Improvement of the Emittance in the Resonant Beam Ejection

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The resonant beam ejection from an AG Synchrotron, by the use of a quadrupole magnet and sextupole magnets, was proposed by Hereward[1] and performed successfully with the CERN PS.[2] A similar method by a current strip was first achieved at CEA.[3] Although many studies have been made on this subject, most of them were made numerically or experimentally. In this paper is given an analytical approach to this subject based on an approximate Hamiltonian.

A radial emittance in the resonant ejection is fairly small, generally. However, according to the predictions of the theory, it can be reduced further. We have investigated numerically motions of particles in a phase space, and found that the reduction of the emittance was really possible.

I. Hamiltonian

Suppose an arrangement shown in Fig. 1, where Q is a quadrupole magnet, K is a kick magnet and M's are non-linear multi-pole magnets. We consider the radial motion in a phase space (x, y), where $y = \frac{\beta}{\sqrt{1+\alpha^2}}\frac{dx}{ds}$, β and α are betatron functions at Q, x is a radial displacement from an equilibrium orbit, and s is a distance along the orbit. Near a m-th resonance, the transfer matrix for one period, that is for m revolutions, from the center of the quadrupole magnet, which will be called the point Q, can be written as

$$\Gamma = \begin{pmatrix} \cos\varepsilon - \alpha\sin\varepsilon & \sqrt{1+\alpha^2}\sin\varepsilon \\ -\sqrt{1+\alpha^2}\sin\varepsilon & \cos\varepsilon + \alpha\sin\varepsilon \end{pmatrix}, \qquad (1)$$

where ε is a small phase shift in one period. Coordinates x_ν and y_ν after ν periods are given by the relation

$$\begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix} = \Gamma \begin{pmatrix} x_{\nu-1} \\ y_{\nu-1} \end{pmatrix} + \begin{pmatrix} \xi_\nu \\ \eta_\nu \end{pmatrix}, \qquad (2)$$

where ξ_ν and η_ν are non-linear displacements during the ν-th period, and are given approximately by

$$\begin{pmatrix} \xi_\nu \\ \eta_\nu \end{pmatrix} = \Gamma \sum_{i=1}^{n} \Gamma_i^{-1} \begin{pmatrix} 0 \\ \frac{\beta}{\sqrt{1+\alpha^2}}\cdot\psi_i \end{pmatrix}, \qquad (3)$$

ψ_i is a deflection due to M_i, Γ_i is the transfer matrix from Q to M_i, and Γ_i^{-1} is its inverse matrix.

Let

$$\psi_i = g_i x_i^{\ell} \qquad (4)$$

where ℓ is a positive integer, g_i is a constant, and x_i is the displacement of a particle from the axis of M_i, and let the matrix Γ_i be

$$\Gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \qquad (5)$$

Then, rewriting eq. (3), we have

$$\xi_\nu \doteqdot \frac{\beta}{\sqrt{1+\alpha^2}} \Sigma g_i \left\{\varepsilon\sqrt{1+\alpha^2}\,a_i - (1-\varepsilon\alpha)b_i\right\}(a_i x_{\nu-1} + b_i x_{\nu-1})^{\ell} \qquad (6)$$

and

$$\eta_\nu \doteqdot \frac{\beta}{\sqrt{1+\alpha^2}} \Sigma g_i \left\{(1+\varepsilon\alpha)a_i + \varepsilon\sqrt{1+\alpha^2}\,b_i\right\}(a_i x_{\nu-1} + b_i y_{\nu-1})^{\ell}. \qquad (7)$$

Changes of the coordinates during one period can be given by the relations

$$\frac{dx}{d\nu} \doteqdot -\varepsilon\alpha x + \varepsilon\sqrt{1+\alpha^2}\,y + \xi \qquad (8)$$

and

$$\frac{dy}{d\nu} \doteqdot -\varepsilon\sqrt{1+\alpha^2}\,x + \varepsilon\alpha y + \eta \qquad (9)$$

Using a transformation defined by

$$x = X(\sqrt{1+\alpha^2}-\alpha)^{1/2}\cos\frac{\pi}{4} + Y(\sqrt{1+\alpha^2}+\alpha)^{1/2}\sin\frac{\pi}{4},$$
$$y = -X(\sqrt{1+\alpha^2}-\alpha)^{1/2}\sin\frac{\pi}{4} + Y(\sqrt{1+\alpha^2}+\alpha)^{1/2}\cos\frac{\pi}{4}, \qquad (10)$$

and neglecting small terms, we have

$$\frac{dX}{d\nu} \doteqdot \varepsilon Y - \frac{\beta(\sqrt{1+\alpha^2}+\alpha)^{\frac{1}{2}}}{\sqrt{2}\sqrt{1+\alpha^2}} \Sigma g_i (a_i + b_i)\left\{\frac{(\sqrt{1+\alpha^2}-\alpha)^{\frac{1}{2}}(a_i - b_i)}{\sqrt{2}} X + \frac{(\sqrt{1+\alpha^2}+\alpha)^{\frac{1}{2}}(a_i + b_i)}{\sqrt{2}} Y\right\}^{\ell} \qquad (11)$$

and

$$\frac{dY}{d\nu} \doteqdot -\varepsilon X + \frac{\beta(\sqrt{1+\alpha^2}+\alpha)^{\frac{1}{2}}}{\sqrt{2}\sqrt{1+\alpha^2}} \Sigma g_i (a_i - b_i)\left\{\frac{(\sqrt{1+\alpha^2}-\alpha)^{\frac{1}{2}}(a_i - b_i)}{\sqrt{2}} X + \frac{(\sqrt{1+\alpha^2}+\alpha)^{\frac{1}{2}}(a_i + b_i)}{\sqrt{2}} X\right\}^{\ell}. \qquad (12)$$

Then, the Hamiltonian of the motion can be given by

$$H = -\frac{\varepsilon}{2}(X^2+Y^2) + \frac{\beta}{(\ell+1)\sqrt{1+\alpha^2}} \Sigma g_i \left\{\frac{(\sqrt{1+\alpha^2}-\alpha)(a_i - b_i)}{\sqrt{2}} X + \frac{(\sqrt{1+\alpha^2}+\alpha)^{\frac{1}{2}}(a_i + b_i)}{\sqrt{2}} Y\right\}^{\ell+1}. \qquad (13)$$

In the (x, y) plane, it can be rewritten as

$$H = -\frac{\varepsilon}{2}\left\{\sqrt{1+\alpha^2}\,x^2 - 2\alpha xy + \sqrt{1+\alpha^2}\,y^2\right\} + \frac{\beta}{(\ell+1)\sqrt{1+\alpha^2}} \Sigma g_i (a_i x + b_i y)^{\ell+1}. \qquad (14)$$

II. Integer Resoance

In an integer or a half-integer resonance, β is infinitely large, and α is infinitely large or infinitesimally small at Q. When |α| is infinitely large, $x \doteqdot y$ for α > 0 and $x \doteqdot -y$ for α < 0, as can be seen in eq. (10). When α is infinitesimally small, b_i's are infinitesimally small. In either cases the Hamiltonian can be rewritten as

$$H \doteqdot -\frac{\varepsilon}{2}(\sqrt{1+\alpha^2}x^2 - 2\alpha xy + \sqrt{1+\alpha^2}y^2) + \frac{\beta x^{\ell+1}}{(\ell+1)\sqrt{1+\alpha^2}} \Sigma g_i (a_i \pm b_i)^{\ell+1}, \qquad (15)$$

where + is for α > 0 and − is for α < 0.

Singular points given by the relations ∂H/∂x = 0 and ∂H/∂y = 0 are called "unstable fixed points" and are given by

$$x_1^{\ell-1} = \frac{\varepsilon}{\beta\Sigma g_i (a_i \pm b_i)^{\ell+1}} \qquad (16)$$

and

$$y_1 = \frac{\alpha}{\sqrt{1+\alpha^2}} x_1. \qquad (17)$$

When ℓ is even, there always exists only one unstable fixed point, whereas when ℓ is odd, there are two or none.

A separatrix, a boundary of stable region, is given by $H(x, y) = H(x_1, y_1)$ and is written as

$$x^2 + (\alpha x - \sqrt{1+\alpha^2}y)^2 = x_1^2\left[\frac{\ell-1}{\ell+1} + \frac{2}{\ell+1}\left(\frac{x}{x_1}\right)^{\ell+1}\right]. \qquad (18)$$

Next, some details of the integer resonance ejection will be investigated for the lowest order non-linearity $\ell = 2$, which is given by a sextupole magnet.

Henceforth, phase space $(x, x' = \frac{dx}{ds}, \frac{\Delta p}{p})$ will be used instead of (x, y), throughout this article. Let the original matrix for one revolution from Q be

$$\Gamma_o = \begin{pmatrix} \cos\mu_o - \alpha_o\sin\mu_o & \beta_o\sin\mu_o & A_{13} \\ -\frac{1+\alpha_o^2}{\beta_o}\sin\mu_o & \cos\mu_o + \alpha_o\sin\mu_o & A_{23} \\ 0 & 0 & 1 \end{pmatrix}, \qquad (19)$$

where

$$A_{13} = \gamma_Q^{\,o}\left\{1 - (\cos\mu_o - \alpha_o\sin\mu_o)\right\} - \kappa_Q^{o}\,\beta_o\,\sin\mu_o, \tag{20}$$

$$A_{23} = \gamma_Q^{\,o}\,\frac{1+\alpha_o^2}{\beta_o}\sin\mu_o + \kappa_Q^{\,o}\left\{1 - (\cos\mu_o + \alpha_o\sin\mu_o)\right\}, \tag{21}$$

and $\gamma_Q^{\,o}$ and $\kappa_Q^{\,o}$ are the momentum compaction factor and its derivative at Q, which are defined by

$$\gamma_Q^{\,o} = \Delta x_Q / \frac{\Delta p}{p} \tag{22}$$

and

$$\kappa_Q^{\,o} = \left(\frac{d\Delta x}{ds}\right)_Q / \frac{\Delta p}{p}, \tag{23}$$

Then, the condition for an integer resonance can be attained by

$$\frac{\phi^2}{\ell_Q} = -\frac{2(1-\cos\mu_o)}{\beta_o\sin\mu_o}, \tag{24}$$

where

$$\phi = \sqrt{\frac{1}{B\rho}\frac{\partial B_z}{\partial x}}\cdot\ell_Q, \tag{25}$$

B is the magnetic field on the central orbit, ρ is its radius of curvature, ℓ_Q is the length of the quadrupole magnet, and $\partial B_z/\partial x$ is its field gradient. The transfer matrix Γo, then changes to

$$\Gamma \doteqdot \begin{pmatrix} 1-\alpha_o\sin\mu_o & \beta_o\sin\mu_o & A_{13} \\ -\frac{1+\alpha_o^2}{\beta_o}\sin\mu_o & 1+\alpha_o\sin\mu_o & A_{23}-\frac{\phi^2}{2\ell_Q}A_{13} \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

When the displacement of the equilibrium orbit is Δx_i, a deflection by the i-th sextupole magnet can be given by

$$\psi_i = g_i(x_i + \Delta x_i)^2 = g_i x_i^2 + 2g_i x_i\Delta x_i + g_i(\Delta x_i)^2, \tag{27}$$

where $g_i x_i^2$ gives a non-linear deflection, $2g_i\Delta x_i$ a shift of the phase advance, and $g_i(\Delta x_i)^2$ a distortion of the equilibrium orbit. Then, the phase advance for one revolution μ for small amplitude oscillations around the equilibrium orbit can be given by

$$\cos\mu \doteqdot 1 + \beta_o\sin\mu_o\,\Sigma g_i\Delta x_i\left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right)^2, \tag{28}$$

where $b_i' = \frac{\beta}{\sqrt{1+\alpha^2}}b_i$. Since $\cos\mu = \cos\varepsilon$, we have

$$\varepsilon^2 \doteqdot -2\beta_o\sin\mu_o\,\Sigma g_i\Delta x_i\left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right)^2, \tag{29}$$

Betatron functions for these small amplitude oscillations are given by

$$\beta \doteqdot \frac{\beta_o\sin\mu_o}{\varepsilon} \tag{30}$$

and

$$\alpha \doteqdot \frac{\alpha_o\sin\mu_o}{\varepsilon}. \tag{31}$$

The displacement of the equilibrium orbit x_{eq} and its derivative x'_{eq} at Q can be given by

$$\begin{pmatrix} x_{eq} \\ x'_{eq} \\ \Delta p/p \end{pmatrix} = \Gamma\begin{pmatrix} x_{eq} \\ x'_{eq} \\ \Delta p/p \end{pmatrix} + \Sigma\Gamma\Gamma_i^{-1}\begin{pmatrix} 0 \\ g_i(\Delta x_i)^2 \\ 0 \end{pmatrix}, \tag{32}$$

where Γ_i is now defined in $(x, x', \Delta p/p)$ plane. Then we have

$$x'_{eq} = \frac{\alpha_o}{\beta_o}x_{eq} \tag{33}$$

and

$$\frac{\Delta p}{p} = -\frac{\beta_o\sin\mu_o}{2(1-\cos\mu_o)\,\gamma_Q^{\,o}}\,\Sigma g_i(\Delta x_i)^2\left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right). \tag{34}$$

Since, near the integer resonance, a wave form of the equilibrium orbit is similar to a free betatron oscillation, we have

$$\Delta x_i \doteqdot \left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right)x_{eq}. \tag{35}$$

Then, from eq. (29), we have

$$x_{eq} = -\frac{\varepsilon^2}{2\beta_o\sin\mu_o\Sigma g_i\left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right)^3}. \tag{36}$$

Since $b_i = \frac{\sqrt{1+\alpha^2}}{\beta}b_i' \doteqdot \pm b_i'\frac{\alpha_o}{\beta_o}$, where + is for $\alpha_o > 0$ and - is for $\alpha_o < 0$, eq.s (16) and (17) can be written as, for $\ell = 2$,

$$x_1 = \frac{\varepsilon^2}{\beta_o\sin\mu_o\Sigma g_i\left(a_i + b_i'\frac{\alpha_o}{\beta_o}\right)^3} \tag{37}$$

and

$$x_1' = \frac{\alpha_o}{\beta_o}x_1. \tag{38}$$

Therefore, we have the following relations

$$x_{eq} = -\frac{1}{2}x_1 \tag{39}$$

and

$$x'_{eq} = -\frac{1}{2}x_1'. \tag{40}$$

When α_o equals zero as usual, as can easily be shown, the maximum amplitude of the oscillation can be obtained at a point K whose phase advance from the exit of the quadrupole magnet is approximately $n\pi + \frac{\mu_o}{2}$, and the amplitude at K is given by

$$x_K \doteqdot x_Q/\cos\frac{\mu_o}{2},$$

where x_Q is the amplitude at Q. The rate of the amplitude blow up at K is given, in the limit ε goes to zero, by

$$\left|\frac{dx_K}{d\nu}\right| = \sqrt{\frac{2}{3}\beta_o\sin\mu_o\cos\frac{\mu_o}{2}\,\Sigma g_i a_i^3\,x_K^3}. \tag{41}$$

In Fig. 2 are shown examples of separatrices. As can be seen there, the accuracy of the theory is fairly good for the integer resonance ejection with sextupole magnets.

III. Third Resonance

Non-linear fields of $\ell = 2$ will be assumed also in this case. In the third resonance ejection, particles will pass each sextupole magnet three times in one period. Therefore, first, we should take summations of contributions from each non-linear field over one period. Then we can rewrite eq. (13) as

$$H = -\frac{\varepsilon}{2}(X^2 + Y^2) + \frac{\beta}{3}\left(AX(X^2 - 3Y^2) + BY(Y^2 - 3X^2)\right), \tag{42}$$

where

$$A = \frac{3}{8\sqrt{2}}\left(\sqrt{1+\alpha^2} - \alpha\right)^{3/2}\Sigma' g_i(a_i - b_i)\left\{(a_i - b_i)^2 - 3\left(\sqrt{1+\alpha^2} + \alpha\right)^2(a_i + b_i)^2\right\} \tag{43}$$

and

$$B = \frac{3}{8\sqrt{2}}\left(\sqrt{1+\alpha^2} + \alpha\right)^{3/2}\Sigma' g_i(a_i + b_i)\left\{(a_i + b_i)^2 - 3\left(\sqrt{1+\alpha^2} - \alpha\right)^2(a_i - b_i)^2\right\}. \tag{44}$$

The summation Σ' in these equations is, now, taken over one period.

Making a rotation of the coordinates by an angle φ given by

$$\tan 3\varphi = -B/A, \tag{45}$$

where 3φ should be taken so that $3\varphi = \pi$ for $\tan 3\varphi = 0$, we can rewrite eq. (42) as

$$H = -\frac{\varepsilon}{2}(X'^2 + Y'^2) + \frac{\beta\sqrt{A^2+B^2}}{3}X'(X'^2 - 3Y'^2). \tag{46}$$

Then, we have following three unstable fixed points;

$$\left.\begin{matrix} X_1' = \frac{\varepsilon}{\sqrt{A^2+B^2}} \\ Y_1' = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} X_2' = -\frac{X_1'}{2} \\ Y_2' = \frac{\sqrt{3}}{2}X_1' \end{matrix}\right\} \text{ and } \left.\begin{matrix} X_3' = -\frac{X_1'}{2} \\ Y_2' = -\frac{\sqrt{3}}{2}X_1' \end{matrix}\right\}. \tag{47}$$

The separatrix in (X', Y') phase space is given by

$$\left(X' + \frac{X_1'}{2}\right)\left(X' - \sqrt{3}Y' - X_1'\right)\left(X' + \sqrt{3}Y' - X_1'\right) \doteqdot 0. \tag{48}$$

Transforming back to the (x, y) plane, we have

$$\left.\begin{aligned} x_1 &= G\cos(\varphi-\chi) \\ y_1 &= G\sin(\varphi-\tfrac{\pi}{2}+\chi), \\ x_2 &= G\cos(\varphi-\chi+\tfrac{2\pi}{3}) \\ y_2 &= G\sin(\varphi+\chi+\tfrac{\pi}{6}), \end{aligned}\right\} \quad (49)$$

and

$$x_3 = G\cos(\varphi-\chi-\tfrac{2\pi}{3})$$

$$y_3 = G\sin(\varphi+\chi-\tfrac{7\pi}{6}),$$

where

$$G = \frac{\varepsilon(1+\alpha^2)^{\frac{1}{4}}}{\beta\sqrt{A^2+B^2}} \quad (50)$$

and

$$\chi = \tan^{-1}(\sqrt{1+\alpha^2}+\alpha). \quad (51)$$

A third resonance can be attained by

$$\frac{\phi^2}{\ell_Q} \doteqdot \frac{1+2\cos\mu_0}{\beta_0\sin\mu_0} \quad (52)$$

instead of eq. (24) in the integer resonance, and betatron functions at Q are given by

$$\left.\begin{aligned} \beta &\doteqdot \frac{2}{\sqrt{3}}\beta_0|\sin\mu_0| \\ a &\doteqdot \frac{2}{\sqrt{3}}\alpha_0|\sin\mu_0| \end{aligned}\right\} \quad (53)$$

By a similar procedure to that in the integer resonance we have

$$\varepsilon \doteqdot \pm 2\sqrt{3}\frac{x_{eq}}{\gamma_Q}\beta_0\sin\mu_0\,\Sigma' g_i\gamma_i\left[(a_i+\frac{\alpha_0}{\beta_0}b_i')^2+\frac{3}{4}(\frac{b_i'}{\beta_0\sin\mu_0})^2\right], \quad (54)$$

where - is for 1/3 resonance and + is for 2/3 resonance. γ_Q and γ_i in eq. (54) are the momentum compaction factors at Q and M_i respectively, and are given by

$$\gamma_Q = \frac{2}{3}(1-\cos\mu_0)\gamma_Q{}^0 \quad (55)$$

and

$$\gamma_i = \gamma_i{}^0 - (a_i+\frac{\alpha_0}{\beta_0}b_i')\frac{(1+2\cos\mu_0)}{3}\gamma_Q{}^0 - \frac{1+2\cos\mu_0}{2\beta_0\sin\mu_0}b_i'\gamma_Q{}^0, \quad (56)$$

where $\gamma_i{}^0$ is the original value of γ_i. In the third resonance, a distortion of the equilibrium orbit caused by non-linear fields will usually be small, we can obtain the equilibrium orbit at Q by the relation

$$\begin{pmatrix} x_{eq} \\ x_{eq}' \\ \Delta p/p \end{pmatrix} = \Gamma \begin{pmatrix} x_{eq} \\ x_{eq}' \\ \Delta p/p \end{pmatrix},$$

where Γ is the transfer matrix for one revolution from Q and is given by

$$\Gamma = \begin{pmatrix} -\frac{1}{2}-\alpha_0\sin\mu_0 & \beta_0\sin\mu_0 & A_{13} \\ \frac{-3+4\alpha_0{}^2\sin\mu_0}{4\beta_0\sin\mu_0} & -\frac{1}{2}+\alpha_0\sin\mu_0 & A_{23}-\frac{1+2\cos\mu_0}{2\beta_0\sin\mu_0}A_{13} \\ 0 & 0 & 1 \end{pmatrix}. \quad (57)$$

Then we have

$$x_{eq} \doteqdot \frac{2(1-\cos\mu_0)}{3}\gamma_Q{}^0\frac{\Delta p}{p}, \quad (58)$$

$$x_{eq}' \doteqdot \left[\kappa_Q{}^0 - \frac{\alpha_0\gamma_Q{}^0}{3\beta_0}(1+2\cos\mu_0)\right]\frac{\Delta p}{p}, \quad (59)$$

and

$$\frac{\Delta p}{p} = \mp\frac{\varepsilon}{2\sqrt{3}\beta_0\sin\mu_0\,\Sigma' g_i\gamma_i\left[(a_i+b_i'\frac{\alpha_0}{\beta_0})^2+\frac{3}{4}(\frac{b_i'}{\beta_0\sin\mu_0})^2\right]}, \quad (60)$$

where - is for 1/3 resonance and + is for 2/3 resonance. In Fig. 3 is shown an example. We can see here that the accuracy of the theory is not so good as in an integer resonance, but it is still fairly good. This good accuracy, however, will be lost when a displacement of the equilibrium orbit becomes large as in a case of minimizing the emittance.

IV. Emittance

(1) Integer resonance

First, the emittance for sextupole magnets will be investigated. From eq. (18), after some calculations, we have a separatrix of the form

$$x' = \frac{\alpha_0}{\beta_0}x \pm \sqrt{\frac{\Sigma g_i(a_i+\frac{\alpha_0}{\beta_0}b_i')^3}{\beta_0\sin\mu_0}x_1{}^3\left[\frac{1}{3}+\frac{2}{3}(\frac{x}{x_1})^3-(\frac{x}{x_1})^2\right]}. \quad (61)$$

Since displacements of the orbit from the central orbit are $x_q = x - \frac{x_1}{2}$ and $x_q' = x - \frac{\alpha_0}{\beta_0}\frac{x_1}{2}$, we can rewrite eq. (61), for $|x_q| \gg x_1$, as

$$x_q' = \frac{\alpha_0}{\beta_0}x_q \pm \sqrt{\frac{2}{3}\frac{\Sigma g_i(a_i+\frac{\alpha_0}{\beta_0}b_i')^3}{\beta_0\sin\mu_0}x_q{}^3\left[1-\frac{3}{8}(\frac{x_1}{x_q})^2\right]}, \quad (62)$$

Therefore, the difference between x_q's for $\varepsilon = 0$ and $\varepsilon = \varepsilon$ is given by

$$\Delta x_q' = (x_q')_\varepsilon - (x_q')_0$$

$$\doteqdot \pm\frac{3}{8}(\frac{x_1}{x_q})^2\sqrt{\frac{2}{3}\frac{\Sigma g_i(a_i+\frac{\alpha_0}{\beta_0}b_i')^3}{\beta_0\sin\mu_0}x_q{}^3}. \quad (63)$$

The change of amplitude in one revolution at Q is given approximately by

$$\Delta x_q = \frac{dx}{d\nu} \doteqdot \pm\sqrt{\frac{2}{3}\beta_0\sin\mu_0\Sigma g_i(a_i+\frac{\alpha_0}{\beta_0}b_i')^3x_q{}^3}. \quad (64)$$

Since $\Delta p/p$ is very small in an integer resonance, the phase space area $\Delta x_q'\Delta x_q$ is approximately equals to $\Delta x_k'\Delta x_k$, where $\Delta x_k'$ and Δx_k are defined at K in the same way as $\Delta x_q'$ and Δx_q. Therefore, the emittance of the ejected beam will be given by

$$E = \Delta x_q'\Delta x_q = \frac{3}{8}(\frac{x_1}{x_q})^2\frac{(\Delta x_q)^2}{\beta_0|\sin\mu_0|} \doteqdot \frac{3}{8}(\frac{x_1}{x_k})^2\frac{(\Delta x_k)^2}{\beta_0|\sin\mu_0|}. \quad (65)$$

As can be seen in eq. (65), the emittance in an integer resonance is independent of arrangements of sextupole magnets.

In order to reduce the emittance we must have some higher order non-linear fields. The theory indicates that this will be possible by the use of octupole magnets in addition to the sextupole magnets or by the use of ten-pole magnets instead of sextupole magnets, but the theory is not so powerful to give quantitative predictions for this problem, because there is a sextupole component which gives large contributions to the motion. Therefore, we have studied this problem numerically, and obtained following results;

(a) The use of octupole magnets is not effective for this purpose, because a decrease of $\Delta x_q'$ is offset by a increase of Δx_q and, moreover, octupole magnets limit the size of separatrices.

(b) The emittance can be appreciably reduced by the use of ten-pole magnets. Examples are shown in Fig. 5 and Fig. 6. In this case, the emittance can be reduced by a factor of three compared with that in an integer resonance, for the same initial amplitude. A disadvantage of the use of ten-pole magnets is that the rate of amplitude blow up changes somewhat rapidly with the amplitude, compared with the case of sextupole magnets. This disadvantage may be removed if the field distribution at a large displacement be shaped so as to show gx^2-like distribution.

(2) Third Resonance

The emittance in a third resonance ejection depends largely on arrangements of an ejection system, and usually it is substantially larger than that in an integer resonance. In the third resonance, however, a size of the separatrix and a displacement of the equilibrium orbit can be determined independently, and in principle it is always possible to realize a condition that an out-going separatrix, a extension of a side of separatrix, passes the origin of the phase space. When such a condition is realized at a point, where the beam leaves the orbit, the emittance of the ejected beam will be reduced substantially. An example is shown in Fig. 7. As illustrated here, the emittance can be reduced to the same order of magnitude with that in an integer resonance. Since, in such cases, a displacement of the equilibrium orbit is large, the theory loses its accuracy and behaviors of the orbit becomes complicated. In order to illustrate the complexity, an example of a relation between the phase space area of the separatrix and the fractional momentum difference is shown in Fig. 8. In this case there are two orbits on which resonance conditions are met and phase space diagrams show complex behaviors. Therefore, considerable amounts of computor works will be required to have the optimum parameters. In spite of such complexities, a third resonance ejection has an obvious advantage that it will give us a longer and more uniform spill of the beam than in an integer resonance. Because, the fractional momentum difference $\Delta p/p$ is more than ten times larger than that in an integer resonance, and the required stability of the guiding magnetic field to have longer spill time is considerably lower than that in an integer resonance.

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References

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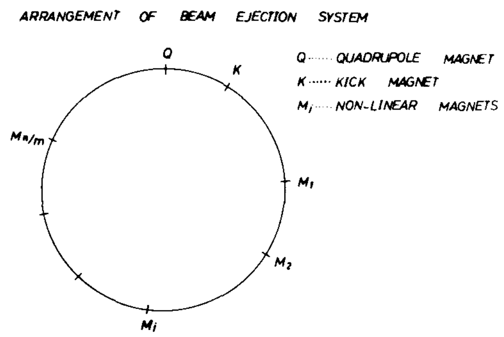

Fig. 1 A schematic diagram showing the arrangement of beam ejection system.

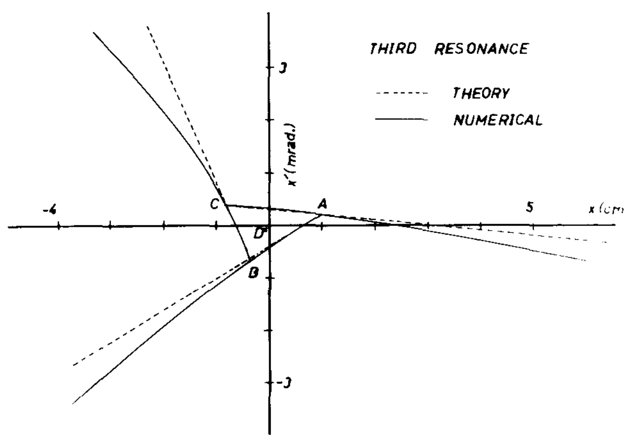

Fig. 3 A separatrix for a third resonance at K_1, where A, B and C are unstable fixed points and D is a stable fixed point.
$g_1 = -0.2102$ and $g_2 = g_3 = 0$

DISCUSSION (condensed and reworded)

M.Q. Barton (BNL): Does Figure 8 represent a numerical or theoretical value?

Kobayashi: This is the result of numerical calculation.

Barton: The non linearity that gave this orbit shift does not cause the separatrices extended to close. Did you have any trouble with the separatrices turning and coming back?

Kobayashi: When the non-linear field is very strong, there is some complicated behavior. In this case the non-linear field is not so strong, so even if the disc diameter converges,it would be larger than the aperture of the donut.

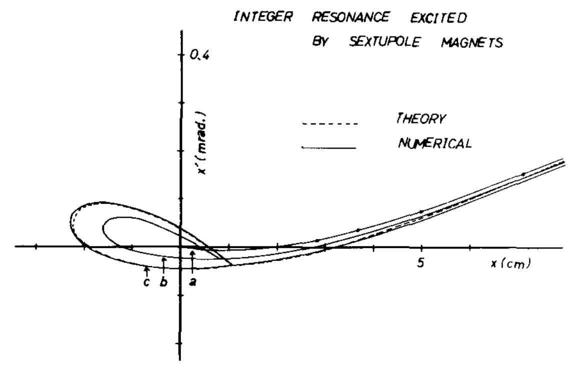

Fig. 2 Separatrices for an integer resonance excited by sextupole magnets. The positions of magnets are indicated in Fig. 4 and separatrices were calculated at K_1. Parameters of sextupoles are $g_1 = g_3 = -0.02102$ and $g_2 = 0.02102$. Phase space area of the separatrices are; a --- 0, b --- 0.117 cm·mrad and c --- 0.280 cm·mrad.

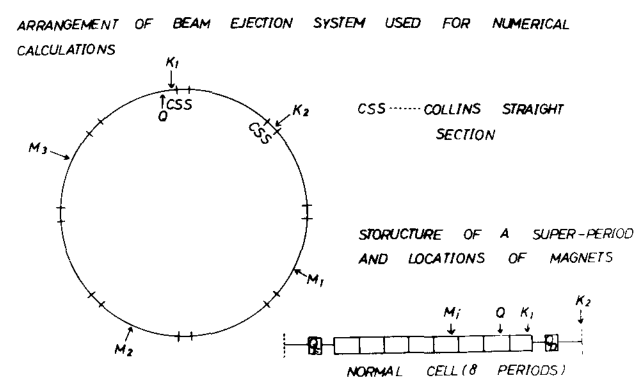

Fig. 4 The arrangement of beam ejection system used for Numerical calculations, where Q is a quadrupole magnet, M_i's are non-linear magnets and K 's are points at which separatrices are obtained. Q, M_i and K_1 are at the center of straight sections between radially focusing sectors.

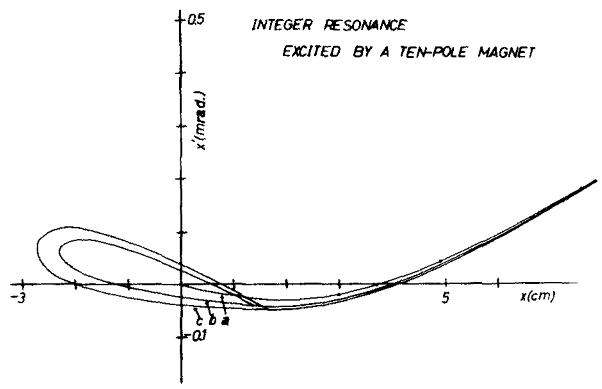

Fig. 5 Separatrices for an integer resonance excited by a ten-pole magnet calculated at K_1. $g_1 = -70$ and $g_2 = g_3 = 0$. Phase space area; a --- 0, b --- 0.181 cm·mrad. and c --- 0.329 cm·mrad.

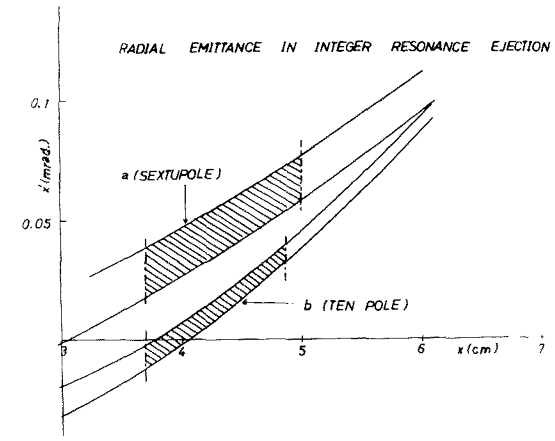

Fig. 6 Emittance of ejected beams in the integer resonance ejection. E = 0.063 cm·mrad for a (the case c in Fig. 2) and E = 0.023 cm·mrad for b (the case c in Fig. 5).

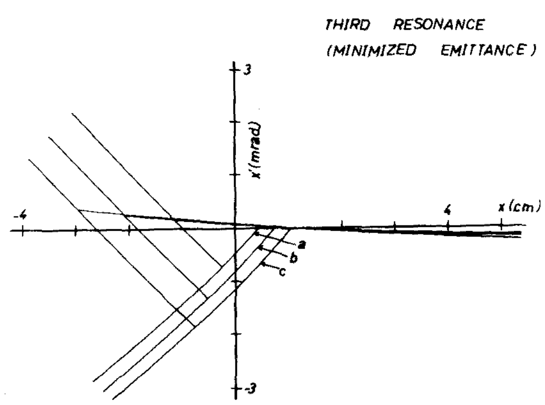

Fig. 7 Separatrices for a third resonance at K_2. $g_1 = -0.1051$, $g_2 = 0$ and $g_3 = 0.06$. Phase space area; a --- 0.56 cm·mrad, b --- 1.63 cm·mrad. and c --- 3.85 cm·mrad..

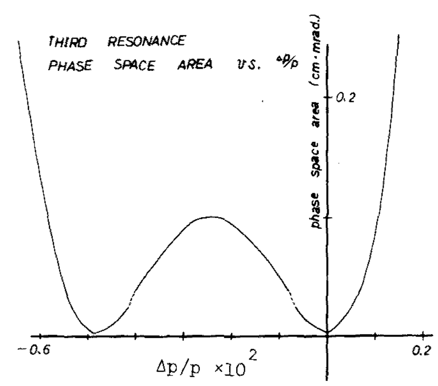

Fig. 8 An example of the relation between the phase space area of the separatrix and the fractional momentum difference. $g_1 = -0.4204$, $g_2 = 0$ and $g_3 = 0.3000$.