Examples of Calabi-Yau threefolds with small Hodge numbers

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Abstract. We observe some suitable examples of Calabi-Yau threefolds for heterotic superstring compactifications. It is reasonable to seek CY threefolds with Euler characteristic equals ±6 because of generation's number. Hosotani mechanism for violations of the gauge group by the Wilson loops requires such CY space has a non-trivial fundamental group. These spaces can be obtained by factoring the complete intersection Calabi-Yau spaces by the free action of some discrete group. Also we shortly discuss cases when discrete groups act with fixed point sets.

1. Introduction

Calabi - Yau (CY) spaces and structures associated with them are one of the most important objects of study in modern mathematical physics. CY spaces appear naturally in areas such as two-dimensional supersymmetric field theories, conformal field theory, topological strings, and mirror symmetry [1]. In string theory [2], [3], [4], [5] CY spaces arise as objects for obtaining reasonable compactifications.

Candelas et all [6] had studied the compactification of $E_8 \times E_8$ heterotic string on $M \times X$. Here M is 4d Minkowski space and X is some six-dimensional manifold. They wondered when such a compactification leads to 4d $N = 1$ supersymmetric field theory. They demonstrated [6] that supersymmetry requires the existence of a covariant constant spinor on X. The presence of such spinors gives the $SU(3)$ holonomy group on X. For any n-dimensional Riemann manifold holonomy group is subgroup of $SO(2n)$. If this manifold is Kähler then holonomy group is subgroup of $U(n)$. For Calabi-Yau manifolds maximal holonomy group reduces to $SU(n)$.

The embedding of the spin connection in the heterotic string leads to the violation of the gauge group from $E_8 \times E_8$ to $E_6 \times E_8$. Adjoint representation of E_8 with dimension 248 is reducible to the group $SU(3) \times E_6$ as

$$248 = (8,1) + (1,78) + (3,27) + (\bar{3}, \overline{27}).$$

The generation number of elementary particles equals difference between the number for generation of particles (3,27) and the number for anti-generation of particles $(\bar{3}, \overline{27})$. These numbers (3,27) and $(\bar{3}, \overline{27})$ equal $h^{2,1}$ and $h^{1,1}$ respectively. Thus it turns out that the number for generations of elementary particles is half of Euler characteristic of X $|h^{2,1} - h^{1,1}| = |\chi|/2$. Then quest of CY threefolds with $\chi = \pm 6$ represents significant interest because of the generation number is expected to be no more than 3. The simplest and most important approach to break E_6 gauge group is Hosotani mechanism [7], [8]. The Hosotani mechanism uses Wilsonian loops. It requires the fundamental group $\pi(X)$ of CY threefold X to be non-trivial.

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2. General information about Calabi-Yau spaces

There are many definitions of Calabi-Yau spaces. It is mathematically correct to treat of Calabi-Yau spaces as Kähler manifolds with a trivial canonical class $K = 0$. This means that the holomorphic form of the highest degree Ω has no poles and zeros anywhere. Physically Calabi-Yau spaces are Ricci- flat (zero first Chern class) Kähler-Einstein manifolds.

The low-dimensional examples of Calabi-Yau spaces are torus and famous *K3*-surface. There are many examples of K3-surface a quartic in $\mathbb{C}P^3$, the intersection of a quadric and a qubic in $\mathbb{C}P^4$, the intersection of three quadrics in $\mathbb{C}P^5$, a Kummer surface and so on. One important example of CY manifolds is elliptic fibrations. *K3*-surface can be obtain, for instance, as elliptic fibration on weighted projective space

$$y^2 = x^3 + f(z)x + g(z).$$

If *f(z)* is polynomial degree 8, *g(z)* is polynomial degree 12, *x* has weight 4 and *y* has weight 6 then this equation defines smooth K3-surface. Also we can obtain some K3-fibration Calabi-Yau threefold if polynomials *f(z,w)* and *g(z,w)* have coordinates *z* and *w* on *K3*-surface.

Let recall that Hodge numbers are dimensions of Dolbeault cohomology groups

$$h^{p,q} = dim\, H^{p,q}_{\bar{\partial}}.$$

For complex manifolds, the numbers $h^{p,q}$ consist some table called the Hodge diamond. The Hodge diamond for CY threefold is defined as

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}.$$

Hence the Euler characteristic χ equals $2(h^{1,1} - h^{2,1})$. Numbers $h^{1,1}$ are called Kähler moduli and $h^{2,1}$ are complex moduli. Mirror manifold for CY threefold with Hodge numbers $(h^{1,1}, h^{2,1})$ has Hodge numbers $(h^{2,1}, h^{1,1})$.

The most known example of CY threefold is quintic in $\mathbb{C}P^4$. Let $(x_0, x_1, x_2, x_3, x_4)$ homogeneous coordinates in $\mathbb{C}P^4$ then the quintic equation has form

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$$

with $h^{1,1} = 1, h^{2,1} = 101$ and $\chi = -200$.

There are few examples of complete intersections CY (CICY) threefolds in $\mathbb{C}P^N$ for some *N*. $\mathbb{C}P_N[d_1, \dots, d_k]_\chi$ denotes complete intersection of k homogeneous polynomials with d_i degrees in $\mathbb{C}P^N$. Then the total Chern class is given by adjunction formula

$$c = \frac{(1+J)^{N+1}}{\prod_{i=1}^{k}(1+d_iJ)}, \tag{1}$$

where J is some 2-form obtained by normalizing the Kähler form. For CY threefolds, the condition zero first Chern class $c_1 = 0$ gives expression

$$\sum_{i=1}^{k} d_i = k + 1.$$

There are five nonlinear examples of such manifolds the quintic $\mathbb{C}P_4[5]_{-200}$ and $\mathbb{C}P_5[2,4]_{-176}$, $\mathbb{C}P_5[3,3]_{-144}$, $\mathbb{C}P_6[2,2,3]_{-144}$, $\mathbb{C}P_7[2,2,2,2]_{-128}$.

Hübsch notation [9] for CICY threefold is defined *N* polynomials in $\mathbb{C}P^{i_1} \times \mathbb{C}P^{i_2} \times \cdots \times \mathbb{C}P^{i_n}$

$$\begin{matrix} P^{i_1} \\ \vdots \\ P^{i_n} \end{matrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ \vdots & \dots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nN} \end{bmatrix}^{h^{1,1},h^{2,1}}_{\chi}.$$

Each column of this table corresponds polynomial in $\mathbb{C}P^{i_1} \times \mathbb{C}P^{i_2} \times \cdots \times \mathbb{C}P^{i_n}$.

The easiest way to lower the Hodge numbers is factorizing manifold X_0 by some discrete group G. If G has free action on X (without fixed points) then the Euler characteristic of quotient manifold $X = X_0/G$ equals

$$\chi(X_0) = \frac{\chi(X_0)}{|G|},$$

where $|G|$ is order of G. This equality is true for spaces of odd dimension, as in our case. It can be obtained using the Atiyah-Bott fixed point formula. It is necessary to check the transformation of the form of highest degree Ω [10]. Number $h^{2,1}$ is usually easy to find. Then the number $h^{1,1}$ of X is given by

$$h^{1,1} = h^{2,1} + \frac{\chi(X_0)}{2|G|}.$$

For example, the quintic is factorized by $\mathbb{Z}_5 \times \mathbb{Z}_5$ has Euler characteristic -8.
Such way the first example of CY threefold with Euler characteristics $|\chi| = 6$ was constructed by S.T. Yau in 1985 [11]. This CY threefold with $\chi = -6$ is called Tian-Yau space. Let us denote quotient complete intersection Calabi-Yau manifold as QCICY.

3. Tian-Yau space and its twins

There exist complete intersection K_0 in $\mathbb{C}P^3 \times \mathbb{C}P^3$ with table

$$\begin{matrix} P^3 \\ P^3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}^{14,23}_{-18}.$$

Let us denote (x_0, x_1, x_2, x_3) four homogeneous coordinates of the first $\mathbb{C}P^3$ and for the second $\mathbb{C}P^3$ homogeneous coordinates (y_0, y_1, y_2, y_3). K_0 is given by polynomials

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,$$
$$y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0,$$
$$x_0 y_0 + x_1 y_1 + x_3 y_3 + x_4 y_4 = 0.$$

There is a group $\mathbb{Z}_3$ with free action. This action is given by $g \in \mathbb{Z}_3$

$$g: (x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \to (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3, y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3),$$

where $\alpha^3 = 1, \alpha \neq 1$. The Tian-Yau space $K_1 = K_0/\mathbb{Z}_3$ has Hodge numbers (6,9) and Hodge diamond

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 6 & & 0 & \\ 1 & & 9 & & 9 & & 1 \\ & 0 & & 6 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} . \quad (2)$$

An explanation of obtaining Hodge numbers for Tian-Yau space can be found in [12]. The fundamental group of Tian-Yau space is $\mathbb{Z}_3$.
Schimmrigk had found another example of factorized CICY with same Hodge numbers and fundamental group [13]. Consider complete intersection in $\mathbb{C}P^2 \times \mathbb{C}P^3$ with table

$$\begin{matrix} P^2 \\ P^3 \end{matrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}^{8,35}_{-54}.$$

This CICY can be factorized by $A \times B$ group. Both group are isomorphic $\mathbb{Z}_3$. Action of group A is specified by element g_1 of order 3

$$g_1 : \mathbb{C}P^2 \times \mathbb{C}P^3 \to \mathbb{C}P^2 \times \mathbb{C}P^3,$$

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This action is free. The group B acts only in $\mathbb{C}P^2$

$$g_2 : (x_0, x_1, x_2, y_0, y_1, y_2, y_3) \to (x_0, \alpha x_1, \alpha^2 x_2, y_0, y_1, y_2, y_3).$$

Here there are three fixed point sets which define three invariant tori

$$(1,0,0) \times \{y_1^3 + y_2^3 + y_3^3 = 0\},$$
$$(0,1,0) \times \{y_0^3 + y_2^3 + y_3^3 = 0\},$$
$$(0,0,1) \times \{y_0^3 + y_1^3 + y_3^3 = 0\}.$$

It is possible to resolve singularities without leaving class of CY threefolds [11].
There is other sample of CY threefold with same Hodge diamond (2). Let us consider bicubic L

$$L = \begin{matrix} P^2 \\ P^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}^{2,83}_{-162}.$$

L is factorized by three groups with fixed point acting groups G_1,G_2 and free acting group G_3 Resolved $L/\widehat{G_1 \times G_2} \times G_3$ has Hodge diamond and fundamental group as Tian-Yau and Schimmrigk spaces.

4. More modern examples of QCICY

Many examples of "three generation" manifolds with $\chi = \pm 6$ was constructed in [15]. Two more samples were built by Braun, Candelas and Davies [16]. CICY is denoted in [16] as $Y^{8,44}$. This manifold is defined by following Hübsch table

$$\begin{matrix} P^2 \\ P^2 \\ P^2 \\ P^2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}^{8,44}_{-72}.$$

Two groups of order 12 act freely on $Y^{8,44}$. One group is abelian $\mathbb{Z}_{12}$. And other group is non-Abelian dicyclic group Dic_3 (semidirect product $\mathbb{Z}_3$ and $\mathbb{Z}_4$). The quotient manifolds for both these groups have Hodge numbers $(h^{1,1}, h^{2,1}) = (1,4)$. Fundamental groups these manifolds are $\mathbb{Z}_{12}$ and Dic_3 respectively.
A huge number of examples of CY threefolds with small Hodge numbers are given in the table article [17]. Many examples with a non-trivial fundamental group provide quotients for complete intersections of four quadrics $\mathbb{C}P_7[2,2,2,2]_{-128}$.

Relevant way to change Hodge number for Calabi-Yau spaces is conifold transition [18], [19], [20]. Usual conifold transition shifts the Hodge numbers as follows

$$\delta(h^{1,1}, h^{2,1}) = (1, -1).$$

The conifold transition does not change the fundamental group. However, there is a hyperconifold transition [21], [22] that can change the fundamental group.
In this text we have given some examples CY threefolds that are interesting for compactifications of heterotic string theory. Mirror symmetry [1] is an actual way to study Calabi-Yau spaces. Euler characteristic for mirror manifold is same. Therefore, from the modern point of view, it is important to explore mirror manifolds with small Hodge numbers.

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