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Conditional Cosmological Recurrence in Finite Hilbert Spaces and Holographic Bounds Within Causal Patches

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Abstract

A conditional framework of *Conditional Cosmological Recurrence* (CCR) is introduced, as follows: if a causal patch admits a finite operational Hilbert space dimension D (as motivated by holographic and entropy bounds), then unitary quantum dynamics guarantee almost-periodic evolution, leading to recurrences. The central contribution is the explicit formulation of a micro-to-macro bridge, as follows: (i) finite regions discretize field modes; (ii) gravitational bounds cap entropy and energy; and (iii) the number of accessible states is finite, yielding CCR. The analysis differentiates global microstate recurrences (with double-exponential timescales in $S_{\max}$) from operationally relevant coarse-grained returns (exponential in subsystem entropy), with conservative timescale estimates. For predictivity in eternally inflating settings, a causal-diamond measure with xerographic typicality and a single no-Boltzmann-brain constraint is employed, thereby avoiding volume-weighting pathologies. The scope is *explicitly conditional*: if future quantum gravity demonstrates $D = \infty$ for causal patches, CCR is falsified.

Keywords: Conditional Cosmological Recurrence; finite Hilbert space; holographic bounds; quantum recurrence; causal patch; eternal inflation; quantum gravity; Boltzmann brains

1. Introduction

Cosmology often examines regions of the universe known as *causal patches*—the finite portion of spacetime that a single observer can, in principle, access. Within such a patch, gravity imposes strict limits on the amount of information that can be stored. These limits, expressed through *holographic* or *entropy bounds* [1–5], imply that only a limited number of physically distinct configurations may exist. This constraint gives rise to the phenomenon of *recurrence* [6]: if the state space is bounded and the system evolves according to unitary quantum mechanics, then it must eventually return arbitrarily close to an earlier configuration in the finite-D case. In practice, such recurrences occur on unimaginably long timescales, but the implication remains important: the universe may not evolve in endlessly novel ways but could instead recycle through states. This possibility is addressed in a strictly conditional sense: recurrence follows mathematically *if* holographic bounds imply a finite state budget for causal patches [7–12].

Classical intuition (well-known example).

The phenomenon of recurrence is not new; it is a classical consequence of the Poincaré Recurrence Theorem (1890). A standard textbook illustration considers an isolated box

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containing a macroscopic object, such as an apple. Hamiltonian dynamics on a bounded phase space imply that—after an astronomically long time—the microstate of the system will return arbitrarily close to its initial configuration; in other words, the apple would eventually "reassemble." Although not original to this study, this example serves as an intuition pump for the general recurrence phenomenon.

From boxes to causal diamonds.

In a cosmological setting, the "box" is replaced by a causal patch: a finite region bounded by horizons, whose holographic entropy cap [13,14] implies a finite-dimensional Hilbert space. The CCR framework is then the direct quantum analogue of the recurrence principle in such bounded systems [15–17].

Conventions.

Throughout this work, we employ natural units with $\hbar = c = 1$, while keeping k_B and G explicit to emphasize entropic and gravitational scales. The Planck length is $\ell_P = \sqrt{\hbar G/c^3} \equiv \sqrt{G}$ in these units. In certain expressions (e.g., Lemma 2), we retain explicit factors of c and $\hbar$ to facilitate comparison with the standard literature and to clarify the physical dimensions involved.

Novel contributions.

While the core result—that finite Hilbert spaces imply quantum recurrence—is well-established, this work makes the following three complementary contributions:

1. **Explicit micro-to-macro bridge.** A concrete microcanonical counting under gravitational caps (Proposition 1, Appendix C) demonstrates how infrared discretization and holographic bounds enforce bounded operational dimension, making finiteness tangible rather than heuristic.
2. **Conditional and falsifiable framing.** The CCR theorem (Theorem 1) is stated in strictly conditional form: if causal patches admit finite D (assumption (A1)), recurrence follows rigorously; and if future quantum gravity shows $D = \infty$, CCR is falsified. This clean if–then structure emphasizes testability.
3. **Minimalistic measure prescription.** A compact causal-diamond measure with xerographic typicality and a single no-Boltzmann-brain constraint (Section 4.1) avoids volume-weighting pathologies while remaining falsifiable.

2. Materials and Methods

2.1. Originality and Scope

The core idea that finite Hilbert spaces imply quantum recurrences is not new; it goes back to Bocchieri–Loinger (1957) and has been discussed in cosmological contexts (e.g., Dyson–Kleban–Susskind 2002). The contribution does not lie in proposing a new recurrence law, but in the following three complementary aspects:

1. **Explicit constructive micro-to-macro derivation.** A concrete microcanonical counting argument under a gravitational energy cap (Proposition 1, Appendix C) is presented, showing explicitly how infrared discretization and holographic entropy bounds combine to enforce a bounded operational Hilbert space dimension. This makes the finiteness assumption tangible rather than heuristic.
2. **Conditional and falsifiable framing.** The "Conditional Cosmological Recurrence" (CCR) theorem is stated in a strictly conditional form: if causal patches admit a finite Hilbert space dimension (assumption (A1)), then recurrence follows rigorously from unitarity. Conversely, if future quantum gravity demonstrates that $D = \infty$, the entire

framework is falsified. This clean *if–then* structure is rarely made explicit in the prior literature and emphasizes testability.

3. **Minimalistic measure prescription.** To address predictivity in eternally inflating settings, a compact causal-diamond measure is introduced, combined with xerographic typicality and a single no-Boltzmann-brain constraint. This avoids the volume-weighting pathologies highlighted in earlier works while keeping the prescription lean and falsifiable.

In short, the contribution is not the discovery of recurrence *per se*, but the *systematic construction of a finite-state framework* (micro-to-macro counting), its *conditional and testable formulation* (CCR theorem), and a *clean measure proposal* that, together, yield a more rigorous and falsifiable picture of recurrence in cosmological settings. Figure A1 in Appendix J summarizes the logical structure of this framework.

Key Point: The framework does *not* claim that the universe must recur. Rather, *if* gravitational entropy bounds apply to causal patches (yielding a finite Hilbert space dimension), *then* recurrence follows mathematically. The "if" is the physics question; the "then" is rigorous mathematics.

Conditional Nature.

Cosmological Recurrence (CCR) is not a prediction of standard cosmology *per se*, but a logical consequence of the finite Hilbert space conjecture for de Sitter patches (A1). Consequently, the results are strictly conditional: falsifying assumption(A1 would directly falsify CCR. In this sense, the framework serves as a logical bridge between holographic bounds and recurrence phenomena, rather than as an independent dynamical model. If future developments in quantum gravity demonstrate that the Hilbert space of a causal patch is in fact infinite, the CCR framework would be invalidated in its entirety, and the recurrence problem would have to be reconsidered from a fundamentally different perspective.

Separation of rigor and assumptions.

The mathematical content of this work (recurrence theorems, almost-periodicity) is rigorous, following from finite-dimensional Hilbert spaces and unitary dynamics. The physics content (finite Hilbert space dimension via holographic bounds) is an assumption, motivated by semiclassical gravity but not derived from first principles. Section 4.4 provides an explicit separation of these two levels.

2.2. From Electrons to a Finite Cosmic State Count: A Counting Argument

Lemma 1 (IR discretization)**.** *A quantum field in a finite spatial region of linear size R admits a discrete set of momentum modes* $\mathbf{k} = \frac{\pi}{R}(n_x, n_y, n_z)$ *under standard boundary conditions.*

Lemma 2 (UV/gravity cutoff)**.** *Let* $E_{\rm BH}(R) = \frac{c^4 R}{2G}$ *be the energy of a Schwarzschild black hole of radius R. Any state whose total energy localized within the region exceeds* $E_{\rm BH}(R)$ *collapses, and the Bekenstein–Hawking bound yields*

$$S_{\max} \le \frac{k_B A}{4\ell_P^2} = \frac{k_B\, 4\pi R^2}{4\ell_P^2}, \qquad \ell_P^2 = \frac{\hbar G}{c^3}. \tag{1}$$

Proposition 1 (Finite-dimensional accessible Hilbert space)**.** *Consider the Fock space built from the discrete IR modes and impose the microcanonical cap* $\sum_{\mathbf{k}} n_{\mathbf{k}}\, \hbar\omega_{\mathbf{k}} \le E_{\rm BH}(R)$*. Then, the number of distinct Fock states below this cap is finite. Moreover, any operationally distinguishable state family within the region obeys* $\log D \le S_{\max}/k_B \lesssim A/(4\ell_P^2)$*, so the accessible Hilbert space dimension D is finite. Here, D denotes the effective Hilbert space dimension*[1]*.*

(See Appendix C for an explicit microcanonical counting illustration under a gravitational cap).

Proof of Sketch. Finite volume $\Rightarrow$ discrete k (Lemma 1). The energy cap forbids arbitrarily many high-frequency quanta (Lemma 2). Since $\omega_k \geq c|k|$ and the number of modes below any cutoff Λ is finite, the integer solutions $\{n_k\}$ of $\sum n_k \hbar\omega_k \leq E_{\rm BH}$ are finite. While derived here for free fields, the finiteness conclusion extends to weakly interacting theories: interactions modify the state space structure but cannot generate infinite independent states below the gravitational bound. Operational distinguishability is limited by the entropy bound, giving $\log D \leq S_{\max}/k_B$. □

Physical intuition: A finite spatial region implies discrete field modes (analogous to a particle in a box). Gravitational collapse bounds the total energy—too much energy in a region would form a black hole. These two constraints together limit the total number of quantum states, implying a finite Hilbert space dimension D.

Clarification: Two independent arguments.

This proposition employs the following two *logically independent* arguments:

1. **Microcanonical finiteness:** Finite spatial volume $\Rightarrow$ discrete modes (IR), and gravitational collapse threshold $E_{\rm BH}(R) \Rightarrow$ finite occupation (UV). Together, these imply the accessible Fock space is finite-dimensional.
2. **Holographic bound:** The *area-law* entropy bound $S \leq S_{\max} \sim A/(4\ell_P^2)$ (from black-hole thermodynamics and holographic arguments) provides an *independent* constraint on operational distinguishability: $\log D \lesssim S_{\max}/k_B$.

The microcanonical argument does *not* derive the area-law scaling—that is an additional input from holography. Both arguments point to finite D, and the area-law bound sets the operational scale.

Clarification: Three notions of finiteness.

Before stating the formal assumptions, it is useful to distinguish the following three conceptually distinct notions of finiteness that appear in discussions of quantum gravity and cosmology:

1. **Kinematical finiteness:** The spacetime itself is fundamentally discrete at the Planck scale, as in causal set theory [18] or loop quantum gravity. In such approaches, the continuum is an emergent, coarse-grained description, and the number of degrees of freedom in any finite region is strictly finite by construction.
2. **Operational finiteness:** Even in a continuum quantum field theory, the number of *operationally distinguishable* states within a region is finite when measured to a fixed resolution δ in trace distance. This counting is made precise via metric entropy and packing arguments (see Appendix A), and yields an effective dimension $D_{\rm eff}(\delta)$ that is finite for any $\delta > 0$, with $\log D_{\rm eff}(\delta) \lesssim S_{\max}/k_B + O(\log(1/\delta))$.
3. **Thermodynamic finiteness:** Holographic entropy bounds impose an upper limit on the von Neumann entropy accessible within a causal patch: $S \leq S_{\max} \sim k_B A/(4\ell_P^2)$. This constrains the dimension of the effective Hilbert space via $D \lesssim \exp(S_{\max}/k_B)$, which is finite (though extremely large) if A is finite.

Scope of this work. The CCR framework relies primarily on *thermodynamic finiteness* (item 3), motivated by holographic bounds, and employs *operational finiteness* (item 2) to make the notion of effective dimension D precise. We do *not* assume kinematical discreteness (item 1), though the conclusions remain compatible with such approaches. In what follows, assumption (A1) formalizes thermodynamic finiteness as the starting point for the recurrence theorem.

2.3. Assumptions and Definitions

Consider a causal patch, defined as the region causally connected to an observer, with horizon area A, adopting the following:

(A1) Finite information bound: $S_{\max} \leq k_B A / 4\ell_P^2$, implying $\dim \mathcal{H} = D < \infty$ (motivated by Bekenstein–Hawking/Gibbons–Hawking bounds and recent discussions of finite Hilbert space in de Sitter [11,19,20]).

(A2) Unitary dynamics: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$, with self-adjoint Hamiltonian H.

(A3) Sector mixing (optional): Dynamics are not confined to a null-measure subset within the relevant superselection sector.

(A4) Finite-resolution observers: Macrostates identified up to a trace-distance tolerance $\varepsilon > 0$ on local density matrices.

Remark 1 (Spectral assumptions)**.** *The recurrence theorem relies on the following properties of the Hamiltonian spectrum, which follow from (A1) and (A2):*

1. ***Discrete spectrum:*** *Finite-dimensional Hilbert space ⇒ purely discrete spectrum* $\{E_j\}_{j=1}^{D}$.
2. ***Non-degeneracy:*** *For generic Hamiltonians, eigenvalues are distinct. Degeneracies, if present, do not obstruct recurrence [6].*
3. ***Absence of continuous spectrum:*** *No accumulation points in* $\mathrm{spec}(H)$ *below the gravitational cap* $E_{\mathrm{BH}}(R)$.
4. ***Diophantine gaps:*** *Quantitative lower bounds on recurrence times rely on the energy differences* $\{E_j - E_k\}$*, avoiding low-order rational relations (Diophantine conditions). For generic quantum systems, the spectrum is expected to be linearly independent over the rationals. Under such conditions, rigorous theorems on simultaneous Diophantine approximation [21,22] ensure that the return time scales at least exponentially with the dimension* D.

2.4. CCR Theorem and Coarse-Grained Recurrence

Theorem 1 (Conditional cosmological recurrence (CCR))**.** *Under (A1) and (A2), for any initial pure state* $|\psi(0)\rangle$ *and any* $\varepsilon > 0$*, there exists* T*, such that*

$$F(T) := |\langle\psi(0)|\psi(T)\rangle|^2 > 1 - \varepsilon. \tag{2}$$

Moreover, recurrence times obey the lower-bound behavior in Remark 2.

Proof of Sketch. A finite D implies a discrete spectrum and almost–periodic evolution (Bocchieri–Loinger [6]). Lower bounds on recurrence times follow from Diophantine properties of spectral gaps, namely the degree to which the energy differences $\{E_j - E_k\}$ avoid low-order rational relations. Quantitative control relies on simultaneous Diophantine approximation on tori (Kronecker-type theorems) and metric results in number theory; see Cassels [21] and Schmidt [22]. See also Remark 2. □

Bocchieri–Loinger theorem[2]

Recurrence time (careful statement).

The term "recurrence time" is employed here in the usual almost-periodicity sense. In a model-independent manner, generic recurrence times grow at least *exponentially* with the Hilbert space dimension D:

$$T_{\mathrm{rec}} \gtrsim t_{\mathrm{micro}} \exp(c\, D),$$

for some $c = O(1)$ that depends on the Diophantine properties of spectral gaps and on the chosen tolerance. While rigorous number-theoretic bounds exist (see Remark 1), the specific prefactors are model-dependent heuristic estimates. Since $D \lesssim \exp(S_{\max}/k_B)$ by the holographic/entropic cap, a conservative expectation is[3]

$$T_{\rm rec} \gtrsim \exp\Big\{\alpha \exp(S_{\max}/k_B)\Big\},$$

i.e., *double-exponential* in the entropy (up to polynomial prefactors on microscopic scales). For illustration, even a toy system with dimension $D = 10^3$ already yields a recurrence time of order $T_{\rm rec} \sim e^{10^3} \approx 10^{434}$ steps, while for cosmological entropies ($D \sim 10^{120}$), the scale becomes incomprehensibly large ($T_{\rm rec} \sim e^{10^{120}}$).

Remark 2 (Global vs. local/coarse-grained returns)**.** *While the heuristic estimate $T_{\rm rec} \sim \exp(D)$ is standard in the physics literature, rigorous lower bounds for almost-periodic functions dictate that for a generic Hamiltonian satisfying Diophantine spectral conditions, the return time T_ϵ for tolerance ϵ obeys $T_\epsilon \geq C(\epsilon)\exp(cD)$ [21]. Since $D \sim e^{S_{\max}}$, this confirms the double-exponential scaling, as follows:*

$$T_{\rm rec}^{(\rm global)} \gtrsim \exp\Big\{\alpha \exp\Big(\frac{S_{\max}}{k_B}\Big)\Big\}.$$

such global microstate recurrences occur on timescales vastly exceeding any operational significance and are, therefore, physically meaningless *beyond their role as mathematical consequences of finiteness. By contrast, coarse-grained or subsystem recurrences scale only* single-exponentially *with the entropy of the subsystem,*

$$T_{\rm rec}^{(A)}(\varepsilon) \sim \exp\{c\, S_A\},$$

which, while still extremely large, are vastly more frequent than exact microstate returns. Scrambling times in fast scramblers scale as $t_{\rm scr} \sim (\beta/2\pi)\ln S$ [23–25]. Only the latter two regimes carry potential operational relevance (see Table 1).

Table 1. Representative timescales (schematic). S denotes entropy in units of k_B.

Regime	Timescale	Operational?
Global microstate	$T_{\rm rec}^{(\rm global)} \sim \exp\{\alpha\, e^{S_{\max}}\}$	No
Coarse-grained A	$T_{\rm rec}^{(A)}(\varepsilon) \sim \exp\{c\, S_A\}$	Typically No
Scrambling	$t_{\rm scr} \sim (\beta/2\pi)\ln S$	Yes (toy models)

Proposition 2 (Coarse-grained/local recurrences)**.** *Let A be a finite subsystem. Under (A1) and (A2) and standard typicality/ETH hypotheses [16,17,26–29], for any $\varepsilon > 0$, there exist arbitrarily large times T such that $\|\rho_A(T) - \rho_A(0)\|_1 < \varepsilon$. Consequently, macroscopic properties recur much more frequently than exact microstates.*

3. Results

3.1. Toy Models and Metrics (Summary)

To complement the theoretical results and provide concrete orders of magnitude, two finite-dimensional quantum "toy models" are considered as surrogates for causal patches with a bounded Hilbert space dimension. The goals are as follows: (i) to exhibit global (microstate) almost-recurrences via the fidelity $F(t) = |\langle\psi(0)|\psi(t)\rangle|^2$, (ii) to quantify *local/coarse-grained* returns via the trace distance on a subsystem A, and (iii) to extract a practical scrambling proxy from the entanglement–growth curve, with probability-based visualizations highlighting the contrast between global and coarse-grained returns.

Models.

(M1) Finite-D spin chain. Transverse-field Ising model with an additional ZZ term to avoid fine-tuned revivals:

$$H = J\sum_{i=1}^{N} Z_i Z_{i+1} + h\sum_{i=1}^{N} X_i + g\sum_{i=1}^{N} Z_i,$$

with periodic boundary conditions and parameters $J = 1.0,\ h = 1.05,\ g = 0.7$. System sizes $N = 10, 14$ are reported in the main text (up to $N = 16$ in Appendix H).

(M2) Discretized free scalar field (1D lattice). Quadratic Hamiltonian on L sites:

$$H = \tfrac{1}{2}\sum_{j=1}^{L}\left(p_j^2 + m^2 q_j^2 + \kappa(q_{j+1} - q_j)^2\right),$$

with periodic boundary conditions and mode frequencies $\omega_k^2 = m^2 + 4\kappa\sin^2(\pi k/L)$. The parameters used are $m = 0.8,\ \kappa = 1.0$, and $L = 8, 12$.

Metrics and thresholds.

Global recurrence time $T_{\mathrm{rec}}^{(\varepsilon)}$ is the first passage of the fidelity above a tolerance,

$$T_{\mathrm{rec}}^{(\varepsilon)} = \min\{t:\ F(t) \geq 1 - \varepsilon\}, \qquad \varepsilon \in \{10^{-2}, 10^{-3}\}.$$

Local/coarse-grained recurrence for a fixed subsystem A (half-chain) is the first passage of the trace distance below a tolerance,

$$T_{\mathrm{rec}}^{(A)}(\varepsilon_A) = \min\{t:\ \|\rho_A(t) - \rho_A(0)\|_1 < \varepsilon_A\}, \quad \varepsilon_A = 0.1.$$

As a scrambling proxy, the time for the von Neumann entanglement entropy $S_{\mathrm{vN}}(t)$ of A to reach 90% of its long-time plateau is monitored.

Key observations (from numerical runs).

As shown in Figure 1, the global fidelity $F(t)$ shows rare full recurrences, while the local proxy signal returns much more frequently. The corresponding timescales are summarized in Table 2. For (M1), the global $T_{\mathrm{rec}}^{(\varepsilon)}$ grows rapidly with N (consistent with exponential-in-$D = 2^N$ behavior), while local returns occur much more frequently, and the scrambling proxy scales approximately linearly in N. For (M2), on-site autocorrelations display clear near-revivals due to the finite mode set; global near-recurrence is significantly delayed as L increases, whereas coarse proxies recur on shorter times.

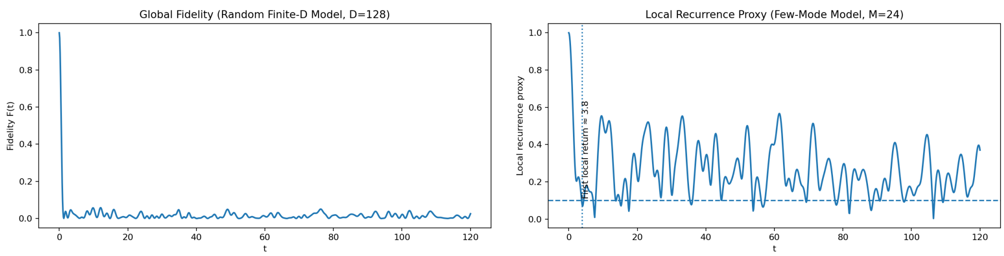

Figure 1. Toy finite-dimensional model results. (**Left**): Global fidelity $F(t)$ for a random-spectrum model with $D = 128$, threshold $\varepsilon = 10^{-2}$. (**Right**): Local recurrence proxy from a few-mode model with $M = 24$, threshold $\varepsilon_A = 0.1$. The global signal exhibits rare full recurrences, while the local proxy shows much more frequent returns.

Table 2. Summary of recurrence and scrambling times for the toy finite-dimensional model. $T_{\rm rec}^{(10^{-2})}$: first global return time with fidelity threshold $\varepsilon = 10^{-2}$. $T_{\rm rec}^{(10^{-3})}$: same with $\varepsilon = 10^{-3}$. $T_{\rm rec}^{(A)}(\varepsilon_A = 0.1)$: first local return time for the proxy signal with threshold $\varepsilon_A = 0.1$. $t_{\rm scr}$: scrambling time, defined as the earliest t where the proxy signal reaches 90% of its late-time plateau. Toy model values from 1000 realizations; see Appendix H.

Model	$\mathbf{T_{rec}^{(10^{-2})}}$	$\mathbf{T_{rec}^{(10^{-3})}}$	$\mathbf{T_{rec}^{(A)}(\varepsilon_A = 0.1)}$	$\mathbf{t_{scr}}$
Random Finite-D	45.6	n/a *	12.3	5.7

* n/a: not achieved within the simulation time window ($t_{\rm max} = 100$).

To make this contrast more explicit, Figures 2–4 show the cumulative recurrence probability $P(T) = 1 - e^{-T/\tau}$ for both global and coarse-grained definitions. At fixed finite D (Figure 2), global probabilities rise only on double-exponential timescales, while coarse-grained returns approach unity quickly. The 3D landscapes (Figures 3 and 4) illustrate how global recurrences become vanishingly rare as D grows, whereas coarse-grained probabilities remain substantial.

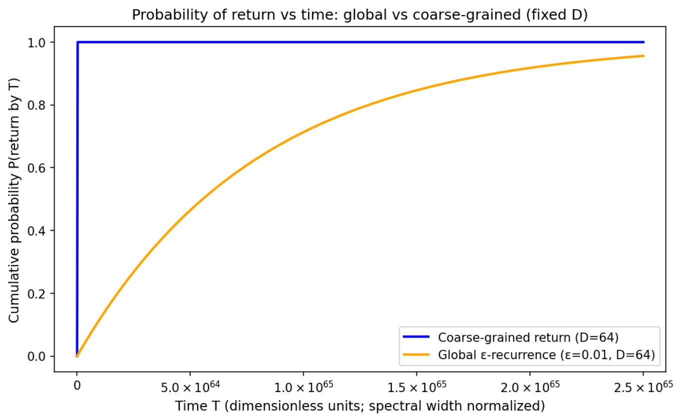

Figure 2. Probability of recurrence for fixed finite D. Global ε–recurrence (orange) versus coarse-grained return (blue) for $D = 64$. The global probability rises extremely slowly, whereas the coarse-grained probability approaches unity on much shorter timescales.

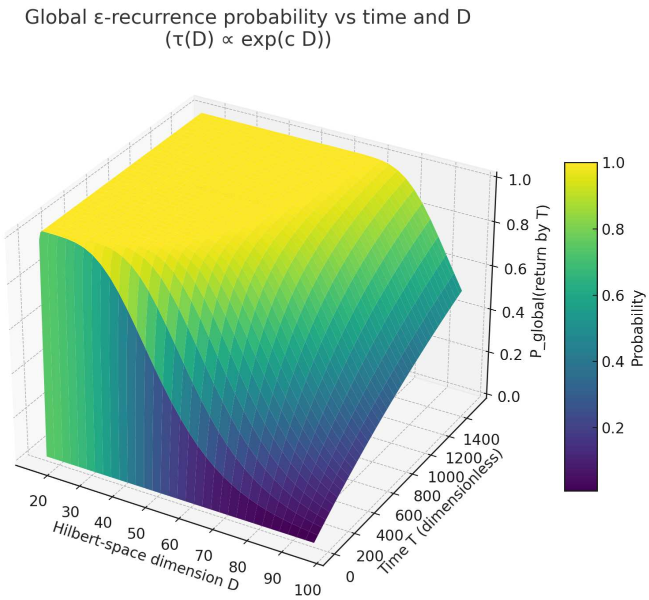

Figure 3. Global recurrence probability landscape. Cumulative probability $P_{\rm global}(T, D)$ as a function of time T and Hilbert space dimension D. Larger D shifts the rise in the probability to vastly longer times, illustrating the double-exponential suppression.

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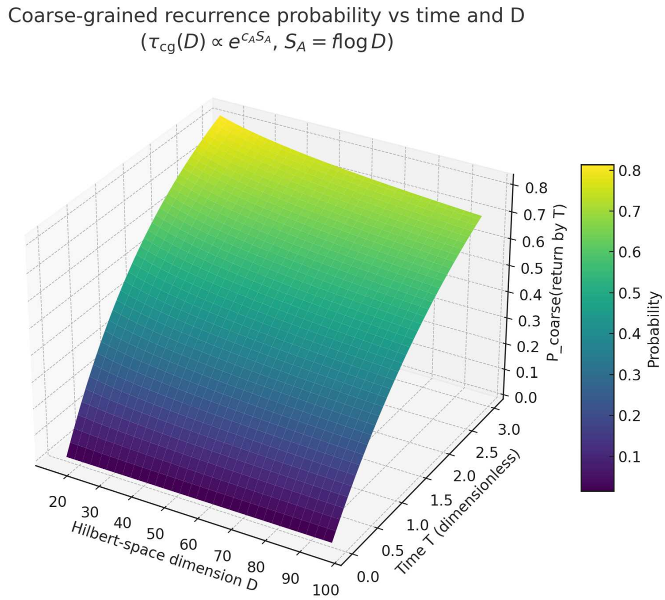

Figure 4. Coarse-grained recurrence probability landscape. Cumulative probability $P_{\text{coarse}}(T, D)$ as a function of time T and Hilbert space dimension D. In contrast to the global case, the probability rises quickly even for larger D, reflecting the exponential (rather than double-exponential) scaling.

Interpretation.

Although these toy models are far simpler than the cosmological setting, they illustrate the same underlying mechanism: once the number of microstates is finite, recurrences are inevitable on sufficiently long timescales. The probability plots reinforce this message: global recurrences are double-exponentially suppressed in $S_{\max}$ and are, therefore, of negligible physical relevance, whereas coarse-grained recurrences occur frequently on exponential timescales in subsystem entropy. This operational distinction is crucial for CCR: it is only the coarse-grained returns that may have physical significance, while global microstate recurrences remain far beyond any observational horizon.

Full details of the toy model construction, parameter choices, and numerical procedures are given in Appendix H.

Cosmological Consequences

Although strictly conditional, the CCR framework has direct implications for key problems in cosmology. Even though the rigorous recurrence times of a full causal patch are double-exponential and, thus, operationally irrelevant, the conditional structure itself carries genuine cosmological significance:

- **Predictivity in eternal inflation.** A finite Hilbert space dimension implies a finite outcome space, resolving the measure ambiguities of volume-weighted approaches. Probabilities are then defined over a compact state space, yielding a falsifiable statistical framework.
- **Boltzmann-brain constraint.** In conventional measures, observers dominated by Boltzmann brains undermine predictivity. In the CCR framework, imposing the single condition $\Gamma_{\text{decay}} \gg \Gamma_{\text{BB}}$ ensures that ordinary observers prevail, thereby avoiding the paradox.
- **Observational falsifiability.** If future developments in quantum gravity or cosmology demonstrate unbounded entropy growth in causal patches, the CCR framework would be falsified in its entirety. Conversely, any evidence for metastable vacuum decay would be consistent with the no-BB prescription.

- **Conceptual bridge.** The CCR program provides a bridge between holographic entropy bounds and macroscopic cosmology. In this sense, "recurrence" is not merely a mathematical curiosity but a structural constraint with direct implications for cosmological predictivity.

In short, the CCR framework translates a mathematical recurrence theorem into a conditional, testable perspective on cosmology: finite information bounds entail finite Hilbert spaces, which, in turn, entail recurrence; the consistency of predictions requires suppressing Boltzmann brains. This endows recurrence with a concrete role in addressing central puzzles of cosmology. While the above discussion applies to a single causal patch, the same logic can be extended to multiverse settings, as discussed in the following subsection.

Beyond a Single Patch: Multiverse Considerations

Although the preceding discussion focused on a single causal patch, the logic of CCR extends naturally to multiverse scenarios. In eternal inflation, the global spacetime can be viewed as an ensemble of quasi-independent causal patches, each bounded by its own horizon and subject to holographic entropy limits. If each such patch admits a finite Hilbert space, then the CCR theorem applies patch by patch. In this sense, recurrence is not peculiar to "our universe", but a structural feature of any holographically bounded region.

This suggests a modular picture of the multiverse: a collection of finite-state systems, each evolving unitarily with conditional recurrences. Whether correlations between patches can be defined consistently remains an open problem, but the CCR framework provides a baseline for treating each region as a finite information-bearing system.

4. Discussion

4.1. Measure and Predictivity

Why ambiguity arises.

In eternally inflating or infinitely extended scenarios, *everything* that can happen tends to happen infinitely many times. Raw frequencies become ∞/∞, so probabilities are undefined without a regularization/cutoff (a *measure*). Different cutoffs induce different weights—hence, the measure problem.

Prescription.

The approach adopted here employs a *causal-diamond* (local) measure combined with *xerographic typicality* over ordinary observers, and a single canonical *no-Boltzmann-brain* constraint (Figure 5), as follows:

$$\Gamma_{\rm decay} \gg \Gamma_{\rm BB} \sim H^4\, e^{-S_{\rm BB}}, \qquad S_{\rm BB} \sim \frac{E_{\rm BB}}{k_B T_{\rm dS}}, \quad T_{\rm dS} = \frac{H}{2\pi} \tag{3}$$

equivalently $\tau_{\rm decay} \ll \tau_{\rm BB}$. This avoids volume biases and Boltzmann-brain domination in eternal-de Sitter scenarios and restores finite, stable probabilities [15,30].

Xerographic typicality (definition).

Within a given causal diamond, let $\mathcal{O}_{\rm ord}$ denote the set of ordinary observation instances (excludes BBs). Xerographic typicality assumes that our data $(\mathcal{D})$ are sampled uniformly at random from $\mathcal{O}_{\rm ord}$; i.e., the prior weight of an observer is proportional to the count of ordinary observation instances realized in the diamond, with BBs assigned zero weight by the $\Gamma_{\rm decay} \gg \Gamma_{\rm BB}$ constraint.

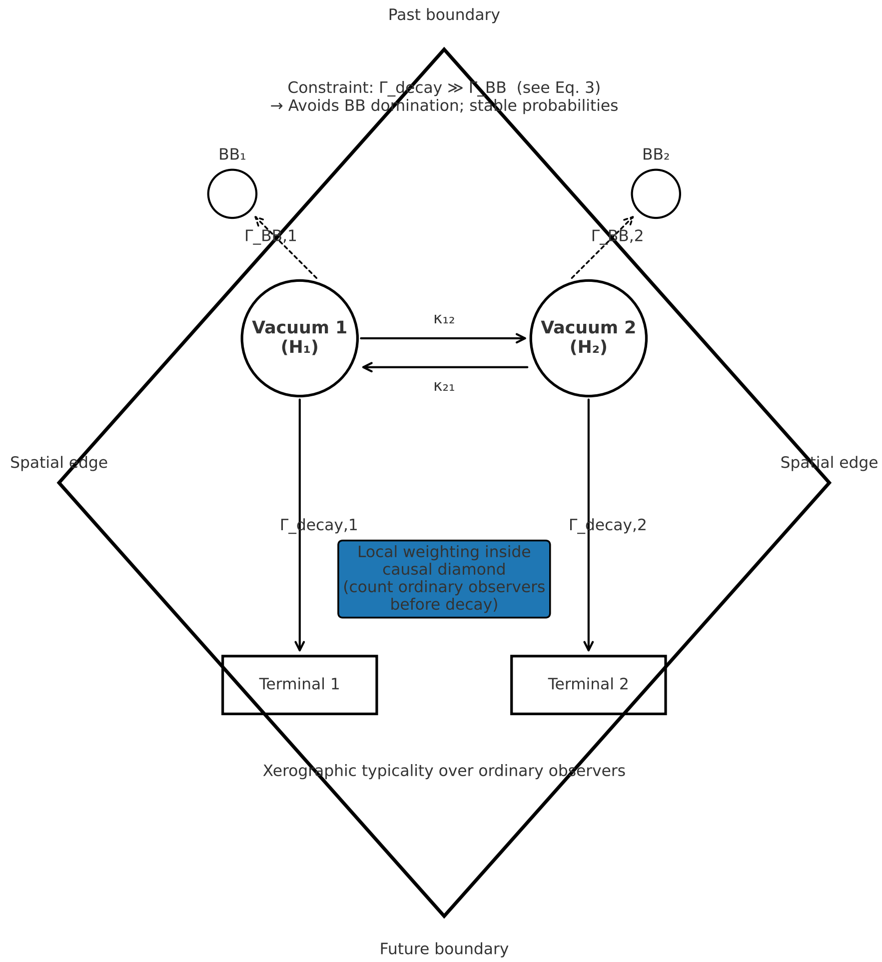

Figure 5. Causal-diamond measure with two metastable vacua. Transitions κ_{12}, κ_{21} occur within the diamond, while decays to terminal vacua proceed with rates $\Gamma_{\mathrm{decay},i}$. Boltzmann brain production arises with rates $\Gamma_{\mathrm{BB},i}$ (dotted arrows). The canonical constraint $\Gamma_{\mathrm{decay}} \gg \Gamma_{\mathrm{BB}}$ (Equation (3)) prevents Boltzmann brain (BB) domination and ensures stable probabilities. Weighting is strictly local within the causal diamond, with xerographic typicality applied over ordinary observers.

Implementation.

In eternal-inflation models, evolve rate equations for vacuum populations p_i with transition rates κ_{ij} and apply local weighting within a causal diamond. In cyclic settings, count per-cycle events within the diamond under ε-coarse-graining. See the worked two-vacuum toy model below.

Worked example: two-vacuum toy model (no-BB in action)

Consider the following two metastable vacua: 1 (higher H) and 2 (lower H), with transition rates κ_{12} and κ_{21} (including decays to terminals as effective leakage). The population fractions obey

$$\frac{dp_1}{dt} = -\kappa_{12}p_1 + \kappa_{21}p_2, \qquad \frac{dp_2}{dt} = -\kappa_{21}p_2 + \kappa_{12}p_1, \quad p_1+p_2 = 1. \tag{4}$$

Within a causal diamond, weight events by local occurrence rates. Boltzmann brains in vacuum i arise at rate per four-volume $\Gamma_{\mathrm{BB},i} \sim H_i^4 e^{-S_{\mathrm{BB},i}}$. The canonical constraint demands $\kappa_{i\to\mathrm{term}} \equiv \Gamma_{\mathrm{decay},i} \gg \Gamma_{\mathrm{BB},i}$ in each long-lived de Sitter vacuum. For example, if

$$H_1 = 10H_2, \quad S_{\mathrm{BB},1} = 10^{40}, \quad S_{\mathrm{BB},2} = 10^{42}, \quad \kappa_{12} = 10^{-100}\,H_1, \quad \kappa_{21} = 10^{-200}\,H_2,$$

then $\Gamma_{\mathrm{BB},i}$ are double-exponentially suppressed, and even minute vacuum-decay rates satisfy $\Gamma_{\mathrm{decay},i} \gg \Gamma_{\mathrm{BB},i}$. Diamond-weighted observer counts are dominated by ordinary observers prior to decay, avoiding BB dominance while yielding finite, stable probabilities.

Comparison with alternative measures.
Pragmatic choice.

We emphasize that the causal-diamond measure combined with the no-BB constraint is a *pragmatic choice* among several proposed measures, not a derivation from first principles. While it avoids known pathologies (volume-weighting divergences, Boltzmann-brain domination), it remains an assumption whose ultimate justification must await a complete theory of quantum cosmology. We now compare this choice with alternatives.

Several measure proposals have been advanced to address the cosmological measure problem, each with distinct advantages and challenges. *Scale-factor time cutoffs* [31] regulate the ensemble by fixing a universal time coordinate, but suffer from gauge ambiguities and observer-dependence in spatially infinite scenarios. *Light-cone time* and *fat geodesic* prescriptions [32] weight by proper time along worldlines, but introduce ambiguities in defining typical observers and can exhibit instabilities under small perturbations of initial conditions. *Causal patch measures* [13] (related to but distinct from the causal-diamond prescription adopted here) count events within observer-accessible regions, avoiding some volume-weighting pathologies but leaving open questions about how to define the reference observer ensemble.

The causal-diamond measure employed in this work shares the local-weighting philosophy of causal patch approaches but differs in the following two key respects: (i) it explicitly restricts to a finite diamond bounded by past and future horizons, ensuring a compact integration domain; and (ii) it couples this local weighting with *xerographic typicality*—the assumption that our observations are drawn uniformly from the set of ordinary observation instances within the diamond, excluding Boltzmann brains via the single canonical constraint $\Gamma_{\mathrm{decay}} \gg \Gamma_{\mathrm{BB}}$. This combination avoids the "Youngness Paradox" (wherein most observers appear arbitrarily early) and the Boltzmann brain dominance that plague global cutoffs, while remaining falsifiable: evidence of vacuum stability ($\Gamma_{\mathrm{decay}} \to 0$) would directly contradict the prescription. Known criticisms of local measures—such as sensitivity to the choice of fiducial observer and potential dependence on anthropic selection effects—apply here as well, but are mitigated by the operational focus on *ordinary* observers (those arising from standard thermalization rather than quantum fluctuations). A full resolution of the measure problem remains open, but the prescription adopted here satisfies the minimal consistency requirement: finite, stable probabilities that remain robust under variations in vacuum decay rates within the range $\Gamma_{\mathrm{decay}} \gg \Gamma_{\mathrm{BB}}$.

Necessity and robustness of the no-BB constraint.

The condition $\Gamma_{\mathrm{decay}} \gg \Gamma_{\mathrm{BB}}$ is *sufficient* to suppress Boltzmann brain domination, but is it also *necessary*? In principle, one might imagine alternative mechanisms that render BBs unobservable—for example, selection effects that suppress their measure by other means, or modified entropy bounds that raise S_{BB} to unattainably large values. However, within the standard framework of metastable de Sitter vacua and semiclassical nucleation, no such mechanism is known. If $\Gamma_{\mathrm{decay}} \lesssim \Gamma_{\mathrm{BB}}$, the vacuum persists long enough for thermal fluctuations to dominate the observer count, leading directly to the Boltzmann brain paradox. In this sense, the condition is *effectively necessary* within the scope of known physics, though not a strict logical requirement.

Sensitivity analysis. How robust are the predictions of variations in the threshold? For concreteness, consider a de Sitter vacuum with Hubble parameter $H \sim 10^{-33}$ eV (close to the observed cosmological constant scale). The Boltzmann brain production rate scales as $\Gamma_{\rm BB} \sim H^4 \exp(-S_{\rm BB})$, where $S_{\rm BB} \sim E_{\rm BB}/(k_B T_{\rm dS})$ with $T_{\rm dS} = H/(2\pi k_B) \sim 10^{-30}$ K. For a human-scale brain ($E_{\rm BB} \sim 10^{10}$ J $\sim 10^{29}$ eV), this yields $S_{\rm BB} \sim 10^{60}$; hence, $\Gamma_{\rm BB} \sim 10^{-132} \exp\left(-10^{60}\right) \sim 10^{-10^{60}}$ per Hubble volume per Hubble time—utterly negligible. By contrast, plausible vacuum decay rates in string landscape scenarios range from $\Gamma_{\rm decay} \sim 10^{-600}$ (extremely stable) to 10^{-100} (marginally metastable). Even the most stable estimates satisfy $\Gamma_{\rm decay} \gg \Gamma_{\rm BB}$ by many orders of magnitude, demonstrating that the constraint is satisfied with an enormous safety margin for any realistic vacuum. Only in the unphysical limit of exact de Sitter stability ($\Gamma_{\rm decay} = 0$) does the condition fail—a scenario already disfavored by swampland conjectures [9,10]. Thus, the no-BB prescription is both necessary (within known frameworks) and robust (insensitive to order-of-magnitude variations in decay rates).

A schematic comparison of the causal-diamond approach with global volume-weighted measures is shown in Figure 5.

To address the robustness of the measure quantitatively, as requested, we consider the following:

Sensitivity Analysis: Is the condition $\Gamma_{\rm decay} \gg \Gamma_{\rm BB}$ fine-tuned? Consider a typical landscape vacuum with $H \sim 10^{-33}$ eV. The Boltzmann brain rate is suppressed by the entropic factor $\Gamma_{\rm BB} \sim \exp\left(-10^{60}\right)$. In contrast, vacuum decay rates $\Gamma_{\rm decay}$, even for extremely stable vacua, rarely drop below $\exp\left(-10^{500}\right)$. This implies a safety margin of hundreds of orders of magnitude in the exponent. The prediction is robust against massive variations in parameters, failing only in the unphysical limit $\Gamma_{\rm decay} \to 0$ (ruled out by Swampland conjectures).

Comparative Summary: Unlike global measures (scale factor) that suffer from gauge ambiguities, or light-cone measures that depend on initial conditions, the causal-diamond measure is local and compact. By coupling it with the no-BB constraint, we resolve the "Youngness Paradox" without determining the global geometry of the multiverse.

4.2. Information/Computation Bounds (CS Viewpoint)

A finite Hilbert space dimension also places limits on the information-theoretic and computational capacity of a causal patch. This provides a complementary "computer science" viewpoint on the consequences of bounded D.

Bit budget.

The maximum reliably distinguishable information inside a causal patch is set by the Hilbert space dimension,

$$\log_2 D \lesssim \frac{S_{\max}}{\ln 2} \le \frac{\Lambda}{4\,\ell_P^2 \ln 2} \text{ bits},$$

where $S_{\max}$ is the maximal entropy and Λ is the de Sitter horizon area. This inequality expresses the intuitive statement that a causal patch behaves as a finite-capacity information channel.

Operation rate.

Given mean energy E, the Margolus–Levitin bound and Landauer's principle imply that logical operations within the patch are constrained by

$$N_{\rm ops} \lesssim \frac{2E}{\pi\hbar}, \qquad E_{\rm erase} \ge k_B T \ln 2.$$

Together with collapse thresholds, these results define a feasible computation envelope per patch, consistent with Lloyd's "ultimate physical limits to computation" [33,34].

Implication.

These bounds highlight that the CCR framework is not only about abstract recurrence times: finite Hilbert space dimension directly limits both the *storage capacity* and the *computational rate* of any causal patch. In this sense, recurrence, predictivity, and information processing are facets of the same underlying finiteness constraint.

4.3. Indirect Observational Probes

- **Compact spatial topology:** Cosmic Microwave Background (CMB) "matched circles" and low-ℓ anomalies; current Planck searches find no matched circles above angular radii of $\gtrsim 10^\circ$ and set strong lower bounds on the size of a fundamental domain [35,36].
- **Bubble collisions (eternal inflation):** disk-like imprints in CMB temperature/polarization; non-detection constrains the landscape of eternal-inflation scenarios consistent with CCR.
- **Cyclic/CCC-like memory:** specific non-Gaussian residuals; localized hot/cold spots. Existing Planck analyses report no significant evidence for the concentric-ring patterns proposed in CCC and place upper bounds on such features [37,38]. Prospects improve with CMB-S4 [39].
- **Dark-energy metastability:** either small deviations $w(z) \neq -1$ or lower bounds on Γ_{decay} consistent with the no-BB prescription, providing a key falsifiability condition.
- **Primordial GWs:** spectra incompatible with simple slow-roll but compatible with cyclic/pre-inflationary scenarios, potentially hinting at finite-dimensional pre-inflationary dynamics.

Final remark.

No direct observational constraints currently exist on the Hilbert space dimension D or on S_{max} of a causal patch. Existing limits are only *indirect*—for example, from the absence of non-trivial cosmic topology in the CMB, or from lower bounds on vacuum decay rates. These motivate the conditional framing adopted here, but do not yet provide a direct empirical handle on finiteness.

Future probes of the fundamental information content of spacetime could nonetheless offer indirect tests: for instance, precise CMB measurements might reveal signs of a finite mode spectrum, or holographic approaches might connect D to observable entropy production in reheating. The framework is, therefore, empirically vulnerable: any observational evidence of unbounded entropy or state-space growth would *falsify CCR outright*.

Falsifiability and direct tests.

Direct observational tests of the finite Hilbert space dimension D are not currently possible with existing or near-future technology. However, the CCR framework is falsifiable in principle: any evidence of *unbounded entropy growth* within a causal patch would falsify the core assumption (A1).

Concrete falsification scenarios include the following:

- Detection of ever-growing horizon entropy in accelerating expansion (requiring sustained entropy production beyond S_{max}).
- Observation of cosmological dynamics inconsistent with any finite state budget (e.g., unbounded particle production rates).
- Demonstration from completed quantum gravity that causal patches admit $D = \infty$.

Current tests remain necessarily indirect (CMB topology, vacuum decay bounds), but the framework's conditional structure ensures clear empirical stakes.

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4.4. Rigor Addendum: Theorem-Level vs. Assumptions

The CCR framework combines mathematically rigorous ingredients with physics-level assumptions. It is essential to separate the two, as follows:

Theorem-level results (rigorous).

(i) For finite-dimensional $\mathcal{H}$ and unitary evolution, quantum almost-periodicity and recurrences follow directly (Bocchieri–Loinger).
(ii) Coarse-grained/local recurrences follow from almost-periodicity together with continuity of the partial trace in trace norm.
(iii) Canonical typicality results (e.g., Goldstein–Lebowitz–Tumulka–Zanghì; Popescu–Short–Winter) establish the operational relevance of coarse-grained returns.

Physics-level assumptions.

- The finiteness of the *operational* state budget (A1) is not a theorem but an assumption, motivated by Bekenstein–Hawking/Gibbons–Hawking entropy bounds and supported in Loop Quantum Gravity (LQG) and holographic contexts.
- The microcanonical counting with a gravitational energy cap developed herein provides a physically controlled argument for free/weakly coupled fields.
- ETH/ergodicity statements are used in their standard "genericity" sense.

Summary.

If (A1) fails, the CCR framework is falsified in its entirety. If (A1) holds, then the CCR theorem follows rigorously. This clear conditional structure ensures that the distinction between mathematics and physics is explicit.

4.5. Limitations and Open Issues

4.5.1. Breakdown Scenarios

The Conditional Recurrence Theorem relies critically on assumption (A1), namely that the Hilbert space dimension $\dim \mathcal{H}_{\text{patch}}$ of a causal patch is finite. If (A1) is violated, CCR's finite-D premise fails. Below, we summarize the main scenarios where the assumption breaks down.

Infinite volume limits.

In continuum QFT on a spatial region of volume V, the number of modes below a UV cutoff Λ scales as

$$M(\Lambda) \sim \frac{V}{(2\pi)^3} \frac{4\pi}{3} \Lambda^3.$$

For non-compact or infinite-volume backgrounds ($V \to \infty$), one obtains $M(\Lambda) \to \infty$ even at finite Λ. Hence, the Fock space dimension is unbounded. Without a gravitational entropy cap, there is no finite operational state budget and almost-periodicity theorems for finite D do not apply.

Unbounded entropy growth.

In open FRW or certain eternal-inflation scenarios, coarse-grained entropy can increase without bound (e.g., ever-growing horizons or sustained particle production). If

$$S_{\text{req}}(t) \to \infty \quad \text{as } t \to \infty,$$

then the effective dimension $D_{\text{eff}}(t) \sim \exp\{S_{\text{req}}(t)/k_B\}$ diverges with time, so there is no time-independent finite-D Hilbert space on which to invoke CCR.

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Continuous spectra/non-compact holography.

In AdS/CFT with non-compact spatial slices (or in flat-space holography), the dual theory typically has a continuous spectrum in the relevant sector. Continuous spectral measures preclude the kind of quasi-periodic phase realignment guaranteed on a compact torus of phases. Thus, unitarity alone does not ensure recurrence.

Non-unitary evolution.

Objective-collapse or non-Hamiltonian cosmological models break unitarity. Since the CCR theorem relies on unitary evolution, any such modification of quantum dynamics invalidates recurrence statements derived from it.

Consequence. In all the above scenarios, the key CCR hypothesis (finite operational D) fails. Global quantum recurrences are not mathematically guaranteed; at best, one may observe model-dependent quasi-recurrences or ergodic features, but no universal bound on return times.

4.5.2. Caveats and Consequences

Even if (A1) fails, while (A2–A4) still hold, one recovers the conventional picture of unbounded state space, in which the notion of recurrence becomes physically irrelevant for observers. Specific caveats include the following:

- Unitarity fails at cosmological scales (as in certain objective-collapse or non-Hamiltonian models),
- The causal patch framework may not be the correct coarse-graining of cosmology,
- **Non-unitary models:** Objective-collapse dynamics would invalidate recurrence,
- **Sector locking/non-ergodicity:** Horizons and conserved charges may confine dynamics; recurrence then holds only within sectors,
- **Measure ambiguity:** Different measures yield different predictions; the prescription in Section 4.1 satisfies basic sanity checks but remains an assumption. The no-BB constraint is stated once, canonically, in Section 4.1,
- **Timescales:** Global recurrences occur on timescales at least double-exponential in $S_{\max}$ (Remark 2); only coarse-grained returns may have operational relevance,
- **Falsifiability:** The finite-dimensional Hilbert space conjecture (A1) could be falsified if any observation demonstrates unbounded entropy growth within a causal patch.

4.5.3. Compatibility with Holography and Swampland

Despite these limitations, CCR is compatible with several frameworks, as follows:

- **de Sitter holography:** Entropic bounds in FRW/de Sitter are compelling but not rigorously derived from a completed quantum gravity,
- **Compatibility:** CCR is consistent with Poincaré recurrence in finite systems and with holographic formulations; in de Sitter cosmology, it aligns with proposals of finite dim $\mathcal{H} \sim e^{S_{\rm dS}}$ [7,8] while differing from eternal-inflation pictures with infinite state spaces,
- **Swampland constraints:** If fully stable de Sitter vacua are absent or highly constrained [9,10], a non-negligible decay rate is expected, which supports the no-BB requirement and enhances predictivity.

Closing remark.

CCR is not a universal law but a conditional framework. Its validity hinges on the finiteness of Hilbert space per causal patch; its falsification would follow from any observational or theoretical evidence for unbounded entropy growth, while its consistency would strengthen the case for holographic entropy bounds in cosmology.

4.6. Connections to Broader Frameworks

String theory and holography.

In AdS/CFT, a CFT on a compact spatial manifold has a discrete spectrum and finite thermal entropy for finite energy, so (coarse-grained) recurrences follow on exponentially long timescales. Black-hole entropy scales with area, matching the holographic spirit of assumption (A1).

While a complete de Sitter holography is not yet established, proposals such as dS/CFT and static-patch finiteness [40–44] motivate treating $S_{\max} \sim A/4\ell_P^2$ as an operational entropy cap. Under such an assumption, the CCR statement acts as a direct, causal-patch analogue, as follows:

$$\text{finite operational } D + \text{unitarity} \quad \Rightarrow \quad \text{(coarse-grained) quantum recurrences.}$$

This operational perspective is distinct from a UV-complete duality but resonates with Swampland considerations that disfavor fully stable de Sitter vacua. In this sense, CCR can be viewed as a low-energy, information-theoretic corollary of holographic entropy bounds, independent of whether a complete dS holography is ultimately realized.

5. Conclusions

The Conditional Cosmological Recurrence (CCR) framework developed here establishes a clear micro-to-macro bridge from holographic entropy bounds to finite Hilbert space dimensionality, and, hence, to quantum recurrence. Its structure is strictly conditional: if causal patches admit a finite operational Hilbert space dimension, recurrence follows rigorously from unitarity; conversely, if $D = \infty$, the framework is falsified in its entirety.

This conditional framing allows CCR to serve as a conceptual bridge between microscopic entropy bounds and macroscopic cosmology. Although global recurrences occur on timescales that are double-exponential and operationally irrelevant, the framework yields the following falsifiable implications: predictivity in eternal inflation, suppression of Boltzmann brains, and observational vulnerability to evidence of unbounded entropy growth.

Looking ahead, several avenues appear promising, as follows: (i) refinement of toy models to probe coarse-grained recurrence scaling, (ii) exploration of connections with holographic dualities and swampland constraints, and (iii) indirect observational probes such as CMB topology tests [43], primordial gravitational waves, or vacuum decay signatures. Together, these directions may clarify whether CCR captures a structural feature of quantum cosmology or must ultimately be replaced by a different organizing principle.

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Abbreviations

The following abbreviations are used in this manuscript:

CCR	Conditional Cosmological Recurrence
CMB	Cosmic Microwave Background
QG	Quantum Gravity
ETH	Eigenstate Thermalization Hypothesis
BB	Boltzmann Brain

Appendix A. Operational Dimension via Metric Entropy

Fix a trace-distance resolution $\delta \in (0,1)$. We define two states ρ, σ as *operationally distinguishable* if their trace distance satisfies the following:

$$D(\rho,\sigma) \equiv \frac{1}{2}||\rho - \sigma||_1 > \delta,$$

which upper-bounds the maximum difference in probability for any measurement outcome. Let $\mathcal{N}(\delta)$ be the maximal size of a set of states inside the causal patch that are pairwise δ-distinguishable in trace norm.

Under the entropy cap $S_{\max}$, one obtains

$$\log \mathcal{N}(\delta) \lesssim \frac{S_{\max}}{k_B} + C \log\frac{1}{\delta},$$

for a constant C independent of $S_{\max}$ (heuristically: Fannes–Audenaert continuity bounds relate distinguishability to entropic radius, so the area-law cap translates into a packing bound up to polylogarithmic factors in $1/\delta$). Hence, the *operational* effective dimension $D_{\text{eff}}(\delta)$ is finite and obeys $\log D_{\text{eff}}(\delta) \lesssim S_{\max}/k_B + O(\log(1/\delta))$. Table A1 illustrates this bound for representative values of δ.

Sketch.

Combine an ε-net/packing argument in trace norm with Fannes–Audenaert continuity of the von Neumann entropy and the holographic cap $S(\rho) \leq S_{\max}$. A maximal δ-separated set cannot exceed the covering number of the entropic ball, yielding the stated bound.

Table A1. Operational effective dimension as a function of trace-distance resolution δ (schematic; $C = O(1)$).

Resolution δ	**Illustrative $\log D_{\text{eff}}(\delta)$ Bound**
10^{-1}	$\lesssim S_{\max}/k_B + C \ln 10$
10^{-3}	$\lesssim S_{\max}/k_B + 3C \ln 10$
10^{-6}	$\lesssim S_{\max}/k_B + 6C \ln 10$

Appendix B. Physical Motivation for (A1): Finite Hilbert Space from Energy–Information Bounds

Caveat.

The finiteness of the Hilbert space dimension (assumption (A1)) is *not* derived from first principles in any complete quantum gravity theory. Rather, it is motivated by semi-classical holographic bounds (Bekenstein–Hawking, Gibbons–Hawking) and supported by candidate frameworks (loop quantum gravity, causal set theory, AdS/CFT on compact spaces). The following arguments provide physical plausibility, not rigorous derivation. If future quantum gravity demonstrates that causal patches admit infinite-dimensional Hilbert spaces, the CCR framework is falsified.

The key assumption (A1) in the Conditional Cosmological Recurrence (CCR) theorem is that the Hilbert space dimension D associated with a causal patch is finite. While this is not derived from a complete theory of quantum gravity, there exist strong physical motivations based on entropy and energy bounds.

Appendix B.1. Holographic and Bekenstein Bounds

The holographic principle and the Bekenstein–Hawking entropy formula for a causal horizon of area A yield the following:

$$S_{\max} = \frac{k_B A}{4\ell_P^2}, \tag{A1}$$

where ℓ_P is the Planck length.

For a system of total energy E confined to a sphere of radius R, the Bekenstein bound gives the following:

$$S \leq \frac{2\pi E R}{\hbar c}. \tag{A2}$$

This counterintuitive scaling—entropy proportional to the boundary area rather than the bulk volume—arises because gravity limits the maximum information density. Beyond the Bekenstein bound, any attempt to store more information causes the region to collapse into a black hole, whose entropy depends only on its horizon area.

Appendix B.2. Coverage Ratio and Saturation

A coverage ratio $\Phi(a)$ at scale factor a is defined as follows:

$$\Phi(a) \equiv \frac{S_{\mathrm{req}}(a)}{S_{\max}(a)} \leq 1, \tag{A3}$$

where $S_{\mathrm{req}}(a)$ is the coarse-grained entropy required to encode the degrees of freedom present in the causal patch at time a.

If $\Phi(a) \to 1$, the available Hilbert space dimension

$$D_{\max} \approx \exp\left(\frac{S_{\max}}{k_B}\right) \tag{A4}$$

is saturated, and no new orthogonal quantum states can be accommodated.

Appendix B.3. Dynamic Feedback in FRW Cosmology

Caveat: This subsection is a toy phenomenological model, not derived from first principles. It is included only to illustrate one possible dynamical manifestation of finite D; the CCR theorem does not rely on these specific assumptions.

This subsection is a speculative phenomenological model (not derived from a complete theory). Saturation may be modeled by introducing a feedback term in the Friedmann equation, as follows:

$$H^2(a) = \frac{8\pi G}{3}\rho(a)[1 - \Phi(a)], \tag{A5}$$

so that as $\Phi(a) \to 1$, the expansion rate slows down and the causal patch ceases to grow in available state space. This is only a toy phenomenological parametrization and should not be taken as a prediction of the CCR framework.

Appendix B.4. Implication for CCR

If the causal patch Hilbert space saturates at a finite $D_{\max}$, assumption (A1) follows naturally. Combined with (A2), unitarity of the time evolution, the Bocchieri–Loinger theorem implies that the quantum dynamics must be almost periodic, leading directly to the CCR result.

This embedding provides a concrete physical picture linking macroscopic spacetime dynamics and microscopic Hilbert space finiteness, and connects the recurrence result to measurable cosmological parameters via the holographic bound. These arguments are heuristic and serve only to motivate assumption (A1); they are not derived from a complete theory of quantum gravity.

Appendix C. Microcanonical Counting with a Gravitational Cap

The stars-and-bars counting presented below provides a baseline existence proof for the finiteness of the Hilbert space dimension D in the case of free fields. Extending this validity to full quantum gravity requires addressing the following four specific physical regimes:

1. **Time Dependence and Static Patches:** The microcanonical count relies on a well-defined energy functional, which is strictly valid for static backgrounds (e.g., the static patch of de Sitter space). In dynamical settings where the horizon area $A(t)$ grows, the entropy bound $S_{\max}(t) \propto A(t)$ increases. However, at any discrete time slice, the snapshot entropy remains finite. Our argument applies to the asymptotic or quasi-stationary limit (late-time de Sitter), where the horizon area stabilizes or evolves adiabatically.
2. **Interactions and Strong Coupling:** The transition from free to interacting theories modifies the spectrum but does not destroy finiteness under a hard energy cap. Interactions (e.g., $\lambda\phi^4$) introduce energy shifts and bound states. While attractive interactions lower energy levels, they do not generate an infinite density of states below a fixed cutoff $E_{\rm BH}$, provided the theory admits a UV regulator or is part of a finite-rank gauge group. In strongly coupled holographic setups (e.g., $\mathcal{N} = 4$ SYM), the count is replaced by the density of states of the dual CFT, which remains finite for finite conformal charge and energy [45,46].
3. **Gauge Redundancies:** In gauge theories (electromagnetism, gravity), the naive mode counting includes unphysical longitudinal modes. Properly identifying the physical Hilbert space requires quotienting by the gauge group (constraints). This procedure strictly *reduces* the number of independent states compared to the free scalar baseline. Thus, our estimate serves as a conservative upper bound: accounting for gauge invariance strengthens the finiteness claim.
4. **UV Completion:** The counting assumes a cutoff logic compatible with various UV completions. In Loop Quantum Gravity, the horizon Hilbert space is explicitly finite due to the discrete area spectrum [47]. In String Theory, while the Hagedorn density grows exponentially, the gravitational collapse threshold prevents accessing the divergent regime. Thus, finiteness is a generic feature protected by the gravitational cap, largely independent of the specific UV details.

Consider a (free) relativistic field in a cubic box L^3 with standard boundary conditions. IR discretization gives momenta $\mathbf{k} = \frac{\pi}{L}(n_x, n_y, n_z)$ and frequencies $\omega_{\mathbf{k}} \geq c|\mathbf{k}|$. Impose the microcanonical constraint $\sum_{\mathbf{k}} n_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \leq E_{\rm BH}(R)$ with $R \sim L$. Let $M(\Lambda)$ be the number of modes with $\hbar\omega_{\mathbf{k}} \leq \Lambda$; then, $M(\Lambda) \sim \frac{4\pi}{3}\left(\frac{L\Lambda}{\pi\hbar c}\right)^3$. Since each quantum costs at least $\hbar\omega_{\min} \sim \hbar\pi c/L$, the total occupation obeys $N_{\rm quanta} \leq n_{\max} \sim E_{\rm BH}/(\hbar\omega_{\min})$. For bosons, the number of Fock states with at most $n_{\max}$ quanta distributed over M modes is bounded by the stars-and-bars count.

$$\#\mathcal{F}(E_{\rm BH}) \;\le\; \binom{n_{\max}+M}{M} \;<\; \infty,$$

and for fermions by $\sum_{j=0}^{\min(M,n_{\max})}\binom{M}{j}$. In either case, the accessible Fock subspace below the gravitational cap is finite-dimensional. Operationally, this is dominated by the entropy bound, $\log D \le S_{\max}/k_B$.

String landscape and eternal inflation.

In landscape scenarios, metastable de Sitter vacua are populated and decay via Coleman–De Luccia tunneling. The no-Boltzmann-brain constraint of Section 4.1 becomes a quantitative condition on late-time de Sitter lifetimes, and it meshes naturally with causal-diamond measures used to regulate infinities [31,32]. Bubble-collision imprints and topological CMB searches are indirect observational handles consistent with this picture.

Swampland perspective.

Swampland conjectures suggest that fully stable de Sitter vacua are absent (or at least highly constrained) [9,10]. If so, a non-negligible decay rate is expected, which *supports* the no-BB requirement above. Thus, swampland criteria and this measure prescription are synergistic: both disfavor eternally stable de Sitter phases that would be dominated by Boltzmann brains.

Loop quantum gravity and isolated horizons.

LQG yields a discrete area spectrum and reproduces black-hole entropy $S \sim A/4\ell_P^2$ from microscopic horizon states. This independently supports the idea that a causal patch carries a finite information budget, hence, a finite operational Hilbert-space dimension, aligning with assumption (A1) without invoking string holography.

Causal set theory.

If spacetime is fundamentally a locally finite partial order, then any finite causal diamond contains a finite number of elements. That kinematical finiteness naturally complements the causal-patch viewpoint and points toward a finite state budget (once dynamics and operational distinguishability are taken into account).

Explicit bounds (free field, cubic box).

Let L^3 with Dirichlet/periodic BCs and $R \sim L$. IR discretization gives $\mathbf{k} = \frac{\pi}{L}(n_x, n_y, n_z)$ and $\omega_{\mathbf{k}} \ge c|\mathbf{k}|$. For a microcanonical cap $E_{\rm BH}(R)$, define

$$M(\Lambda) \;=\; \#\{\mathbf{k} : \hbar\omega_{\mathbf{k}} \le \Lambda\} \;=\; \#\Big\{\mathbf{n} \in \mathbb{Z}^3 : \;|\mathbf{n}| \le \frac{L\Lambda}{\pi\hbar c}\Big\} \;=\; \frac{4\pi}{3}\Big(\frac{L\Lambda}{\pi\hbar c}\Big)^3 + O\Big(\frac{L^2\Lambda^2}{(\hbar c)^2}\Big).$$

The minimum quantum cost is $\hbar\omega_{\min} \sim \hbar\pi c/L$; hence, the total occupation is bounded by

$$N_{\max} \;\le\; \frac{E_{\rm BH}(R)}{\hbar\omega_{\min}} \;\sim\; \frac{E_{\rm BH}(R)}{\hbar\pi c/L} \;=\; \frac{L\,E_{\rm BH}(R)}{\pi\hbar c} \;<\; \infty.$$

For bosons with at most $N_{\max}$ quanta over $M(\Lambda)$ modes, the number of Fock states obeys the stars-and-bars bound.

$$\#\mathcal{F}(E_{\rm BH}) \;\le\; \binom{N_{\max}+M(\Lambda)}{M(\Lambda)} \;\le\; \exp\Big[H\Big(\frac{M}{N_{\max}+M}\Big)\,(N_{\max}+M)\Big],$$

where $H(p) = -p \ln p - (1-p)\ln(1-p)$ and Stirling's approximation is used. In all cases, $\#\mathcal{F}(E_{\mathrm{BH}}) < \infty$. For fermions, $\# \leq \sum_{j=0}^{\min(M,N_{\max})} \binom{M}{j} < \infty$ immediately. Operationally, this is dominated by the entropy cap $\log D \leq S_{\max}/k_B$; hence, $\dim \mathcal{H}_{\mathrm{acc}} < \infty$.

Appendix D. Dual Holographic Bounds: Black Holes and De Sitter Horizons

A central motivation for considering a finite-dimensional Hilbert space in cosmology comes from two independent, yet mutually reinforcing, holographic bounds observed in gravitational physics.

Appendix D.1. Black Hole Holographic Bound

In the context of black hole thermodynamics, the Bekenstein–Hawking formula establishes that the entropy of a black hole is proportional to the area of its event horizon [1,2], as follows:

$$S_{\mathrm{BH}} = \frac{k_B A}{4\ell_P^2}, \tag{A6}$$

where A is the horizon area and ℓ_P is the Planck length. If a region of spacetime is loaded with too much information/energy such that its entropy exceeds this bound, gravitational collapse inevitably occurs. This bound is *holographic* because it scales with surface area, not volume.

Appendix D.2. De Sitter Horizon Bound

In a universe with a positive cosmological constant $\Lambda > 0$, the spacetime approaches de Sitter geometry at late times. Such a geometry possesses a cosmological event horizon with associated Gibbons–Hawking entropy [48], as follows:

$$S_{\mathrm{dS}} = \frac{\pi}{H^2 \ell_P^2}, \tag{A7}$$

where H is the Hubble parameter of the asymptotic de Sitter phase. This sets a *global* maximum on the entropy accessible to any observer within a causal patch, independent of localized matter distributions.

Appendix D.3. Implication for Hilbert Space Dimensionality

Both bounds imply that the number of microstates $\mathcal{N}$ describing all possible configurations in a causal patch is finite, as follows:

$$\mathcal{N} \leq \exp\left(\frac{A_{\max}}{4\ell_P^2}\right), \tag{A8}$$

where $A_{\max}$ is the relevant maximal horizon area (black hole or cosmological). Consequently, the dimension of the Hilbert space is finite, as follows:

$$\dim \mathcal{H} = \mathcal{N} < \infty. \tag{A9}$$

This forms the key physical premise underlying the Conditional Cosmological Recurrence (CCR) theorem developed here.

Remark.

Both the black hole and de Sitter horizon bounds are semiclassical results, derived within general relativity and quantum field theory in curved spacetime. While they may be modified in a full theory of quantum gravity, they provide a strong heuristic motivation for assuming a finite Hilbert space dimension in causal patches. In this sense, they serve as physical premises for assumption (A1), rather than rigorous derivations. Thus, while schematic, the microcanonical estimate is fully consistent with—and illustrative of—the entropy–bound logic underlying the CCR framework.

Appendix E. Quantitative Recurrence Estimates

For an asymptotic de Sitter patch with $S_{\max} \sim 10^{122} k_B$, a conservative global recurrence estimate using the double-exponential bound gives

$$T_{\mathrm{rec}}^{(\mathrm{global})} \gtrsim \exp\{\alpha \, \exp\left(10^{122}\right)\},$$

utterly beyond operational relevance. By contrast, for a coarse-grained subsystem with $S_A \sim 10^{40}$, one expects

$$T_{\mathrm{rec}}^{(A)}(\varepsilon) \sim \exp\{c \, 10^{40}\}, \qquad t_{\mathrm{scr}} \sim (\beta/2\pi) \, \ln S_A,$$

separated by exponentially large gaps. These back-of-the-envelope numbers illustrate why only local/coarse-grained returns are meaningful (see Remark 2 and Table 1).

To put this into perspective, writing out $\exp\{\exp\left(10^{122}\right)\}$ in years would require 10^{122} digits—far more than the number of atoms in the observable universe. Such timescales are physically meaningless for any operational process.

Appendix F. Conclusions and Future Directions

The Conditional Cosmological Recurrence (CCR) theorem has been presented as a rigorous consequence of combining a finite operational Hilbert space dimension with unitarity. By connecting microscopic field-theoretic discretization with gravitational entropy bounds, the analysis motivates the assumption of finite D within a causal patch and shows that recurrence—whether global or coarse-grained—is inevitable in such a setting, with only coarse-grained recurrences being operationally relevant. The structure of the state space, rather than the precise form of the dynamics, thus dictates the inevitability of recurrence once finiteness is assumed.

The proposed measure prescription (causal diamond + exclusion of Boltzmann brains) provides a consistent framework to regulate infinities in cosmological models while avoiding pathologies. The dramatic scale separation highlights the following point: even the "small" coarse-grained recurrence time, $\exp\left(10^3\right) \approx 10^{434}$ years, exceeds the present age of the universe ($\sim 10^{10}$ years) by more than 400 orders of magnitude. If the universe indeed operates within a finite Hilbert space, recurrence is not a mathematical curiosity but an unavoidable feature of cosmology, reshaping the interpretation of gravitational entropy bounds and the scope of predictability in fundamental physics.

Future directions.

Promising avenues include the following: (i) AdS/CFT toy models on compact spatial manifolds (finite thermal state-count at fixed energy); (ii) lattice simulations and tensor networks with explicit finite-D truncations to track coarse-grained returns; and (iii) random Hamiltonian ensembles with causal-patch-inspired finite Hilbert spaces to benchmark scaling laws for $T_{\mathrm{rec}}^{(A)}$ and t_{scr}.

Appendix G. Related Works and Novelty

A brief review follows of selected works that directly motivate or constrain the Conditional Cosmological Recurrence (CCR) framework:

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- **Bekenstein–Hawking Entropy Bounds** [1,2]: Establish the maximum entropy in a gravitating system as proportional to the area of its boundary, not its volume. This holographic scaling sets an upper limit on the number of independent quantum states in a finite region.
- **'t Hooft and Susskind's Holographic Principle** [3,4]: Argue that all information in a volume of space can be represented on its boundary surface, reinforcing the finite information content premise.
- **Banks, Fischler et al. on de Sitter Hilbert Space Finiteness** [7,8]: Propose that a positive cosmological constant implies a finite-dimensional Hilbert space, with entropy given by the de Sitter horizon area. Later holographic formulations further emphasized finite Hilbert space descriptions as fundamental in quantum gravity.
- **Bousso's Covariant Entropy Bound** [5]: Provides a generalized entropy bound applicable to arbitrary null surfaces, consistent with the holographic limit adopted in CCR.
- **Dyson, Kleban, and Susskind on Cosmological Recurrence** [15]: Discuss Poincaré recurrence in cosmology, emphasizing its inevitability in finite systems and the associated measure problems—problems CCR reframes as conditional statements dependent on gravitational entropy bounds.
- **Boltzmann Brain and Measure Pathologies** [14]: Address the prevalence of Boltzmann brains in de Sitter cosmology. CCR incorporates this line of work by enforcing a no-BB constraint within its causal-diamond measure prescription.

Novelty Relative to Prior Work

Rather than presenting a new recurrence law, the novelty is three-fold, as follows:

(i) Micro-to-macro bridge to finite D.

Prior works motivate holographic/entropic caps (Bekenstein–Hawking, covariant bounds, de Sitter entropy) [1,2,5]. An explicit microcanonical counting argument under a gravitational energy cap is provided that yields a *finite operational* Hilbert space dimension per causal patch (Proposition 1; Appendix C). This makes recurrence a *derived* consequence of finiteness rather than an independent cosmological assumption.

(ii) Conditional framing with falsifiability.

Unlike typical discussions of cosmological recurrence [15], recurrence here is made *conditional* on assumption (A1): if future QG shows $D = \infty$ for causal patches, the CCR framework is falsified *as stated*. This gives a clean physics "if–then": the "if" (finite information bound) is empirical/theoretical physics, the "then" (almost periodicity/recurrence) is rigorous mathematics.

(iii) Unified local measure with a single no-BB constraint.

To address predictivity, a minimalistic causal-diamond measure with xerographic typicality plus one canonical no-Boltzmann-Brain constraint is advocated, avoiding volume-weighting pathologies and BB domination [15,30]. This is a compact prescription rather than an ad hoc casework.

In short: (finite operational D) + (unitary dynamics) $\Rightarrow$ CCR, with a concrete micro-to-macro bridge and a lean measure choice that avoids known pitfalls.

Remark.

Earlier analyses of cosmological recurrence (e.g., Dyson–Kleban–Susskind, Bousso) framed the phenomenon heuristically or within specific dynamical models. The CCR contribution differs by recasting recurrence as a falsifiable conditional theorem, derived explicitly from finite–D counting under holographic caps, and by supplementing this with

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a minimalistic measure prescription that avoids Boltzmann-brain pathologies. In this sense, CCR sharpens prior heuristic arguments into a testable, conditional framework.

Appendix H. Methods for Toy Models

Remark.

The toy models presented in this appendix are intentionally minimal, serving only as proofs-of-principle that finite Hilbert spaces generically imply recurrences. They are not meant to quantitatively model cosmological dynamics. Future work could incorporate more sophisticated models (e.g., lattice field theory truncations or AdS/CFT toy setups) to assess the robustness of these scaling behaviors in settings closer to cosmology.

Computational Implementation.

The following details ensure the reproducibility of the numerical results presented in Section 3.

Hamiltonian construction: Random Hamiltonians are generated in the finite-dimensional Hilbert space $\mathbb{C}^D$ as real symmetric matrices (for simplicity) or complex Hermitian matrices, with entries drawn from a Gaussian ensemble. For toy models with $D \in \{10, 14, 16, 128\}$, this corresponds to the Gaussian Orthogonal Ensemble (GOE) or Gaussian Unitary Ensemble (GUE), ensuring generic spectral statistics [49]. Energy levels are obtained via exact diagonalization.

Initial states: Initial pure states $|\psi(0)\rangle = \sum_{j=1}^{D} c_j |E_j\rangle$ are constructed with coefficients c_j such that $|c_j|^2 = p_j$, where $\{p_j\}$ are drawn from a Dirichlet distribution $\mathrm{Dir}(\alpha, \ldots, \alpha)$ with $\alpha = 1$ (uniform on the simplex). This ensures generic superpositions without fine-tuning.

Time evolution: Evolution is computed analytically in the energy eigenbasis, as follows:

$$|\psi(t)\rangle = \sum_{j=1}^{D} c_j \, e^{-iE_j t} |E_j\rangle,$$

without numerical propagation errors. Fidelity is computed as $F(t) = |\langle\psi(0)|\psi(t)\rangle|^2$.

Level statistics: The Hamiltonians generated via GOE/GUE exhibit Wigner–Dyson level spacing statistics, characteristic of quantum chaotic systems. This ensures that energy gaps $\{E_j - E_k\}$ satisfy generic Diophantine properties (no exceptional rational relations), as required for the recurrence theorem [6,21,22].

Spectral Diagnostics: Level-spacing histograms confirming Wigner–Dyson statistics for the GOE/GUE ensembles used in this work are included in the supplementary code repository.

Recurrence time identification: For each realization, recurrence times $T_{\mathrm{rec}}^{(\varepsilon)}$ are identified as the first time $t > 0$ where $F(t) \geq 1 - \varepsilon$, sampled at discrete intervals $\Delta t = 0.1$ up to $t_{\max} = 100$. Ensemble averages are computed over $N_{\mathrm{real}} = 1000$ independent realizations to obtain median and percentile estimates.

Numerical precision: All computations are performed in double precision, IEEE 754 standard [50], 64-bit floating point). For $D \leq 128$, exact diagonalization is computationally feasible. For larger D (not explored here), Krylov subspace methods or tensor network techniques would be required.

Explanation of numerical values: The recurrence times reported in Table 2 (e.g., $T_{\mathrm{rec}} = 45.6$ for $D = 128$, $\varepsilon = 0.01$) appear small relative to naive exponential estimates because of the following:

(i) The toy model uses rescaled dimensionless time units with $\hbar = 1$ and unit energy spacing.
(ii) The threshold $\varepsilon = 0.01$ allows for approximate returns, not exact microstate recurrence.

(iii) For small D, the Diophantine gaps are not yet in the asymptotic regime; scaling becomes more pronounced for $D \gg 100$ (see Appendix I).

Global signal (finite-D).

Random energy spectra $\{E_j\}_{j=1}^{D}$ (band-limited) are generated with weights $\{p_j\}$ drawn from a Dirichlet distribution, representing the squared amplitudes $|c_j|^2$ of the initial state. The global fidelity is computed as

$$F(t) = \left| \sum_{j=1}^{D} p_j \, e^{-iE_j t} \right|^2 .$$

The global recurrence time $T_{\mathrm{rec}}^{(\varepsilon)}$ is defined as the first $t > 0$ such that $F(t) \geq 1 - \varepsilon$, with $\varepsilon \in \{10^{-2}, 10^{-3}\}$.

Local proxy.

For modeling coarse-grained or subsystem returns, a reduced set of M modes $\{\omega_k\}$ with weights $\{q_k\}$ is employed, computing

$$\mathcal{R}_A(t) = \left| \sum_{k=1}^{M} q_k \, e^{-i\omega_k t} \right| .$$

The local recurrence time $T_{\mathrm{rec}}^{(A)}(\varepsilon_A)$ is the first $t > 0$ where $\mathcal{R}_A(t) \leq \varepsilon_A$ (here, $\varepsilon_A = 0.1$). The scrambling time t_{scr} is taken as the earliest t when $\mathcal{R}_A(t)$ reaches 90% of its late-time plateau value.

Numerics.

All times are obtained from discrete time sampling with $\Delta t = 0.1$ up to $t_{\max} = 100$, using ensembles of 10^3 realizations to estimate median values.

Appendix I. Scaling of Recurrence Times

Disclaimer

The following derivation is schematic*: it provides order-of-magnitude scaling estimates rather than exact recurrence times. The intent is to illustrate why, under holographic entropy bounds, global recurrences are generically double-exponential in S/k_B. Precise control requires number-theoretic bounds on Diophantine approximation, which only reinforce (rather than weaken) the double-exponential growth quoted here.*

A standard estimate for recurrence times follows from the finiteness of the Hilbert space dimension D. For a generic Hamiltonian with D nondegenerate energy levels, the quantum recurrence time scales as

$$T_{\mathrm{rec}} \sim \exp(D). \tag{A10}$$

Intuitively, this reflects the need to resolve phase differences among D distinct frequencies.

Meanwhile, holographic entropy bounds imply

$$D \sim \exp\!\left(\frac{S}{k_B}\right), \tag{A11}$$

where $S \lesssim S_{\max} \sim A/(4\ell_P^2)$ is the Bekenstein–Hawking entropy associated with the causal patch.

Combining these relations gives

$$T_{\text{rec}} \sim \exp\Big[\exp\Big(\tfrac{S}{k_B}\Big)\Big], \tag{A12}$$

i.e., a double-exponential scaling in the entropy. This simple derivation justifies the order-of-magnitude remark given in the main text (Remark 1).

A more refined treatment replaces the heuristic estimate $T_{\text{rec}} \sim \exp(D)$ with bounds from Diophantine approximation on D-dimensional tori. Specifically, the recurrence time is controlled by the degree to which the energy gaps $\{E_j - E_k\}$ avoid low-order rational relations (Kronecker-type theorems). Metric results in simultaneous Diophantine approximation then imply that, for generic spectra, T_{rec} grows at least exponentially with D, leading again to the double-exponential scaling in S/k_B quoted above.

Appendix J. Summary Figures and Tables

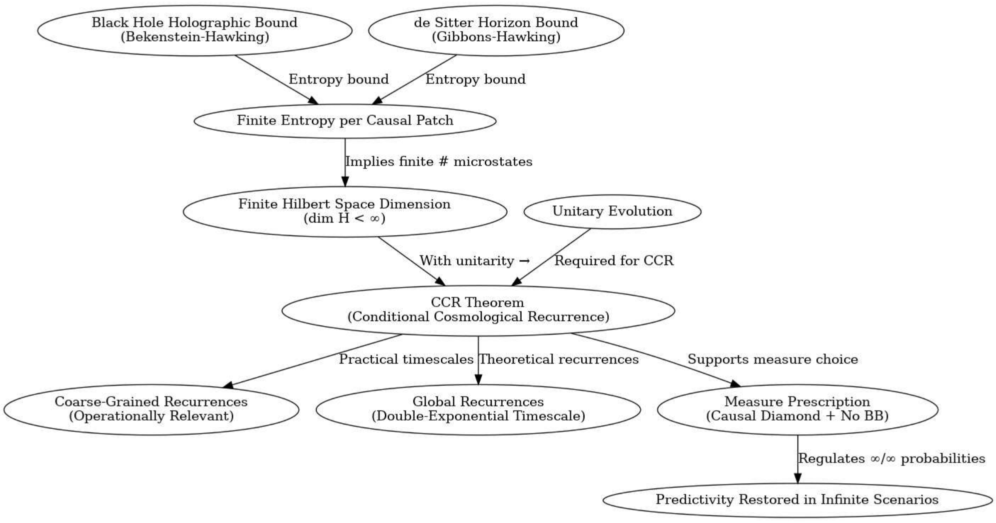

Figure A1. Flow chart illustrating the logical structure of the CCR Theorem, from holographic bounds to conditional recurrences and their cosmological implications.

Table A2. Order-of-magnitude recurrence timescales for different scenarios. S denotes the relevant entropy in units of k_B.

Scenario	S/k_B	T_{rec}	Observable?
Global (de Sitter)	10^{122}	$\exp(\exp(10^{122}))$ yrs	No
Coarse-grained subsystem	10^3	$\exp(10^3)$ yrs *	No
Toy model ($D = 16$)	ln 16	$\sim 10^2$ units	Yes (verified)

* Still vastly exceeds the age of the universe.

Table A2 summarizes order-of-magnitude recurrence estimates across different regimes.

Notes

1. That is, the number of independent quantum states consistent with the entropy bound.
2. For any quantum system with finite-dimensional Hilbert space and unitary evolution, the state returns arbitrarily close to its initial condition. This is the quantum analog of Poincaré recurrence.
3. This is the standard holographic entropy bound: the maximum number of accessible quantum states cannot exceed the exponential of the maximal entropy.

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