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Analysis of Overlapping Resonances with Unitary Breit–Wigner and K-Matrix Approaches

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Abstract: We compare two methods for obtaining the parameters of overlapping resonances. The convenience of the Breit–Wigner (BW) approach is based on the fact that it operates with the masses and widths of the states. For several resonances with the same quantum numbers, a sum of BW functions violates the unitarity of the S-matrix. However, unitarity can be maintained by introducing interference phases to a BW implementation of scattering matrix formalism. A background can be added to the BW amplitudes in the standard way by using background phases. The *K*-matrix method is often used to analyze data related to several resonances with the same quantum numbers. It guarantees the unitarity of the *S*-matrix, but its parameters can be considered as resonance masses and widths only for well-spaced states. It also does not allow the separation of the resonant and background contributions in scattering amplitudes, which is critically important for determining parameters of wide resonances. To demonstrate the features of these methods, we consider several examples using simulated data.

Keywords: overlapping resonances; hadrons

1. Introduction

The Breit–Wigner function [1] describes partial amplitudes in a form which directly contains the mass and width of resonances. The BW function for one resonance satisfies the unitarity condition; a problem arises, however, when one needs to construct the unitary *S*-matrix for several resonances with the same quantum numbers.

A scattering operator connecting an initial and a final state, $S_{if} = \langle f|S|i \rangle$, must be unitary

$$SS^\dagger = I \qquad (1)$$

and symmetric, $S_{if} = S_{fi}$, to satisfy the time-reversal invariance.

The results of an analysis of any *isolated* resonance are eventually compared to the BW function (M is the number of channels)

$$S_{ij} = \delta_{ij} + 2i\frac{\sqrt{\Gamma_{ri}\Gamma_{rj}}/2}{E_r - E - i\Gamma_r/2}, \ \Gamma_r = \sum_i^M \Gamma_{ri} \qquad (2)$$

(we start with a form without a background), or with the variable $s = E^2$,

$$S_{ij} = \delta_{ij} + 2i\frac{m_r\sqrt{\Gamma_{ri}\Gamma_{rj}}}{m_r^2 - s - im_r\Gamma_r} \qquad (3)$$

The unitarity of these expressions, along with a form of the wave function of an unstable state, $\psi(t) = \psi(0)e^{-i(E_r - i\Gamma_r/2)t/\hbar}$, $|\psi(t)|^2 = |\psi(0)|^2 e^{-t/\tau}$, where $\tau = \hbar/\Gamma_r$ is the life-time of a resonance, supports the argument for writing the partial amplitudes in the BW form.

The idea of writing the *S*-matrix as a sum of resonant terms in a general way, which must satisfy the unitarity constraints, is due to work [2]. We demonstrate that this can be achieved in the form

$$S_{ij} = \delta_{ij} + 2i \sum_{r=1}^{N} e^{i\varphi_{ij}^r} \frac{m_r \sqrt{\Gamma_{ri}\Gamma_{rj}}}{m_r^2 - s - im_r\Gamma_r} \tag{4}$$

where the interference between the states is taken into account by the phases φ_{ij}^r. In the original work [2], this expression was written with complex numerators (residues) instead of such phases—both forms are equivalent. This scheme was realized in work [3] for two resonances with constant (energy-independent) widths.

To consider energy-dependent widths, and to take into account channel thresholds, the *S*-matrix should be given in the form

$$S_{ij} = \delta_{ij} + 2iF_{ij} = \delta_{ij} + 2i\sqrt{\rho_i}T_{ij}\sqrt{\rho_j} \tag{5}$$

where T_{ij} are transition amplitudes and ρ_i are phase-space factors. The conditions which the unitarity imposes on the widths $\Gamma_{ri}(s)$ and (constant) phases φ_{ij}^r are formulated in Section 2.

In the *K*-matrix approach [4], the *S*-matrix is expressed as

$$S = \frac{I + i\rho K}{I - i\rho K} \tag{6}$$

From the unitarity and symmetry of the *S*-matrix, it follows that *K* is a real and symmetric operator. From Equation (6), it follows that

$$T = (I - i\rho K)^{-1} K \tag{7}$$

In the case of one channel, $S = e^{2i\delta}$, where δ is the scattering phase; thus, $K = \tan\delta$. At the resonance position, $\delta = \pi/2$; thus, *K* has a pole. A single pole parametrization,

$$K = \frac{m_1\Gamma_1}{m_1^2 - s} \tag{8}$$

gives the standard BW function for the partial amplitude:

$$F = \frac{\rho m_1\Gamma_1}{m_1^2 - s - i\rho m_1\Gamma_1} \tag{9}$$

Thus, m_1 and Γ_1 in Equation (8) are the resonance's mass and the width in this simple situation.

For several poles and channels, a commonly used parametrization is [5]

$$K_{ij} = \sum_{\alpha} \frac{m_\alpha \Gamma_\alpha^0 \gamma_{\alpha i}\gamma_{\alpha j}}{m_\alpha^2 - s} \tag{10}$$

For *N* poles and *M* channels, the number of free parameters in the *K*-matrix method is $N(M+1)$ [6]. Parameter m_α is often called the "nominal" mass and parameter $\gamma_{\alpha i}$ is called the "coupling constant" of the state α to the decay channel *i*. To enhance the similarity with the BW formula, they are usually normalized: $\sum_i \gamma_{\alpha i}^2$, and $\gamma_{\alpha i}^2$ are considered as resonance branching ratios. Notice that all these statements are based on the comparison with the BW expression for an isolated resonance.

The relationship between the *K*- and the BW- methods and the appearance of relative phases in formula (4), can be demonstrated by considering two states in one channel. With

$$K = \frac{\gamma_1^2}{m_1 - E} + \frac{\gamma_2^2}{m_2 - E} \tag{11}$$

(here, and below, we use variable E and omit ρ only to simplify the expressions illustrating the comparison between the two methods), we obtain the scattering amplitude:

$$F = \frac{\gamma_1^2 m_2 + \gamma_2^2 m_1 - E(\gamma_1^2 + \gamma_2^2)}{E^2 - E[m_1 + m_2 - i(\gamma_1^2 + \gamma_2^2)] + m_1 m_2 - i(\gamma_1^2 m_2 + \gamma_2^2 m_1)} \tag{12}$$

Denoting complex roots of the denominator as μ_1 and μ_2, F can be expressed as a sum of two BW functions:

$$F = \frac{A_1}{\mu_1 - E} + \frac{A_2}{\mu_2 - E} \tag{13}$$

where $A_{1,2}$ are complex quantities:

$$A_1 = \frac{\mu_1(\gamma_1^2 + \gamma_2^2) - \gamma_1^2 m_2 - \gamma_2^2 m_1}{\mu_2 - \mu_1}, A_2 = \frac{\mu_2(\gamma_1^2 + \gamma_2^2) - \gamma_1^2 m_2 - \gamma_2^2 m_1}{\mu_1 - \mu_2}. \tag{14}$$

The real parts of $\mu_{1,2}$ give energies (masses) of resonances, the imaginary parts give their widths:

$$E_{1,2} = \mathrm{Re}\mu_{1,2} = \frac{m_1 + m_2}{2} \pm \frac{1}{2\sqrt{2}}\sqrt{(m_1 - m_2)^2 - (\gamma_1^2 + \gamma_2^2)^2 + W} \tag{15a}$$

$$\Gamma_{1,2} = 2\mathrm{Im}\mu_{1,2} = \frac{\gamma_1^2 + \gamma_2^2}{2} \pm \frac{1}{\sqrt{2}} \frac{(m_1 - m_2)(\gamma_1^2 - \gamma_2^2)}{\sqrt{(m_1 - m_2)^2 - (\gamma_1^2 + \gamma_2^2)^2 + W}} \tag{15b}$$

where

$$W = \sqrt{\left[(m_1 - m_2)^2 + (\gamma_1^2 + \gamma_2^2)^2\right]^2 + 16(m_1 - m_2)^2 \gamma_1^2 \gamma_2^2} \tag{15c}$$

Thus, the amplitude F in the *K*-matrix method is equivalent to the sum of BW functions with *complex residues*; in other words, with the *relative phase*, as seen in Equation (4).

The expression for this phase is easy to find. For

$$F = \sum_{n=1}^{2} \frac{\alpha_n/2}{E_n - E - i\Gamma_n/2} \tag{16}$$

S is unitary if

$$\frac{\alpha_1}{\alpha_2} = \frac{\Gamma_1}{\Gamma_2} e^{2i\varphi}, \text{ where } \varphi = -\arctan\frac{\Gamma_1 + \Gamma_2}{2(E_1 - E_2)}. \tag{17}$$

When m_1 and m_2 are *well separated*, i.e., $|m_1 - m_2| \gg \gamma_1^2 + \gamma_2^2$, then $E_i \Rightarrow m_i$, $\Gamma_i \Rightarrow 2\gamma_i^2$ and $\varphi \Rightarrow 0$. The fact that the amplitude F in the *K*-method can be presented as a sum of the BW functions is often taken as the justification that the *K*-matrix pole parameters are close to those of the physical resonances. However, 'good separation' is a rather subjective argument; the unitarity of the *S*-matrix is very crudely violated, even when resonances are well spaced, as shown in Figure 1.
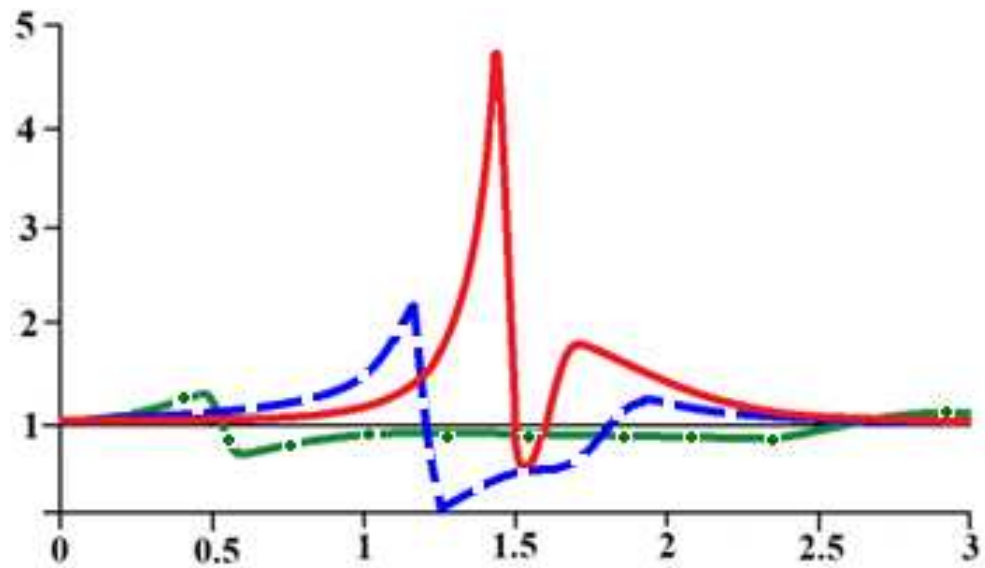
Figure 1. Plot of $|S(E)|^2$ for a sum of two BW functions. Green dashed-dotted line $E_1 = 0.5$, $\Gamma_1 = 0.1$, $E_2 = 2.5$, $\Gamma_2 = 0.3$; blue dashed line $E_1 = 1.2$, $\Gamma_1 = 0.1$, $E_2 = 1.8$, $\Gamma_2 = 0.3$; red solid line $E_1 = 1.45$, $\Gamma_1 = 0.1$, $E_2 = 1.65$, $\Gamma_2 = 0.3$ (all in GeV).

2. Brief Description of the Unitary BW Method

Here, we briefly describe the method of constructing the unitary S-matrix [7]. The regular procedure is presented in Appendix A.

The unitarity condition, $S^{\dagger}S = I$, with $S = I + 2iF = I + 2i\sqrt{\rho}T\sqrt{\rho}$, gives

$$T^{\dagger} - T = iT^{\dagger}\rho T \tag{18}$$

The unitarization procedure becomes technically simpler if, following the idea of works [2,3], to introduce complex vectors of partial widths, $\vec{g}_r$. With $g_{ri} = e^{i\varphi_{ri}}|g_{ri}|$, $i = 1, \ldots, M$, $r = 1, \ldots, N$, the S-matrix is:

$$S_{ij} = \delta_{ij} + 2i\sqrt{\rho_i(s)}\left(\sum_{r=1}^{N} e^{i\varphi_{ij}^{(r)}} \frac{m_r\Gamma_r^0|g_{ri}| \cdot |g_{rj}|}{m_r^2 - s - im_r\Gamma_r(s)}\right)\sqrt{\rho_j(s)} \tag{19}$$

and the T-matrix is:

$$T_{ij}(s) = \sum_{r=1}^{N} \frac{m_r\Gamma_r^0 g_{ri}g_{rj}}{m_r^2 - s - im_r\Gamma_r(s)} \tag{20}$$

Here, $\varphi_{ij}^r = \varphi_{ri} + \varphi_{rj}$ are energy-independent phases; factors $\Gamma_r^0 = \Gamma_r(m_r^2)$ are introduced to keep vectors g_{ri} dimensionless. Expression (19) is time-invariant.

The decay branching ratio of a resonance r in channel i is

$$B_{ri} = \frac{|g_{ri}|^2}{\left|\vec{g}_r\right|^2} = \frac{|g_{ri}|^2}{\sum_{k=1}^{M} |g_{rk}|^2} \tag{21}$$

The partial widths are $\Gamma_{ri} = \Gamma_r B_{ri}$. The total and partial widths can depend on energy,
$$\Gamma_r(s) = \sum_i^M \Gamma_{ri}(s).$$

Formulas in the rest of this section become technically simpler if the variable E is used rather than $s = E^2$—all the expressions and algorithms can be rewritten in terms of s. For

$$T_{ij} = \sum_{r=1}^N \frac{\Gamma_r^0 g_{ri} g_{rj}/2}{m_r - E - i\Gamma_r(E)/2}, \tag{22}$$

let us formulate the conditions that should be imposed on vectors $\vec{g}_r$ to maintain $S_{ij}(E)$ unitarity. From Equation (18) we have

$$i \sum_{r=1}^N \left[\frac{g_{ri}^* g_{rj}^*}{m_r - E + i\Gamma_r(E)/2} - \frac{g_{ri} g_{rj}}{m_r - E + i\Gamma_r(E)/2} \right]$$
$$+ \sum_{l=1}^N \sum_{r=1}^N V_{lr} \frac{g_{ri} g_{lj}^*}{(m_r - E + i\Gamma_r(E)/2)(m_r - E + i\Gamma_r(E)/2)} \equiv 0, \tag{23}$$

where the following notation is used:

$$V_{lr} = \sqrt{\Gamma_l^0 \Gamma_r^0} \sum_{k=1}^M \rho_k(E) g_{lk}^* g_{rk} \tag{24}$$

The constraints Equation (23) involves complicated non-linear conditions. To resolve them, we use a method which allows substantial technical difficulties to be overcome; these previously restricted the approach to a maximum of two [3], or three [8] resonances, even when $\rho_k \equiv 1$. The method is based on the construction of vectors $\vec{g}_r = \vec{g}_r^x + i\vec{g}_r^y$ in such a form that their imaginary parts $\vec{g}_r^y$ are combinations of their real parts $\vec{g}_r^x$: $\vec{g}_r^y = U\vec{g}_r^x$, or

$$\vec{g}_r^y = u_{r1} \vec{g}_1^x + u_{r2} \vec{g}_2^x + \ldots + u_{rN} \vec{g}_N^x \tag{25}$$

U is a real anti-symmetric matrix which has a simple form for any particular N and M (see Appendix A). Next, instead of trying to find all components of vectors $\vec{g}_r$, we only find their real parts, and then obtain their imaginary parts using matrix U. Notice that the number of free parameters, $N(M+1)$, is the *same as in the K-matrix method*.

When $|m_r - m_{r'}| \gg \Gamma_r + \Gamma_{r'}$, the matrix elements $u_{rk} \to 0$ and vectors $\vec{g}_r$ become real and orthogonal, $\left(\vec{g}_r, \vec{g}_q\right) = 0$, and we return to a simple sum of the BW functions without relative phases.

The constraints on vectors $\vec{g}_r$ are the following:

$$\sum_{k=1}^M \Gamma_r^0 \rho_k(E) |g_{rk}|^2 = -\frac{1}{S}[S + 2Q_r]\Gamma_r, \tag{26}$$

$$\sum_{k=1}^M \Gamma_r^0 \rho_k(E) \mathrm{Re}\left(g_{qk}^* g_{rk}\right) = -\frac{1}{S}\left[2F_{qr}(m_q - m_r) - iG_{qr}(\Gamma_q + \Gamma_r)\right], \tag{27}$$

$$\sum_{k=1}^M \Gamma_r^0 \rho_k(E) \mathrm{Im}\left(g_{qk}^* g_{rk}\right) = -\frac{1}{S}\left[2G_{qr}(m_q - m_r) + iF_{qr}(\Gamma_q + \Gamma_r)\right], \tag{28}$$

$$r = 1, \ldots, N; q = r + 1, \ldots, N.$$

Constant coefficients S, Q_r, F_{qr}, G_{qr} are determined via the elements of matrix U.

The widths are given by Equation (26):

$$\Gamma_r(E) = -\frac{S}{S + 2Q_r}\Gamma_r^0 \sum_{k=1}^{M} \rho_k(E)|g_{rk}|^2 \tag{29}$$

Formulas (25)–(29) provide the algorithm for the method. For any particular case, $N = 2$, $N = 3$, etc.; the coefficients S, Q_r, F_{qr}, G_{qr} are given by simple expressions, presented in Appendix A.

In the case of one resonance, this approach reduces to the traditional BW function:

$$S_{ij} = \delta_{ij} + 2i\sqrt{\rho_i\rho_j}\frac{\Gamma_1^0 g_{1i}g_{1j}/2}{m_1 - E - i\Gamma_1(E)/2}, i, j = 1, \ldots, M. \tag{30}$$

Vector $\vec{g}_1 = \vec{g}_1^x$ is real, and $\Gamma_1^0 g_{1i}g_{1j} = \sqrt{\Gamma_{ri}\Gamma_{rj}}$. With Equation (29), it gives the analogue of Flatte's formula:

$$T_{ij} = \frac{\Gamma_1^0 g_{i1}g_{1j}/2}{m_1 - E - \frac{i}{2}\Gamma_1^0 \sum\limits_{k=1}^{M} \rho_k(E)\left(g_{1k}^x\right)^2} \tag{31}$$

or in variable s,

$$T_{ij} = \frac{m_1 \Gamma_1^0 g_{i1}g_{1j}}{m_1^2 - s - im_1\Gamma_1^0 \sum\limits_{k=1}^{M} \rho_k(s)\left(g_{1k}^x\right)^2} \tag{32}$$

In a fitting procedure, the free parameters are: mass m_1 and the components of vector $\vec{g}_1 = \vec{g}_1^x$.

If a resonance is lying above all the thresholds, its mass is simply m_1 and its width is

$$\Gamma_1(E) = \Gamma_1^0 \sum_{k=1}^{M} \rho_k(E)(g_{1k}^x)^2. \tag{33}$$

If a resonance is lying between the thresholds L and $(L+1)$, there are two options: to set $\rho_k(E) = 0$ in the energy region below the corresponding threshold, $E < E_k$, or to continue it as $\rho_k => i\sigma_k|\rho_k|$ (σ_k can take the values ± 1 to consider different Riemann sheets). The effective mass is then

$$\tilde{m}_1(E) = m_1 + \frac{\Gamma_1^0}{2}\sum_{k=L+1}^{M}\sigma_k|\rho_k(E)|(g_{1k}^x)^2 \tag{34}$$

and the width is

$$\Gamma_1(E) = \Gamma_1^0\sum_{k=1}^{L}\rho_k(E)(g_{1k}^x)^2. \tag{35}$$

For the case of *two resonances and one channel*, this algorithm leads to the same expression for the amplitude T as in the *K*-matrix method (as in Section 1).

Let us illustrate the method with the example of *two resonances and two channels*, $N = 2$, $M = 2$. The goal is to find vectors $\vec{g}_r$.

For two channels, matrix U is

$$U = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} \tag{36}$$

(α is the real parameter, $0 \leq \alpha < 1$), i.e.,

$$\vec{g}_1^{\,y} = -\alpha \vec{g}_2^{\,X}, \quad \vec{g}_2^{\,y} = \alpha \vec{g}_1^{\,X}. \tag{37}$$

The coefficients in the unitary constraint Equations (26)–(28) are:

$$S = 1 - \alpha^2, Q_1 = Q_2 = \alpha^2, F_{12} = -\alpha, G_{12} = 0. \tag{38}$$

Six real quantities, for example, masses m_1 and m_2, parameter α, and $g_{11}^x, g_{12}^x, g_{22}^x$ can be taken as independent parameters in data fitting. The remaining real component g_{21}^x is determined from Equations (27) and (28).

When both resonances lie above the 2nd threshold, $E_2 < m_1 < m_2$,

$$g_{21}^x = \frac{1}{\rho_1(m_2)g_{11}^x} \left[\frac{2\alpha}{\Gamma_1^0(1-\alpha^2)^2}(m_1 - m_2) - \rho_2(m_2)g_{12}^x g_{22}^x \right] \tag{39}$$

After this, the components $\vec{g}_r^{\,y}$ can be calculated with Equation (37). This completes the construction of the amplitudes

$$T_{ij} = \frac{\Gamma_1^0 g_{1i}g_{1j}/2}{m_1 - E - i\Gamma_1(E)/2} + \frac{\Gamma_2^0 g_{2i}g_{2j}/2}{m_2 - E - i\Gamma_2(E)/2} \tag{40}$$

The functions $\Gamma_r(E)$ are given by expression (29), their values at $E = m_r$,

$$\Gamma_r(m_r) = \frac{1-\alpha^2}{1+\alpha^2}\Gamma_r^0 \sum_{k=1}^{2} \rho_k(m_r)|g_{rk}|^2 \tag{41}$$

can be considered as resonance "widths".

If the 2nd threshold is located between m_1 and m_2, i.e., $m_1 < E_2 < m_2$, the mass of the 1st resonance, $\tilde{m}_1$, is determined by the zero of the real part of the denominator in the first term in Equation (40), i.e., by the root of the equation

$$E - \tilde{m}_1 - \frac{1-\alpha^2}{2(1+\alpha^2)}\Gamma_1^0|\rho_2(E)| \cdot |g_{12}|^2 = 0 \tag{42}$$

so this resonance width is

$$\tilde{\Gamma}_1(\tilde{m}_1) = \frac{1-\alpha^2}{1+\alpha^2}\Gamma_1^0\rho_2(\tilde{m}_1)|g_{12}|^2 \tag{43}$$

The component g_{21}^x can be found from Equation (39) with $\tilde{m}_1$ instead of m_1.

3. Examples: K-Matrix vs. Unitary BW Approach

Let us consider two resonances in two channels, $S = I + 2iF = I + 2i\sqrt{\rho}T\sqrt{\rho}$, where ρ is a diagonal 2×2 matrix. In the unitary BW expression,

$$S_{ij} = \delta_{ij} + 2i\sqrt{\rho_i\rho_j}\sum_{r=1}^{2}\frac{m_r\Gamma_r(m_r)g_{ri}g_{rj}}{m_r^2 - s - im_r\Gamma_r(s)}, \qquad (44)$$

there are six independent parameters: $m_1, m_2, \alpha, g_{11}^x, g_{12}^x, g_{22}^x$. (The product $g_{ri}g_{rj} = e^{i\varphi_{ij}^{(r)}}\left|g_{ri}\right|\left|g_{rj}\right|$).

The *K*-matrix pole terms in expression (10),

$$K_{ij} = \gamma_{1i}\gamma_{1j}\frac{m_1\Gamma_1^0}{m_1^2 - s} + \gamma_{2i}\gamma_{2j}\frac{m_2\Gamma_2^0}{m_2^2 - s}, \qquad (45)$$

have the same number of independent parameters: m_r, Γ_r^0, and two γ_{ri} (with γ being normalized, $\sum_i \gamma_{ri}^2 = 1$). The *K*-matrix amplitudes are

$$F = \frac{\sqrt{\rho}K\sqrt{\rho}}{1 - i\rho K} = \frac{1}{1 - \rho_1\rho_2 D - i(\rho_1 K_{11} + \rho_2 K_{22})}\begin{pmatrix} \rho_1(K_{11} - i\rho_2 D) & \sqrt{\rho_1\rho_2}K_{12} \\ \sqrt{\rho_1\rho_2}K_{21} & \rho_2(K_{22} - i\rho_1 D) \end{pmatrix}, \quad (46)$$

where $D = K_{11}K_{22} - K_{12}^2$.

Let us start with the important observation that the K-matrix amplitudes T_{ij} become zero (or are very close to zero values) between the m_1 and m_2 locations. This feature directly follows from expression (7), it is, therefore, retained in any modification of the *K*-matrix approach. In the case of two resonances, equation $T_{12}(E) = 0$ (i.e., $K_{12}(E) = 0$) is a linear equation; in the case of three states, it is a quadratic equation—these results directly follow from expressions (10) and (46). The zeros in amplitudes T_{11} and T_{22} are slightly shifted from the zero position in T_{12}. Thus, resonances in scattering amplitudes are, in fact, well spaced (a polynomial added to expression (10) for K_{ij} does not change this observation, see Appendix B). As a result, this method can work well only when describing experimental data with sufficiently well-separated states.

Since the purpose of this article is to demonstrate the features of the two methods, it is more useful to consider simple examples than discuss a real physical problem in which there are always ambiguous questions, such as the choice of the most significant channels, the number of resonances, etc. Particular physical problems are beyond the scope of this paper and will be studied separately.

Let us start by considering two *overlapping* states (peaks) in the amplitudes $\left|F_{ij}(E)\right|^2$ located near 1.3 and 1.6 GeV, with widths about $0.15 \div 0.3$ GeV. We also consider different positions of the 2nd channel threshold: below both states, at $E_2 = 1.22$, and between the states, at $E_2 = 1.38$ (the position of the 1st threshold is fixed at $E_1 = 0.5$; all quantities here and below are in GeV). Next, we generate "data" which qualitatively correspond to these situations—the details about such data generation are given in Appendix B. (Note that these data are similar to the data on scattering amplitudes in the case of vector ω' resonances.) To do this, we draw smooth curves (different peak heights and widths can be considered), then discretize and randomize these curves and introduce dispersions (error bars at each point are generated randomly with the upper limit of 0.05). Examples of such data are presented in Figure 2 for $E_2 = 1.22$ and Figure 3 for $E_2 = 1.38$. The peaks overlap and, at least in one channel, are not well resolved, as often occurs in real situations. We then fit these data with the unitary BW formulas (44)—red lines, and *K*-matrix (46)—blue lines.
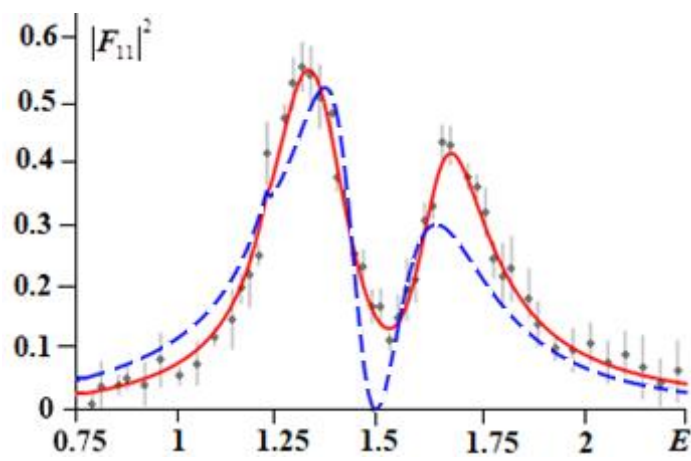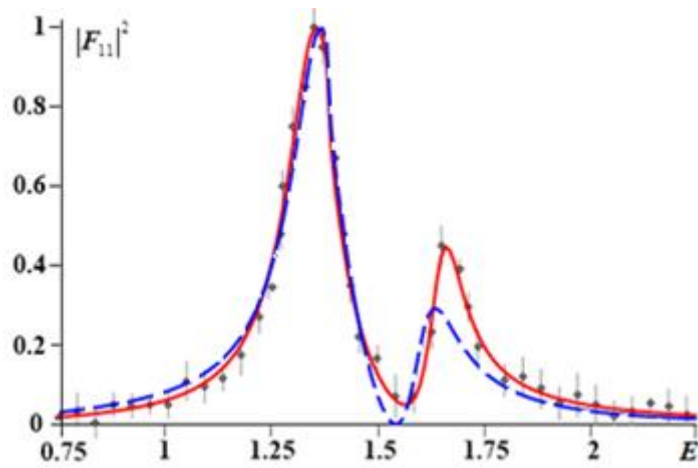
(a) Amplitude $\left|F_{11}(E)\right|^2$
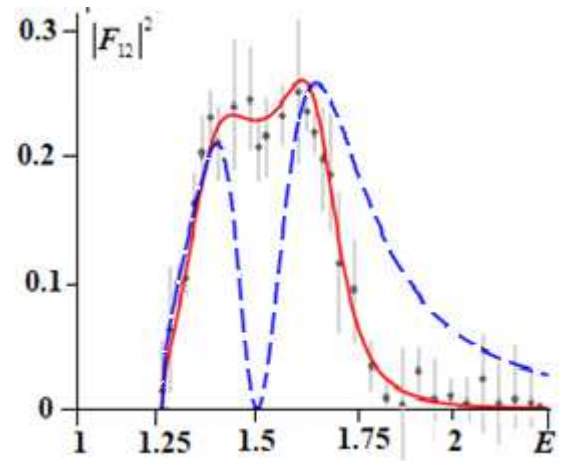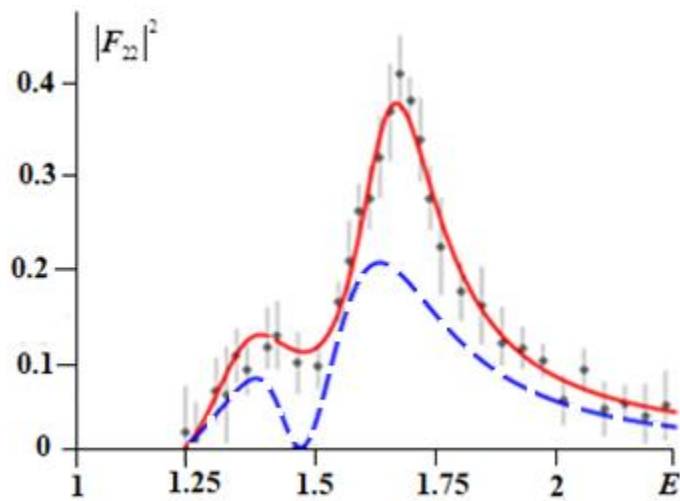
(b) Amplitude $\left|F_{12}(E)\right|^2$

(c) Amplitude $\left|F_{22}(E)\right|^2$

Figure 2. Panel (**a**) $|F_{11}(E)|^2$, panel (**b**) $|F_{12}(E)|^2$, panel (**c**) $|F_{22}(E)|^2$. Red solid lines—BW formula (44); blue dashed lines—*K*-matrix Equation (46); threshold $E_2 = 1.22$.

(a) Amplitude $\left|F_{11}(E)\right|^2$

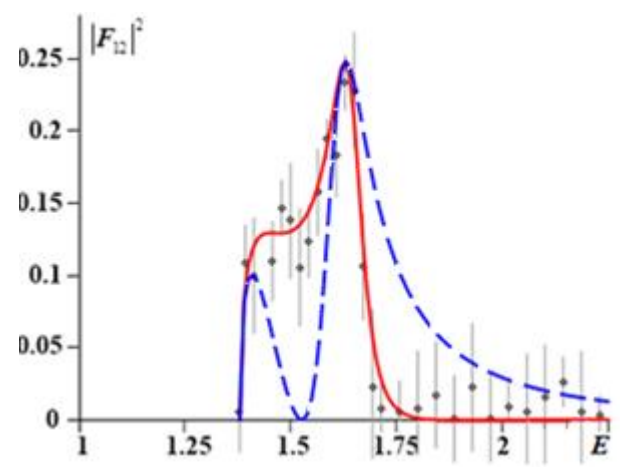(b) Amplitude $\left|F_{12}(E)\right|^2$
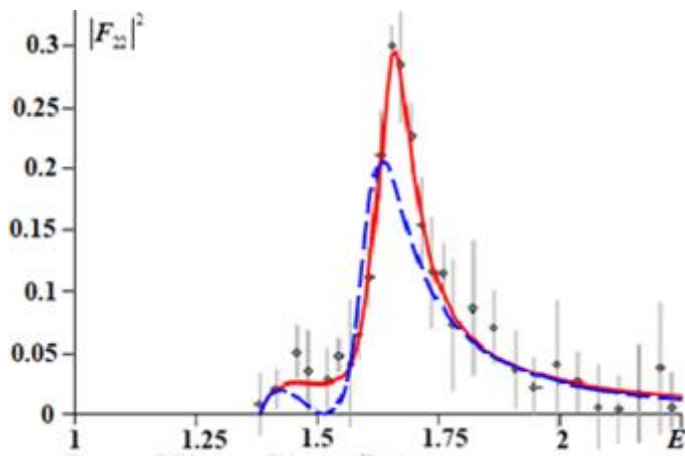
(c) Amplitude $\left|F_{22}(E)\right|^2$

Figure 3. Panel (**a**) $|F_{11}(E)|^2$, panel (**b**) $|F_{12}(E)|^2$, panel (**c**) $|F_{22}(E)|^2$. Red solid lines—BW formula (44), blue dashed lines—*K*-matrix Equation (46); threshold $E_2 = 1.38$.

In calculations, we use the phase factors $\rho_i(s) = \sqrt{(s - s_i)/s}$ (these can be modified by including the barrier factors). The continuation $\rho_2(s) => \pm i |\rho_2(s)|$ below the E_2 threshold can be used in expression (32) for the amplitudes T_{ij}.

In the unitary BW formula (44) for the threshold at $E_2 = 1.22$, the values of the six independent parameters are:

$$m_1 = 1.35, m_2 = 1.65, \alpha = 0.07, g_{11}^x = -0.43, g_{12}^x = 0.41, g_{22}^x = 0.49$$

the fit gives $\chi^2/d = 0.49$ (d is degrees of freedom number).

For the threshold at $E_2 = 1.38$, the values of these parameters are:

$$m_1 = 1.36, m_2 = 1.65, \alpha = 0.08, g_{11}^x = -0.42, g_{12}^x = 0.38, g_{22}^x = 0.36$$

the fit gives $\chi^2/d = 0.53$. The remaining parameters can be calculated using the six independent parameters. (After the g_{ri} values are found, the value of a 'technical' parameter α is no longer required.)

Table 1 shows the BW resonance parameters—the branching ratios B_{ri} are calculated with formula (21), the phases φ_{ij}^r can be obtained from the g_{ri} values, and $\Gamma_r = \Gamma_r(m_r)$. By the direct substitution of these g_{ri} in formula (44), it can be confirmed that matrix $S(E)$ is unitary for any E.

Table 1. BW parameters for two threshold positions.

Threshold $E_2=1.22$						Threshold $E_2=1.38$					
m_r	Γ_r	B_{r1}	B_{r2}	g_{r1}	g_{r2}	m_r	Γ_r	B_{r1}	B_{r2}	g_{r1}	g_{r2}
1.32	0.26	53	47	$-0.43 - i0.03$	$0.41 - i0.04$	1.36	0.19	55	45	$-0.43 - i0.02$	$-0.38 - i0.03$
1.65	0.318	44	56	$0.43 - i0.03$	$0.49 + i0.03$	1.65	0.14	42	58	$0.30 - i0.03$	$0.36 - i0.03$

The fit based on the K-matrix formula (46) for the threshold $E_2 = 1.22$, gives the following values for the six independent parameters:

$$m_1 = 1.36, m_2 = 1.63, \Gamma_1 = 0.27, \Gamma_2 = 0.37, \gamma_{11} = 0.77, \gamma_{21} = 0.68$$

resulting in $\chi^2/d = 8.72$.

For the threshold $E_2 = 1.38$, the parameters are:

$$m_1 = 1.37, m_2 = 1.63, \Gamma_1 = 0.32, \Gamma_2 = 0.19, \gamma_{11} = 0.73, \gamma_{21} = 0.63$$

resulting in $\chi^2/d = 4.75$.

Table 2 contains the K-matrix parameters. The branching ratios are $B_{ri} = \gamma_{ri}^2$.

Table 2. K-matrix parameters for two threshold positions.

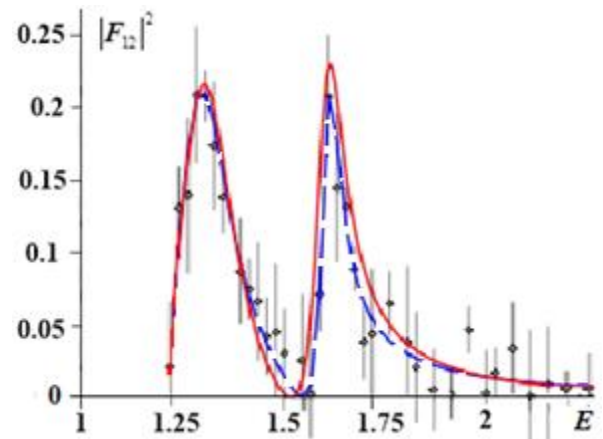|	Threshold E_2=1.22				Threshold E_2=1.38		
m_r	Γ_r	B_{r1}	B_{r2}	m_r	Γ_r	B_{r1}	B_{r2}
1.36	0.27	59	41	1.37	0.32	54	46
1.63	0.37	47	53	1.63	0.20	40	60

Large values of χ^2 reflect the fact that the *K*-matrix method cannot adequately describe the data in situations when wide resonances are not well spaced, such as in the channel $1 \rightarrow 2$, in Figures 2 and 3. However, this method works perfectly well when resonances are well resolved, such as in Figure 4, where the data have substantial dips between the resonances in all three channels (here, for brevity, we consider only one threshold location, $E_2 = 1.22$). The quality of the fits in both methods is practically the same, $\chi^2/d \approx 0.5$. The resonance parameters for Figure 4 are shown in Table 3.

Table 3. BW and K-matrix parameters for data in Figure 4.

			BW						K	
m_r	Γ_r	B_{r1}	B_{r2}	g_{r1}	g_{r2}	m_r	Γ_r	B_{r1}	B_{r2}
1.30	0.19	49	51	$-0.39 - i0.02$	$-0.37 - i0.03$	1.30	0.28	45	55
1.60	0.09	18	82	$-0.11 - i0.05$	$-0.25 + i0.06$	1.60	0.12	27	73
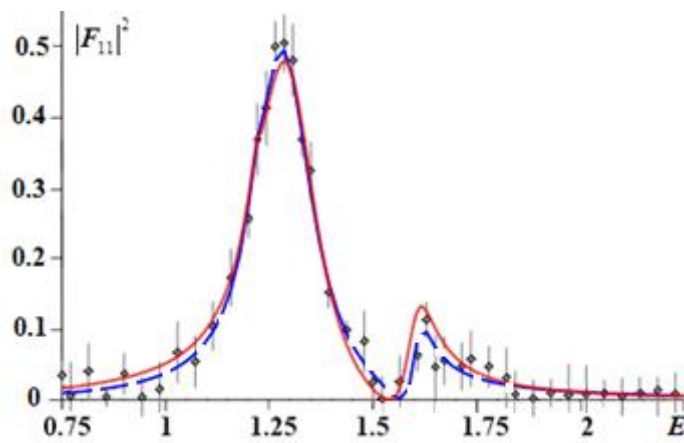
(a) Amplitude $\left|F_{11}(E)\right|^2$
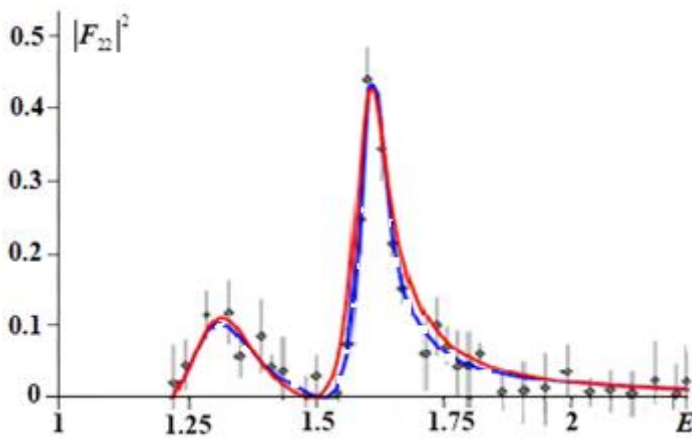
(b) Amplitude $\left|F_{12}(E)\right|^2$

(c) Amplitude $\left|F_{22}(E)\right|^2$

Figure 4. Panel (**a**) $|F_{11}(E)|^2$, panel (**b**) $|F_{12}(E)|^2$, panel (**c**) $|F_{22}(E)|^2$. Red solid lines—BW formula (44), blue dashed lines–K-matrix Equation (46); threshold $E_2 = 1.22$.

The Argand diagrams corresponding to the amplitude F_{11} in Figures 2 and 4 are shown in Figure 5.

(a)   (b)
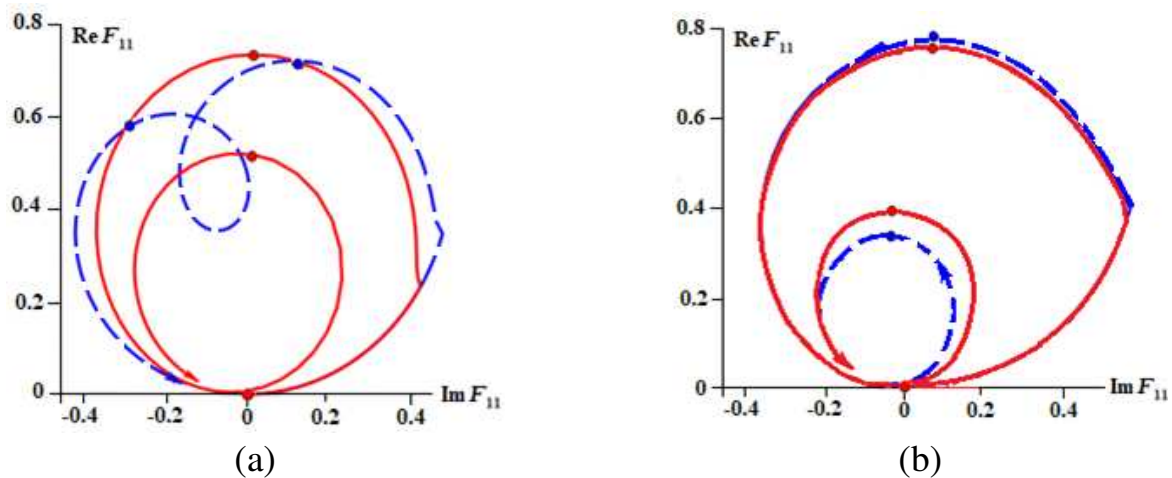
Figure 5. Argand diagrams for the amplitude F_{11} in Figure 2 (**a**), and in Figure 4 (**b**). Red solid lines-BW formula (44), blue dashed lines -*K*-matrix Equation (46).

4. Background

Taking background into account in a rigorous way is especially important when several resonances are not well resolved, or their existence is not well established. Even a small background can hide a resonance [9,10], as in the situation with a vector $\rho(1250)$ meson overlapping with other wide vector states, $\rho(1450)$ and $\rho(1600)$.

Usually, to account for the background in the K-matrix description, polynomials are added to the pole terms in K_{ij}:

$$K_{ij} = \sum_a \frac{\gamma_{ai}\gamma_{aj}m_a\Gamma_a^0}{m_a^2 - s} + (A_{ij} + B_{ij}s) \tag{47}$$

The form (47) does not allow the background to be presented in terms of background phases, $S_{ij}^B = e^{i(\beta_i + \beta_j)}$; in this form, here, even the number of parameters is different to that in (47): for example, for two channels and an energy-independent background, there are two parameters β_1 and β_2, versus three A_{ij} in K_{ij}. Thus, polynomials in Equation (47) only allow the use of additional fitting parameters besides the pole parameters.

Another important fact is that the K-matrix resonances remain well spaced even when polynomials in K_{ij} are taken into account; for details see Appendix B.

In the BW scheme, the *S*-matrix can be separated in a sum of the resonant and background terms from the very beginning:

$$S_{ij} = B_{ij} + 2i\sqrt{\rho_i\rho_j}\sum_{r=1}^{N} \frac{m_r\Gamma_r^0 A_{ri}A_{rj}}{m_r^2 - s - im_r\Gamma_r(s)} \tag{48}$$

The unitary and symmetric background matrix B can be constructed as described below. If matrix B is *diagonal*, i.e., it is non-zero only in channels with $i = j$,

$$B_{ij} = \delta_{ij} e^{i\beta_i} \quad (i, j = 1, \ldots, M),$$

vectors $\vec{A}_r$ in (48) can be presented as

$$A_{rk} = e^{i\beta_k} g_{rk} = (\cos \beta_k + i \sin \beta_k) \left(g_{rk}^x + i g_{rk}^y \right),$$

where g_{ri} can be found with the procedure described in the previous sections. Matrix S, in such a situation, is

$$S_{ij} = e^{i(\beta_i + \beta_j)} \left[\delta_{ij} + 2i \sqrt{\rho_i \rho_j} \sum_{r=1}^{N} e^{i(\varphi_{ri} + \varphi_{rj})} \frac{m_r \Gamma_r^0 |g_{ri}| \cdot |g_{rj}|}{m_r^2 - s - i m_r \Gamma_r} \right] \tag{49}$$

or

$$S_{ij} = e^{i(\beta_i + \beta_j)} \left[\delta_{ij} + 2i \sqrt{\rho_i \rho_j} \sum_{r=1}^{N} e^{i\varphi_{ij}^r} \frac{m_r \sqrt{\Gamma_{ri} \Gamma_{rj}}}{m_r^2 - s - i m_r \Gamma_r} \right] \tag{50}$$

If matrix B is *non-diagonal*, i.e., a background potentially is not zero in all channels, it is helpful to present it in form [3], $B = W e^{2i\vec{\beta}} W^T$, where W is a real orthogonal matrix, $W^{-1} = W^T$ and $e^{2i\vec{\beta}}$ is a diagonal matrix ($W = I$ when matrix B is diagonal). Writing matrix B as $B = bb^T$ with $b = W e^{i\vec{\beta}}$, we define $\vec{g}_r = b^T \vec{A}_r$; thus, $\vec{A}_r = b \vec{g}_r$ and matrix S can be expressed as $S = b \tilde{S} b^T$, where $\tilde{S}$ does not contain a background:

$$\tilde{S}_{ij} = \delta_{ij} + 2i \sqrt{\rho_i \rho_j} \sum_{r=1}^{N} \frac{m_r \Gamma_r^0 g_{ri} g_{rj}}{m_r^2 - s - i m_r \Gamma_r} \tag{51}$$

Obviously, S is unitary if matrix $\tilde{S}$ is unitary. Therefore, we can independently find vectors g_{ri} in $\tilde{S}$, and then return to the matrix S.

Matrix W can be constructed as a sequence of $M(M-1)/2$ rotations:

$$W = R_1 \cdot R_2 \cdot \ldots \cdot R_{M(M-1)/2} \tag{52}$$

Matrix $R_k = \left\{ r_{ij}^{(k)} \right\}$ $(i, j = 1, \ldots, M)$ is convenient to write in the form:

$$r_{pp}^{(k)} = r_{qq}^{(k)} = \cos \psi_k, \quad r_{pq}^{(k)} = -r_{qp}^{(k)} = \sin \psi_k, \tag{53}$$

$p = 1, \ldots, M, q = p + 1, \ldots, M$; here $\psi_k \in [0, 2\pi)$ are the rotation angles.

For instance, for three channels

$$b = W \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}$$

where

$$W = R_1 R_2 R_3 = \begin{pmatrix} \cos\psi_1 & \sin\psi_1 & 0 \\ -\sin\psi_1 & \cos\psi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_2 & \sin\psi_2 \\ 0 & -\sin\psi_2 & \cos\psi_2 \end{pmatrix} \begin{pmatrix} \cos\psi_3 & 0 & \sin\psi_3 \\ 0 & 1 & 0 \\ -\sin\psi_3 & 0 & \cos\psi_3 \end{pmatrix}$$

The quantities β_k, ψ_k can be taken as fitting parameters (they can be functions of energy). We, then, have

$$F_{ij} = (S_j - I)/2i = (B_F)_{ij} + \sqrt{\rho_i \rho_j} \sum_{r=1}^{N} \frac{m_r \Gamma_r^0 A_{ri} A_{rj}}{m_r^2 - s - i m_r \Gamma_r(s)} \tag{54}$$

with the matrix

$$B_F = W \begin{pmatrix} \sin\beta_1 e^{i\beta_1} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \sin\beta_M e^{i\beta_M} \end{pmatrix} W^T \tag{55}$$

Thus, the background in the unitary BW method can be described by formulas (54) and (55).

To demonstrate this scheme, let us take the case of two resonances and three channels using the resonance parameters in Table 4.

Table 4. BW pole functions.

r	m_r	Γ_r	B_{r1} (%)	B_{r2} (%)	B_{r3} (%)
Resonance 1	1.30	0.10	40	40	20
Resonance 2	1.70	0.30	20.8	59.4	19.8

Next, we include a background in these resonant unitary BW amplitudes to see how it modifies the scattering amplitudes. In Figure 6, graphs of $|F_{11}(E)|^2$ and $|F_{12}(E)|^2$ (chosen as examples) are shown for the cases of diagonal and non-diagonal backgrounds. The assigned background parameters (the choice of these values is determined only for the purpose of illustrating the complex role of the background in the scattering amplitudes) are listed in the Figure 6 caption. For brevity, we use an energy-independent background.

(a) Amplitude $\left|F_{11}(E)\right|^2$ (b) Amplitude $\left|F_{12}(E)\right|^2$
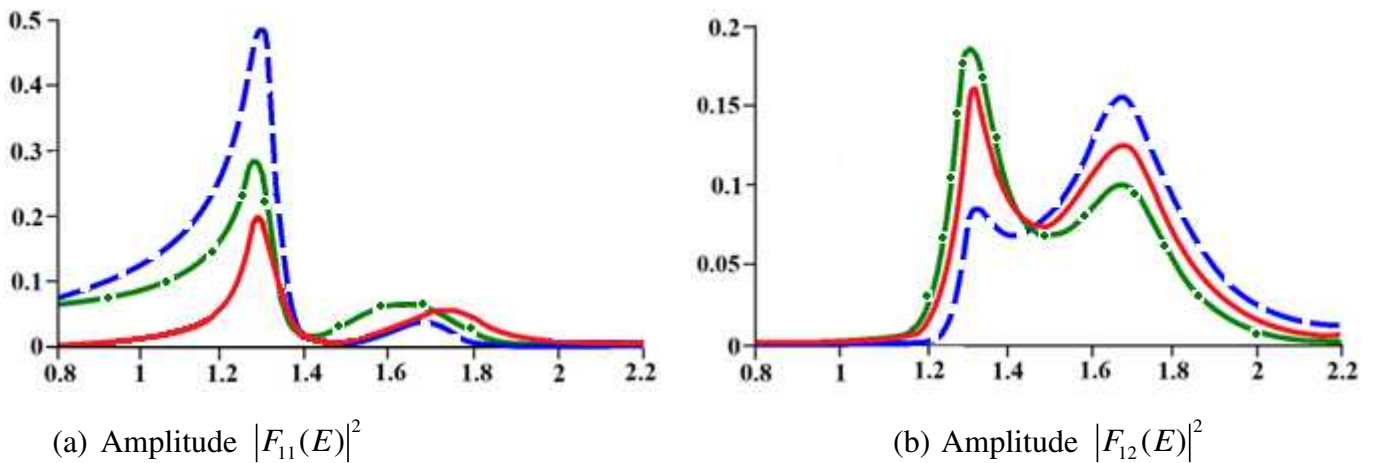
Figure 6. Panel (**a**): $|F_{11}(E)|^2$. Red solid line—no background, $\beta_k = 0$, $\psi_k = 0$ ($k = 1,2,3$); green dashed-dotted line—$\beta_k = \pi/18$, $\psi_k = 0$; blue dashed line—$\beta_k = \pi/18$, $\psi_k = \pi/18$. Panel (**b**): $|F_{12}(E)|^2$. Red solid line—$\psi_k = 0$ (no background); green dashed-dotted line—$\psi_k = \pi/18$; blue dashed line—$\psi_k = -\pi/18$ (β_k do not contribute to non-diagonal amplitudes).

It is seen that even a small 10° diagonal background modifies the resonance shapes. The non-diagonal background affects even the peak locations. For example, the second peak in $|F_{11}(E)|^2$ is shifted by about 0.15 from m_2, the "correct" one. Since a background contribution may be very different to the intuitively expected smooth plateau in each of the channels, an adequate treatment of the background is absolutely necessary when describing overlapping multichannel states.

Because a *priori* there are no criteria for choosing the background parametrization, an additional uncertainty to resonance parameters is added. If some particular parametrization fits the data substantially better than the others, it can be considered as a practical criterion. If this is not the case, then a simplified approach may be the following: first, evaluate the background by smoothing ("smearing") the data [11,12]—this procedure will remove all the peaks; then, subtract the background obtained in this way from the data and analyze the remainder using methods that preserve unitarity—a similar approach was used in work [13] to analyze overlapping $\omega'(1400)$ and $\omega'(1600)$ resonances (in the approximation of constant widths).

5. Conclusions

The parameters of broad inelastic resonances depend on the way the scattering data are analyzed. Obviously, the interference between resonances is the central part of an analysis and interpretation. In this respect, the usual Breit–Wigner parametrization which does not satisfy unitarity, is very unreliable (as shown in Figure 1); for example, it can possibly lead to deviations in mass of more than 100 MeV in the case of ρ' mesons [14].

There are different unitarization techniques to describe resonant states, both theoretical and phenomenological. Many analyses describe multi-channel resonances as poles on unphysical Riemann sheets for scattering amplitudes, as in works [10,15]. Resonances can be given by poles and corresponding zeros on an uniformization plane, which allows the description of broad multi-channel resonances [16]. The inverse amplitude method is another unitarization technique to present resonances and to enlarge the energy applicability region of effective Lagrangians [17].

We compare the results of the unitary BW approach and the standard widely used *K*-matrix method [5]. Both methods lead to similar results when resonances in *all channels* are well resolved, with a better ability of the BW method to resolve *wide overlapping resonances*. Another advantage of the BW method is that scattering amplitudes can be separated in the resonant and background contributions. A background, in terms of the phase shifts, does not necessarily form a smooth plateau in scattering amplitudes, and its adequate treatment is critically important for describing overlapping multichannel states.

Papers on overlapping resonances often include statements that because a sum of the BW functions violates the unitarity, a BW description cannot be used by default. Here, we draw attention to the fact that the unitary BW form for several states is possible when the BW functions are taken with the proper phase factors. From a theoretical point of view, it worth knowing that the problem formulated a long time ago [2], to present the unitary *S*-matrix as a sum of BW terms, has a regular solution. This approach can be used in fitting procedures, along with the *K*-matrix and other methods. Despite the *K*-matrix description being perfectly adequate as *unitary parametrization* for relatively narrow resonances, the names of its parameters, e.g., "masses", "widths", "and branching ratios" are only borrowed from the BW description through the comparison with a single resonance situation. A unitary BW model-independent method to describe overlapping resonances, with the parameters having direct physical meaning and the background being treated in a regular quantum mechanics form [18], can be a good addition to other unitarization techniques.

Author Contributions: V.H.–theory; T.B.–calculations. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: Authors declare no conflict of interest.

Appendix A. Construction of Unitary *S* Matrix in BW Approach

Despite the formulas below looking somewhat sophisticated, their final form for a particular num;ber of channels and states is rather simple.

For *N* resonances and *M* channels, the *S* matrix is given by (background contribution is considered in Section 4) the expression

$$S(E) = I + 2i\sqrt{\rho}\,T\sqrt{\rho} \tag{A1}$$

with (variable *E* is used only to simplify the formulas)

$$T(E) = \sum_{r=1}^{N} \frac{\vec{g}_r \vec{g}_r / 2}{\varepsilon_r(E) - E}. \tag{A2}$$

The unitarity condition $S^{\dagger}(E)S(E) \equiv I$ gives

$$T - T^{+} = 2iT\rho T^{+},$$

or

$$i\sum_{r=1}^{N}\left[\frac{g_{ri}g_{rj}}{\varepsilon_r - E} - \frac{g_{ri}^*g_{rj}^*}{\varepsilon_r^* - E}\right] + \sum_{l=1}^{N}\sum_{r=1}^{N}V_{lr}\frac{g_{ri}g_{lj}^*}{(\varepsilon_r - E)(\varepsilon_l^* - E)} \equiv 0, \tag{A3}$$

where

$$V_{lr} = \sqrt{\Gamma_l^0\Gamma_r^0}\sum_{k=1}^{M}\rho_k(E)g_{lk}^*g_{rk} \tag{A4}$$

Multiplying Equation (A3) by the product $\prod_{k=1}^{N}(\varepsilon_k - E)(\varepsilon_k^* - E)$, we obtain

$$\begin{aligned}
&\sum_{r=1}^{N}B_r\left[-ig_{ri}^*g_{rj}^*(\varepsilon_r - E) + ig_{ri}g_{rj}(\varepsilon_r^* - E) + V_{rr}g_{ri}g_{rj}^*\right]\\
&+\sum_{r=1}^{N-1}\sum_{l=r+1}^{N}B_{rl}[V_{lr}g_{ri}g_{lj}^*(\varepsilon_l - E)(\varepsilon_r^* - E) + V_{rl}g_{li}g_{rj}^*(\varepsilon_l^* - E)(\varepsilon_r - E)] \equiv 0,
\end{aligned} \tag{A5}$$

with $B_r \equiv \prod_{k\neq r}^{N}(E - \varepsilon_k)(E - \varepsilon_k^*)$, $B_{rl} \equiv \prod_{k\neq r,l}^{N}(E - \varepsilon_k)(E - \varepsilon_k^*)$.

To satisfy the unitarity relation (A3), it is necessary and sufficient that coefficients of the polynomial (A5) be zero for all powers of E^k. The coefficient of E^k is

$$\begin{aligned}
&\sum_{r=1}^{N}\left\{B_r^{(k)}V_{rr}g_{ri}g_{rj}^* + 2B_r^{(k-1)}\text{Im}(g_{ri}g_{rj}) + 2B_r^{(k)}\left[-\varepsilon_r^x\text{Im}(g_{ri}g_{rj}) + \varepsilon_r^y\text{Re}(g_{ri}g_{rj})\right]\right\}\\
&+\sum_{r=1}^{N-1}\sum_{l=r+1}^{N}\left\{B_{rl}^{(k-2)}\left[V_{lr}g_{ri}g_{lj}^* + V_{lr}^*g_{li}g_{rj}^*\right] - B_{rl}^{(k-1)}\left[V_{lr}g_{ri}g_{lj}^*(\varepsilon_l + \varepsilon_r^*)\right.\right.\\
&\left.\left.+V_{lr}^*g_{li}g_{rj}^*(\varepsilon_l^* + \varepsilon_r)\right] + B_{rl}^{(k)}\left[V_{lr}g_{ri}g_{lj}^*\varepsilon_l\varepsilon_r^* + V_{lr}^*g_{li}g_{rj}^*\varepsilon_l^*\varepsilon_r\right]\right\} = 0.
\end{aligned} \tag{A6}$$

Here, $B_r^{(k)}$ and $B_{rl}^{(k)}$ are coefficients at the powers E^k in the polynomials B_r and B_{rl}, respectively.

Equating the coefficient to zero at the highest degree E^{2N-1}, taking into account that B_r is the polynomial of degree $(2N - 2)$ and B_{rl} is the polynomial of degree $(2N - 4)$, we obtain:

$$\sum_{r=1}^{N}\text{Im}(g_{ri}g_{rj}) = 0 \text{ or } \sum_{r=1}^{N}\left[g_{ri}^xg_{rj}^y + g_{ri}^yg_{rj}^x\right] = 0 \tag{A7}$$

where $i,j = 1,\ldots,M$, $\vec{g}_r = \vec{g}_r^x + i\vec{g}_r^y$.

The key moment of the method is that we construct the imaginary parts $\vec{g}_r^y$ of vectors $\vec{g}_r$ as linear combinations of their real parts $\vec{g}_r^x$ ($r = 1,\ldots,N$):

$$\vec{g}_r^y = u_{r1}\vec{g}_1^x + u_{r2}\vec{g}_2^x + \ldots + u_{rN}\vec{g}_N^x \tag{A8}$$

Substituting expression (A8) into relation (A7), we obtain $u_{rr} = 0$ and $u_{rk} = -u_{kr}$; thus, U is an anti-symmetric matrix. (If there is no time-reversal symmetry restriction, matrix U can take a more general form, not necessarily anti-symmetric.) Equation (A8) gives NM relations involving $N(N-1)/2$ parameters (matrix elements u_{ij}).

Next, equate the coefficient to zero at E^{2N-2}:

$$
\sum_{r=1}^{N} \left\{ 2B_r^{(2N-3)} \mathrm{Im}(g_{ri}g_{rj}) + B_r^{(2N-2)} \left[V_{rr}g_{ri}g_{rj}^* - 2\varepsilon_r^x \mathrm{Im}(g_{ri}g_{rj}) + 2\varepsilon_r^y \mathrm{Re}(g_{ri}g_{rj}) \right] \right\}
$$
$$
+ \sum_{r=1}^{N-1} \sum_{l=r+1}^{N} \left\{ B_{rl}^{(2N-4)} \left[V_{lr}g_{ri}g_{lj}^* + V_{lr}^* g_{li}g_{rj}^* \right] - B_{rl}^{(2N-3)} \left[V_{lr}g_{ri}g_{lj}^* (\varepsilon_l + \varepsilon_r^*) \right. \right. \tag{A9}
$$
$$
\left. \left. + V_{lr}^* g_{li}g_{rj}^* (\varepsilon_l^* + \varepsilon_r) \right] + B_{rl}^{(2N-2)} \left[V_{lr}g_{ri}g_{lj}^* \varepsilon_l \varepsilon_r^* + V_{lr}^* g_{li}g_{rj}^* \varepsilon_l^* \varepsilon_r \right] \right\} = 0.
$$

Because $B_r^{(2N-3)} = -2\sum\limits_{k \neq r} \varepsilon_k^x$, $B_r^{(2N-2)} = B_{rl}^{(2N-4)} = 1$, $B_{rl}^{(2N-2)} = B_{rl}^{(2N-3)} = 0$,
we obtain

$$
\sum_{r=1}^{N} \left[-\sum_{k \neq r} 4\varepsilon_k^x \mathrm{Im}(g_{ri}g_{rj}) + 2\varepsilon_r^y \mathrm{Re}(g_{ri}g_{rj}) - 2\varepsilon_r^x \mathrm{Im}(g_{ri}g_{rj}) + V_{rr}g_{ri}g_{rj}^* \right]
$$
$$
+ \sum_{r=1}^{N-1} \sum_{l=r+1}^{N} \left[V_{lr}g_{ri}g_{lj}^* + V_{lr}^* g_{li}g_{rj}^* \right] = 0. \tag{A10}
$$

Taking into account expression (A8), we can state

$$
\begin{aligned}
g_{ri}g_{rj} &= \left[g_{ri}^x g_{rj}^x - \sum_{k=1}^{N} \sum_{v=1}^{N} u_{rk}u_{rv}g_{ri}^x g_{vj}^x \right] + i\sum_{k=1}^{N} u_{rk} \left[g_{ri}^x g_{kj}^x + g_{rj}^x g_{ki}^x \right], \\
g_{ri}g_{rj}^* &= \left[g_{ri}^x g_{rj}^x + \sum_{k=1}^{N} \sum_{v=1}^{N} u_{rk}u_{rv}g_{ki}^x g_{vj}^x \right] + i\sum_{k=1}^{N} u_{rk} \left[-g_{ri}^x g_{kj}^x + g_{rj}^x g_{ki}^x \right], \\
g_{ri}g_{lj}^* &= \left[g_{ri}^x g_{lj}^x + \sum_{k=1}^{N} \sum_{v=1}^{N} u_{rk}u_{lv}g_{ki}^x g_{vj}^x \right] + i\sum_{k=1}^{N} \left[-u_{lk}g_{ri}^x g_{kj}^x + u_{rk}g_{lj}^x g_{ki}^x \right], \\
g_{li}g_{rj}^* &= \left[g_{li}^x g_{rj}^x + \sum_{k=1}^{N} \sum_{v=1}^{N} u_{lk}u_{rv}g_{ki}^x g_{vj}^x \right] + i\sum_{k=1}^{N} \left[-u_{rk}g_{li}^x g_{kj}^x + u_{lk}g_{rj}^x g_{ki}^x \right].
\end{aligned} \tag{A11}
$$

In order for the coefficient in (A10) at E^{2N-2} to be zero, it is necessary to set the coefficients of all products $g_{\mu i}^x g_{\nu j}^x$ to zero, $(\mu, \nu = 1, \dots, N)$. We substitute expressions (A11) into relation (A10) and equate the real and imaginary parts of these coefficients to zero. This yields:

$$
2\varepsilon_\mu^y - 2\sum_{r \neq \mu} \varepsilon_\mu^y u_{r\mu}^2 + V_{\mu\mu} + \sum_{r \neq \mu} V_{rr}u_{r\mu}^2 + 2\sum_{r=1}^{N-1} \sum_{l=r+1}^{N} V_{lr}^x u_{r\mu}u_{l\mu} + 2\sum_{r \neq \mu} V_{r\mu}^y u_{r\mu} = 0, \tag{A12}
$$

$$
-2\sum_{r=1}^{N} \varepsilon_r^y u_{r\mu}u_{r\nu} + 2u_{\mu\nu}(\varepsilon_\mu^x - \varepsilon_\nu^x) + \sum_{r=1}^{N} V_{rr}u_{r\mu}u_{r\nu} + V_{\mu\nu}^x
$$
$$
+ \sum_{r=1}^{N-1} \sum_{l=r+1}^{N} V_{lr}^x [u_{r\mu}u_{l\nu} + u_{l\mu}u_{r\nu}] + \sum_{r \neq \mu,\nu} [V_{r\mu}^y u_{r\nu} + V_{r\nu}^y u_{r\mu}] = 0, \tag{A13}
$$

$$u_{\nu\mu}(V_{\mu\mu} + V_{\nu\nu}) + V_{\nu\mu}^y + \sum_{r=1}^{N-1}\sum_{l=r+1}^{N} V_{lr}^y[u_{r\mu}u_{l\nu} - u_{l\mu}u_{r\nu}] + \sum_{r\neq\mu,\nu}^{N} [V_{r\nu}^x u_{r\mu} + V_{r\mu}^x u_{r\nu}] = 0. \quad \text{(A14)}$$

Thus, we obtain N^2 simultaneous equations linear in the scalar products V_{rr}, V_{lr}^x and V_{lr}^y: N equations (A12), $N(N-1)/2$ equations (A13), and $N(N-1)/2$ equations (A14). The coefficients at $g_{\mu i}^x g_{\nu j}^x$ and $g_{\nu i}^x g_{\mu j}^x$ are identical because equations (A13) and (A14) are symmetrical under the μ and ν interchange.

Solving these equations, we obtain (recall that $V_{lr} = \sqrt{\Gamma_l^0\Gamma_r^0}\sum_{k=1}^{M}\rho_k(E)g_{lk}^*g_{rk}$):

$$
\begin{aligned}
V_{rr} &= \Gamma_r^0\sum_{k=1}^{M}\rho_k(E)\left[(g_{rk}^x)^2 + (g_{rk}^y)^2\right] = -\tfrac{2}{S}[S + 2Q_r]\varepsilon_r^y,\\
V_{lr}^x &= \sqrt{\Gamma_l^0\Gamma_r^0}\sum_{k=1}^{M}\rho_k(E)\left[g_{lk}^x g_{rk}^x + g_{lk}^y g_{rk}^y\right] = -\tfrac{2}{S}\left[F_{lr}(\varepsilon_l^x - \varepsilon_r^x) + G_{lr}\left(\varepsilon_l^y + \varepsilon_r^y\right)\right],\\
V_{lr}^y &= \sqrt{\Gamma_l^0\Gamma_r^0}\sum_{k=1}^{M}\rho_k(E)\left[g_{lk}^x g_{rk}^y - g_{lk}^y g_{rk}^x\right] = -\tfrac{2}{S}\left[G_{lr}(\varepsilon_l^x - \varepsilon_r^x) - F_{lr}\left(\varepsilon_l^y + \varepsilon_r^y\right)\right],\\
&\quad r = 1,\dots,N; l = r+1,\dots,N,
\end{aligned}
\quad\text{(A15)}
$$

where

$$S = 1 + \sum_{p=1}^{[\frac{N}{2}]} (-1)^p \sum_{i_1=1}^{N-2p+1}\sum_{i_2>i_1}^{N-2p+2}\cdots\sum_{i_{2p}>i_{2p-1}}^{N}\begin{vmatrix} i_1 & i_2 & \cdots & i_{2p} \\ i_1 & i_2 & \cdots & i_{2p} \end{vmatrix},$$

$$Q_r = \sum_{p=1}^{[\frac{N}{2}]} (-1)^p \sum_{i_1=1}^{N-2p+2}\sum_{i_2>i_1}^{N-2p+3}\cdots\sum_{i_{2p-1}>i_{2p-2}}^{N}\begin{vmatrix} r & i_1 & i_2 & \cdots & i_{2p-1} \\ r & i_1 & i_2 & \cdots & i_{2p-1} \end{vmatrix},$$
$$(i_1,\dots,i_{2p-1} \neq r),$$

$$F_{lr} = u_{lr} + \sum_{p=1}^{[\frac{N}{2}]-1} (-1)^{p+1} \sum_{i_1=1}^{N-2p+1}\sum_{i_2>i_1}^{N-2p+2}\cdots\sum_{i_{2p}>i_{2p-1}}^{N}\begin{vmatrix} l & i_1 & i_2 & \cdots & i_{2p} \\ r & i_1 & i_2 & \cdots & i_{2p} \end{vmatrix},$$
$$(i_1,\dots,i_{2p} \neq r,l),$$

$$G_{lr} = \sum_{p=1}^{[\frac{N+1}{2}]-1} (-1)^{p+1} \sum_{i_1=1}^{N-2p+2}\sum_{i_2>i_1}^{N-2p+3}\cdots\sum_{i_{2p-1}>i_{2p-2}}^{N}\begin{vmatrix} l & i_1 & i_2 & \cdots & i_{2p-1} \\ r & i_1 & i_2 & \cdots & i_{2p-1} \end{vmatrix},$$
$$(i_1,\dots,i_{2p-1} \neq r,l).$$

Here,

$$\begin{pmatrix} l & i_1 & i_2 & \cdots & i_k \\ r & i_1 & i_2 & \cdots & i_k \end{pmatrix} = \begin{pmatrix} u_{lr} & u_{li_1} & \cdots & u_{li_k} \\ u_{i_1 r} & u_{i_1 i_1} & \cdots & u_{i_1 i_k} \\ \cdots & \cdots & \cdots & \cdots \\ u_{i_k r} & u_{i_k i_1} & \cdots & u_{i_k i_k} \end{pmatrix}$$

is the minor with the rows $l, i_1, i_2,\dots, i_k$ and the columns $r, i_1, i_2,\dots, i_k$ of matrix $\boldsymbol{u}$ $(i_1,\dots,i_{2p} \neq r,l)$. Notations such as $[N/2]$ mean an integer part of the expression in the brackets.

Relations (A8) and the scalar products (A15) completely define the conditions imposed on vectors $\vec{g}_r$ ($r = 1, \ldots, N$). It can be shown that if vectors $\vec{g}_r$ satisfy the relations (A8) and (A15), then the coefficients at lower degrees of polynomial (A5) are identically equal to zero.

For practical purposes, it is convenient to express the scalar products (A15) in terms of real vectors $\vec{g}_r^{\,x}$:

$$
\begin{aligned}
\Gamma_r^0 \sum_{k=1}^M \rho_k \left(g_{rk}^x\right)^2 &= \frac{2}{S^2}\left\{ -(S+Q_r)^2 \varepsilon_r^y + \sum_{i\neq r}\left[2F_{ri}G_{ri}\varepsilon_i^x + \left(F_{ri}^2 - G_{ri}^2\right)\varepsilon_i^y \right] \right\} \\
\sqrt{\Gamma_l^0 \Gamma_r^0} \sum_{k=1}^M \rho_k g_{lk}^x g_{rk}^x &= \frac{2}{S^2}\left\{ F_{lr}\left[(S+Q_r)\varepsilon_r^x - (S+Q_l)\varepsilon_l^x\right] \right. \\
&\quad - G_{lr}\left[(S+Q_r)\varepsilon_r^y + (S+Q_l)\varepsilon_l^y\right] \\
&\quad \left. + \sum_{i\neq l,r}\left[(F_{li}G_{ri} + F_{ri}G_{li})\varepsilon_i^x + (F_{li}F_{ri} - G_{li}G_{ri})\varepsilon_i^y \right] \right\}, \\
&\quad r = 1, \ldots, N; l = r+1, \ldots, N.
\end{aligned}
\tag{A16}
$$

The domain of the elements of the matrix U is defined by the conditions

$$
0 \leq \frac{F_{lr}^2 + G_{lr}^2}{(S + 2Q_r)(S + 2Q_l)} \leq \frac{\Gamma_r \Gamma_l}{4(E_r - E_l)^2 + (\Gamma_r + \Gamma_l)^2} \leq 1
$$

which follows from the Cauchy–Schwarz inequality $\left(\vec{a}^{\,*}, \vec{b}\right)^2 \leq \left(\vec{a}^{\,*}, \vec{a}\right)\left(\vec{b}^{\,*}, \vec{b}\right)$.

It is seen from these inequalities that if resonances are very far from each other, then $u_{rl} \to 0$ and vectors $\vec{g}_r$ become real and orthogonal. In this situation, the lengths of vectors $\vec{g}_r$ are equal to the resonance widths Γ_r, but, in general, these lengths are larger than Γ_r:

$$
\frac{\left|\vec{g}_r\right|^2}{\Gamma_r} = \frac{V_{rr}}{\Gamma_r} = 1 + 2\frac{Q_r}{S}.
$$

There is also a sum rule for vectors $\vec{g}_r$:

$$
\sum_{r=1}^N \left(\vec{g}_r, \vec{g}_r\right) = \sum_{r=1}^N \Gamma_r.
\tag{A17}
$$

The branching ratios for the decay of the r-th resonance in the i-th channel is

$$
B_{ri} = \frac{\left|g_{ri}\right|^2}{\left|\vec{g}_r\right|^2} = \frac{\left|g_{ri}\right|^2}{\sum_{k=1}^M \left|g_{rk}\right|^2}.
\tag{A18}
$$

For $F = (S - I)/2i$, we have:

$$F_{ij} = \sum_{r=1}^{N} e^{i(\varphi_{ri} + \varphi_{rj})} \frac{\left|\vec{g}_r\right|^2 \sqrt{B_{ri}B_{rj}}/2}{E_r - E - i\Gamma_r/2},$$ (A19)

where φ_{ri} and φ_{rj} are real phases of vectors g_{ri} and g_{rj}, $g_{rk} = e^{i\varphi_{rk}}|g_{rk}|$.

In a potential fitting procedure there are $2N(M + 1)$ real parameters: $E_r, \Gamma_r, \vec{g}_r = \vec{g}_r^x + i\vec{g}_r^y$. Besides these, the relations (A15) contain $N(N - 1)$ elements of real anti-symmetric matrix U ($|u_{rl}| \leq 1$). (These u_{ij} are 'technical' and do not appear in the final expression for the S-matrix.) Not all these parameters are independent. The NM relations (A6) connect the real and imaginary parts of vectors $\vec{g}_r$. Moreover, the real parts, $\vec{g}_r^x$, are connected by $N(N + 1)/2$ relations (A15). Thus, in total there are $N(M + 1)$ free real parameters:

$$\left[2N(M + 1) + \frac{N(N - 1)}{2}\right] - NM - \frac{N(N + 1)}{2} = N(M + 1)$$

the same number as in the K-matrix parametrization.

Even though the formulas (A15) look somewhat sophisticated, the resonance parameters can be determined using a straightforward regular algorithm; the unitarity of S can be confirmed at every value of E. In this Appendix, we use the energy variable E to avoid unnecessary complications in the formulas; however, the algorithm can be rewritten in terms of the variable $s = E^2$, as in Section 1 and in one of the Examples in Section 3.

For *two resonances*, the algorithm is presented in the main text.

For *three resonances*, $N = 3$, matrix $S(E)$ is

$$S_{ij} = \delta_{ij} + 2i\sqrt{\rho_i \rho_j} \left[\frac{\Gamma_1^0 g_{1i} g_{1j}/2}{\varepsilon_1 - E} + \frac{\Gamma_2^0 g_{2i} g_{2j}/2}{E - \varepsilon_2} + \frac{\Gamma_3^0 g_{3i} g_{3j}/2}{\varepsilon_3 - E}\right], \quad (i, j = 1, \ldots, M).$$

Matrix U, relating the real and imaginary parts of vectors $\vec{g}_1, \vec{g}_2, \vec{g}_3$, is

$$U = \begin{pmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & -\gamma \\ \beta & \gamma & 0 \end{pmatrix},$$ (A20)

i.e.,

$$\vec{g}_1^y = -\alpha \vec{g}_2^x - \beta \vec{g}_3^x, \ \vec{g}_2^y = \alpha \vec{g}_1^x - \gamma \vec{g}_3^x, \ \vec{g}_3^y = \beta \vec{g}_1^x + \gamma \vec{g}_2^x.$$

Real parameters α, β, γ are restricted by the relation $\alpha^2 + \beta^2 + \gamma^2 < 1$.

The coefficients in formulas (A16) are:

$$\begin{aligned}
&S = 1 - \alpha^2 - \beta^2 - \gamma^2, \\
&Q_1 = \alpha^2 + \beta^2, Q_2 = \alpha^2 + \gamma^2, Q_3 = \beta^2 + \gamma^2, \\
&F_{12} = -\alpha, F_{13} = -\beta, F_{23} = -\gamma; \\
&G_{12} = \beta\gamma, G_{13} = -\alpha\gamma, G_{23} = \alpha\beta.
\end{aligned}$$ (A21)

Substituting these in formulas (A16), we obtain the relations needed to construct vectors $\vec{g}_1$, $\vec{g}_2$ and $\vec{g}_3$:

$$\Gamma_1^0 \sum_{k=1}^{M} \rho_k(g_{1k}^{x\,2}) = \frac{2}{S^2}[2\alpha\beta\gamma(\varepsilon_3^x - \varepsilon_2^x) - \varepsilon_1^y(1 - \gamma^{2^2}) + \varepsilon_2^y(\alpha^2 - \beta^2\gamma^2) + \varepsilon_3^y(\beta^2 - \alpha^2\gamma^2)],$$

$$\Gamma_2^0 \sum_{k=1}^{M} \rho_k(g_{2k}^x)^2 = \frac{2}{S^2}[2\alpha\beta\gamma(\varepsilon_1^x - \varepsilon_3^x) + \varepsilon_1^y(\alpha^2 - \beta^2\gamma^2) - \varepsilon_2^y(1 - \beta^2)^2 + \varepsilon_3^y(\gamma^2 - \alpha^2\beta^2)],$$

$$\Gamma_3^0 \sum_{k=1}^{M} \rho_k(g_{3k}^x)^2 = \frac{2}{S^2}[2\alpha\beta\gamma(\varepsilon_2^x - \varepsilon_1^x) + \varepsilon_1^y(\beta^2 - \alpha^2\gamma^2) + \varepsilon_2^y(\gamma^2 - \alpha^2\beta^2) - \varepsilon_3^y(1 - \alpha^2)^2],$$

$$\sqrt{\Gamma_1^0\Gamma_2^0} \sum_{k=1}^{M} \rho_k g_{1k}^x g_{2k}^x = \frac{2}{S^2}\left\{\alpha\left[\varepsilon_1^x(1 - \gamma^2) - \varepsilon_2^x(1 - \beta^2) - \varepsilon_3^x(\beta^2 - \gamma^2)\right]\right.$$
$$\left. + \beta\gamma\left[-\varepsilon_1^y(1 - \gamma^2) - \varepsilon_2^y(1 - \beta^2) + \varepsilon_3^y(1 + \alpha^2)\right]\right\},$$

$$\sqrt{\Gamma_1^0\Gamma_3^0} \sum_{k=1}^{M} \rho_k g_{1k}^x g_{3k}^x = \frac{2}{S^2}\left\{\beta\left[\varepsilon_1^x(1 - \gamma^2) - \varepsilon_2^x(\alpha^2 - \gamma^2) - \varepsilon_3^x(1 - \alpha^2)\right]\right.$$
$$\left. + \alpha\gamma\left[\varepsilon_1^y(1 - \gamma^2) - \varepsilon_2^y(1 + \beta^2) + \varepsilon_3^y(1 - \alpha^2)\right]\right\},$$

$$\sqrt{\Gamma_2^0\Gamma_3^0} \sum_{k=1}^{M} \rho_k g_{2k}^x g_{3k}^x = \frac{2}{S^2}\left\{\gamma\left[-\varepsilon_1^x(\alpha^2 - \beta^2) + \varepsilon_2^x(1 - \beta^2) - \varepsilon_3^x(1 - \alpha^2)\right]\right.$$
$$\left. + \alpha\beta\left[\varepsilon_1^y(1 + \gamma^2) - \varepsilon_2^y(1 - \beta^2) - \varepsilon_3^y(1 - \alpha^2)\right]\right\}.$$

The widths are

$$\Gamma_1 = \Gamma_1(\hat{\varepsilon}_1) = \frac{1 - \alpha^2 - \beta^2 - \gamma^2}{2(1 + \alpha^2 + \beta^2 - \gamma^2)}\Gamma_1^0 \sum_{k=1}^{M} \rho_k(\hat{\varepsilon}_1)|g_{1k}|^2,$$

$$\Gamma_2 = \Gamma_2(\hat{\varepsilon}_2) = \frac{1 - \alpha^2 - \beta^2 - \gamma^2}{2(1 + \alpha^2 - \beta^2 + \gamma^2)}\Gamma_2^0 \sum_{k=1}^{M} \rho_k(\hat{\varepsilon}_2)|g_{2k}|^2, \tag{A22}$$

$$\Gamma_3 = \Gamma_3(\hat{\varepsilon}_3) = \frac{1 - \alpha^2 - \beta^2 - \gamma^2}{2(1 - \alpha^2 + \beta^2 + \gamma^2)}\Gamma_3^0 \sum_{k=1}^{M} \rho_k(\hat{\varepsilon}_3)|g_{3k}|^2,$$

where $\hat{\varepsilon}_i$ are zeros of real parts of denominators in the corresponding BW terms.

Let us very briefly describe the case of *four resonances*, $N = 4$. Matrix U is

$$U = \begin{pmatrix} 0 & -\alpha & -\beta & -\delta \\ \alpha & 0 & -\gamma & -\eta \\ \beta & \gamma & 0 & -\omega \\ \delta & \eta & \omega & 0 \end{pmatrix} \tag{A23}$$

thus,

$$\vec{g}_1^{\,y} = -\alpha\vec{g}_2^{\,x} - \beta\vec{g}_3^{\,x} - \delta\vec{g}_4^{\,x},$$
$$\vec{g}_2^{\,y} = \alpha\vec{g}_1^{\,x} - \gamma\vec{g}_3^{\,x} - \eta\vec{g}_4^{\,x},$$
$$\vec{g}_3^{\,y} = \beta\vec{g}_1^{\,x} + \gamma\vec{g}_2^{\,x} - \omega\vec{g}_4^{\,x}, \tag{A24}$$
$$\vec{g}_4^{\,y} = \delta\vec{g}_1^{\,x} + \eta\vec{g}_2^{\,x} + \omega\vec{g}_3^{\,x}.$$

The lengths of vectors $\overset{\rightarrow x}{g}_1$, $\overset{\rightarrow x}{g}_2$, $\overset{\rightarrow x}{g}_3$, $\overset{\rightarrow x}{g}_4$ and the angles between them are determined by the scalar products (A15) for $N = 4$. From that we have for the coefficients in formulas (A16):

$$S = 1 - \alpha^2 - \beta^2 - \delta^2 - \gamma^2 - \eta^2 - \omega^2 + \widetilde{D}^2$$

$$Q_1 = \alpha^2 + \beta^2 + \delta^2 - \widetilde{D}^2, \; Q_2 = \alpha^2 + \gamma^2 + \eta^2 - \widetilde{D}^2,$$

$$Q_3 = \beta^2 + \gamma^2 + \omega^2 - \widetilde{D}^2, \; Q_4 = \delta^2 + \eta^2 + \omega^2 - \widetilde{D}^2,$$

$$F_{21} = \alpha - \omega\widetilde{D}, \; F_{31} = \beta + \eta\widetilde{D}, \; F_{41} = \delta - \gamma\widetilde{D}$$

$$F_{32} = \gamma - \delta\widetilde{D}, \; F_{42} = \eta + \beta\widetilde{D}, \; F_{43} = \omega - \alpha\widetilde{D},$$

$$F_{12} = -\alpha + \omega\widetilde{D}, \; F_{13} = -\beta - \eta\widetilde{D}, \; F_{14} = -\delta + \gamma\widetilde{D},$$

$$F_{23} = -\gamma + \delta\widetilde{D}, \; F_{42} = -\eta - \beta\widetilde{D}, \; F_{34} = -\omega + \alpha\widetilde{D}.$$

$$G_{21} = \beta\gamma + \delta\eta, \; G_{12} = \beta\gamma + \delta\eta, \; G_{31} = -\alpha\gamma + \delta\omega,$$

$$G_{41} = -\alpha\eta - \beta\omega, \; G_{32} = \alpha\beta + \eta\omega,$$

$$G_{42} = \alpha\delta - \gamma\omega, \; G_{43} = \beta\delta + \gamma\eta,$$

where $\widetilde{D} = \alpha\omega - \beta\eta + \gamma\delta$.

Formulas (A15) give the scalar products—as an example we present two of these here:

$$\begin{aligned}
\Gamma_1^0 \sum_{k=1}^{M} \rho_k (g_{1k}^x)^2 = \frac{2}{S^2} \cdot \Big\{ &-2(\beta\gamma + \delta\eta)\left(\alpha - \omega\widetilde{D}\right) \cdot \varepsilon_2^x + 2(\alpha\gamma - \delta\omega)\left(\beta + \eta\widetilde{D}\right) \cdot \varepsilon_3^x \\
&+2(\alpha\eta + \beta\omega)\left(\delta - \gamma\widetilde{D}\right) \cdot \varepsilon_4^x - (1 - \gamma^2 - \eta^2 - \omega^2)^2 \cdot \varepsilon_1^y \\
&+\left[\left(\alpha - \omega\widetilde{D}\right)^2 - (\beta\gamma + \delta\eta)^2\right] \cdot \varepsilon_2^y + \left[\left(\beta + \eta\widetilde{D}\right)^2 - (\alpha\gamma - \delta\omega)^2\right] \cdot \varepsilon_3^y \\
&+\left[\left(\delta - \gamma\widetilde{D}\right)^2 - (\alpha\eta + \beta\omega)^2\right] \cdot \varepsilon_4^y \Big\},
\end{aligned}$$

$$\begin{aligned}
\sqrt{\Gamma_2^0 \Gamma_1^0} \sum_{k=1}^{M} \rho_k g_{2k}^x g_{1k}^x = \frac{2}{S^2} \Big\{ &\left(1 - \gamma^2 - \eta^2 - \omega^2\right)\left(\alpha - \omega\widetilde{D}\right) \varepsilon_1^x \\
&-\left(1 - \beta^2 - \delta^2 - \omega^2\right)\left(\alpha - \omega\widetilde{D}\right) \varepsilon_2^x \\
&-\left[(-\alpha\gamma + \delta\omega)(\gamma - \delta\widetilde{D}) + (\alpha\beta + \eta\omega)(\beta + \eta\widetilde{D})\right] \varepsilon_3^x \\
&-\left[(-\alpha\eta - \beta\omega)(\eta + \beta\widetilde{D}) + (\alpha\delta - \gamma\omega)(\delta - \gamma\widetilde{D})\right] \varepsilon_4^x \\
&-(\beta\gamma + \delta\eta)\left(1 - \gamma^2 - \eta^2 - \omega^2\right) \varepsilon_1^y \\
&-(\beta\gamma + \delta\eta)\left(1 - \beta^2 - \delta^2 - \omega^2\right) \varepsilon_2^y \\
&+\left[(\beta + \eta\widetilde{D})(\gamma - \delta\widetilde{D}) - (-\alpha\gamma + \delta\omega)(\alpha\beta + \eta\omega)\right] \varepsilon_3^y \\
&+\left[(\delta - \gamma\widetilde{D})(\eta + \beta\widetilde{D}) - (-\alpha\eta - \beta\omega)(\alpha\delta - \gamma\omega)\right] \varepsilon_4^y \Big\}.
\end{aligned}$$

Appendix B. *K*-Matrix Interactive Software Description

We created an interactive software to work with *K*-matrix scattering amplitudes T_{ij}, which helps to see the main features of this method. This Appendix contains its brief description. This software can be provided upon request to interested colleagues.

The *S*-matrix:

$$S = \frac{1 + i\rho K}{1 - i\rho K} = 1 + 2i\sqrt{\rho}T\sqrt{\rho} = 1 + 2iF \tag{A25}$$

From this, the scattering amplitudes are:

$$T = (I - i\rho K)^{-1}K \tag{A26}$$

We use the standard *K*-matrix parametrization [5]:

$$K_{ij} = \sum_a \frac{\gamma_{ai}\gamma_{aj}m_a\Gamma_a^0}{m_a^2 - s} + (A_{ij} + B_{ij}s), \sum_i \gamma_{ai}^2 = 1. \tag{A27}$$

The feature which directly follows from (A26) and (A27) is that resonance peaks in $|T_{ij}|^2$ are well resolved, the scattering amplitudes have zero, or very close to zero, values between the pole locations m_a. (The introduction of Adler's zero in expression (A27) does not change this feature.) For instance, in the case of two resonances, the zero of equation $T_{12}(E) = 0$ (i.e., $K_{12}(E) = 0$) is located between the poles m_1 and m_2, and is given by a linear equation; in the case of three states, this is given by a quadratic equation. The zeros in T_{11} and T_{22} are slightly shifted from T_{12} zero. In general, for $N \geq 2$, when a polynomial contribution in (A27) is substantial, instead of exact zeros, there are deep minima in $|T_{ij}|^2$ between the peaks—the resonances in the K-matrix method are always well separated.

This particular software has three sections:

(a) two channels, two resonances;
(b) two channels, three resonances;
(c) three channels, two resonances.

The phase factors are:

$\rho_i(s) = \sqrt{(s - s_i)/s}$, where $s_i = E_i^2$; the resonances lie above the opening threshold, i.e., $m_\alpha > E_1$.

Appendix B.1. Two States, Two Channels

In the case of *two channels*, matrix *T* is:

$$T = \frac{1}{1 - \rho_1\rho_2 D - i(\rho_1 K_{11} + \rho_2 K_{22})} \begin{pmatrix} K_{11} - i\rho_2 D & K_{12} \\ K_{21} & K_{22} - i\rho_1 D \end{pmatrix} \tag{A28}$$

where $D = K_{11}K_{22} - K_{12}K_{21}$.

So

$$\begin{aligned} F &= \begin{pmatrix} \sqrt{\rho_1} & 0 \\ 0 & \sqrt{\rho_2} \end{pmatrix} T \begin{pmatrix} \sqrt{\rho_1} & 0 \\ 0 & \sqrt{\rho_2} \end{pmatrix} = \frac{\sqrt{\rho}K\sqrt{\rho}}{1 - i\rho K} \\ &= \frac{1}{1 - \rho_1\rho_2 D - i(\rho_1 K_{11} + \rho_2 K_{22})} \begin{pmatrix} \rho_1(K_{11} - i\rho_2 D) & \sqrt{\rho_1\rho_2}K_{12} \\ \sqrt{\rho_1\rho_2}K_{21} & \rho_2(K_{22} - i\rho_1 D) \end{pmatrix}. \end{aligned} \tag{A29}$$

For *two poles* in K_{ij}, the independent parameters are: six in the pole terms (their values can be set by *using the interface scroll bars*):
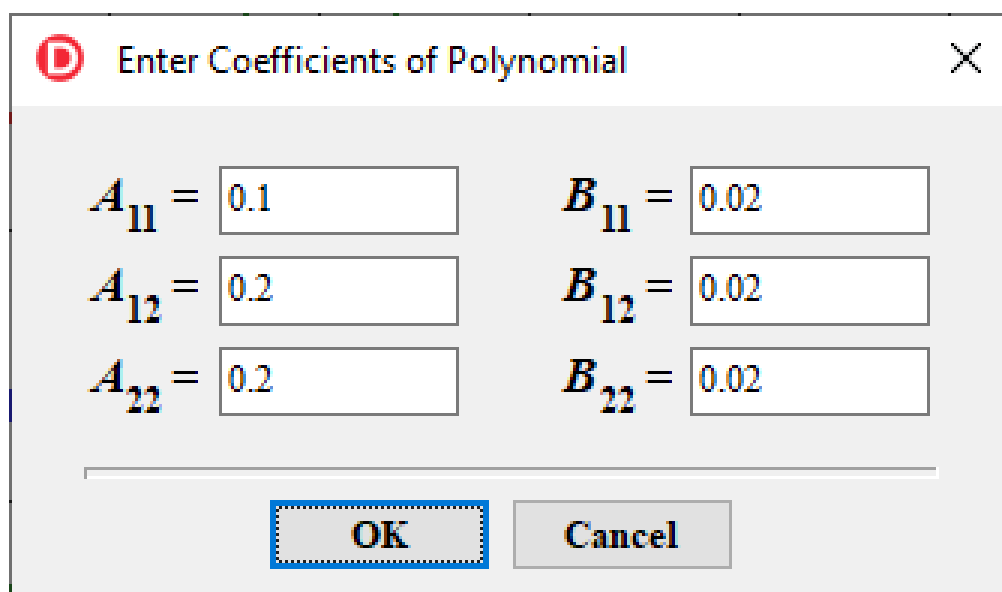
$$m_1, m_2, \Gamma_1, \Gamma_2, \gamma_{11}, \gamma_{21}$$

and six coefficients $A_{11}, A_{12}, A_{22}, B_{11}, B_{12}, B_{22}$ (their values can be *entered interactively*).

An obvious restriction on polynomial coefficients is that resonance peaks should not be wiped out.
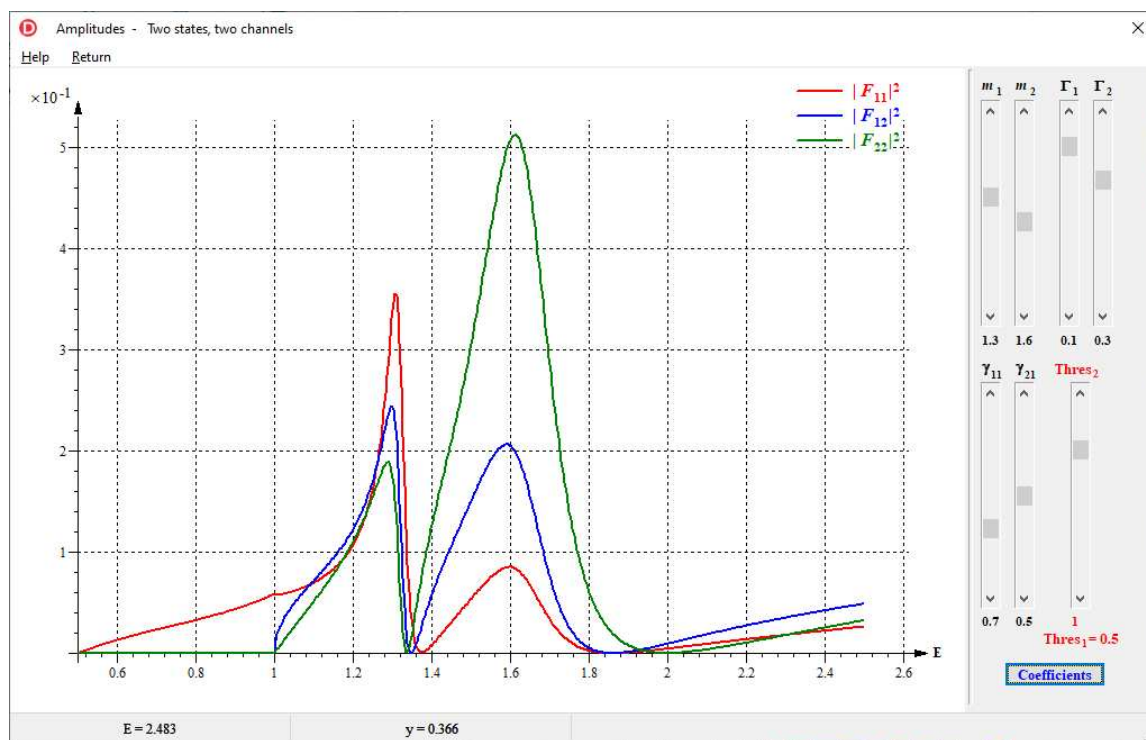
Example

Enter the polynomial coefficients in the Table:

Set the poles parameters and the threshold positions, E_1 and E_2 by using scroll bars—see the screenshot in Figure A1:
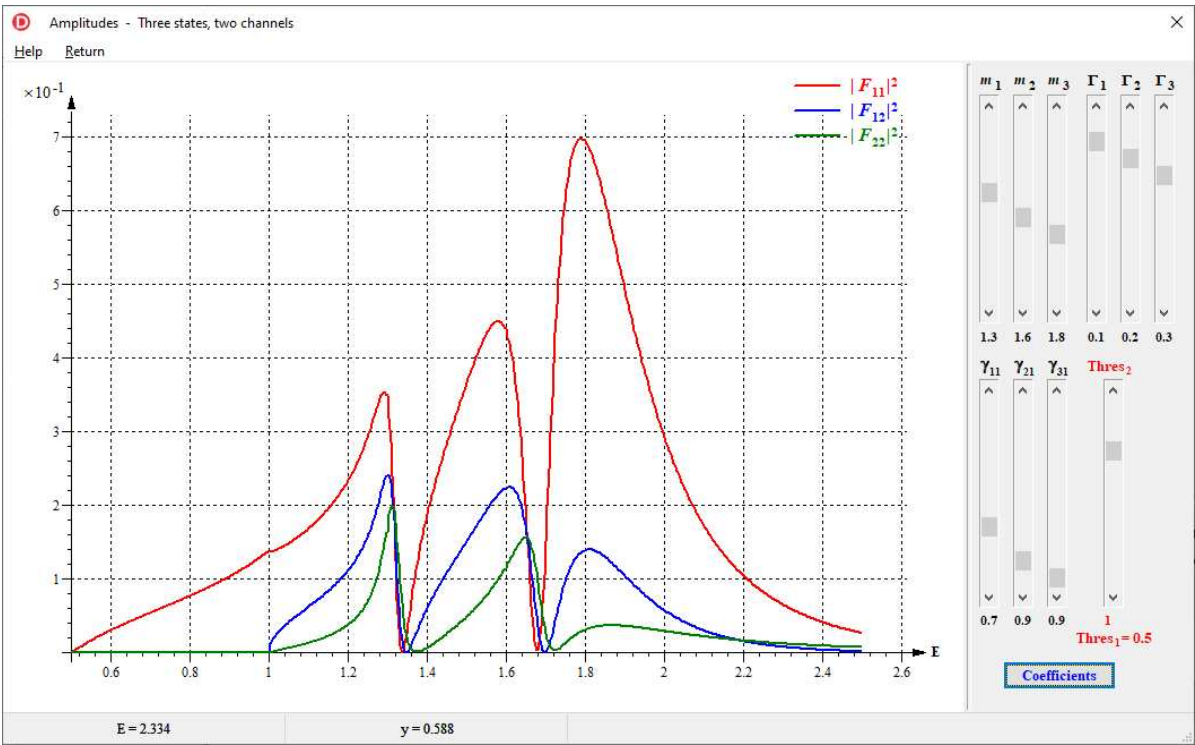
Figure A1. K-matrix scattering amplitudes.

The parameters in the screenshot in Figure A1 are:

$$m_1 = 1.3, \ m_2 = 1.6, \ \Gamma_1 = 0.1, \ \Gamma_2 = 0.3, \ \gamma_{11} = 0.7, \ \gamma_{21} = 0.5$$

$$Thres_1 = 0.5, \ Thres_2 = 1; \ \text{here } Tres_i \equiv E_i.$$

Appendix B.2. Three States, Two Channels

For three poles and two channels, the independent parameters are:

$$m_1, \ m_2, \ m_3, \ \Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \gamma_{11}, \ \gamma_{21}, \ \gamma_{31}, \ A_{11}, \ A_{12}, \ A_{22}, \ B_{11}, \ B_{12}, \ B_{22}.$$

Example

Enter the polynomial coefficients in the Table:

Set the poles parameters and the threshold positions, E_1 and E_2 by using scroll bars—see the screenshot in Figure A2:

Figure A2. K-matrix scattering amplitudes.

The parameters in the screenshot in Figure A2 are:

$$m_1 = 1.3,\ m_2 = 1.6,\ m_3 = 1.8,\ \Gamma_1 = 0.1,\ \Gamma_2 = 0.2,\ \Gamma_2 = 0.3,$$

$$\gamma_{11} = 0.7, \gamma_{21} = 0.9,\ \gamma_{31} = 0.9,\ \text{thresholds } Thres_1 = 0.5,\ Thres_2 = 1.$$

Appendix B.3. Three Channels, Two States

$$T = \frac{K}{1 - iK} = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 - iK_{11} & -iK_{12} & -iK_{13} \\ -iK_{21} & 1 - iK_{22} & -iK_{23} \\ -iK_{31} & -iK_{32} & 1 - iK_{33} \end{pmatrix}^{-1} \tag{A30}$$

(In this part of the software, in order do not overcomplicate the interface, we take $\rho_i(s) = 1$.)

The eight independent parameters are:

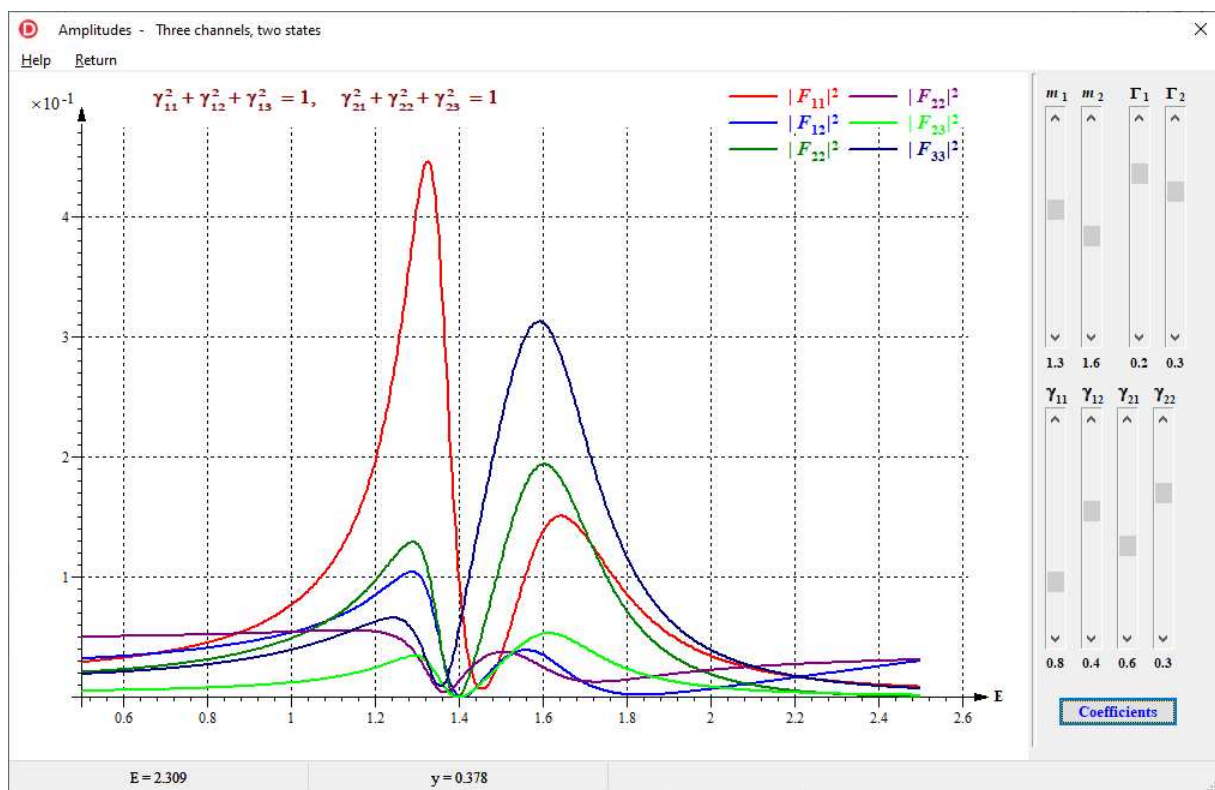$$m_1,\ m_2,\ \Gamma_1,\ \Gamma_2,\ \gamma_{11},\ \gamma_{12},\ \gamma_{21},\ \gamma_{22}$$

(γ is normalized, $\gamma_{a1}^2 + \gamma_{a2}^2 + \gamma_{a3}^2 = 1$).

Example

Enter the polynomial coefficients in the Table:

Set the poles parameters by using scroll bars—see the screenshot in Figure A3:

Figure A3. K-matrix scattering amplitudes.

The parameters in the screenshot in Figure A3 are:

$$m_1 = 1.3, \ m_2 = 1.6, \ \Gamma_1 = 0.2, \ \Gamma_2 = 0.3,$$

$$\gamma_{11} = 0.8, \gamma_{12} = 0.4, \ \gamma_{21} = 0.6, \ \gamma_{22} = 0.3$$

The purpose of the following example is to demonstrate that the K-matrix amplitudes can be always fitted with the unitary BW method, but the opposite is not true if amplitudes in all channels do not have zeros between resonance peaks, in other words, when the peaks actually *overlap*.

Consider the K-matrix amplitudes in the screen-shot in Figure A4.

The values of the K-matrix parameters in the screenshot in Figure A4 are:

$$m_1 = 1.3, \ m_2 = 1.6, \ \Gamma_1 = 0.3, \ \Gamma_2 = 0.3, \gamma_{11} = 0.7, \gamma_{21} = 0.5$$
$$(Thres_1 = 0.5, Thres_2 = 1.0)$$

Then we do the fit to these K-matrix curves by using the unitary BW formulas. In the first step we use the BW software to obtain qualitatively similar result to the K-matrix curves. The results of this first step (a visual approximation) are shown in Figure A5 (BW software screenshot).
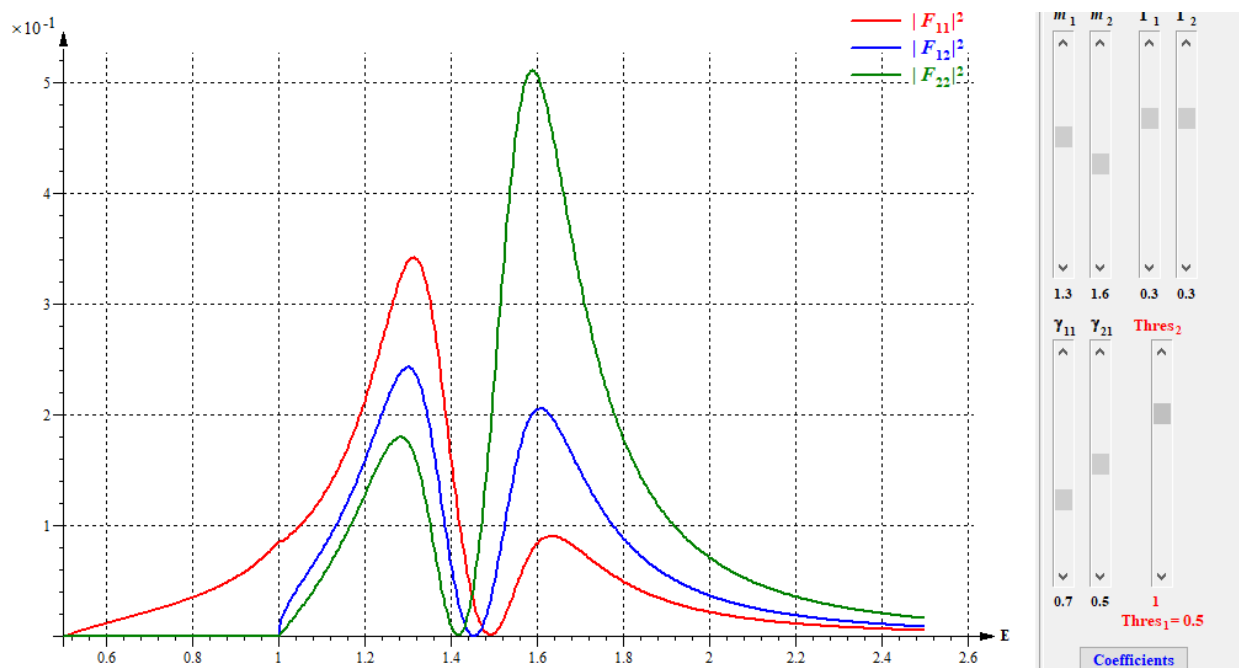
Figure A4. K-matrix scattering amplitudes.

The values of the BW parameters (entered with scroll-bars) in the screenshot in Figure A5 are:

$$m_1 = 1.3, m_2 = 1.6, \alpha = 0.39, g^x_{11} = -0.38, g^x_{12} = 0.41, g^x_{22} = -0.62$$

It gives the following BW physical parameters:

Branching ratios

$$Br_{11} = \frac{|g_{11}|^2}{|g_{11}|^2 + |g_{12}|^2} = 0.419, \ Br_{12} = \frac{|g_{12}|^2}{|g_{11}|^2 + |g_{12}|^2} = 0.581$$

$$Br_{21} = \frac{|g_{21}|^2}{|g_{21}|^2 + |g_{22}|^2} = 0.262, \ Br_{22} = \frac{|g_{22}|^2}{|g_{21}|^2 + |g_{22}|^2} = 0.738$$

BW widths

$$\Gamma_1(m_1) = \frac{1-\alpha^2}{1+\alpha^2}\left[\rho_1(m_1)|g_{11}|^2 + \rho_2(m_1)|g_{12}|^2\right] = 0.217$$

$$\Gamma_2(m_2) = \frac{1-\alpha^2}{1+\alpha^2}\left[\rho_1(m_2)|g_{21}|^2 + \rho_2(m_2)|g_{22}|^2\right] = 0.338$$

After that preliminary step (which is not compulsory but helpful) we do fitting to the K-matrix curves for a more accurate finding of BW parameter values.
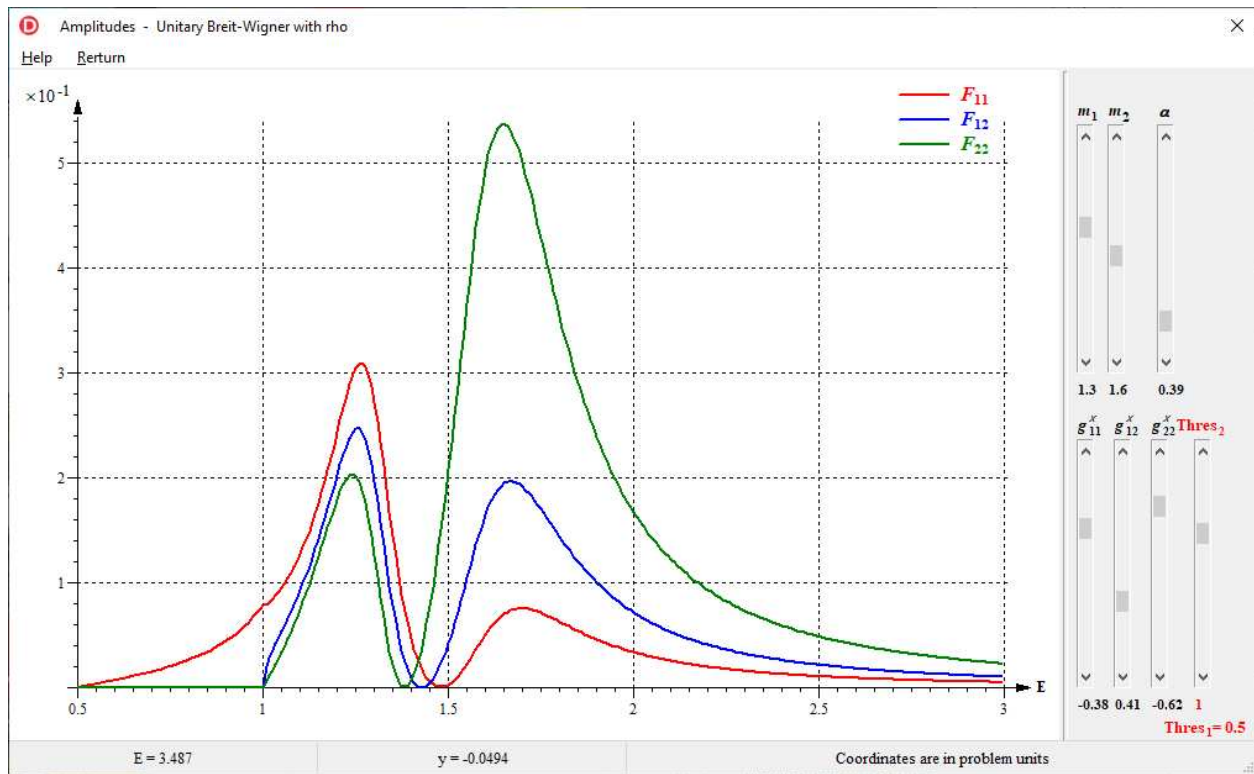

Figure A5. Unitary BW scattering amplitudes.

Appendix B.4. Fitting to K-Matrix Curves Using the Unitary BW Formulas

"Experimental points" represent the K-matrix curves – emphasize that this is a 100% bias situation for the K-matrix method. The result of the fitting is shown in Figure A6. (Technically, to do χ^2 fitting we assign error bars.)

Fitting BW parameters in Figure A6 are:

$$m_1 = 1.32, m_2 = 1.58, \alpha = 0.349, g_{11}^x = -0.387, g_{12}^x = 0.379, g_{22}^x = -0.462$$

BW Physical Parameters:

Branching ratios

$$Br_{11} = \frac{|g_{11}|^2}{|g_{11}|^2 + |g_{12}|^2} = 0.478, Br_{12} = \frac{|g_{12}|^2}{|g_{11}|^2 + |g_{12}|^2} = 0.522$$

$$Br_{21} = \frac{|g_{21}|^2}{|g_{21}|^2 + |g_{22}|^2} = 0.522, Br_{22} = \frac{|g_{22}|^2}{|g_{21}|^2 + |g_{22}|^2} = 0.723$$

BW widths

$$\Gamma_1(m_1) = \frac{1 - \alpha^2}{1 + \alpha^2} \left[\rho_1(m_1)|g_{11}|^2 + \rho_2(m_1)|g_{12}|^2 \right] = 0.212$$

$$\Gamma_2(m_2) = \frac{1 - \alpha^2}{1 + \alpha^2} \left[\rho_1(m_2)|g_{21}|^2 + \rho_2(m_2)|g_{22}|^2 \right] = 0.198$$

With the error bar shown in the Figures, $err = 0.025$ (this value is taken arbitrarily—it affects only the value of χ^2/d), $d = N_{total} - N_{param} - 1 = 146$, $\chi^2/d = 0.39$.

Therefore, the data which can be successfully fitted with the K-matrix method, can also be fitted with the unitary BW approach. Both methods lead to similar results when resonances in *all channels* are well resolved. However, the opposite is not true if amplitudes in all channels do not have zeros between resonance peaks; in other words, when the peaks actually *overlap*. Such a case is shown in Figure A7 (BW software screenshot)—the amplitudes, chosen simply as an example, cannot be fitted with the K-matrix approach with any reasonable value of the χ^2.
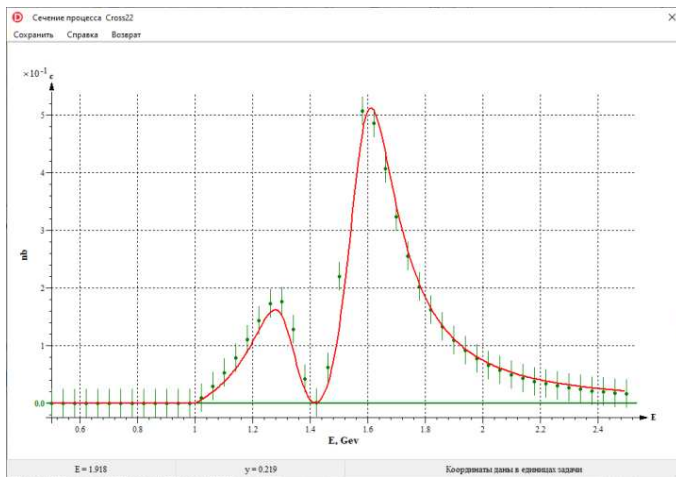
$$Thres_1 = 0.5, Thres_2 = 1.0$$
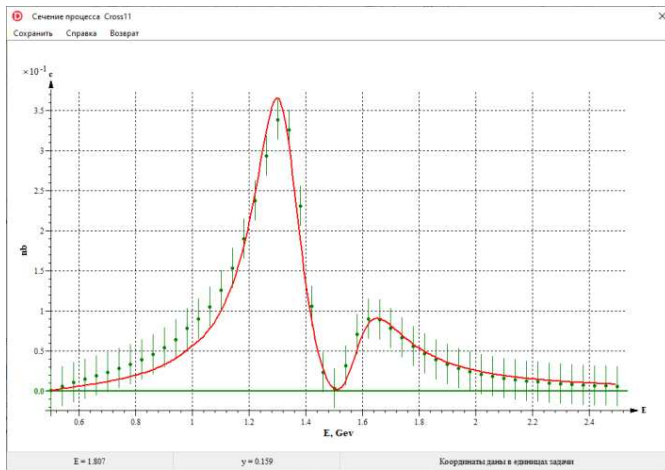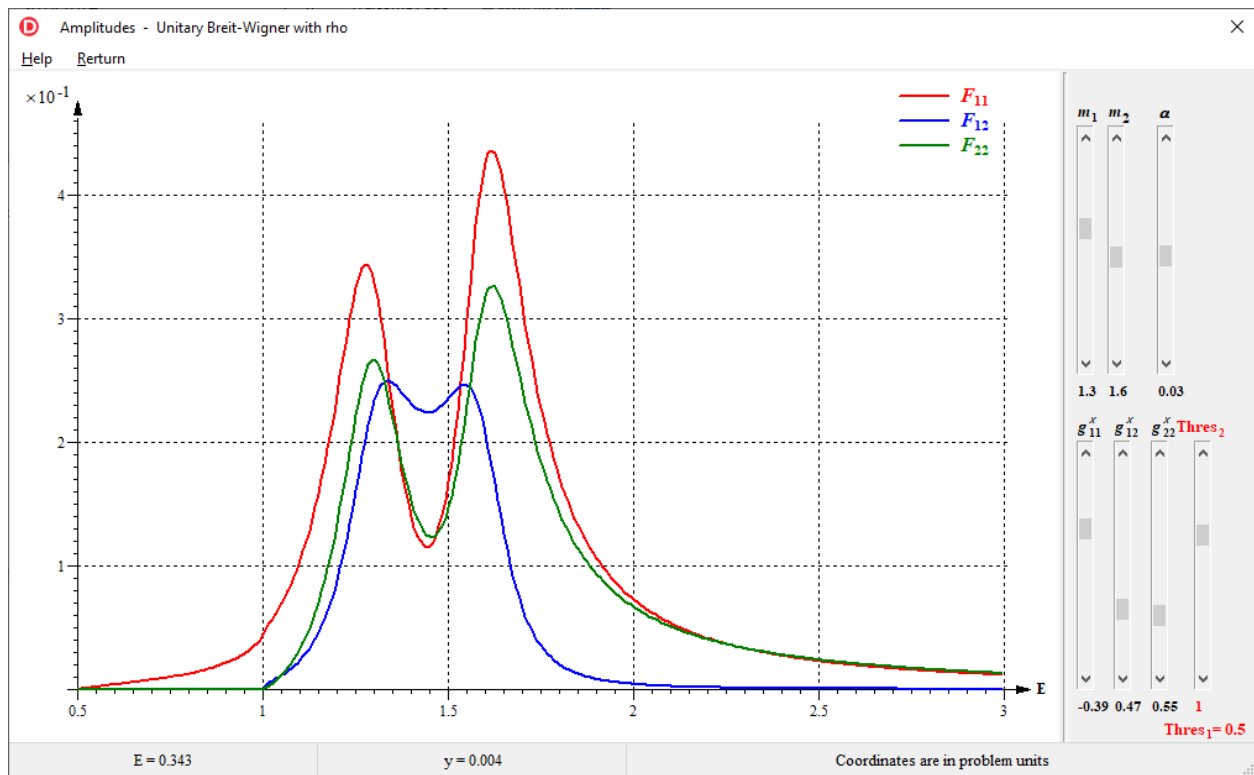
(a) Amplitude $\left|F_{11}(E)\right|^2$
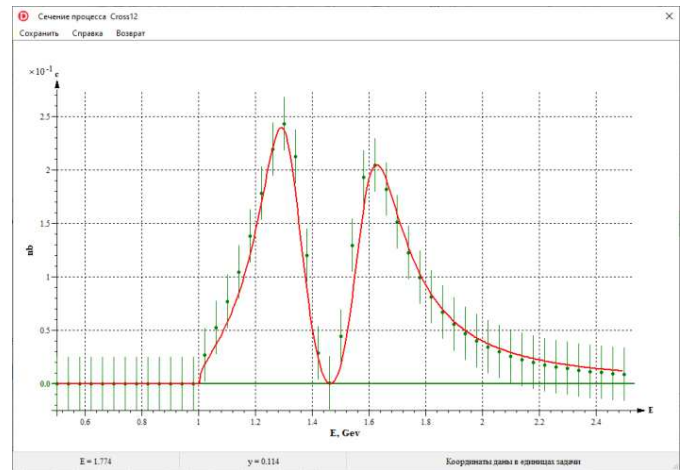
(b) Amplitude $\left|F_{12}(E)\right|^2$

(c) Amplitude $\left|F_{22}(E)\right|^2$

Figure A6. Unitary BW scattering amplitudes.

Figure A7. Unitary BW scattering amplitudes.

BW parameters in Figure A7 are:

$$m_1 = 1.3, m_2 = 1.6, \, \alpha = 0.03, g_{11}^x = -0.39, \, g_{12}^x = 0.47, \, g_{22}^x = 0.55$$

Clearly, there could be data for which both methods cannot give good fitting results for all channels scattering amplitudes; technically, this is related to insufficient number of fitting parameters to describe *all F_{ij}, $i, j = 1, 2 \ldots M$.* (Obviously, when the data for some channels are missing, as often occurs, the fitting task is much easier.) Physically, poor fitting may be a signal of incompleteness or incompatibility of data.

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