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**Lorenzo Sebastiani and Sergio Zerbini \***

Dipartimento di Fisica, Università di Trento, Via Sommarive 14, 38123 Povo, Italy
\* Correspondence: sergio.zerbini@unitn.it

**Abstract:** A short review of spherically symmetric static regular black holes and spherically symmetric non-singular cosmological space-time is presented. Several models, including new ones, of regular black holes are considered. First, a large class of regular black holes having an inner de Sitter core with the related issue of a Cauchy horizon is investigated. Then, Black Bounce space-times, where the Cauchy horizon and therefore the related instabilities are absent, are discussed as valid alternatives to regular black holes with inner de Sitter cores. Friedman–Lemaître–Robertson–Walker space-times admitting regular bounce solutions are also discussed. In the general analysis concerning the presence or absence of singularities in the equations of motion, the role of a theorem credited to Osgood is stressed.

**Keywords:** regular black holes; non-singular models; Sakarov Criterium

## 1. Introduction

As is well known, fundamental achievements in gravity research have recently been obtained: the first detection of gravitational waves from binary systems of black holes (BHs) and the "multimessenger" signals obtained from the first observation of the collision of two relativistic neutron stars [1–4]. Furthermore, the images of BH shadows by the Event Horizon Telescope have been reproduced (see [5] and references therein) with further important information about the nature of these astrophysical compact objects. Other important analyses on ultra-compact stars can be found in [6–10].

The main result of these studies is that the Kerr nature of these compact objects is confirmed within a small uncertainty. Since the Kerr BH solution is a mathematical vacuum solution of General Relativity (GR), one may conclude that it might be a good, although approximate, description of reality, and the possible existence of other compact objects with horizons may not be ruled out. We should also stress that many conceptual problems still exist regarding the nature of Kerr BHs, namely, the central singularity problem and so on. In fact, Kerr BH is the unique vacuum solution of GR and admits a non-physical singularity. This is a consequence of the GR singularity theorems.

Thus, one needs alternative "stuffs" to avoid the problem: for example, regular black holes associated with exotic sources in order to bypass GR singularity theorems. We also should note that regular BHs exist in a vacuum (absence of matter) if one goes beyond GR in a modified gravity framework.

With regard to this issue, maybe it is necessary to clarify that in our paper, non-singular space-times are space-times free of curvature divergences. We remind that in GR, the singularity theorems state that the presence of geodesic incompleteness is associated with the appearance of physical singularities. For us, the presence of severe pathologies in the curvature invariants are a sufficient condition to regard the space-times as singular, being aware that geodesic incompleteness and curvature singularities might be, in some cases, not equivalent concepts.

In our paper, we discuss several aspects of regular black holes, namely, BHs where the central singularity is absent due to the presence of a de Sitter (dS) core or the presence of a fundamental length.

With regard to this, we first analyze a class of solutions with a delta-regularized source. These solutions are interesting since they present Schwarzschild-like behaviour at large distances, and for suitable choices of parameters, the horizons (event and Cauchy horizons) may be absent. The related metrics are asymptotically flat. In the BH case, however, the regular behaviour of the metric is not a sufficient condition to avoid singularities in the curvature invariants and their derivatives. Thus, we discuss the Sakarov criterion, which allows us to identify a restricted class of regular BH solutions for which the scalar invariants and their covariant derivatives are everywhere bounded.

Furthermore, in general in the BH case, the presence of a dS core also brings the existence of an inner Cauchy horizon, which may lead to some instability. In this respect, a class of alternative BHs is given by Black Bounce space-times, metrics which are asymptotically flat and for which the Cauchy horizon is absent. For these metrics, the central singularity is absent thanks to the introduction of a minimal length scale in the metric components.

We also show how various BHs can be recovered in the framework of regularized Lovelock Lagrangians, a class of Lagrangians that have been an object of recent interest.

Finally, we give attention to non-singular cosmological models, investigating the absence of singularities in cosmology with some general considerations of the proprieties of field equations of the theories under investigation.

The paper is organized as follows. In Section 2, we revisit the formalism of Spherically Symmetric (SS) space-time and focus our attention on static black hole and wormhole (WH) solutions. In Section 3, we discuss a special class of black holes with an inner de Sitter core in the presence of a delta-like regularized source, and some applications of the covariant Sakarov Criterion. In Section 4, alternative black holes described by Black Bounce space-time are analysed as valid alternatives to non-singular black holes without Cauchy horizons. In Section 5, black holes in four-dimensional regularized Lovelock Lagrangians are discussed, while Section 6 is devoted to cosmological models free of space-time singularity. Conclusions and final remarks are given in Section 7, and Appendix A reports additional material on Painlevè gauge and its application to Hawking radiation regarding generic static BHs and WHs.

In our convention, the speed of light $c = 1$ and the Newton Constant $G_N = 1$; thus, the Planck mass is $M_{Pl}^2 = 1/8\pi$. We also adopt the "mostly plus" metric convention.

## 2. Black Holes and Wormholes in Static Spherical Symmetric Space-Time

Here, we recall the Kodama–Hayward invariant formalism (see, for example, [11–14]) for a generic four-dimensional Spherical Symmetric Space-time (SSS). The related metric reads

$$ds^2 = \gamma_{ab}dx^a dx^b + r(x^a)^2 dS^2 \,, \tag{1}$$

where $dS^2$ is the metric of a two-dimensional sphere, $\gamma_{ab}$ is the metric tensor of the two-dimensional space-time (the normal metric) with coordinates $x^a$, $a = 0, 1$, and $r \equiv r(x^a)$ is the areal radius and is a scalar quantity that depends on the coordinates of the normal space-time. Another relevant scalar, related to the variation of a surface with radius $r$, is given by

$$\chi(x^a) = \gamma^{ab}\partial_a r \partial_b r \,, \tag{2}$$

which defines a (dynamical) trapping horizon by

$$\chi(x_H) = 0 \,, \quad \partial_a \chi(x_H) > 0 \,, \quad a = 0, 1 \,. \tag{3}$$

The second condition is required in order to preserve the metric signature out of the horizon. A related scalar quantity is the Hayward surface gravity,

$$\kappa_H = \frac{1}{2\sqrt{-\gamma}}\partial_a\left(\sqrt{-\gamma}\gamma^{ab}\partial_b r\right)_H, \tag{4}$$

where $\gamma$ is the determinant of the two-metric $\gamma_{ab}$, and the pedex '$H$' denotes a quantity evaluated on the horizon.

Finally, the Kodama vector is

$$K^a = \frac{\epsilon^{ab}}{\sqrt{-\gamma}}\partial_a r, \tag{5}$$

where $\epsilon^{ab}$ is the volume-form associated with the two-metric $\gamma_{ab}$. The Kodama vector is orthogonal to the normal space-time and is covariant conserved in a generic SS space-time, namely

$$\nabla_\mu K^\mu = 0. \tag{6}$$

Note that $\sqrt{-\gamma}$ has to be well-defined such that $-\gamma > 0$.

### 2.1. The Static Case

The Static SSS case is well understood and investigated. In the Schwarzschild gauge, the metric reads

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 dS^2, \tag{7}$$

where $A \equiv A(r)$ and $B \equiv B(r)$ are metric functions of the radial coordinate $r$ only. In what follows, we denote with the prime index the derivative with respect to $r$.

Note that the determinant of the metric $g$ is given by $\sqrt{-g} = \sqrt{\frac{A(r)}{B(r)}}r^2$, such that $\frac{A(r)}{B(r)} > 0$. The invariant quantity $\chi \equiv \chi(x^a)$ and the Kodama vector are well defined and read

$$\chi = B(r), \quad K^\mu = \left(\sqrt{\frac{B(r)}{A(r)}}, 0, 0, 0\right). \tag{8}$$

The Kodama energy $\omega$ associated with a test particle with four-momentum $p_\mu = \partial_\mu I$, $I$ being the action is given by

$$\omega = -K^\mu p_\mu = \sqrt{\frac{B}{A}}E, \tag{9}$$

where $E = -\partial_t I$ is the test-particle Killing energy.

Black holes and wormholes possess a trapping (event) horizon with $\chi_H = 0$. If $A(r_H) = B(r_H) = 0$, one has a BH. If $B(r_H) = 0$, but $A(r_H) \neq 0$, one has to deal with a WH [15–24]. If $\chi$ is never vanishing, one obtains an horizonless compact object (HCO).

The Kodama vector may provide an invariant way to distinguish between BHs and WHs [25]. In the Schwarzschild static gauge, the trapping horizon is defined by

$$B(r_H) = 0. \tag{10}$$

Thus, we may define a static black hole as a SSS solution where the Kodama energy in (9) evaluated on the horizon is not vanishing (see also Appendix A for a derivation of the Hawking temperature). This is the case for a black hole where $A(r)$ is proportional to $B(r)$, and $A(r_H) = 0$, $A'(r_H) > 0$ (event horizon).

The Hayward surface gravity associated with a trapping horizon is

$$\kappa_H = \frac{B'(r_H)}{2}. \tag{11}$$

In the static case, one can define a time-like Killing vector field with an associated Killing surface gravity,

$$\kappa_H^K = \frac{\sqrt{B'(r_H)A'(r_H)}}{2} . \tag{12}$$

In general, the Killing surface gravity differs from the Hayward surface gravity due to a different renormalization of the Killing vector with respect to the Kodama vector. However, for most known BHs, $A(r) = B(r)$, and they coincide.

We recall that if $A(r) \neq 0$, in particular on the horizon, one has a static regular wormhole a la Morris–Thorne. Thus, $\omega_H$ is vanishing. The Hayward surface gravity

$$\kappa_H = \frac{B'(r_H)}{4} , \tag{13}$$

implies a minimum value $r_H$ for $r$, the throat or mouth of a traversable wormhole ($\kappa_H > 0$), such that $r > r_H$. Moreover, if $A(r)$ is vanishing at some point, one has singular WHs. A well-known example is the Brans–Dicke WH [26–33].

Another interesting example is given by a variant of the Damour and Solodhukin metric [34],

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m+b^2 r}{r}\right)} + r^2 dS^2 , \tag{14}$$

where $m$ is a mass parameter, and $b^2$ is a dimensionless arbitrarily small parameter. Here, $r_H = \frac{2m}{1-b^2}$, but $A(r_H) = b^2 > 0$, and the Kodama energy vanishes on the throat, namely, $\omega_H = 0$, and we are describing a static wormhole. Since $r > r_H$, the singularity $r = 0$ is not present.

Static WHs are non-singular objects, but in General Relativity, they can be obtained only in the presence of a source of exotic matter that violates the energy conditions, and they are generally unstable (see also the recent work in [35], where vacuum WH solutions are found in the presence of trace anomaly contributions).

## 2.2. Effective Fluid Models in General Relativity

The simplest way to go beyond GR is to consider Einstein's equations in the presence of effective relativistic anisotropic fluids. Einstein's equations are

$$G^{\mu\nu} = 8\pi T^{\mu\nu} , \tag{15}$$

with

$$T^{\mu\nu} = (\rho + p_T)u^\mu u^\nu + p_T g_{\mu\nu} + (p_r - p_T)C^\mu C^\nu , \tag{16}$$

where $u^\mu u_\mu = -1$ is a time-like vector, and $C^\mu C_\mu = 1$ is the anisotropy space-like vector. Moreover, $\rho$ is the energy density of fluid, $p_r$ is the radial pressure, and $p_T$ is the trasversal pressure. Equations of motion read

$$rB' + B - 1 = -8\pi r^2 \rho , \tag{17}$$

$$\frac{A'}{A} - \frac{B'}{B} = \frac{8\pi r(\rho + p_r)}{B} . \tag{18}$$

The Tolman–Oppenheimer–Volkov equation leads to

$$p_r' + \frac{\rho + p_r}{2}\frac{A'}{A} = \frac{2(p_T - p_r)}{r} . \tag{19}$$

In principle, given the energy density $\rho$, the metric function $B(r)$ is computable. Chosing the fluid Equation of State (EoS) $p_r = p_r(\rho)$, the metric function $A(r)$ is also computable. Finally, the Tolman–Oppenheimer–Volkov equation gives the form of $p_T$. Alternatively,

choosing $A(r)$ and $B(r)$, $\rho, p_r, p_T$ are computable, and we can reconstruct the fluid that supports a given SSS metric.

### 3. Regular Black Hole Solutions with Inner de Sitter Core

Since space-time singularities are very problematic to treat, a lot of investigations have been carried out on non-singular black hole solutions, where the central singularity that characterizes the Schwarzschild metric is removed and substituted with a De Sitter core. The resulting solution is a non-vacuum solution of Einstein's equations. Explicit examples have been provided by Bardeen, Hayward and many others (for a partial list see [36–54] and references therein). Other related results can be found in [55–57].

Since on a BH horizon, the conditions $A(r) = 0$ and $B(r) = 0$ must be satisfied for the same value of $r = r_H$, a direct consequence of Equations (17) and (18) is that $p_H = -\rho_H$. Thus, a simple way to obtain non-vacuum BH solutions is given by the generalization of this condition, namely, one assumes the EoS $p = -\rho$ such that $A(r) = B(r)$. Furthermore, it is convenient to write the metric functions in the form

$$A(r) = B(r) = 1 - \frac{2m(r)}{r} , \tag{20}$$

where $m \equiv m(r)$ is a mass function depending on $r$. For $m(r) = m$, one gets the Schwarzschild solution. The Ricci scalar reads

$$R = \frac{2}{r^2}(1 - A(r)) - 4\frac{A'(r)}{r} - A''(r) = \frac{4m'(r)}{r^2} + 2\frac{m''(r)}{r} , \tag{21}$$

while the other curvature invariants are

$$R_{\mu\nu}R^{\mu\nu} = 8\frac{m'(r)^2}{r^4} + 2\frac{m''(r)^2}{r^2} , \tag{22}$$

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 48\frac{m(r)^2}{r^6} + O\left(\frac{m''(r)^2}{r^2}, \frac{m'(r)}{r^2}\right) . \tag{23}$$

As a result, for $r \to 0$, in order to avoid the central singularity, one has to take $m(r) = r^3C + O(r^4)$, $m'(r) = 3r^2C + O(r^3)$, $m''(r) = 6rC + O(r^2)$, and $C$ being a constant, and the curvature invariants are finite at $r = 0$. In this way, one gets small values of $r$,

$$A(r) = 1 - 2Cr^2 + \dots \tag{24}$$

This is the so-called Sakarov Criterion: an interior de Sitter core leads to a BH with finite curvature invariants. Related to this is the Limiting Curvature Conjecture (see [58–60] and references therein).

A very popular example is given by a Poisson–Israel–Hayward BH [38]:

$$A(r) = B(r) = 1 - \frac{2mr^2}{r^3 + \ell^3} , \tag{25}$$

and

$$A(r) = B(r) = 1 - \frac{2mr^2}{r^3 + \ell^2 m} , \tag{26}$$

where $m$ and $\ell$ are constants. The metric exhibits a de Sitter core for small values of $r$, while for large values of $r$, one recovers the behaviour of Schwarzschild space-time. In order to deal with a BH, the mass $m$ has to satisfy the inequality $m > m_c = 3^{3/2}\ell/4$. For $m < m_c$, there is not a horizon, and one has a simple example of a horizonless compact object (HCO), very similar to the Gravastar of Mazur–Mottola, which is characterized by a dS core with a stiff matter shell (interior) and Schwarschild behaviour at large distances (exterior).

Unfortunately, solutions with $A(r) = B(r)$ and a dS core suffer two problems. The first one is that the square of the radial speed of sound $v_s^2$ is negative, a possible signal of instability. This is a direct consequence of the fact that $p_r = -\rho$. Thus, $v_s^2 = \frac{dp_r}{d\rho} = -1 < 0$.

The second problem is related to the presence of an inner Cauchy horizon with associated instability related to mass inflation [61] and kink instability [62]. Mass inflation is the result of the exponential growth of the mass parameter of the solution due to the crossflow of infalling and outgoing radiation perturbations, and it is strictly connected to the presence of a Cauchy horizon (see the recent papers and references therein [63,64]). Moreover, regarding the mass inflation issue, see also [65–67], and [68] for quadratic gravity.

Very recently, Visser and co-workers [52] constructed a regular BH with a dS inner core that was asymptotically flat, working in the Schwarzschild gauge (7) with metric functions $A(r) = B(r)$ such that

$$A(r) = \frac{(r - r_H)(r - r_C)c^3}{N(r)},\tag{27}$$

with $r_C < r_H$, and $N(r)$ a suitable function such that there exists a dS core, and $A(r) = 1 - \frac{2M}{r} + \dots$ for large values of $r$. There are two horizons, the event horizon at $r = r_H$ and a Cauchy horizon at $r = r_C$, which is, however, a triple zero of $A(r)$. As a consequence, the surface gravity associated with it is vanishing and the mass inflation is absent. This result has some similarities with the absence of mass inflation in higher-order derivative gravity obtained in [68].

### 3.1. Regular Black Holes with Delta-like Regularized Sources

Among regular BHs with de Sitter cores, there exists a specific class we are going to discuss. The main idea, well-justified by physical considerations, is to deal with densities that are related to regularization of Dirac delta distribution in spatial dimension $D = 3$.

With regard to this issue, we recall that if we have an even integrable positive function, namely $f(-\vec{x}) = f(\vec{x})$, $f(\vec{x}) > 0$, and with

$$\int_{R^3} d\vec{x} f(\vec{x}) = C < \infty,\tag{28}$$

then,

$$\lim_{\epsilon \to 0} \frac{1}{C\epsilon^3} f\left(\frac{\vec{x}}{\epsilon}\right) = \delta(\vec{x}).\tag{29}$$

Let us use the Schwarzschild gauge (7) and now take the simple EoS $p = -\rho$ such that $A(r) = B(r)$, as previously discussed. We assume the form (20) for our metric functions. Moreover, we take as density

$$\rho = \frac{M}{C\epsilon^3} f\left(\frac{r}{\epsilon}\right).\tag{30}$$

Here $f(r)$ is an even integrable positive function which satisfies (28), $M$ is a mass parameter, and $\epsilon$ is the regularization parameter, such that $\rho$ is a delta-like regularized energy density source.

By using (17), the mass function results are defined as

$$m(r) = 4\pi \int_0^r dy y^2 \rho(y).\tag{31}$$

Now it is convenient to introduce a new function $g(r)$, dubbed "g-function", defined by $2m(r) = r^3 g(r)$. Then, one has

$$g(r) = \frac{8\pi}{r^3} \int_0^r dy y^2 \rho(y), \quad A(r) = B(r) = 1 - r^2 g(r).\tag{32}$$

Therefore, the g-function associated with a delta-regularized density

$$g(r) = \frac{8\pi}{r^3} \int_0^r dy y^2 \frac{M}{C\varepsilon^3} f\left(\frac{y}{\varepsilon}\right),$$
(33)

namely,

$$g(r) = \frac{8\pi M}{Cr^3} \int_0^{r/\varepsilon} dy y^2 f(y).$$
(34)

It is easy to show that one recovers the Schwarzschild limit for large values of $r$, namely, $r^2 g_G = \frac{2M}{r} + O(1/r)$. In order to obtain a (hyper) dS core for a small $r$, one has to assume smooth behaviour for $f(r)$ at $r = 0$, namely, $f(r) \simeq r^a(c_0 + c_1 r + \ldots)$, $a > 0$ and $c_{0,1,\ldots}$ are constants. Thus,

$$g(r) \simeq \frac{8\pi M r^a}{(3+a)C\varepsilon^{3+a}}(c_0 + O(r)).$$
(35)

For $a = 0$, one has a dS core. Moreover, since $A(r = 0) = 1$ and $A(r \to \infty) = 1$, an even number of zeros exist. For example, according to the values of parameters $\varepsilon$, $a$ and $M$, there are two horizons or no horizons. Let us see some examples.

As a first example, one may take the family of the well-known Gaussian delta-like generating functions $f(\vec{x}) = |\vec{x}|^a e^{-\vec{x}^2}$ with $a$ a non-negative real number. Then,

$$C_a \equiv \int_0^\infty r^{a+2} e^{-r^2} dr = 2\pi\Gamma\left(\frac{3+a}{2}\right),$$
(36)

where $\Gamma(\alpha)$ is the Gamma function, the related energy density is well-defined, and there is a dS core,

$$\rho_G = \frac{M}{C_a \varepsilon^3}\left(\frac{r}{\varepsilon}\right)^a e^{-\frac{r^2}{\varepsilon^2}}.$$
(37)

In fact, the associated g-function reads

$$g_G(r) = \frac{2M}{\Gamma\left(\frac{3+a}{2}\right)r^3}\gamma\left(\frac{3+a}{2}, \frac{r^2}{\varepsilon^2}\right),$$
(38)

where now $\gamma(\alpha, z)$ is the (lower) incomplete Gamma function. For example, for $a = 1$, one simply obtains

$$g_G(r) = \frac{M}{r^3}\left(1 - e^{-\frac{r^2}{\varepsilon^2}}\left(1 + \frac{r^2}{\varepsilon^2}\right)\right).$$
(39)

We can check the presence of a dS core for small $r$ when $g_G(r)$ in (38) assumes the form

$$g_G(r) \simeq \frac{2M}{\Gamma\left(\frac{3+a}{2}\right)}\frac{r^a}{\varepsilon^{a+3}}\left(c_o + c_1\frac{r^2}{\varepsilon^2} + c_2 + \frac{r^4}{\varepsilon^4} + \ldots\right),$$
(40)

with $c_{0,1,2,\ldots}$ constant coefficients depending on $a$. In the case of $a = 1$, one has

$$g_G(r) \simeq \frac{Mr}{\varepsilon^4}.$$
(41)

For $a = 0$, the related BH has been discussed in [41].

Another class of regular BHs may be constructed by making use of the other well-known delta-generating function, the Lorentzian, namely, $f(\vec{x}) = \frac{|\vec{x}|^a}{(x^2+1)^{N/2}}$, where $a$ is a real number, and $N$ is a natural number. The integrability in $D = 3$ requires $N > 3 + a > 0$, and we take $a > 0$. The normalization constant results in

$$C_N = 4\pi \int_0^\infty \frac{r^{2+a}}{(r^2+1)^{N/2}} dr = 2\pi\Gamma\left(\frac{3+a}{2}\right)\frac{\Gamma\left(\frac{N-3-a}{3}\right)}{\Gamma(N/2)}.$$
(42)

The associated energy density is

$$\rho_N = \frac{M}{C_N} \frac{\varepsilon^{N-3-a}}{(r^2 + \varepsilon^2)^{N/2}}. \tag{43}$$

The related g-function reads

$$g_N(r) = \frac{8\pi M}{C_N r^3} \int_0^{r/\varepsilon} \frac{y^{2+a}}{(y^2+1)^{N/2}} dy = \frac{8\pi M \varepsilon^{-a-3}}{c_N(3+a)} r^a F\left(\frac{3+a}{2}, \frac{N}{2}; \frac{5+a}{2}, -\frac{r^2}{\varepsilon^2}\right), \tag{44}$$

where $F(\alpha, \beta; \gamma, z)$ is the Gauss hypergeometric function. For general $N > 3 + a$, it is easy to show that for a small $r$, there is a dS core,

$$g_N(r) \simeq \frac{8\pi M \varepsilon^{-a-3}}{C_N(3+a)} r^a \left(1 - \frac{(3+a)N}{2a+10} \frac{r^2}{\varepsilon^2} + O(r^4)\right). \tag{45}$$

On the other hand, for large values of $r$, one has $r^2 g_N(r) = \frac{2M}{r} + \ldots$, and $M$ can be identified with the BH mass.

Let us consider some specific values for $N$ and $a$. We start with $a = 0, N = 4$, such that

$$g_4(r) = \frac{4M}{\pi r^3} \left(\arctan\left(\frac{r}{\varepsilon}\right) - \frac{\varepsilon r}{r^2 + \varepsilon^2}\right). \tag{46}$$

This corresponds to a BH discussed by Dymnikova [40].

The next example is $N = 5, a = 0$. With this choice, one has

$$g_5(r) = \frac{M}{(r^2 + \varepsilon^2)^{3/2}}. \tag{47}$$

This corresponds to the well-known Bardeen BH [36].

Of course, one may continue, and for $N = 6, a = 0$, we get

$$g_6(r) = \frac{4M}{\pi r^3} \left(\arctan\frac{r}{\varepsilon} - \frac{\varepsilon r(r^2 - \varepsilon^2)}{(r^2 + \varepsilon^2)^2}\right). \tag{48}$$

The cases $N = 7, 8, 9, a = 0$ present no difficulties.

We conclude this subsection with two other examples. The first example is related to the following choice:

$$f(\vec{x}) = \frac{3}{(1 + |\vec{x}|^3)^2}. \tag{49}$$

The normalization constant is $C = 4\pi$, and the g-function turns out to be

$$g(r) = \frac{24\pi M}{C r^3} \int_0^{r/\varepsilon} dy \frac{y^2}{(1 + y^3)^2}. \tag{50}$$

Thus,

$$g(r) = \frac{2M}{r^3 + \varepsilon^3}, \tag{51}$$

which corresponds to Poisson–Israel–Hayward BH in (25) and (26).

For the second example, we take

$$f(\vec{x}) = \frac{3}{(1 + |\vec{x}|)^4}. \tag{52}$$

The normalization constant is $C = 4\pi$, and the g-function reads

$$g(r) = \frac{24\pi M}{Cr^3} \int_0^{r/\varepsilon} dy \frac{y^2}{(1+y)^4} \, . \tag{53}$$

Thus,

$$g(r) = \frac{2M}{(r+\varepsilon)^3} \, , \tag{54}$$

associated with a regular BH investigated by Fan and Wang [51].

With respect to the validity and violation of the several energy conditions associated with these regular BHs, a detailed discussion can be found in a recent paper by Maeda [69].

*3.2. GR Coupled with Non-Linear Electrodynamics*

For sake of completeness, in the following, we shortly describe regular BH solutions with an inner dS core that may be obtained coupling GR with Non-Linear Electrodynamics (NLE) [37]. We follow [48]. However, for a recent critical discussion concerning this approach, see [70], in which further references can be found.

The NED gravitational model is based on the following action:

$$S = \int d^4x \, \sqrt{-g} \left( \frac{R}{2} - 2\Lambda - \mathscr{L}(I) \right) , \tag{55}$$

where $R$ is the Ricci scalar, $\Lambda$ is a cosmological constant, and $I = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is an electromagnetic-like tensor, with $\mathscr{L}(I)$ a suitable function of it. Recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will only deal with gauge invariant quantities, and we put $\Lambda = 0$ because its contribution can be easily restored. The equations of motion read

$$G_\mu^\nu = -F_{\alpha\nu} \partial \mathscr{L} F^{\mu\alpha} + \mathscr{L} \delta_\mu^\nu \tag{56}$$

$$\nabla^\mu (F_{\mu\nu} \partial_I \mathscr{L}) = 0 \, . \tag{57}$$

Another equivalent approach is called the dual $P$ approach, and it is based on two new gauge invariant quantities

$$P_{\mu\nu} \equiv F_{\mu\nu}(\partial_I \mathscr{L}(I)) \, , P \equiv \frac{1}{4} P_{\mu\nu} P^{\mu\nu} \, , \tag{58}$$

and

$$\mathscr{H} \equiv 2I(\partial_I \mathscr{L}(I)) - \mathscr{L}(I) \, , \tag{59}$$

$$\nabla^\mu P_{\mu\nu} = 0 \, . \tag{60}$$

In the following, we make use only of the traditional approach based on Equations (56) and (57).

Within the static spherically symmetric ansatz and from (57), one has

$$\partial_r \left( r^2 \partial_I \mathscr{L} F^{0r} \right) = 0 \, , \tag{61}$$

Since $I = \frac{1}{2} F_{0r} F^{0r} = -\frac{1}{2} F_{0r}^2$, one gets

$$r^2 \partial_I \mathscr{L} = \frac{Q}{\sqrt{-2I}} \, , \tag{62}$$

where $Q$ is a constant of integration. As a result, within this NED approach, one may solve the generalized Maxwell equation. We make use of this equation and the the $(t,t)$ component of the Einstein equation, which reads

$$G_t^t = \frac{rf' + f - 1}{r^2} = 8\pi(-2I\partial_I \mathscr{L} + \mathscr{L}) = -8\pi\rho \, . \tag{63}$$

Introducing the more convenient quantity $X$

$$X = Q\sqrt{-2I}, \tag{64}$$

one may rewrite Equation (62) as

$$r^2 \partial_X \mathcal{L} = 1. \tag{65}$$

Furthermore, we have

$$\rho = X \partial_X \mathcal{L} - \mathcal{L} = \frac{X}{r^2} - \mathcal{L}. \tag{66}$$

Thus, when $\mathcal{L}(X)$ is given, then, making use of (62), one may obtain $\rho = \rho(r)$.

As an example of this direct approach, let us investigate the following class of NED models

$$\mathcal{L}(X) = \frac{1}{\alpha \gamma}\left((1 - \frac{\gamma}{2}X^2)^\alpha - 1\right), \quad \partial_X \mathcal{L} = -X(1 - \frac{\gamma}{2}X^2)^{\alpha-1}. \tag{67}$$

in which $\alpha$ and $\gamma$ are two parameters. Note that for small $\gamma$, one has the Maxwell Lagrangian.

Let us show that for $\gamma$ not vanishing, we may obtained specific exact black hole solutions. In fact, from (67), one gets

$$(1 - \frac{\gamma}{2}X^2)^{-2\alpha+2} = r^4 X^2. \tag{68}$$

Again, for $\alpha = 1$, one has the usual Maxwell case. Thus, we consider $\alpha \neq 1$.

The choice $\alpha = \frac{1}{2}$ leads the to well-known Einstein–Born–Infeld case. With this choice, one has

$$X^2 = \frac{1}{r^4 + \frac{\gamma}{2}}. \tag{69}$$

This means that the static electric field is regular at $r = 0$, similar to the Born–Infeld model. Furthermore, since

$$\mathcal{L} = -\frac{2}{\gamma}\left(1 + r^2 X\right), \tag{70}$$

the effective density reads

$$r^2 \rho(r) = \frac{2r^2}{\gamma} + \frac{2\sqrt{r^4 + \frac{\gamma}{2}}}{\gamma}. \tag{71}$$

In order for this object to satisfy the Weak Energy Condition (WEC), it is necessary to require $\gamma > 0$, since $\rho$ is ill-defined in the limit $\gamma \to 0$, and no solution associated with a vanishing electromagnetic field might exist. However, since $\gamma$ is an external parameter and not an integration constant, there is no trouble fixing it to be positive, so that the Lagrangian (67) satisfies the WEC.

When $\gamma > 0$, the solution is

$$A(r) = 1 - \frac{8\pi}{r}\int_0^r r_1^2 \rho(r_1) dr_1, \tag{72}$$

and may be expressed in terms of an Elliptic function, but it is easy to show there is no strictly de Sitter core for $r \to 0$

$$\lim_{r \to 0} r^2 \rho(r) = \sqrt{2/\gamma} + \frac{2r^2}{\gamma} + O(r^4). \tag{73}$$

The presence of the non-vanishing constant $\sqrt{2/\gamma}$ means that a conical singularity is present.

As a last example, let us consider the generalized Maxwell Lagrangian

$$\mathcal{L}(X) = -\frac{1}{\xi^2}\left(\sqrt{b} - \sqrt{X}\right)^2, \quad b = \frac{a}{4\pi\xi^2}, \tag{74}$$

with $a$ and $\xi$ given parameters. One easily gets

$$\sqrt{X} = \frac{\sqrt{b}\,r^2}{r^2 + \xi^2}, \quad \rho = \frac{b}{r^2 + \xi^2}. \tag{75}$$

Here, the WEC is satisfied as long as $b$ is positive. However, in this case also, $b$ is just an external parameter; its being positive rests on the condition $a \geq 0$, but this is a safe condition, since we are allowed to impose it into the Lagrangian.

For this model, one has no conical singularity in the origin, namely, a de Sitter core, and the particular solution reads

$$A(r) = 1 - \frac{2a}{\xi^2} + \frac{2a}{\xi r}\arctan\left(\frac{r}{\xi}\right). \tag{76}$$

However, this solution is not asymptotically Minkoskian.

### 3.3. The Covariant Sakarov Criterion

Let us work in a generic SS space-time. We have already remarked that the areal radius $r$ is a scalar quantity, as well as $\chi = \gamma^{ab}\partial_a r \partial_b r$ in (2). Thus, one may introduce another invariant,

$$Z = \frac{1 - \chi}{r^2}. \tag{77}$$

In [48], the authors propose a so-called Sakarov (covariant) Criterion, which states that a sufficient condition to deal with a generic non-singular SS is to assume $Z$ and its covariant derivatives are uniformly bounded everywhere. In the static case for small values of $r$ and in the the Schwarzschild gauge with $A(r) = B(r)$, this leads to $A(r) = 1 + cr^2 + \ldots$, namely, to the existence of a dS core. We have already observed that this condition renders the curvature invariants finite at $r = 0$.

We observe that in SSS space-time where $\chi = B(r)$, $Z$ is nothing other than the g-function (32). Then, by assuming a power series expansion in $r$, we get

$$Z(r) = g(r) = \sum_{0}^{\infty} g_n r^n = g_0 + g_1 r + g_2 r^2 + g_3 r^3 + \ldots. \tag{78}$$

For the sake of simplicity, we consider here only the presence of a dS core, namely, $g_0 \neq 0$. Furthermore, in order to have an asymptotically flat SSS, for large $r$, we know one has to assume

$$g(r \to \infty) = \frac{2M}{r^3}. \tag{79}$$

One may further test the regularity of the SSS solution, checking the regularity of other invariants associated with $Z$. All the invariants built with the contractions of the four-vector $\nabla_\mu Z$ must be regular at $r = 0$. Let us consider the invariant $Z_1 = g^{\mu\nu}\nabla_\mu\nabla_\nu Z$, namely, the d'Alambertian of $Z$. One has (here, $A(r) = B(r)$)

$$Z_1 = \frac{2g'(r)}{r} - 2rg(r)g'(r) + A'(r)g'(r) + A(r)g''(r) = \frac{2g'(r)}{r} + Z_{12}. \tag{80}$$

Here $Z_{12}$ is a smooth function of $r$. The first term reads

$$\frac{2g'(r)}{r} = \frac{2g_1}{r} + 4g_2 + 6g_3 r + \ldots \tag{81}$$

As a result, one has to deal with $g_1 = 0$; thus, the scalar $Z_1$ is uniformly bounded for every $r$.

We can continue, considering the invariant $Z_2 = g^{\mu\nu}\nabla_\mu\nabla_\nu Z_1$, and we get

$$Z_2 = -\frac{2g'(r)}{r} + Z_{22}\,.\tag{82}$$

Here $Z_{22}$ is a smooth function of $r$, while the first term reads

$$-\frac{2g'(r)}{r} = 6g_3 r + O(r)\,.\tag{83}$$

As a result, if we require the scalar $Z_2$ to be uniformly bounded for every $r$, one should assume

$$g_3 = 0\,.\tag{84}$$

Continuing in this way, requiring $Z_n = (g^{\mu\nu}\nabla_\mu\nabla_\nu)^n Z$ to be uniformly bounded for every $r$, it follows that $g_{2n+1} = 0$, namely, $g(-r) = g(r)$. This result is in agreement with the one obtained recently in [71]. Thus, a BH admitting a dS core is regular only if the g-function is an even function of $r$.

Let us consider some explicit examples. The well-known Hayward BH (26) does not fulfil this requirement, as has also been stressed in [71], where other BH examples have been investigated.

In our example of delta-like regularized BHs (38), one has to take $a$ as an even number (see (40)). Thus, the relate g-functions are even functions in $r$. For example, Bardeen and Dynmikova BHs belong to this class of regular BHs.

Finally, we mention the class of regular BHs, the one with an inner Minkowsky core, recently re-proposed by Simpson and Visser [72–76]. In the Schwarzschild gauge, the metric reads

$$ds^2 = -\left(1 - \frac{2me^{-\ell/r}}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2me^{-\ell/r}}{r}\right)} + r^2 dS^2\,,\tag{85}$$

where $\ell$, $m$ are constants. The g-function associated with this class reads

$$g(r) = \frac{2M}{r^3}e^{-\frac{\ell}{r}}\,.\tag{86}$$

When $r \to 0$, all the curvature invariants vanish.

One might think that the further requirement we have stressed to adopt in order to deal with a regular BH having an inner dS core, namely, $g(-r) = g(r)$, depends on the fact we are working in the Schwarschild gauge. In fact, for example, performing the coordinate change $r = \sigma^2$, one obtains

$$ds^2 = -A(\sigma^2)dt^2 + \frac{4\sigma^2 d\sigma^2}{A(\sigma^2)} + \sigma^4 dS^2\,.\tag{87}$$

In this way, all the components of the metric are always even functions of $\sigma$, and also the invariant $Z = g(\sigma^2)$. However, if $g(r)$ is not an even function, the divergences in the invariant are still present.

At this point, we may take an example, namely, the BH in [51] with

$$A(r) = B(r) = 1 - \frac{r^2}{(r+\varepsilon)^3}\,,\tag{88}$$

such that

$$g(r) = \frac{1}{(r+\varepsilon)^3}\,,\tag{89}$$

is not an even function, and we expect divergence in $Z_1$.

After the identification $r = \sigma^2$, we have

$$g(\sigma^2) = \frac{1}{(\sigma^2 + \varepsilon)^3} \, . \tag{90}$$

Now, if we compute the invariant $Z_1 = g^{\mu\nu}\nabla_\mu\nabla_\nu Z$, we derive

$$Z_1 = \frac{1}{4\sigma^2}\left(3\frac{g'(\sigma^2)}{\sigma} + g''(\sigma^2)\right) + O(\sigma) \, . \tag{91}$$

Therefore,

$$Z_1 = -\frac{6}{\sigma^2}\frac{1}{\varepsilon^4} + O(\sigma) \, , \tag{92}$$

and we see that the result in the Schwarschild gauge is still valid: if $g(r)$ is not even function, there exists a divergent invariant when $r \to 0$.

## 4. Alternative Regular Black Holes: Black Bounce Space-Times

These regular space-times have been dubbed "Black Bounce space-times" by Visser et al. [77–79]. In the examples of static metrics we are going to discuss, no interior dS core is present, and the central singularity is avoided thanks to the introduction of a minimal length scale.

The starting point is the following metric

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)}\frac{1}{(1 - f_\ell(r))} + r^2 d\Omega^2 \, . \tag{93}$$

with $f_\ell(r)$ a positive function of $r$ such that $f_\ell(r) \to 0$ for $\ell \to 0$, and $f_\ell(r) \to 0$ for $r \to \infty$. Therefore, $\ell$ is a small length parameter, for example, on the order of the Planck length.

Since in the Schwarzschild gauge (7), the metric function $B(r)$ is given by

$$B(r) = A(r)(1 - f_\ell(r)) \, , \tag{94}$$

it follows that the range of $r$ is restricted by the condition $\frac{A(r)}{B(r)} > 0$, namely, $0 < f_\ell(r) < 1$. The possible horizons satisfy

$$A(r)(1 - f_\ell(r)) = 0 \, , \tag{95}$$

and are located at $r = r_H, r_0$ such that

$$A(r_H) = 0 \, , \quad f_\ell(r_0) = 1 \, . \tag{96}$$

When $r_0$ is smaller than $r_H$, one has a BH, and $r_0$ is identified with a sort of minimal length scale for the metric. In the other case, one is dealing with a WH; $r_0$ represents the WH throat, and $r_H$ may also be absent. Note that in general, $r_0 \equiv r_0(\ell)$.

Within $F(R)$-modified gravity models with the on-shell condition $F(0) = F_R(0) = 0$, an attempt to deal with such BHs has been presented by Bertipagani et al. [68], but in this case a Cauchy horizon is generically present unless one makes a specific choice for $\ell$.

We observe that in the the Eddington–Filkenstein (EF) gauge, one has

$$ds^2 = -A(r)dv^2 + 2\sqrt{\frac{1}{(1 - f_\ell(r))}}dvdr + r^2 d\Omega^2 \, , \tag{97}$$

and the singularity at $(1 - f_\ell(r))$ is harmless and may be removed by a suitable change of coordinates.

Coming back to the framework of GR in the presence of fluid, the simplest and physically relevant choice for $A(r)$ is given by

$$A(r) = 1 - \frac{C}{r}, \quad C = 2m,$$ (98)

where $m$, as usual, is a constant mass parameter. Thus, making use of Equation (17), the associated energy density reads

$$\rho = \frac{f_\ell(r) + (r - C)f'_\ell(r)}{8\pi r^2}.$$ (99)

From Equation (18), one derives

$$p_r = -\frac{f_\ell(r)}{8\pi r^2},$$ (100)

while from Equation (19), we get

$$p_T = \frac{(C/2 - r)f'_\ell(r)}{16\pi r^2}.$$ (101)

Thus,

$$\rho + p_r = \frac{(r - C)f'_\ell(r)}{8\pi r^2}.$$ (102)

As a consequence, the Null Energy Condition (NEC) may be violated, but due to the restrictions $1 > f_\ell(r) > 0, r > r_0(\ell)$, all the physical quantities and curvature invariants are bounded. For example, the Ricci curvature is

$$R = 16\pi\rho + \frac{Cf'_\ell(r)}{2r^2}.$$ (103)

Let us see some examples of these non-singular BH metrics. The simplest choice for $f_\ell(r)$ is

$$f_\ell(r) = \frac{\ell}{r}.$$ (104)

This has been used by D'Ambrosio–Rovelli [80,81] and Bertipagani et al. [68]. The static version of the D'ambrosio– Rovelli solution reads

$$ds^2 = -\left(1 - \frac{C}{r}\right)dt^2 + \frac{dr^2}{(1 - C/r)(1 - \ell/r)} + r^2 d\Omega^2.$$ (105)

The Simpson–Visser choice is [77]

$$f_\ell(r) = \frac{\ell^2}{r^2},$$ (106)

with the related metric

$$ds^2 = -\left(1 - \frac{C}{r}\right)dt^2 + \frac{dr^2}{(1 - C/r)(1 - \ell^2/r^2)} + r^2 dS^2.$$ (107)

In both cases, we have the restriction on $r$ as $r > \ell$. The possible horizons are $r_H = C$ and $r = \ell$. When $\ell < C$, one has a BH; for $\ell > C$, one is dealing with a WH. These metrics are regular for $\ell > 0$, namely, $r > \ell$, and the singularity in $r = 0$ is avoided.

In order to remove the coordinate singularity, it is convenient to make use of another radial coordinate, namely, $r = \sqrt{\sigma^2 + \ell^2}$, with $-\infty < \sigma < +\infty$. With regard to this new radial coordinate, the most-natural choice for $f_\ell$ is [77]

$$f_\ell(r) = \frac{\ell^2}{r^2},$$ (108)

and we get

$$ds^2 = -A(\sigma)dt^2 + \frac{d\sigma^2}{A(\sigma)} + (\sigma^2 + \ell^2)d\Omega^2 \,. \tag{109}$$

With the choice from (104), the metric reads, in terms of $\sigma$,

$$ds^2 = -A(\sigma)dt^2 + \frac{d\sigma^2}{A(\sigma)}\left(1 + \frac{\ell}{\sqrt{\sigma^2 + \ell^2}}\right) + (\sigma^2 + \ell^2)d\Omega^2 \,. \tag{110}$$

For a generic metric, one has a regular BH as soon as the function $A(\sigma)$ and its derivatives with respect to $\sigma$ are finite everywhere.

For example, the Simpson–Visser BH (107) assumes the simple form

$$ds^2 = -\left(1 - \frac{C}{\sqrt{\sigma^2 + \ell^2}}\right)dt^2 + \frac{d\sigma^2}{\left(1 - \frac{C}{\sqrt{\sigma^2 + \ell^2}}\right)} + (\sigma^2 + \ell^2)d\Omega^2 \,. \tag{111}$$

The horizon is given by $\sigma_H = \sqrt{C^2 - \ell^2}$, and for large $\sigma$, one gets $A(\sigma) = B(\sigma) = 1 - \frac{C}{\sigma} - \frac{C\ell^2}{2\sigma^3} \times \dots$

For the D'ambrosio–Rovelli metric (105) one has,

$$ds^2 = -\left(1 - \frac{C}{\sqrt{\sigma^2 + \ell^2}}\right)dt^2 + \frac{d\sigma^2}{\left(1 - \frac{C}{\sqrt{\sigma^2 + \ell^2}}\right)}\left(1 + \frac{\ell}{\sqrt{\sigma^2 + \ell^2}}\right) + (\sigma^2 + \ell^2)d\Omega^2 \,. \tag{112}$$

The horizon is given by $\sigma_H = \sqrt{C^2 - \ell^2}$, and for large $\sigma$, we have $A(\sigma) = 1 - \frac{C}{\sigma} - \frac{C\ell^2}{2\sigma^3} + \dots$, and $B(\sigma) = 1 - \frac{C+\ell}{\sigma} - \frac{C\ell^2+\ell^3}{2\sigma^3} + \dots$.

A further example is the Peltola–Kunstetter BH, motivated by Loop Quantum Gravity (LQG) [82,83]. The related space-time is given by

$$ds^2 = -\left(\sqrt{1 - \ell^2/r^2} - \frac{C}{r}\right)dt^2 + \frac{dr^2}{\left(\sqrt{1 - \frac{\ell^2}{r^2}} - C/r\right)(1 - \ell^2/r^2)} + r^2 dS^2 \,, \tag{113}$$

or, in the free-coordinate singularity form,

$$ds^2 = -\frac{(\sigma - C)}{\sqrt{\sigma^2 + \ell^2}}dt^2 + \frac{d\sigma^2}{\frac{(\sigma-C)}{\sqrt{\sigma^2+\ell^2}}} + (\sigma^2 + \ell^2)d\Omega^2 \,. \tag{114}$$

The horizon is located at $\sigma_H = C$, and for large $\sigma$, one has $A(\sigma) = B(\sigma) = 1 - \frac{C}{\sigma} - -\frac{\ell^2}{\sigma^2} - \frac{C\ell^2}{2\sigma^3} + \dots$.

Related to these examples, there is the BH solution found by Modesto within the Loop Quantum Gravity (LQG) approach (see [46] and references therein).

These metrics are quite interesting examples of regular BHs that are asymptotically flat and without Cauchy horizons. The issue of their stability can be investigated by studying the Quasi Normal Modes (QNMs). The other important issue is generalization to the rotating case. For the Simpson–Visser BH, this has been done, and the result is the Kerr rotating metric with the new radial $\sqrt{\sigma^2 + \ell^2}$ replacing $r$ [84,85]. With regard to this issue, the possible physical relevance for these metrics has been subjected to several investigations; see [8], in which further references can be found.

It should be noted that some of these metrics can be derived by suitable Non-Polynomial Lagrangian (NPL) (see [48], which also discusses a class of metrics describing regular BHs without the Cauchy horizon issue).

### 5. Black Holes in Four-Dimensional Regularized Lovelock Models

Recently, there has been increasing interest in an approach initiated by Tomozawa [86], further revisited and extended to flat Friedman–Lemaître–Robertson–Walker (FLRW) cosmological models in [87], and recently rediscovered and extended in [88].

Several aspects have been considered in [89,90]. It should be stressed that such a procedure has been criticized in the literature; see, for example, [91]. However, in [92], one can find a recent and very complete review, containing a vast bibliography on this issue.

The basic idea is to bypass the Lovelock theorem, which states that the only gravitational theory admitting second-order equations of motion in four dimension ($D = 4$) is the Einstein–Hilbert, plus a cosmological constant term. On the other hand, in $D > 4$, the so-called Lovelock contributions are possible, and the equations of motion are still second-order. However, by making use of a suitable regularization procedure, it is possible to include additional non-trivial Lovelock contributions in $D = 4$ also.

Let us start with the simplest case. The starting point is the following gravitational action in the generic dimension $D$,

$$I = \int d^D x \sqrt{-g} \left( R - \xi \frac{G}{D-4} \right), \tag{115}$$

where $\xi$ is real, and $G$ is the Gauss–Bonnet Lovelock contribution, namely,

$$G = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. \tag{116}$$

It is well-known that in $D = 4$, the Gauss–Bonnet is a topological invariant, and it does not contribute to the equations of motion.

The trick (motivated by dimensional regularization), consists in including the factor $\frac{1}{D-4}$, regularizing the Gauss–Bonnet coupling constant. As a result, evaluating the equations of motion in a $D$-dimensional SSS space-time and then taking the limit $D \to 4$, one has, for four-dimensional SSS space-time (7), the solution [86–88]

$$A(r) = B(r) = 1 - \frac{r^2}{\xi} \left( 1 - \sqrt{1 - \frac{8\xi m}{r^3}} \right), \tag{117}$$

where $m$ is an integration constant, and we assume $\xi > 0$. The solution represents a BH because $A(r) = 0$ gives only a positive root,

$$r_H = m + \sqrt{m^2 + \xi}. \tag{118}$$

We note that the other root $r = m - \sqrt{m^2 + \xi}$ is negative when $\xi > 0$, and there is no Cauchy horizon. Furthermore, for large values of $r$, one has $A(r) = 1 - \frac{2m}{r} + \ldots$, and $m$ can be identified with the mass of the BH.

Finally, there exists a restriction on the values of $r$, since the solution above is real as soon as [86]

$$r > r_c = 2(m\xi)^{1/3}. \tag{119}$$

Therefore, one gets an asymptotic flat BH without the Cauchy horizon problem. One may take $\xi = \gamma m^2$ with the dimensionless parameter $\gamma << 1$. The critical radius becomes $r_c = 2m(\gamma)^{1/3}$, and the horizon is located at

$$r_H = m(1 + \sqrt{1 + \gamma}). \tag{120}$$

However, also with the above restriction, since the derivatives of $A(r)$ are ill-defined at the critical radius, if follows that the curvature invariants are ill-defined there, and one is not dealing with a regular BH.

This regularization procedure may be generalized to higher-order Lovelock gravity, including a suitable dimensional-dependent factor in all higher-order Lovelock coupling constants [93–96]. Again, the starting point is the following gravitational action in *D*-dimension,

$$I = \int d^D x \sqrt{-g} \left( R + \sum_p a_p L_p \right), \tag{121}$$

in which there is no cosmological constant, and the Einstein–Hilbert action leads to $a_p = 0$. All the other higher-curvature terms may be dimensionally regularized, including a suitable factor $(\prod_{(} kD - k))^{-1}$ in all higher-order Lovelock coupling constants [93], and $L_p$ are the related Lovelock higher-curvature invariants.

In the limit $D \to 4$ and in a vacuum, the solution of the regularized EOMs is

$$\frac{C}{r^3} = \sum_p c_p g(r)^p = G(r), \quad g(r) = \frac{1 - A(r)}{r^2}, \tag{122}$$

with $g(r)$ a function of $r$, and $C$ the integration constant of the solution. When the sum is over a finite number (depending on $D$), one may determine $g(r)$ by solving an algebraic equation, and then $A(r)$ is found. In general, the resulting BH solutions are singular. For example, in GR, $G(r) = g(r) = \frac{C}{r^3}$, and one has $A(r) = 1 - \frac{C}{r}$. In the Gauss–Bonnet case, $G(g) = g(r) - \xi g(r)^2$, and one finds the regularized Gauss–Bonnet model as discussed above, since

$$\frac{C}{r^3} = g(r) - \xi g(r)^2, \tag{123}$$

the solution being

$$g(r) = \frac{1}{2\xi} \left( 1 - \sqrt{1 - \frac{4\xi C}{r^3}} \right), \tag{124}$$

and we recover (117) once $\xi \to \xi/2$. However, we may consider an infinite number of suitable dimensional-regularized Lovelock terms [93,95,96], and if the arbitrary coupling constants left are properly chosen, the sum may be considered the expansion of a function $G(r) \equiv G(g(r))$ within its radius of convergence, and one derives

$$\frac{C}{r^3} = G(g(r)) \quad g(r) = \frac{1 - A(r)}{r^2}. \tag{125}$$

As an interesting application within this infinite regularized Lovelock model, we present two non-trivial examples. The first one is related to the choice

$$G(g(r)) = \frac{2Mh(r)}{(1 - \varepsilon^2 h(r)^{2/3})^{3/2}}, \quad h(r) = \frac{g(r)}{2M}, \tag{126}$$

where $\varepsilon$ is an arbitrarily small constant parameter. Thus,

$$h(r) = \frac{1}{(r^2 + \varepsilon^2)^{3/2}}, \tag{127}$$

and one gets the Bardeen BH,

$$A(r) = 1 - \frac{2Mr^2}{(r^2 + \varepsilon^2)^{3/2}}. \tag{128}$$

The second choice is [93,96]

$$G(g(r)) = \frac{h(r)}{1 - b^3 h(r)}, \quad h(r) = \frac{g(r)}{2M}, \tag{129}$$

where $b$ is a constant. Thus,

$$h(r) = \frac{1}{r^3 + b^3},$$ (130)

and one gets the Poisson–Israel BH (25) with $b^3 = \ell^3$ or the Hayward BH (26) with $b^3 = \ell^2 m$. Thus, we have shown that Bardeen and Hayward and other regular BHs may be also derived as vacuum solutions within this Lagrangian framework.

## 6. Non-Singular Cosmological Models

In this section, we discuss some non-singular cosmological models. We work within flat FLRW models whose metric reads

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2,$$ (131)

where $a \equiv a(t)$ stands for the expansion factor and is a function of the cosmological time $t$. Moreover, in what follows, we use the Hubble parameter $H \equiv H(t) = \frac{\dot{a}(t)}{a(t)}$, where the dot is the derivative with respect to time. In GR, the first Friedmann equation and matter conservation law read

$$3H^2 = 8\pi\rho,$$ (132)

$$\frac{d\rho}{dt} + 3H(\omega + 1)\rho = 0,$$ (133)

where $\rho$, $p$ are the energy density and pressure of matter contents of the Universe, respectively, and we assuming the EoS $p = \omega\rho$ with the EoS parameter $\omega$ constant. As is well known, matter conservation has a solution $\rho = \rho_0 a^{-3(1+\omega)}$ ($\rho_0$ is the energy density when $a(t) = 1$) and if $\omega \neq -1$, these equations lead to the well-known Big Bang (or Big Rip) singularity when $a(t)$ vanishes.

If we apply the Covariant Sakarov Criterion (see Section 3.3), we have

$$\chi = 1 - H^2, \quad Z = \frac{1 - \chi}{r^2} = H^2.$$ (134)

Thus, one has to deal with $Z$ and therefore $H(t)$ and its derivatives, and must require them to be uniformly bounded. In a FLRW Universe, if $H$ and $\dot{H}$ are uniformly bounded, then the space-time is causally geodesically complete (see [97–99] and references therein).

However, in this cosmological framework, we may investigate the absence of singularities working directly on the equations of motion as follows.

Loop Quantum Cosmology (LQC) [100], modified gravity and mimetic gravitational models [101,102], NPL models [103] amd Lovelock-regularized models [93,95,96] lead to a first generalized Friedmann equation of the type

$$3H^2 = F(\rho),$$ (135)

or

$$G(H^2) = \frac{\rho}{3},$$ (136)

where the form of $F(\rho)$ and $G(H^2)$ depends on the model under consideration, and the on-shell are functions of $\rho$ or $H^2$. Moreover, the matter conservation law (133) is still valid.

### 6.1. The $3H^2 = F(\rho)$ Case

When $F(\rho)$ is a positive or negative function, in general, the Big Bang singularity is present. This may be understood as a consequence of the Osgood Criterion (OC) (see, for example, [104]): given $y \equiv y(t)$, if $f(y)$ is never vanishing (always positive or negative) and one has the initial value problem

$$\dot{y} = f(y(t)) \quad y(t_0) = y_0,$$ (137)

then there exists a finite singularity if

$$\int_{y_0}^{\infty} \frac{dy}{f(y)} < \infty. \tag{138}$$

In our case from Equations (133) and (135), we get

$$\dot{\rho} = -\sqrt{3F(\rho)}(1 + \omega)\rho, \tag{139}$$

and if $F(\rho) > 0$, we do not have any restriction on the range of $\rho$. We assume $\omega \neq -1$ and $\rho(t_0) = \rho_0$. Now, if

$$\frac{1}{\sqrt{3}(1 + \omega)} \int_{\rho_0}^{\infty} \frac{d\rho}{\rho\sqrt{F(\rho)}} < \infty, \tag{140}$$

the OC guarantees the existence of a finite-time singularity. For example, the string brane correction leads to

$$3H^2 = \rho + \alpha\rho^2, \quad \alpha > 0. \tag{141}$$

Thus, $F(\rho) = (\rho + \alpha\rho^2)$, no restriction on $\rho$ is a priori present, $\frac{1}{\rho\sqrt{F(\rho)}}$ is summable, and OC leads to the existence of a finite-time singularity. One derives the exact solution

$$\rho = \frac{\rho_0}{-\alpha + \frac{3}{4}(1 + \omega)^2 t^2}, \tag{142}$$

and the related scale factor becomes

$$a(t) = \left(-\alpha + \frac{9}{4}(1 + \omega)^2 t^2\right)^{\frac{1}{3(1+\omega)}}. \tag{143}$$

As a consequence, $a(t)$ is vanishing at $t_s^2 = \frac{4\alpha}{3(1+\omega)^2}$, and $\rho$ and $H^2$ diverge there. For $\alpha = 0$, we recover the GR case. If $\omega > -1$, the singularity is present at $t = 0$; otherwise $a(t)$ is well-defined by replacing $(1 + \omega)t \to -(1 + \omega)(t^* - t)$, where $t^*$ is an integration constant corresponding to the time of future singularity [105].

The situation changes in the case of Loop Quantum Cosmology (LQC) or other similar cases. In fact, one has

$$3H^2 = \rho - \alpha\rho^2, \quad \alpha > 0. \tag{144}$$

Now, since $H^2 > 0$, the OC cannot be applied due to the constraint $\rho < \frac{1}{\alpha} = \rho_c$, and the exact solution confirms the non-existence of the time singularity, and reads

$$a(t) = (\alpha + \frac{3}{4}(1 + \omega)^2 t^2)^{\frac{1}{3(1+\omega)}}. \tag{145}$$

Here, $a(t)$ is always positive, and $\rho$ and $H^2$ are bounded. Thus, all the curvature invariants are bounded.

In general, when $F(\rho)$ is not positive definite and admits a (positive) fixed point, one may argue as follows [101]. Let us rewrite Equation (135) as

$$\left(\frac{da}{dt}\right)^2 = \frac{a^2}{3} F(\rho(a)) = Y(a). \tag{146}$$

We assume that there exists a positive fixed point $F(\rho(a_*)) = 0$, $a_* = a(t_*)$, namely, $Y(a_*) = 0$. By expanding $Y(a)$ near this fixed point, we obtain,

$$Y(a) = \frac{dY(a^*)}{da}(a - a_*). \tag{147}$$

Solving the differential equation above, one gets

$$a(t) = a_* + \frac{1}{4}\frac{dY(a^*)}{da}t^2 \,. \tag{148}$$

The chain rule together with the solution of the conservation law (133), $\rho = \rho_0 a^{-3(1+\omega)}$ lead to

$$a(t) = a_* - \frac{(1+\omega)}{4}a_*^{-(2+3\omega)}\left(\frac{\partial F}{\partial\rho}\right)_{t_*}t^2 \,. \tag{149}$$

As a result, if $\left(\frac{\partial F}{\partial\rho}\right)_{t_*} < 0$, there is no Big Bang-like singularity.

As a check, in the case of LQC, $F(\rho) = \rho - \alpha\rho^2$ with a fixed point $\rho(a_*) = \frac{1}{\alpha}$, and $\left(\frac{\partial F}{\partial\rho}\right)_{t_*} = -1$, such that there is no Big Bang singularity.

A simple generalization is given by $F(\rho) = \rho - \alpha\rho^2 - \beta\rho^3$ with $\alpha > 0$ and $\beta > 0$. The positive fixed point is

$$\rho(a_*) = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta} \,, \tag{150}$$

and

$$\left(\frac{\partial F}{\partial\rho}\right)_{t_*} = -\rho_*(\alpha + 2\beta\rho_*) < 0 \,. \tag{151}$$

Again, no Big Bang singularity is present.

*6.2. The $G(H^2) = \frac{\rho}{3}$ Case*

As in the previous case, when $G(H^2)$ is always positive, the Big Bang-like singularity is present. In fact, we may again argue as follows. First, taking the derivative with respect to $t$ of Equation (136) and making use of the matter conservation law (133), one obtains

$$\frac{dH}{dt} = -\frac{3(1+\omega)}{2}\frac{G(H^2)}{G_{H^2}(H^2)} = Y(H^2), \quad G_{H^2}(H^2) = \frac{\partial G(H^2)}{\partial H^2} \,. \tag{152}$$

Now, by introducing $H_0^2 \equiv H^2(t_0)$, and if $\frac{G(H^2)}{G_{H^2}(H^2)}$ is positive or negative definite,

$$-\frac{2}{3(1+\omega)}\int_{H_0^2}^{\infty}dH^2\frac{G_{H^2}(H^2)}{G(H^2)} < \infty \,. \tag{153}$$

Then, OC says that a finite-time singularity is present. For example, if $G(H^2) = H^2 + \xi^2 H^4$ and $\xi^2$ is a positive constant, the OC hypotheses are satisfied and there is singularity.

If $G(H^2)$ is not positive definite, then there exists a fixed point $H_*^2$ for which $G(H_*^2) = 0$, and by expanding around the positive fixed point, we get

$$\frac{dG(H^2)}{dH^2}(H^2 - H_*^2) = \frac{\rho}{3} \,. \tag{154}$$

By using the fact that

$$\rho = -\frac{\dot{\rho}}{3H(1+\omega)} = -\frac{2}{3(1+\omega)}\frac{dG(H^2)}{H^2}\dot{H} \,, \tag{155}$$

we obtain

$$\dot{H} = -\frac{9(1+\omega)}{2}\left(H^2 - H_*^2\right) \,. \tag{156}$$

Thus, one has, for small $t$,

$$H(t) = H_* \tanh\left(\frac{9(1+\omega)}{2}H_* t\right). \tag{157}$$

From this it follows $H^2(t) < H_*^2$, and

$$a(t) = \left(\cosh\left(\frac{9(1+\omega)}{2}H_* t\right)\right)^{\frac{2}{9(1+\omega)}}, \tag{158}$$

namely, the Big Bang singularity is absent.

As an interesting example, let us consider $G(H^2) = H^2 - \xi H^4$, $\xi > 0$. The fixed point is $H_*^2 = \frac{1}{\xi} > 0$. Thus, there is no Big Bang singularity, in agreement with [87].

This is a quite general result concerning an approximate solution of a generalized Friedmann equation, since it is a consequence only of the existence of a positive fixed point. In this case, $G(H^2)$ cannot be always positive or negative.

When $G(H^2)$ is always positive/negative, a negative/positive fixed point $H_*^2$ may exist, formally an imaginary $H_*$. One may proceed as above, making the expansion around this fixed point, and for small $t$, the solution for $a(t)$ is now

$$a(t) = \left(\cos\left(\frac{9(1+\omega)}{2}|H_*|t\right)\right)^{\frac{2}{9(1+\omega)}}. \tag{159}$$

As a result, a singular cyclic universe is present, in agreement with the numerical results presented in [92].

We conclude with this remark: As in the static case, we may generalize the Lovelock-regularized model in $D = 4$ in the cosmological setting considering an infinite number of Lovelock contributions. Thus, the first generalized Friedmann equation reads

$$G(H^2) = \sum_p c_p H^{2p} = \frac{\rho}{3}, \tag{160}$$

with $G(H^2)$ a suitable function of $H^2$ depending on the coefficients $c_p$ (for example, $c_0 = 0, c_1 = 1$).

An interesting example is given by

$$G(H^2) = \frac{1}{2\alpha}\left(1 - \sqrt{1 - 12\alpha H^2}\right), \tag{161}$$

with $\alpha > 0$, which leads to the LQC-modified equation

$$3H^2 = \rho - \alpha\rho^2. \tag{162}$$

Other examples can be found in [96].

### 7. Conclusions

In this paper, the problem of singularity in spherically symmetric space-times has been investigated. Specifically, some aspects of regular BHs and non-singular cosmological models have been discussed.

We have constructed a class of regular BH solutions in the framework of GR and in the presence of a delta-like regularized source. The metric components have been expressed as $A(r) = B(r) = 1 - r^2 g(r)$, namely, via the "g-function" $g(r)$. This category of metrics is asymptotically flat and show Schwarzschild-like behaviour at large distances. In the case of regular BHs with a dS core, the central singularity is absent. However, even though all the curvature invariants are bounded, for the class of models that do not admit an even-numbered g-function, namely, $g(-r) \neq g(r)$, there exist invariants built with covariant derivatives that are not bounded when $r \to 0$. This result can be achieved by making

use of the Sakarov Criterium, which offers a method that permits discrimination between singularity-free solutions and solutions with singularities.

We may add this remark: Regular BHs for which $g(r)$ is an even function (for example, the Bardeen BH) admit an extension to negative values of $r$. As a consequence, the curvature invariants have singularities only for imaginary values of $r$. On the other hand, regular BHs for which $g(r)$ is a non-even function (for example, the Hayward BH) have curvature singularities for real (negative) values of $r$. This fact is also related to the determination of the QNM asymptotics of regular BHs via the monodromy approach (see [106]).

We should note that this first kind of regular BHs may suffer from an instability issue associated with the presence of the Cauchy inner horizon. In this respect, it has been noticed that, if the Cauchy horizon surface gravity is vanishing, then Cauchy horizon instability may be avoided [52,68]. In the model investigated in [68], this is equivalent to the introduction of a minimal length scale in the metric, for example in the order of the Planck scale, and the resulting metric falls within the other class of regular BHs, the so-called Black Bounce space-times.

We have also shown that many regular BH solutions may also be viewed as vacuum solutions of four-dimensional regularized Lovelock models. As a consequence, there exist alternative Lagrangian methods, such as the NPL approach [48] and the Lovelock-regularized approach discussed in this paper, which permit derivation of regular BHs besides those found using the Non-Linear Electrodynamics approach [89].

In the final part of our work, we have presented a general approach to the problem of finite-time singularities in flat FLRW space-time cosmological models, making use of the so-called Osgood Criterion.

**Author Contributions:** The authors contributed to the manuscript equally, in details: Conceptualization and supervision S.Z.; methodology, L.S. and S.Z.; formal analysis, L.S. and S.Z.; investigation, L.S. and S.Z.; writing-original draft preparation, L.S. and S.Z.; final editing original draft L.S.;writing-review and editing, L.S. and S.Z. All authors have read and agreed to the published version of the manuscript.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflicts of interest.

## Appendix A. A Note on Painlevè Gauge and Hawking Temperature

When one deals with SSS space-time, it may be convenient to adopt different metric gauges with respect to the Schwarzschild one. For example, by using the Eddington–Filkenstein gauge with $\frac{A(r)}{B(r)} > 0$, we can rewrite the static metric (7) as

$$ds^2 = -A(r)dv^2 + 2\sqrt{\frac{A(r)}{B(r)}}dvdr + r^2dS^2 \,. \tag{A1}$$

Here $v$ is the advanced time coordinate, and $v = cost$ represents radial in-going null geodesics. In this coordinate system, the metric is explicitly non-singular at the BH radius.

Another useful reference system where there is no coordinate singularity at the BH event horizon is the Painlevè gauge, with

$$dt = dT - \sqrt{\frac{(1 - B(r))}{A(r)B(r)}}dr \,, \tag{A2}$$

such that the metric (7) can be rewritten as

$$ds^2 = -A(r)dT^2 + 2\sqrt{\frac{A(r)}{B(r)}(1 - B(r))}dTdr + dr^2 + r^2dS^2, \tag{A3}$$

and a further restriction is present since $B(r) < 1$. This means that Painlevè gauge cannot be extended to all ranges of the radial coordinate $r$ for every SSS metric.

For example, a static Kottler solution with

$$A(r) = B(r) = 1 - \frac{2m}{r} - H_0 r^2, \tag{A4}$$

where $m$, $H_0$ are constants, leads to

$$1 - B(r) = \frac{2m}{r} + H_0^2 r^2 > 0, \tag{A5}$$

and the metric exists for any value of $r$. However, if one considers the Schwarzshild Anti-De Sitter (AdS) BH solution with

$$B(r) = 1 - \frac{2m}{r} + H_0^2 r^2, \tag{A6}$$

we get

$$1 - B(r) = \frac{2m}{r} - H_0^2 r^2, \tag{A7}$$

and there is a restriction on the range of $r$. The same fact is also present in the Reissner–Nordström BH solution, where

$$A(r) = B(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \tag{A8}$$

where $Q$ is a constant. Thus,

$$1 - B(r) = \frac{2m}{r} - \frac{Q^2}{r^2}, \tag{A9}$$

which is not positive-defined for any value of $r$.

We recall a possible way to deal with this issue [107]. One may introduce a generalized Painlevè time in (7) as

$$dT = dt + \sqrt{\frac{(1 - B(r)g(r))}{A(r)B(r)}}dr, \tag{A10}$$

with $g(r) > 0$ an arbitrary function such that $(1 - B(r)g(r)) > 0$. The associated metric is

$$ds^2 = -A(r)dT^2 + 2\sqrt{\frac{A(r)}{B(r)}(1 - g(r)B(r))}dTdr + g(r)dr^2 + r^2dS^2. \tag{A11}$$

Thus, for a BH with $A(r) = B(r) = 1 - \frac{2m}{r} + Z(r)$, $Z(r) > 0$, which has a restriction with the usual Painlevè gauge, one may make the choice $G(r) = \frac{1}{1+Z(r)}$, and the (new) metric is

$$ds^2 = -\left(1 - \frac{2m}{r} + Z(r)\right)dT^2 + 2\sqrt{\frac{2m}{r}\frac{1}{(1 + Z(r))}}dTdr + \frac{dr^2}{1 + Z(r)} + r^2dS^2, \tag{A12}$$

which is well defined for any value of $r$. Is it easy to show that all the invariant quantities do not depend on the positive function $g(r)$.

As an application of the generalized Painlevè gauge, we review the Hawking radiation for static BHs and WHs, making use of the so-called tunnelling method [82] in its covariant variant of the Hamilton–Jacobi (HJ) tunnelling method [108].

We begin with massless test particle action,

$$I = \int_{\gamma} \partial_{\mu} I dx^{\mu} \,, \tag{A13}$$

where $\gamma$ represents a path crossing the horizon. The action satisfies the HJ relativistic equation,

$$g^{\mu\nu} \partial_{\mu} I \partial_{\nu} I = 0 \,. \tag{A14}$$

Thus, the radial trajectory of a massless particle is given by

$$\gamma^{ab} \partial_a I \partial_b I = 0 \,. \tag{A15}$$

The relevant two-dimensional normal metric in the generalized Pailevè gauge (A11) reads

$$d\gamma^2 = -A(r)dT^2 - 2Q(r)dTdr + g(r)dr^2 \,, \quad Q(r) = \sqrt{\frac{A(r)}{B(r)} - g(r)A(r)} \,. \tag{A16}$$

Introducing the particle energy $E = -\partial_T I$, one has

$$\partial_r I = -\frac{E}{A(r)} \left( Q(r) + \sqrt{Q^2(r) + g(r)A(r)} \right) \,. \tag{A17}$$

Thus,

$$I = -E \int_{\gamma} \frac{1}{A(r)} \left( Q(r) + \sqrt{(Q^2(r) + g(r)A(r))} \right) dr \,, \tag{A18}$$

with the integration variable $r$ crossing the horizon. We remember that the trapping horizon is located at $r = r_H$ with $B(r_H) = 0$, $B'(r_H) > 0$. First, if we are dealing with a regular WH, only $B(r_H) = 0$, $A(r_H) \neq 0$, and $Q(r)$ has a integrable singularity at $r = r_H$. Thus, the action is finite and real. On the other side, in the BH case, $A(r_H)$ is also vanishing at the horizon, and one has

$$B(r) = B'(r_H)(r - r_H) + \dots \,, \quad A(r) = A'(_H)(r - r_H) + \dots \,. \tag{A19}$$

As a consequence, $Q(r_H) = \sqrt{A'(r_H)/B'(r_H)}$. One may split the integration over $r$ and write

$$I = -E \int_{\gamma} \frac{1}{A'(r_H)(r - r_H + i\varepsilon)} \left( Q(r) + \sqrt{(Q^2(r) + g(r)A(r))} \right) dr + I_1 \,, \tag{A20}$$

where $I_1$ is a finite, real contribution, and in the first integral, the horizon divergence is present and has been cured by deforming the integration in $r$ in a suitable way. As a result, an imaginary part of the action appears as

$$\mathrm{Im} I = \frac{2\pi E}{\sqrt{A'(r_H)B'(r_H)}} \,. \tag{A21}$$

Since the tunnelling probability is given by

$$\Gamma = e^{-2Im[I]} \,, \tag{A22}$$

one derives the Hawking radiation formula,

$$\Gamma = e^{-\frac{4\pi E}{\sqrt{A'(r_H)B'(r_H)}}} \,, \tag{A23}$$

with Hawking temperature,

$$T_H = \frac{\sqrt{A'(r_H)B'(r_H)}}{4\pi} \,. \tag{A24}$$

It should be noted that the function $g(r)$ does not enter into the final result.

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