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Quantum geometrodynamics of Einstein and conformal (Weyl-squared) gravity

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Abstract. We discuss the canonical quantization of general relativity and Weyl-squared gravity. We present the classical and quantum constraints and discuss their similarities and differences. We perform a semiclassical expansion and discuss the emergence of time for the two theories. While in the first case semiclassical time has a scale and a shape part, in the second case it only has a shape part.

1. Introduction

The quantization of gravity is perhaps the most important open problem in theoretical physics [1]. Among the oldest approaches is quantum geometrodynamics ("Wheeler-DeWitt equation"). This is still a promising approach because it is very conservative: its equations are found if one searches for quantum wave equations that directly lead to Einstein's equations in the semiclassical (WKB) limit [2]. In 1926, Erwin Schrödinger found his wave equation by formulating classical mechanics in Hamilton-Jacobi form and "guessing" a wave equation that leads to this form in the limit of geometric optics. We recall what he wrote in 1926 [3]:

> *We know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures.* Perhaps this failure is in strict analogy with the failure of geometrical optics . . . that becomes evident as soon as the obstacles or apertures are no longer large compared with the real, finite, wavelength. . . . Then it becomes a question of searching for an undulatory mechanics, and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.[1]

The same procedure can be applied to Einstein's equations [2]. If written into Hamilton-Jacobi form, it can be "translated" into the Wheeler-DeWitt equation and the diffeomorphism (momentum) constraints. These equations are timeless, that is, there is no external time parameter or external spacetime background. One can introduce an *intrinsic time* by the structure of the Wheeler-DeWitt equation whose kinetic term is locally hyperbolic; this intrinsic time is related to $\sqrt{h}$, where h is the determinant of the three-metric h_{ab}. An *extrinsic time* is sometimes used in the alternative approach of reduced quantization; a typical choice is York's time (see e.g. [1], p. 148), which is related to the trace K of the extrinsic curvature.

[1] *Wir wissen doch heute, daß unsere klassische Mechanik bei sehr kleinen Bahndimensionen und sehr starken Bahnkrümmungen versagt.* Vielleicht ist dieses Versagen eine volle Analogie zum Versagen der geometrischen Optik . . . , das bekanntlich eintritt, sobald die 'Hindernisse' oder 'Öffnungen' nicht mehr groß sind gegen die wirkliche, endliche Wellenlänge. . . . Dann gilt es, eine 'undulatorische Mechanik' zu suchen – und der nächstliegende Weg dazu ist wohl die wellentheoretische Ausgestaltung des Hamiltonschen Bildes.

Starting with quantum geometrodynamics as the fundamental framework, one can derive the limit of quantum (field) theory in an external background spacetime by a Born-Oppenheimer type of approximation scheme with respect to the Planck mass [1, 2]. At leading order, the semiclassical or "WKB time" is recovered as an emergence concept from the timeless quantum equations.

In this contribution, we want to shed more light on the nature of emergent time. We will study the quantum geometrodynamics corresponding to general relativity (GR) and the quantum geometrodynamics corresponding to conformal (Weyl-squared) gravity, briefly called Weyl theory. The latter serves as a model for a gravitational theory devoid of scale. We will discuss the common features of and the differences between those theories.

For this purpose, we will employ a unimodular decomposition of the three-metric. In quantum GR (section 2), we can identify the scale and the shape part of the WKB time; the scale part is related to the intrinsic time in the vacuum Wheeler-DeWitt equation. So the WKB time needs for its "passing" both scale and shape.

In section 3, we present a model of quantum geometrodynamics based on Weyl-squared gravity. We find there equations similar to the equations of quantum GR, but with a different explicit structure.

In section 4, we discuss the semiclassical limit of these equations, which gives rise to a "Weyl-WKB time" (WWKB time). It has the property that the evolution of the scale degree of freedom does not contribute to it; this is because the WWKB time inherits the conformal symmetry of the Weyl-squared action from which the intrinsic time is absent. WWKB time is thus scale-less and conformally invariant; we call it the *shape time*. In section 5, we present our conclusions. Further details and references can be found in our papers [4, 5, 6].

2. Canonical quantum general relativity

Instead of the usual canonical formulation of GR and its quantization, we formulate the quantum geometrodynamics here in variables which manifestly reveal the conformal behaviour of the theory. We are interested, in particular, in identifying the three-metric volume $\sqrt{h}$ and the trace of the extrinsic curvature K as canonical variables carrying a single degree of freedom. This allows us to keep track of "intrinsic time" and "extrinsic time" by investigating what happens to these canonical degrees of freedom. These variables are called *unimodular-conformal variables* and are discussed in detail in [4, 5, 6]. To formulate GR in these variables, we will briefly review here section 2 from [7].

The extrinsic curvature and its trace are given by

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - 2D_{(i}N_{j)}\right), \quad K = h^{ij}K_{ij} = \frac{1}{N}\left(\frac{\dot{\sqrt{h}}}{\sqrt{h}} - D_iN^j\right), \tag{1}$$

where D_i is the covariant derivative with respect to the three-metric. We decompose these as well as the metric variables into scale and shape part, called unimodular-conformal decomposition:

$$a := (\sqrt{h})^{1/3}, \quad \bar{h}_{ij} := a^{-2}h_{ij}, \tag{2}$$

$$\bar{N}^i := N^i, \quad \bar{N}_i := a^{-2}N_i, \quad \bar{N} := a^{-1}N, \tag{3}$$

$$\bar{K}^{\mathrm{T}}_{ij} := a^{-1}K^{\mathrm{T}}_{ij} = \frac{1}{2\bar{N}}\left(\dot{\bar{h}}_{ij} - 2\left[\bar{D}_{(i}\bar{N}_{j)}\right]^{\mathrm{T}}\right), \quad \bar{K} := \frac{aK}{3} = \frac{1}{\bar{N}}\left(\frac{\dot{a}}{a} - \frac{1}{3}D_iN^i\right), \tag{4}$$

where $\bar{D}_i$ is the conformal part of the covariant derivative depending only on $\bar{h}_{ij}$. Note that intrinsic time $\sqrt{h}$ is here captured by the scale density a.

The Einstein-Hilbert (EH) action in its ADM formulation can be written in unimodular-conformal

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variables as follows:

$$\mathcal{S}^{\mathrm{E}} = \int \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{L}^{\mathrm{E}},$$

$$\mathcal{L}^{\mathrm{E}} = \frac{1}{2\kappa} N\sqrt{h}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) = \frac{1}{2\kappa}\bar{N}a^2\left(a^{2\,(3)}R + \bar{K}^{\mathrm{T}2}_{ij} - 6\bar{K}^2\right), \tag{5}$$

where $\kappa = 8\pi G$ ($c = 1$), and we keep in mind that ${}^{(3)}R$ can also be decomposed if needed (see the Appendix in [5]). Since the configuration variables are a and $\bar{h}_{ij}$, the conjugate momenta will be p_a and $\bar{p}^{ij}$, respectively,

$$p_a = \frac{\partial \mathcal{L}^{\mathrm{E}}}{\partial \dot{a}} = -\frac{6a}{\kappa}\bar{K}\,, \quad \bar{p}^{ij} = \frac{\partial \mathcal{L}^{\mathrm{E}}}{\partial \dot{\bar{h}}_{ij}} = \frac{a^2}{2\kappa}\bar{K}^{\mathrm{T}}_{ab}\bar{h}^{ia}\bar{h}^{jb}\,. \tag{6}$$

Using the definitions (2)–(4) and recalling that the ADM momentum is given by $p^{ij} = \sqrt{h}\left(K^{ij} - h^{ij}K\right)/(2\kappa)$, the momenta (6) can be related to the trace and traceless part of the ADM momentum, respectively, by

$$p_a = \frac{2}{a}\,p_{\mathrm{ADM}}\,, \quad \bar{p}^{ij} = a^2\,p^{ij\mathrm{T}}_{\mathrm{ADM}}, \tag{7}$$

so that $p^{ij}_{\mathrm{ADM}} = p^{ij\mathrm{T}}_{\mathrm{ADM}} + h^{ij}p_{\mathrm{ADM}}/3$. Performing a Legendre transformation, we see that the total Hamiltonian is simply a linear combination of first class constraints,

$$H^{\mathrm{E}} = \int \mathrm{d}^3x \left\{\bar{N}\mathcal{H}^{\mathrm{E}}_{\perp} + N^i\mathcal{H}^{\mathrm{E}}_i + \lambda_{\bar{N}}p_{\bar{N}} + \lambda^i p_i\right\}. \tag{8}$$

The momenta with respect to the lapse density $\bar{N}$ and shift N^i are the primary constraints $p_{\bar{N}} \approx 0$ and $p_i \approx 0$, respectively. They give rise to secondary constraints: the Hamiltonian constraint $\mathcal{H}^{\mathrm{E}}_{\perp} \approx 0$ and the momentum constraints $\mathcal{H}^{\mathrm{E}}_i \approx 0$, which in unimodular-conformal variables have the following form:

$$\mathcal{H}^{\mathrm{E}}_{\perp} = -\frac{\kappa}{12}\,p_a^2 + \frac{2\kappa}{a^2}\bar{h}_{ik}\bar{h}_{jl}\bar{p}^{ij}\bar{p}^{kl} - \frac{a^4}{2\kappa}\left({}^{(3)}R - 2\Lambda\right) \approx 0, \tag{9}$$

$$\mathcal{H}^{\mathrm{E}}_i = -2\bar{D}_k\left(\bar{h}_{ij}\bar{p}^{jk}\right) - \frac{1}{3}D_i\left(a\,p_a\right) \approx 0. \tag{10}$$

For completeness, we have added the cosmological constant Λ.

It is important to notice that the constraints explicitly depend on the scale density, which is related to intrinsic time $\sqrt{h}$, recall (2). It is this and *only* this variable that is responsible for the conformal non-invariance of GR [7].

Canonical quantization now proceeds by promoting the canonical variables into operators acting on wave functionals $\Psi\left[\bar{h}_{ab}(\mathbf{x}), a(\mathbf{x})\right]$:

$$\hat{\bar{h}}_{ab}(\mathbf{x})\Psi = \bar{h}_{ab}(\mathbf{x})\cdot\Psi, \qquad \hat{a}(\mathbf{x})\Psi = a(\mathbf{x})\cdot\Psi, \tag{11}$$

$$\hat{\bar{p}}^{cd}(\mathbf{x})\Psi = \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta\bar{h}_{cd}(\mathbf{x})}\Psi, \qquad \hat{p}_a(\mathbf{x})\Psi = \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta a(\mathbf{x})}\Psi. \tag{12}$$

From the classical Poisson brackets, one gets the standard commutators. The constraints are then promoted to operators annihilating the wave functional, leading to the Wheeler-DeWitt equation and the momentum constraints, respectively,

$$\hat{\mathcal{H}}_{\perp}\Psi = \left[+\frac{\kappa\hbar^2}{12}\frac{\delta^2}{\delta a^2} - \frac{2\kappa\hbar^2}{a^2}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta^2}{\delta\bar{h}_{ij}\delta\bar{h}_{kl}} - \frac{a^4}{2\kappa}\left({}^{(3)}R - 2\Lambda\right) + \hat{\mathcal{H}}^{\mathrm{m}}_{\perp}\right]\Psi \approx 0, \tag{13}$$

$$\hat{\mathcal{H}}_i\Psi = 2\,\mathrm{i}\hbar\bar{D}_k\left(\bar{h}_{ij}\frac{\delta\Psi}{\delta\bar{h}_{jk}}\right) + \frac{\mathrm{i}\hbar}{3}D_i\left(a\,\frac{\delta\Psi}{\delta a}\right) + \hat{\mathcal{H}}^{\mathrm{m}}_a\Psi \approx 0. \tag{14}$$

We have added the parts of the Hamiltonian and momentum constraints $\hat{\mathcal{H}}_\perp^{\rm m}$ and $\hat{\mathcal{H}}_a^{\rm m}$ coming from quantized matter (non-gravitational fields), so that the wave functional now depends on matter fields, too, $\Psi \equiv \Psi\left[\bar{h}_{ab}, a, \phi\right]$.

It is evident from (13) that the Wheeler-DeWitt equation has an indefinite kinetic term, with a playing the role of "intrinsic time". This intrinsic time is, however, constructed from the three-metric, so the Wheeler-DeWitt equation is timeless in the sense of absence of spacetime. That spacetime has disappeared is in full analogy to the absence of classical trajectories in quantum mechanics. The absence of spacetime is symbolized in Figure 1 by the absence of the pointer in the clock. (But the scale is present, waiting to be measured in the semiclassical limit.)

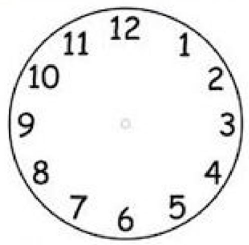

Figure 1. Absence of time in full quantum gravity

The notion of spacetime can be recovered from full canonical quantum GR by a Born-Oppenheimer type of expansion with respect to the Planck-mass squared, $m_{\rm P}^2 = \hbar/G$. In this limit, we obtain the well known theories of quantum field theory in curved spacetime (see e.g. [1] for all technical details). We write the total wave functional in the form

$$\Psi[\bar{h}_{ab}, a, \phi] \equiv \exp\left(\frac{\rm i}{\hbar} S[\bar{h}_{ab}, a, \phi]\right), \tag{15}$$

where ϕ stands for 'matter field', and perform an expansion of S (which is a complex function) with respect to the Planck-mass squared,

$$S[\bar{h}_{ab}, a, \phi] = m_{\rm P}^2 S_0^{\rm E} + S_1^{\rm E} + m_{\rm P}^{-2} S_2^{\rm E} + \ldots .$$

We insert this expansion into the Wheeler-DeWitt equation and compare different orders of $m_{\rm P}^2$. At highest order, $m_{\rm P}^4$, we find that $S_0^{\rm E}$ is independent of ϕ,

$$\frac{\delta S_0^{\rm E}}{\delta \phi} = 0 \quad \Rightarrow \quad S_0^{\rm E} = S_0^{\rm E}[\bar{h}_{ij}, a]. \tag{16}$$

At order $m_{\rm P}^2$, one obtains an equation for $S_0^{\rm E}$:

$$-\frac{1}{12}\left(\frac{\delta S_0^{\rm E}}{\delta a}\right)^2 + \frac{2}{a^2}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\rm E}}{\delta \bar{h}_{ij}}\frac{\delta S_0^{\rm E}}{\delta \bar{h}_{kl}} - \frac{a^4}{2}\left({}^{(3)}R - 2\Lambda\right) = 0\,. \tag{17}$$

Rewriting this in terms of the original variables and using the chain rule for the functional derivative with respect to the metric,

$$\frac{\delta}{\delta h_{ij}} = \frac{\delta a}{\delta h_{ij}}\frac{\delta}{\delta a} + \frac{\delta \bar{h}_{kl}}{\delta h_{ij}}\frac{\delta}{\delta \bar{h}_{kl}} = \frac{1}{6a}\bar{h}^{ij}\frac{\delta}{\delta a} + \frac{1}{a^2}\frac{\delta}{\delta \bar{h}_{ij}}, \tag{18}$$

one realizes that (17) is just the Einstein-Hamilton-Jacobi equation,

$$\frac{2\kappa}{\sqrt{h}}\mathcal{G}_{ikjl}\frac{\delta S^{\mathrm{E}}}{\delta h_{ij}}\frac{\delta S^{\mathrm{E}}}{\delta h_{kl}} - \frac{\sqrt{h}}{2\kappa}\,{}^{(3)}R = 0, \tag{19}$$

for the Hamilton-Jacobi functional $S^{\mathrm{E}} := S_0^{\mathrm{E}}/\kappa$, where $\mathcal{G}_{ikjl} = \left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)/2$ is the inverse of the DeWitt supermetric [1]. Since this equation is equivalent to all Einstein equations [8], one has recovered classical GR at this order of semicalssical expansion.

At the next order, m_{P}^0, an equation for S_1^{E} is obtained that can be simplified by writing

$$\psi^{(1)} \equiv D[\bar{h}_{ij}, a]\exp\left(\frac{\mathrm{i}}{\hbar}S_1^{\mathrm{E}}\right) \tag{20}$$

and demanding for D the standard WKB prefactor equation to hold [1].

After some manipulations, one arrives at the following equation for $\psi^{(1)}$:

$$\mathrm{i}\hbar\left[-\frac{1}{24}\frac{\delta S_0^{\mathrm{E}}}{\delta a}\frac{\delta}{\delta a} + \frac{1}{a^2}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\mathrm{E}}}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}\right]\psi^{(1)} \equiv \mathrm{i}\hbar a\mathcal{G}_{ijkl}\frac{\delta S_0^{\mathrm{E}}}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}\psi^{(1)} = \hat{\mathcal{H}}_{\perp}^{\mathrm{m}}\psi^{(1)}. \tag{21}$$

The form of this quation reminds one of the Tomonaga-Schwinger equation for $\psi^{(1)}$. Using again (18), the left-hand side of this equation has been rewritten in terms of the original variables (middle expression in (21)), which is well known from the literature [1].

To make the Tomonaga-Schwinger form explicit, we define

$$\frac{\delta}{\delta\tau(\mathbf{x})} := \mathcal{G}_{ijkl}\frac{\delta S_0^{\mathrm{E}}}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}, \tag{22}$$

thus introducing a local "bubble" (Tomonaga-Schwinger) time functional $\tau(\mathbf{x})$. We can thus write (21) in the form

$$\mathrm{i}\hbar\frac{\delta\psi^{(1)}}{\delta\tau} = \hat{\mathcal{H}}_{\perp}^{\mathrm{m}}\psi^{(1)}, \tag{23}$$

also known as Tomonaga-Schwinger equation. Note that $\hat{\mathcal{H}}_{\perp}^{\mathrm{m}}$ in (23) differs from $\hat{\mathcal{H}}_{\perp}^{\mathrm{m}}$ in (22) by a factor of a, which also induces the same relative rescaling of the bubble time. This is just a consequence of using unimodular-conformal variables.

It is of interest to have a closer look at the bubble time in unimodular-conformal variables,

$$\frac{\delta}{\delta\tau(\mathbf{x})} := -\frac{1}{24a}\frac{\delta S_0^{\mathrm{E}}}{\delta a}\frac{\delta}{\delta a} + \frac{1}{a^3}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\mathrm{E}}}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}. \tag{24}$$

We see from this that there are two main components, a derivative along the direction (in functional configuration space) of the scale density a and a component along the five directions of the conformal (shape) part $\bar{h}_{ij}$. The scale part is the part that corresponds to intrinsic time in the full Wheeler-DeWitt equation (13). We will call the two *independent* contributions to the bubble time as "scale time" and "shape time", respectively. Figure 2 symbolizes the emergence of time (the pointer) in the semiclassical approximation. While the picture shows only scale time, the full semiclassical time contains, of course, also the shape part.

We finally note that τ is not a scalar function, because this would be in contradiction to the commutator of the matter Hamiltonian densities at different space point [9]. A functional Schrödinger equation can be obtained from the Tomonaga-Schwinger equation after choosing a particular foliation and integrating over space. The next order of the Born-Oppenheimer expansion, m_{P}^{-2}, leads to genuine quantum gravitational effects; for example, corrections to the power spectrum of the CMB anisotropies [10].

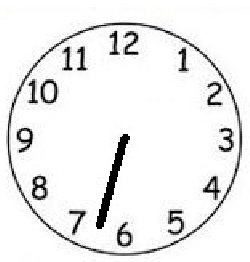

Figure 2. Emergence of semiclassical time

3. Quantum Weyl-squared gravity and the absence of scale

In this section, we address Weyl-squared gravity as a model of a conformally invariant gravitational theory. It is defined by the action

$$S^{\mathrm{W}} := -\frac{\alpha_{\mathrm{W}}\hbar}{4}\int \mathrm{d}^4x\sqrt{-g}\, C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}, \tag{25}$$

where $C_{\mu\nu\lambda\rho}$ is the Weyl tensor. This action is invariant under the Weyl transformations

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x). \tag{26}$$

We have written this action in a form that is appropriate for discussing the semiclassical approximation of its quantum version: α_{W} is a dimensionless constant, and it is the quantity $S^{\mathrm{W}}/\hbar$ that is relevant for the semiclassical expansion. We do not expect the Weyl theory to be a classical alternative to GR, but it may be of relevance in the early universe and at the Planck scale, where scales may become irrelevant [11].

We note that (25) defines a theory with higher derivatives, which demands an enlargement of the configuration space (see [5] for details and references to earlier work). Here, we have to add the extrinsic curvature K_{ab} as an additional independent canonical variable besides the three-metric h_{ab}. This can be accomplished by adding a constraint to the original Lagrangian which takes care of the definition (1). In unimodular-conformal variables, the new Lagrangian becomes [5]

$$\mathcal{L}_c^{\mathrm{W}} = \mathcal{L}^{\mathrm{W}} - a^5\lambda^{ij\mathrm{T}}\left[2\bar{K}_{ij}^{\mathrm{T}} - \frac{1}{\bar{N}}\left(\dot{\bar{h}}_{ij} - 2\left[\bar{D}_{(i}\bar{N}_{j)}\right]^{\mathrm{T}}\right)\right] - 2a^3\lambda\left[\bar{K} - \frac{1}{\bar{N}}\left(\frac{\dot{a}}{a} - \frac{1}{3}D_aN^a\right)\right]\Bigg\}, \tag{27}$$

where "T" stands for the traceless part of the corresponding quantity. For the canonical momenta, we find from (27):

$$p_{\bar{N}} = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{\bar{N}}} \approx 0\,, \quad p^i = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{N}^i} \approx 0\,, \quad \bar{P} = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{\bar{K}}} \approx 0\,, \tag{28}$$

$$\bar{p}^{ij} = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{\bar{h}}_{ij}} = a^5\lambda^{ij\mathrm{T}}\,, \quad p_a = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{a}} = 2a^2\lambda\,, \tag{29}$$

$$\bar{P}^{ij} = \frac{\partial\mathcal{L}_c^{\mathrm{W}}}{\partial\dot{\bar{K}}_{ij}^{\mathrm{T}}} = -\alpha_{\mathrm{W}}\hbar\,\bar{h}^{ia}\bar{h}^{jb}\bar{C}_{ab}^{\mathrm{T}}. \tag{30}$$

Note that the momenta $\bar{p}^{ij}$ and $\bar{P}^{ij}$ are traceless. The quantity $\bar{C}_{ab}^{\mathrm{T}}$ is the traceless part of the "electric" part of the Weyl tensor [5].

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The constraint analysis proceeds in the standard way. In addition to the primary constraints with respect to lapse and shift (as known from GR), we have here an additional primary constraint given by

$$\bar{P} = \frac{\partial \mathcal{L}_c^{\mathrm{W}}}{\partial \dot{\bar{K}}} \approx 0.$$

The presence of this constraint suggests that $\bar{K}$ is arbitrary, in the same manner as $p_{\bar{N}} \approx 0$ and $p_i \approx 0$ suggest that $\bar{N}$ and N^i are arbitrary. The secondary constraints follow from the conservation of the primary constraints and read (attention is here restricted to the vacuum case)

$$\mathcal{H}_\perp^{\mathrm{W}} = -\frac{\bar{h}_{ik}\bar{h}_{jl}\bar{P}^{ij}\bar{P}^{kl}}{2\alpha_{\mathrm{W}}\hbar} + \left({}^{(3)}\bar{R}_{ij}^{\mathrm{T}} + \partial_i \bar{D}_j\right)\bar{P}^{ij} + 2\bar{K}_{ij}^{\mathrm{T}}\bar{p}^{ij} - \alpha_{\mathrm{W}}\hbar\bar{C}_{ijk}^2 \approx 0, \tag{31}$$

$$\mathcal{H}_i^{\mathrm{W}} = -2\bar{D}_k\left(\bar{h}_{ij}\bar{p}^{jk}\right) - 2\bar{D}_k\left(\bar{K}_{ij}^{\mathrm{T}}\bar{P}^{jk}\right) + \bar{P}^{jk}\bar{D}_i\bar{K}_{jk}^{\mathrm{T}} \tag{32}$$

$$\mathcal{Q}^{\mathrm{W}} = a p_a \approx 0. \tag{33}$$

A combination of the secondary constraint $\mathcal{Q}^{\mathrm{W}}$ and the primary constraint $\bar{P}$ generates conformal transformations (of a and $\bar{K}$) [12, 5]. Due to the enlargement of the configuration space, the structure of these constraints differs from those of GR, see (9) and (10). We note, in particular, that the intrinsic time in (9), which is related to the determinant of the three-metric, is absent here because the scale density is arbitrary due to conformal invariance. This will later be connected with the emergence of pure shape time in the semiclassical limit.

Canonical quantization now proceeds in a manner analogously to (11) and (12), taking into account the enlarged configuration space. Adding a (conformally coupled) matter part symbolized by ϕ, we arrive at a quantum wave functional $\Psi[\bar{h}_{ij}, a, \bar{K}_{ij}^{\mathrm{T}}, \bar{K}, \phi]$, which is subject to the following constraints:

$$\hat{\mathcal{H}}_\perp^{\mathrm{W}}\Psi = 0\,, \quad \hat{\mathcal{H}}_i^{\mathrm{W}}\Psi = 0\,, \quad \hat{\bar{P}}\Psi = 0\,, \quad \hat{\mathcal{Q}}^{\mathrm{W}}\Psi = 0. \tag{34}$$

The first equation is the "Weyl-Wheeler-DeWitt equation" (WWDW equation) and reads

$$\left[\frac{\hbar}{2\alpha_{\mathrm{W}}}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta^2}{\delta\bar{K}_{ij}^{\mathrm{T}}\delta\bar{K}_{kl}^{\mathrm{T}}} - \mathrm{i}\hbar\left({}^{(3)}\bar{R}_{ij} + \partial_i\bar{D}_j\right)^{\mathrm{T}}\frac{\delta}{\delta\bar{K}_{ij}^{\mathrm{T}}} - 2\mathrm{i}\hbar\bar{K}_{ij}^{\mathrm{T}}\frac{\delta}{\delta\bar{h}_{ij}} - \alpha_{\mathrm{W}}\hbar\bar{C}_{ijk\perp}^2 + \hat{\mathcal{H}}_\perp^m\right]\Psi = 0\,. \tag{35}$$

The new quantum momentum constraints are given by

$$\mathrm{i}\hbar\left[2\bar{D}_k\left(\bar{h}_{ij}\frac{\delta\Psi}{\delta\bar{h}_{jk}}\right) + 2\bar{D}_k\left(\bar{K}_{ij}^{\mathrm{T}}\frac{\delta\Psi}{\delta\bar{K}_{jk}^{\mathrm{T}}}\right) - \bar{D}_i\bar{K}_{jk}^{\mathrm{T}}\frac{\delta\Psi}{\delta\bar{K}_{jk}^{\mathrm{T}}}\right] + \hat{\mathcal{H}}_i^{\mathrm{m}}\Psi = 0. \tag{36}$$

As for the classical constraints, the WWDW equation is structurally different from the Wheeler-DeWitt equation (13). Its dynamics is not only determined by the three-metric, but also by the extrinsic curvature (which classically corresponds to the time evolution of the three-metric). But there is no scale a and thus no intrinsic time. This is symbolized in Figure 3. We also note that in the vacuum case $\hbar$ drops out of these equations.

The third and fourth of the above constraints show that the wave functional does not depend on a and $\bar{K}$,

$$\frac{\delta\Psi}{\delta\bar{K}} = 0\,, \quad a\frac{\delta\Psi}{\delta a} = 0, \tag{37}$$

and is thus conformally invariant (apart possibly from a phase). This is the most important difference to GR, especially in the light of the semiclassical approximation which we will discuss in the next section.

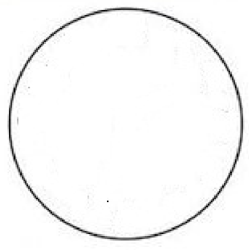

Figure 3. Absence of scale and absence of time

4. Recovery of (shape) time in the semiclassical limit

In this section, we will discuss the semiclassical approximation scheme for the WWDW equation [4]. While in quantum GR (section 2) m_{P}^{-2} was used as the expansion parameter, this role will be played here by α_{W}^{-1}.

Let us consider Weyl gravity with a conformally coupled matter field ϕ. We restrict ourselves to this case (massless vector or conformally coupled scalar field) because the constraints will remain first class and a and $\bar{K}$ will remain arbitrary. The general case is discussed in [6]. Writing

$$\Psi\left[\bar{h}_{ij}, \bar{K}^{\mathrm{T}}_{ij}, \phi\right] \equiv \exp\left(\frac{\mathrm{i}}{\hbar} S\left[\bar{h}_{ij}, \bar{K}^{\mathrm{T}}_{ij}, \phi\right]\right),$$

and performing an expansion of S with respect to the (dimensionless) variable α_{W}^{-1},

$$S = \alpha_{\mathrm{W}} \sum_{n=0}^{\infty} \left(\frac{1}{\alpha_{\mathrm{W}}}\right)^n S_n^{\mathrm{W}} \tag{38}$$

we find the following equations at consecutive orders of α_{W}:

- α_{W}^2: S_0^{W} is independent of ϕ, that is,

$$\left(\left(\hat{\mathcal{H}}_{\perp}^m \Psi\right)/\Psi\right)^{(2)} = 0 \quad \Rightarrow \quad \frac{\delta S_0^{\mathrm{W}}}{\delta \phi} = 0. \tag{39}$$

- α_{W}^1: This gives the Weyl-Hamilton-Jacobi equation for S_0^{W}. We first find

$$-\frac{1}{2\hbar}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{K}^{\mathrm{T}}_{ij}}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{K}^{\mathrm{T}}_{kl}} + \left({}^{(3)}\bar{R}_{ij} + \partial_i \bar{D}_j\right)^{\mathrm{T}}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{K}^{\mathrm{T}}_{ij}} + 2\bar{K}^{\mathrm{T}}_{ij}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{h}_{ij}} - \hbar \bar{C}^2_{ijk} = 0. \tag{40}$$

Relabelling $S^{\mathrm{W}} \equiv \alpha_{\mathrm{W}} S_0$, this is recognized as the Hamilton-Jacobi (HJ) equation for the Weyl theory after substituting

$$\bar{P}^{ij} \to \frac{\delta S^{\mathrm{W}}}{\delta \bar{K}^{\mathrm{T}}_{ij}}, \quad \bar{p}^{ij} \to \frac{S^{\mathrm{W}}}{\delta \bar{h}_{ij}} \tag{41}$$

into the Hamiltonian constraint (31). The classical solutions to the HJ equation are thus determined by the conformal parts of both the three-metric and the extrinsic curvature. It would be interesting to show (as expected) that the Weyl-HJ equation is equivalent to the Bach equations (equations of motion for the Weyl theory) in analogy to Gerlach's [8] work for GR, but we assume this equivalence here without proof.

- α_{W}^0: The equation for S_1^{W} can be simplified by introducing

$$\psi^{(1)} \equiv D[\bar{h}_{ij}, \bar{K}_{ij}^{\mathrm{T}}] \exp\left(\frac{\mathrm{i}}{\hbar} S_1^{\mathrm{W}}\right) \tag{42}$$

and demanding the "WKB prefactor equation" to hold for D, in analogy to the procedure in quantum GR. After the elimination of D, one is left with the following equation for $\psi^{(1)}$:

$$\mathrm{i}\hbar\left[-\frac{1}{2\hbar}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{K}_{kl}^{\mathrm{T}}}\frac{\delta}{\delta \bar{K}_{ij}^{\mathrm{T}}} + \left({}^{(3)}\bar{R}_{ij} + \partial_i \bar{D}_j\right)^{\mathrm{T}}\frac{\delta}{\delta \bar{K}_{ij}^{\mathrm{T}}} + 2\bar{K}_{ij}^{\mathrm{T}}\frac{\delta}{\delta \bar{h}_{ij}}\right]\psi^{(1)} = \hat{\mathcal{H}}_{\perp}^{\mathrm{m}}\psi^{(1)}, \tag{43}$$

which should be compared with (21) above. We can now define the following "bubble Weyl time"

$$\frac{\delta}{\delta\tau_{\mathrm{W}}(\mathbf{x})} := -\frac{1}{2\hbar}\bar{h}_{ik}\bar{h}_{jl}\frac{\delta S_0^{\mathrm{W}}}{\delta \bar{K}_{kl}^{\mathrm{T}}}\frac{\delta}{\delta \bar{K}_{ij}^{\mathrm{T}}} + \left({}^{(3)}\bar{R}_{ij} + \partial_i \bar{D}_j\right)^{\mathrm{T}}\frac{\delta}{\delta \bar{K}_{ij}^{\mathrm{T}}} + 2\bar{K}_{ij}^{\mathrm{T}}\frac{\delta}{\delta \bar{h}_{ij}}, \tag{44}$$

and one arrives at the "Weyl-Tomonaga-Schwinger" equation for $\psi^{(1)}$,

$$\mathrm{i}\hbar\frac{\delta\psi^{(1)}}{\delta\tau_{\mathrm{W}}} = \hat{\mathcal{H}}_{\perp}^{\mathrm{m}}\psi^{(1)}. \tag{45}$$

Comparing the Weyl bubble time τ_{W} with the τ in (22), we immediately recognize the absence of a and $\bar{K}$. But Weyl time has additional components compared with (22): time passes as a result of the dynamics in the extended configuration space. Since there is no scale part, the Weyl time is scale-less or, in other words, it is a pure shape time. This will be different if, for example, an R^2 term is added; the additional term leads to a scale time, so that the full action gives the total "higher-derivative" bubble time. The emergence of a scale-less time in the Weyl theory is symbolized in Figure 4. Here, time is only defined from the semiclassical *shape* degrees of freedom.

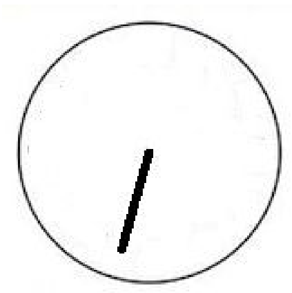

Figure 4. Emergence of scale-less time

We note that, as in the case of quantum GR, τ_{W} is not a scalar function [9]. We also note that $\psi^{(1)}$ is conformally invariant and that higher orders in α_{W}^{-1} give quantum gravitational correction terms.

An important consequence from these discussions is that semiclassical time is ambiguous with respect to the underlying theory. Both quantum GR and quantum Weyl gravity action can give rise (formally) to semiclassical time. The qualitative difference between the two times cannot be determined solely from the (functional) Schrödinger equation; one needs to go to higher orders for a (in principle) measurable distinction to emerge.

Another interesting question is if scale-ful and shape-ful times can emerge from a scale-less quantum gravity theory. This possibility would imply a breaking of conformal symmetry in the transition from Planckian to lower energy scales. One can represent this question visually as in Figure 5. This question will be discussed in [6].

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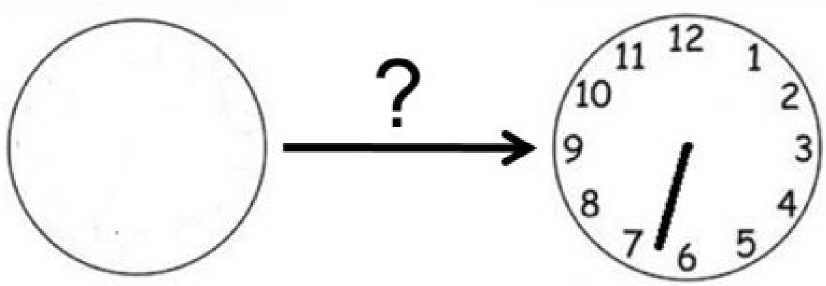

Figure 5. Emergence of full WKB time from a scale-less quantum theory of gravity?

5. Conclusion

Let us briefly summarize our main points as follows:

- The canonical quantization of Weyl gravity can be performed analogously to GR, but the structure of the configuration space is different: it consists of both the three-metric and the extrinsic curvature.
- We have performed a consistent decomposition of variables into scale part and conformally invariant part. The constraints can be rewritten in terms of the conformally invariant parts.
- We have displayed and discussed the Weyl-Wheeler-DeWitt equation and the Weyl diffeomorphism constraints. There are also new constraints which show that the wave functional is conformally invariant.
- We have discussed the semiclassical expansion and the recovery of time for both quantum GR and the quantum Weyl theory. Whereas in the former we get both a scale and a shape time, the latter only leads to a shape time.

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