Two conserved angular momenta in Schwarzschild spacetime geodesics

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Abstract. The present study investigated the geodesic paths in the 3+1 dimensional Schwarzschild spacetime. Four conserved parameters were found: the first is the conserved total energy: the second is the coordinate-invariant metrics: and the final two are the angular momenta (P_θ and P_ϕ) in the spherical coordinate. For $\theta = \pi/2$ and when excluding a $1/c^2$ term in the equation of motion, we recover the orbit equation of the two-body problem. But when not excluding that term, we recover the orbit precession, e.g. the perihelion precession of Mercury. When the value θ is not fixed, we found the equation of motion to be the radius $r(\theta)$ as a function of θ, which is similar to the function $r(\phi)$ for a fixed value of θ.

1. Introduction

Since its proposal by Albert Einstein 104 years ago, the theory of general relativity [1] continues to provide new ways visualizing the nature of spacetimes and the universe, e.g. massive objects curving spacetimes [2], black hole singularities [3], the contradiction between relativity and quantum mechanics [4], and the correspondence between the anti de Sitter spacetime and the conformal field theory [5]. The present study analytically solves geodesic equations in a curved spacetime [6]. To simplify the problem the researchers studied in the 3+1 Schwarzschild spacetime [7]. Problem-solving procedures of orbit precessions due to the effect of general relativity are offered. Moreover the study results are compared with a non-relativistic case, i.e. the two-body problem and the numerical calculations of planet perihelion precessions. The study also investigates the case of non-fixing value angle θ.

The paper is organized as follows. Section 2 studies the four geodesic equations where the four constants of motion are found. In section 3 finds the geodesic path solutions for both non-relativistic approximation (the orbits of the two-body problem) and the relativistic case (the perihelion precession of planet orbits). In section 4 then explores the non-fixing value case of the angle θ. Section 5 then provides a conclusion.

2. Four constants of motion in geodesic trajectories in 3+1 dimensions

According to the theory of general relativity, spacetime is curved by any object that has mass, charge and, or angular momentum. The present study only considers mass M which causes a curvature in a spacetime. Schwarzschilds solution is the simple solution to Einsteins equation in the presence of only mass M [7], which can be written in terms of the metrics of the time and the spherical coordinate space variables, (t, r, θ, ϕ) as

$$ds^2 = -f(r)c^2dt^2 + f^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \qquad f(r) = 1 - \frac{r_h}{r} \tag{1}$$

$r_h = 2GM/c^2$ is the horizon radius, G is Newtons constant, and c is the speed of light.

Geodesic paths refer to the trajectory of objects in a curved spacetime. Various studies have researched these geodesic paths in the Schwarzschild spacetime, both analytically and numerically. The present study offers another new analytical calculation in 3+1dimensions and shows that the system contains four constants of motion from the four geodesic equations. The first constant of motion is the total energy, the second is the invariance of metrics, and the third and fourth are the two conserved angular momenta in the directions of θ and ϕ .

Start from the geodesic equation [6]:

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2}$$

where τ is the proper time of the orbiting object, x^μ is the coordinate in the metrics, and $\Gamma^\lambda_{\sigma\nu}$ is the Christoffel symbol,

$$\Gamma^\lambda_{\sigma\nu} = \frac{g^{\lambda\gamma}}{2}\left(\partial_\nu g_{\gamma\sigma} + \partial_\sigma g_{\nu\gamma} - \partial_\gamma g_{\sigma\nu}\right)$$

There are four cases $(t, r, \theta, \phi\ )$ to calculate:

(i) $\lambda = t$ The metric can be written in matrices as:

$$g_{\mu\nu} = \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & f^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} -f^{-1} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix}$$

The non-zero Christoffel symbols in this case are $\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2f}\partial_r f$ and substituted into equation (2)

$$\frac{d^2t}{d\tau^2} + \frac{1}{f}\frac{df}{dr}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0 \tag{3}$$

Divides equation (3) by $dt/d\tau$, one can reduce equation (3) to:

$$\frac{d}{d\tau}\ln\left(\frac{dt}{d\tau}f\right) = 0 \tag{4}$$

One can therefore define a constant of motion as:

$$\frac{dt}{d\tau}f \equiv k \tag{5}$$

This constant k is related to the total energy, which can be seen in the comparison between equation (14) and equation (15).

(ii) $\lambda = \phi$ The Christoffel symbols are $\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$ and $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$

$$\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau}\frac{d}{d\tau}\left(\ln r^2\right) + \frac{d\phi}{d\tau}\frac{d}{d\tau}\left(\ln\sin^2\theta\right) = 0 \tag{6}$$

Divides equation (6) by $d\phi/d\tau$, one can reduce equation (6) to:

$$\frac{d}{d\tau}\left(\ln r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0 \tag{7}$$

One can therefore define a constant of motion, an angular momentum long ϕ direction as:

$$\ln r^2 \sin^2\theta \frac{d\phi}{d\tau} \equiv P_\phi \tag{8}$$

(iii) $\lambda = \theta$ The Christoffel symbols are $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$ and $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$. Substitutes the constant of motion P_ϕ from equation (8), $d\phi/d\tau \equiv P_\phi/r^2\sin^2\theta$ into the geodesic equation (2), which obtains:

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \frac{\cos\theta P_\theta^2}{r^4\sin^3\theta} = 0 \tag{9}$$

where the term $-\frac{\cos\theta}{\sin^3\theta} = \frac{1}{2}\frac{d}{d\theta}\frac{1}{\sin^2\theta} = \frac{1}{2}\frac{d\tau}{d\theta}\frac{d}{d\tau}\left(\frac{1}{\sin^2\theta}\right)$ in equation (9), and multiples equation (9) with $2d\theta/d\tau$, which obtains

$$\frac{d}{d\tau}\left(r^4\left(\frac{d\theta}{d\tau}\right)^2 + \frac{P_\phi^2}{\sin^2\theta}\right) = 0 \tag{10}$$

We can therefore define a constant of motion, another angular momentum along θ as:

$$r^4\left(\frac{d\theta}{d\tau}\right)^2 + \frac{P_\phi^2}{\sin^2\theta} \equiv P_\theta^2 \tag{11}$$

(iv) in this case the geodesic equation (2) can be reduced to equation (5) with a constant of motion $f(dt/d\tau) \equiv k$ with the help of the metric in equation (1). This implies the fourth conserved parameter, i.e., the coordinate invariance metric .

3. Geodesic paths

In this section, some assumptions and approximations were made in order to compare this study to the non-relativistic two-body problem, or the Kepler problem. For the constant value of $\theta = \pi/2$ [8], the equation (11) is reduced to $P_\phi^2 = P_\theta^2$. Equation (8) can be rewritten as:

$$d\tau = \frac{r^2\sin^2\theta d\phi}{P_\phi} \tag{12}$$

Divided the metric equation (1) by $ds^2 = -c^2 d\tau^2$, equation (1) changes to:

$$1 = f\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2 f}\left(\frac{dr}{d\tau}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2 = \frac{k^2}{f} - \frac{1}{c^2 f}\left(\frac{dr}{d\tau}\right)^2 - \frac{P_\phi^2}{c^2 r^2} \tag{13}$$

$$d\tau = \frac{dr}{\sqrt{c^2k^2 - \frac{fP_\phi^2}{r^2} - c^2 f}} \tag{14}$$

Therefore, from equation (12) and equation (13), one can deduce to equation (14) below:

$$\frac{r^2 d\phi}{P_\phi} = \frac{dr}{\sqrt{c^2k^2 - \frac{f(r)P_\phi^2}{r^2} - c^2 f(r)}} = \frac{dr}{\sqrt{c^2(k^2-1) + \frac{2GM}{r} - \frac{P_\phi^2}{r^2} - \frac{2GMP_\phi^2}{c^2r^3}}} \tag{15}$$

where we substitute $f = 1 - \frac{r_h}{r}$ from equation (1) into equation (14). One then can compare equation (14) to the radius (r)angle(ϕ) differential relation in the Kepler problem [8]:

$$\frac{r^2 d\phi}{p_\phi} = \frac{dr}{\sqrt{\frac{2E_{\text{total}}}{\mu} - \frac{p_\phi^2}{r^2} + \frac{2GM}{r}}} \tag{16}$$

$p_\phi = r^2 d\phi/dt$ is the conserved angular momentum and μ is the reduced mass in the Kepler problem. In equation (14), one can make certain approximations and assumptions in order to reduce to equation (15). The approximations and assumptions in equation (14) are that the first term inside the squared root is approximated as $c^2(k^2-1) \approx 2E_{\text{total}}/\mu$, and the last term inside the squared root is proportional $1/c^2$, much smaller than the rest of the terms inside the squared root and can be excluded from the equation for the non-relativistic mechanic limit.

A new variable should be defined to integrate equation (14):

$$v \equiv \frac{r_h}{r} \tag{17}$$

$$\frac{P_\theta \sin^2\theta d\phi}{P_\phi} = \frac{-dv}{\sqrt{v^3 - v^2 + sv + s'}} \tag{18}$$

$$v^3 - v^2 + \frac{c^2 r_h^2}{P_\theta^2} v + \frac{c^2 r_h^2}{P_\theta^2}(k^2-1) = (v - v_1)(v - v_2)(v - v_3) \tag{19}$$

where $s \equiv \frac{c^2 r_h^2}{P_\theta^2}$ and $s' \equiv \frac{c^2 r_h^2}{P_\theta^2}(k^2-1)$. The roots of the cubic equation inside the squared root in equation (17) are:

$$v_1 = \frac{1}{3} + A + B, \quad v_2 = \frac{1}{3} - \frac{1}{2}(A+B) + \frac{i\sqrt{3}}{2}(A-B), \quad v_3 = \frac{1}{3} - \frac{1}{2}(A+B) - \frac{i\sqrt{3}}{2}(A-B) \tag{20}$$

where $A, B = \left[-\frac{1}{2}\left(-\frac{2}{3^2} + \frac{s}{3} + s'\right) \pm \sqrt{\frac{1}{2^2}\left(-\frac{2}{3^2} + \frac{s}{3} + s'\right)^2 + \frac{1}{3^2}\left(-\frac{1}{3} + s\right)^3}\right]^{1/3}$. Next, a new variable should be defined:

$$\cos\alpha \equiv \frac{\sqrt{v - v_2}}{\sqrt{v_3 - v_2}} \tag{21}$$

and substitute $\cos\alpha$, v_1, v_2, and v_3 back into equation (17), which changes to

$$\frac{P_\theta \sin^2\theta d\phi}{P_\phi} = \frac{1}{\sqrt{v_1 - v_2}} \frac{2d\alpha}{\sqrt{1 - m\cos^2\alpha}} \tag{22}$$

where $m \equiv \frac{v_3 - v_2}{v_1 - v_2} \approx \frac{4GM}{cP_\theta}\sqrt{k^2-1} \propto \frac{1}{c}$ and $\frac{1}{\sqrt{v_1 - v_2}} \approx 1 + \frac{1}{2}\sqrt{s(k^2-1)} \propto 1 + \frac{1}{c} + ...$ their leading expansions are 1/c and 1 respectively. Equation (20) is actually the first kind of the ellipse integral [9]. Therefore, in the non-relativistic limit, for $\theta = \pi/2$, $\phi = 2\alpha$, or:

$$\cos\phi = \cos 2\alpha = \frac{2v - (v_2 + v_3)}{v_3 - v_2} \approx \frac{\frac{1}{r} - \frac{GM}{p_\phi^2}}{\sqrt{\frac{2E}{\mu p_\phi^2} + \left(\frac{GM}{p_\phi^2}\right)^2}} \tag{23}$$

Equation (21) is the trajectory equation of the two body system, where the orbits are stable. For example, for an ellipse orbit the principle and minor axes do not move. However, for $m \neq 0$ ($m \propto 1/c$), when including the relativistic effect, one can approximate the integral in equation (20) by a series expansion:

$$\left[1 - m\cos^2\alpha\right]^{-1/2} = 1 + \frac{1}{2}\frac{m}{2^2}\left[e^{i\alpha} + e^{-i\alpha}\right]^2 + \frac{3}{2^3}\frac{m^2}{2^4}\left[e^{i\alpha} + e^{-i\alpha}\right]^4 + \; ... \tag{24}$$

$$\phi = \frac{1}{v_3 - v_1}\int_0^{2\pi}\left\{1 - \frac{1}{2}m\cos^2\frac{\alpha}{2} + \frac{1\cdot 3}{2^2\cdot 2!}m^2\cos^4\frac{\alpha}{2} + \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}m^3\cos^6\frac{\alpha}{2} + \; ...\right\}d\alpha \tag{25}$$

The limit of integration is $0 \leq \alpha < \pi$, a full turn of a single orbit. We use the identity:

$$\int_0^{\pi} e^{i2n\alpha}d\alpha = \begin{cases} 0, & n \neq 0, \text{ integer} \\ \pi, & n = 0 \end{cases} \tag{26}$$

to obtain the angle . The effect of non-zero causes the precession of the orbiting axis, for example the perihelion precession of Mercury. Therefore, let define the precession angle of one turn of orbit as $\Delta\phi = \phi - 2\pi$:

$$\Delta\phi = \frac{2\pi}{\sqrt{v_3 - v_1}}\sum_{k=0}^{\infty}\frac{\left[(2k)!\right]^2}{\left(2^k\cdot k!\right)^4}m^k - 2\pi \tag{27}$$

To obtain the numeric value of equation (23), some orbit data is required, for instance eccentricity, average velocity, average radius, plus the Sun mass and Newton constant. For the case of Mercury [10], $c^2(k^2 - 1) = -2.29129 \times 10^9$ m^2/s^2 and $\Delta\phi = 5.02158 \times 10^{-7}$ radian and 1 arcsecond $= 4.848136811 \times 10^{-6}$ radian, one century on the Earth equals to 415.20261071 turns of Mercury around the Sun, therefore the arcsecond per century is:

$$\Delta\phi = \frac{5.02158 \times 10^{-7} \times 415.20261071}{4.848136811 \times 10^{-6}} = 43.005 \text{ arcsecond/century} \tag{28}$$

The numerical calculation yielded 42.9195 arcsecond/century [11]. The other planet precessions are calculated analytically and compared with numerical results in Table 1 [12, 13].

Table 1. Comparison between the analytical and numerical perihelion planet precessions

Planets	Analytical Precession (arcsecond/century)	Numerical Precession (arcsecond/century)
Mercury	43.0	42.9195
Venus	8.61	8.6186
Earth	3.83	8.6186
Mars	1.35	1.3502
Jupiter	0.0622	0.0623
Saturn	0.0137	0.0137

4. Non-fixing values of θ trajectories

By following the procedures outlined in sections 2 and 3, we can study the system when θ is not fixed. Begin by rewriting equation (10) to:

$$\frac{1}{r^2}d\tau = \frac{\sin\theta d\theta}{\sqrt{P_\theta^2 - P_\phi^2}} \tag{29}$$

From equation (13), the relation between θ and r is:

$$\frac{\sin\theta d\theta}{\sqrt{P_\theta^2 - P_\phi^2}} = \frac{dr}{r^2\sqrt{c^2(k^2-1) - \frac{P_\theta^2}{r^2} + \frac{2GM}{r} + \frac{2GMP_\theta^2}{c^2r^3}}} \tag{30}$$

The left-hand side of equation (26) can be integrated as:

$$\beta \equiv \cos^{-1}\left(\frac{\cos\theta}{\sqrt{1-P_\phi^2/P_\theta^2}}\right) = P_\theta \int \frac{\sin\theta d\theta}{\sqrt{P_\theta^2 \sin^2\theta - P_\phi^2}} \tag{31}$$

The new variable β can also be written down as:

$$d\beta = \frac{P_\theta d\tau}{r^2} = \frac{P_\theta dr}{r^2\sqrt{c^2(k^2-1) - \frac{P_\theta^2}{r^2} + \frac{2GM}{r} + \frac{2GMP_\theta^2}{c^2r^3}}} \tag{32}$$

Equation (28) is actually the same as equation (14), for $\theta = \pi/2$, which is the first kind of ellipse integral. From equation (11), the constants of motion P_θ and P_ϕ provide the opportunity to study the relation between the variables θ and τ by changing the variable $u = \cos\theta$ and using $\beta = P_\theta d\tau/r^2$:

$$\left(\frac{du}{d\beta}\right)^2 + u^2 = 1 - \frac{P_\phi^2}{P_\theta^2} \tag{33}$$

and differentiating equation (29) with $\frac{d}{du}$. One would obtain, when $du/d\beta \neq 0$:

$$\frac{d^2u}{d\beta^2} + u = 0 \tag{34}$$

$$u = \cos\theta = ae^{i\beta} + be^{-i\beta} \tag{35}$$

where a and b are unknown constants with the relationship $4ab = 1 - P_\phi^2/P_\theta^2$. It would be interesting to study the relationship between the variables θ and ϕ through the variables and in equation (31).

5. Conclusion

The present study solves geodesic equations for trajectories in 3+1 Schwarzschild spacetime. The study found four constants of motion: the total energy: the invariant metric: and two angular momenta along the directions θ and ϕ. Moreover, this research offers analytical calculation procedures to solve geodesic equations, the approximation methods for the non-relativistic limit to obtain Keplers problem orbits and for relativistic cases, and the methods of calculating the perihelion precessions of planets. Section 4 studied the orbit for $\theta \neq \pi/2$. The same solutions were derived as in the ϕ case, but with the different values of θ. In addition, the relation between the angles θ and ϕ were also studied.

There are a number of interesting issues for further study. For example, the relationships between the system variables and parameters ,(τ, r, θ, ϕ) (total energy, P_θ, P_ϕ). It would also be interesting to take the procedures and methods presented in this research to study other systems, for example the Kerr spacetime with various kinds of particles.

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