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Varying Physical Constants, Astrometric Anomalies, Redshift and Hubble Units

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Abstract: We have developed a cosmological model by allowing the speed of light c, gravitational constant G and cosmological constant Λ in the Einstein filed equation to vary in time, and solved them for Robertson-Walker metric. Assuming the universe is flat and matter dominant at present, we obtain a simple model that can fit the supernovae 1a data with a single parameter almost as well as the standard ΛCDM model with two parameters, and which has the predictive capability superior to the latter. The model, together with the null results for the variation of G from the analysis of lunar laser ranging data determines that at the current time G and c both *increase* as $\mathrm{d}G/\mathrm{d}t = 5.4GH_0$ and $\mathrm{d}c/\mathrm{d}t = 1.8cH_0$ with H_0 as the Hubble constant, and Λ decreases as $\mathrm{d}\Lambda/\mathrm{d}t = -1.2\Lambda H_0$. This variation of G and c is all what is needed to account for the Pioneer anomaly, the anomalous secular increase of the moon eccentricity, and the anomalous secular increase of the astronomical unit. We also show that the Planck's constant $\hbar$ increases as $\mathrm{d}\hbar/\mathrm{d}t = 1.8\hbar H_0$ and the ratio D of any Hubble unit to the corresponding Planck unit increases as $\mathrm{d}D/\mathrm{d}t = 1.5DH_0$. We have shown that it is essential to consider the variation of all the physical constants that may be involved directly or indirectly in a measurement rather than only the one whose variation is of interest.

Keywords: astroparticle physics; astrometric anomaly; supernovae redshift; cosmology theory; variable physical constants; VSL

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1. Introduction

Variation of physical constants is a subject that is marred with semantics: What exactly is varying and how is it being measured? There is an ongoing debate about dimensionful and dimensionless constants (e.g., Uzan [1,2], Duff [3], Chiba [4]). Our approach therefore would be to work mostly with easily comprehensible dimensionful constants and later on see if a meaningful relationship can be established for a common dimensionless parameter and how it evolves with time. The physical constants considered in this work are primarily the speed of light c, the Newton's gravitational constant G, the Einstein's cosmological constant Λ, the Planck's constant $\hbar$, the Hubble constant H_0, and the fine structure constant α. There is a plethora of literature discussing the variation, or lack thereof, of these physical constants and others, and there are excellent reviews on the subject [1–6]. We will therefore limit ourselves to a selected few with direct relevance to our work. In addition, we will focus only on the time variation of physical constants in the spirit of the cosmological principle, which assumes the universe to be isotropic and homogeneous in space at large scale.

Varying physical constant theories gained traction after Dirac [7,8] in 1937 suggested such variation based on his large number hypothesis that related ratios of certain scales in the universe to that of the forces of nature. Magueijo [6] reviewed the variable speed of light (VSL) theories and their limitations in 2003 that included theories based on hard breaking of Lorentz invariance, biometric models, local Lorentz invariance, color dependent speed of light, extra dimension (e.g., brane-world) induced variation, and field functions. Farrell and Dunning-Davis [9] discussed in 2004 the VSL

theories that were used as alternatives to the inflationary model of the universe and reviewed evidence for the same.

Maharaj and Naidoo [10] introduced variable G and Λ in Einstein field equation using Robertson-Walker metric in 1993. Belenchon and Chakrabarty [11] added the variation of c to develop a perfect fluid cosmological model in 2003. Recently (2017) Franzmann [12] developed an approach that included space as well as time dependence of the constants. More recently Barrow and Magueijo [13] proposed that the constants be considered as quantum observables in a kinematical Hilbert space. These works are mostly theoretical and do not directly offer how much exactly they vary and if they can directly explain some observations or measurements. Our focus will be to develop a model that can be used to explain certain anomalies, hitherto not explained satisfactorily, as well as the redshift vs distance modulus data on supernovae 1a better than alternative models.

The possible variation of the fine structure constant α has been of great interest as it is perhaps the most basic dimensionless constant in physics. Rosenband et al. [14] have put a constraint on the $\dot{\alpha}/\alpha = (-1.6 \pm 2.3) \times 10^{-17}\text{yr}^{-1}$ derived from the constancy of the ratio of aluminum and mercury single-ion optical clock frequencies. More recently Gohar [15], using his entropic model of the universe and data on supernovae 1a, baryon acoustic oscillations, and cosmic microwave background, has established even more stringent constraint on the variation of α. Additionally, he states that in his model G and c should be increasing with the evolution of the universe, which corroborates our findings in this work. Similar constraints on α were shown by Songaila and Cowie [16] from the observation of narrow quasar absorption lines at redshift $z > 1.5$. There is a significant amount of work on the subject, most of it can be found referenced in the papers cited above.

If α does vary, no matter how small the variation, it is normal to ask what causes its variation—electric charge e, c or $\hbar$? We will show that since c and $\hbar$ variations cancel out, it is e that should be considered responsible for the variation of α if there is any.

We will solve the Einstein field equation with varying c, G and Λ with Robertson-Walker metric in Section 2, and show that $\dot{\Lambda}/\Lambda = -1.2H$ and $\dot{H}/H = -0.6H$ where H is the Hubble parameter. Based on the Hofmann and Müller's [17] determination of a very tight constraint on the variation of G from the analysis of laser lunar ranging data of more than 40 years, we will establish in Section 3 that $\dot{G}/G = 5.4H$ and $\dot{c}/c = 1.8H$. Section 4 is devoted to the derivation of the expression for distance modulus μ of an intergalactic light emitting source in terms of its redshift z. Section 5 delineates the methodology for fitting the $\mu - z$ data and applying the same to the new model, the variable c, G and Λ (VcGΛ) model, and for comparison also to the standard ΛCDM model. Having shown that the VcGΛ model fits the supernovae 1a $\mu - z$ data almost as well as the ΛCDM model and has predictive capability better than the latter, we will proceed to demonstrate that the model can explain the three astrometric anomalies that have not yet been explained satisfactorily. All we need to explain these anomalies is $\dot{G}/G = 5.4H_0$ and $\dot{c}/c = 1.8H_0$ at current time with H_0 as the Hubble constant.

The first anomaly we will consider here is the Pioneer anomaly, which refers to the near constant acceleration back towards the sun, observed when a spacecraft cruises on a hyperbolic path away from the solar system (Anderson et al., 1998 [18]). Many explanations have been given for such an anomaly but none appears to be satisfactory and they are difficult to incorporate in the models used for real time spacecraft astrodynamics. Principal among these explanations are as follows: (a) Turyshev et al. [19] in 2012 tried to explain the anomaly as being due to the recoil force associated with an anisotropic emission of thermal radiation off the spacecraft. However, it is not clear why it should be the same for Pioneer 10/11, Galileo and Ulysses spacecrafts. (b) Feldman and Anderson [20] in 2015 used "the theory of inertial centers" [21] to develop a model to compute the anomaly. (c) Kopeikim [22] in 2012 used Hubble expansion of the universe to address the anomaly and gave a reason why one should see deceleration rather than acceleration of the spacecraft due the expansion of the universe. These approaches are rather circuitous and depend on many assumptions to explain the anomaly. Feldman and Anderson [20] allocated 12% of the total anomalous acceleration of $8.74 \pm 1.33 \times 10^{-10}\ \text{m s}^{-2}$ to various thermal contributions, leaving $7.69 \pm 1.17 \times 10^{-10}\ \text{m s}^{-2}$ that requires other explanations. In Section 6 we will try to explain this unexplained Pioneer acceleration.

The lunar laser ranging technique has improved to the extent that it can determine the lunar orbit with an accuracy of better than a centimeter. The moon's orbit has an eccentricity that depends on the tidal forces due to surficial and the geophysical processes interior to Earth and the moon. After all the known sources responsible for the eccentricity e were included, Williams and Dickey [23] in 2003 estimated that there remained a discrepancy of $\Delta\dot{e} = (16 \pm 5) \times 10^{-12}\ \mathrm{yr}^{-1}$ between the observed and calculated values. This value was revised downward by Williams and Boggs [24] in 2009 to $\Delta\dot{e} = (9 \pm 3) \times 10^{-12}\ \mathrm{yr}^{-1}$ and by Williams, Turyshev and Boggs [25] in 2014 to $\Delta\dot{e} = (5 \pm 2) \times 10^{-12}\ \mathrm{yr}^{-1}$ with updated data and tidal models. With additional terrestrial tidal modeling, William and Boggs [26] in 2016 were able to further reduce the number and stated that it might even be negative. While these authors possibly felt that unexplained secular increase of the eccentricity was due to the deficiency in their model and therefore a better tidal modeling should eliminate it, others feel that the anomaly may be pointing to some unknown physical process. There have been attempts to resolve the anomaly using Newtonian, relativistic and modified gravity approaches [27–29] as well as using some unfamiliar gravitational effects [27]. Reviews by Anderson and Nieto [30] in 2009 and Iorio [31] in 2015 have covered the above and additional attempts to solve the problem. It appears that none of the models secularly affect the lunar eccentricity. Attempts of cosmological origin were also not successful [22,29,32,33]. We attempt to explain this anomaly in Section 7 with the varying G and c approach developed here.

The anomalous secular increase of astronomical unit AU was first reported by Krasinsky and Brumberg [34] in 2004 as dAU/dt = (15 ± 4) m cy^{-1} from the analysis of all radiometric measurements of distances between Earth and the major planets they had available over the period 1971–2003, which included the observations of Martian landers and orbiters. They noted that unexplained secular increase in AU might point to some fundamental features of space time that are beyond the current cosmological understanding according to which the Hubble expansion yields dAU/dt = 1 km cy^{-1}. This value is almost two orders of magnitude higher than observed. Their theoretical analysis revealed that the relativistic calculations that included the gravitational shift of proper time gave null results. Anderson and Nieto [30] in 2009 corroborated Krasinsky and Brumberg's [34] findings. They showed that the effect of the loss of solar mass on AU is miniscule and will cause the AU to shrink rather than increase (dAU/dt = −0.34 cm cy^{-1}). Iorio [31] in 2015 reviewed the status of the AU anomaly in significant details and concluded that, considering the various unsatisfactory attempts to explain the anomaly and the new IAU definition of astronomical unit, the anomaly no longer exists (just by virtue of new definition). We show in Section 8 that the AU anomaly based on the old definition can be easily explained with the new approach.

Section 9 shows how we obtain the variation of $\hbar$ from the null result on the variation of fine structure constant. We explore the relationship between Planck units and Hubble units in Section 10, and show that all units have the same constant relating them, and then determine how this constant evolves in time. Section 11 is devoted to discussion and Section 12 to conclusions.

2. Evolutionary Constants Model

We will develop our model in the general relativistic domain starting from the Robertson-Walker metric with the usual coordinates x^{μ} (ct, r, θ, ϕ):

$$ds^2 = c^2dt^2 - a(t)^2\left[\frac{dr^2}{1 - k_s r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (1)$$

where $a(t)$ is the scale factor and k_s determines the spatial geometry of the universe: $k_s = -1$ (closed), 0 (flat), +1 (open). The Einstein field equations may be written in terms of the Einstein tensor $G^{\mu\nu}$, metric tensor $g^{\mu\nu}$, energy-momentum tensor $T^{\mu\nu}$, cosmological constant Λ, gravitational constant G and speed of light c, as:

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu} \quad (2)$$

When solved for the Robertson-Walker metric, we get the following non-trivial equations for the flat universe ($k_S = 0$) of interest to us here, with p as the pressure and ε as the energy density [10]:

$$\frac{\ddot{a}}{a}+\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2=-\frac{4\pi G}{c^2}p+\frac{1}{2}\Lambda \tag{3}$$

$$\frac{\dot{a}^2}{a^2}=\frac{8\pi G}{3c^2}\varepsilon+\frac{1}{3}\Lambda \tag{4}$$

If we do not regard G, c and Λ to be constant and define $K\equiv\frac{G}{c^2}$, we may easily derive the continuity equation by taking time derivative of Equation (4) and substituting in Equation (3) (see Appendix A):

$$\dot{\varepsilon}+\frac{3\dot{a}}{a}(\varepsilon+p)+\frac{\dot{K}}{K}\varepsilon+\frac{\dot{\Lambda}}{8\pi K}=0 \tag{5}$$

This reduces to the standard continuity equation when K and Λ are held constant. And since the Einstein field equations require that the covariant derivative of the energy-momentum tensor $T^{\mu\nu}$ be zero, we can interpret Equation (5) as comprising of two continuity equations [10], viz:

$$\dot{\varepsilon}+\frac{3\dot{a}}{a}(\varepsilon+p)=0 \tag{6}$$

$$8\pi\varepsilon\dot{K}+\dot{\Lambda}=0 \tag{7}$$

This separation simplifies the solution of the field equations (Equations (3) and (4)). Equation (6) yields the standard solution for the energy density $\varepsilon=\varepsilon_0 a^{-3(1+w)}$. Here w is the equation of state parameter defined as $p\equiv w\varepsilon$ with $w=0$ for matter, 1/3 for radiation and -1 for Λ.

As has been explicitly delineated by Magueijo in several of his papers (e.g., reference [35]), this approach is not generally Lorentz invariant albeit relativistic. Strictly speaking we should have used the Einstein-Hilbert action to obtain correct Einstein equations with variable G and c as scalar fields. Thus, one may consider the current formulation quasi-phenomenological.

Since the expansion of the universe is determined by $H(t)\equiv\dot{a}/a$, it is natural to assume the time dependence of any time dependent parameter to be proportional to $\dot{a}/a$ (the so called Machian scenario—Magueijo [6]). Let us therefore write:

$$\frac{\dot{K}}{K}=k\left(\frac{\dot{a}}{a}\right),\ \frac{\dot{\Lambda}}{\Lambda}=l\left(\frac{\dot{a}}{a}\right)\ \text{and}\ \frac{\dot{H}}{H}=m\left(\frac{\dot{a}}{a}\right),\ \text{i.e.,} \tag{8}$$

$$K=K_0a^k,\ \Lambda=\Lambda_0a^l\ \text{and}\ H=H_0a^m. \tag{9}$$

where k, l and m are the proportionality constants, and subscript zero indicates the parameter value at present $(t=t_0)$. With this substitution in Equation (4) we may write:

$$\frac{\dot{a}^2}{a^2}=H_0^2a^{2m}=\frac{8\pi}{3}(K_0a^k)\varepsilon_0a^{-3(1+w)}+\frac{1}{3}\Lambda_0a^l. \tag{10}$$

Comparing the exponents of the only time dependent parameter a of all the terms, we may write $2m=k-3-3w=l$, and with $w=0$ for matter, we have $2m=k-3=l$. Thus, if we know k, we know l and m.

We can now have a closed analytical solution of Equation (10) as follows (since $a(t_0)\equiv1$):

$$a(t)=\frac{a(t)}{a(t_0)}=\left(\frac{t}{t_0}\right)^{\frac{2}{3+3w-k}};\ \frac{\dot{a}}{a}=\frac{2}{3+3w-k}t^{-1}; \tag{11}$$

$$\frac{\ddot{a}}{\dot{a}}=\left(\frac{\dot{a}}{a}\right)\left(1-\frac{3+3w-k}{2}\right);\ \text{and}\ -q\equiv\frac{\ddot{a}a}{\dot{a}^2}=\frac{-1-3w+k}{2} \tag{12}$$

where q is the deceleration parameter. It may be noticed that q does not depend on time, i.e., $q_0=q$. As we know the radiation energy density is negligible at present, and dark energy Λ is implicitly included in the above formulation, so we need to be concerned with the matter only solutions, i.e., with $w=0$.

The deceleration parameter q_0 has been analytically determined on the premise that expansion of the universe and the tired light phenomena are jointly responsible for the observed redshift, especially in the limit of very low redshift [36]. One could see it as if the tired light effect is superimposed on the Einstein de Sitter's matter only universe rather than the cosmological constant

[37]. By equating the expressions for the proper distance of the source of the redshift for the two, one gets $q_0 = -0.4$. Then from Equation (12) we get $k = 1.8$, and also $l = -1.2$ and $m = -0.6$. We thus have from Equation (8) $\dot{K}/K = 1.8H$, $\dot{\Lambda}/\Lambda = -1.2H$ and $\dot{H}/H = -0.6H$.

3. Varying G and c Formulation

Having determined the value of $k = 1.8$, and since the Hubble parameter is defined as $H = \dot{a}/a$, we may write from Equations (8) and (9):

$$K = K_0 a^{1.8}, \text{ and } \frac{\dot{K}}{K} = 1.8H. \tag{13}$$

We may also write explicitly:

$$\frac{\dot{K}}{K} = \frac{\dot{G}}{G} - \frac{2\dot{c}}{c} = 1.8H. \tag{14}$$

Taking H at the present time as $H_0 \simeq 70$ km s^{-1} Mpc^{-1} (2.27×10^{-18} s^{-1}) we get $\frac{\dot{K}}{K} = 4.09 \times 10^{-18}$ s^{-1} $= 1.29 \times 10^{-10}$ yr^{-1}.

The findings from the Lunar Laser Ranging (LLR) data analysis provides the limits on the variation of $\dot{G}/G$ ($7.1 \pm 7.6 \times 10^{-14}$) [17], which is considered to be about three orders of magnitude lower than was expected [7,8,38]. However, the LLR data analysis is based on the assumption that the distance measuring tool, i.e., the speed of light, is constant and non-evolutionary. If this constraint were dropped then the finding would be very different.

As is well known [39], a time variation of G should show up as an anomalous evolution of the orbital period P of astronomical bodies expressed by Kepler's 3rd law:

$$P^2 = \frac{4\pi^2 r^3}{GM}, \tag{15}$$

where r is semi-major axis of the orbit, G is the gravitational constant and M is the mass of the bodies involved in the orbital motion considered. If we take time derivative of Equation (15), divide by P^2 and rearrange, we get:

$$\frac{\dot{G}}{G} = \frac{3\dot{r}}{r} - \frac{2\dot{P}}{P} - \frac{\dot{M}}{M} \tag{16}$$

If we write $r = ct$ then $\frac{\dot{r}}{r} = \frac{1}{t} + \frac{\dot{c}}{c}$. Here t may be considered associated with the Hubble time (i.e., $1/H$), as are other quantities. We may now rewrite Equation (16) as:

$$\frac{\dot{G}}{G} - \frac{3\dot{c}}{c} = \frac{3}{t} - \frac{2\dot{P}}{P} - \frac{\dot{M}}{M}. \tag{17}$$

Since LLR measures the time of flight of the laser photons, it is the right hand side of Equation (17) that is determined from LLR data analysis [17] to be $7.1 \pm 7.6 \times 10^{-14}$ and not the right hand side of Equation (16).

Then, taking the right hand side of Equation (17) as 0 and combining it with Equation (14), one can solve the two equations and get $\dot{G}/G = 5.4H$ and $\dot{c}/c = 1.8H$. It should be emphasized that both $\dot{G}/G$ and $\dot{c}/c$ are positive and thus both of them are *increasing* with time rather than decreasing, as is generally believed (e.g., [7,8,40]). This may be considered as the most significant observational finding of cosmological consequences just by studying the Earth–moon system.

4. Redshift vs. Distance Modulus

The distance d of a light emitting source in a distant galaxy is determined from the measurement of its bolometric flux f and comparing it with a known luminosity L. The luminosity distance d_L is defined as:

$$d_L = \sqrt{\frac{L}{4\pi f}} \tag{18}$$

In a flat universe the measured flux could be related to the luminosity L with an inverse square relation $f = L/(4\pi d^2)$. However, this relation needs to be modified to take into account the flux losses due to the expansion of the universe through the scale factor a, the redshift z and all other phenomena that can result in the loss of flux. Generally accepted flux loss phenomena are as follows [41]:

a. Increase in the wavelength causes a flux loss proportional to $1/(1+z)$.
b. In an expanding universe, an increase in detection time between two consecutive photons emitted from a source leads to a reduction of flux proportional to a, i.e., proportional to $1/(1+z)$.

Therefore, in an expanding universe the necessary flux correction required is proportional to $1/(1+z)^2$. The measured bolometric flux f_B and the luminosity distance d_L may thus be written as:

$$f_B = L/[4\pi d^2(1+z)^2], \text{ and} \tag{19}$$

$$d_L = d(1+z). \tag{20}$$

How does d compare with and without varying c? Let us first consider the case of non-expanding universe. The distance from the point of emission at time t_e to the point of observation at time t_0 may be written as $d_c = \int_{t_e}^{t_0} c\, dt$. Therefore for constant $c = c_0$:

$$d_{c_0} = c_0 t_0 \left(1 - \frac{t_e}{t_0}\right) \tag{21}$$

When $c = c_0 a^{1.8}$, and since $a = \left(\frac{t}{t_0}\right)^{\frac{2}{1.2}}$ from Equation (11), we may write:

$$d_c = c_0 \int_{t_e}^{t_0} \left(\frac{t}{t_0}\right)^3 dt = \frac{c_0}{t_0^3}\int_{t_e}^{t_0} t^3 dt = \frac{1}{4} c_0 t_0 \left(1 - \frac{t_e^4}{t_0^4}\right). \tag{22}$$

The ratio of the two distances may be considered the normalization factor F when using the variable c in calculating the proper distance of a source. Since $a \equiv 1/(1+z)$, we may write for the source of redshift z with emission time t_e:

$$\frac{t_e}{t_0} = a(z)^{0.6}, \text{ or} \tag{23}$$

$$\frac{t_e}{t_0} = (1+z)^{-0.6}. \tag{24}$$

Now the proper distance of the source with variable c may be defined as [41] (page 105):

$$\begin{aligned} d_{P_c} &= \int_{t_e}^{t_0} \left(\frac{c}{a}\right) dt = \int_{t_e}^{t_0} \left(\frac{c_0 a^{1.8}}{a}\right) dt = c_0 \int_{t_e}^{t_0} a^{0.8}\, dt \\ &= c_0 \int_{t_e}^{t_0} \left(\frac{t}{t_0}\right)^{\frac{4}{3}} dt = \frac{3}{7} c_0 t_0 \left[1 - \left(\frac{t_e}{t_0}\right)^{\frac{7}{3}}\right]. \end{aligned} \tag{25}$$

From Equation (11) $H_0 \equiv \dot{a}/a = (2/1.2)t_0^{-1}$. Therefore:

$$d_{P_c} = \frac{1}{1.4}(c_0/H_0)[1 - (1+z)^{-1.4}]. \tag{26}$$

Thus the expression for d to be substituted in Equation (20) to determine the luminosity distance of the source is $d = d_{P_c} F$.

Since the observed quantity is distance modulus μ rather than the luminosity distance d_L, we will use the relation:

$$\mu = 5\log(d_L) + 25,$$
$$= 5\log(\frac{1}{1.4}R_0(1-(1+z)^{-1.4}) + 5\log\big(F(z)\big) \quad (27)$$
$$+5\log(1+z) + 25. \quad (28)$$

where $R_0 \equiv c_0/H_0$ and all the distances are in Mpc. It is the only free parameter in Equation (28).

We will compare the new model, hereafter referred to as the VcGΛ (variable c, G and Λ) model, with the standard ΛCDM model, which is the most accepted model for explaining cosmological phenomena, and thus may be considered the reference models for all the other models. Ignoring the contribution of radiation density at the current epoch, we may write the distance modulus μ for redshift z in a flat universe for the ΛCDM model as follows [42]:

$$\mu = 5\log[R_0\int_0^z du/\sqrt{\Omega_{m,0}(1+u)^3+1-\Omega_{m,0}}] \quad (29)$$
$$+5\log(1+z) + 25.$$

Here $\Omega_{0,m}$ is the current matter density relative to critical density and $1-\Omega_{m,0} \equiv \Omega_{\Lambda,0}$ is the current dark energy density relative to critical density.

5. Supernovae Ia z-μ Data Fit

We tried the VcGΛ model developed here to see how well it fits the best supernovae Ia data [43] as compared to the standard ΛCDM model. The data fit is shown in Figure 1. The VcGΛ model requires only one parameter to fit all the data ($H_0 = 68.90 \pm 0.26$ km s^{-1} Mpc^{-1}), whereas the ΛCDM model requires two parameters ($H_0 = 70.16 \pm 0.42$ km s^{-1} Mpc^{-1} and $\Omega_{m,0} = 0.2854 \pm 0.0245$).

The data used in this work is the so-called Pantheon Sample of 1048 supernovae Ia in the range of $0.01 < z < 2.3$ [43]. The data is in terms of the apparent magnitude and we added 19.35 to it to obtain normal luminosity distance numbers as suggested by Scolnic [43]. To test the fitting and predictive capability of the two models, we divided the data in 6 subsets: (a) $z < 0.5$; (b) $z < 1.0$; (c) $z < 1.5$; (d) $z > 0.5$; (e) $z > 1.0$; and (f) $z > 1.5$. The idea is to parameterize a model with a low redshift data subset and then see how the model, using parameters thus obtained, fits the remaining redshift data. In addition, we considered the fits for the whole data. The models were parameterized with subsets (a), (b) and (c). The parameterized models were then tried to fit the data in the subsets that contained data with z values higher than in the parameterized subset. For example if the models were parameterized with data subset (a) $z < 0.5$, then the models were fitted with the data subsets (d) $z > 0.5$, (e) $z > 1.0$ and (f) $z > 1.5$ to examine the models' predictive capability.

The Matlab curve fitting tool was used to fit the data by minimizing χ^2 and the latter was used for determining the corresponding χ^2 probability [44] P. Here χ^2 is the weighted summed square of residual of μ:

$$\chi^2 = \sum_{i=1}^{N} w_i\big[\mu(z_i; R_0, p_1, p_2 \ldots) - \mu_{obs,i}\big]^2, \quad (30)$$

where N is the number of data points, w_i is the weight of the ith data point $\mu_{obs,i}$ determined from the measurement error $\sigma_{\mu_{Obs,i}}$ in the observed distance modulus $\mu_{obs,i}$ using the relation $w_i = 1/\sigma^2_{\mu_{Obs,i}}$, and $\mu(z_i; R_0, p_1, p_2..)$ is the model calculated distance modulus dependent on parameters R_0 and all other model dependent parameter p_1, p_2, etc. As an example, for the ΛCDM models considered here, $p_1 \equiv \Omega_{m,0}$ and there is no other unknown parameter.

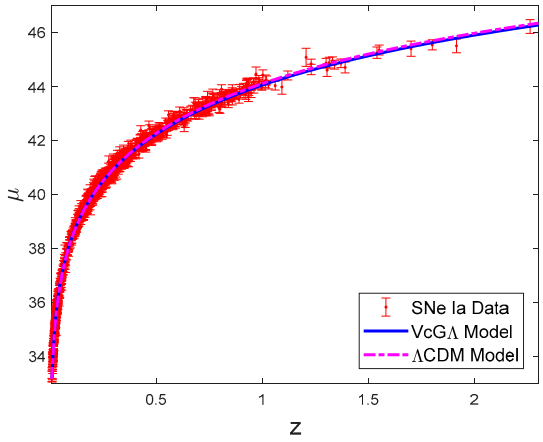

Figure 1. Supernovae Ia redshift z vs. distance modulus μ data fit using the variable c, G and Λ (VcGΛ) model as compared to the fit using the ΛCDM model.

We then quantified the goodness-of-fit of a model by calculating the χ^2 probability for a model whose χ^2 has been determined by fitting the observed data with known measurement error as above. This probability P for a χ^2 distribution with n degrees of freedom (DOF), the latter being the number of data points less the number of fitted parameters, is given by:

$$P(\chi^2, n) = \left(\frac{1}{\Gamma\left(\frac{n}{2}\right)}\right) \int_{\frac{\chi^2}{2}}^{\infty} e^{-u} u^{\frac{n}{2}-1} du, \quad (31)$$

where Γ is the well know gamma function that is generalization of the factorial function to complex and non-integer numbers. The lower the value of χ^2, the better the fit, but the real test of the goodness-of-fit is the χ^2 probability P; the higher the value of P for a model, the better the model's fit to the data. We used an online calculator to determine P from the input of χ^2 and DOF [45]. Our primary findings are presented in Table 1. The unit of the Hubble distance R_0 is Mpc and that of the Hubble constant H_0 is km s^{-1} Mpc^{-1}. The table is divided into four categories vertically and four categories horizontally. Vertical division is based on the parameterizing data subset indicated in the second row and discussed above. The parameters determined for each model are in the first horizontal category. The remaining horizontal categories show the goodness-of-fit parameters for higher redshift subsets than those used for parameterizing the models. Thus this table shows the relative predictive capability of the two models. The model cells with the highest probability in each category are shown in bold and highlighted.

Table 1. Parameterizing and prediction table for the two models. This table shows how well a model is able to fit the data that is not used to determine the model parameters. The unit of R_0 is Mpc and of H_0 is km s^{-1} Mpc^{-1}. P% is the $\chi 2$ probability in percent that is used to assess the best model for each category; the higher the $\chi 2$ probability P, the better the model fits to the data. R^2 is the square of the correlation between the response values and the predicted response values. RMSE is the root mean square error. Highest P% value in each category is shown in bold and the cell highlighted.

Action/Item	**ΛCDM**	**VcGΛ**	**ΛCDM**	**VcGΛ**	**ΛCDM**	**VcGΛ**	**ΛCDM**	**VcGΛ**
Parameterized	**Model Dataset z < 0.5; 832 Points**		**Model Dataset z < 1.0; 1025 Points**		**Model Dataset z < 1.5; 1042 Points**		**Model Dataset all; 1048 Points**	
R_0	4259 ± 34	4337 ± 18	4269 ± 27	4351 ± 17	4271 ± 26	4352 ± 17	4273 ± 26	4351 ± 16
$\Omega_{m,0}$	0.2601 ± 0.0457	NA	0.2793 ± 0.0261	NA	0.2818 ± 0.0249	NA	0.2845 ± 0.0245	NA
H_0	70.39 ± 0.56	69.13 ± 0.29	70.23 ± 0.44	68.90 ± 0.27	70.19 ± 0.42	68.89 ± 0.27	70.16 ± 0.42	68.90 ± 0.25
$\chi 2$	863.5	889.4	1018	1060	1033	1074	1036	1076
DOF	830	831	1023	1024	1040	1041	1046	1047
P%	**20.39**	7.83	**53.82**	21.15	**55.53**	23.26	**58.11**	26.02
R^2	0.9961	0.9960	0.9969	0.9968	0.9970	0.9969	0.9970	0.9969
RMSE	1.020	1.035	0.9977	1.017	0.9965	1.016	0.9951	1.014
Model Fit	**Dataset z > 0.5; 216 points**		**Dataset z > 0.5; 216 points**		**Dataset z > 0.5; 216 points**		**Dataset z > 0.5; 216 points**	
$\chi 2$	176.9	190	NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL		NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL		NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL	
DOF	216							
P%	**97.59**	89.84						
R^2	0.9605	0.9575						
RMSE	0.905	0.938						
Model Fit	**Dataset z > 1.0; 23 points**				NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL		NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL	
$\chi 2$	19.54	17.01	17.59	16.75				
DOF	23							
P%	66.94	**80.43**	77.93	**82.13**				
R^2	0.8741	0.8904	0.8867	0.8921				
RMSE	0.9216	0.86	0.8746	0.8533				
Model Fit	**Dataset z > 1.5; 6 points**						NOT APPLICABLE SINCE THIS DATASET INCLUDES THE DATASET USED TO PARAMETERIZE THE MODEL	
$\chi 2$	4.090	1.946	3.167	1.983	3.076	1.986		
DOF	6							
P%	66.44	**92.45**	78.76	**92.12**	79.92	**92.09**		
R^2	0.5993	0.8093	0.6897	0.8057	0.6986	0.8054		
RMSE	0.8256	0.5696	0.7265	0.5749	0.716	0.5754		

6. Pioneer Anomaly

Having determined the values of $\dot{G}/G$ and $\dot{c}/c$ we can now proceed to calculate the anomalous acceleration towards the sun of Pioneer 10 and 11 spacecraft [20]. Since the gravitational pull of the sun on the spacecraft decreases according the inverse square law, $\frac{\dot{G}}{G}$ cannot be expected to give a constant acceleration independent of the distance of the spacecraft. If the acceleration is denoted by f, one can easily work out, using the Newtonian relation $f = GM/r^2$, that $\dot{f}/f = -3.74 \times 10^{-19}\, s^{-1}$, which yields negligible anomalous acceleration. Thus, we need to only consider the effect of $\dot{c}/c$ from a different perspective. If the spacecraft is at a distance r_0 from Earth then the signal from Earth will have a two way transit time Δt given by $2r_0 = c_0\Delta t$, assuming c_0 as the speed of light. But, if the speed of light is evolving as $\dot{c}/c = 1.8H_0$ near $t = t_0$, i.e., as $c = c_0 e^{1.8H_0(t-t_0)}$ during the transit time, then the actual transit time will be shorter than Δt (since $c > c_0$ for $t > t_0$). Because of the shorter actual transit time, an observer would consider the spacecraft to be nearer to Earth than it actually is and thus would think that there is a deceleration of the spacecraft due to some unaccounted-for cause.

We could write the proper distance of the spacecraft r_p and its apparent distance r_a as:

$$\begin{gathered} 2r_a = c_0\Delta t, \text{ and} \\ 2r_p = c_0\int_0^{\Delta t} e^{1.8H_0 t}dt = \tfrac{c_0}{1.8H_0}(e^{1.8H_0\Delta t} - 1), \\ \text{and since } 1.8H_0\Delta t \ll 1, \\ 2r_p = \tfrac{c_0}{1.8H_0}[\left(1 + 1.8H_0\Delta t + \tfrac{1}{2}(1.8H_0)^2\Delta t^2 \ldots.\right) - 1], \text{ or} \\ r_p = \tfrac{1}{2}c_0\Delta t + \tfrac{1.8H_0}{4}c_0\Delta t^2 = r_a + \tfrac{1}{2}(0.9H_0c_0)\Delta t^2, \text{ or} \end{gathered} \tag{32}$$

$$r_a = r_p - \tfrac{1}{2}(6.129 \times 10^{-10}\ \mathrm{m\ s^{-2}})\Delta t^2. \tag{33}$$

Thus the acceleration is -6.129×10^{-10} $\mathrm{ms^{-2}}$, and since it is negative, it is towards the observer at Earth.

Out of $7.69 \pm 1.17 \times 10^{-10}\ \mathrm{ms^{-2}}$ anomalous acceleration of Pioneer 10 and 11 towards the sun (truly towards Earth) we are able to analytically account for $6.129 \times 10^{-10}\ \mathrm{ms^{-2}}$, leaving only $1.56 \pm 1.17 \times 10^{-10}\ \mathrm{ms^{-2}}$ as the anomaly.

It should be mentioned that Kopeikin [22] has obtained essentially the same result and explained it as due to the cosmological effect of quadratic divergence between the electromagnetic and atomic time scales governing the propagation of radio waves in the Doppler tracking system and the atomic clock on Earth, respectively. However, his approach is not conducive to explaining the other two anomalies.

7. The Moon's Eccentricity Anomaly

The eccentricity e of the orbit of the moon may be written as [46]:

$$e = \sqrt{1 + \tfrac{2\epsilon h^2}{\mu^2}}, \text{ or } e^2 - 1 = \tfrac{2\epsilon h^2}{\mu^2}. \tag{34}$$

where $\epsilon = -\mu/2a_m$ is the specific orbital energy, $\mu = G(m_e + m_m)$; here the gravitational parameter for the Earth–moon system, $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}\, m_e/M_r$ is the specific relative angular momentum, a_m is the semi-major axis of the orbit, m_e is mass of Earth, m_m is the mass of the moon, $\boldsymbol{r}$ is the radius vector and $\boldsymbol{v}$ is the velocity vector of the moon, and $M_r = m_e m_m/(m_e + m_m)$ is the reduced mass. Taking $a_m = r$ and assuming $\boldsymbol{r}$ is normal to $\boldsymbol{v}$, we may write Equation (34) as:

$$1 - e^2 = \tfrac{rv^2}{G}\left(\tfrac{m_e + m_m}{m_e^2}\right). \tag{35}$$

Differentiating this equation with respect to time, assuming the mass factor to be constant, and dividing by the same equation, we get:

$$-\frac{e\dot{e}}{1-e^2} = \frac{\dot{r}}{r} + \frac{2\dot{v}}{v} - \frac{\dot{G}}{G} \tag{36}$$

and, since $e \ll 1$:

$$e\dot{e} = \frac{\dot{G}}{G} - \frac{\dot{r}}{r} - \frac{2\dot{v}}{v}. \tag{37}$$

Since r is measured by electromagnetic waves, the measuring tool for distance, the speed of light, enters in it, i.e., $r = ct$, or $\dot{r} = v = \dot{c}t + c\dot{t}$, or $\frac{\dot{r}}{r} = \frac{\dot{c}}{c} + \frac{1}{t}$. Since all the parameters are expressed at current time, t in the denominator must be expressed in terms of the Hubble time $1/H_0$. However, it is better to write $t = pH_0^{-1}$ where $(1-p)$ is the small factor very close to 0 that may be considered to correct for the approximations made in our model (p here is not pressure). We can also determine $\frac{\dot{v}}{v}$: $\dot{v} = \ddot{r} = \ddot{c}t + \dot{c} + \dot{c} = 2\dot{c}$, assuming $\dot{c}$ as constant. Thus $\frac{\dot{v}}{v} = \frac{2\dot{c}}{\dot{c}t+c} = \frac{2\dot{c}}{c}/(\frac{\dot{c}}{c}t + 1)$. We may therefore write Equation (37) as:

$$e\dot{e} = \frac{\dot{G}}{G} - \left(\frac{\dot{c}}{c} + \frac{1}{t}\right) - \frac{4\dot{c}}{c}/(\frac{\dot{c}}{c}t + 1), \text{ or} \tag{38}$$

$$\frac{e\dot{e}}{H_0} = 5.4 - \left(1.8 + \frac{1}{p}\right) - \frac{7.2}{1.8p+1}. \tag{39}$$

For $p = 1, e\dot{e}/H_0 = 0.0285715$ and taking $H_0 = 0.716 \times 10^{-10}$ yr^{-1}, and $e = 0.0549$ for the moon, we get $\dot{e} = 37 \times 10^{-12}$ yr^{-1}. This is about twice the original value of the anomalous rate of eccentricity increase.

The value determined is very sensitive to the value of the parameter p. We have therefore plotted dimensionless eccentricity variation $e\dot{e}/H_0$ against p in Figure 2. It can be approximated near $p = 1$ with the expression:

$$\frac{e\dot{e}}{H_0} = 0.0285714 + 2.65306(p-1). \tag{40}$$

There are three values of $\dot{e}$ that are significant here:

a. $\dot{e} = 16 \pm 5 \times 10^{-12}$ yr^{-1} originally estimated by Williams and Dickey in 2003 [23]; it gives $p = 0.993855 \pm 0.001445$.
b. $\dot{e} = 9 \pm 3 \times 10^{-12}$ yr^{-1}, the updated value using more data and 'better' tidal effect model by Williams and Boggs in 2009 [24]; it gives $p = 0.991832 \pm 0.000867$.
c. $\dot{e} = 5 \pm 2 \times 10^{-12}$ yr^{-1}, the updated value with even more data and 'even better' tidal effect model by Williams et al. in 2014 [25]; it gives $p = 0.990676 \pm 0.000578$.

All the values of p are very close to 1, indicating that our model is a very good approximation to the exact solution of the Einstein field equations, at least locally, with variable c and G. Even lower and negative values of $\dot{e}$ derived by Williams and Boggs in 2016 [26] can be easily explained with this approach. The question remains—is it the tidal model's deficiency that is being corrected or is it presumed that there could be no other cause for the anomaly?

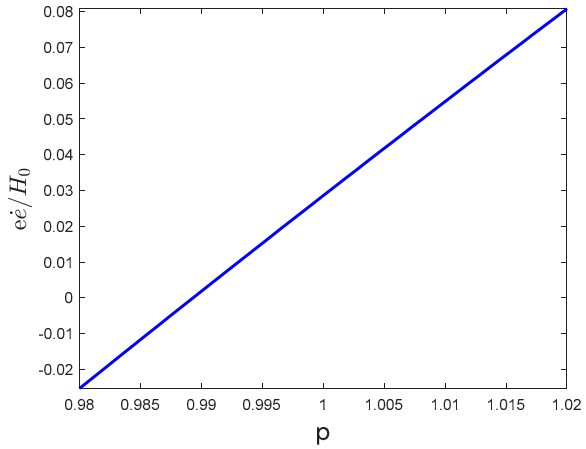

Figure 2. Dimensionless eccentricity variation $e\dot{e}/H_0$ plotted against age of the universe parameter p.

8. Astronomical Unit (AU) Anomaly

The orbit of Earth around the sun is Keplerian and thus is governed by Equation (15). A radiometric measurement will therefore yield a null result in our approach using evolutionary G and c. However, the eccentricity evolution is equally valid for the moon and Earth. Thus applying Equation (39) to Earth's orbit, and assuming $p = 1$ and $e = 0.0167$, we get $\dot{e} = 122.5 \times 10^{-12}$ yr^{-1}. This can be translated easily into AU increase as follows. The semi-major axis a and semi-minor axis b of an orbit may be written as [46]:

$$a = \frac{p_s}{1-e^2}, \text{ and } b = \frac{p_s}{\sqrt{1-e^2}}. \tag{41}$$

where, $p_s = a = b$ defines a circle when $e = 1$. AU may then be written as:

$$\text{AU} = \frac{a+b}{2} = \frac{p_s}{2}\left(\frac{1}{1-e^2} + \frac{1}{\sqrt{1-e^2}}\right) = p_s\left(1 + \frac{3}{4}e^2 + \frac{11}{16}e^4 + O(e^6)\right). \tag{42}$$

Suppose now that the eccentricity e increases by Δe to e' in a time period Δt. Then $e' = e + \Delta e$ and $e'^2 = e^2 + 2e^2\Delta e$ when we ignore higher order terms in Δe. We may now write the increase in AU as ΔAU:

$$\Delta\text{AU} = \frac{3}{2}p_s e^2 \Delta e, \text{ or } \frac{dAU}{dt} = \frac{3}{2}e^2\dot{e} \times \text{AU}. \tag{43}$$

Here we have approximated $p_s = \text{AU}$ since $e^2 \ll 1$. Taking $\text{AU} = 1.496 \times 10^{11}$ m and using Equation (39) for $e\dot{e}$ with $p = 1$, we get $\text{dAU/dt} = 0.77$ m cy^{-1} against its measured value of 1.5 m cy^{-1}. As can be seen from Equation (40) and Figure 2, if we took $p = 1.010$ instead of 1, we would get the desired value. The reason could be the same as discussed at the end of the previous section. Alternatively, there may be other phenomena contributing to the anomalous AU.

It should be mentioned that recently the AU has been redefined (e.g., [47]) and just by definition the AU anomaly has been eliminated. One has to resort to the old definition of AU to appreciate the AU anomaly and its resolution.

9. Variation of Planck's Constant ħ

The variation of the fine structure constant $\alpha = (1/4\pi\epsilon_0)\,e^2/\hbar c$ (here ϵ_0 is the vacuum permittivity and e is electron charge) has been studied extensively. Since $\epsilon_0 = 1/\mu_0 c^2$, where μ_0 is the vacuum permeability, $\alpha = (\mu_0/4\pi)e^2 c/\hbar$. Recent estimates put a very low value on $\dot{\alpha}$ [14,15]. We may write $\frac{\dot{\alpha}}{\alpha} = 2\frac{\dot{e}}{e} - \frac{\dot{\hbar}}{\hbar} + \frac{\dot{c}}{c}$. If $\dot{\alpha}/\alpha$ and $\dot{e}/e$ are zero, or varying very little compared to $\dot{\hbar}/\hbar$ and $\dot{c}/c$, and μ_0 is a constant, then it is implied that $\dot{\hbar}/\hbar = \dot{c}/c = 1.8H$. This possibly answers the question posed by Magueijo et al. [48]: Is it e or is it c?

10. Planck Units and Hubble Units

We will now review the Plank and Hubble units of time, length, mass, charge and temperature and then explore how the dimensionless constant relating the two units evolves in time.

Plank time $t_P = \sqrt{\frac{\hbar G}{c^5}}$

Planck length $l_P = \sqrt{\frac{\hbar G}{c^3}}$

Planck mass $m_P = \sqrt{\frac{\hbar c}{G}}$

Planck charge $q_P = \sqrt{4\pi\epsilon_0 \hbar c}$

Planck temperature $T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}}$

Hubble time $t_H = 1/H_0$

Hubble length $l_H = c/H_0$

Hubble mass $m_H = \frac{c^3}{GH_0}$

Hubble charge $q_H = \sqrt{\frac{4\pi\epsilon_0 c^6}{GH_0^2}}$

Hubble temperature $T_H = \frac{c^5}{GH_0 k_B}$

where ϵ_0 is the permittivity of space, $\hbar$ is the Planck's constant and k_B is the Boltzmann constant. If we divide any of the Hubble units by the corresponding Planck unit we always get the dimensionless quantity, say D, as:

$$D \equiv \sqrt{\frac{c^5}{H_0^2 G\hbar}} = 0.818 \times 10^{61}. \quad (44)$$

Every Hubble unit is 61 orders of magnitude larger than the corresponding Planck unit. Taking time derivative of the equation and dividing by itself, we get:

$$\frac{\dot{D}}{D} = \frac{1}{2}\left(5\frac{\dot{c}}{c} - \frac{2\dot{H}_0}{H_0} - \frac{\dot{\hbar}}{\hbar} - \frac{\dot{G}}{G}\right) = 1.5H_0. \quad (45)$$

This means that the dimensionless D that relates Hubble units and Planck units is increasing in time.

11. Discussion

As should be expected, the two-parameter ΛCDM model is able to fit any data set better than the one-parameter VcGΛ model. What is unexpected is that when parameterized with a relatively low redshift data the VcGΛ model is able to fit the higher redshift data better than the ΛCDM model. This shows that the second parameter in the latter, while trying to fit a limited dataset as best as possible, compromises the model fit for data not used for parameterizing. This means that the ΛCDM model does not have as good a predictive capability (i.e., the capability to fit the data that is not included for determining the model parameters) as the VcGΛ model, despite having twice as many parameters as the VcGΛ model. In addition, the VcGΛ model has the analytical expression for the distance modulus μ unlike the ΛCDM model, which must be evaluated numerically.

One would notice that while R_0 (and hence H_0) values are relatively stable with the parameterizing dataset containing higher and higher redshift values, varying no more than 0.35%, the variation in the $\Omega_{m,0}$ is up to 9.4%, i.e., 27 time larger. This confirms that the $\Omega_{m,0}$ parameter, and hence Λ through $\Omega_{\Lambda,0}$, is an artificially introduced parameter to fit the data rather than being fundamental to the ΛCDM model. In contrast, Λ is an integral part of the VcGΛ model. Since K ($\equiv G/c^2$) and Λ are related through Equation (7), one could easily derive that the Λ term contributes 60% for the VcGΛ model against 70% for the ΛCDM model.

We have established that the supernovae 1a data is compatible with the variable constants proposition. This is contrary to the findings of Mould and Uddin [49] in 2014 who considered only the variation of G in their work. We believe most of the negative findings on the variation of physical

constants are possibly due to the variation of a constant being considered in isolation rather than holistically for all the constants involved. We have established that the physical constants not only vary but also how much they vary: $\dot{c}/c = 1.8H$, $\dot{G}/G = 5.4H$, $\dot{\Lambda}/\Lambda = -1.2H$ and $\dot{H}/H = -0.6\,H$. In addition, from the null results on the variation of the fine structure constant [14,15], we have shown that $\dot{\hbar}/\hbar = 1.8H$. We urge that they be used in union rather than in isolation. This indeed was not possible until now when one knows the exact form of the variation of each as above.

One basic question naturally arises—what is the consequences of the findings here? It is clear from the above that at time $t = 0$ the dark energy parameter Λ was infinity, whereas c, G and $\hbar$ were zero. Existence of any baryonic matter and radiation was irrelevant since they did not provide any energy density due to c, G and $\hbar$ all being zero. We may need to explore how the universe would evolve from such a state against the state assumed in the standard model.

One may wonder how the physical constants' variation could be measured experimentally. The most accurate device developed to date to measure the variation of fine structure constant α is atomic clock based on the hyperfine trasitions of certain atoms at microwave and optical frequencies. The transitions are also used for tests of quantum electrodynamics, general relativity and the equivalence principle, searches for dark matter, dark energy and extra forces, and tests of the spin-statistics theorem [50–52]. However, tests related to the variation of c, G and $\hbar$ as presented here are not possible using the atomic transitions since the latter are dependent on the variation of α, which is already assumed to be zero in our theory.

Spinning bodies cause spacetime to rotate around it causing the nearby angular momentum vector to precess. This so-called frame-dragging phenomenon causes the electromagnetic signal from an orbiting spacecraft to register a redshift Δz, given by [53]:

$$\Delta z \sim \frac{GM}{c^3 P},$$

where M is the mass of the spinning body and P is its spinning period. If c and G evolve in time as determined in this paper then Δz will not vary in time due to the variation of these constants, and therefore this method is not suitable for measuring their variation.

If we could isolate all the perturbative and relativistic effects on a high eccentricity satellite orbit then any residual increase in its eccentricity and orbit size may be attributed to the variation of G and c, and Equations (40) and (43) may be adapted to the satellite parameters. In addition, any spacecraft receding from Earth should experience anomalous deceleration similar to Pioneer's. The spacecraft may be designed to eliminate or minimize the thermal radiation anisotropy. One could possibly design other experiments that could test the variability of constants when all the constants discussed here are simultaneously varying.

Existence of the parameter p in estimating eccentricity increase can be seen as a deficiency of the quasi-phenomenological model we have used. Since the Moon eccentricity involves Earth and Moon whereas the AU increase involves Earth and Sun, and since the masses of the two systems are enormously different, the parameter p may be considered to take this difference into account. We will need to develop a fully relativistic theory to eliminate this arbitrariness in p for estimating the two anomalies with varying c and G. Until then it would be prudent to leave $p = 1$ and just be contended that the variable c and G theory is able to estimate the anomalies within a factor of 2.

12. Conclusions

Salient points of the finding in this work are:

1. The single-parameter VcGΛ model fits the supernovae 1a data almost as well as the two-parameter standard ΛCDM model. The VcGΛ model has better predictive capability than the ΛCDM model.
2. One could see that the approach taken here to explain the three anomalies is based on a very simple analytically derived expression for the evolution of the speed of light and gravitational constant. Thus one could infer that the Occam's razor principle would favour the new approach over other approaches. In the case of the Pioneer anomaly, it should be rather easy to implement it in the real time modeling of the astrodynamics of long-range spacecraft.

3. The expression of eccentricity variation involves G and c in a manner that the contributions of $\dot{G}/G$ and $\dot{c}/c$ almost cancel each other except for a small residual. This may be compared with the case of using Kepler's Equation (15) when the two contributions cancel out entirely.
4. Based on G variation alone, one can see from Equations (37)–(39) that calculated $\dot{e}$ is more than two orders of magnitude higher than measured $\dot{e}$. Current work corrects this by including $\dot{c}$ variation.
5. There may be bias factors in the models (say in favour of tidal effects) used for data analysis, since a model is not considered good enough unless it can account for all the observed value. We believe that one may be able to remove this bias by the inclusion of the local effect of cosmology, as presented here, in the data analysis models.
6. As mentioned above, both $\dot{G}/G$ and $\dot{c}/c$ are positive and thus both of them are increasing with time rather than decreasing. The simple model presented above is effectively inclusive of the cosmological constant. The existence of cosmological constant Λ in standard ΛCDM model leads to a continuous addition of dark energy to the universe as the universe expands, i.e., it causes the total energy of the universe to increase. The same is achieved by the increase of G and c through the second continuity equation (Equation (7)).
7. Variability of all the constants is expressed in terms of the Hubble parameter $H(t)$, and at the present time, relative to the Hubble constant H_0. In summary, the physical constants evolve as follows: $\dot{c}/c = 1.8H$, $\dot{G}/G = 5.4H$, $\dot{\hbar}/\hbar = 1.8H$, $\dot{\Lambda}/\Lambda = -1.2H$, and $\dot{H}/H = -0.6H$.
8. There is an exact proportionality between the quantum Planck units and cosmological Hubble units and the proportionality is evolutionary. All Hubble units are 61 orders of magnitude larger than the corresponding Planck units. The factor determining the same, i.e., $D \equiv \sqrt{[c^5/(H_0^2 G\hbar)]}$, varies as $\dot{D}/D = 1.5H_0 \simeq 1.07 \times 10^{-10}\ \mathrm{yr}^{-1}$.
9. The model limitation is that it does not use Einstein-Hilbert action with c and G as scalar fields rather than constants to determine variable constants compliant with field equations [54]. Thus, the VcGΛ model, albeit simple, can only be considered quasi-phenomenological as it does not fully account for the variability of c, G, and Λ.

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Conflicts of Interest: The author declares no conflict of interest.

Appendix A

We may write Equation (4) with $K \equiv G/c^2$ as follows:

$$\dot{a}^2 = \left(\tfrac{8\pi K\varepsilon}{3} + \tfrac{1}{3}\Lambda\right) a^2. \tag{A1}$$

Differentiating it with respect to time gives:

$$2\dot{a}\ddot{a} = \left(\tfrac{8\pi K\varepsilon}{3} + \tfrac{1}{3}\Lambda\right) 2a\dot{a} + \left(\tfrac{8\pi \dot{K}\varepsilon}{3} + \tfrac{8\pi K\dot{\varepsilon}}{3} + \tfrac{1}{3}\dot{\Lambda}\right) a^2. \tag{A2}$$

Dividing it by $2\dot{a}a$ yields:

$$\frac{\ddot{a}}{a} = \left(\frac{8\pi K\varepsilon}{3} + \frac{1}{3}\Lambda\right) + \left(\frac{8\pi \dot{K}\varepsilon}{3} + \frac{8\pi K\dot{\varepsilon}}{3} + \frac{1}{3}\dot{\Lambda}\right)\left(\frac{a}{2\dot{a}}\right) \tag{A3}$$

Substituting $\ddot{a}/a$ from Equation (3):

$$-\frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 - 4\pi Kp + \frac{1}{2}\Lambda = \left(\frac{8\pi K\varepsilon}{3} + \frac{1}{3}\Lambda\right) + \left(\frac{8\pi \dot{K}\varepsilon}{6} + \frac{8\pi K\dot{\varepsilon}}{6} + \frac{1}{6}\dot{\Lambda}\right)\left(\frac{a}{\dot{a}}\right) \tag{A4}$$

Substituting $\left(\frac{\dot{a}}{a}\right)^2$ from Equation (4):

$$-\left(\frac{8\pi K\varepsilon}{6}+\frac{1}{6}\Lambda\right)-4\pi Kp+\frac{1}{2}\Lambda-\left(\frac{8\pi K\varepsilon}{3}+\frac{1}{3}\Lambda\right)-\left(\frac{8\pi\dot{K}\varepsilon}{6}+\frac{8\pi K\dot{\varepsilon}}{6}+\frac{1}{6}\dot{\Lambda}\right)\left(\frac{a}{\dot{a}}\right)=0, \text{ or} \quad \text{(A5)}$$

$$4\pi K\varepsilon+4\pi Kp+\left(\frac{a}{\dot{a}}\right)\left(\frac{8\pi}{6}\dot{K}\varepsilon+\frac{8\pi}{6}K\dot{\varepsilon}+\frac{1}{6}\dot{\Lambda}\right)=0, \text{ or} \quad (46)$$

$$\frac{\dot{a}}{a}(24\pi K\varepsilon+24\pi Kp)+8\pi\dot{K}\varepsilon+8\pi K\dot{\varepsilon}+\dot{\Lambda}=0. \quad (47)$$

Dividing it by $8\pi K$ and rearranging we get:

$$\dot{\varepsilon}+3\left(\frac{\dot{a}}{a}\right)(\varepsilon+p)+\frac{\dot{K}}{K}\varepsilon+\frac{\dot{\Lambda}}{8\pi K}=0. \quad (48)$$

This is the same equation as Equation (5).

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