Fréchet-algebraic deformation quantizations

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Abstract. In this review I present some recent results on the convergence properties of formal star products. Based on a general construction of a Fréchet topology for an algebra with countable vector space basis I discuss several examples from deformation quantization: the Wick star product on the flat phase space $\mathbb{R}^{2n}$ gives a first example of a Fréchet algebraic framework for the canonical commutation relations. More interesting, the star product on the Poincaré disk can be treated along the same lines, leading to a non-trivial example of a convergent star product on a curved Kähler manifold.

1. Introduction

In this review we recall some recent progress in the understanding of the passage from formal deformation quantization [1] to a more analytic framework for star products as obtained in [2].

Let us recall the basic definitions of formal deformation quantization from [1], see also [3] for a pedagogical introduction. We consider a Poisson manifold (M, π) where $\pi \in \Gamma^\infty(\Lambda^2 TM)$ is the Poisson bivector field. This means that $\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g)$ defines a Poisson bracket for $f, g \in \mathscr{C}^\infty(M)$. We will use complex-valued functions throughout this paper. The Jacobi identity for this bracket is equivalent to the condition $[\![\pi, \pi]\!] = 0$ using the Schouten bracket. Then a *star product* $\star$ for (M, π) is a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication for $\mathscr{C}^\infty(M)[[\lambda]]$ written as

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g) \tag{1.1}$$

such that $C_0(f, g) = fg$ is the pointwise product,

$$C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\} \tag{1.2}$$

gives the Poisson bracket in the first order commutator, the C_r are bidifferential operators, and $1 \star f = f = f \star 1$ for all $f \in \mathscr{C}^\infty(M)[[\lambda]]$.

The classification and existence of such star products is by now well-established: starting with the early works of deWilde and Lecomte [4] and Fedosov [5] in the symplectic case, Kontsevich proved the existence and classification in the general Poisson case in [arXiv:q-alg/9709040] that later appeared as [6]. In addition to these existence and classification results one has many explicit examples, a good understanding of the quantization of symmetries and phase space reduction in this context, and a well-working representation theory for the resulting algebras, see e.g [7] for a review.

However, from a physical point of view the situation is still unsatisfactory: the deformation parameter λ plays the role of Planck's constant $\hbar$ which one would like to substitute by a positive real number. Since one can easily show that the series (1.1) typically diverges for all $\lambda \neq 0$, one faces a serious convergence problem.

There are several different approaches to overcome this problem. Most notably is the idea to replace the deformation aspect as above by the notion of a continuous field of topological algebras. Here mainly continuous fields of C^*-algebras have been considered, see e.g Rieffel's definition of strict deformation quantization [8]. In this context one knows various examples and some general constructions based on actions of $\mathbb{R}^d$ and some more general groups [9–13] but a general picture seems to be missing.

In this review we present a different approach, namely to investigate the convergence of the series (1.1) more directly within a Fréchet-algebraic framework. The reason for this choice is that for $M = \mathbb{R}^{2n}$ with the usual Weyl-Moyal star product the polynomials form a subalgebra where convergence is trivially fulfilled as the series simply terminate. To extend the star product beyond polynomials one would like to establish a topology for which it is continuous. Then the completion with respect to this topology should yield a hopefully large and interesting class of functions. However, since the canonical commutation relations will be realised in this larger algebra, no Banach topology will be possible.

In this work we do not get any general existence or classification results as this may even be too much to ask for. Instead, we focus on some non-trivial example, the star product on the Poincaré disk. Topologically, the disk is still trivial but from the Kähler geometry point of view, we have curvature compared to the case of the flat $\mathbb{R}^{2n}$.

Though we only have examples so far, we can at least split the question about convergence of star products in two separate problems. The first will be a general construction of a locally convex topology for an algebra having a countable basis. This construction will not work in general but requires some rather technical conditions on the multiplication to be satisfied. Nevertheless, it is conceptually rather easy and clear. The second step will consist in finding good examples where the above construction actually works. Surprisingly, we have many interesting examples even from beyond deformation quantization which stress that the general construction can be used to produce examples of locally convex algebras in various areas of mathematics.

The paper is organized as follows: in the next section we present the general construction of a Fréchet topology for a complex algebra with countable vector space basis in two versions. We discuss several simple examples where the necessary technical conditions for the constructions are fulfilled. In Section 3 we pass to the first non-trivial example from deformation quantization: the Wick star product on $\mathbb{R}^{2n}$. Already here we get a quite non-trivial completion and a Fréchet algebraic framework for the canonical commutation relations. Section 4 contains a brief review of the explicit construction of a star product of Wick type on the Poincaré disk. In Section 5 we show that the general construction can be applied to the star product on the disk. In fact, the whole phase space reduction of the star product can be formulated within our Fréchet algebraic context. The resulting Fréchet algebra on the disk enjoys many nice features some of which we explain.

2. The general construction

Consider a complex algebra $\mathcal{A}$ with a vector space basis $\{e_\alpha\}_{\alpha \in I}$ such that the index set I is *countable*. Then the product $\star$ of $\mathcal{A}$ is completely determined by its values on the basis vectors

$$e_\alpha \star e_\beta = \sum\nolimits_\gamma C^\gamma_{\alpha\beta} e_\gamma \tag{2.1}$$

Here we introduced the structure constants $C^\gamma_{\alpha\beta}$. For given α and β we have only finitely many γ for which $C^\gamma_{\alpha\beta} \neq 0$. A general element $a \in \mathcal{A}$ can be written as $a = \sum_\alpha a_\alpha e_\alpha$ with only finitely many a_α being different from zero, thereby defining the evaluation functionals $e^\alpha \colon a \mapsto a_\alpha$.

The aim is now to find a locally convex topology on $\mathcal{A}$ such that $\star$ becomes continuous *and* all the evaluation functionals e^α become continuous, too.

The first step is therefore to consider the seminorm $\|a\|_{0,0,\gamma} = |a_\gamma|$. which has to be among the continuous seminorms of the searched-for topology if we want e^γ to be continuous. While this makes e^γ continuous, the product $\star$ has no reason to be continuous with respect to all the $\|\cdot\|_{0,0,\gamma}$. In order to see the possible failure of continuity we estimate

$$\|a \star b\|_{0,0,\gamma} = \Big|\sum\nolimits_{\alpha,\beta} C^\gamma_{\alpha\beta} a_\alpha b_\beta\Big| \leq \sqrt{\sum\nolimits_{\alpha,\beta} |a_\alpha|^2 |C^\gamma_{\alpha\beta}|}\sqrt{\sum\nolimits_{\alpha,\beta} |b_\beta|^2 |C^\gamma_{\alpha\beta}|} \tag{2.2}$$

Thus we get the following simple candidate for additional seminorms in order to control the continuity of $\star$. We define

$$h_{1,0,\gamma})(a) = \sum\nolimits_\alpha |a_\alpha|^2 C^\gamma_{\alpha,\cdot} \quad \text{with} \quad C^\gamma_{\alpha,\cdot} = \sum\nolimits_\beta |C^\gamma_{\alpha\beta}| \tag{2.3}$$

and

$$h_{1,1,\gamma})(b) = \sum\nolimits_\beta |b_\beta|^2 C^\gamma_{\cdot,\beta} \quad \text{with} \quad C^\gamma_{\cdot,\beta} = \sum\nolimits_\alpha |C^\gamma_{\alpha\beta}| \tag{2.4}$$

Then $\|a\|_{1,\ell,\gamma} = \sqrt{h_{1,\ell,\gamma}(a)}$ for $\ell = 0, 1$ will be seminorms estimating the continuity requirement for $\star$ with respect to $\|\cdot\|_{0,0,\gamma}$. But then we have to iterate this procedure to get continuity estimates for $\star$ with respect to $\|\cdot\|_{1,\ell,\gamma}$ and so on. Thus we define recursively for $m = 0, 1, \ldots$, $\ell = 0, \ldots 2^m - 1$, and $\gamma \in I$

$$h_{m+1,2\ell,\gamma}(a) = \sum\nolimits_\alpha h_{m,\ell,\alpha}(a)^2 C^\gamma_{\alpha,\cdot} \quad h_{m+1,2\ell+1,\gamma}(a) = \sum\nolimits_\beta h_{m,\ell,\beta}(a)^2 C^\gamma_{\cdot,\beta} \tag{2.5}$$

for every $a \in \mathcal{A}$ and set $\|a\|_{m,\ell,\gamma} = \sqrt[2^m]{h_{m,\ell,\gamma}(a)}$. By construction, it is now clear that we have the continuity estimate

$$\|a \star b\|_{m,\ell,\gamma} \leq \|a\|_{m+1,\ell,\gamma} \|b\|_{m+1,2^m+\ell,\gamma} \tag{2.6}$$

However, it may very well happen that all these quantities diverge to $+\infty$. In particular, the numbers $C^\gamma_{\alpha,\cdot}$ and $C^\gamma_{\cdot,\beta}$ need not to be finite. In this situation our construction *fails*. In any case, we consider those elements of $\mathcal{A}$ where we have convergence of all the above series and set

$$\mathcal{A}_{\text{nice}} = \Big\{a \in \mathcal{A} \;\Big|\; \|a\|_{m,\ell,\gamma} < \infty \text{ for all } m, \ell, \gamma\Big\} \tag{2.7}$$

It is then easy to see that $\mathcal{A}_{\text{nice}}$ is a subalgebra of $\mathcal{A}$ on which the product is continuous with respect to the locally convex topology induced by all the seminorms $\|\cdot\|_{m,\ell,\gamma}$. Nevertheless, it might happen that $\mathcal{A}_{\text{nice}} = \{0\}$ is trivial.

Theorem 2.1 *Suppose we have* $\mathcal{A} = \mathcal{A}_{\text{nice}}$.

i.) The completion $\widehat{\mathcal{A}}$ *of* $\mathcal{A}$ *becomes a Fréchet algebra.*

ii.) The evaluation functionals $e^\gamma : \mathcal{A} \longrightarrow \mathbb{C}$ *are continuous and extend continuously to* $\widehat{\mathcal{A}}$.

iii.) The vectors $\{e_\alpha\}_{\alpha \in I}$ *form an unconditional Schauder basis of* $\widehat{\mathcal{A}}$, *i.e* $a = \sum_\gamma e^\gamma(a) e_\gamma$ *converges unconditionally.*

Let us comment on this construction: the first problem is of course that we have to choose a basis and the construction *depends* on this choice in a rather obscure way. In many examples we will see that there are good reasons for such a choice, so this will be only a minor drawback. More severe is that it might happen that $\mathcal{A}_{\rm nice} = \{0\}$ is trivial. In this case, it will turn out that sometimes a rescaling of the given basis yields better convergence properties and can make the construction work. It turns out that by an appropriate rescaling one even can control properties of the resulting topology.

The remarkable feature of the construction is that it does not require an *associative* algebra at all: it would be very interesting to apply this to some of the standard candidates of infinite-dimensional Lie algebras with countable vector space basis like e.g the Witt algebra or the Virasoro algebra. We also note that there are alternative constructions of locally convex topologies for algebras having a countable vector space basis, see [14, Prop. 2.1].

While the above construction already yields several interesting examples, we can still refine it. The idea is that on $\mathcal{A}$ there might be a collection of linear functionals Ω of special interest. We are now interested in constructing a finer topology such that the functionals $\omega \in \Omega$ also become continuous. Of course, we can just add the seminorms $\|a\|_\omega = |\omega(a)|$ to achieve the continuity of ω and then start the recursion again, in order to guarantee the continuity of the product. However, this will not be necessary as the seminorms

$$\|a\|_{m,\ell,\omega} = \sqrt[2^m]{\sum\nolimits_\gamma h_{m,\ell,\gamma}(a)|\omega(e_\gamma)|} \tag{2.8}$$

will do the job. Again, we have the problem that these new seminorms might all diverge to $+\infty$. In this case the construction fails again. Thus we consider

$$\mathcal{A}_{\Omega\text{-nice}} = \left\{a \in \mathcal{A}_{\rm nice} \;\middle|\; \|a\|_{m,\ell,\omega} < \infty \text{ for all } m, \ell, \omega\right\} \tag{2.9}$$

which turns out to be a subalgebra of $\mathcal{A}_{\rm nice}$. Under the assumption $\mathcal{A}_{\Omega\text{-nice}} = \mathcal{A}$ we get the same conclusions as in Theorem 2.1 also in this case.

We conclude this section with some examples where the above general construction has been applied successfully:

Example 2.2 (polynomials I) Consider $\mathcal{A} = \mathbb{C}[z]$ with the standard basis $\{e_n = z^n\}_{n\in\mathbb{N}_0}$. In this case the construction can be show to work. The resulting completion is the algebra of formal power series $\mathbb{C}[[z]]$ with its usual Fréchet algebra structure. Now we can no longer interpret the elements of the completion as functions on $\mathbb{C}$. Thus we pass to the second version of the construction and take $\Omega = \{\delta_p\}$ for a point $p \in \mathbb{C}$ different from 0. Denote by $R = |p| > 0$ its absolute value. For $R \geq 1$ the second version of the construction fails but for $R < 1$ we get a non-trivial completion which is a subalgebra of the Banach algebra of holomorphic functions on the closed disk $\mathrm{B}_R(0)^{\rm cl} \subseteq \mathbb{C}$, endowed with a Fréchet topology strictly finer than the Banach space topology of uniform convergence.

Example 2.3 (polynomials II) Again, we consider $\mathcal{A} = \mathbb{C}[z]$ but now with the basis $e_n = \frac{z^n}{n!}$. The first version of the construction still works the same way and yields the same completion $\widehat{\mathcal{A}} = \mathbb{C}[[z]]$. However, the second version changes drastically. The resulting Fréchet algebra $\widehat{\mathcal{A}}$ does not depend on $R > 0$ at all and it is explicitly given by those series $a = \sum_n \tilde{a}_n z^n/n!$ with

$$\|a\|_\epsilon = \sup_n \frac{|\tilde{a}_n|}{n!^\epsilon} < \infty \tag{2.10}$$

for all $\epsilon > 0$. With other words, the Taylor coefficients of a have *sub-factorial* growth. In fact, the (quite complicated) seminorms for the second version of the construction can be shown to lead to the same topology as the one described by the seminorms (2.10). This example can be extended to the case of Laurent polynomials $\mathbb{C}[z^{-1}, z]$ where one gets a similar completion: the Laurent coefficients are required to have sub-factorial growth, too. This is again a strongly nuclear Köthe space with the monomials being an absolute Schauder basis.

Example 2.4 (matrices) The next example starts with the noncommutative non-unital algebra $\mathrm{M}_\infty(\mathbb{C})$ of matrices with only finitely many non-zero entries. Here we have an obvious basis to start with: the elementary matrices E_{ij} where we have a 1 at the (i,j)-th position and zeros elsewhere. Taking them as a first try, the construction fails for this basis. A rescaling to $\hat{E}_{ij} = \frac{1}{\sqrt{i!j!}}E_{ij}$ makes the construction work. The completion is a Fréchet algebra containing matrices with entries having sub-factorial growth. Moreover, the trace functional is continuous and hence the completion can be seen as a particular subalgebra of the trace class operators. The alternative we investigated starts with the basis $\tilde{E}_{ij} = \frac{1}{ij}E_{ij}$ instead. Also in this case the construction works and we get a larger completion: the completion contains at least those infinite matrices A whose coefficients $\tilde{A}_{ij}$ with respect to $\tilde{E}_{ij}$ are bounded. The trace functional is not continuous and the completion does contain operators which are not of trace class.

Example 2.5 (group algebras) A last class of examples is given by the group algebra $\mathbb{C}[G]$ of a finitely generated group. A first try suggests to take the group elements $g \in G$ as the basis, but then the construction fails. A rescaling can be done by means of the length $L(g)$ of $g \in G$ with respect to a fixed choice of generators of the group. Then the basis $e_g = 1/(L(g)!)^\epsilon$ makes the construction work. Here $\epsilon > 0$ is a parameter of the construction. The resulting completion gives a Fréchet algebra inside $\ell^1(G)$. Many properties can be determined explicitly. A further study of this group algebra is work in progress [Cahen M, Gutt S and Waldmann S].

As a conclusion one can say that the procedure described in Theorem 2.1 and in its alternative version gives a rather general framework to produce interesting Fréchet algebras, even though for the particular example there is typically still some work to be done in order to identify the precise topology.

3. The Wick star product on $\mathbb{C}^n$

We come now to a first non-trivial example from deformation quantization: on $\mathbb{C}^n$ one considers the Wick star product

$$f \star_{\mathrm{Wick}} g = \sum_{I,J=0}^{\infty} \frac{(2\hbar)^{|I|}}{I!} g^{IJ} \frac{\partial^{|I|} f}{\partial z^I} \frac{\partial^{|J|} g}{\partial \overline{z}^J} \tag{3.1}$$

where $f, g \in \mathcal{A}_{\mathrm{Wick}} = \mathbb{C}[z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n]$ are polynomials, $I, J \in \mathbb{N}_0^n$ are multiindices, and $g = \mathrm{diag}(+1, \ldots, +1)$ is the canonical Kähler structure on $\mathbb{C}^n$. Clearly, the Wick star product converges for polynomials as the series is actually a finite sum. We take now the following basis

$$e_{I,J} = \frac{z^I \overline{z}^J}{(2\hbar)^{|I|+|J|} I! J!} \tag{3.2}$$

to start our construction. Here $I, J \in \mathbb{N}_0^n$ are as before and $a \in \mathcal{A}_{\mathrm{Wick}}$ will be written as $a = \sum_{I,J} a_{IJ} e_{IJ}$. The choice for this normalization of the basis is motivated on one hand by physics as the $(2\hbar)^{|I|+|J|}$ in the denominator will ensure that the basis is "dimensionless", and on the other hand by the Taylor formula. It is now a quite involved computation of all the relevant structure constants to show that the first version of the construction actually works. The resulting completion can be described in a rather simple way: first we introduce the following seminorms measuring again a sub-factorial growth of the coefficients a_{IJ}, i.e

$$\|a\|_\epsilon = \sup_{I,J} \frac{|a_{IJ}|}{(|I|+|J|)!^\epsilon} \tag{3.3}$$

Then the completion can be viewed as the subset of all formal series $a = \sum_{IJ} a_{IJ} e_{IJ}$ with finite seminorms $\|a\|_\epsilon < \infty$. In fact, we have the following theorem:

Theorem 3.1 *The completion $\widehat{\mathcal{A}}_{\mathrm{Wick}}$ of the Wick star product algebra $\mathcal{A}_{\mathrm{Wick}} = (\mathbb{C}[z^1, \ldots, \overline{z}^n], \star_{\mathrm{Wick}})$ with respect to the construction of Theorem 2.1 is a strongly nuclear Fréchet algebra with absolute Schauder basis $\{e_{IJ}\}_{I,J\in\mathbb{N}_0^n}$ consisting of real-analytic functions on $\mathbb{R}^{2n}$ with Taylor coefficients a_{IJ} having sub-factorial growth. An equivalent system of seminorms is given by $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$.*

Remark 3.2 The (completed) Wick star product algebra has been studied in detail in [2, 15] resulting in many additional properties some of which we list here:

i.) Having an absolute Schauder basis, the completion is a Köthe sequence space. This can also be seen directly, as the isomorphism to a Köthe sequence space is given by mapping a to the sequence of Taylor coefficients $\{a_{IJ}\}_{I,J\in\mathbb{N}_0^n}$. This gives also the proof of the strong nuclearity thanks to the Grothendieck-Pietsch criterion.

ii.) The Fréchet topology is independent of $\hbar$. Moreover, the Wick star product $f \star_{\mathrm{Wick}} g$ of two functions in the completion $\widehat{\mathcal{A}}_{\mathrm{Wick}}$ depends holomorphically on $\hbar$. In fact, the series expression (3.1) converges absolutely in the Fréchet topology. This shows that we have a holomorphic (even an entire) deformation in the sense of [16].

iii.) The $\star_{\mathrm{Wick}}$-exponential of linear polynomials converges absolutely. This is non-trivial since the completion has *no* general entire functional calculus, it is *not* locally multiplicatively convex. In fact, due to the presence of elements obeying the canonical commutation relations this can not be the case.

iv.) The additive group $(\mathbb{R}^{2n}, +)$ acts on the completion by continuous algebra automorphism in a smooth way. In fact, the smooth topology with respect to this action coincides with the original topology. Moreover, the action is inner by means of the $\star_{\mathrm{Wick}}$-exponentials of linear polynomials.

v.) For $\hbar > 0$ the δ-functionals are continuous positive linear functionals and the corresponding GNS construction with respect to δ_0 is the usual Bargmann-Fock representation in normal ordering of the z's and $\overline{z}$'s.

vi.) Since all evaluation functionals at points in $\mathbb{R}^{2n}$ are continuous, we can view the elements of the completion as functions. They are real-analytic and have a holomorphic/anti-holomorphic extension to $\mathbb{C}^n \times \mathbb{C}^n$. The Fréchet topology of $\widehat{\mathcal{A}}_{\mathrm{Wick}}$ is then finer as the Fréchet topology of locally uniform convergence of the extensions on $\mathbb{C}^n \times \mathbb{C}^n$.

As a last remark we would like to mention that the construction of the topology by means of Theorem 2.1 of course looks very ad-hoc. However, *a posteriori*, we showed in [17] that there is a much more conceptual approach to the Wick algebra, based on an arbitrary continuous bilinear form on a locally convex space: this enlarges the construction to infinite dimensions and proves some very nice functorial properties.

4. The star product on the Poincaré disk

We come now to the main example for a convergent star product beyond the flat situation. We base the construction on the results from [18, 19] where a star product on $\mathbb{CP}^n$ and on the

Poincaré disk $\mathbb{D}_n$ was constructed by means of a phase space reduction. We have to recall some parts of this construction in order to introduce some notation.

Consider $\mathbb{C}^{n+1}$ with the pseudo Kähler metric defined by $g = \mathrm{diag}(-1, +1, \ldots, +1)$ and define the Kähler potential

$$y(z) = -g(z, \overline{z}) = |z^0|^2 - |z^1|^2 - \cdots - |z^n|^2 \tag{4.1}$$

Then one considers the open cone

$$C^+_{n+1} = \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; y(z) > 0 \right\} \tag{4.2}$$

We use y as a momentum map for the canonical $\mathrm{U}(1)$-action on $\mathbb{C}^{n+1}$ to construct the Marsden-Weinstein quotient for the momentum value 1. Thus let $C = \{z \in C^+_{n+1} \mid y(z) = 1\}$ be the constraint surface and let $M_{\mathrm{red}} = C/\mathrm{U}(1)$ be the reduced phase space. It turns out that this is the Poincaré disk $M_{\mathrm{red}} = \mathbb{D}_n$ with the symplectic structure being the one from its canonical Kähler structure of constant negative holomorphic curvature. Alternatively, we can view $\mathbb{D}_n$ as an open subset of $\mathbb{CP}^n$ by $\mathbb{D}_n = \{[z] \in \mathbb{CP}^n \mid z \in C^+_{n+1}\}$ where $[z]$ denotes the complex line through the point $z \in \mathbb{C}^{n+1} \setminus \{0\}$. This gives the correct complex structure on $\mathbb{D}_n$ but of course not the correct Kähler structure. Note that in this particular situation we also have a direct map going from the big phase space C^+_{n+1} to the reduced one, i.e the holomorphic quotient map $\pi \colon C^+_{n+1} \longrightarrow \mathbb{D}_n$ by mapping z to $[z]$.

Using the metric g on C^+_{n+1} this gives us a Wick star product $\star_{\mathrm{Wick}}$ by the very same formula (3.1), now of course involving some additional signs due to the changed g. This Wick star product inherits the correct $\mathrm{SU}(1, n)$ symmetry from the pseudo Kähler structure of C^+_{n+1}. The basic idea is to modify the Wick star product on C^+_{n+1} in a suitable way such that it passes to the quotient $\mathbb{D}_n$.

To this end, one first observes that the $\mathrm{U}(1)$-invariant functions form a subalgebra with respect to the Wick star product. Next, one calls a function $R \in \mathscr{C}^\infty(C^+_{n+1})$ *radial* if it depends only on the "radius" y, i.e if there is a smooth function $\varrho \in \mathscr{C}^\infty(\mathbb{R}^+)$ with $R = \varrho \circ y$. Clearly, a radial function is $\mathrm{U}(1)$-invariant. Moreover, a function $F \in \mathscr{C}^\infty(C^+_{n+1})$ is called *homogeneous* if it is of the form $F = \pi^* u$ with $u \in \mathscr{C}^\infty(\mathbb{D}_n)$. Finally, we introduce the global vector field

$$\frac{\partial}{\partial y} = \frac{1}{2y} \sum_{k=0}^{n} \left(z^k \frac{\partial}{\partial z^k} + \overline{z}^k \frac{\partial}{\partial \overline{z}^k} \right) \tag{4.3}$$

which on radial functions indeed acts as the partial derivative by y. A homogeneous function F can be characterized infinitesimally by $\frac{\partial}{\partial y} F = 0 = X_y F$, where X_y is the Hamiltonian vector field of y.

A simple computation shows that for a radial function R and a $\mathrm{U}(1)$-invariant function F one has

$$R \star_{\mathrm{Wick}} F = F \star_{\mathrm{Wick}} R = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} y^r \frac{\partial^r R}{\partial y^r} \frac{\partial^r F}{\partial y^r} \tag{4.4}$$

This shows that inside the $\mathrm{U}(1)$-invariant functions the radial ones are central. However, the product is not the pointwise product. Hence a restriction to the constraint surface C is not possible directly: the pointwise ideal generated by $y - 1$ is not the $\star_{\mathrm{Wick}}$-ideal. Thus the Wick star product does not restrict directly to $\mathbb{D}_n$. Now this flaw can be cured by passing to an equivalent star product $\tilde{\star}$ such that for this new star product the $\mathrm{U}(1)$-invariant functions are still a subalgebra and the $\tilde{\star}$-product with a radial function becomes the pointwise product. There is an explicit formula for the equivalence transformation and the resulting formula for $\tilde{\star}$ is

$$F \mathbin{\tilde{\star}} G = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{2\hbar}{y}\right)^r \prod_{k=1}^{r} \left(1 - k\frac{2\hbar}{y}\right)^{-1} y^r g^{i_1 j_1} \cdots g^{i_r j_r} \frac{\partial^r F}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r G}{\partial \overline{z}^{j_1} \cdots \partial \overline{z}^{j_r}} \tag{4.5}$$

for $F, G \in \mathscr{C}^\infty(C^+_{n+1})^{\mathrm{U}(1)}[[\hbar]]$ being U(1)-invariant. Now the radial functions behave like scalars and hence the $\tilde{\star}$-ideal generated by $y-1$ coincides with the pointwise ideal, i.e the vanishing ideal of C. Hence we can induce a star product $\star_{\mathbb{D}_n}$ on the disk by the identification

$$\mathscr{C}^\infty(\mathbb{D}) \cong \mathscr{C}^\infty(C^+_{n+1})^{\mathrm{U}(1)} / (y-1)\mathscr{C}^\infty(C^+_{n+1})^{\mathrm{U}(1)} \tag{4.6}$$

This gives the very explicit formula for $u, v \in \mathscr{C}^\infty(\mathbb{D}_n)$: we take $F = \pi^* u$ and $G = \pi^* v$, compute their star product upstairs according to (4.5) and set $y = 1$ afterwards. Then the resulting function on the constraint surface C is the pull-back of $u \star_{\mathbb{D}_n} v$.

We use now this star product $\star_{\mathbb{D}_n}$ on the disk to implement our construction of a Fréchet topology according to Theorem 2.1.

5. The basis and the algebra on the disk

The first step is of course to find a subalgebra of $\mathscr{C}^\infty(C^+_{n+1})^{\mathrm{U}(1)}$ with a countable vector space basis such that the product $\tilde{\star}$ is well-defined for some $\hbar \neq 0$. Then we can implement our construction, still on the big phase space, and pass to the quotient by the same identification as in (4.6) after having established the topological context.

We start with the U(1)-invariant monomials

$$\mathsf{e}_{P,Q,\alpha} = \left(z^0\right)^{\alpha-|P|} \left(\overline{z}^0\right)^{\alpha-|Q|} z^P \overline{z}^Q \tag{5.1}$$

where $P, Q \in \mathbb{N}_0^n$ are multiindices for the last n coordinates and $\alpha \geq |P|, |Q|$. We call such (P, Q, α) an *index triple*. Then we consider the functions

$$\mathsf{f}_{P,Q,\alpha} = \frac{1}{P!(\alpha-|P|)!Q!(\alpha-|Q|)!} \left(\frac{y}{2\hbar}\right)_\alpha \frac{\mathsf{e}_{P,Q,\alpha}}{y^\alpha} \tag{5.2}$$

where $(x)_\alpha = x(x+1)\cdots(x+\alpha-1)$ is the Pochhammer symbol. Here and in the following $2\hbar$ will be a complex number from the set

$$2\hbar \in \mathbb{C} \setminus \{0, -1, -1/2, -1/3, \ldots\} \tag{5.3}$$

which guarantees that the Pochhammer symbol $\left(\frac{y}{2\hbar}\right)_\alpha$ will be different from zero on the constraint surface for all $\alpha \in \mathbb{N}_0$.

Proposition 5.1 *The functions* $\{\mathsf{f}_{P,Q,\alpha}\}_{P,Q,\alpha}$ *form a basis of a* $\tilde{\star}$*-subalgebra* $\mathcal{A}_\hbar(C^n_{n+1})$ *where the star product is convergent. The structure constants can be computed explicitly and one obtains* $\mathcal{A}_\hbar(C^n_{n+1}) = (\mathcal{A}_\hbar(C^n_{n+1}))_{\mathrm{nice}}$.

While the computation of the structure constants and the relevant estimates to guarantee $\mathcal{A}_\hbar(C^n_{n+1}) = (\mathcal{A}_\hbar(C^n_{n+1}))_{\mathrm{nice}}$ according to Theorem 2.1 are quite involved, the above first version of our construction is not yet sufficient. The reason is that the evaluation functionals at points $w \in C^+_{n+1}$ are *not* continuous. In fact, one can show that the topology given by Theorem 2.1 on the subalgebra spanned by the basis $\{\mathsf{f}_{P,Q,\alpha}\}_{P,Q,\alpha}$ is just the Cartesian product topology with respect to the index triples. Thus it is much too coarse to interpret the elements of the completion as functions on C^+_{n+1}.

The second version of our general construction provides a way out of this: we enforce the continuity of all the δ-functionals at points in C^+_{n+1} by considering the set $\Omega = \{\delta_w \mid w \in C^+_{n+1}\}$. It is now again a nontrivial estimation to see that the procedure works and yields a reasonable Fréchet topology. Indeed, the result is again very simple. Consider the seminorms

$$\|a\|_\epsilon = \sup_{P,Q,\alpha} |a_{P,Q,\alpha}|/\alpha!^\epsilon \tag{5.4}$$

where the supremum is taken over all index triples and the $a_{P,Q,\alpha}$ are the coefficients of a with respect to the basis $\{\mathsf{f}_{P,Q,\alpha}\}_{P,Q,\alpha}$. Then one has the following result:

Theorem 5.2 *For Ω as above we have $\mathcal{A}_\hbar(C^n_{n+1}) = (\mathcal{A}_\hbar(C^n_{n+1}))_{\Omega\text{-nice}}$ and the induced topology can equivalently be described by the seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$. The completion with respect to this topology yields a Fréchet algebra such that every evaluation functional δ_w for $w \in C^+_{n+1}$ is continuous. The completion is a strongly nuclear Köthe space with $\{\mathsf{f}_{P,Q,\alpha}\}_{P,Q,\alpha}$ being an absolute Schauder basis, consisting of a subalgebra of real-analytic functions on C^+_{n+1}.*

The last step is now to pass to the disk: as before in the formal setting we can simply quotient by the $\tilde{\star}$-ideal generated by the function $y - 1$ which coincides with the vanishing ideal of the constraint surface C inside $\mathcal{A}_\hbar(C^n_{n+1})$. This will only be possible for the allowed values of $\hbar$ according to (5.3). Since the δ-functionals are continuous the ideal is closed and hence we obtain a Fréchet algebra $\mathcal{A}_\hbar(\mathbb{D}_n)$ as quotient. While this works by abstract arguments we also have a more explicit description of the elements of $\mathcal{A}_\hbar(\mathbb{D}_n)$ as functions on the disk $\mathbb{D}_n$.

In fact, we have again a vector space basis becoming a absolute Schauder basis after completion given by the functions

$$\mathsf{f}_{P,Q}(v) = [\mathsf{f}_{P,Q,\alpha}](v) = \frac{1}{P!(\alpha - |P|)!Q!(\alpha - |Q|)!}\left(\frac{1}{2\hbar}\right)_\alpha \frac{v\overline{v}^Q}{(1 - |v|^2)^\alpha} \tag{5.5}$$

where $P, Q \in \mathbb{N}^n_0$ are multiindices, $\alpha = \max(|P|, |Q|)$, and $v = \frac{z}{z^0} \in \mathbb{D}_n$ is a point in the disk corresponding to the complex line $[z]$. These functions form a linearly independent set and we write $[a] = \sum_{P,Q} a_{P,Q}\mathsf{f}_{P,Q}$ for an element in the span of them, where $a \in \mathcal{A}_\hbar(C^n_{n+1})$ is a representative upstairs. Then we define the seminorms

$$\|[a]\|_\epsilon = \sup_{P,Q} \frac{|a_{P,Q}|}{\max(|P|, |Q|)!^\epsilon} \tag{5.6}$$

again controlling a sub-factorial growth of the "Taylor coefficients" $a_{P,Q}$. The final result is then the following:

Theorem 5.3 *The algebra $\mathcal{A}_\hbar(\mathbb{D}_n)$ consists of those real-analytic functions $[a]$ on the disk with $\|[a]\|_\epsilon < \infty$ for all $\epsilon > 0$. It is a strongly nuclear Köthe space with absolute Schauder basis $\{\mathsf{f}_{P,Q}\}_{P,Q}$ and all evaluation functionals at $v \in \mathbb{D}_n$ are continuous.*

Remark 5.4 We list some further properties of the algebra on the disk:

i.) The class of functions we obtain is a subalgebra of those real-analytic functions on $\mathbb{D}_n$ which allow for a holomorphic/anti-holomorphic extension to $\mathbb{D}_n \times \mathbb{D}_n$ and not just to a small open neighbourhood of the diagonal. Then the topology of $\mathcal{A}_\hbar(\mathbb{D}_n)$ is (strictly) finer than the Fréchet topology of locally uniform convergence on $\mathbb{D}_n \times \mathbb{D}_n$ of the extensions.

ii.) The symmetry group $\mathrm{SU}(1, n)$ of the Kähler structure of $\mathbb{D}_n$ acts via pull-backs also on the algebra $\mathcal{A}_\hbar(\mathbb{D}_n)$ by continuous automorphisms. Moreover, the action is smooth and the smooth topology coincides with the original topology of $\mathcal{A}_\hbar(\mathbb{D}_n)$. The classical momentum map for the corresponding Lie algebra action of $\mathfrak{su}(1,n)$ provides also a *quantum momentum map*, i.e the $\star_{\mathbb{D}_n}$-commutators with the components of the classical momentum map generate the $\mathfrak{su}(1,n)$-action.

iii.) If $\hbar$ is real then the complex conjugation provides a continuous *-involution for the algebra $\mathcal{A}_\hbar(\mathbb{D}_n)$. The $\mathrm{SU}(1,n)$-symmetry acts by *-automorphisms in this case.

iv.) Let $\hbar$ be in the set (5.3) of allowed values. Then the Fréchet space $\mathcal{A}_\hbar(\mathbb{D}_n)$ does *not* depend on $\hbar$. This is not true upstairs: here even the vector spaces $\mathcal{A}_\hbar(C^n_{n+1})$ viewed as subspaces of $\mathscr{C}^\infty(C^+_{n+1})$ do depend on $\hbar$. Since the topology of $\mathcal{A}_\hbar(\mathbb{D}_n)$ is independent of $\hbar$, the following statement makes sense: for every two functions $a, b \in \mathcal{A}_\hbar(\mathbb{D}_n)$ the map $\hbar \mapsto a \star_{\mathbb{D}_n} b$ is holomorphic for the allowed values of $\hbar$. Thus we have a holomorphic deformation of the functions of the disk in the sense of [16]. Note however, that the classical case $\hbar = 0$ is *not* in the domain where we have a holomorphic dependence. This shows that the singularities for the classical limit $\hbar \longrightarrow 0$ are much more subtle than in the flat case discussed in Remark 3.2.

v.) For $\hbar > 0$ the evaluation functionals δ_v at points $v \in \mathbb{D}_n$ are continuous positive functionals. Thanks to the transitive action of $\mathrm{SU}(1,n)$ it will be sufficient to consider one of them, say δ_0. The Gel'fand ideal can be computed explicitly. It has a complementary closed subspace

$$\mathfrak{D}_\hbar = \left\{ \psi = \sum\nolimits_Q \psi_Q \mathsf{f}_{0,Q} \;\middle|\; (\psi_Q)_{Q \in \mathbb{N}_0} \text{ has sub-factorial growth} \right\} \quad \subseteq \mathcal{A}_\hbar(\mathbb{D}_n) \tag{5.7}$$

in $\mathcal{A}_\hbar(\mathbb{D}_n)$ which can be identified with the GNS pre-Hilbert space. The inner product as well as the GNS representation can be computed explicitly. It is an ongoing project to identify the corresponding Hilbert space with more familiar Hilbert spaces from coherent states quantizations like in [20].

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