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The alternating central extension for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Abstract

This paper is about the positive part U_q^+ of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. The algebra U_q^+ has a presentation with two generators A, B that satisfy the cubic q-Serre relations. Recently we introduced a type of element in U_q^+, said to be alternating. Each alternating element commutes with exactly one of A, B, $qBA - q^{-1}AB$, $qAB - q^{-1}BA$; this gives four types of alternating elements. There are infinitely many alternating elements of each type, and these mutually commute. In the present paper we use the alternating elements to obtain a central extension $\mathcal{U}_q^+$ of U_q^+. We define $\mathcal{U}_q^+$ by generators and relations. These generators, said to be alternating, are in bijection with the alternating elements of U_q^+. We display a surjective algebra homomorphism $\mathcal{U}_q^+ \to U_q^+$ that sends each alternating generator of $\mathcal{U}_q^+$ to the corresponding alternating element in U_q^+. We adjust this homomorphism to obtain an algebra isomorphism $\mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$ where $\mathbb{F}$ is the ground field and $\{z_n\}_{n=1}^{\infty}$ are mutually commuting indeterminates. We show that the alternating generators form a PBW basis for $\mathcal{U}_q^+$. We discuss how $\mathcal{U}_q^+$ is related to the work of Baseilhac, Koizumi, Shigechi concerning the q-Onsager algebra and integrable lattice models.

1. Introduction

The q-Onsager algebra $\mathcal{O}_q$ is often used to investigate integrable lattice models [1,2,4–8,10]. In [6] Baseilhac and Koizumi introduced a current algebra $\mathcal{A}_q$ for $\mathcal{O}_q$, in order to solve boundary

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integrable systems with hidden symmetries. In [10, Definition 3.1] Baseilhac and Shigechi gave a presentation of $\mathcal{A}_q$ by generators and relations. The presentation is a bit complicated, and the precise relationship between $\mathcal{A}_q$ and $\mathcal{O}_q$ is presently unknown. However see [5, Conjectures 1, 2] and [20, Conjectures 4.5, 4.6]. Hoping to shed light on the above relationship, in the present paper we consider a limiting case in which the technical details are less complicated. Following [21, Section 1] we replace $\mathcal{O}_q$ by the positive part U_q^+ of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. We introduce an algebra $\mathcal{U}_q^+$ that is related to U_q^+ in roughly the same way that $\mathcal{A}_q$ is related to $\mathcal{O}_q$. We describe in detail how $\mathcal{U}_q^+$ is related to U_q^+. We will summarize our results after a few comments.

We now give some background information about U_q^+. The algebra U_q^+ has a presentation with two generators A, B that satisfy the cubic q-Serre relations; see Definition 2.1 below. In [21] we introduced a type of element in U_q^+, said to be alternating. As we showed in [21, Lemma 5.11], each alternating element commutes with exactly one of A, B, $qBA - q^{-1}AB$, $qAB - q^{-1}BA$. This gives four types of alternating elements, denoted

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}, \quad \{G_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}.$$

By [21, Lemma 5.11] the alternating elements of each type mutually commute.

The alternating elements arise naturally in the following way. Start with the free algebra $\mathbb{V}$ on two generators x, y. The standard (linear) basis for $\mathbb{V}$ consists of the words in x, y. In [14,15] M. Rosso introduced an algebra structure on $\mathbb{V}$, called a q-shuffle algebra. For $u, v \in \{x, y\}$ their q-shuffle product is $u \star v = uv + q^{\langle u,v\rangle} vu$, where $\langle u, v\rangle = 2$ (resp. $\langle u, v\rangle = -2$) if $u = v$ (resp. $u \neq v$). Rosso gave an injective algebra homomorphism $\natural$ from U_q^+ into the q-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$. By [21, Definition 5.2] the map $\natural$ sends

$$\begin{array}{llll} W_0 \mapsto x, & W_{-1} \mapsto xyx, & W_{-2} \mapsto xyxyx, & \ldots \\ W_1 \mapsto y, & W_2 \mapsto yxy, & W_3 \mapsto yxyxy, & \ldots \\ G_1 \mapsto yx, & G_2 \mapsto yxyx, & G_3 \mapsto yxyxyx, & \ldots \\ \tilde{G}_1 \mapsto xy, & \tilde{G}_2 \mapsto xyxy, & \tilde{G}_3 \mapsto xyxyxy, & \ldots \end{array}$$

In [21] we used $\natural$ to obtain many relations involving the alternating elements; see Lemmas 2.3, 2.4 below. These relations resemble the defining relations for $\mathcal{A}_q$ found in [10, Definition 3.1]. We will say more about Lemmas 2.3, 2.4 shortly. In [21, Section 10] we used the alternating elements to obtain some PBW bases for U_q^+. For instance, in [21, Theorem 10.1] we showed that the elements in order

$$\{W_{-k}\}_{k\in\mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{W_{k+1}\}_{k\in\mathbb{N}}$$

give a PBW basis for U_q^+, said to be alternating [21, Definition 10.3].

We now summarize the main results of the present paper. We define an algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}} \tag{1}$$

and the relations in Lemmas 2.3, 2.4. The generators (1) are called alternating. By construction there exists a surjective algebra homomorphism $\mathcal{U}_q^+ \to U_q^+$ that sends

$$\mathcal{W}_{-k} \mapsto W_{-k}, \qquad \mathcal{W}_{k+1} \mapsto W_{k+1}, \qquad \mathcal{G}_k \mapsto G_k, \qquad \tilde{\mathcal{G}}_k \mapsto \tilde{G}_k$$

for $k \in \mathbb{N}$. As we will see, this map is not injective. Denote the ground field by $\mathbb{F}$ and let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \dots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$. We display an algebra isomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$ that sends

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$. In particular φ sends

$$\mathcal{W}_0 \mapsto W_0 \otimes 1, \qquad \mathcal{W}_1 \mapsto W_1 \otimes 1.$$

We use φ to obtain the following results. Let $\mathcal{Z}$ denote the center of $\mathcal{U}_q^+$. We show that $\mathcal{Z}$ is generated by $\{Z_n^\vee\}_{n=1}^\infty$, where

$$Z_n^\vee = \sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}.$$

We show that for $n \geq 1$, φ sends $Z_n^\vee \mapsto 1 \otimes z_n^\vee$ where $z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}$. We show that $\{Z_n^\vee\}_{n=1}^\infty$ are algebraically independent. Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\mathcal{W}_0, \mathcal{W}_1$. We show that the algebra $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ is isomorphic to U_q^+. We show that the multiplication map

$$\begin{aligned} \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} &\to \mathcal{U}_q^+ \\ w \otimes z &\mapsto wz \end{aligned}$$

is an algebra isomorphism. We show that the alternating generators in order

$$\{\mathcal{W}_{-k}\}_{k \in \mathbb{N}}, \qquad \{\mathcal{G}_{k+1}\}_{k \in \mathbb{N}}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k \in \mathbb{N}}, \qquad \{\mathcal{W}_{k+1}\}_{k \in \mathbb{N}}$$

give a PBW basis for $\mathcal{U}_q^+$. Motivated by the above results, near the end of the paper we give some conjectures concerning $\mathcal{A}_q$ and $\mathcal{O}_q$.

The paper is organized as follows. In Section 2 we give some background information about U_q^+. In Section 3 we introduce the algebra $\mathcal{U}_q^+$ and describe its basic properties. In Section 4 we obtain some results about the polynomial algebra $\mathbb{F}[z_1, z_2, \dots]$ that will be used in later sections. In Section 5 we show that the map φ is an algebra isomorphism. In Section 6 we describe the center of $\mathcal{U}_q^+$ and also the subalgebra of $\mathcal{U}_q^+$ generated by $\mathcal{W}_0, \mathcal{W}_1$. In Section 7 we describe several ideals of $\mathcal{U}_q^+$, and in Section 8 we describe some symmetries of $\mathcal{U}_q^+$. In Section 9 we describe a grading of $\mathcal{U}_q^+$, that gets used in Section 10 to establish a PBW basis for $\mathcal{U}_q^+$. In Section 11 we give some conjectures concerning $\mathcal{A}_q$ and $\mathcal{O}_q$. Appendix A contains some technical details.

2. The algebra U_q^+

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. Let $\mathbb{F}$ denote a field. We will be discussing vector spaces, tensor products, and algebras. Each vector space and tensor product discussed is over $\mathbb{F}$. Each algebra discussed is associative, over $\mathbb{F}$, and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{Z}.$$

For elements X, Y in any algebra, define their commutator and q-commutator by

$$[X, Y] = XY - YX, \qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$

Definition 2.1. (See [13, Corollary 3.2.6].) Define the algebra U_q^+ by generators A, B and relations

$$[A, [A, [A, B]_q]_{q^{-1}}] = 0, \qquad [B, [B, [B, A]_q]_{q^{-1}}] = 0. \tag{2}$$

We call U_q^+ the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$. The relations (2) are called the *q-Serre relations*.

We will be discussing automorphisms and antiautomorphisms. For an algebra $\mathcal{A}$, an *automorphism* of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}$. The *opposite algebra* $\mathcal{A}^{\text{opp}}$ consists of the vector space $\mathcal{A}$ and multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ba$. An *antiautomorphism* of $\mathcal{A}$ is an algebra isomorphism $\mathcal{A} \to \mathcal{A}^{\text{opp}}$.

Lemma 2.2. *There exists an automorphism σ of U_q^+ that swaps A, B. There exists an antiautomorphism S of U_q^+ that fixes each of A, B.*

We mention a grading for the algebra U_q^+. The q-Serre relations are homogeneous in both A and B. Therefore the algebra U_q^+ has an $(\mathbb{N} \times \mathbb{N})$-grading for which A and B are homogeneous, with degrees $(1, 0)$ and $(0, 1)$ respectively. The *trivial* homogeneous component of U_q^+ has degree $(0, 0)$ and is equal to $\mathbb{F}1$.

The alternating elements of U_q^+ were introduced in [21]. There are four types, denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}. \tag{3}$$

As we will review in Lemma 2.9, the above elements are obtained from A, B using a recursive procedure with initial conditions $W_0 = A$ and $W_1 = B$.

In [21] we displayed many relations satisfied by the alternating elements of U_q^+. In the next three lemmas we list some of these relations.

Lemma 2.3. (See [21, Proposition 5.7].) *For $k \in \mathbb{N}$ the following holds in U_q^+:*

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{4}$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1}, \tag{5}$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}. \tag{6}$$

Lemma 2.4. (See [21, Proposition 5.9].) *For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:*

$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \tag{7}$$

$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \tag{8}$$

$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \tag{9}$$

$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \tag{10}$$

$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \tag{11}$$

$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \tag{12}$$

$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \tag{13}$$

$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0. \tag{14}$$

Lemma 2.5. (See [21, Proposition 5.10].) *For $k, \ell \in \mathbb{N}$ the following relations hold in U_q^+:*

$$[W_{-k}, G_\ell]_q = [W_{-\ell}, G_k]_q, \qquad [G_k, W_{\ell+1}]_q = [G_\ell, W_{k+1}]_q, \tag{15}$$

$$[\tilde{G}_k, W_{-\ell}]_q = [\tilde{G}_\ell, W_{-k}]_q, \qquad [W_{\ell+1}, \tilde{G}_k]_q = [W_{k+1}, \tilde{G}_\ell]_q, \tag{16}$$

$$[G_k, \tilde{G}_{\ell+1}] - [G_\ell, \tilde{G}_{k+1}] = q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q, \tag{17}$$

$$[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_\ell, G_{k+1}] = q[W_{\ell+1}, W_{-k}]_q - q[W_{k+1}, W_{-\ell}]_q, \tag{18}$$

$$[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q = q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}], \tag{19}$$

$$[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q = q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}]. \tag{20}$$

Note 2.6. By [3, Propositions 3.1, 3.2] the relations in Lemma 2.5 are implied by the relations in Lemmas 2.3, 2.4. For this reason we will give Lemma 2.5 less emphasis than Lemmas 2.3, 2.4.

Note 2.7. The relations in Lemmas 2.3, 2.4 resemble the defining relations for $\mathcal{A}_q$ found in [10, Definition 3.1].

Consider the four sequences in (3). By (7), (13) the elements of each sequence mutually commute. According to [21, Lemma 5.11],

(i) an alternating element commutes with A if and only if it is among $\{W_{-k}\}_{k\in\mathbb{N}}$;
(ii) an alternating element commutes with B if and only if it is among $\{W_{k+1}\}_{k\in\mathbb{N}}$;
(iii) an alternating element commutes with $[B, A]_q$ if and only if it is among $\{G_{k+1}\}_{k\in\mathbb{N}}$;
(iv) an alternating element commutes with $[A, B]_q$ if and only if it is among $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$.

For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$.

Lemma 2.8. (See [21, Proposition 8.1].) *For $n \geq 1$ the following hold in U_q^+:*

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k}, \tag{21}$$

$$\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{n-1-2k}, \tag{22}$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{2k+1-n}, \tag{23}$$

$$\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{2k+1-n}. \tag{24}$$

Lemma 2.9. (See [21, Proposition 8.2].) *Using the equations below, the alternating elements in U_q^+ are recursively obtained from A, B in the following order:*

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad W_{-2}, \quad W_3, \quad \ldots$$

We have $W_0 = A$ and $W_1 = B$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n W_0 - W_0 W_n}{(1+q^{-2n})(1-q^{-2})}, \tag{25}$$

$$\tilde{G}_n = G_n + \frac{W_0 W_n - W_n W_0}{1 - q^{-2}}, \tag{26}$$

$$W_{-n} = \frac{q W_0 G_n - q^{-1} G_n W_0}{q - q^{-1}}, \tag{27}$$

$$W_{n+1} = \frac{q G_n W_1 - q^{-1} W_1 G_n}{q - q^{-1}}. \tag{28}$$

Lemma 2.10. (See [21, Proposition 5.3].) *The maps σ, S from Lemma 2.2 act on the alternating elements as follows. For $k \in \mathbb{N}$,*

(i) *the map σ sends*

$$W_{-k} \mapsto W_{k+1}, \qquad W_{k+1} \mapsto W_{-k}, \qquad G_k \mapsto \tilde{G}_k, \qquad \tilde{G}_k \mapsto G_k;$$

(ii) *the map S sends*

$$W_{-k} \mapsto W_{-k}, \qquad W_{k+1} \mapsto W_{k+1}, \qquad G_k \mapsto \tilde{G}_k, \qquad \tilde{G}_k \mapsto G_k.$$

Lemma 2.11. (See [21, Section 5].) *The alternating elements of U_q^+ are homogeneous, with degrees shown below:*

Alternating element	Degree
W_{-k}	$(k+1, k)$
W_{k+1}	$(k, k+1)$
G_k	(k, k)
$\tilde{G}_k$	(k, k)

3. The algebra $\mathcal{U}_q^+$

Motivated by Lemmas 2.3, 2.4 and [10, Definition 3.1], we now introduce the algebra $\mathcal{U}_q^+$.

Definition 3.1. We define the algebra $\mathcal{U}_q^+$ by generators

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \quad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \quad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}} \tag{29}$$

and relations

$$[\mathcal{W}_0, \mathcal{W}_{k+1}] = [\mathcal{W}_{-k}, \mathcal{W}_1] = (1-q^{-2})(\tilde{\mathcal{G}}_{k+1} - \mathcal{G}_{k+1}), \tag{30}$$

$$[\mathcal{W}_0, \mathcal{G}_{k+1}]_q = [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_0]_q = (q-q^{-1})\mathcal{W}_{-k-1}, \tag{31}$$

$$[\mathcal{G}_{k+1}, \mathcal{W}_1]_q = [\mathcal{W}_1, \tilde{\mathcal{G}}_{k+1}]_q = (q-q^{-1})\mathcal{W}_{k+2}, \tag{32}$$

$$[\mathcal{W}_{-k}, \mathcal{W}_{-\ell}] = 0, \qquad [\mathcal{W}_{k+1}, \mathcal{W}_{\ell+1}] = 0, \tag{33}$$

$$[\mathcal{W}_{-k}, \mathcal{W}_{\ell+1}] + [\mathcal{W}_{k+1}, \mathcal{W}_{-\ell}] = 0, \tag{34}$$

$$[\mathcal{W}_{-k}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{-\ell}] = 0, \tag{35}$$

$$[\mathcal{W}_{-k}, \tilde{\mathcal{G}}_{\ell+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{-\ell}] = 0, \tag{36}$$

$$[\mathcal{W}_{k+1}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \mathcal{W}_{\ell+1}] = 0, \tag{37}$$

$$[\mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] + [\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{\ell+1}] = 0, \tag{38}$$

$$[\mathcal{G}_{k+1}, \mathcal{G}_{\ell+1}] = 0, \qquad [\tilde{\mathcal{G}}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] = 0, \tag{39}$$

$$[\tilde{\mathcal{G}}_{k+1}, \mathcal{G}_{\ell+1}] + [\mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}] = 0. \tag{40}$$

The generators (29) are called *alternating*. For notational convenience define $\mathcal{G}_0 = 1$ and $\tilde{\mathcal{G}}_0 = 1$.

Lemma 3.2. *For $k, \ell \in \mathbb{N}$ the following relations hold in $\mathcal{U}_q^+$:*

$$[\mathcal{W}_{-k}, \mathcal{G}_\ell]_q = [\mathcal{W}_{-\ell}, \mathcal{G}_k]_q, \qquad [\mathcal{G}_k, \mathcal{W}_{\ell+1}]_q = [\mathcal{G}_\ell, \mathcal{W}_{k+1}]_q, \tag{41}$$

$$[\tilde{\mathcal{G}}_k, \mathcal{W}_{-\ell}]_q = [\tilde{\mathcal{G}}_\ell, \mathcal{W}_{-k}]_q, \qquad [\mathcal{W}_{\ell+1}, \tilde{\mathcal{G}}_k]_q = [\mathcal{W}_{k+1}, \tilde{\mathcal{G}}_\ell]_q, \tag{42}$$

$$[\mathcal{G}_k, \tilde{\mathcal{G}}_{\ell+1}] - [\mathcal{G}_\ell, \tilde{\mathcal{G}}_{k+1}] = q[\mathcal{W}_{-\ell}, \mathcal{W}_{k+1}]_q - q[\mathcal{W}_{-k}, \mathcal{W}_{\ell+1}]_q, \tag{43}$$

$$[\tilde{\mathcal{G}}_k, \mathcal{G}_{\ell+1}] - [\tilde{\mathcal{G}}_\ell, \mathcal{G}_{k+1}] = q[\mathcal{W}_{\ell+1}, \mathcal{W}_{-k}]_q - q[\mathcal{W}_{k+1}, \mathcal{W}_{-\ell}]_q, \tag{44}$$

$$[\mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{\ell+1}]_q - [\mathcal{G}_{\ell+1}, \tilde{\mathcal{G}}_{k+1}]_q = q[\mathcal{W}_{-\ell}, \mathcal{W}_{k+2}] - q[\mathcal{W}_{-k}, \mathcal{W}_{\ell+2}], \tag{45}$$

$$[\tilde{\mathcal{G}}_{k+1}, \mathcal{G}_{\ell+1}]_q - [\tilde{\mathcal{G}}_{\ell+1}, \mathcal{G}_{k+1}]_q = q[\mathcal{W}_{\ell+1}, \mathcal{W}_{-k-1}] - q[\mathcal{W}_{k+1}, \mathcal{W}_{-\ell-1}]. \tag{46}$$

Proof. By Note 2.6. □

The algebras $\mathcal{U}_q^+$ and U_q^+ are related as follows.

Lemma 3.3. *There exists an algebra homomorphism $\gamma : \mathcal{U}_q^+ \to U_q^+$ that sends*

$$\mathcal{W}_{-n} \mapsto W_{-n}, \qquad \mathcal{W}_{n+1} \mapsto W_{n+1}, \qquad \mathcal{G}_n \mapsto G_n, \qquad \tilde{\mathcal{G}}_n \mapsto \tilde{G}_n$$

for $n \in \mathbb{N}$. Moreover γ is surjective.

Proof. By Definition 3.1. □

The kernel of γ is described in Section 7.

Definition 3.4. Let $\{z_n\}_{n=1}^{\infty}$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \ldots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \ldots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

Lemma 3.5. *There exists an algebra homomorphism $\eta : \mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \ldots]$ that sends*

$$\mathcal{W}_{-n} \mapsto 0, \qquad \mathcal{W}_{n+1} \mapsto 0, \qquad \mathcal{G}_n \mapsto z_n, \qquad \tilde{\mathcal{G}}_n \mapsto z_n \tag{47}$$

for $n \in \mathbb{N}$. Moreover η is surjective.

Proof. Use Definition 3.1. $\square$

The kernel of η is described in Section 7.

We have indicated how $\mathcal{U}_q^+$ is related to U_q^+ and $\mathbb{F}[z_1, z_2, \ldots]$. Next we consider how $\mathcal{U}_q^+$ is related to $U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$.

Lemma 3.6. *There exists an algebra homomorphism $\varphi : \mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ that sends*

$$\mathcal{W}_{-n} \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k, \qquad \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$$

$$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k, \qquad \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$$

for $n \in \mathbb{N}$. In particular φ sends

$$\mathcal{W}_0 \mapsto W_0 \otimes 1, \qquad \mathcal{W}_1 \mapsto W_1 \otimes 1. \tag{48}$$

Proof. Use Lemmas 2.3, 2.4 and Definition 3.1. $\square$

In Section 5 we show that φ is an isomorphism.

Next we consider how γ is related to φ. There exists an algebra homomorphism $\theta : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}$ that sends $z_n \mapsto 0$ for $n \geq 1$. The map θ is surjective. Consequently the vector space $\mathbb{F}[z_1, z_2, \ldots]$ is the direct sum of $\mathbb{F}1$ and the kernel of θ. This kernel is the ideal of $\mathbb{F}[z_1, z_2, \ldots]$ generated by $\{z_n\}_{n=1}^{\infty}$.

Lemma 3.7. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\ \gamma \downarrow & & \downarrow \text{id}\otimes\theta \\ U_q^+ & \xrightarrow[x \mapsto x \otimes 1]{} & U_q^+ \otimes \mathbb{F} \end{array} \qquad \text{id} = \text{identity map}$$

Proof. Chase each alternating generator of $\mathcal{U}_q^+$ around the diagram, using Lemmas 3.3, 3.6 and the definition of θ. $\square$

Next we consider how η is related to φ. Since U_q^+ is generated by A, B and the q-Serre relations are homogeneous, there exists an algebra homomorphism $\vartheta : U_q^+ \to \mathbb{F}$ that sends $A \mapsto$

0 and $B \mapsto 0$. The map ϑ is surjective, so U_q^+ is the direct sum of $\mathbb{F}1$ and the kernel of ϑ. The following are the same: (i) the kernel of ϑ; (ii) the two-sided ideal of U_q^+ generated by A, B; (iii) the sum of the nontrivial homogeneous components of U_q^+. By Lemma 2.11 the map ϑ sends

$$W_{-k} \mapsto 0, \qquad W_{k+1} \mapsto 0, \qquad G_{k+1} \mapsto 0, \qquad \tilde{G}_{k+1} \mapsto 0 \tag{49}$$

for $k \in \mathbb{N}$.

Lemma 3.8. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots] \\ \eta \downarrow & & \downarrow \vartheta \otimes \mathrm{id} \\ \mathbb{F}[z_1, z_2, \dots] & \xrightarrow[x \mapsto 1 \otimes x]{} & \mathbb{F} \otimes \mathbb{F}[z_1, z_2, \dots] \end{array}$$

Proof. Chase each alternating generator of $\mathcal{U}_q^+$ around the diagram, using Lemmas 3.5, 3.6 and (49). □

Next we describe some symmetries of $\mathcal{U}_q^+$.

Lemma 3.9. *There exists an automorphism σ of $\mathcal{U}_q^+$ that sends*

$$\mathcal{W}_{-k} \mapsto \mathcal{W}_{k+1}, \qquad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{-k}, \qquad \mathcal{G}_k \mapsto \tilde{\mathcal{G}}_k, \qquad \tilde{\mathcal{G}}_k \mapsto \mathcal{G}_k$$

for $k \in \mathbb{N}$. There exists an antiautomorphism S of $\mathcal{U}_q^+$ that sends

$$\mathcal{W}_{-k} \mapsto \mathcal{W}_{-k}, \qquad \mathcal{W}_{k+1} \mapsto \mathcal{W}_{k+1}, \qquad \mathcal{G}_k \mapsto \tilde{\mathcal{G}}_k, \qquad \tilde{\mathcal{G}}_k \mapsto \mathcal{G}_k$$

for $k \in \mathbb{N}$.

Proof. Use Definition 3.1. □

Next we introduce a grading for $\mathcal{U}_q^+$.

Lemma 3.10. *The algebra $\mathcal{U}_q^+$ has an $(\mathbb{N} \times \mathbb{N})$-grading for which the alternating generators are homogeneous, with degrees shown below:*

Alternating generator	Degree
$\mathcal{W}_{-k}$	$(k+1, k)$
$\mathcal{W}_{k+1}$	$(k, k+1)$
$\mathcal{G}_k$	(k, k)
$\tilde{\mathcal{G}}_k$	(k, k)

Proof. The defining relations for $\mathcal{U}_q^+$ are homogeneous with respect to the above degree assignment. □

4. The polynomial algebra $\mathbb{F}[z_1, z_2, \ldots]$

Recall the algebra $\mathbb{F}[z_1, z_2, \ldots]$ from Definition 3.4. In this section we obtain some results about $\mathbb{F}[z_1, z_2, \ldots]$ that will be used in later sections.

Definition 4.1. For $n \in \mathbb{N}$ define

$$z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}. \tag{50}$$

Note that $z_0^\vee = 1$.

Example 4.2. We have

$$\begin{aligned} z_1^\vee &= (q + q^{-1}) z_1, \\ z_2^\vee &= (q^2 + q^{-2}) z_2 + z_1^2, \\ z_3^\vee &= (q^3 + q^{-3}) z_3 + (q + q^{-1}) z_1 z_2. \end{aligned}$$

Lemma 4.3. *For $n \geq 1$ the element $z_n^\vee$ is a homogeneous polynomial of total degree n in $z_1, z_2, \ldots, z_n$, where we view each z_k as having degree k. For this polynomial the coefficient of z_n is $q^n + q^{-n}$.*

Proof. By (50). □

For $n \geq 1$, we now seek to express z_n as a polynomial in $z_1^\vee, z_2^\vee, \ldots, z_n^\vee$. Towards this goal, we first express z_n as a polynomial in $z_n^\vee$ and $z_1, z_2, \ldots, z_{n-1}$.

Lemma 4.4. *For $n \geq 1$,*

$$z_n = \frac{z_n^\vee - \sum_{k=1}^{n-1} z_k z_{n-k} q^{n-2k}}{q^n + q^{-n}}. \tag{51}$$

Proof. Solve (50) for z_n. □

For $n \geq 1$, we use Lemma 4.4 and induction on n to express z_n as a polynomial in $z_1^\vee, z_2^\vee, \ldots, z_n^\vee$.

Example 4.5. We have

$$\begin{aligned} z_1 &= \frac{z_1^\vee}{q + q^{-1}}, \\ z_2 &= \frac{(q + q^{-1})^2 z_2^\vee - (z_1^\vee)^2}{(q + q^{-1})^2 (q^2 + q^{-2})}, \\ z_3 &= \frac{(q + q^{-1})^2 (q^2 + q^{-2}) z_3^\vee - (q + q^{-1})^2 z_1^\vee z_2^\vee + (z_1^\vee)^3}{(q + q^{-1})^2 (q^2 + q^{-2})(q^3 + q^{-3})}. \end{aligned}$$

Lemma 4.6. *For $n \geq 1$ the element z_n is a homogeneous polynomial of total degree n in $z_1^\vee, z_2^\vee, \ldots, z_n^\vee$, where we view each $z_k^\vee$ as having degree k. For this polynomial the coefficient of $z_n^\vee$ is $(q^n + q^{-n})^{-1}$.*

Proof. By (51) and induction on n. □

Corollary 4.7. *The elements $\{z_n^\vee\}_{n=1}^\infty$ are algebraically independent and generate $\mathbb{F}[z_1, z_2, \ldots]$.*

Proof. The first assertion follows from Lemma 4.3 and since $\{z_n\}_{n=1}^\infty$ are algebraically independent. The second assertion follows from Lemma 4.6 and since $\{z_n\}_{n=1}^\infty$ generate $\mathbb{F}[z_1, z_2, \ldots]$. □

Corollary 4.8. *There exists an automorphism of the algebra $\mathbb{F}[z_1, z_2, \ldots]$ that sends $z_n \mapsto z_n^\vee$ for $n \geq 1$.*

Proof. This is a reformulation of Corollary 4.7. □

5. The map φ is an isomorphism

Recall the map φ from Lemma 3.6. In this section we show that φ is an isomorphism. The following definition is motivated by (21).

Definition 5.1. For $n \geq 1$ define

$$Z_n^\vee = \sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}. \tag{52}$$

For notational convenience define $Z_0^\vee = 1$.

For any algebra $\mathcal{A}$, an element in $\mathcal{A}$ is *central* whenever it commutes with every element of $\mathcal{A}$.

Lemma 5.2. *For $n \in \mathbb{N}$ the element $Z_n^\vee$ is central in $\mathcal{U}_q^+$.*

The proof of Lemma 5.2 is slightly technical, and contained in the Appendix.

Note 5.3. The central elements (52) resemble the central elements for $\mathcal{A}_q$ given in [5, Lemma 2.1].

Lemma 5.4. *For $n \in \mathbb{N}$ the map φ sends $Z_n^\vee \mapsto 1 \otimes z_n^\vee$.*

Proof. Expand $\varphi(Z_n^\vee)$ using Lemma 3.6 and Definition 5.1. Evaluate the result using (21) and (50). □

Definition 5.5. Let $\mathcal{Z}$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\{Z_n^\vee\}_{n=1}^\infty$.

For an algebra $\mathcal{A}$, its central elements form a subalgebra called the *center* of $\mathcal{A}$. By Lemma 5.2 the subalgebra $\mathcal{Z}$ is contained in the center of $\mathcal{U}_q^+$. In Section 6 we show that $\mathcal{Z}$ is equal to the center of $\mathcal{U}_q^+$.

Next we introduce some elements $\{Z_n\}_{n\in\mathbb{N}}$ in $\mathcal{Z}$, that are related to $\{Z_n^\vee\}_{n\in\mathbb{N}}$ in the same way that $\{z_n\}_{n\in\mathbb{N}}$ are related to $\{z_n^\vee\}_{n\in\mathbb{N}}$. The elements $\{Z_n\}_{n\in\mathbb{N}}$ are defined recursively.

Definition 5.6. Define $Z_0 = 1$ and for $n \geq 1$,

$$Z_n = \frac{Z_n^\vee - \sum_{k=1}^{n-1} Z_k Z_{n-k} q^{n-2k}}{q^n + q^{-n}}. \tag{53}$$

Lemma 5.7. *For $n \in \mathbb{N}$,*

$$Z_n^\vee = \sum_{k=0}^{n} Z_k Z_{n-k} q^{n-2k}. \tag{54}$$

Proof. Solve (53) for $Z_n^\vee$. □

Lemma 5.8. *The subalgebra $\mathcal{Z}$ from Definition 5.5 is generated by $\{Z_n\}_{n=1}^\infty$.*

Proof. By Definition 5.5 and Lemma 5.7. □

Lemma 5.9. *The map φ sends $Z_n \mapsto 1 \otimes z_n$ for $n \in \mathbb{N}$.*

Proof. We use induction on n. The result holds for $n = 0$, since $Z_0 = 1$ and $z_0 = 1$. Next assume $n \geq 1$. Using Lemma 5.4 and (51), (53) along with induction,

$$\begin{aligned}
\varphi(Z_n) &= \frac{\varphi(Z_n^\vee) - \sum_{k=1}^{n-1} \varphi(Z_k)\varphi(Z_{n-k}) q^{n-2k}}{q^n + q^{-n}} \\
&= \frac{1 \otimes z_n^\vee - \sum_{k=1}^{n-1} (1 \otimes z_k)(1 \otimes z_{n-k}) q^{n-2k}}{q^n + q^{-n}} \\
&= 1 \otimes \frac{z_n^\vee - \sum_{k=1}^{n-1} z_k z_{n-k} q^{n-2k}}{q^n + q^{-n}} \\
&= 1 \otimes z_n. \qquad \square
\end{aligned}$$

We have a comment.

Lemma 5.10. *The map φ sends $\mathcal{Z}$ onto $\mathbb{F} \otimes \mathbb{F}[z_1, z_2, \ldots]$.*

Proof. By Lemmas 5.8, 5.9. □

Next we show that the algebra $\mathcal{U}_q^+$ is generated by $\mathcal{W}_0, \mathcal{W}_1, \mathcal{Z}$.

Lemma 5.11. *Using the equations below, the alternating generators of $\mathcal{U}_q^+$ are recursively obtained from $\mathcal{W}_0, \mathcal{W}_1, Z_1^\vee, Z_2^\vee, \ldots$ in the following order:*

$$\mathcal{W}_0, \quad \mathcal{W}_1, \quad \mathcal{G}_1, \quad \tilde{\mathcal{G}}_1, \quad \mathcal{W}_{-1}, \quad \mathcal{W}_2, \quad \mathcal{G}_2, \quad \tilde{\mathcal{G}}_2, \quad \mathcal{W}_{-2}, \quad \mathcal{W}_3, \quad \ldots$$

For $n \geq 1$,

$$\mathcal{G}_n = \frac{Z_n^\vee + q\sum_{k=0}^{n-1} \mathcal{W}_{-k}\mathcal{W}_{n-k}q^{n-1-2k} - \sum_{k=1}^{n-1} \mathcal{G}_k\tilde{\mathcal{G}}_{n-k}q^{n-2k}}{q^n + q^{-n}} + \frac{\mathcal{W}_n\mathcal{W}_0 - \mathcal{W}_0\mathcal{W}_n}{(1+q^{-2n})(1-q^{-2})}, \tag{55}$$

$$\tilde{\mathcal{G}}_n = \mathcal{G}_n + \frac{\mathcal{W}_0\mathcal{W}_n - \mathcal{W}_n\mathcal{W}_0}{1-q^{-2}}, \tag{56}$$

$$\mathcal{W}_{-n} = \frac{q\mathcal{W}_0\mathcal{G}_n - q^{-1}\mathcal{G}_n\mathcal{W}_0}{q-q^{-1}}, \tag{57}$$

$$\mathcal{W}_{n+1} = \frac{q\mathcal{G}_n\mathcal{W}_1 - q^{-1}\mathcal{W}_1\mathcal{G}_n}{q-q^{-1}}. \tag{58}$$

Proof. Equation (56) is from (30). To obtain (55), add q^n times (56) to (52), and solve the resulting equation for $\mathcal{G}_n$. Equations (57), (58) are from (31), (32). □

Corollary 5.12. *The algebra $\mathcal{U}_q^+$ is generated by $\mathcal{W}_0, \mathcal{W}_1, \mathcal{Z}$.*

Proof. By Lemma 5.11 and since $\{Z_n^\vee\}_{n=1}^\infty$ generate $\mathcal{Z}$. □

Note 5.13. Lemma 5.11 and Corollary 5.12 resemble the results for $\mathcal{A}_q$ given in [5, Proposition 3.1] and [5, Corollary 3.1].

Lemma 5.14. *In $\mathcal{U}_q^+$ we have*

$$[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_q]_{q^{-1}}] = 0, \tag{59}$$

$$[\mathcal{W}_1, [\mathcal{W}_1, [\mathcal{W}_1, \mathcal{W}_0]_q]_{q^{-1}}] = 0. \tag{60}$$

Proof. Consider (59). Setting $n = 1$ in (55), (57) we obtain

$$\mathcal{G}_1 = \frac{Z_1^\vee + q\mathcal{W}_0\mathcal{W}_1}{q+q^{-1}} + \frac{[\mathcal{W}_1, \mathcal{W}_0]}{(1+q^{-2})(1-q^{-2})} = \frac{Z_1^\vee}{q+q^{-1}} - q\frac{[\mathcal{W}_0, \mathcal{W}_1]_{q^{-1}}}{q^2-q^{-2}} \tag{61}$$

and

$$\mathcal{W}_{-1} = \frac{[\mathcal{W}_0, \mathcal{G}_1]_q}{q-q^{-1}}. \tag{62}$$

The elements $\mathcal{W}_0, \mathcal{W}_{-1}$ commute by (33), and $Z_1^\vee$ is central by Lemma 5.2. By these comments and (61), (62) we obtain

$$0 = [\mathcal{W}_0, \mathcal{W}_{-1}] = \frac{[\mathcal{W}_0, [\mathcal{W}_0, \mathcal{G}_1]_q]}{q-q^{-1}}$$

$$
\begin{aligned}
&= \frac{[\mathcal{W}_0, [\mathcal{W}_0, \mathcal{G}_1]]_q}{q - q^{-1}} \\
&= -q \frac{[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_{q^{-1}}]]_q}{(q - q^{-1})(q^2 - q^{-2})} \\
&= -q \frac{[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_q]_{q^{-1}}]}{(q - q^{-1})(q^2 - q^{-2})},
\end{aligned}
$$

which implies (59). The equation (60) is similarly obtained. □

Note 5.15. Lemma 5.14 resembles a result for $\mathcal{A}_q$ given in [5, Line (3.7)].

Lemma 5.16. *There exists an algebra homomorphism $\phi : U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \to \mathcal{U}_q^+$ that sends*

$$
W_0 \otimes 1 \mapsto \mathcal{W}_0, \qquad W_1 \otimes 1 \mapsto \mathcal{W}_1, \qquad 1 \otimes z_n \mapsto Z_n, \qquad n \geq 1.
$$

Proof. By Lemma 5.14 and since $\{Z_n\}_{n=1}^\infty$ are central in $\mathcal{U}_q^+$. □

Theorem 5.17. *The maps φ, ϕ are inverses. Moreover they are bijections.*

Proof. By Lemma 5.8 and Corollary 5.12, the algebra $\mathcal{U}_q^+$ is generated by $\mathcal{W}_0$, $\mathcal{W}_1$, $\{Z_n\}_{n=1}^\infty$. Each of these generators is fixed by the composition $\phi \circ \varphi$, in view of (48) and Lemmas 5.9, 5.16. Therefore $\phi \circ \varphi$ is the identity map on $\mathcal{U}_q^+$. By Definition 2.1 and the construction, the algebra $U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ is generated by $W_0 \otimes 1$, $W_1 \otimes 1$, $\{1 \otimes z_n\}_{n=1}^\infty$. Each of these generators is fixed by $\varphi \circ \phi$, in view of (48) and Lemmas 5.9, 5.16. Therefore $\varphi \circ \phi$ is the identity map on $U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$. By these comments the maps φ, ϕ are inverses, and hence bijections. □

6. Two subalgebras of $\mathcal{U}_q^+$

In this section we describe two subalgebras of $\mathcal{U}_q^+$: the center of $\mathcal{U}_q^+$ and the subalgebra generated by $\mathcal{W}_0$, $\mathcal{W}_1$.

To obtain the center of $\mathcal{U}_q^+$ we will use the following fact.

Lemma 6.1. (See [22].) *The center of U_q^+ is $\mathbb{F}1$.*

Recall the subalgebra $\mathcal{Z}$ of $\mathcal{U}_q^+$ described in Definition 5.5 and Lemma 5.8.

Proposition 6.2. *The following* (i)–(iv) *hold:*

(i) *$\mathcal{Z}$ is the center of $\mathcal{U}_q^+$;*
(ii) *there exists an algebra isomorphism $\mathbb{F}[z_1, z_2, \ldots] \to \mathcal{Z}$ that sends $z_n \mapsto Z_n$ for $n \in \mathbb{N}$;*
(iii) *the above isomorphism sends $z_n^\vee \mapsto Z_n^\vee$ for $n \in \mathbb{N}$;*
(iv) *the inverse isomorphism is the restriction of η to $\mathcal{Z}$.*

Proof. (i) By Lemma 6.1 and the construction, the center of $U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ is equal to $\mathbb{F} \otimes \mathbb{F}[z_1, z_2, \ldots]$. Applying $\phi = \varphi^{-1}$ and Lemma 5.10, we find that the center of $\mathcal{U}_q^+$ is equal to $\mathcal{Z}$.

(ii), (iv) Since $\{Z_n\}_{n=1}^{\infty}$ mutually commute, there exists an algebra homomorphism $\sharp : \mathbb{F}[z_1, z_2, \dots] \to \mathcal{U}_q^+$ that sends $z_n \mapsto Z_n$ for $n \geq 1$. The map $\sharp$ has image $\mathcal{Z}$ by Lemma 5.8. By Lemma 3.8 and Lemma 5.9, the map η sends $Z_n \mapsto z_n$ for $n \geq 1$. So the restriction of η to $\mathcal{Z}$ is the inverse of $\sharp$. These maps are invertible and hence isomorphisms.

(iii) Compare (50), (54) and use (ii) above. □

Corollary 6.3. *The elements* $\{Z_n\}_{n=1}^{\infty}$ *are algebraically independent. Moreover the elements* $\{Z_n^{\vee}\}_{n=1}^{\infty}$ *are algebraically independent.*

Proof. The first assertion follows from Proposition 6.2(ii). The second assertion follows from Corollary 4.7 and Proposition 6.2(iii). □

Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\mathcal{W}_0, \mathcal{W}_1$.

Proposition 6.4. *The following* (i), (ii) *hold:*

(i) *there exists an algebra isomorphism* $U_q^+ \to \langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ *that sends* $W_0 \mapsto \mathcal{W}_0$ *and* $W_1 \mapsto \mathcal{W}_1$*;*
(ii) *the inverse isomorphism is the restriction of* γ *to* $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$*.*

Proof. By Definition 2.1 and Lemma 5.14, there exists an algebra homomorphism $\flat : U_q^+ \to \mathcal{U}_q^+$ that sends $W_0 \mapsto \mathcal{W}_0$ and $W_1 \mapsto \mathcal{W}_1$. By Lemma 3.3 the map γ sends $\mathcal{W}_0 \mapsto W_0$ and $\mathcal{W}_1 \mapsto W_1$. So the restriction of γ to $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ is the inverse of $\flat$. These maps are invertible and hence isomorphisms. □

Next we describe how the subalgebras $\mathcal{Z}$ and $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ are related.

Proposition 6.5. *The multiplication map*

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \to \mathcal{U}_q^+$$
$$w \otimes z \mapsto wz$$

is an algebra isomorphism.

Proof. Let m denote the above multiplication map. The map m is an algebra homomorphism since $\mathcal{Z}$ is central in $\mathcal{U}_q^+$. Let γ_{rest} denote the restriction of γ to $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$. The map $\gamma_{\text{rest}} : \langle \mathcal{W}_0, \mathcal{W}_1 \rangle \to U_q^+$ is an algebra isomorphism by Proposition 6.4(ii). Let η_{rest} denote the restriction of η to $\mathcal{Z}$. The map $\eta_{\text{rest}} : \mathcal{Z} \to \mathbb{F}[z_1, z_2, \dots]$ is an algebra isomorphism by Proposition 6.2(iv). By these comments and Theorem 5.17, the composition

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \xrightarrow[\gamma_{\text{rest}} \otimes \eta_{\text{rest}}]{} U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots] \xrightarrow[\phi]{} \mathcal{U}_q^+ \tag{63}$$

is an algebra isomorphism. The composition (63) is equal to m, since it agrees with m on the generators $\mathcal{W}_0 \otimes 1$, $\mathcal{W}_1 \otimes 1$, $\{1 \otimes Z_n\}_{n=1}^{\infty}$. It follows that m is an algebra isomorphism. □

7. The kernels of γ and η

Recall the maps γ and η from Section 3. In this section we describe their kernels. We begin with γ.

Proposition 7.1. *The following are the same:*

(i) *the kernel of* γ*;*
(ii) *the 2-sided ideal of* $\mathcal{U}_q^+$ *generated by* $\{Z_n\}_{n=1}^{\infty}$*;*
(iii) *the 2-sided ideal of* $\mathcal{U}_q^+$ *generated by* $\{Z_n^{\vee}\}_{n=1}^{\infty}$*.*

Proof. (i), (ii) In the diagram of Lemma 3.7, the two horizontal maps are bijections. So the kernel of γ is the φ-preimage of the kernel of $\mathrm{id} \otimes \theta$. The kernel of $\mathrm{id} \otimes \theta$ is obtained using the description of θ above Lemma 3.7.

(iii) Use (53), (54) and (ii) above. □

Proposition 7.2. *The vector space* $\mathcal{U}_q^+$ *is the direct sum of the following:*

(i) *the kernel of* γ*;*
(ii) *the subalgebra* $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$*.*

Proof. By Proposition 6.4(ii) and linear algebra. □

We turn our attention to η.

Proposition 7.3. *The following are the same:*

(i) *the kernel of* η*;*
(ii) *the 2-sided ideal of* $\mathcal{U}_q^+$ *generated by* $\mathcal{W}_0$*,* $\mathcal{W}_1$*.*

Proof. In the diagram of Lemma 3.8, the two horizontal maps are bijections. So the kernel of η is the φ-preimage of the kernel of $\vartheta \otimes \mathrm{id}$. The kernel of $\vartheta \otimes \mathrm{id}$ is obtained using the description of ϑ above Lemma 3.8. □

Proposition 7.4. *The vector space* $\mathcal{U}_q^+$ *is the direct sum of the following:*

(i) *the center* $\mathcal{Z}$ *of* $\mathcal{U}_q^+$*;*
(ii) *the kernel of* η*.*

Proof. By Proposition 6.2(iv) and linear algebra. □

8. The automorphism σ and antiautomorphism S

In Lemma 2.2 we gave an automorphism σ of U_q^+ and an antiautomorphism S of U_q^+. In Lemma 3.9 we gave the analogous maps for $\mathcal{U}_q^+$. In this section we describe how the maps in Lemmas 2.2 and 3.9 are related.

Lemma 8.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\ \sigma \downarrow & & \downarrow \sigma\otimes \text{id} \\ \mathcal{U}_q^+ & \xrightarrow[\varphi]{} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \end{array}$$

Proof. Each map in the diagram is an algebra homomorphism. To check that the diagram commutes, it suffices to chase each alternating generator of $\mathcal{U}_q^+$ around the diagram. This chasing is routinely accomplished using Lemma 3.6 for the horizontal maps and Lemmas 2.10, 3.9 for the vertical maps. □

Lemma 8.2. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_q^+ & \xrightarrow{\varphi} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\ S \downarrow & & \downarrow S\otimes \text{id} \\ \mathcal{U}_q^+ & \xrightarrow[\varphi]{} & U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \end{array}$$

Proof. Each horizontal (resp. vertical) map in the diagram is an algebra homomorphism (resp. antiautomorphism). To check that the diagram commutes, it suffices to chase each alternating generator of $\mathcal{U}_q^+$ around the diagram. This chasing is routinely accomplished using Lemma 3.6 for the horizontal maps and Lemmas 2.10, 3.9 for the vertical maps. □

Proposition 8.3. *Referring to the algebra $\mathcal{U}_q^+$,*

(i) *the automorphism σ fixes everything in $\mathcal{Z}$;*
(ii) *the antiautomorphism S fixes everything in $\mathcal{Z}$.*

Proof. (i) By Lemma 5.8, it suffices to check that $\sigma(Z_n) = Z_n$ for $n \geq 1$. This checking is routinely accomplished by chasing Z_n around the diagram in Lemma 8.1, using the fact that φ sends $Z_n \mapsto 1 \otimes z_n$ by Lemma 5.9, and $\sigma \otimes \text{id}$ sends $1 \otimes z_n \mapsto 1 \otimes z_n$ by construction.

(ii) By Lemma 5.8 and since $\mathcal{Z}$ is commutative, it suffices to check that $S(Z_n) = Z_n$ for $n \geq 1$. This checking is routinely accomplished by chasing Z_n around the diagram in Lemma 8.2, using the fact that φ sends $Z_n \mapsto 1 \otimes z_n$ and $S \otimes \text{id}$ sends $1 \otimes z_n \mapsto 1 \otimes z_n$ by construction. □

Corollary 8.4. *Referring to the algebra $\mathcal{U}_q^+$, for $n \geq 1$ the element $Z_n^\vee$ is equal to each of the following:*

$$\sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k}, \tag{64}$$

$$\sum_{k=0}^{n} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{2k-n} - q \sum_{k=0}^{n-1} \mathcal{W}_{n-k} \mathcal{W}_{-k} q^{n-1-2k}, \tag{65}$$

$$\sum_{k=0}^{n} \tilde{\mathcal{G}}_k \mathcal{G}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_{n-k} \mathcal{W}_{-k} q^{2k+1-n}, \tag{66}$$

$$\sum_{k=0}^{n} \tilde{\mathcal{G}}_k \mathcal{G}_{n-k} q^{2k-n} - q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{2k+1-n}. \tag{67}$$

Proof. By Definition 5.1 the element $Z_n^\vee$ is equal to the element (64). By Proposition 8.3 the element $Z_n^\vee$ is fixed by σ and S. By Lemma 3.9 the map σ sends the elements

$$(64) \leftrightarrow (66), \qquad (65) \leftrightarrow (67)$$

and S sends the elements

$$(64) \leftrightarrow (65), \qquad (66) \leftrightarrow (67).$$

The result follows. □

It is illuminating to compare Lemma 2.8 with Corollary 8.4.

9. The grading for $\mathcal{U}_q^+$

In Lemma 3.10 we introduced an $(\mathbb{N} \times \mathbb{N})$-grading for $\mathcal{U}_q^+$. In this section we compute the dimension of each homogeneous component, and express the answer using a generating function. Throughout this section let λ, μ denote commuting indeterminates.

We start with some comments about the $(\mathbb{N} \times \mathbb{N})$-grading for U_q^+. This grading was mentioned below Lemma 2.2 and described further in Lemma 2.11.

Definition 9.1. Define a generating function

$$H(\lambda, \mu) = \prod_{n=1}^{\infty} \frac{1}{1-\lambda^n \mu^{n-1}} \frac{1}{1-\lambda^n \mu^n} \frac{1}{1-\lambda^{n-1} \mu^n}. \tag{68}$$

Note 9.2. In the product (68) we expand each factor using $(1-x)^{-1} = 1 + x + x^2 + \cdots$ to express $H(\lambda, \mu)$ as a formal power series in λ, μ. We will do something similar for all the generating functions encountered in this section.

Lemma 9.3. (See [21, Corollary 3.7].) *For $(i, j) \in \mathbb{N} \times \mathbb{N}$ the following are the same:*

(i) *the dimension of the (i, j)-homogeneous component of U_q^+;*
(ii) *the coefficient of $\lambda^i \mu^j$ in $H(\lambda, \mu)$.*

Recall the algebra $\mathbb{F}[z_1, z_2, \ldots]$ from Definition 3.4. Shortly we will endow this algebra with an $(\mathbb{N} \times \mathbb{N})$-grading. The grading is motivated by the following result concerning the $(\mathbb{N} \times \mathbb{N})$-grading of $\mathcal{U}_q^+$.

Lemma 9.4. *For $n \geq 1$ the elements $Z_n, Z_n^\vee$ are homogeneous with degree (n, n).*

Proof. For $Z_n^\vee$ use Lemma 3.10 and Definition 5.1. For Z_n use (53) and induction on n. □

Definition 9.5. We endow the algebra $\mathbb{F}[z_1, z_2, \ldots]$ with an $(\mathbb{N} \times \mathbb{N})$-grading such that z_n is homogeneous with degree (n, n) for $n \geq 1$.

Definition 9.6. Define a generating function

$$Z(\lambda, \mu) = \prod_{n=1}^{\infty} \frac{1}{1 - \lambda^n \mu^n}.$$

Lemma 9.7. *For $(i, j) \in \mathbb{N} \times \mathbb{N}$ the following are the same:*

(i) *the dimension of the (i, j)-homogeneous component of $\mathbb{F}[z_1, z_2, \dots]$;*
(ii) *the coefficient of $\lambda^i \mu^j$ in $Z(\lambda, \mu)$.*

Proof. This is routinely checked. □

We have been discussing an $(\mathbb{N} \times \mathbb{N})$-grading of U_q^+ and an $(\mathbb{N} \times \mathbb{N})$-grading of $\mathbb{F}[z_1, z_2, \dots]$. We now combine these gradings to get an $(\mathbb{N} \times \mathbb{N})$-grading of $U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$.

Lemma 9.8. *The algebra $U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$ has a unique $(\mathbb{N} \times \mathbb{N})$-grading with the following property: for all homogeneous elements $f \in U_q^+$ and $g \in \mathbb{F}[z_1, z_2, \dots]$, the element $f \otimes g \in U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$ is homogeneous with $\deg(f \otimes g) = \deg(f) + \deg(g)$. With respect to this grading each of $W_0 \otimes 1$, $W_1 \otimes 1$, $\{1 \otimes z_n\}_{n=1}^{\infty}$ is homogeneous with degree shown below:*

Element	Degree
$W_0 \otimes 1$	$(1, 0)$
$W_1 \otimes 1$	$(0, 1)$
$1 \otimes z_n$	(n, n)

Proof. By construction. □

Definition 9.9. Define a generating function

$$\begin{aligned}\mathcal{H}(\lambda, \mu) &= H(\lambda, \mu) Z(\lambda, \mu) \\ &= \prod_{n=1}^{\infty} \frac{1}{1 - \lambda^n \mu^{n-1}} \frac{1}{(1 - \lambda^n \mu^n)^2} \frac{1}{1 - \lambda^{n-1} \mu^n}.\end{aligned}$$

Lemma 9.10. *For $(i, j) \in \mathbb{N} \times \mathbb{N}$ the following are the same:*

(i) *the dimension of the (i, j)-homogeneous component of $U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$;*
(ii) *the coefficient of $\lambda^i \mu^j$ in $\mathcal{H}(\lambda, \mu)$.*

Proof. By Lemmas 9.3, 9.7 and Definition 9.9. □

Recall the isomorphism $\phi : U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots] \to \mathcal{U}_q^+$ from Lemma 5.16 and Theorem 5.17.

Lemma 9.11. *For $(i, j) \in \mathbb{N} \times \mathbb{N}$ the isomorphism ϕ sends the (i, j)-homogeneous component of $U_q^+ \otimes \mathbb{F}[z_1, z_2, \dots]$ to the (i, j)-homogeneous component of $\mathcal{U}_q^+$.*

Proof. Use Lemmas 3.10, 5.16, 9.4, 9.8. □

Proposition 9.12. *For $(i, j) \in \mathbb{N} \times \mathbb{N}$ the following are the same:*

(i) *the dimension of the (i, j)-homogeneous component of $\mathcal{U}_q^+$;*
(ii) *the coefficient of $\lambda^i \mu^j$ in $\mathcal{H}(\lambda, \mu)$.*

Proof. By Lemmas 9.10, 9.11. □

Example 9.13. For $0 \le i, j \le 6$ the dimension of the (i, j)-homogeneous component of $\mathcal{U}_q^+$ is given in the (i, j)-entry of the matrix below:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 4 & 4 & 4 & 4 \\ 1 & 4 & 10 & 13 & 14 & 14 & 14 \\ 1 & 4 & 13 & 27 & 36 & 39 & 40 \\ 1 & 4 & 14 & 36 & 69 & 91 & 101 \\ 1 & 4 & 14 & 39 & 91 & 161 & 213 \\ 1 & 4 & 14 & 40 & 101 & 213 & 361 \end{pmatrix}.$$

10. A PBW basis for $\mathcal{U}_q^+$

In this section we obtain a PBW basis for $\mathcal{U}_q^+$. First we clarify our terms.

Definition 10.1. (See [11, p. 299].) Let $\mathcal{A}$ denote an algebra. A *Poincaré-Birkhoff-Witt* (or *PBW*) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω, such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega, \qquad a_1 \le a_2 \le \cdots \le a_n.$$

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

Before proceeding, we have a comment about our approach. Shortly we will apply [21, Propositions 6.2, 7.2]. The results in [21, Propositions 6.2, 7.2] are about U_q^+. However the proofs of [21, Propositions 6.2, 7.2] use only Lemmas 2.3, 2.4. Therefore the results in [21, Propositions 6.2, 7.2] apply to $\mathcal{U}_q^+$ as well as U_q^+. We will apply [21, Propositions 6.2, 7.2] to $\mathcal{U}_q^+$.

Theorem 10.2. *A PBW basis for $\mathcal{U}_q^+$ is obtained by its alternating generators*

$$\{\mathcal{W}_{-i}\}_{i\in\mathbb{N}}, \qquad \{\mathcal{G}_{j+1}\}_{j\in\mathbb{N}}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{W}_{\ell+1}\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathcal{W}_{-i} < \mathcal{G}_{j+1} < \tilde{\mathcal{G}}_{k+1} < \mathcal{W}_{\ell+1} \qquad i, j, k, \ell \in \mathbb{N}.$$

Proof. Let Ω denote the set of alternating generators for $\mathcal{U}_q^+$. Consider the following vectors in $\mathcal{U}_q^+$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega, \qquad a_1 \le a_2 \le \cdots \le a_n. \tag{69}$$

We will show that the vectors in (69) form a basis for the vector space $\mathcal{U}_q^+$. We first show that the vectors in (69) span $\mathcal{U}_q^+$. To each element of Ω we assign a weight as follows. The elements $\{\mathcal{W}_{-i}\}_{i\in\mathbb{N}}$ (resp. $\{\mathcal{G}_{j+1}\}_{j\in\mathbb{N}}$) (resp. $\{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$) (resp. $\{\mathcal{W}_{\ell+1}\}_{\ell\in\mathbb{N}}$) get weight 0 (resp. 1) (resp. 2) (resp. 3). Any two elements of Ω commute if they have the same weight. Let $\mathcal{S}$ denote the subspace of $\mathcal{U}_q^+$ spanned by (69). Note that $\mathcal{S}$ is spanned by the vectors

$$a_1a_2\cdots a_n \qquad n\in\mathbb{N}, \qquad a_1,a_2,\ldots,a_n\in\Omega, \qquad \mathrm{wt}(a_1)\le \mathrm{wt}(a_2)\le\cdots\le \mathrm{wt}(a_n). \tag{70}$$

The algebra $\mathcal{U}_q^+$ is generated by Ω. Therefore the vector space $\mathcal{U}_q^+$ is spanned by

$$a_1a_2\cdots a_n \qquad n\in\mathbb{N}, \qquad a_1,a_2,\ldots,a_n\in\Omega. \tag{71}$$

For any product $a_1a_2\cdots a_n$ in (71), define its *defect* to be $\sum_{i=1}^n (n-i)\mathrm{wt}(a_i)$. We assume that $\mathcal{S}\ne\mathcal{U}_q^+$ and get a contradiction. There exists a product in (71) that is not contained in $\mathcal{S}$. Let D denote the minimum defect of all such products. Pick a product $a_1a_2\cdots a_n$ in (71) that is not contained in $\mathcal{S}$ and has defect D. The product $a_1a_2\cdots a_n$ is not listed in (70). Therefore there exists an integer s $(2\le s\le n)$ such that $\mathrm{wt}(a_{s-1})>\mathrm{wt}(a_s)$. Using [21, Propositions 6.2, 7.2] we express the product $a_{s-1}a_s$ as a linear combination of products $a'_{s-1}a'_s$ such that $a'_{s-1},a'_s\in\Omega$ and $\mathrm{wt}(a'_{s-1})<\mathrm{wt}(a'_s)$ and $\mathrm{wt}(a_{s-1})+\mathrm{wt}(a_s)=\mathrm{wt}(a'_{s-1})+\mathrm{wt}(a'_s)$. Replacing $a_{s-1}a_s$ by $a'_{s-1}a'_s$ in $a_1a_2\cdots a_n$ we obtain a product with defect less than D, and hence contained in $\mathcal{S}$. We have now expressed $a_1a_2\cdots a_n$ as a linear combination of products, each contained in $\mathcal{S}$. Consequently $a_1a_2\cdots a_n$ is contained in $\mathcal{S}$, for a contradiction. We have shown that the vectors in (69) span $\mathcal{U}_q^+$. We can now easily show that the vectors in (69) form a basis for $\mathcal{U}_q^+$. Each element in Ω is homogeneous with respect to the $(\mathbb{N}\times\mathbb{N})$-grading of $\mathcal{U}_q^+$. So each vector in (69) is homogeneous with respect to the $(\mathbb{N}\times\mathbb{N})$-grading of $\mathcal{U}_q^+$. Consequently for $(i,j)\in\mathbb{N}\times\mathbb{N}$ the vectors in (69) that have degree (i,j) span the (i,j)-homogeneous component of $\mathcal{U}_q^+$. The number of such vectors is equal to the coefficient of $\lambda^i\mu^j$ in $\mathcal{H}(\lambda,\mu)$, and by Proposition 9.12 this coefficient is equal to the dimension of the (i,j)-homogeneous component of $\mathcal{U}_q^+$. By these comments and linear algebra, the vectors in (69) that have degree (i,j) form a basis for the (i,j)-homogeneous component of $\mathcal{U}_q^+$. We conclude that the vectors in (69) form a basis for the vector space $\mathcal{U}_q^+$. The result follows. □

11. Directions for future research

In this section we give some conjectures concerning the q-Onsager algebra $\mathcal{O}_q$ and its current algebra $\mathcal{A}_q$. We will use the notation of Definition 3.4 and [10, Definition 3.1].

Conjecture 11.1. *There exists an algebra isomorphism* $\mathcal{A}_q\to\mathcal{O}_q\otimes\mathbb{F}[z_1,z_2,\ldots]$.

Conjecture 11.2. *Let* $\mathcal{Z}$ *denote the center of* $\mathcal{A}_q$. *Then the algebra* $\mathcal{Z}$ *is isomorphic to* $\mathbb{F}[z_1,z_2,\ldots]$.

Conjecture 11.3. *Let* $\langle\mathcal{W}_0,\mathcal{W}_1\rangle$ *denote the subalgebra of* $\mathcal{A}_q$ *generated by* $\mathcal{W}_0,\mathcal{W}_1$. *Then the algebra* $\langle\mathcal{W}_0,\mathcal{W}_1\rangle$ *is isomorphic to* $\mathcal{O}_q$.

Conjecture 11.4. *The multiplication map*

$$\langle\mathcal{W}_0,\mathcal{W}_1\rangle\otimes\mathcal{Z}\to\mathcal{A}_q$$

$$w \otimes z \mapsto wz$$

is an algebra isomorphism.

Conjecture 11.5. *The generators in order*

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{G}_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{W}_{k+1}\}_{k\in\mathbb{N}}$$

give a PBW basis for $\mathcal{A}_q$.

See [5] for results that support the above conjectures. For more general information on $\mathcal{A}_q$ and $\mathcal{O}_q$, see [6,10] for a mathematical physics point of view, and [9,12,16–20] for an algebraic point of view.

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Appendix A. The $Z_n^\vee$ are central in $\mathcal{U}_q^+$

Our goal here is to prove Lemma 5.2. It will be convenient to use generating functions.

Definition A.1. We define some generating functions in an indeterminate t:

$$\mathcal{G}(t) = \sum_{n\in\mathbb{N}} \mathcal{G}_n t^n, \qquad \tilde{\mathcal{G}}(t) = \sum_{n\in\mathbb{N}} \tilde{\mathcal{G}}_n t^n,$$

$$\mathcal{W}^-(t) = \sum_{n\in\mathbb{N}} \mathcal{W}_{-n} t^n, \qquad \mathcal{W}^+(t) = \sum_{n\in\mathbb{N}} \mathcal{W}_{n+1} t^n.$$

By (33) we have

$$[\mathcal{W}_0, \mathcal{W}^-(t)] = 0, \qquad [\mathcal{W}_1, \mathcal{W}^+(t)] = 0. \tag{72}$$

Next we express the relations (30)–(32) in terms of generating functions.

Lemma A.2. *We have*

$$[\mathcal{W}_0, \mathcal{W}^+(t)] = [\mathcal{W}^-(t), \mathcal{W}_1] = (1-q^{-2})t^{-1}\big(\tilde{\mathcal{G}}(t) - \mathcal{G}(t)\big),$$

$$[\mathcal{W}_0, \mathcal{G}(t)]_q = [\tilde{\mathcal{G}}(t), \mathcal{W}_0]_q = (q-q^{-1})\mathcal{W}^-(t),$$

$$[\mathcal{G}(t), \mathcal{W}_1]_q = [\mathcal{W}_1, \tilde{\mathcal{G}}(t)]_q = (q-q^{-1})\mathcal{W}^+(t).$$

Next we express the relations (33)–(40) in terms of generating functions. Let s denote an indeterminate that commutes with t.

Lemma A.3. *We have*

$$[\mathcal{W}^-(s), \mathcal{W}^-(t)] = 0, \qquad [\mathcal{W}^+(s), \mathcal{W}^+(t)] = 0,$$
$$[\mathcal{W}^-(s), \mathcal{W}^+(t)] + [\mathcal{W}^+(s), \mathcal{W}^-(t)] = 0,$$
$$s[\mathcal{W}^-(s), \mathcal{G}(t)] + t[\mathcal{G}(s), \mathcal{W}^-(t)] = 0,$$
$$s[\mathcal{W}^-(s), \tilde{\mathcal{G}}(t)] + t[\tilde{\mathcal{G}}(s), \mathcal{W}^-(t)] = 0,$$
$$s[\mathcal{W}^+(s), \mathcal{G}(t)] + t[\mathcal{G}(s), \mathcal{W}^+(t)] = 0,$$
$$s[\mathcal{W}^+(s), \tilde{\mathcal{G}}(t)] + t[\tilde{\mathcal{G}}(s), \mathcal{W}^+(t)] = 0,$$
$$[\mathcal{G}(s), \mathcal{G}(t)] = 0, \qquad [\tilde{\mathcal{G}}(s), \tilde{\mathcal{G}}(t)] = 0,$$
$$[\tilde{\mathcal{G}}(s), \mathcal{G}(t)] + [\mathcal{G}(s), \tilde{\mathcal{G}}(t)] = 0.$$

Next we express the relations (41)–(46) in terms of generating functions.

Lemma A.4. *We have*

$$[\mathcal{W}^-(s), \mathcal{G}(t)]_q = [\mathcal{W}^-(t), \mathcal{G}(s)]_q, \qquad [\mathcal{G}(s), \mathcal{W}^+(t)]_q = [\mathcal{G}(t), \mathcal{W}^+(s)]_q,$$
$$[\tilde{\mathcal{G}}(s), \mathcal{W}^-(t)]_q = [\tilde{\mathcal{G}}(t), \mathcal{W}^-(s)]_q, \qquad [\mathcal{W}^+(s), \tilde{\mathcal{G}}(t)]_q = [\mathcal{W}^+(t), \tilde{\mathcal{G}}(s)]_q,$$
$$t^{-1}[\mathcal{G}(s), \tilde{\mathcal{G}}(t)] - s^{-1}[\mathcal{G}(t), \tilde{\mathcal{G}}(s)] = q[\mathcal{W}^-(t), \mathcal{W}^+(s)]_q - q[\mathcal{W}^-(s), \mathcal{W}^+(t)]_q,$$
$$t^{-1}[\tilde{\mathcal{G}}(s), \mathcal{G}(t)] - s^{-1}[\tilde{\mathcal{G}}(t), \mathcal{G}(s)] = q[\mathcal{W}^+(t), \mathcal{W}^-(s)]_q - q[\mathcal{W}^+(s), \mathcal{W}^-(t)]_q,$$
$$[\mathcal{G}(s), \tilde{\mathcal{G}}(t)]_q - [\mathcal{G}(t), \tilde{\mathcal{G}}(s)]_q = qt[\mathcal{W}^-(t), \mathcal{W}^+(s)] - qs[\mathcal{W}^-(s), \mathcal{W}^+(t)],$$
$$[\tilde{\mathcal{G}}(s), \mathcal{G}(t)]_q - [\tilde{\mathcal{G}}(t), \mathcal{G}(s)]_q = qt[\mathcal{W}^+(t), \mathcal{W}^-(s)] - qs[\mathcal{W}^+(s), \mathcal{W}^-(t)].$$

So far in this section we displayed many relations involving the generating functions from Definition A.1. In the next two lemmas we express these relations in a more convenient form.

Lemma A.5. *We have*

$$\mathcal{W}^-(t)\mathcal{W}_0 = \mathcal{W}_0\mathcal{W}^-(t),$$
$$\mathcal{W}^+(t)\mathcal{W}_0 = \mathcal{W}_0\mathcal{W}^+(t) + (1-q^{-2})t^{-1}\big(\mathcal{G}(t) - \tilde{\mathcal{G}}(t)\big),$$
$$\mathcal{G}(t)\mathcal{W}_0 = q^2\mathcal{W}_0\mathcal{G}(t) + (1-q^2)\mathcal{W}^-(t),$$
$$\tilde{\mathcal{G}}(t)\mathcal{W}_0 = q^{-2}\mathcal{W}_0\tilde{\mathcal{G}}(t) + (1-q^{-2})\mathcal{W}^-(t)$$

and

$$\mathcal{W}_1\mathcal{W}^+(t) = \mathcal{W}^+(t)\mathcal{W}_1,$$
$$\mathcal{W}_1\mathcal{W}^-(t) = \mathcal{W}^-(t)\mathcal{W}_1 + (1-q^{-2})t^{-1}\big(\mathcal{G}(t) - \tilde{\mathcal{G}}(t)\big),$$
$$\mathcal{W}_1\mathcal{G}(t) = q^2\mathcal{G}(t)\mathcal{W}_1 + (1-q^2)\mathcal{W}^+(t),$$
$$\mathcal{W}_1\tilde{\mathcal{G}}(t) = q^{-2}\tilde{\mathcal{G}}(t)\mathcal{W}_1 + (1-q^{-2})\mathcal{W}^+(t).$$

Proof. These are reformulations of (72) and Lemma A.2. □

Lemma A.6. *We have*

$$\mathcal{G}(s)\mathcal{W}^-(t) = q\,\frac{(qs - q^{-1}t)\mathcal{W}^-(t)\mathcal{G}(s) - (q - q^{-1})s\mathcal{W}^-(s)\mathcal{G}(t)}{s-t},$$

$$\tilde{\mathcal{G}}(s)\mathcal{W}^-(t) = q^{-1}\,\frac{(q^{-1}s - qt)\mathcal{W}^-(t)\tilde{\mathcal{G}}(s) + (q - q^{-1})s\mathcal{W}^-(s)\tilde{\mathcal{G}}(t)}{s-t},$$

$$\mathcal{W}^+(s)\mathcal{G}(t) = q\,\frac{(q^{-1}s - qt)\mathcal{G}(t)\mathcal{W}^+(s) + (q - q^{-1})t\mathcal{G}(s)\mathcal{W}^+(t)}{s-t},$$

$$\mathcal{W}^+(s)\tilde{\mathcal{G}}(t) = q^{-1}\,\frac{(qs - q^{-1}t)\tilde{\mathcal{G}}(t)\mathcal{W}^+(s) - (q - q^{-1})t\tilde{\mathcal{G}}(s)\mathcal{W}^+(t)}{s-t}$$

and

$$\mathcal{W}^+(s)\mathcal{W}^-(t) = \mathcal{W}^-(t)\mathcal{W}^+(s) + (1-q^{-2})\frac{\mathcal{G}(s)\tilde{\mathcal{G}}(t) - \mathcal{G}(t)\tilde{\mathcal{G}}(s)}{s-t},$$

$$\tilde{\mathcal{G}}(s)\mathcal{G}(t) = \mathcal{G}(t)\tilde{\mathcal{G}}(s) + (1-q^2)st\,\frac{\mathcal{W}^-(t)\mathcal{W}^+(s) - \mathcal{W}^-(s)\mathcal{W}^+(t)}{s-t}.$$

Proof. To obtain the first equation of the lemma statement, consider the relations

$$s[\mathcal{W}^-(s), \mathcal{G}(t)] + t[\mathcal{G}(s), \mathcal{W}^-(t)] = 0,$$
$$[\mathcal{W}^-(s), \mathcal{G}(t)]_q = [\mathcal{W}^-(t), \mathcal{G}(s)]_q$$

from Lemmas A.3, A.4. These relations give a system of linear equations in two unknowns $\mathcal{G}(s)\mathcal{W}^-(t)$, $\mathcal{G}(t)\mathcal{W}^-(s)$. Solve this system using linear algebra to obtain the first equation in the lemma statement. The next three equations in the lemma statement are similarly obtained. To obtain the last two equations in the lemma statement, consider the last four relations in Lemma A.4. These relations give a system of linear equations in four unknowns

$$\mathcal{W}^+(s)\mathcal{W}^-(t), \qquad \mathcal{W}^+(t)\mathcal{W}^-(s), \qquad \tilde{\mathcal{G}}(s)\mathcal{G}(t), \qquad \tilde{\mathcal{G}}(t)\mathcal{G}(s).$$

Solve this system using linear algebra to obtain the last two equations in the lemma statement. □

The relations in Lemmas A.5, A.6 will be called *reduction rules*.

Definition A.7. Define the generating function

$$Z^\vee(t) = \sum_{n\in\mathbb{N}} Z_n^\vee t^n.$$

Lemma A.8. *We have*

$$Z^\vee(t) = \mathcal{G}(q^{-1}t)\tilde{\mathcal{G}}(qt) - qt\mathcal{W}^-(q^{-1}t)\mathcal{W}^+(qt).$$

Proof. By Definitions 5.1, A.1, A.7. □

Lemma A.9. *We have*

$$[\mathcal{W}_0, Z^\vee(t)] = 0, \qquad [\mathcal{W}_1, Z^\vee(t)] = 0.$$

Proof. To verify each equation, eliminate $Z^\vee(t)$ using Lemma A.8, and evaluate the result using the reduction rules. □

Lemma A.10. *We have*

$$[\mathcal{G}(s), Z^\vee(t)] = 0, \qquad [\tilde{\mathcal{G}}(s), Z^\vee(t)] = 0,$$
$$[\mathcal{W}^-(s), Z^\vee(t)] = 0, \qquad [\mathcal{W}^+(s), Z^\vee(t)] = 0.$$

Proof. To verify $[\mathcal{G}(s), Z^\vee(t)] = 0$, eliminate $Z^\vee(t)$ using Lemma A.8, and evaluate the result using the reduction rules. To obtain the remaining three equations in the lemma statement, use Lemmas A.2, A.9. □

We can now easily prove Lemma 5.2. Let $n \in \mathbb{N}$ be given. By Lemma A.10, $Z_n^\vee$ commutes with every alternating generator of $\mathcal{U}_q^+$, and hence everything in $\mathcal{U}_q^+$. In other words, $Z_n^\vee$ is central in $\mathcal{U}_q^+$.

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