hence, in fact there is no such symmetry. It starts from $N \geq 2$ in which case the chiral transformation is non-Abelian and nonanomalous. Consideration of the chiral fermion symmetries in the heterotic $CP(N-1)$ model was started by Tong [69].

[8] The SU$(N-1)$ rotations of ψ_L^i of which we speak here are rather peculiar since we deal with the nonflat target space. The fermions ψ are defined on the tangent space, which is different for different points on the target manifold. Therefore, the SU$(N-1)$ rotations cannot be global. For all terms other than kinetic, this is unimportant. To properly define the chiral symmetry that leaves the kinetic term of ψ_L invariant we have to impose an additional constraint. See App. G for more details.

[9] In the notation of [67] the symmetry enhancement occurs at $|\tilde{\gamma}|^2 = 1$.

At $\rho = 1/2$ the fourth line in Eq. (5.38) vanishes. Other terms in the Lagrangian are invariant under the SU$(N-1)$ rotations of the left-handed fermions Eq. (5.48). (For the kinetic term of ψ_L's see the discussion in App. G.)

Thus, at $\rho = 1/2$ the symmetry of the heterotic $CP(N-1)$ Lagrangian is enhanced. It seems likely that this enhancement (and, hence, the fixed point at $\rho = 1/2$ which goes with it) will hold to all orders in the coupling constant. Indeed, if one remembers about the origin of the heterotic $CP(N-1)$ model as the world-sheet theory on the strings supported in μA^2 deformed $\mathcal{N} = 2$ SQCD, one can try to relate the above symmetry enhancement at $\rho = 1/2$ with that in the bulk theory [69]. In the limit $\mu \to \infty$ the bulk theory becomes $\mathcal{N} = 1$ SQCD acquiring a chiral symmetry absent at finite μ. Remarkably, the $\mu \to \infty$ limit corresponds to $\rho \to 1/2$ on the world sheet [67]. Thus, we expect the β function β_ρ to be proportional to $2\rho - 1$ to all orders.

Now we would like to argue that the solution Eq. (5.47) is, in fact, *valid to all* orders in perturbation theory in the (planar) limit of large N, and so is Eq. (5.44) for β_ρ. Indeed, the heterotic $CP(N-1)$ model was solved in the large-N (planar) limit. The heterotic deformation parameter determining a number of physical quantities (e.g. the vacuum energy density) is [67]

$$u = \frac{\rho}{(1-2\rho)\,(Ng^2)}\,; \tag{5.49}$$

it must be renormalization-group invariant. In addition, u does not scale with N. Substituting the solution Eq. (5.47) in Eq. (5.49) we indeed get

$$u = \frac{1}{c}\,, \tag{5.50}$$

quod erat demonstrandum. The normalization-point independence and N^0 scaling law are explicit in Eq. (5.50).

5.8 Conclusions

In this chapter we started the study of perturbation theory in the recently found $\mathcal{N} = (0,2)$ $CP(N-1)$ sigma models. We carried out explicit calculations of both relevant β functions at one loop and demonstrated that the theory is asymptotically free much in the same way as the unperturbed $\mathcal{N} = (2,2)$ $CP(N-1)$ models provided the initial condition for γ^2 is chosen in a self-consistent way (i.e. c is positive). The β function for the ratio $\rho = \gamma^2/g^2$ exhibits an IR fixed point at $\rho = 1/2$. Formally this fixed point lies outside the validity of the one-loop approximation. We argued, however, basing on additional considerations, that the fixed point at $\rho = 1/2$ may survive to all orders. The reason is the enhancement of symmetry (restoration of a chiral fermion flavor symmetry) at $\rho = 1/2$. Moreover, we argued that Eq. (5.47) for β_ρ formally obtained at one loop, is in fact *exact to all orders* in the large-N (planar) approximation. Thus, in this approximation the fixed point at $\rho = 1/2$ is firmly established.

In addition to the above quantitive results, we also got insights on field-theoretical aspects of the heterotic model. Using the $\mathcal{N} = (0,2)$ superfield language, we saw that in $CP(1)$ both fields ψ_R and ζ_R get the same renormalization at one loop level. This is due to an unexpectated and unusual SU(2) symmetry between ψ_R and ζ_R. The novelty of this symmetry is quite obvious because it mixes chiral and anti-chiral superfields, and does not commute with the target space symmetry.

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Appendix A

Background field method with auxiliary gauge field

Arkady Vainshtein suggested that after we fixed the background fields, we could "gauge away" some of the terms in our Lagrangian by introducing an auxiliary gauge field. This could greatly reduce the amount of calculation. We include here some discussion about this very nice trick.

So we start with the Lagrangian Eq. 2.21, and have in mind that the background field is taken to be plane-wave function as usual. Then in the Lagrangian that is quadratic in quantum fields, $\mathcal{L}_{\phi(2)}$, we have some terms that looks like a mixing of kinetic term and mass term, namely, the terms involves one quantum field and one derivative of the quantum field, which are involved in many diagrams. Vainshtein's trick is roughly a technique to "diagonalize" the Lagrangian and hence reduce the number of mixing terms.

We start by thinking about a $U(1)$ gauged bosonic field, with the kinetic term $\mathcal{D}_\mu\phi\mathcal{D}^\mu\bar{\phi}$, where we have $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ when acted on ϕ. So we compare the structure with $\mathcal{L}_{\phi(2)}$, and define our A_μ to be $-i\frac{G^{0,1}}{G}\partial_\mu\bar{\phi}$. Thus we have $G\partial_\mu q\partial^\mu\bar{q} + G^{0,1}\bar{q}\partial_\mu q\partial^\mu\bar{\phi} + h.c. = G\partial_\mu(qe^{-iA\cdot x})\partial^\mu(\bar{q}e^{iA\cdot x}) - Gq\bar{q}A^\mu A_\mu$. Now we could re-define our field $q \to qe^{-iA\cdot x}$. Then the mixing term $q\partial_\mu\bar{q}$ and $\bar{q}\partial_\mu q$ automatically vanishes.

The importance of such trick is that it reduces the number of diagrams dramatically, while the calculation of each diagrams does not get complicated. It is exactly due to all the terms that are quadratic in quantum field while linear in $\partial_\mu\phi_0$ or $\partial_\mu\bar{\phi}_0$, that complicated

Diagram	I^2 term	IJ term
TB1	$-\frac{f\bar{f}(f^2\bar{f}^2-2f\bar{f}+3)g^2k^2}{(1+f\bar{f})^2}I^2$	0
TB3	$\frac{2f\bar{f}(1-2f\bar{f})^2g^2k^2}{(1+f\bar{f})^2}I^2$	$\frac{2f\bar{f}(1-2f\bar{f})^2g^2k^2}{(1+f\bar{f})^2}IJ$
SB1	$\frac{1}{2}f\bar{f}g^2k^2I^2 - \frac{1}{4}f\bar{f}g^2k^2(d-2)I^2$	0
SB2	$-\frac{4f^2\bar{f}^2g^2k^2}{(1+f\bar{f})^2}I^2$	0
SB3	$\frac{6f^2\bar{f}^2g^2k^2}{(1+f\bar{f})^2}I^2$	$\frac{4f^2\bar{f}^2g^2k^2}{(1+f\bar{f})^2}IJ$
SB8	$f\bar{f}^2(1-f\bar{f})g^2k^2(1+f\bar{f})^2I^2 + \frac{f\bar{f}(1-f\bar{f})g^2k^2}{2(1+f\bar{f})}(d-2)I^2$	0
SB9	$-f\bar{f}^2(1-f\bar{f})g^2k^22(1+f\bar{f})^2I^2 - \frac{f\bar{f}(1-f\bar{f})g^2k^2}{4(1+f\bar{f})}(d-2)I^2$	0

Table A.1: Two-loop calculation for bosonic $CP(1)$ sigma model using Vainshtein's trick

our situation. So we think this trick is a very good complementation of our method. To give the readers a better understanding of how this trick worked, we attached the result of two loop calculations for the bosonic model, which could be compared with Table. 2.2 and Table. 2.3. We note here that one could use it to calculate supersymmetric model and the heterotic model very easily.

Appendix B

Sub-diagram divergences of two-loop calculation

B.1 Vacuum diagrams

In this section we are going to give some examples on the calculation of vacuum diagrams and the comparison of our approach with the one by introducing Feynman parameters and doing the loop integration explicitly, which at the same time could show us the validity of doing so.

TF1

The relevant vertices are $\Gamma^{1,1}q\bar{q}i\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\phi_0$ and $\Gamma i\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\phi_0$, and T-product is calculated as follows.

$$
\begin{aligned}
&\langle T(\bar{\psi}\gamma^{\mu}\psi)_x(q\bar{q}\bar{\psi}\gamma^{\nu}\psi)_y\rangle = (-1)\mathrm{Tr}[\langle Tq_y\bar{q}_y\rangle\gamma^{\mu}\langle T\psi_x\bar{\psi}_y\rangle\gamma^{\nu}\langle T\psi_y\bar{\psi}_x\rangle] \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}\int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{-i}{D2}\mathrm{Tr}\left[\gamma^{\mu}i\frac{\not{p}+m}{D1}\gamma^{\nu}i\frac{\not{p}+m}{D1}\right] \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}\left(\frac{d^dr}{(2\pi)^d}\frac{-i^3}{D2}\right)\int \frac{d^dp}{(2\pi)^d}(\frac{4}{d}-2)\frac{p^2}{D1^2}+2g^{\mu\nu}\frac{m^2}{D1^2} \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}ig^{\mu\nu}\left(\frac{4-2d}{d}I^2+\frac{4-2d}{d}IJ+2IJ\right) \\
= \;& ig^{\mu\nu}\left(\frac{4-2d}{d}I^2+\frac{4-2d}{d}IJ+2IJ\right)\delta(x-y).
\end{aligned}
\tag{B.1}
$$

During the calculation we used our previous argument to set the external momentum to be zero, otherwise the second line does not follow. We note that in the calculation of T-product, we dropped the propagator normalizer $\frac{2}{g^2(1+f\bar{f})^2}$, which can be easily recovered by counting the number of quantum field line in our diagrams. Now let us look at the result. I^2 is a double pole, which diverges when both p and r are large. The coefficient $4-2d$ lowers the degree of divergence, due to the conserved current $\bar{\psi}\gamma^{\mu}\psi$. So we identify $(4-2d)I^2$ as a genuine single pole. On the other hand, the part IJ means the fermion loop is UV finite. This contribution should be a manifestation of one-loop effect.

We are going to show another example where the integrations with respect to the two loops are not independent.

SF1

Here the two vertices that involved are $\Gamma^{1,0}qi\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\phi_0$ and $\Gamma^{0,1}\bar{q}i\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\phi_0$.

$$\begin{aligned}
&\langle T(q\bar{\psi}\gamma^{\mu}\psi)_x(\bar{q}\bar{\psi}\gamma^{\nu}\psi)_y\rangle = (-1)\mathrm{Tr}[\langle Tq_x\bar{q}_y\rangle\gamma^{\mu}\langle T\psi_x\bar{\psi}_y\rangle\gamma^{\nu}\langle T\psi_y\bar{\psi}_x\rangle] \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}\int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{-i}{D2}\mathrm{Tr}\left[\gamma^{\mu}i\frac{\not{p}+m}{D1}\gamma^{\nu}i\frac{\not{p}-\not{r}+m}{D3}\right] \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}\int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}i\frac{(\frac{4}{d}-2)g^{\mu\nu}p\cdot(p-r)+2g^{\mu\nu}m^2}{D1\cdot D2\cdot D3} \\
= &\int \frac{d^2k}{(2\pi)^2}e^{-ik\cdot(x-y)}\int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}i\frac{(\frac{2}{d}-1)g^{\mu\nu}(D1-D2+D3+m^2)+2g^{\mu\nu}m^2}{D1\cdot D2\cdot D3} \\
= &\; i\left(\frac{2}{d}-1\right)I^2\delta(x-y).
\end{aligned} \tag{B.2}$$

All the equations hold up to a finite constant. We use the fact that changing integration variable or shifting it with Jacobian of absolute value 1 does not change the result. That means $\int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{1}{D1\cdot D2} = \int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{1}{D1\cdot D3} = \int \frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\frac{1}{D2\cdot D3} = I^2$. There is no sub-diagram divergence in this diagram, which can be seen immediately by the fact that $\bar{\psi}\gamma^{\mu}\psi$ is the conserved current. So if we integrate the fermion loop first, we have a structure proportional to $(d-2)(\frac{r^{\mu}r^{\nu}}{r^2}-g^{\mu\nu})\ln r^2$, and it is clear that the second loop integration is free of sub-diagram divergence.

B.2 Diagrams with momentum insertion

We are going to give some examples on how to distinguish between genuine single poles and sub-diagram divergences for diagrams with momentum insertion.

SF11

At first glance of this diagram one should notice that there is no mass insertion, hence no sub-diagram divergence. In this case all the single poles contributes to β function. The result is indeed this because of more than one reasons. Take, for example, the one involves

$\bar{\psi}\gamma^\mu\partial_\mu q\psi$ and $\bar{\psi}\gamma^\mu\partial_\mu\phi\psi\bar{q}$.

$$\langle \mathrm{T}(\bar{\psi}\gamma^\mu\psi\bar{q})_x(\bar{\psi}\gamma^\nu\partial_\nu q\psi)_y\rangle = (-1)\mathrm{Tr}[\gamma^\mu\langle \mathrm{T}\psi_x\bar{\psi}_y\rangle\gamma^\nu\langle \mathrm{T}\psi_y\bar{\psi}_x\rangle]\langle \mathrm{T}\partial_\nu q_y\bar{q}_x\rangle$$

$$= \int\frac{d^2l}{(2\pi)^2}e^{-il\cdot(x-y)}\int\frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}\cdot$$

$$(-1)\mathrm{Tr}\left[\gamma^\mu i\frac{\not{p}-\not{r}+m}{D3}\gamma^\nu i\frac{\not{p}+m}{D1}\right]\frac{-r^\nu-l^\nu}{(r+l)^2-m^2} \tag{B.3}$$

The integration along the fermion loop is transversal due to the conserved current $\bar{\psi}\gamma^\mu\psi$, which means that it has the structure $\frac{r^\mu r^\nu}{r^2}-g^{\mu\nu}$. Then the integration

$$\int\frac{d^dp}{(2\pi)^d}\frac{d^dr}{(2\pi)^d}(-1)\mathrm{Tr}\left[\gamma^\mu i\frac{\not{p}-\not{r}+m}{D3}\gamma^\nu i\frac{\not{p}+m}{D1}\right]\frac{-r^\nu}{(r+k)^2-m^2}$$

must be trivial. The remaining part, being at lease linear in k^μ, can be calculated by simply setting $(r+k)^2-m^2$ on the denominator to be r^2-m^2, due to dimensional reasons.

This method applies easily to all such diagrams with fermion loops. Thanks to the background we have chosen, we can always arrange the integration in such a way that the external momentum p flows through the boson propagators. we integrate fermion loop first, then it is both UV and IR regularized by a factor $d-2$. So it is finite. Notice that the ambiguity of expanding integrand around small p is a finite quantity. It will not manifest itself at two loop level as long as the other loop is finite.

When, generally we are dealing with diagrams that an external has to flow into fermion loops, or when we are calculation boson diagrams, we need to do more, as shown in the following example.

SB7

Note that there are actually more than one calculations correspond to this diagram. Specifically, we are talking about the one made up of the vertices $G^{0,1}\bar{q}\partial^\mu\phi_0\partial_\mu\bar{q}$ and $G^{1,0}q\partial^\mu\bar{q}\partial_\mu q$.This is about three point Green function and our choice of external field indicates that there are momentum insertion flowing from y to x and z to x. But notice that the vertex x is more or less like a mass insertion, so there are ambiguities of whether it is m^2 or the external momentum that regularizes the IR behavior. Graphically we should see that in this diagram there is no external momentum flowing from z to y, so there is certain prediction that the external momentum is not going to regularize the integration. Put it in another words, all the poles should be due to the sub-diagram divergence. Our

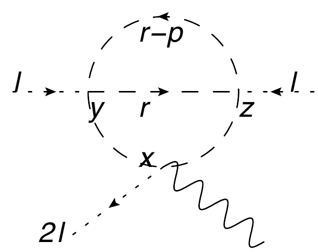

Figure B.1: Two-loop diagram SB7 (see Figure 2.3) with only bosonic loops contributing to the two-loop β function. The wavy lines stand for the background field ϕ_0, the dotted lines for momentum insertion.

strategy is, we are going to expand the integrand around small external momentum. We keep track of the generation of structure $\frac{1}{D1^2}$ to see if it comes from the expansion of a single propagator carrying momentum $q+p$ or it is because of the mass insertion. The T-product is as follows:

$$\langle \mathrm{T}(\bar{q}\partial^\mu \bar{q})_x (q\partial^\nu \bar{q}\partial_\nu q)_y (q\partial^\sigma \bar{q}\partial_\sigma q)_z \rangle = \int \frac{d^d l}{(2\pi)^d} e^{il\cdot(y+z-2x)} \int \frac{d^d p}{(2\pi)^d} \frac{d^d r}{(2\pi)^d}$$

$$2i^3 l^\mu \frac{(r\cdot(r-p))^2 - r\cdot(p-l) r\cdot(p-r) - (p-l)\cdot(r-p) r\cdot(p+l)}{[(p+l)^2-m^2][(p-l)^2-m^2] D2\cdot D3} \tag{B.4}$$

Since we are interested in only the term proportional to p^μ, and there is no need to expand the denominator, it is clear that the loop integrand equals to

$$(2\pi)^d 2i^3 l^\mu \frac{(r\cdot(r-p))^2 - r\cdot p r\cdot(p-r) - p\cdot(r-p) r\cdot p}{D1^2\cdot D2\cdot D3}.$$

So it is clearly that all the $\frac{1}{D1^2}$ structure is because of the mass insertion vertex and hence are regularized by m^2. Thus within this typer of diagram there is no genuine single pole. Everything is simply the remnant of sub-diagram divergence.

Appendix C

Mass terms in $\mathcal{N} = (2,2)$ $CP(1)$ model

To fully understand the infrared regulator of the perturbative calculation, we give a complete calculation for the mass renormalization of boson and fermion masses and show the covariance structure explicitly. We keep the undetermined constants c and c' in our calculation explicitly.

For boson mass, we have the diagrams shown in Fig. C.1. It is straight-forward to show that due to these diagrams, $Z_{m^2} = 1 + (1-c)ig^2 I + 2c' ig^2 I$. For fermion mass, we

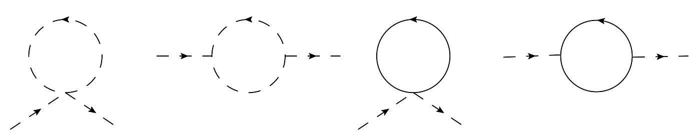

Figure C.1: One-loop correction for the mass term of quantum fields q and $\bar{q}$. The notation follows that of previous diagrams.

have the diagrams as in Figure C.2, where the contribution gives $Z_m = 1 + ig^2 I - c' ig^2 I$.

On the other hand combining the contributions from bosonic and fermionic mass terms, we can see

$$\Delta\mathcal{L}_{\text{two loop}} = \frac{2}{g^2(1+\bar{\phi}\phi)^2}\partial^\mu\phi\partial_\mu\bar{\phi}(1-c+2c')g^4 IJ\,, \tag{C.1}$$

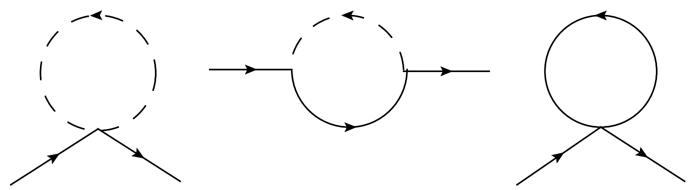

Figure C.2: One-loop correction for the mass term of quantum fields ψ and $\bar{\psi}$. The notation follows that of previous diagrams.

which is exactly canceled by one loop effects from Z_{m^2}. But in order to maintain supersymmetry, we need $Z_{m^2} = Z_m^2$, which gives that $c' = \frac{1+c}{4}$. This gives us the fine tuning relation.

Appendix D

Superspace notations

D.1 Minkowski spacetime

In this appendix we give a description of $\mathcal{N} = (0,2)$ $D = 1+1$ superspace and fix the notations.

The space-time coordinate $x^\mu = \{t, z\}$ can be promoted to superspace by adding a complex Grassmann variable θ_R and its complex conjugate $\theta_R^\dagger$. Wherever our expressions are dependent on the representation of Clifford algebra, we use the following convention.

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = \gamma^0\gamma^1 . \tag{D.1}$$

Under this representation the Dirac fermion is expressed as

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} . \tag{D.2}$$

We define the left moving and right moving derivatives as

$$\partial_{LL} = \partial_t + \partial_z \,, \qquad \partial_{RR} = \partial_t - \partial_z \,, \tag{D.3}$$

and use the following definition for the superderivatives:

$$D_L = \frac{\partial}{\partial\theta_R} - i\theta_R^\dagger \partial_{LL} \,, \qquad \bar{D}_L = -\frac{\partial}{\partial\theta_R^\dagger} + i\theta_R \partial_{LL} \,. \tag{D.4}$$

Their commutator gives $\{D_L, \bar{D}_L\} = 2i\partial_{LL}$.

To change between ordinary coordinates and the lightcone coordinates, we have, for supercurrent:

$$S_L^0 = S_{RRL} + S_{LLL}\,, \quad S_L^1 = S_{LLL} - S_{RRL}\,. \tag{D.5}$$

And for $T^{\mu\nu}$:

$$T_{LLLL} = T_{00} + T_{10} + T_{11} + T_{01}\,, \tag{D.6}$$

$$T_{LLRR} = T_{00} + T_{10} - T_{11} - T_{01}\,, \tag{D.7}$$

$$T_{RRLL} = T_{00} - T_{10} - T_{11} + T_{01}\,, \tag{D.8}$$

$$T_{RRRR} = T_{00} - T_{10} + T_{11} - T_{01}\,. \tag{D.9}$$

The shifted space-time coordinates that satisfy the chiral condition are

$$y^0 = t + i\theta_R^\dagger \theta_R\,, \qquad y^1 = z + i\theta_R^\dagger \theta_R\,. \tag{D.10}$$

The antichiral counterparts are

$$\tilde{y}^0 = t - i\theta_R^\dagger \theta_R\,, \qquad \tilde{y}^1 = z - i\theta_R^\dagger \theta_R\,. \tag{D.11}$$

Under supersymmetric transformation $\delta_\epsilon + \delta_{\bar{\epsilon}}$

$$\begin{aligned} \theta_R &\to \theta_R + \epsilon\,, & \theta_R^\dagger &\to \theta_R^\dagger + \bar{\epsilon}\,, \\ y^\mu &\to y^\mu + 2i\bar{\epsilon}\theta_R\,, & \tilde{y}^\mu &\to \tilde{y}^\mu - 2i\theta_R^\dagger \epsilon\,, \end{aligned} \tag{D.12}$$

where $\mu = 0, 1$.

We can now define the chiral $\mathcal{N} = (0, 2)$ bosonic and fermionic superfields in our model,

$$\begin{aligned} A(y^\mu, \theta_R) &= \phi(y^\mu) + \sqrt{2}\theta_R \psi_L(y^\mu)\,, \\ B(y^\mu, \theta_R) &= \psi_R(y^\mu) + \sqrt{2}\theta_R F(y^\mu)\,. \end{aligned} \tag{D.13}$$

Here ϕ, ψ_L and ψ_R describe physical degrees of freedom, while F will enter without derivatives and, thus, can be eliminated by virtue of equations of motion.

D.2 Euclidean spacetime

Here we summarize the notation involved in instanton analysis, where one has to work in the context of Euclidean field theory.

The Wick rotation is defined by

$$x^1 = x\,, \quad x^2 = it\,. \tag{D.14}$$

Gamma matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \gamma^3 = i\gamma^1\gamma^2\,. \tag{D.15}$$

We define

$$\partial_z = \partial_1 - i\partial_2\,, \quad \partial_{\bar{z}} = \partial_1 + i\partial_2\,, \tag{D.16}$$

and the supercharges are given by

$$Q = \frac{\partial}{\partial\theta} + i\theta^\dagger\partial_{\bar{z}}\,, \quad \bar{Q} = -\frac{\partial}{\partial\theta^\dagger} - i\theta\partial_{\bar{z}}\,, \tag{D.17}$$

together with the commutation relation

$$\{Q, \bar{Q}\} = -2i\partial_{\bar{z}}\,. \tag{D.18}$$

Correspondingly, superderivatives are given by

$$D = \frac{\partial}{\partial\theta} - i\theta^\dagger\partial_{\bar{z}}\,, \quad \bar{D} = -\frac{\partial}{\partial\theta^\dagger} + i\theta\partial_{\bar{z}}\,. \tag{D.19}$$

It is easy to verify that

$$D(z) = \bar{D}(z) = D(\bar{z}_{ch}) = 0\,, \tag{D.20}$$

where $\bar{z}_{ch} = \bar{z} - 2i\theta^\dagger\theta$.

We define the Dirac fermion to be

$$\psi = \begin{pmatrix} \psi_z \\ \psi_{\bar{z}} \end{pmatrix}\,. \tag{D.21}$$

Now the bosonic chiral superfield is defined as

$$A = \phi(\bar{z} + 2i\theta^\dagger\theta, z) + \sqrt{2}\theta\psi_{\bar{z}}(\bar{z} + 2i\theta^\dagger\theta, z)\,. \tag{D.22}$$

The fermionic chiral superfield can be written down in a similar manner.

Appendix E

Fermionic one loop

In this appendix we analyze the infrared behavior of the finite fermion loop. In the limit of small m the one-loop fermion contribution is determined by the diagram depicted in Fig. E.1. Let us start from $m = 0$. Then the fermion Green's function is

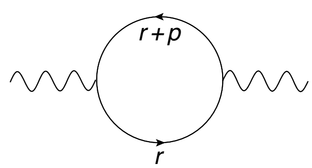

Figure E.1: One-loop contribution by fermionic loop. The solid lines denote fermionic propagators, and the wave lines are bosonic background fields.

$$\left\langle T\,\psi(x)\,\bar{\psi}(0)\right\rangle = -\frac{i}{2\pi}\frac{\not{x}}{x^2}\,. \tag{E.1}$$

Correspondingly,

$$\begin{aligned}\left\langle T\,\bar{\psi}(x)\gamma^\mu\psi(x)\;\bar{\psi}(0)\gamma^\nu\psi(0)\right\rangle &= \frac{1}{2\pi^2}\left(x^2 g^{\mu\nu} - 2x^\mu x^\nu\right)\frac{1}{x^4} \\ &\to \frac{i}{\pi}\left(\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu}\right).\end{aligned} \tag{E.2}$$

This expression is singular at $p \to 0$. However, if we keep a small IR regularizing mass, then it must be multiplied by a function $f(p^2/m^2)$ which is proportional to p^2/m^2 at small p^2. Thus the fermion loop in Fig. E.1 vanishes at $p \to 0$.

Appendix F

One-loop renormalization of four-fermion interaction terms in the heterotic $CP(1)$ model

To show that the renormalization respects the original symmetry of the heterotic model, we calculate the renormalization of the four-fermion interaction terms as a direct check. To simplify the calculation, we only collect the covariant contribution, and take the target space symmetry of the theory for granted. First we calculate one-loop correction to $(\bar{\psi}\psi)^2$. The relevant diagrams are shown in Fig. F.1. Finally we have

$$\Delta\mathcal{L}_{(\bar{\psi}\psi)^2} = i\frac{2}{(1+\phi^\dagger\phi)^4}(-1+\frac{\gamma^2}{g^2}+\frac{\gamma^4}{g^4})I(\bar{\psi}\psi)^2\,. \tag{F.1}$$

In order to see whether there are new structures, we recall that previously we have

$$\mathcal{L}_{0,(\bar{\psi}\psi)^2} = \frac{2}{g_0^2(1+\phi^\dagger\phi)^4}(\bar{\psi}_0\psi_0)^2 - \frac{2\gamma_0^2}{g_0^4(1+\phi^\dagger\phi)^4}(\bar{\psi}_0\psi_0)^2\,. \tag{F.2}$$

So if the structure remains the same after one-loop renormalization, we should have

$$\begin{aligned}\mathcal{L}_{\text{eff},(\bar{\psi}\psi)^2} &= \frac{2}{g^2(1+\phi^\dagger\phi)^4}(\bar{\psi}\psi)^2 - \frac{2\gamma^2}{g^4(1+\phi^\dagger\phi)^4}(\bar{\psi}\psi)^2 \\ &= Z_{g^2}^{-1}Z_\psi\frac{2}{g_0^2(1+\phi^\dagger\phi)^4}(\bar{\psi}_0\psi_0)^2 - Z^2_{\frac{\gamma}{g^2}}Z_\psi\frac{2\gamma_0^2}{g_0^4(1+\phi^\dagger\phi)^4}(\bar{\psi}_0\psi_0)^2\,, \end{aligned}\tag{F.3}$$

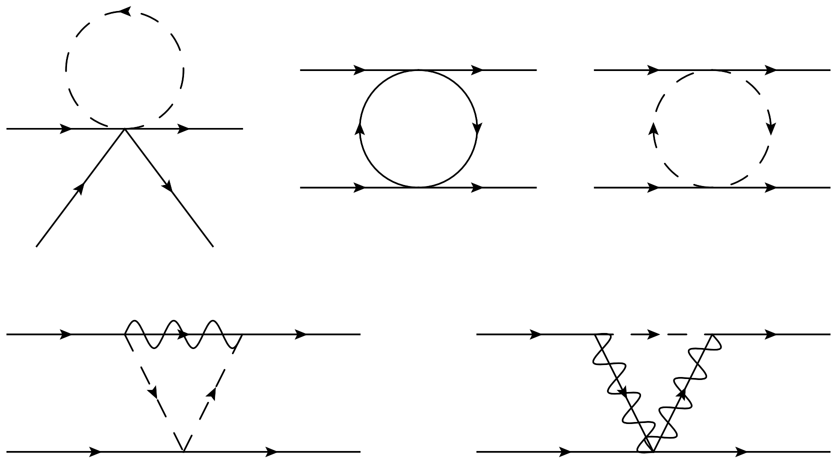

Figure F.1: One-loop corrections to $(\bar{\psi}\psi)^2$ term.

where

$$Z_{g^2} = \frac{g^2}{g_0^2} = 1 + iIg^2\,, \qquad Z_{\frac{\gamma}{g^2}} = \frac{\gamma g_0^2}{\gamma_0 g^2} = 1 - iI\gamma^2\,,$$

$$Z_\psi = 1 + iI\gamma^2\,. \tag{F.4}$$

The last one is because $\bar{\psi}\psi$ is linear in ψ_R(or $\psi_R^\dagger$). So if we plug in Eq. (F.3) the known result Eq. (F.4), we should expect that

$$\Delta\mathcal{L}_{(\bar{\psi}\psi)^2} = i\frac{2}{(1+\phi^\dagger\phi)^4}(-1 + \frac{\gamma^2}{g^2} + \frac{\gamma^4}{g^4})I(\bar{\psi}\psi)^2\,. \tag{F.5}$$

And it is precisely the case.

We can also calculate one-loop correction to $\zeta_R^\dagger\zeta_R\psi_L^\dagger\psi_L$, and the relevant diagrams are given in Fig. F.2. Finally we have

$$\Delta\mathcal{L}_{\zeta_R^\dagger\zeta_R\psi_L^\dagger\psi_L} = i\frac{2}{(1+\phi^\dagger\phi)^2}\frac{\gamma^2}{g^2}(g^2-\gamma^2)I\zeta_R^\dagger\zeta_R\psi_L^\dagger\psi_L\,. \tag{F.6}$$

Following a similar analysis one can show that this is also consistent with our expectation.

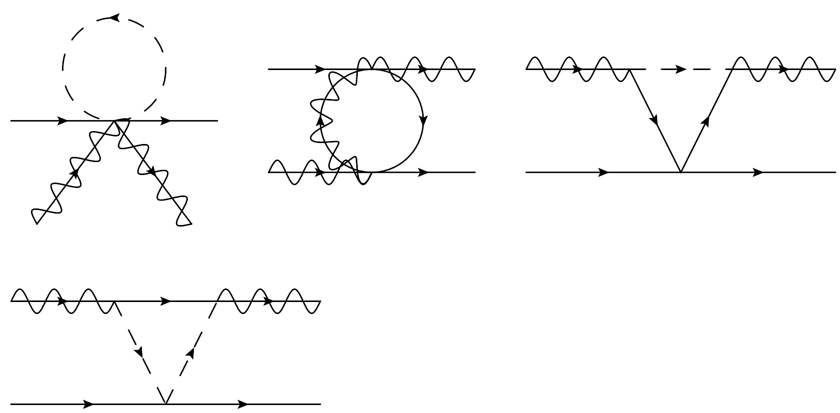

Figure F.2: One-loop corrections to $\zeta_R^\dagger \zeta_R \psi_L^\dagger \psi_L$ term.

Appendix G

Chiral fermion flavor symmetry in heterotic $CP(N-1)$ models

In this appendix we introduce vielbeins E_i^a, used in Sec. 5.7 to describe the flavor symmetry of the left-handed fermions ψ_L. The ψ fermions live in the tangent space of the target manifold. Vielbeins make it clear that locally we could find a coordinate frame that is as good as the one for flat spaces. However, such choice of coordinates varies from point to point, so one should expect that the fermions have nontrivial connections.

Let us look at the fermionic part of the Lagrangian, which is given by Eq. (5.38). It is convenient to write the Kähler metric as $G_{\bar{j}i}$. Then all vectors that carry the barred indices must be understood as lines, and all that carry the unbarred indices as columns. We consider the following representation of the metric:

$$\left(E^{\dagger}\right)_{\bar{j}}^{a} E_{a,i} = G_{\bar{j}i} \,. \tag{G.1}$$

Rising or lowering of the a index is done by the identity matrix, so we can be loose about its position. The above equation does not uniquely determine the matrix E. Rather, we start with $(N-1)\times(N-1)$ complex matrix and impose $(N-1)^2$ real conditions. The remaining freedom (the ambiguity can be represented by a constant $\mathrm{U}(N-1)$ matrix) is non-physical and one can fix the ambiguity by imposing further compatible conditions. After that, we can define our "flat" fermions as

$$\psi^a = E_i^a \psi^i \,. \tag{G.2}$$

On the other hand, in any case the kinetic term for fermions needs some adjustments. Previously we had $\not{D}\psi^i \equiv \not{\partial}\psi^i + \Gamma^i_{lk}\not{\partial}\phi^l\psi^k$, and now, in order to require the covariant derivatives to be the same both before and after we apply Eq. (G.2), we must have

$$\not{D}\psi^a = E^a_i \not{D}\psi^i \,. \tag{G.3}$$

These conditions determine that

$$D_\mu\psi^a = \partial_\mu\psi^a + \partial_\mu(E^a_i)\psi^i - E^a_i\Gamma^i_{lk}\partial_\mu\phi^l\psi^k \,. \tag{G.4}$$

Generally speaking, it is impossible to choose the set of the vielbeins E^a_i to reduce $D_\mu\psi^a$ to $\partial_\mu\psi^a$. The reason is simple: the change of local coordinate frames from one point on the target space to another is not trivial. In a sence, the $\mathrm{U}(N-1)$ symmetries in choosing E^a_i's are similar to a gauge symmetry.

The symmetry we demonstrate here, is seen by replacing ψ^i's by ψ^a's. Now the fermion part of the Lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_{\mathrm{CP}(N-1)} \quad &= i\psi^{\dagger a}_R D_L \psi^a_R + i\psi^{\dagger a}_L D_R \psi^a_L \\
&+ i\zeta^\dagger_R \partial_L \zeta_R + \left[\gamma\zeta_R \left(i\partial_L\phi^{\dagger\bar{j}} E^\dagger_{\bar{j}a}\right)\psi^a_R + \text{H.c.}\right] + \gamma^2\left(\zeta^\dagger_R\zeta_R\right)\left(\psi^{\dagger a}_L\psi^a_L\right) \\
&- \frac{g^2}{2}\left(\psi^{\dagger a}_R\psi^a_R\right)\left(\psi^{\dagger b}_L\psi^b_L\right) \\
&+ \frac{g^2}{2}\left(1 - 2\frac{\gamma^2}{g^2}\right)\left(\psi^{\dagger a}_R\psi^a_L\right)\left(\psi^{\dagger b}_L\psi^b_R\right) \,,
\end{aligned} \tag{G.5}$$

where D_R and D_L are defined from D_μ in the way similar to the replacement of ∂_μ by ∂_R and ∂_L.

As was emphasized before, since D_L and D_R do not reduce to ∂_L and ∂_R, we cannot yet apply the flavor rotation to ψ^a_L's: the corresponding kinetic term in the Lagrangian will not be invariant. But it will be invariant, if we further assume that ϕ's are only dependent on $t + z$. By doing so, $D_R \to \partial_R$, since the connection part is always linear in $\partial_R\phi^i$'s. Now we can see that the last line in Eq. (G.5) is the only term that is noninvariant under the $\mathrm{SU}(N-1)$ flavor rotation of ψ^a_L. Needless to say, the symmetry is restored when $\rho = 1/2$.