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Memory Tensor for Non-Markovian Dynamics with Random Hamiltonian

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Abstract: In the theory of open quantum systems, the Markovian approximation is very widespread. Usually, it assumes the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation for density matrix dynamics and quantum regression formulae for multi-time correlation functions. Nevertheless, now, quantum non-Markovianity is being actively studied, especially the non-Markovianity of multi-time correlations. In this work, we consider dynamics with a random Hamiltonian, which can lead to GKSL dynamics of the density matrix for some special cases, but correlation functions generally do not satisfy the quantum regression formulae. Despite the fact that random Hamiltonians have been actively studied, dynamics with such Hamiltonians has been little discussed from the viewpoint of multi-time correlations. For specific models with a random Hamiltonian, we provide the formulae for multi-time correlations which occur instead of the usual regression formulae. Moreover, we introduce and calculate the memory tensor, which characterizes multi-time correlations against the Markovian ones. We think that, despite being applied to specific models, the methods developed in this work can be used in a much broader setup.

Keywords: random Hamiltonian; quantum stochastic process; multi-time correlation functions; regression formula

MSC: 81S22; 81Q15; 60B20

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1. Introduction

Markovian quantum dynamics is very important in the theory of open quantum systems ([1], Section 3.2). Even if the dynamics is not exactly Markovian, the Markovian approximation usually plays the role of zero-order approximation of some perturbation theory [2–4] or the dynamics becomes Markovian after the inclusion of some degrees of freedom of the heat bath into the new system [5–12]. Several recent results [13–17] also suggest that the dynamics of open quantum systems can become Markovian at long times with very high accuracy under very generic conditions.

There are many distinct approaches to the definition of quantum Markovianity [18]. Most of them assume at least the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation for density matrix dynamics. Sometimes, only this assumption is included in the definition of quantum Markovianity and its measures [19–21]. But even in the classical case, the Markovian master equation for one-time dynamics is not enough for the classical stochastic process to be Markov [22,23]. One of the most natural approaches to define the quantum Markov stochastic process in such a way that it generalizes classical Markovian dynamics is to assume that the multi-time correlation functions are defined by regression formulae. Nevertheless, now, quantum non-Markovianity is being actively studied [24–27], especially the non-Markovianity of multi-time correlations [28–32].

In this work, we consider dynamics with a random Hamiltonian. It can lead to both GKSL and non-GKSL dynamics of the density matrix [33–36]. But even for GKSL dynamics, the correlation functions do not satisfy the quantum regression formulae in this case. There

are some different examples in the literature with the same property, but dynamics with a random Hamiltonian seems to be very natural from the physical point of view. On the one hand, random Hamiltonians ([37], Chapters 13–14) and unitary dynamics with random Hamiltonians [38,39] have been actively studied. On the other hand, we think that it is little understood from the viewpoint of multi-time correlations for such dynamics.

Despite the fact that the usual regression formulae are not valid in such a case, we show that it is still possible to calculate multi-time correlations explicitly. Namely, we calculate the so-called process tensor, which defines such correlations. Moreover, we represent the process tensor as a product of the process tensor for Markovian dynamics and another tensor, which we call the memory tensor and which characterizes the Markovianity violation for multi-time correlations.

We use the characterization of multi-time correlations of a quantum system in terms of the process tensor, which has become widespread in modern works on open quantum systems [40–42]. So, in Section 2, we introduce all necessary definitions. There are also several different approaches in the physical and mathematical literature to characterizing the multi-time correlations. So, in Section 2, we briefly discuss their relation to the approach that we use. In Sections 2.1 and 2.2, we specify the process tensors for closed and Markovian dynamics, respectively.

In Section 3, we consider the model of dynamics with a random Hamiltonian with multiplicative dependence on a classical random variable. We calculate the one-time dynamics in this case. We introduce and calculate the memory tensor for such a model. And we discuss its physical meaning. To illustrate our results, we provide explicit formulae for the Gaussian and Poisson cases in Sections 3.1 and 3.2, respectively.

In Section 4, we discuss a more complicated model. We consider the model from Section 3 as a free dynamics. And we perturbate it with a deterministic interaction Hamiltonian. We do not assume that the interaction Hamiltonian commutes with the free Hamiltonian or satisfies some simple commutation relation with it. So, for such a general case, it seems hard to obtain some non-perturbative formulae. But we develop systematic asymptotic expansions for one-time dynamics and the memory tensor.

In Conclusions, we summarize our results and discuss possible directions of further study.

2. Process Tensor and Regression Formulae

We consider only finite-dimensional matrices and maps throughout the work. We refer to linear maps between finite-dimensional matrices as super-operators. We also assume that the readers know the basic definitions from open quantum systems theory, quantum information theory and related areas of science such as density matrices and positive and completely positive maps (see, e.g., [43]). But let us recall the definition of a quantum instrument, because we will need some generalizations of it in our work.

Definition 1. *Let X be a finite set (of outcomes). Then, the function $\mathcal{A}(x)$ from X to completely positive, trace non-increasing maps is called an* ***instrument*** *if*

$$\sum_{x\in X}\mathcal{A}(x)$$

is a trace-preserving map.

From the point of view of quantum information theory, a completely positive, trace-preserving map is usually called a quantum channel ([43], Section 6.3).

Definition 2. *Let $\vec{t}\in\mathbb{R}^{k+1}$ with elements t_j, $j=0,\dots,k$, such that $t_0\leqslant\ldots\leqslant t_k$. We call $\vec{t}$ a* ***vector of measurement times****. Let $\mathcal{A}^j(x_j), x_j\in X_j$ be instruments, $j=0,\dots,k$. Let us define a* ***k-time factorized instrument*** *as*

$$\mathfrak{A}(\vec{x}) := \mathcal{A}^k(x_k)\otimes\cdots\otimes\mathcal{A}^0(x_0), \tag{1}$$

where $\vec{x} \in X_0 \times \cdots \times X_k$.

Despite the fact that the definitions of the vector of measurement times and the k-time factorized instrument are not couples, we combine them in one definition. We do so due to the fact that from the physical point of view the k-time factorized instrument $\mathfrak{A}(\vec{x})$ is interpreted as an independent application of instruments $\mathcal{A}^j(x_j)$ to the quantum system at times t_j, forming the vector of measurement times.

Similarly to [41], let us introduce the following definition.

Definition 3. *A linear map* $\mathcal{T}$ *from the set of all* $(k-1)$*-time factorized instruments with fixed* $k \in \mathbb{N}$ *to the matrices of the system is called a* k*-time* ***process tensor*** *if*

$$\sum_{x_0,\ldots,x_{k-1}} \mathcal{T}[\mathfrak{A}(\vec{x})]$$

is a density matrix for any $(k-1)$*-time factorized instrument* $\mathfrak{A}(\vec{x})$.

From the physical point of view, this density matrix is interpreted as a final state of the system at the time t_k, when we have applied instruments $\mathcal{A}^j(x_j)$ non-selectively to the quantum system at times t_j, $j = 1, \ldots, k-1$. If we do not need to specify k explicitly, we will write just "multi-time factorized instrument" and "process tensor".

Sometimes ([40], Section V.D), the term "process tensor" is used for a linear function from the set of all k-time (in contrast to $(k-1)$-time in the definition below) factorized instruments to the segment $[0, 1]$. In the present notation, such a function has the form

$$\operatorname{Tr} \mathcal{A}^k(x_k)\mathcal{T}[\mathfrak{A}(\vec{x})],$$

where $\mathfrak{A}(\vec{x})$ is a $(k-1)$-time factorized instrument and $\mathcal{A}^k(x_k) \otimes \mathfrak{A}(\vec{x})$ is a k-time factorized instrument. From the physical point of view, this function gives a probability that we have obtained outcomes x_j as a result of applying instrument $\mathcal{A}^j_{x_j}$ to the quantum system at time t_j, $j = 1, \ldots, k$.

Usually, the process tensor depends on the vector of measurement times as a parameter, so we will write $\mathcal{T}(\vec{t})$ to emphasize this dependence.

Until recently, it has been common in the physics literature ([1], Section 3.2.4) to define time-ordered correlation functions of von Neumann observables instead of process tensors. If, in Formula (1), instead of instruments one formally (which is possible due to the linearity of $\mathcal{T}(\vec{t})$) sets

$$\mathcal{A}^k(x_k) = A^{(k)}_L \cdot A^{(k)}_R, \tag{2}$$

where $A^{(k)}_L$ and $A^{(k)}_R$ are Hermitian operators (von Neumann observables), then the time-ordered correlation functions can be defined as

$$\langle A^{(0)}_R(t_0) \cdot \ldots \cdot A^{(k)}_R(t_k) A^{(k)}_L(t_k) \cdot \ldots \cdot A^{(0)}_L(t_0) \rangle := \operatorname{Tr} \mathcal{A}^k(x_k)\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})], \tag{3}$$

where the left-hand side is written in usual physical notation ([1], Section 3.2.4). By $A_L \cdot A_R$ for arbitrary (not necessarily Hermitian) matrices $A_{L,R}$, hereinafter we will denote a super-operator, which acts on a matrix M as $(A_L \cdot A_R)(M) := A_L M A_R$.

Let us note that due to the fact that (2) are not instruments, the operational meaning of such correlation functions is not so direct and it differs from its classical analogs. Nevertheless, it is possible to represent super-operator $A^{(k)}_L \cdot A^{(k)}_R$ as a linear combination of instruments ([18], Section 3.5.2). Thus, the tensor product of them, which defines (2), can also be represented as a linear combination of multi-time factorized instruments. But it is not a multi-time factorized instrument itself. So, from the physical point of view, the measurement procedure behind (3) assumes time-correlated measurement rather than separate instruments at the specified times. And this fact is a bit hidden in the usual physical

notation $\langle A_R^{(0)}(t_0) \cdot \ldots \cdot A_R^{(k)}(t_k) A_L^{(k)}(t_k) \cdot \ldots \cdot A_L^{(0)}(t_0) \rangle$ of multi-time correlation functions. Actually, multi-time correlation functions are usually measured by some interference setups, which are indeed not just sequences of separated measurements. Such setups are usual for spectroscopy [44,45] or for measurements based on homodyne detection [46] such as quantum tomography [47,48], which is still actively being developed [49,50].

If the process tensors are given for all possible sets of times, this fact can be considered as a possible definition of the quantum stochastic process in the sense of [51]. But let us remark that, more widely used, the mathematical literature definition [52] of a quantum stochastic process includes more information than the process tensor. The "gap" between these definitions is discussed, e.g., in [53]. This "gap" occurs due to the fact that Formula (3) does not allow one to calculate out-of-time-ordered correlation functions, but these functions have attracted much attention of researchers and, in particular, the regression formulae for them have been studied [54].

We will not dwell further on the definition of quantum stochastic processes and will consider only the process tensors.

Let us remark that we consider finite sets of outcomes and finite-dimensional spaces in this work only for simplicity. The notion of instrument can be generalized to infinite sets of outcomes (but the set of outcomes should have a structure of a measurable set) and for infinite-dimensional Hilbert spaces ([53], Section 4.1.1). This allows one to generalize Definitions 2 and 3 as well. If one considers only instruments for discrete observables, such a generalization is quite straightforward. But for continuous observables, there are known issues with the mathematical theory of their repeated measurements ([53], Section 4.1.5).

The main idea of such issues can be caught from the following non-rigorous consideration. Let one measure a coordinate of the system, obtaining some fixed result, and want to measure momentum just after that. Then, the "distribution" of possible momenta, after the first measurement is done, is homogeneous on the whole real axis. But such "distributions" do not fit the usual (Kolmogorov) probability theory, as they are not σ-additive. There are several approaches [55,56] to deal with such issues, but we will not dwell on these directions further, restricting overselves to the finite-dimensional case with a finite set of outcomes.

2.1. Regression Formulae for Closed Dynamics

For unitary dynamics of closed quantum systems, it is usually assumed that

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \mathcal{U}_{t_k,t_{k-1}} \mathcal{A}^{k-1}(x_{k-1}) \ldots \mathcal{A}^1(x_1) \mathcal{U}_{t_1,t_0} \mathcal{A}^0(x_0) \rho_{t_0}, \tag{4}$$

where $\mathfrak{A}(\vec{x}) = \mathcal{A}^{k-1}(x_{k-1}) \otimes \ldots \otimes \mathcal{A}^0(x_0)$ is a $(k-1)$-time factorized instrument and

$$\mathcal{U}_{t,s} := U_{t,s} \cdot U_{t,s}^\dagger, \tag{5}$$

where U_{t,t_0} is a solution of the Cauchy problem

$$\frac{d}{dt} U_{t,s} = -iH(t) U_{t,s}, \qquad U_{t,s} = I.$$

Usually, outside of open quantum systems theory, Formula (4) is not written explicitly, but it is equivalent to the assumption that at the left-hand side of Equation (3) $A_{L,R}^{(k)}(t)$ are just dynamics of von Neumann observables in the Heisenberg representation and the symbol $\langle \cdot \rangle$ means just the average with respect to the initial density matrix $\langle \cdot \rangle := \mathrm{Tr}(\cdot \rho_{t_0})$. That is why no significant discussion usually occurs when defining multi-time correlation functions for closed systems.

Let us remark that Equation (4) can be used not only directly but also for multi-time generalization of the concept of decoherence-free subspaces [57–59]. We can do so because, informally speaking, decoherence-free subspaces are such subspaces on which dynamics is

effectively unitary. But to formalize it in our setup, we need to formulate this concept in terms of instruments instead of density matrices.

Definition 4. *A linear subspace is called a* ***support of instrument*** $\mathcal{A}(x)$ *if it is a union over all density matrices* ρ *and over all* $x \in X$ *of supports of matrices*

$$\mathcal{A}(x)\rho.$$

From the physical point of view, it is a union of supports of matrices, which can be "prepared" by the instrument $\mathcal{A}(x)$ from some density matrix ρ.

Definition 5. *A linear subspace is called a* ***support of*** k***-time factorized instrument*** (1) *if it is a union of supports of the instruments* $\mathcal{A}^j(x_j)$.

So, from the physical point of view, we "prepare" a state with such a support at least at one of the measurement times (but not necessarily all of them).

Definition 6. *A linear subspace is called a* ***decoherence-free subspace under*** $k-1$***-time factorized instruments*** *if for any* $k-1$*-time factorized instrument* $\mathfrak{A}(\vec{x})$ *with support* $\mathcal{H}$ *the process tensor can be computed by Formula* (4) *for some* $\mathcal{U}_{t,s}$ *of the form* (5).

The usual definition ([59], Defintion 1) of a decoherence-free space occurs for 0-time factorized instruments with a given support, which from the physical point of view are just all possible "preparations" of the initial states with a given support from an arbitrary state.

2.2. Regression Formulae for Markovian Dynamics

For general Markovian dynamics, the process tensor takes the form ([40], Section D)

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \Phi_{t_k,t_{k-1}}\mathcal{A}^{k-1}(x_{k-1})\dots\mathcal{A}^1(x_1)\Phi_{t_1,t_0}\mathcal{A}^0(x_0)\rho_{t_0}, \tag{6}$$

where

$$\frac{d}{dt}\Phi_{t,s} = \mathcal{L}(t)\Phi_{t,s}, \qquad \Phi_{t,s} = I, \qquad t \geqslant s \geqslant t_0$$

and $\mathcal{L}(t)$ has the (possibly time-dependent) GKSL form [60,61]

$$\mathcal{L}(t)\rho = -i[H(t),\rho] + \sum_i \left(L_i(t)\rho L_i^\dagger(t) - \frac{1}{2}L_i^\dagger(t)L_i(t)\rho - \frac{1}{2}L_i^\dagger(t)L_i(t)\rho \right), \tag{7}$$

where $H(t) = H^\dagger(t)$. Formula (6) generalizes Formula (4) for closed dynamics. In physics, the formulae for correlation functions (3) defined by process tensors (6) are usually called regression formulae (or generalized regression formulae, using the term "regression formulae" only for the case $k = 1$ or $k = 2$ with a trivial instrument at time t_0), so we use this term for (4) and (6) as well.

For general quantum dynamics, if the one-time dynamics of the density matrix is given by

$$\rho_t = \Psi_{t,t_0}\rho_{t_0}, \qquad \Psi_{t_0,t_0} = I, \tag{8}$$

such that Ψ_{s,t_0} is invertible, then one can define

$$\Phi_{t,s} = \Psi_{t,t_0}(\Psi_{s,t_0})^{-1}. \tag{9}$$

If the super-operator $\Phi_{t,s}$ is completely positive, then dynamics is called CP-divisible ([26], Definition 5.1). $\Phi_{t,s}$ satisfies the propagator property $\Phi_{t,s} = \Phi_{t,\tau}\Phi_{\tau,s}$ for $t \geqslant \tau \geqslant s \geqslant t_0$, which is not necessarily true for Ψ_{t,t_0}. In such a general case, we will use Formula (6) as the definition of the Markovian process tensor $\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})]$.

Formula (6), informally speaking, means that multi-time dynamics is fully characterized by one-time dynamics. In particular, if each instrument $\mathcal{A}^{j-1}(x_{j-1})$ in the tensor (6) has a support included in decoherence-free subspace in the usual sense, then $\Phi_{t_j,t_{j-1}}$ is reduced to some unitary maps $\mathcal{U}_{t_j,t_{j-1}}$ on such a support. Hence, the whole tensor (6) reduces to the tensor of the form (4) for a $(k-1)$-time factorized instrument $\mathfrak{A}(\vec{x})$ with such a support. Thus, in the Markovian case (6), the usual decoherence-free subspaces are also decoherence-free subspaces under $(k-1)$-time factorized instruments for any $k \in \mathbb{N}$. And, in particular, these spaces are fully defined by the properties of generator (7). But this is not necessarily the case for non-Markovian process tensors. This is important due to the fact that in many experimental setups [44,45] the quantum coherence is measured via multi-time correlation functions. So, we should remark that it can be related to quantum coherence in terms of density matrix dynamics only in the Makovian case.

In our calculation, we will rewrite Formula (4) in another form. To do it, we will use Definition 1 from [62], so let us recall it in the following form.

Definition 7. *Let us define the contraction map* $\mathfrak{C}$ *as a linear map such that for any super-operators* $\mathcal{A}^1, \ldots, \mathcal{A}^k$ *one has*

$$\mathfrak{C}(\mathcal{A}^k \otimes \ldots \otimes \mathcal{A}^1) = \mathcal{A}^k \ldots \mathcal{A}^1.$$

Lemma 1. *Let* $\mathcal{T}(\vec{t})$ *be defined by Equation* (6)*. Then,*

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \mathfrak{C}(\mathfrak{F}(\vec{t})\mathfrak{A}(\vec{x}))\rho_{t_0}, \tag{10}$$

where

$$\mathfrak{F}(\vec{t}) := \Phi_{t_k,t_{k-1}} \otimes \ldots \otimes \Phi_{t_1,t_0}.$$

Proof. Using Definition 7 of the contraction map, we represent Equation (6) in the form

$$\begin{aligned}\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] &= \Phi_{t_k,t_{k-1}}\mathcal{A}^{k-1}(x_{k-1}) \ldots \mathcal{A}^1(x_1)\Phi_{t_1,t_0}\mathcal{A}^0(x_0)\rho_{t_0} \\ &= \mathfrak{C}\Big(\Phi_{t_k,t_{k-1}}\mathcal{A}^{k-1}(x_{k-1}) \otimes \ldots \otimes \Phi_{t_1,t_0}\mathcal{A}^0(x_0)\Big)\rho_{t_0} \\ &= \mathfrak{C}\Big((\Phi_{t_k,t_{k-1}} \otimes \ldots \otimes \Phi_{t_1,t_0})(\mathcal{A}^{k-1}(x_{k-1}) \otimes \ldots \otimes \mathcal{A}^0(x_0))\Big)\rho_{t_0}\end{aligned}$$

Thus, using the definitions of $\mathfrak{A}(\vec{x}))$ and $\mathfrak{F}(\vec{t})$, we obtain (10). □

We will see that Formula (10) is useful for explicit calculations.

3. Dynamics with Random Hamiltonian

We consider finite-dimensional random unitary dynamics with a time-dependent Hamiltonian of the following form:

$$H(t) = \xi\dot{\lambda}_t B,$$

where ξ is a real random variable and B is a Hermitian $n \times n$ complex matrix.

Let us define U_{t,t_0} as a solution to the Cauchy problem

$$\frac{d}{dt}U_{t,t_0} = -i\xi\dot{\lambda}_t B U_{t,t_0}, \qquad U_{t_0,t_0} = I,$$

or more explicitly as

$$U_{t,t_0} = e^{-i\xi B(\lambda_t - \lambda_{t_0})}. \tag{11}$$

Proposition 1. *Let us define the averaged propagator* Ψ_{t,t_0} *as*

$$\Psi_{t,t_0} := \mathbb{E}(U_{t,t_0} \cdot U_{t,t_0}^\dagger), \tag{12}$$

where $\mathbb{E}$ *is the mathematical expectation and* U_{t,t_0} *is defined by* (11)*. Then,*

$$\Psi_{t,t_0} = h(-[B,\ \cdot\ ](\lambda_t - \lambda_{t_0})), \tag{13}$$

where h *is the characteristic function of the random variable* ξ*. The function of the super-operator* $[B,\ \cdot\ ]$ *is defined in terms of the spectral decomposition.*

Proof. As B is Hermitian, then it can be represented in terms of the spectral decomposition ([43], p. 18)

$$B = \sum_{b \in \operatorname{spec} B} b P_b,$$

where spec B are the eigenvalues of matrix B and P_b are orthogonal projectors ($P_{b_1} P_{b_1} = \delta_{bb'} P_b$, $P_b^\dagger = P_b$) to eigenspaces correspondent to these eigenvalues. For any $X \in \mathbb{C}^{n\times n}$, using $\sum_{b\in \operatorname{spec} B} P_b = I$, we have

$$\begin{aligned}
[B, X] &= \left[\sum_{b\in \operatorname{spec} B} b P_b, X\right] = \sum_{b_1 \in \operatorname{spec} B} b_1 P_{b_1} X - \sum_{b_2 \in \operatorname{spec} B} X b_2 P_{b_2} \\
&= \sum_{b_1, b_2 \in \operatorname{spec} B} b_1 P_{b_1} X P_{b_2} - \sum_{b_1, b_2 \in \operatorname{spec} B} P_{b_1} X b_2 P_{b_2} = \sum_{b_1, b_2 \in \operatorname{spec} B} (b_1 - b_2) P_{b_1} X P_{b_2} \\
&= \sum_{b\in \operatorname{spec} B, \Delta b} \Delta b P_{b+\Delta b} X P_b,
\end{aligned}$$

where the sum with respect to Δb runs over all the differences of the eigenvalues of B. Such differences are usually called Bohr frequencies (see, e.g., [63], p. 122) and they are nothing else but eigenvalues of the super-operator $[B,\ \cdot\ ]$. Thus, we have

$$[B,\ \cdot\ ] = \sum_{\Delta b} \Delta b \mathcal{P}_{\Delta b},$$

where

$$\mathcal{P}_{\Delta b} = \sum_{b\in \operatorname{spec} B} P_{b+\Delta b}\ \cdot\ P_b.$$

As for any $\Delta b_1, \Delta b_2 \in \operatorname{spec}[B,\ \cdot\ ]$,

$$\begin{aligned}
\mathcal{P}_{\Delta b_1} \mathcal{P}_{\Delta b_2} &= \sum_{b_1 \in \operatorname{spec} B} \sum_{b_2 \in \operatorname{spec} B} P_{b_1+\Delta b_1} P_{b_2+\Delta b_2}\ \cdot\ P_{b_2} P_{b_1} \\
&= \sum_{b_1 \in \operatorname{spec} B} \sum_{b_2 \in \operatorname{spec} B} \delta_{b_1+\Delta b_1, b_2+\Delta b_2} \delta_{b_1, b_2} P_{b_2+\Delta b_2}\ \cdot\ P_{b_1} \\
&= \sum_{b_1 \in \operatorname{spec} B} \delta_{b_1+\Delta b_1, b_1+\Delta b_2} P_{b_1+\Delta b_2}\ \cdot\ P_{b_1} \\
&= \sum_{b_1 \in \operatorname{spec} B} \delta_{\Delta b_1, \Delta b_2} P_{b_1+\Delta b_1}\ \cdot\ P_{b_1} = \delta_{\Delta b_1, \Delta b_2} \mathcal{P}_{\Delta b_1}
\end{aligned}$$

and as $\mathcal{P}_{\Delta b}$ is self-adjoint with respect to the Hilbert–Schmidt scalar product $\langle\langle A|B\rangle\rangle := \operatorname{Tr} A^\dagger B$ of matrices A and B, then $\mathcal{P}_{\Delta b}$ are the eigenprojectors of the super-operator $[B,\ \cdot\ ]$. Then, we have

$$\begin{aligned}
\Psi_{t,t_0} := \mathbb{E} e^{-i\xi [B,\ \cdot\ ](\lambda_t - \lambda_{t_0})} &= \sum_{\Delta b \in \operatorname{spec}[B,\ \cdot\ ]} \mathbb{E} e^{-i\xi \Delta b (\lambda_t - \lambda_{t_0})} \mathcal{P}_{\Delta b} \\
&= \sum_{\Delta b \in \operatorname{spec}[B,\ \cdot\ ]} h(-\Delta b(\lambda_t - \lambda_{t_0})) \mathcal{P}_{\Delta b} = h(-[B,\ \cdot\ ](\lambda_t - \lambda_{t_0})),
\end{aligned}$$

where the function of the super-operator $[B,\ \cdot\ ]$ is defined in terms of the spectral decomposition. So, we have obtained Equation (13). □

If we assume that the initial condition ρ_{t_0} is deterministic, then the density matrix averaged by a random variable ξ has the form

$$\rho_t = \mathbb{E}(U_{t,t_0}\rho_{t_0}U_{t,t_0}^{\dagger}) = \Psi_{t,t_0}\rho_{t_0} \tag{14}$$

with Ψ_{t,t_0} defined by (12). Then, using (8) and (9), we immediately obtain the following proposition.

Proposition 2. *Let $h(u) \neq 0$. Then,*

$$\Phi_{t,s} = h(-[B, \cdot\,](\lambda_t - \lambda_{t_0}))(h(-[B, \cdot\,](\lambda_s - \lambda_{t_0})))^{-1}. \tag{15}$$

Let, in addition, h be continuously differentiable. Then,

$$\frac{d}{dt}\Phi_{t,s} = \mathcal{L}(t)\Phi_{t,s}, \tag{16}$$

where

$$\mathcal{L}(t) := -\dot{\lambda}_t[B, \cdot\,]h'(-[B, \cdot\,](\lambda_t - \lambda_{t_0}))(h(-[B, \cdot\,](\lambda_t - \lambda_{t_0}))^{-1}. \tag{17}$$

If one assumes such a physical setup that only observations averaged with respect to a random variable ξ are accessible for given experimental capabilities (or only such averages are interesting for some other reasons), then, similarly to (4) and (14), it is natural to define the process tensor as

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] := \mathbb{E}\mathcal{U}_{t_k,t_{k-1}}\mathcal{A}^{k-1}(x_{k-1})\ldots\mathcal{A}^{1}(x_1)\mathcal{U}_{t_1,t_0}\mathcal{A}^{0}(x_0)\rho_{t_0}, \tag{18}$$

where $\mathcal{U}_{t,s}$ is defined by (5) and (11).

Let us remark that (14) and (18) can be represented in a more usual open quantum systems form by changing $\mathbb{E}$ to the partial trace with respect to a reservoir Hilbert space, ξ to the multiplication operator in a reservoir Hilbert space and ρ_{t_0} to $\rho_{t_0} \otimes \rho_B$, where ρ_B is a diagonal density matrix.

The Markovian process tensor $\mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})]$ is defined by Formula (6), i.e.,

$$\mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})] := \Phi_{t_k,t_{k-1}}\mathcal{A}^{k-1}(x_{k-1})\ldots\mathcal{A}^{1}(x_1)\Phi_{t_1,t_0}\mathcal{A}^{0}(x_0)\rho_{t_0}, \tag{19}$$

where $\Phi_{t,s}$ is defined by Formula (15).

In the next theorem, we will also need to introduce

$$B_j := \left(\bigotimes_{i=1}^{j-1} I\right) \otimes B \otimes \left(\bigotimes_{i=j+1}^{k} I\right), \tag{20}$$

where $j = 1, \ldots, k$, i.e., B_j acts as B in the j-th tensor multiplicand and as I in all other tensor multiplicands.

Theorem 1. *Let $\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})]$ be defined by (18), with $\mathcal{U}_{t,t_0}$ defined by (11), where ξ is a real random variable with the characteristic function $h(u)$ such that $h(u) > 0$. And let $\mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})]$ be defined by (6), where $\Phi_{t,s}$ is defined by Formula (15). Then,*

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \mathcal{T}_M(\vec{t})[\mathfrak{M}(\vec{t})\mathfrak{A}(\vec{x})], \tag{21}$$

*where $\mathfrak{M}(\vec{t})$, which we call a **memory tensor**, is defined by*

$$\mathfrak{M}(\vec{t}) := \frac{h(-[B_k(\lambda_{t_k} - \lambda_{t_{k-1}}) + \ldots + B_1(\lambda_{t_1} - \lambda_{t_0}), \cdot\,])}{\prod_{j=1}^{k}\left(h(-[B_j, \cdot\,](\lambda_{t_j} - \lambda_{t_0}))(h(-[B_j, \cdot\,](\lambda_{t_{j-1}} - \lambda_{t_0})))^{-1}\right)}. \tag{22}$$

Here, the function is defined in terms of the spectral decomposition.

Proof. Using Lemma 1, one can rewrite Formula (18) as

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \mathfrak{C}(\mathbb{E}\mathfrak{F}(\vec{t})\mathfrak{A}(\vec{x}))\rho_{t_0}, \tag{23}$$

where

$$\begin{aligned}\mathfrak{F}(\vec{t}) &= \mathcal{U}_{t_k,t_{k-1}} \otimes \ldots \otimes \mathcal{U}_{t_1,t_0} = e^{-i\xi[B,\,\cdot\,](\lambda_{t_k}-\lambda_{t_{k-1}})} \otimes \ldots \otimes e^{-i\xi[B,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0})} \\ &= \exp\big(-i\xi[B_k(\lambda_{t_k}-\lambda_{t_{k-1}}) + \ldots + B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,]\big).\end{aligned}$$

Then,

$$\mathbb{E}\mathfrak{F}(\vec{t}) = h(-[B_k(\lambda_{t_k}-\lambda_{t_{k-1}}) + \ldots + B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,]), \tag{24}$$

where the function is defined in terms of the spectral decomposition, which can be justified similarly to the proof of Proposition 1.

Similarly, by Lemma 1, Formula (19) takes the form

$$\mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})] = \mathbb{E}\mathfrak{C}(\mathfrak{F}_M(\vec{t})\mathfrak{A}(\vec{x}))\rho_{t_0}, \tag{25}$$

where

$$\begin{aligned}\mathfrak{F}_M(\vec{t}) &= \Phi_{t_k,t_{k-1}} \otimes \ldots \otimes \Phi_{t_1,t_0} \\ &= \bigotimes_{j=1}^{k}\Big(h(-[B,\,\cdot\,](\lambda_{t_j}-\lambda_{t_0}))(h(-[B,\,\cdot\,](\lambda_{t_{j-1}}-\lambda_{t_0})))^{-1}\Big) \\ &= \prod_{j=1}^{k}\Big(h(-[B_j,\,\cdot\,](\lambda_{t_j}-\lambda_{t_0}))(h(-[B_j,\,\cdot\,](\lambda_{t_{j-1}}-\lambda_{t_0})))^{-1}\Big).\end{aligned}$$

Then, using Formula (22), we have

$$\mathbb{E}\mathfrak{F}(\vec{t}) = \mathfrak{M}(\vec{t})\mathfrak{F}_M(\vec{t}).$$

Thus, taking into account Formulae (23) and (25), we obtain (21). □

To clarify Formula (22), let us write it for $k = 1$ and $k = 2$ explicitly. For $k = 1$, Formula (22) takes the form

$$\mathfrak{M}(\vec{t}) = \frac{h(-[B,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))}{h(-[B,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))(h(-[B,\,\cdot\,](\lambda_{t_0}-\lambda_{t_0})))^{-1}} = I.$$

So, for $k = 1$, it is trivial. For $k = 2$, Formula (22) takes the form

$$\mathfrak{M}(\vec{t}) = \frac{h(-[B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_1}) - [B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))}{h(-[B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_0}))(h(-[B_2,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0})))^{-1}h(-[B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))}. \tag{26}$$

Let us remark that to calculate quantum multi-time correlations, we only need to use (23) along with (24), but we represented $\mathcal{T}(\vec{t})$ in terms of the Markovian process tensor $\mathcal{T}_M(\vec{t})$ and the memory tensor $\mathfrak{M}(\vec{t})$. This is because we think it is natural to consider non-Markovian dynamics against the background of Markovian dynamics because the Markovian assumption about correlation functions is usually implicitly embedded in many physical setups. In particular, for Markovian dynamics, the memory tensor is trivial $\mathfrak{M}(\vec{t}) = I^{\otimes k}$.

Formula (21) can also be interpreted as follows. The dynamics of multi-time correlations could be considered Markovian, but the sequence of the measurements is not factorized, which is described by the memory tensor $\mathfrak{M}(\vec{t})$. It can be considered as transformation into a "Markovian picture", in a certain sense analogous to the transformation from a Shroedinger picture to a Heisenberg picture.

Moreover, for $h(u) > 0$, the memory tensor $\mathfrak{M}(\vec{t})$ is invertible, so due to the linearity of the process tensor, one can rewrite (21) as

$$\mathcal{T}(\vec{t})[(\mathfrak{M}(\vec{t}))^{-1}\mathfrak{A}(\vec{x})] = \mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})].$$

From the physical point of view, it could be interpreted as the fact that the non-Markovian dynamics described by the process tensor $\mathcal{T}(\vec{t})$ is seen for the observer to be a Markovian one if the observer performs some time-correlated measurements described by $(\mathfrak{M}(\vec{t}))^{-1}\mathfrak{A}(\vec{x})$ but thinks that they are uncorrelated. And let us remark that, as we have discussed in Section 2, even the measurement procedure for the usual correlation function is actually time-correlated, so time-correlated instruments can occur in physics quite naturally.

Of course, a deeper study of the physical realizability of such series of measurements described by $(\mathfrak{M}(\vec{t}))\mathfrak{A}(\vec{x})$ or by $(\mathfrak{M}(\vec{t}))^{-1}\mathfrak{A}(\vec{x})$ should be performed to justify their physical meaning. But even at the present level, our results suggest that as other things in quantum theory the Markovianity should be considered not as a property of the system only, but rather as some observer-dependent property of the system.

Let us also remark that, similarly to Equations (16) and (17), one could write differential equations for $\mathbb{E}\mathfrak{F}(\vec{t})$

$$\frac{\partial}{\partial t_k}\mathbb{E}\mathfrak{F}(\vec{t}) = [\dot{\lambda}_{t_k}B_k, \cdot\,]\frac{h'(-[B_k(\lambda_{t_k}-\lambda_{t_{k-1}})+\ldots+B_1(\lambda_{t_1}-\lambda_{t_0}),\ \cdot\,])}{h(-[B_k(\lambda_{t_k}-\lambda_{t_{k-1}})+\ldots+B_1(\lambda_{t_1}-\lambda_{t_0}),\ \cdot\,])}\mathbb{E}\mathfrak{F}(\vec{t}), \tag{27}$$

$$\frac{\partial}{\partial t_j}\mathbb{E}\mathfrak{F}(\vec{t}) = [\dot{\lambda}_{t_j}(B_j-B_{j+1}), \cdot\,]\frac{h'(-[B_k(\lambda_{t_k}-\lambda_{t_{k-1}})+\ldots+B_1(\lambda_{t_1}-\lambda_{t_0}),\ \cdot\,])}{h(-[B_k(\lambda_{t_k}-\lambda_{t_{k-1}})+\ldots+B_1(\lambda_{t_1}-\lambda_{t_0}),\ \cdot\,])}\mathbb{E}\mathfrak{F}(\vec{t}), \tag{28}$$

where $j = 1,\ldots,k-1$, with the initial conditions

$$\mathbb{E}\mathfrak{F}(\vec{t})_{t_j=t_0} = I^{\otimes k}, \qquad j = 1,\ldots,k.$$

It can be considered as muti-time generalization of quantum master equations.

To illustrate our results, let us consider some special cases.

3.1. Gaussian Case

Let us assume that the random variable ξ has the standard normal distribution, i.e., with $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = 1$. Then, we have

$$h(u) = e^{-\frac{1}{2}u^2}.$$

Equation (13) takes the form

$$\Psi_{t,t_0} = e^{-\frac{1}{2}[B,[B,\ \cdot\,]](\lambda_t-\lambda_{t_0})^2}$$

and Equation (15) takes the form

$$\Phi_{t,s} = \Psi_{t,t_0}(\Psi_{s,t_0})^{-1} = e^{-\frac{1}{2}[B,[B,\ \cdot\,]]((\lambda_t-\lambda_{t_0})^2-(\lambda_s-\lambda_{t_0})^2)}$$

Let us remark that, in particular, this is an explicit example showing that in general it can be $\Phi_{t,s} \neq \Psi_{t,s}$.

Equation (17) for the generator takes the form

$$\mathcal{L}(t) = -[B,[B,\ \cdot\,]](\lambda_t-\lambda_{t_0})\dot{\lambda}_t, \qquad t \geqslant t_0.$$

In particular, for any monotonically non-decreasing function λ_t, $\mathcal{L}(t)$ is a time-dependent GKSL generator (7) with $H(t) = 0$ and $L(t) = \sqrt{2(\lambda_t-\lambda_{t_0})\dot{\lambda}_t}B$, where $t \geqslant t_0$. Let us remark that if this is not the case, then Ψ_{s,t_0} is still invertible but not CP-divisible. It differs

from some widely used physical models ([1], Section 10.1) for which *CP*-divisiblity becomes broken exactly by non-invertibility of Ψ_{s,t_0}.

From the physical point of view, the most important cases are

$$\mathcal{L}(t) = \begin{cases} -\frac{1}{2}[B,[B,\,\cdot\,]], & \lambda_t = \sqrt{t}, t_0 = 0, \\ -[B,[B,\,\cdot\,]]t, & \lambda_t = t, t_0 = 0. \end{cases}$$

The first case leads to time-independent GKSL dynamics, despite the fact that the initial random Hamiltonian depends non-trivially on time. Applications of such generators can be found, e.g., in [64]. The second case for $B = \sqrt{\gamma}\sigma_z$ leads to Gaussian decay of the off-diagonal elements of the density matrix, which occurs in physical applications ([65], p. 14) but is usually explained by non-trivial environment spectral density. But our approach explaining it by a random Hamiltonian seems to be even more natural. And it predicts that the difference between our explanation and the one mentioned occurs in multi-time correlations, which can be tested experimentally.

Using Equation (26), the memory tensor for $k = 2$ takes the form

$$\begin{aligned}\mathfrak{M}(\vec{t}) = \exp\Big(&-\frac{1}{2}([B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_1}) + [B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))^2 \\ &+ \frac{1}{2}([B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_0}))^2 - \frac{1}{2}([B_2,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))^2 + \frac{1}{2}([B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0}))^2\Big) \\ = \exp\Big(&(\lambda_{t_1}-\lambda_{t_0})(\lambda_{t_2}-\lambda_{t_0})[B_2,[B_2,\,\cdot\,]]\Big)\exp\Big((\lambda_{t_2}-\lambda_{t_1})(\lambda_{t_1}-\lambda_{t_0})[B_2,[B_1,\,\cdot\,]]\Big).\end{aligned}$$

So, it is an explicit non-trivial example with $\mathfrak{M}(\vec{t}) \neq I^{\otimes 2}$.

For $k = 2$, multi-time master Equations (27) and (28) take the form

$$\begin{aligned}\frac{\partial}{\partial t_2}\mathbb{E}\mathfrak{F}(\vec{t}) &= -[B_2\dot{\lambda}_{t_2}, [B_2(\lambda_{t_2}-\lambda_{t_1}) + B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,]]\mathbb{E}\mathfrak{F}(\vec{t}), \\ \frac{\partial}{\partial t_1}\mathbb{E}\mathfrak{F}(\vec{t}) &= -[(B_1-B_2)\dot{\lambda}_{t_1}, [B_2(\lambda_{t_2}-\lambda_{t_1}) + B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,]]\mathbb{E}\mathfrak{F}(\vec{t}).\end{aligned}$$

3.2. Poisson Case

Let us now assume that the random variable ξ has the Poisson distribution with $\mathbb{E}\xi = 1$. Then, we have

$$h(u) = \exp(e^{iu} - 1).$$

Equation (13) takes the form

$$\Psi_{t,t_0} = \exp\Big(e^{-i[B,\,\cdot\,](\lambda_t-\lambda_{t_0})} - 1\Big)$$

and Equation (15) takes the form

$$\Phi_{t,s} = \exp\Big(e^{-i[B,\,\cdot\,](\lambda_t-\lambda_{t_0})} - e^{-i[B,\,\cdot\,](\lambda_s-\lambda_{t_0})}\Big).$$

Equation (17) for the generator takes the form

$$\mathcal{L}(t) = -i\dot{\lambda}_t[B,\,\cdot\,]e^{-i[B,\,\cdot\,](\lambda_t-\lambda_{t_0})}.$$

In particular, for the special case $B = \sigma_z$, it takes the form

$$\mathcal{L}(t) = i\dot{\lambda}_t\cos 2(\lambda_t-\lambda_{t_0})[\sigma_z,\rho] - \frac{1}{2}\dot{\lambda}_t\sin 2(\lambda_t-\lambda_{t_0})[\sigma_z,[\sigma_z,\rho]].$$

Thus, for $\dot{\lambda}_t \sin 2(\lambda_t - \lambda_{t_0}) = -\frac{1}{2}\frac{d}{dt}\cos 2(\lambda_t - \lambda_{t_0}) \geqslant 0$, the generator $\mathcal{L}(t)$ has GKSL form (7) with $H(t) = -\dot{\lambda}_t \cos 2(\lambda_t - \lambda_{t_0})\sigma_z$ and $L(t) = \sqrt{\dot{\lambda}_t \sin 2(\lambda_t - \lambda_{t_0})}\sigma_z$. This condition is satisfied, e.g., for $t_0 = 0$ and $\lambda_t = \frac{\pi}{4}(1 - e^{-t})$.

Using Equation (26), the memory tensor for $k = 2$ takes the form

$$\mathfrak{M}(\vec{t}) = \exp\Big(e^{-i[B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_1})-i[B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0})} - e^{-i[B_2,\,\cdot\,](\lambda_{t_2}-\lambda_{t_0})} + e^{-i[B_2,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0})} - e^{-i[B_1,\,\cdot\,](\lambda_{t_1}-\lambda_{t_0})}\Big).$$

For $k = 2$, multi-time master Equations (27) and (28) take the form

$$\frac{\partial}{\partial t_2}\mathbb{E}\mathfrak{F}(\vec{t}) = -i[B_2\dot{\lambda}_{t_2},\,\cdot\,]e^{-i[B_2(\lambda_{t_2}-\lambda_{t_1})+B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,])}\mathbb{E}\mathfrak{F}(\vec{t}),$$
$$\frac{\partial}{\partial t_1}\mathbb{E}\mathfrak{F}(\vec{t}) = -i[(B_1 - B_2)\dot{\lambda}_{t_1},\,\cdot\,]e^{-i[B_2(\lambda_{t_2}-\lambda_{t_1})+B_1(\lambda_{t_1}-\lambda_{t_0}),\,\cdot\,])}\mathbb{E}\mathfrak{F}(\vec{t}).$$

4. Asymptotic Expansion of Dynamics with Random Free Hamiltonian

Now, consider random unitary dynamics with a Hamiltonian of the form

$$H(\alpha) = \xi B + \alpha V, \tag{29}$$

where, similarly to the previous section, ξ is a real random variable, B and V are Hermitian $n \times n$ complex matrices and α is a real parameter. Instead of (11), we have

$$U_{t,t_0} := e^{-iH(\alpha)(t-t_0)}. \tag{30}$$

Proposition 3. *Let us define the averaged propagator* Ψ_{t,t_0} *as*

$$\Psi_{t,t_0}(\alpha) := \mathbb{E}(U_{t,t_0}\ \cdot\ U^{\dagger}_{t,t_0}), \tag{31}$$

where $\mathbb{E}$ *is the mathematical expectation and* U_{t,t_0} *is defined by* (29) *and* (30)*. Then, the following asymptotic expansion in the limit* $\alpha \to 0$ *holds.*

$$\Psi_{t,t_0}(\alpha) = \sum_{m=0}^{\infty}(-i\alpha)^m \mathfrak{C}\big(g_m(-[B_{m+1},\,\cdot\,],\ldots,-[B_1,\,\cdot\,];t-t_0)([V,\,\cdot\,]^{\otimes m}\otimes I)\big), \tag{32}$$

where

$$g_m(b_m,\ldots,b_1;t) := \int_0^t ds_m \ldots \int_0^{s_2} ds_1 h(b_{m+1}(t-s_m) + \ldots + b_1 s_1). \tag{33}$$

The function of the super-operators $[B_j,\,\cdot\,]$ *is defined in terms of the spectral decomposition.*

Proof. First of all, let us define

$$\mathcal{W}_t(\alpha) := e^{i[H(0),\,\cdot\,]t}e^{-i[H(\alpha),\,\cdot\,]t}. \tag{34}$$

(From the physical point of view, $\mathcal{W}_t(\alpha)$ describes the dynamics in the interaction picture). Then, $\mathcal{W}_t(\alpha)$ satisfies the Cauchy problem

$$\frac{d}{dt}\mathcal{W}_t(\alpha) = -i\alpha[V(t),\,\cdot\,]\mathcal{W}_t(\alpha), \qquad \mathcal{W}_0(\alpha) = I,$$

where $V(t) := e^{i[H(0),\,\cdot\,]t}V = e^{i\xi[B,\,\cdot\,]t}V$. Then, the Dyson expansion for $\mathcal{W}_t(\alpha)$ takes the form

$$\mathcal{W}_t(\alpha) = \sum_{m=0}^{\infty}(-i\alpha)^m \int_0^t ds_m \ldots \int_0^{s_2} ds_1 [V(s_m),\,\cdot\,]\ldots[V(s_1),\,\cdot\,]$$

Taking into account

$$[V(t),\ \cdot\ ] = [e^{i\xi[B,\ \cdot\ ]t}V,\ \cdot\ ] = [e^{i\xi Bt}Ve^{-i\xi Bt},\ \cdot\ ]$$
$$= e^{i\xi Bt}[V, e^{-i\xi Bt}\ \cdot e^{i\xi Bt}\ ]e^{-i\xi Bt} = e^{i\xi[B,\ \cdot\ ]t}[V,\ \cdot\ ]e^{-i\xi[B,\ \cdot\ ]t},$$

we have

$$\mathcal{W}_t(\alpha) = \sum_{m=0}^{\infty}(-i\alpha)^m\int_0^t ds_m\ldots\int_0^{s_2} ds_1 e^{i\xi[B,\ \cdot\ ]s_m}[V,\ \cdot\ ]e^{-i\xi[B,\ \cdot\ ](s_m-s_{m-1})}\ldots[V,\ \cdot\ ]e^{-i\xi[B,\ \cdot\ ]s_1}.$$

So, taking into account (34), we obtain

$$U_{t,t_0}\ \cdot\ U^{\dagger}_{t,t_0} = e^{-i[H(\alpha),\ \cdot\ ]t} = e^{-i[H(0),\ \cdot\ ]t}\mathcal{W}_t(\alpha)$$
$$= \sum_{m=0}^{\infty}(-i\alpha)^m\int_0^t ds_m\ldots\int_0^{s_2} ds_1 e^{-i\xi[B,\ \cdot\ ](s-s_m)}[V,\ \cdot\ ]e^{-i\xi[B,\ \cdot\ ](s_m-s_{m-1})}\ldots[V,\ \cdot\ ]e^{-i\xi[B,\ \cdot\ ]s_1}$$
$$= \sum_{m=0}^{\infty}(-i\alpha)^m\mathfrak{C}\left(\int_0^t ds_m\ldots\int_0^{s_2} ds_1 e^{-i\xi[B_{m+1}(t-s_m)+\ldots+B_1s_1,\ \cdot\ ]}([V,\ \cdot\ ]^{\otimes m}\otimes I)\right).$$

Substituting it into (31), we have

$$\Psi_{t,t_0}(\alpha) = \mathbb{E}e^{-i[H(\alpha),\ \cdot\ ](t-t_0)} = \sum_{m=0}^{\infty}(-i\alpha)^m\times$$
$$\times\,\mathfrak{C}\left(\int_0^{t-t_0} ds_m\ldots\int_0^{s_2} ds_1 h(-[B_{m+1}(s-s_m)+\ldots+B_1(s_1-s_0),\ \cdot\ ])([V,\ \cdot\ ]^{\otimes m}\otimes I)\right)$$

Thus, taking into account (33), we obtain (32). □

Proposition 4. *Let $\Phi_{t,s}(\alpha)$ be defined (for sufficiently small α) by (9), with $\Psi_{t,t_0}(\alpha)$ defined in the previous proposition. Then, the following asymptotic expansion in the limit $\alpha\to 0$ holds.*

$$\Phi_{t,s}(\alpha) = \sum_{m=0}^{\infty}(-i\alpha)^m\sum_{q=0}^{m-1}(-1)^q\sum_{\sum_{j=0}^{q}m_j=m,m_j\geqslant 1} G_{m_0}(t-t_0)G_{m_1}(s-t_0)\ldots G_{m_q}(s-t_0), \quad (35)$$

where

$$G_m(t) := \mathfrak{C}(g_m(-[B_{m+1},\ \cdot\ ],\ldots,-[B_1,\ \cdot\ ];t)[V,\ \cdot\ ]^{\otimes m}\otimes I), \quad (36)$$

and let $\Phi_{t,s}(\alpha)$ be differentiable with respect to t. Then,

$$\frac{d}{dt}\Phi_{t,s}(\alpha) = \mathcal{L}(t;\alpha)\Phi_{t,s}(\alpha), \quad (37)$$

where the following asymptotic expansion in the limit $\alpha\to 0$ holds.

$$\mathcal{L}(t;\alpha) := \sum_{m=1}^{\infty}(-i\alpha)^m\sum_{q=0}^{m-1}(-1)^q\sum_{\sum_{j=0}^{q}m_j=m,m_j\geqslant 1}\dot{G}_{m_0}(t-t_0)G_{m_1}(t-t_0)\ldots G_{m_q}(t-t_0). \quad (38)$$

Proof. Taking into account Definition (36), the asymptotic expansion (32) for $\Psi_{s,t_0}(\alpha)$ takes the form

$$\Psi_{s,t_0}(\alpha) = \sum_{m=0}^{\infty}(-i\alpha)^m G_m(s-t_0).$$

Using Lemma 1 from [62], we have

$$(\Psi_{s,t_0}(\alpha))^{-1} = \sum_{m=0}^{\infty}(-i\alpha)^m \sum_{q=1}^{m}(-1)^q \sum_{\sum_{j=1}^{q} m_j=m, m_j\geqslant 1} G_{m_1}(s-t_0)\dots G_{m_q}(s-t_0)$$

As $\Phi_{t,s}(\alpha) = \Psi_{t,t_0}(\alpha)(\Psi_{s,t_0}(\alpha))^{-1}$, then by multiplying the series for $\Psi_{t,t_0}(\alpha)$ and $(\Psi_{s,t_0}(\alpha))^{-1}$, we obtain the asymptotic expansion (35).

From (9), we have $\Phi_{t,t}(\alpha) = I$. Hence, from Equation (37), we have

$$\mathcal{L}(t;\alpha) = \left(\frac{d}{dt}\Phi_{t,s}(\alpha)\right)_{s=t}.$$

Taking into account the expansion (35), we obtain (38). □

Let us remark upon the similarity between the expansion (35) and the expansion of $\mathcal{K}(t)$ in ([62], Theorem 1).

For the next theorem, we need to expand notation (20), defining

$$B_{j,l} := (I^{\otimes(m+1)})^{\otimes(l-1)} \otimes (I^{\otimes(j-1)} \otimes B \otimes I^{\otimes(m+1-j)}) \otimes (I^{\otimes(m+1)})^{\otimes(k-l)}$$

for $j = 1,\dots,m+1$ and $l = 1,\dots,k$. And we denote $\mathfrak{C}_1$ as the contraction operator $\mathfrak{C}$, which acts on indices j only, e.g.,

$$\mathfrak{C}_1([B_{j_1,l_1},\cdot\,][B_{j_2,l_2},\cdot\,]) = [B_{l_1},\cdot\,][B_{l_2},\cdot\,].$$

Theorem 2. *Let $\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})]$ be defined by (18), with $\mathcal{U}_{t,t_0}$ defined by (5) and (30), where ξ is a real random variable with the characteristic function $h(u)$. And let $\mathcal{T}_M(\vec{t})[\mathfrak{A}(\vec{x})]$ be defined by (6), where $\Phi_{t,s}(\alpha)$ is defined in the previous proposition. Then,*

$$\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})] = \mathcal{T}_M(\vec{t})[\mathfrak{M}(\vec{t})\mathfrak{A}(\vec{x})], \tag{39}$$

where the memory tensor $\mathfrak{M}(\vec{t})$ has the following asymptotic expansion in the limit $\alpha \to 0$.

$$\mathfrak{M}(\vec{t}) = \sum_{m=0}^{\infty}(-i\alpha)^m \sum_{\sum_{r=0}^{k} m_r=m} G_{[m_0]}(\vec{t}) \prod_{r=1}^{k}\Bigg(\sum_{q_r=0}^{m_r-1}(-1)^{q_r}\times$$
$$\times \sum_{\sum_{j_r=0}^{q_r} m_{r,j_r}=m_r, m_{r,j_r}\geqslant 1} G_{m_{r,0},r}(t_{r-1}-t_0)G_{m_{r,1},r}(t_r-t_0)\dots G_{m_{q_r},r}(t_r-t_0)\Bigg). \tag{40}$$

Here, the following notation is introduced.

$$G_{[m]}(\vec{t}) := (-i\alpha)^m \mathfrak{C}_1\Big(g_m(-[(\vec{B}_{m+1},\Delta\vec{t}),\cdot\,],\dots,-[(\vec{B}_1,\Delta\vec{t}),\cdot\,]);1)([V,\cdot\,]^{\otimes m}\otimes I)^{\otimes k}\Big), \tag{41}$$

where g_m is defined by Formula (33),

$$(\vec{B}_j,\Delta\vec{t}) := B_{j,k}(t_k - t_{k-1}) + \dots + B_{j,1}(t_1 - t_0), \qquad (\Delta\vec{t})_k := t_k - t_{k-1},$$

and $G_{m,r}(t)$ is defined similarly to (20) acting as $G_m(t)$ defined by (36) in the r-th tensor multiplicand and as I in other ones.

Proof. Analogously to the proof of Theorem 1, let us rewrite $\mathcal{T}(\vec{t})[\mathfrak{A}(\vec{x})]$ using Lemma 1 in the form of (23), where

$$\begin{aligned}\mathfrak{F}(\vec{t}) &= \exp(-i[H_k(\alpha)(t_k - t_{k-1}) + \ldots + H_1(\alpha)(t_1 - t_0), \cdot\,]) \\ &= \exp\Big(-i\xi[B_k(t_k - t_{k-1}) + \ldots + B_1(t_1 - t_0), \cdot\,] \\ &\qquad - i\alpha[V_k(t_k - t_{k-1}) + \ldots + V_1(t_1 - t_0), \cdot\,]\Big).\end{aligned}$$

Similarly to the proof of Proposition 3, but changing B to $B_k(t_k - t_{k-1}) + \ldots + B_1(t_1 - t_0)$ and V to $V_k(t_k - t_{k-1}) + \ldots + V_1(t_1 - t_0)$ and $t - t_0$ to 1, we obtain

$$\begin{aligned}\mathbb{E}\mathfrak{F}(\vec{t}) &= \sum_{m=0}^{\infty}(-i\alpha)^m \mathfrak{C}_1\Big(g_m(-[(\vec{B}_{m+1}, \Delta\vec{t}), \cdot\,], \ldots, -[(\vec{B}_1, \Delta\vec{t}), \cdot\,]); 1)([V, \cdot\,]^{\otimes m} \otimes I)^{\otimes k}\Big) \\ &= \sum_{m=0}^{\infty}(-i\alpha)^m G_{[m]}(\vec{t}),\end{aligned} \tag{42}$$

where $G_{[m]}(\vec{t})$ is defined by Formula (41).

Also, by Lemma 1, Formula (19) takes the form (25), where

$$\mathfrak{F}_M(\vec{t}) = \Phi_{t_k,t_{k-1}} \otimes \ldots \otimes \Phi_{t_1,t_0} = \prod_{r=1}^{k}(\Phi_{t_r,t_{r-1}}(\alpha))_r,$$

where $(\Phi_{t_r,t_{r-1}}(\alpha))_r$ is defined similarly to (20) acting as $\Phi_{t_r,t_{r-1}}(\alpha)$ defined by (36) in the r-th tensor multiplicand and as I in other ones. So, we have

$$(\mathfrak{F}_M(\vec{t}))^{-1} = \prod_{r=1}^{k}(\Phi_{t_r,t_{r-1}}(\alpha))_r^{-1}, \tag{43}$$

Due to (35), we have

$$\begin{aligned}(\Phi_{t_r,t_{r-1}}(\alpha))_r^{-1} &= \Phi_{t_{r-1},t_r}(\alpha) \\ &= \sum_{m_r=0}^{\infty}(-i\alpha)^{m_r}\sum_{q_r=0}^{m_r-1}(-1)^{q_r}\times \\ &\quad\times \sum_{\sum_{j_r=0}^{q_r} m_{r,j_r}=m_r, m_{r,j_r}\geqslant 1} G_{m_{r,0},r}(t_{r-1} - t_0)G_{m_{r,1},r}(t_r - t_0)\ldots G_{m_{q_r},r}(t_r - t_0).\end{aligned}$$

Substituting it into (43), we obtain

$$\begin{aligned}(\mathfrak{F}_M(\vec{t}))^{-1} = \sum_{m=0}^{\infty}(-i\alpha)^m \sum_{\sum_{r=1}^{k} m_r=m}\prod_{r=1}^{k}\Big(\sum_{q_r=0}^{m_r-1}(-1)^{q_r}\times \\ \times \sum_{\sum_{j_r=0}^{q_r} m_{r,j_r}=m_r, m_{r,j_r}\geqslant 1} G_{m_{r,0},r}(t_{r-1} - t_0)G_{m_{r,1},r}(t_r - t_0)\ldots G_{m_{q_r},r}(t_r - t_0)\Big).\end{aligned}$$

Substituting it and (42) into $\mathfrak{M}(\vec{t}) = (\mathbb{E}\mathfrak{F}(\vec{t}))(\mathfrak{F}_M(\vec{t}))^{-1}$, we obtain (40). Thus, taking into account Formulae (23) and (25), we obtain (39). □

5. Conclusions

In this work, we have introduced the memory tensor, which characterizes multi-time correlations in quantum systems against Markovian ones. In Theorem 1, we have calculated it explicitly for a simple model with a random Hamiltonian, with multiplicative dependence on a random variable. In Theorem 2, we have calculated it perturbatively for a model with a random free Hamiltonian and a deterministic interaction Hamiltonian. But we think that the introduced concepts and developed approaches can be used in a much broader context.

First of all, the concept of the memory tensor itself can be applied to other quantum problems and is not restricted to these specific models. Its interpretation in Section 3 in terms of transformation into the "Markovian picture" is also general.

The analytical approach based on characteristic functions can be directly generalized for an arbitrary commutative random Hamiltonian. And it seems to be generalizable for more complex situations, when the randomness is quantum and is described by non-commuting reservoir variables in the Hamiltonian, but the system variables in the Hamiltonian are still commuting with each other. In such a case, the characteristic functions of quantum observables ([43], Section 12.3.1) will occur.

Despite the fact that the explicit perturbative formulae in Theorem 2 are a bit sophisticated, the trick which was used in its proof allows one to calculate the process tensor perturbatively, similarly to the perturbative calculation of one-time dynamics.

For several physical models [31,32], in a usual open quantum system setup it is shown that only for flat spectral densities the regression formulae are satisfied, i.e., the memory is trivial. Thus, when one assumes corrections to weak coupling limit GKSL equations (e.g., Redfield without secular approximation, next-order time-convolutionless cumulant corrections, etc.), it automatically leads to a non-trivial memory tensor. The results of [14,16] suggest that under conditions of sufficiently fast decaying heat bath correlation functions, the memory tensor becomes constant after bath correlation time in a usual weak coupling limit setup. So, the systematic perturbative calculation of the memory tensor in the weak coupling limit setup is one of natural directions for further study.

We think that the multi-time master Equations (27) and (28) introduced in Section 2.2 can be interesting for perturbative expansions based on cumulant expansions [2–4] or on effective generators [66–69]. Namely, the perturbative derivation of such master equations in the setup of Section 4 can be considered as a resummation of the asymptotic series from Theorem 2. And analogous multi-time master equations can be useful in the usual weak coupling limit setup in open quantum systems theory to derive corrections to regression formulae, e.g., their renormalization in the Bogolubov–van Hove limit [14,16,70].

Let us remark that similarly to the specific model considered in this work, one can assume that the reservoir spectral density has a random parameter and experimentally we have only access to the averaged measurements on the systems. Then, even in the weak coupling limit, one can obtain a non-trivial memory tensor by averaging weak coupling limit Markovian dynamics. Similarly, it is natural to assume that only averaged reservoir spectral density is experimentally observable. In particular, the same experimentally observable reservoir spectral density can be obtained by a deterministic random variable leading to a trivial memory tensor in the weak coupling limit and many other ones with non-trivial memory tensors.

A similar thing can be done for the multi-bath setup for which usually the calculation of the heat flows is interesting [71–73]. One can introduce a random parameter in such a way that the average heat flows will be the same for its different distributions, but this difference can be detected only by multi-time correlations of heat flows.

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Abbreviations

The following abbreviations are used in this manuscript:

CP	completely positive
GKSL	Gorini–Kossakowski–Sudarshan–Lindblad

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