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The Structure of Parafermion Vertex Operator Algebras: General Case

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Abstract The structure of the parafermion vertex operator algebra associated to an integrable highest weight module for any affine Kac-Moody algebra is studied. In particular, a set of generators for this algebra has been determined.

1 Introduction

This paper is a continuation of our study of the parafermion vertex operator algebra associated to an integrable highest weight module for an arbitrary affine Kac-Moody algebra. We determine a set of generators for these algebras. If the affine Kac-Moody algebra is $A_1^{(1)}$, this result was obtained previously in (6).

The parafermion algebra was first studied in (27) in the context of conformal field theory. It was clarified in (7) that the parafermion algebras are essentially the Z-algebras introduced and studied earlier in (19; 20; 21) in the process of studying the representation theory for the affine Kac-Moody Lie algebras. As it proved in (7), the parafermion algebras generate certain generalized vertex operator algebras. The partition functions for the parafermion conformal field theory have been given in (13) and (12) in connection with the partition functions associated to the integrable representations for the affine Kac-Moody Lie algebras. We refer the reader to (1; 12; 13; 15; 24; 26; 27) for various aspects of parafermion conformal field theory.

The parafermion vertex operator algebras which have roots in the parafermion conformal field theory are realized as commutants of the Heisenberg vertex operator subalgebras in the vertex operator algebras associated to the integrable highest

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weight modules for the affine Kac-Moody Lie algebra (11; 14; 18). More precisely, let $L(k,0)$ be the level k integrable highest weight module for affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ associated to a finite dimensional simple Lie algebra $\mathfrak{g}$. Then $L(k,0)$ has a vertex operator subalgebra $M_{\widehat{\mathfrak{h}}}(k,0)$ generated by the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The commutant $K(\mathfrak{g},k)$ of $M_{\widehat{\mathfrak{h}}}(k,0)$ in $L(k,0)$ is called the parafermion vertex operator algebra. Although the parafermion field theory has been studied for more than two decades, the mathematical investigation of the parafermion vertex operator algebras have been limited by a lack of understanding of the structural theory of these algebras. The goals of this paper and (4; 5; 6) are to alleviate this situation.

More importantly, it is widely believed that $K(\mathfrak{g},k)$ should give a new class of rational, C_2-cofinite vertex operator algebras although this can only been proved in the case $\mathfrak{g} = sl_2$ and $k \leq 6$ (5). Most well known vertex operator algebras such as lattice vertex operator algebras (2; 3; 10), the affine vertex operator algebras (7; 11; 23) and the Virasoro vertex operator algebras (11; 25) can be understood well using the underline lattices or Lie algebras. Unfortunately, the structures of parafermion vertex operator algebras with weight one subspaces being zero are much more complicated. It seems that a determination of a set of generators is the first step in understanding parafermion vertex operator algebras and their representation theory.

It is well known that $L(k,0)$ is the irreducible quotient of the generalized Verma module $V(k,0)$ (see Sect. 2). So the structure of $L(k,0)$ can be determined by studying the maximal submodule of $V(k,0)$ or the maximal ideal of $V(k,0)$ which is also a vertex operator algebra. The same idea can also be applied to the study of parafermion vertex operator algebras. In fact, the Heisenberg vertex operator algebra $M_{\widehat{\mathfrak{h}}}(k,0)$ is also a subalgebra of $V(k,0)$ and the parafermion vertex operator algebra $K(\mathfrak{g},k)$ is the simple quotient of the commutant $N(\mathfrak{g},k)$ of $M_{\widehat{\mathfrak{h}}}(k,0)$ in $V(k,0)$. The main part of this paper is to determine a set of generators for $V(k,0)(0)$ which is the weight zero subspace of $V(k,0)$ under the action of the Cartan subalgebra $\mathfrak{h}$. The generators for $N(\mathfrak{g},k)$ and $K(\mathfrak{g},k)$ will be found easily then. Also, the maximal ideal of $N(\mathfrak{g},k)$ is generated by one vector. This result is similar to that for the maximal ideal of $V(k,0)$.

It is worth pointing out that the structure theory for the parafermion vertex operator algebra is similar to the structure theory for the finite dimensional Lie algebras or Kac-Moody Lie algebras. The building block of the Kac-Moody Lie algebras is the 3-dimensional simple Lie algebra sl_2 associated to any real root. The generator results for the parafermion vertex operator algebras given in this paper and (6) show that the parafermion vertex operator algebras associated to the affine Lie algebra $A_1^{(1)}$ are also the building block of general parafermion vertex operator algebras. We hope this fact will be important in the future study of the representation theory for the parafermion vertex operator algebra. So a complete understanding of representation theory of $K(\mathfrak{g},k)$ in the case $\mathfrak{g} = sl_2$ becomes necessary.

The paper is organized as follows. In Sect. 2, we give the construction of the vertex operator algebra $V(k,0)$ associated to the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ from (11). $V(k,0)$ has a vertex operator subalgebra $V(k,0)(0)$ which is the space of $\mathfrak{h}$-invariants of $V(k,0)$. A foundational result in this section is to determine a set of

of generators for $V(k,0)(0)$. In Sect. 3, we give a set of generators for the vertex operator algebra $N(\mathfrak{g},k)$ which is the commutant of the Heisenberg vertex operator algebra $M_{\widehat{\mathfrak{h}}}(k,0)$ in $V(k,0)$. We also discuss the vertex operator subalgebra $\widehat{P}_\alpha$ generated by $\omega_\alpha, W^3_\alpha$ (which is defined in Sect. 3) associated to any positive root α and prove that $\widehat{P}_\alpha$ is isomorphic to $N(sl_2,k_\alpha)$, where $k_\alpha = \frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}k$ and θ is the highest root. In fact, $N(\mathfrak{g},k)$ is generated by $\widehat{P}_\alpha$ for positive roots α. In Sect. 4, we give a set of generators for the parafermion vertex operator algebra $K(\mathfrak{g},k)$ which is the simple quotient of $N(\mathfrak{g},k)$. We prove that the maximal ideal of $N(\mathfrak{g},k)$ is generated by the vector $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$, where x_θ is the root vector associated to θ (see Sect. 2). We also show that the image P_α of $\widehat{P}_\alpha$ in $K(\mathfrak{g},k)$ is isomorphic to $K(sl_2,k_\alpha)$ and $K(\mathfrak{g},k)$ is generated by P_α for positive roots α. That is, $K(sl_2,k_\alpha)$ are the building blocks of $K(\mathfrak{g},k)$.

We expect the reader to be familiar with the elementary theory of vertex operator algebras as found, for example, in (10) and (18).

We thank the referees for many excellent suggestions which improve and simplify the proof of Theorem 21 greatly.

2 Vertex Operator Algebras $V(k,0)$ and $V(k,0)(0)$

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Let Δ be the corresponding root system and Q the root lattice. Let $\langle,\rangle$ be an invariant symmetric nondegenerate bilinear form on $\mathfrak{g}$ such that $\langle\alpha,\alpha\rangle = 2$ if α is a long root, where we have identified $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle,\rangle$. As in (15), we denote the image of $\alpha\in\mathfrak{h}^*$ in $\mathfrak{h}$ by t_α. That is, $\alpha(h) = \langle t_\alpha, h\rangle$ for any $h\in\mathfrak{h}$. Fix simple roots $\{\alpha_1,\cdots,\alpha_l\}$ and denote the highest root by θ.

Let $\mathfrak{g}_\alpha$ denote the root space associated to the root $\alpha\in\Delta$. For $\alpha\in\Delta_+$, we fix $x_{\pm\alpha}\in\mathfrak{g}_{\pm\alpha}$ and $h_\alpha = \frac{2}{\langle\alpha,\alpha\rangle}t_\alpha\in\mathfrak{h}$ such that $[x_\alpha,x_{-\alpha}] = h_\alpha, [h_\alpha,x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$. That is, $\mathfrak{g}^\alpha = \mathbb{C}x_\alpha + \mathbb{C}h_\alpha + \mathbb{C}x_{-\alpha}$ is isomorphic to sl_2 by sending x_α to $\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $x_{-\alpha}$ to $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and h_α to $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Then $\langle h_\alpha,h_\alpha\rangle = 2\frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}$ and $\langle x_\alpha,x_{-\alpha}\rangle = \frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}$ for all $\alpha\in\Delta$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ be the corresponding affine Lie algebra. Let $k\geq 1$ be an integer and

$$V(k,0) = V_{\widehat{\mathfrak{g}}}(k,0) = Ind^{\widehat{\mathfrak{g}}}_{\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}K}\mathbb{C}$$

the induced $\widehat{\mathfrak{g}}$-module such that $\mathfrak{g}\otimes\mathbb{C}[t]$ acts as 0 and K acts as k on $\mathbb{1} = 1$.

We denote by $a(n)$ the operator on $V(k,0)$ corresponding to the action of $a\otimes t^n$. Then

$$[a(m),b(n)] = [a,b](m+n) + m\langle a,b\rangle\delta_{m+n,0}k$$

for $a,b\in\mathfrak{g}$ and $m,n\in\mathbb{Z}$.

Let $a(z) = \sum_{n\in\mathbb{Z}}a(n)z^{-n-1}$. Then $V(k,0)$ is a vertex operator algebra generated by $a(-1)\mathbb{1}$ for $a\in\mathfrak{g}$ such that $Y(a(-1)\mathbb{1},z) = a(z)$ with the vacuum vector $\mathbb{1}$ and

the Virasoro vector

$$\omega_{\mathrm{aff}} = \frac{1}{2(k+h^\vee)}\left(\sum_{i=1}^{l} h_i(-1)h_i(-1)\mathbb{1} + \sum_{\alpha\in\Delta}\frac{\langle\alpha,\alpha\rangle}{\langle\theta,\theta\rangle}x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1}\right)$$

of central charge $\frac{k\dim\mathfrak{g}}{k+h^\vee}$ (e.g. (17; 18, Sect. 6.2)), where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $\{h_i|i=1,\dots,l\}$ is an orthonormal basis of $\mathfrak{h}$.

We will use the standard notation for the component operators of $Y(u,z)$ for $u\in V(k,0)$. That is, $Y(u,z)=\sum_{n\in\mathbb{Z}}u_nz^{-n-1}$. From the definition of vertex operators, we immediately see that $(a(-1)\mathbb{1})_n = a(n)$ for $a\in\mathfrak{g}$. So in the rest of paper, we will use both $a(n)$ and $(a(-1)\mathbb{1})_n$ for $a\in\mathfrak{g}$ and use u_n only for general u without further explanation.

For $\lambda\in\mathfrak{h}^*$, set

$$V(k,0)(\lambda)=\{v\in V(k,0)|h(0)v=\lambda(h)v,\forall\, h\in\mathfrak{h}\}.$$

Then we have

$$V(k,0)=\oplus_{\lambda\in Q}V(k,0)(\lambda). \tag{2.1}$$

Since $[h(0),Y(u,z)]=Y(h(0)u,z)$ for $h\in\mathfrak{h}$ and $u\in V(k,0)$, from the definition of affine vertex operator algebra, we see that $V(k,0)(0)$ is a vertex operator subalgebra of $V(k,0)$ with the same Virasoro vector ω_{aff} and each $V(k,0)(\lambda)$ is a module for $V(k,0)(0)$.

Our first theorem is on a set of generators for $V(k,0)(0)$.

Theorem 21 *The vertex operator algebra $V(k,0)(0)$ is generated by vectors $\alpha_i(-1)\mathbb{1}$ and $x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1}$ for $1\le i\le l,\alpha\in\Delta_+$.*

Proof First note that $V(k,0)(0)$ is spanned by the vectors

$$a_1(-m_1)\cdots a_s(-m_s)x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)\cdots x_{\beta_t}(-n_t)\mathbb{1},$$

where $a_i\in\mathfrak{h},\beta_j\in\Delta,m_i>0,n_j>0$ and $\beta_1+\beta_2+\cdots+\beta_t=0$. Let U be the vertex operator subalgebra generated by $\alpha_i(-1)\mathbb{1}$ and $x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1}$ for $1\le i\le l,\alpha\in\Delta_+$. Clearly, $\alpha_i(-1)\mathbb{1}$ and $x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1}\in V(k,0)(0)$ for $1\le i\le l,\alpha\in\Delta_+$. It suffices to prove that $V(k,0)(0)\subset U$.

Since $(h(-1)\mathbb{1})_n=h(n)$ for $h\in\mathfrak{h}$, we see that $h(n)U\subset U$ for $h\in\mathfrak{h}$ and $n\in\mathbb{Z}$. So we only need to prove $u=x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)\cdots x_{\beta_t}(-n_t)\mathbb{1}\in U$ with $\beta_1+\beta_2+\cdots+\beta_t=0$. We will prove it by induction on t.

Clearly, $t\ge 2$. If $t=2$, it follows from Theorem 2.1 in (6) that

$$x_{-\alpha}(-m)x_\alpha(-n)\mathbb{1}\in U$$

for $m,n>0$. Note that if $m\ge 0$, then

$$x_{-\alpha}(m)x_\alpha(n)\mathbb{1}=-h_\alpha(m+n)\mathbb{1}+mk\langle x_\alpha,x_{-\alpha}\rangle\delta_{m+n,0}\mathbb{1}\in U.$$

We claim that $x_{-\alpha}(m)x_\alpha(n)U\subset U$ for all $m,n\in\mathbb{Z}$. Let $u\in U$. From Proposition 4.5.7 of (18), there exist nonnegative integers p,q such that

$$x_{-\alpha}(m)x_\alpha(n)u=\sum_{i=0}^{p}\sum_{j=0}^{q}\binom{m-q}{i}\binom{q}{j}(x_{-\alpha}(m-q-i+j)x_\alpha(-1)\mathbb{1})_{n+q+i-j}u.$$

Since $x_{-\alpha}(m-q-i+j)x_\alpha(-1)\mathbb{1} \in U$, the claim follows.

From now on, we assume that $t > 2$ and that

$$x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)\cdots x_{\beta_\nu}(-n_\nu)\mathbb{1} \in U$$

with $\beta_1+\beta_2+\cdots+\beta_\nu = 0$ for $2 \le \nu \le t-1$ and $n_i > 0$. We have to show that

$$x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)\cdots x_{\beta_t}(-n_t)\mathbb{1} \in U$$

with $\beta_1+\beta_2+\cdots+\beta_t = 0$. We divide the proof into two cases.
Case 1. There exist $1 \le i, j \le t$ such that $\beta_i+\beta_j \in \Delta$. Note that if

$$x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)\cdots x_{\beta_t}(-n_t)\mathbb{1} \in U,$$

then

$$x_{\beta_{i_1}}(-n_{i_1})x_{\beta_{i_2}}(-n_{i_2})\cdots x_{\beta_{i_t}}(-n_{i_t})\mathbb{1} \in U$$

by the induction assumption, where $(i_1,\dots,i_t)$ is any permutation of $(1,\dots,t)$. Without loss of generality, we may assume that $\beta_1+\beta_2 \in \Delta$.

Let k,n be positive integers such that $-k+n = -n_2$ and $n > n_i$ for $i \ge 3$. Let $w = x_{\beta_1+\beta_2}(-k)x_{\beta_3}(-n_3)\cdots x_{\beta_t}(-n_t)\mathbb{1}$ with $\beta_1+\beta_2+\cdots+\beta_t = 0$. Then $w \in U$ by the induction assumption and $x_{\beta_1}(-n_1)x_{-\beta_1}(n)w \in U$ by the claim.

Let $[x_{-\beta_1}, x_{\beta_1+\beta_2}] = \lambda x_{\beta_2}$ for some nonzero λ. Then

$$\begin{aligned} x_{\beta_1}(-n_1)x_{-\beta_1}(n)w &= \lambda x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)x_{\beta_3}(-n_3)\cdots x_{\beta_t}(-n_t)\mathbb{1} \\ &+ x_{\beta_1}(-n_1)x_{\beta_1+\beta_2}(-k)[x_{-\beta_1}, x_{\beta_3}](n-n_3)x_{\beta_4}(-n_4)\cdots x_{\beta_t}(-n_t)\mathbb{1} \\ &+ \cdots + x_{\beta_1}(-n_1)x_{\beta_1+\beta_2}(-k)x_{\beta_3}(-n_3)\cdots[x_{-\beta_1}, x_{\beta_t}](n-n_t)\mathbb{1}. \end{aligned}$$

Since $n-n_i > 0$ for $i \ge 3$, we see that

$$\begin{aligned} &x_{\beta_1}(-n_1)x_{\beta_1+\beta_2}(-k)[x_{-\beta_1}, x_{\beta_3}](n-n_3)x_{\beta_4}(-n_4)\cdots x_{\beta_t}(-n_t)\mathbb{1} \\ &\quad + \cdots + x_{\beta_1}(-n_1)x_{\beta_1+\beta_2}(-k)x_{\beta_3}(-n_3)\cdots[x_{-\beta_1}, x_{\beta_t}](n-n_t)\mathbb{1} \end{aligned}$$

lies in U by the induction assumption. As a result, $x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)x_{\beta_3}(-n_3)$ $\cdots x_{\beta_t}(-n_t)\mathbb{1} \in U$.

Case 2. For any $1 \le i, j \le t$, $\beta_i+\beta_j \notin \Delta$. We claim that there exist $1 \le i', j' \le t$ such that $\beta_{i'}+\beta_{j'} = 0$. Otherwise, $\beta_i+\beta_j \ne 0$ for all i, j. This implies that $\langle \beta_i, \beta_j\rangle \ge 0$ for all i, j, thus $\langle \beta_1, \sum_{j=2}^t \beta_j\rangle \ge 0$. On the other hand, since $\sum_{j=2}^t \beta_j = -\beta_1$, we have $\langle \beta_1, \sum_{j=2}^t \beta_j\rangle < 0$, a contradiction.

Without loss of generality, we may assume $\beta_1+\beta_2 = 0$. Then $\beta_3+\cdots+\beta_t = 0$. By the induction assumption, $x_{\beta_3}(-n_3)\cdots x_{\beta_t}(-n_t)\mathbb{1} \in U$. It is immediate that

$$x_{\beta_1}(-n_1)x_{\beta_2}(-n_2)x_{\beta_3}(-n_3)\cdots x_{\beta_t}(-n_t)\mathbb{1} \in U$$

from the claim. The proof is complete. □

Remark 22 Theorem 2.1 has been obtained in (6) previously in the case $\mathfrak{g} = sl_2$. The proof given here simplifies the proof of Theorem 2.1 in (6).

Next we discuss some automorphisms of vertex operator algebras $V(k,0)$ and $V(k,0)(0)$ for later purposes. It is well known that the automorphism group $Aut(V(k,0))$ is isomorphic to the automorphism group $Aut(\mathfrak{g})$. In fact, if $\sigma \in Aut(\mathfrak{g})$, then σ lifts to an automorphism of $V(k,0)$ in the following way:

$$\sigma(x_1(-n_1)\cdots x_s(-n_s)\mathbb{1}) = (\sigma x_1)(-n_1)\cdots(\sigma x_s)(-n_s)\mathbb{1}$$

for $x_i \in \mathfrak{g}$ and $n_i > 0$. Let $W(\mathfrak{g})$ be the Weyl group of $\mathfrak{g}$. Then $W(\mathfrak{g})$ can naturally be regarded as a subgroup of $Aut(\mathfrak{g})$ (16). It is easy to see that if $\sigma(\mathfrak{h}) = \mathfrak{h}$, then $\sigma(V(k,0)(0)) = V(k,0)(0)$ and the restriction of σ to $V(k,0)(0)$ gives an automorphism of $V(k,0)(0)$. In particular, any Weyl group element gives an automorphism of $V(k,0)(0)$. This fact will be used in later sections.

3 Vertex Operator Algebra $N(\mathfrak{g},k)$

Let $V_{\widehat{\mathfrak{h}}}(k,0)$ be the vertex operator subalgebra of $V(k,0)$ generated by $h(-1)\mathbb{1}$ for $h \in \mathfrak{h}$ with the Virasoro element

$$\omega_{\mathfrak{h}} = \frac{1}{2k}\sum_{i=1}^{l} h_i(-1)h_i(-1)\mathbb{1}$$

of central charge l, where $\{h_1,\cdots h_l\}$ is an orthonormal basis of $\mathfrak{h}$ as before. For $\lambda \in \mathfrak{h}^*$, let $M_{\widehat{\mathfrak{h}}}(k,\lambda)$ denote the irreducible highest weight module for $\widehat{\mathfrak{h}}$ with a highest weight vector v_λ such that $h(0)v_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$. Then $V_{\widehat{\mathfrak{h}}}(k,0)$ is identified with $M_{\widehat{\mathfrak{h}}}(k,0)$.

Recall $V(k,0)(\lambda)$ from Sect. 2. Both $V(k,0)$ and $V(k,0)(\lambda), \lambda \in Q$ are completely reducible $V_{\widehat{\mathfrak{h}}}(k,0)$-modules. That is,

$$V(k,0) = \oplus_{\lambda\in Q} M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes N_\lambda, \tag{3.1}$$

$$V(k,0)(\lambda) = M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes N_\lambda, \tag{3.2}$$

where

$$N_\lambda = \{v \in V(k,0)\,|\,h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}$$

is the space of highest weight vectors with highest weight λ for $\widehat{\mathfrak{h}}$.

Note that $N(\mathfrak{g},k) = N_0$ is the commutant (11, Theorem 5.1) of $V_{\widehat{\mathfrak{h}}}(k,0)$ in $V(k,0)$. The commutant $N(\mathfrak{g},k)$ is a vertex operator algebra with the Virasoro vector $\omega = \omega_{\text{aff}} - \omega_{\mathfrak{h}}$ whose central charge is $\frac{k\dim\mathfrak{g}}{k+h^\vee} - l$.

Recall from Sect. 2, the 3-dimensional subalgebra $\mathfrak{g}^\alpha$ for $\alpha \in \Delta_+$. Then the restriction $\langle,\rangle_{\mathfrak{g}^\alpha}$ of the bilinear form $\langle,\rangle$ to $\mathfrak{g}^\alpha$ is equal to $\frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}(,)$, where $(,)$ is the standard nondegenerate symmetric invariant bilinear form on $\mathfrak{g}^\alpha$ such that $(h_\alpha,h_\alpha) = 2$. As a result, $V(k,0)$ is a module for $\widehat{\mathfrak{g}^\alpha} = \mathfrak{g}^\alpha\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$ of level $k_\alpha = \frac{\langle\theta,\theta\rangle}{\langle\alpha,\alpha\rangle}k$ as we regard $V(k,0)$ as a module for the subalgebra $\widehat{\mathfrak{g}^\alpha}$ of $\widehat{\mathfrak{g}}$. In other words, $V(k,0)$ is a $\widehat{\mathfrak{g}^\alpha}$-module of level $2k$ or $3k$ if α is a short root.

Following (5), we let

$$\omega_\alpha = \frac{1}{2k(k+2)}(-kh_\alpha(-2)\mathbb{1} - h_\alpha(-1)^2\mathbb{1} + 2kx_\alpha(-1)x_{-\alpha}(-1)\mathbb{1}), \tag{3.3}$$

$$\begin{aligned} W^3_\alpha = {} & k^2h_\alpha(-3)\mathbb{1} + 3kh_\alpha(-2)h_\alpha(-1)\mathbb{1} + 2h_\alpha(-1)^3\mathbb{1} \\ & - 6kh_\alpha(-1)x_\alpha(-1)x_{-\alpha}(-1)\mathbb{1} \\ & + 3k^2x_\alpha(-2)x_{-\alpha}(-1)\mathbb{1} - 3k^2x_\alpha(-1)x_{-\alpha}(-2)\mathbb{1} \end{aligned} \tag{3.4}$$

if $\alpha \in \triangle_+$ is a long root. If α is a short root, we also define $\omega_\alpha, W^3_\alpha$ as in (3.3) and (3.4) by replacing k by k_α.

Let $\widehat{P}_\alpha$ be the vertex operator subalgebra of $N(\mathfrak{g},k)$ generated by ω_α and W^3_α. Then $\widehat{P}_\alpha$ is isomorphic to the W-algebra $W(2,3,4,5)$ (1) by (6, Theorem 3.1) with k replaced by k_α, i.e., $\widehat{P}_\alpha$ is isomorphic to $N(sl_2,k_\alpha)$.

The first main theorem of this paper is about the generators of $N(\mathfrak{g},k)$.

Theorem 31 *The vertex operator algebra $N(\mathfrak{g},k)$ is generated by $\dim\mathfrak{g} - l$ vectors ω_α and W^3_α for $\alpha \in \Delta_+$. That is, $N(\mathfrak{g},k)$ is generated by subalgebras $\widehat{P}_\alpha$ for $\alpha \in \Delta_+$.*

Proof We first prove that $V(k,0)(0)$ is generated by vectors $\alpha_i(-1)\mathbb{1}, \omega_\alpha$ and W^3_α for $i = 1,\dots,l$ and $\alpha \in \Delta_+$. In fact, let U be the vertex operator subalgebra generated by $h(-1)\mathbb{1}, \omega_\alpha$ and W^3_α for $h \in \mathfrak{h}$ and $\alpha \in \Delta_+$. Then $x_{-\alpha}(-1)x_\alpha(-1)\mathbb{1} \in U$ and $\omega_{\mathrm{aff}} \in U$. Moreover, from the expression of W^3_α, we see that $x_{-\alpha}(-1)x_\alpha(-2)\mathbb{1} - x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1} \in U$. Set $L_{\mathrm{aff}}(n) = (\omega_{\mathrm{aff}})_{n+1}$; we have

$$[L_{\mathrm{aff}}(m), a(n)] = -na(m+n)$$

for $m,n \in \mathbb{Z}, a \in \mathfrak{g}$. Thus,

$$L_{\mathrm{aff}}(-1)x_{-\alpha}(-1)x_\alpha(-1)\mathbb{1} = x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1} + x_{-\alpha}(-1)x_\alpha(-2)\mathbb{1} \in U.$$

Together with $x_{-\alpha}(-1)x_\alpha(-2)\mathbb{1} - x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1} \in U$, we get $x_{-\alpha}(-2)x_\alpha(-1)\mathbb{1} \in U$, and so U is equal to $V(k,0)(0)$ by Theorem 21.

Next we show that $\omega_\alpha, W^3_\alpha \in N(\mathfrak{g},k)$ for $\alpha \in \Delta$. Since $\langle h_\alpha, h_\alpha\rangle \neq 0$, we have decomposition $\mathfrak{h} = \mathbb{C}h_\alpha \oplus (\mathbb{C}h_\alpha)^\perp$, where $(\mathbb{C}h_\alpha)^\perp$ is the orthogonal complement of $\mathbb{C}h_\alpha$ with respect to $\langle,\rangle$. From (6), we know that $h_\alpha(n)\omega_\alpha = h_\alpha(n)W^3_\alpha = 0$ for $n \geq 0$. If $u \in (\mathbb{C}h_\alpha)^\perp$, we clearly have $u(n)\omega_\alpha = u(n)W^3_\alpha = 0$ for $n \geq 0$. This implies that $\omega_\alpha, W^3_\alpha \in N(\mathfrak{g},k)$.

Notice that $Y(u,z_1)Y(v,z_2) = Y(v,z_2)Y(u,z_1)$ for $u \in M_{\widehat{\mathfrak{h}}}(k,0)$ and $v \in N(\mathfrak{g},k)$. Since $V(k,0)(0) = M_{\widehat{\mathfrak{h}}}(k,0) \otimes N(\mathfrak{g},k), h(-1)\mathbb{1} \in M_{\widehat{\mathfrak{h}}}(k,0)$ for $h \in \mathfrak{h}$, and $\omega_\alpha, W^3_\alpha \in N(\mathfrak{g},k)$ for $\alpha \in \Delta_+$, we conclude that $N(\mathfrak{g},k)$ is generated by $\omega_\alpha, W^3_\alpha$ for $\alpha \in \Delta_+$. □

Remark 32 Using the Z-algebra introduced and studied in (19) and (20; 21), we can rewrite ω_α and W^3_α in terms of Z-operators $Z_\alpha(m)$ and $Z_{-\alpha}(n)$. It is not too hard to see that $\omega_\alpha = a_\alpha Z_\alpha(-1)Z_{-\alpha}(-1)\mathbb{1}$ and $W^3_\alpha = b_\alpha Z_\alpha(-2)Z_{-\alpha}(-1)\mathbb{1} + c_\alpha Z_{-\alpha}(-2)Z_\alpha(-1)\mathbb{1}$ for some constants $a_\alpha, b_\alpha, c_\alpha \in \mathbb{C}$. One could determine these constants explicitly using the definition of Z-operators.

Remark 33 The vertex operator algebra $N(\mathfrak{g},k)$ and its quotient $K(\mathfrak{g},k)$ are of moonshine type. That is, their weight zero subspaces are 1-dimensional and weight one subspaces are zero.

Remark 34 We already know that each ω_α is a Virasoro element and how to compute the Lie brackets $[Y(\omega_\alpha,z_1),Y(W^3_\alpha,z_2)],[Y(W^3_\alpha,z_1),Y(W^3_\alpha,z_2)]$ for $\alpha\in\Delta_+$. It is important to calculate the Lie brackets for vertex operators associated to vectors in different $\widehat{P_\alpha}$. This will be done in a sequel to this paper where the representation theory will be investigated.

Following the discussion given at the end of Sect. 2, we see that any Weyl group element gives an automorphism of $N(\mathfrak{g},k)$.

4 Parafermion Vertex Operator Algebras $K(\mathfrak{g},k)$

It is well known that the vertex operator algebra $V(k,0)$ has a unique maximal ideal $\mathscr{J}$ generated by a weight $k+1$ vector $x_\theta(-1)^{k+1}\mathbb{1}$ (17), where θ is the highest root of $\mathfrak{g}$. The quotient vertex operator algebra $L(k,0)=V(k,0)/\mathscr{J}$ is a simple, rational vertex operator algebra associated to the affine Lie algebra $\widehat{\mathfrak{g}}$. Again, the Heisenberg vertex operator algebra $V_{\widehat{\mathfrak{h}}}(k,0)$ generated by $h(-1)\mathbb{1}$ for $h\in\mathfrak{h}$ is a simple subalgebra of $L(k,0)$ and $L(k,0)$ is a completely reducible $V_{\widehat{\mathfrak{h}}}(k,0)$-module. We have a decomposition

$$L(k,0)=\oplus_{\lambda\in Q}M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes K_\lambda \tag{4.1}$$

as modules for $V_{\widehat{\mathfrak{h}}}(k,0)$, where

$$K_\lambda=\{v\in L(k,0)\,|\,h(m)v=\lambda(h)\delta_{m,0}v \text{ for } h\in\mathfrak{h},m\geq 0\}.$$

Set $K(\mathfrak{g},k)=K_0$. Then $K(\mathfrak{g},k)$ is the commutant of $V_{\widehat{\mathfrak{h}}}(k,0)$ in $L(k,0)$ and is called the parafermion vertex operator algebra associated to the irreducible highest weight module $L(k,0)$ for $\widehat{\mathfrak{g}}$. As we mentioned in the Introduction, $K(\mathfrak{g},k)$ are conjectured to be rational, C_2-cofinite vertex operator algebras.

As a $V_{\widehat{\mathfrak{h}}}(k,0)$-module, $\mathscr{J}$ is completely reducible. From (3.1),

$$\mathscr{J}=\oplus_{\lambda\in Q}M_{\widehat{\mathfrak{h}}}(k,\lambda)\otimes(\mathscr{J}\cap N_\lambda).$$

In particular, $\widetilde{\mathscr{J}}=\mathscr{J}\cap N(\mathfrak{g},k)$ is an ideal of $N(\mathfrak{g},k)$ and $K(\mathfrak{g},k)\cong N(\mathfrak{g},k)/\widetilde{\mathscr{J}}$. Following the same proof as (5, Lemma 3.1), we know that $\widetilde{\mathscr{J}}$ is the unique maximal ideal of $N(\mathfrak{g},k)$. Thus $K(\mathfrak{g},k)$ is a simple vertex operator algebra.

We still use $\omega_{\mathrm{aff}},\omega_{\mathfrak{h}},\omega_\alpha,W^3_\alpha$ to denote their images in $L(k,0)=V(k,0)/\mathscr{J}$.

Remark 41 In the case $k=1$, it follows from the construction of $L(1,0)$ (9; 10) that $\omega=0$ and $K(\mathfrak{g},k)=\mathbb{C}$ if $\mathfrak{g}$ is of ADE type.

The following result is a direct consequence of Theorem 31.

Theorem 42 *The simple vertex operator algebra $K(\mathfrak{g},k)$ is generated by ω_α,W^3_α for $\alpha\in\Delta_+$.*

Next, we study the ideal $\widetilde{\mathscr{I}}$ of $N(\mathfrak{g},k)$ in detail. The vector $x_\theta(-1)^{k+1}\mathbb{1} \notin N(\mathfrak{g},k)$. From (5, Theorem 3.2) we know that $h_\theta(n)x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1} = 0$ for $n \geq 0$. It is clear that if $h \in \mathfrak{h}$ satisfies $\langle h_\theta, h\rangle = 0$, then $h(n)x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1} = 0$ for $n \geq 0$. So we have proved the following

Lemma 43 $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1} \in \widetilde{\mathscr{I}}$.

Furthermore, we have

Proposition 44 *The maximal ideal $\widetilde{\mathscr{I}}$ of $N(\mathfrak{g},k)$ is generated by $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$.*

Proof The proof is similar to that of (6, Theorem 4.2 (1)). Recall $\mathfrak{g}^\theta = \mathbb{C}x_\theta + \mathbb{C}h_\theta + \mathbb{C}x_{-\theta}$ is a subalgebra of $\mathfrak{g}$ isomorphic to sl_2.$V(k,0)$ is an $\mathfrak{g}^\theta$-module where $a \in \mathfrak{g}^\theta$ acts as $a(0)$. Each weight subspace of the vertex operator algebra $V(k,0)$ is a finite dimensional $\mathfrak{g}^\theta$-module and $V(k,0)$ is completely reducible as a module for $\mathfrak{g}^\theta$. Consider the $\mathfrak{g}^\theta$-submodule X of $V(k,0)$ generated by $x_\theta(-1)^{k+1}\mathbb{1}$. Since $x_\theta(0)x_\theta(-1)^{k+1}\mathbb{1} = 0$ and $h_\theta(0)x_\theta(-1)^{k+1}\mathbb{1} = 2(k+1)x_\theta(-1)^{k+1}\mathbb{1}, x_\theta(-1)^{k+1}\mathbb{1}$ is a highest weight vector with highest weight $2(k+1)$ for $\mathfrak{g}^\theta$. Then X is an irreducible $\mathfrak{g}^\theta$-module with basis $x_{-\theta}(0)^i x_\theta(-1)^{k+1}\mathbb{1}, 0 \leq i \leq 2(k+1)$ from the representation theory of sl_2. This implies that the ideal $\mathscr{J}$ of the vertex operator algebra $V(k,0)$ can be generated by any nonzero vector in X. In particular, $\mathscr{J}$ is generated by $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$. Then $\mathscr{J}$ is spanned by $u_n x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$ for $u \in V(k,0)$ and $n \in \mathbb{Z}$ by (8, Cor. 4.2) or (22, Prop. 4.1).

Since $v_m u \in V(k,0)(\lambda+\mu)$ for $v \in V(k,0)(\lambda), u \in V(k,0)(\mu), \lambda, \mu \in Q$ and $m \in \mathbb{Z}$, we see that $\mathscr{J} \cap V(k,0)(0)$ is spanned by vectors of the form $u_n x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$ with $u \in V(k,0)(0)$. Let $u = v \otimes w \in V(k,0)(0) = M_{\widehat{\mathfrak{h}}}(k,0) \otimes N(\mathfrak{g},k)$ with $v \in M_{\widehat{\mathfrak{h}}}(k,0)$ and $w \in N(\mathfrak{g},k)$. Then $Y(u,z) = Y(v,z) \otimes Y(w,z)$ acts on $M_{\widehat{\mathfrak{h}}}(k,0) \otimes N(\mathfrak{g},k)$. As a result, we have that $\widetilde{\mathscr{I}}$ is spanned by $w_n x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$ for $w \in N(\mathfrak{g},k)$ and $n \in \mathbb{Z}$. That is, the ideal $\widetilde{\mathscr{I}}$ of the vertex operator algebra $N(\mathfrak{g},k)$ is generated by $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}$. The proof is complete. □

For $\alpha \in \Delta_+$, we let P_α be the vertex operator subalgebra of $K(\mathfrak{g},k)$ generated by ω_α and W^3_α. Then P_α is a quotient of $\widehat{P}_\alpha$. A natural question is whether or not P_α is a simple vertex operator algebra. For this purpose, we recall our discussion earlier on the automorphisms of the vertex operator algebra $V(k,0)$ and $N(\mathfrak{g},k)$. That is, any Weyl group element gives an automorphism of $V(k,0)$ and $N(\mathfrak{g},k)$.

Clearly, any automorphism σ of $V(k,0)$ induces an automorphism of $L(k,0)$ as σ maps the unique maximal ideal $\mathscr{J}$ to $\mathscr{J}$. If $\sigma \in W(\mathfrak{g})$, then σ preserves the unique maximal ideal $\widetilde{\mathscr{I}}$ and σ gives an automorphism of the parafermion vertex operator algebra $K(\mathfrak{g},k)$, Now let $\alpha \in \Delta_+$ be a long root. Then there exists $\sigma \in W(\mathfrak{g})$ such that $\sigma\theta = \alpha$ (16). As a result,

$$\sigma(x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}\mathbb{1}) = a x_{-\alpha}(0)^{k+1}x_\alpha(-1)^{k+1}\mathbb{1}$$

for some constant a. This implies from Lemma 43 that $x_{-\alpha}(0)^{k+1}x_\alpha(-1)^{k+1}\mathbb{1} \in \widetilde{\mathscr{I}}$. Using (6, Theorem 4.2) we obtain:

Proposition 45 *For any long root $\alpha \in \Delta_+$, the vertex operator subalgebra P_α of $K(\mathfrak{g},k)$ is a simple vertex operator algebra isomorphic to the parafermion vertex operator algebra $K(sl_2,k)$ associated to sl_2.*

We next deal with short roots $\alpha \in \Delta_+$. As we mentioned already $V(k,0)$ is a level k_α-module for the affine algebra $\widehat{\mathfrak{g}^\alpha}$. We need a different method to prove the following which is a generalization of Proposition 45.

Proposition 46 *Let $\alpha \in \Delta_+$. Then the vertex operator subalgebra P_α of $K(\mathfrak{g},k)$ is a simple vertex operator algebra isomorphic to the parafermion vertex operator algebra $K(sl_2,k_\alpha)$ associated to sl_2.*

Proof As in the proof of Proposition 45 we only need to prove that

$$x_{-\alpha}(0)^{k_\alpha+1}x_\alpha(-1)^{k_\alpha+1}\mathbb{1} \in \widetilde{\mathscr{I}}.$$

Clearly, $L(k,0)$ is an integrable module for $\widehat{\mathfrak{g}^\alpha}$ as $x_\alpha(-1)$ is locally nilpotent on $L(k,0)$. In particular, the vertex operator subalgebra U of $L(k,0)$ generated by $\mathfrak{g}^\alpha$ is an integrable highest weight module. That is, U is isomorphic to $L(k_\alpha,0)$ associated to the affine algebra $\widehat{\mathfrak{g}^\alpha}$. As a result, we have $x_\alpha(-1)^{k_\alpha+1}\mathbb{1} \in \mathscr{J}$. It follows then immediately that $x_{-\alpha}(0)^{k_\alpha+1}x_\alpha(-1)^{k_\alpha+1}\mathbb{1} \in \widetilde{\mathscr{I}}$, as desired. □

Remark 47 We expect from Proposition 46 that the role of $K(sl_2,k_\alpha)$ played in the theory of the parafermion vertex operator algebra is similar to the role of sl_2 played in the theory of Kac-Moody Lie algebras. So a study of structural and representation theory for $K(sl_2,k_\alpha)$ becomes extremely important for general parafermion vertex operator algebras.

References

1. R. Blumenhagen W. Eholzer A. Honecker K. Hornfeck R. Hübel (1995) Coset realization of unifying W-algebras *Int. J. Mod. Phys.* **A10** 2367 – 2430
2. R.E. Borcherds (1986) Vertex algebras, Kac-Moody algebras, and the Monster *Proc. Natl. Acad. Sci. USA* **83** 3068 – 3071
3. C. Dong (1993) Vertex algebras associated with even lattices *J. Algebra* **160** 245 – 265
4. Dong, C., Lam, C.H., Yamada, H.: W-algebras in lattice vertex operator algebras. In: *Lie Theory and Its Applications in Physics VII*, ed. by H.-D. Doebner, V. K. Dobrev, Proc. of the VII International Workshop, Varna, Bulgaria, 2007, *Bulgarian J. Phys.* **35** supplement, 25–35 (2008)
5. C. Dong C.H. Lam H. Yamada (2009) W-algebras related to parafermion algebras *J. Algebra* **322** 2366 – 2403
6. C. Dong C.H. Lam Q. Wang H. Yamada (2010) The structure of parafermion vertex operator algebras *J. Algebra* **323** 371 – 381
7. Dong, C., Lepowsky, J.: *Generalized Vertex Algebras and Relative Vertex Operators*. Progress in Math., Vol. **112**, Boston, MA: Birkhäuser, 1993

8. C. Dong G. Mason (1997) On quantum Galois theory *Duke Math. J.* **86** 305 – 321
9. I.B. Frenkel V. Kac (1980) Basic representations of affine Lie algebras and dual resonance models *Invent. Math* **62** 23 – 66
10. Frenkel, I.B., Lepowsky, J., Meurman, A.: *Vertex Operator Algebras and the Monster*. Pure and Applied Math., Vol. **134**, Boston, MA: Academic Press, 1988
11. I.B. Frenkel Y. Zhu (1992) Vertex operator algebras associated to representations of affine and Virasoro algebras *Duke Math. J.* **66** 123 – 168
12. D. Gepner (1987) New conformal field theory associated with Lie algebras and their partition functions *Nucl. Phys.* **B290** 10 – 24
13. D. Gepner Z. Qiu (1987) Modular invariant partition functions for parafermionic field theories *Nucl. Phys.* **B285** 423 – 453
14. P. Goddard A. Kent D. Olive (1986) Unitary representations of the Virasoro and super-Virasoro algebras *Commun. Math. Phys.* **103** 105 – 119
15. K. Hornfeck (1993) W-algebras with set of primary fields of dimensions $(3,4,5)$ and $(3,4,5,6)$ *Nucl. Phys.* **B407** 237 – 246
16. Humphreys, J.: *Introduction to Lie Algebras and Representation Theory.* New York: Spring-Verlag, 1987
17. Kac, V.G.: *Infinite-dimensional Lie Algebras*. 3rd ed., Cambridge: Cambridge University Press, 1990
18. Lepowsky, J., Li, H.: *Introduction to Vertex Operator Algebras and Their Representations*. Progress in Math., Vol. **227**, Boston, MA: Birkhäuser, 2004
19. Lepowsky, J., Primc, M.: Structure of the standard modules for the affine Lie algebra $A_1^{(1)}$. *Contemp. Math.* **46**, Providence, RI: Amer. Math. Soc., 1985
20. J. Lepowsky R.L. Wilson (1981) A new family of algebras underlying the Rogers-Ramanujan identities and generalizations *Proc. Natl. Acad. Sci. USA* **78** 7245 – 7248
21. J. Lepowsky R.L. Wilson (1984) The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities *Invent. Math.* **77** 199 – 290
22. Li, H.: *An approach to tensor product theory for representations of a vertex operator algebra*, Ph.D. thesis, Rutgers University, 1994
23. A. Meurman M. Primc (1996) Vertex operator algebras and representations of affine Lie algebras. Representation of Lie groups, Lie algebras and their quantum analogues *Acta Appl. Math.* **44** 207 – 215
24. P. Wang X. Ding (1995) W-algebra constructed from $\mathbb{Z}_k$-parafermion through Nahm's normal ordering product *Comm. Theor. Phys.* **4** 155 – 189
25. W. Wang (1993) Rationality of Virasoro vertex operator algebras *Int. Math. Res. Notices* **7** 197 – 211
26. P. West (1994) W strings and cohomology in parafermionic theories *Phys. Lett. B* **329** 199 – 209
27. A.B. Zamolodchikov V.A. Fateev (1985) Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in Z_N-symmetric statistical systems *Sov. Phys. JETP* **62** 215 – 225