Chiral trace relations in Ω-deformed $\mathcal{N} = 2$ theories

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Chiral trace relations in Ω-deformed $\mathcal{N} = 2$ theories

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Abstract. We study chiral trace relations in $\mathcal{N} = 2$ supersymmetric theories. Applying localization formulae for chiral observables, we derive closed chiral trace relations relating the vacuum expectation values of chiral ring elements. In this setting, we discuss how the Ω-background breaks the polynomial nature of such relations. These results are interpreted in the light of AGT duality, thus making contact with the integrable structure of conformal field theories on Riemann surfaces.

1. Introduction

Instanton calculus in gauge theories with extended supersymmetry is a useful tool to investigate non-perturbative effects. As is well-known [1], the exact solution of the Wilsonian effective theory of $SU(2)$ $\mathcal{N} = 2$ model without matter hypermultiplet (given in term of an holomorphic function called the prepotential) is fully encoded in an hyperelliptic curve capturing both perturbative and non-perturbative contributions. The low-energy effective theory is given implicitely in terms of contour integrals of the Seiberg-Witten differential along non-contractible cycles encircling the cuts of this curve. This result was then extended to generic mass content and gauge groups [2]. SW solution shed also light on the relation between $\mathcal{N} = 2$ supersymmetric theories and integrable systems, with the Seiberg-Witten curve playing the role of the spectral curve of its integrable counterpart [3, 4, 5, 6].

Despite this suggestive picture and the presence of several undirect checks of SW solution, a direct test of non-perturbative contributions was difficult since instanton moduli space is not compact and requires a systematical regularization. Investigations in this sense (see for instance [7]) culminated in the work by Nekrasov introducing the so-called Ω-deformation [8], see also [9]. It consists in a 2-parametric Poincaré-breaking regularization which allows to use explicit localization formulae to compute the partition function order by order in the instanton number. The regularized partition function plays a central role in the AGT duality [10] relating $\mathcal{N} = 2$ supersymmetric theories and Liouville CFT on 2d Riemann surfaces.[1] In more quantitative terms, AGT relation links the instanton partition function of the deformed gauge theory with the conformal blocks of correlation functions with a suitable operator insertion in the CFT.

Recently, the interest in supersymmetric gauge theories has been extended to specific observables in the chiral ring [17], *i.e.* trace operators $\mathrm{Tr}\,\varphi^n$ involving powers of the scalar field

[1] The original AGT duality is formulated only for $\mathcal{N} = 2$ superconformal $SU(2)$ quivers, see also [11]. Similar correspondence also holds for asymptotically free theories [12]. Finally, there exist AGT relations for bigger quivers, see for example [13, 14, 15, 16].

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in the vector multiplet. Chiral observables play an important role in integrable properties of supersymmetric gauge theories [18, 19] and in the physics of surface operators [20, 21, 22, 23, 24]. For further details on the topic of the present paper, we refer to our original work [25].

As it is well-known, vacuum expectation values of chiral trace operators $\mathbf{t}_n = \langle \mathrm{Tr}\, \varphi^n \rangle$ are not independent at the classical level [17]. This is a consequence of gauge group matrix properties, so it is possible to find simple relations involving them. In the quantum theory, such relations get modified non-perturbatively with contributions involving the instanton counting parameter q. The next natural question is how such identities are further modified when the Ω-background is switched on. The aim of this paper is to discuss chiral trace relations in the pure $\mathcal{N} = 2$ deformed supersymmetric theory for various gauge groups.

2. Chiral trace relations in undeformed Seiberg-Witten theories

Chiral trace relations are easily found in undeformed theories with resolvent techniques. In pure $SU(N)$ gauge theories, the resolvent reads [17]

$$\left\langle \mathrm{Tr}\, \frac{1}{z-\varphi} \right\rangle = \frac{P'(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}. \tag{1}$$

Here, $P(z) = z^N - \sum_{l=2}^{N} \frac{\boldsymbol{u}_l}{l} z^{N-l}$ with the $\boldsymbol{u}_l$ parametrizing the $(N-1)$-dimensional quantum moduli space, while Λ is the dynamically generated scale related to the instanton counting parameter as $q \sim \Lambda^{2N}$. Chiral trace relations are obtained expanding (1) at large z. For the $SU(2)$ gauge group, the moduli space is parametrized by the quantity $\mathbf{t}_2 \equiv \boldsymbol{u}$. From (1), we get the trace relations

$$\mathbf{t}_4 = 4q + \frac{\boldsymbol{u}^2}{2}, \qquad \mathbf{t}_6 = 6q\boldsymbol{u} + \frac{\boldsymbol{u}^3}{4}, \qquad \mathbf{t}_8 = 12q^2 + 6q\boldsymbol{u}^2 + \frac{\boldsymbol{u}^4}{8}, \quad \dots \tag{2}$$

while odd traces identically vanish. Increasing the rank of the gauge group, chiral trace relations are less trivial, since more independent quantities appear, but their structure is totally similar. For the $SU(3)$ gauge group, they are $\mathbf{t}_2 = \boldsymbol{u}_2$ and $\mathbf{t}_3 = \boldsymbol{u}_3$. Then:

$$\begin{aligned} &\mathbf{t}_4 = \frac{\mathbf{t}_2^2}{2}, \qquad &&\mathbf{t}_5 = \frac{5\mathbf{t}_2\mathbf{t}_3}{6}, \qquad &&\mathbf{t}_6 = -6q + \frac{\mathbf{t}_2^3}{4} + \frac{\mathbf{t}_3^2}{3}, \\ &\mathbf{t}_7 = \frac{7}{12}\mathbf{t}_2^2\mathbf{t}_3, \qquad &&\mathbf{t}_8 = -8q\mathbf{t}_2 + \frac{\mathbf{t}_2^4}{8} + \frac{4}{9}\mathbf{t}_2\mathbf{t}_3^2, \qquad &&\dots \end{aligned} \tag{3}$$

Finally, for the $SU(4)$ gauge group, the independent quantities are $\mathbf{t}_2 = \boldsymbol{u}_2$, $\mathbf{t}_3 = \boldsymbol{u}_3$ and $\mathbf{t}_4 = \boldsymbol{u}_4 + \frac{\boldsymbol{u}_2^2}{2}$. The other relations read

$$\begin{aligned} &\mathbf{t}_5 = \frac{5\mathbf{t}_2\mathbf{t}_3}{6}, \qquad &&\mathbf{t}_6 = -\frac{\mathbf{t}_2^3}{8} + \frac{\mathbf{t}_3^2}{3} + \frac{3\mathbf{t}_2\mathbf{t}_4}{4}, \\ &\mathbf{t}_7 = \frac{7\mathbf{t}_2^2\mathbf{t}_3}{24} + \frac{7\mathbf{t}_3\mathbf{t}_4}{12}, \qquad &&\mathbf{t}_8 = 8q - \frac{\mathbf{t}_2^4}{16} + \frac{4\mathbf{t}_2\mathbf{t}_3^2}{9} + \frac{\mathbf{t}_2^2\mathbf{t}_4}{4} + \frac{\mathbf{t}_4^2}{4}, \\ &\dots \end{aligned} \tag{4}$$

In pure Seiberg-Witten theories, chiral trace relations are clearly coherent with the natural gradation $[\varphi] = 1$ (which also implies that $[\mathbf{t}_n] = n$) and $[q] = 2N$. The latter condition means that quantum contributions to trace relations start at the order $n = 2N$.

3. Chiral trace relations in Ω-deformed theories

Chiral trace relations obviously change when Ω-background is switched on. One of the most important reasons is that the original Poincaré symmetry is here downsized by the presence of the gravitational background. This has remarkable effects on chiral operators, since factorizations of the form $\langle \mathrm{Tr}\,\varphi^n\,\mathrm{Tr}\,\varphi^m\rangle \to \langle \mathrm{Tr}\,\varphi^n\rangle\langle \mathrm{Tr}\,\varphi^m\rangle$ no longer holds [26]. As a consequence, chiral trace relations are much more intricated in deformed theories. Moreover, resolvent techniques are not available in this setup, so localization formulae [27, 28] have to be adopted. Applying them, we get the vacuum expectation values of chiral trace operators in power series in the instanton counting parameter, with coefficients being functions of the scalar field vacuum expectation values $a_1, \dots, a_N$, the deformation parameters $\epsilon_{1,2}$ and eventually the masses of matter fields. The problem is therefore to express higher trace in terms of the independent ones in closed form. This is of very simple solution in the undeformed limit, since it only requires the inversion $\mathbf{t}_i(a_1, \dots, a_N)$ in favor of $a_i(\mathbf{t}_1, \dots, \mathbf{t}_N)$ with $i = 1, \dots, N$. For instance, computing $\langle \mathrm{Tr}\,\varphi^2\rangle$ in the $SU(2)$ gauge theory and then sending the deformation parameters to zero, we easily get

$$\boldsymbol{u} = 2a^2 + \frac{q}{a^2} + \frac{5q^2}{16a^6} + \frac{9q^3}{32a^{10}} + \frac{1469q^4}{4096a^{14}} + \frac{4471q^5}{8192a^{18}} + \mathcal{O}(q^6), \tag{5}$$

Inverting this equality, we express the scalar field vev in power series of the moduli $\boldsymbol{u}$ in the form

$$a = \sqrt{\frac{\boldsymbol{u}}{2}} - \frac{q}{\sqrt{2}\left(\sqrt{\boldsymbol{u}}\right)^3} - \frac{15q^2}{4\sqrt{2}\left(\sqrt{\boldsymbol{u}}\right)^7} - \frac{105q^3}{4\sqrt{2}\left(\sqrt{\boldsymbol{u}}\right)^{11}} - \frac{15015q^4}{64\sqrt{2}\left(\sqrt{\boldsymbol{u}}\right)^{15}} - \frac{153153q^5}{64\sqrt{2}\left(\sqrt{\boldsymbol{u}}\right)^{19}} + O\left(\boldsymbol{u}^{-10}\right). \tag{6}$$

Repeating the computation for higher order traces, we find

$$\begin{aligned} \mathbf{t}_4 &= 2a^4 + 6q + \frac{9q^2}{8a^4} + \frac{7q^3}{8a^8} + \frac{2145q^4}{2048a^{12}} + \frac{1575q^5}{1024a^{16}} + \mathcal{O}(q^6), \\ \mathbf{t}_6 &= 2a^6 + 15a^2 q + \frac{135q^2}{16a^2} + \frac{125q^3}{32a^6} + \frac{16335q^4}{4096a^{10}} + \frac{44343q^5}{8192a^{14}} + \mathcal{O}(q^6), \\ &\dots \end{aligned} \tag{7}$$

It is not hard to verify that, inserting the expansion (6) into higher order traces, we recover the relations (10). This method can be in principle used also in the general Ω-background. In this case, we get the more intricated quantity

$$\begin{aligned} \mathbf{t}_2 &= 2a^2 + \frac{q}{a^2} + \frac{5q^2}{16a^6} + \frac{9q^3}{32a^{10}} + \frac{1469q^4}{4096a^{14}} + \dots \\ &+ \epsilon_1\epsilon_2\left(-\frac{q^2}{8a^8} - \frac{q^3}{2a^{12}} - \frac{1647q^4}{1024a^{16}} + \dots\right) \\ &+ (\epsilon_1+\epsilon_2)^2\left(\frac{q}{4a^4} + \frac{21q^2}{32a^8} + \frac{55q^3}{32a^{12}} + \frac{18445q^4}{4096a^{16}} + \dots\right) \\ &+ (\epsilon_1\epsilon_2)^2\left(\frac{11q^2}{256a^{10}} + \frac{351q^3}{512a^{14}} + \frac{171201q^4}{32768a^{18}} + \dots\right) + \dots \end{aligned} \tag{8}$$

We can now try to invert this equation and express again a as a function of $\langle \mathrm{Tr}\,\varphi^2\rangle = \boldsymbol{u}$. If we replace this expansion in higher order traces, say for example $\mathbf{t}_4$, we get the involved expression

$$\begin{aligned} \mathbf{t}_4 &= \frac{\boldsymbol{u}^2}{2} + \frac{\left(8\boldsymbol{u} - 4\left(\epsilon_1^2 + 3\epsilon_1\epsilon_2 + \epsilon_2^2\right)\right)}{2\boldsymbol{u} - (\epsilon_1+\epsilon_2)^2} q \\ &+ \frac{8\epsilon_1\epsilon_2\left(36\boldsymbol{u}^2 + 9\epsilon_1^4 + 50\epsilon_1^3\epsilon_2 + 86\epsilon_1^2\epsilon_2^2 + 50\epsilon_1\epsilon_2^3 + 9\epsilon_2^4 - 4\boldsymbol{u}\left(9\epsilon_1^2 + 13\epsilon_1\epsilon_2 + 9\epsilon_2^2\right)\right)}{\left(-2\boldsymbol{u} + (\epsilon_1+\epsilon_2)^2\right)^3\left(4\boldsymbol{u}^2 + \left(2\epsilon_1^2 + 5\epsilon_1\epsilon_2 + 2\epsilon_2^2\right)^2 - 2\boldsymbol{u}\left(5\epsilon_1^2 + 8\epsilon_1\epsilon_2 + 5\epsilon_2^2\right)\right)} q^2 + \mathcal{O}(q^3). \end{aligned} \tag{9}$$

Clearly, this method would not give closed trace relations. Moreover, even in the pure case similar expressions do not truncate at any order in the instanton number, as opposite to the undeformed case in which quantum corrections are polynomial in q.

These problems could discourage in proceeding further, but we can ascribe it *a posteriori* to the unnecessary requirement that chiral trace relations will depend polynomially on $\boldsymbol{u}$ in a general Ω-background. In fact, relaxing this assumption and allowing for the presence of derivatives of $\boldsymbol{u}$ with respect to the instanton counting parameter, we are able to derive closed trace relations! Although it can appear somewhat unexpected, this new feature has a nice AGT justification, see Section 4. The procedure we used to generate closed relation is simple. For each $\mathbf{t}_n$, we express it as combinations of independent traces and their derivatives, then we fix the coefficients in order to get agreement with localization formulae. Denoting $\boldsymbol{u}' = q\partial_q\boldsymbol{u}$, we get for the $SU(2)$ pure case

$$
\begin{aligned}
\mathbf{t}_3 &= 0,\\
\mathbf{t}_4 &= \frac{1}{2}\boldsymbol{u}^2 - \epsilon_1\epsilon_2\boldsymbol{u}' + 4q,\\
\mathbf{t}_5 &= 10\,(\epsilon_1+\epsilon_2)\,q,\\
\mathbf{t}_6 &= \frac{1}{4}\boldsymbol{u}^3 - \frac{3}{2}\epsilon_1\epsilon_2\boldsymbol{u}\boldsymbol{u}' + \epsilon_1^2\epsilon_2^2\boldsymbol{u}'' + 6q\boldsymbol{u} + 6q\left(3\epsilon_1^2+4\epsilon_1\epsilon_2+3\epsilon_2^2\right),\\
\mathbf{t}_7 &= (\epsilon_1+\epsilon_2)\left(21q\boldsymbol{u} + 7q\left(4\epsilon_1^2+3\epsilon_1\epsilon_2+4\epsilon_2^2\right)\right),\\
\mathbf{t}_8 &= \frac{1}{8}\boldsymbol{u}^4 + 6q\boldsymbol{u}^2 + 12q^2 - \epsilon_1^3\epsilon_2^3\boldsymbol{u}''' + 2\epsilon_1^2\epsilon_2^2\boldsymbol{u}\boldsymbol{u}'' + \frac{3}{2}\epsilon_1^2\epsilon_2^2\boldsymbol{u}'^2 - 12\epsilon_1\epsilon_2 q\boldsymbol{u}'\\
&\quad - \frac{3}{2}\epsilon_1\epsilon_2\boldsymbol{u}^2\boldsymbol{u}' + (52\epsilon_1^2+72\epsilon_1\epsilon_2+52\epsilon_2^2)q\boldsymbol{u} + 8q(5\epsilon_1^4+11\epsilon_1^3\epsilon_2+15\epsilon_1^2\epsilon_2^2+11\epsilon_1\epsilon_2^3+5\epsilon_2^4),\\
&\quad \ldots
\end{aligned}
\tag{10}
$$

For the $SU(3)$ gauge group, the modification coming from Ω-background reads

$$
\begin{aligned}
\mathbf{t}_4 &= \frac{1}{2}\mathbf{t}_2^2 - \epsilon_1\epsilon_2\mathbf{t}_2',\\
\mathbf{t}_5 &= \frac{5}{6}\mathbf{t}_2\mathbf{t}_3 - \frac{5}{3}\epsilon_1\epsilon_2\mathbf{t}_3',\\
\mathbf{t}_6 &= -6q + \frac{\mathbf{t}_{3,3}}{3} + \frac{1}{4}\mathbf{t}_2^3 - \frac{3}{2}\epsilon_1\epsilon_2\mathbf{t}_2\mathbf{t}_2' + \epsilon_1^2\epsilon_2^2\mathbf{t}_2'',\\
\mathbf{t}_7 &= \frac{7\mathbf{t}_2^2\mathbf{t}_3}{12} - 21q(\epsilon_1+\epsilon_2) - \frac{7}{6}\epsilon_1\epsilon_2\mathbf{t}_3\mathbf{t}_2' - \frac{7}{3}\epsilon_1\epsilon_2\mathbf{t}_2\mathbf{t}_3' + \frac{7}{3}\epsilon_1^2\epsilon_2^2\mathbf{t}_3'',\\
&\ldots
\end{aligned}
\tag{11}
$$

while for the $SU(4)$ gauge group, we find

$$
\begin{aligned}
\mathbf{t}_5 &= \frac{5\mathbf{t}_2\mathbf{t}_3}{6} - \frac{5}{3}\epsilon_1\epsilon_2\mathbf{t}_3',\\
\mathbf{t}_6 &= \frac{3\mathbf{t}_{2,4}}{4} - \frac{\mathbf{t}_2^3}{8} + \frac{\mathbf{t}_{3,3}}{3} + \frac{3}{4}\epsilon_1\epsilon_2\mathbf{t}_2\mathbf{t}_2' - \frac{1}{2}\epsilon_1^2\epsilon_2^2\mathbf{t}_2'',\\
\mathbf{t}_7 &= \frac{7\mathbf{t}_2^2\mathbf{t}_3}{24} + \frac{7\mathbf{t}_{3,4}}{12} - \frac{7}{12}\epsilon_1\epsilon_2\mathbf{t}_3\mathbf{t}_2' - \frac{7}{6}\epsilon_1\epsilon_2\mathbf{t}_2\mathbf{t}_3'' + \frac{7}{6}\epsilon_1^2\epsilon_2^2\mathbf{t}_3'',\\
&\ldots
\end{aligned}
\tag{12}
$$

We checked these expressions up to $\sim q^8$ instanton order.

To conclude this section, we would like to do some comments. First, gravitational corrections also hold in the so-called Nekrasov-Shatashvili limit $\epsilon_2 \to 0$, but in this case the differential structure of chiral ring is lost. Moreover, in this case odd traces no longer vanish (unless in the limit $\epsilon_1 = -\epsilon_2$). Finally, a relevant feature of the gravitational background is the appearence of new universal terms (*i.e.* $-\epsilon_1\epsilon_2\boldsymbol{u}'$ for $n = 4$, $-\frac{3}{2}\epsilon_1\epsilon_2\boldsymbol{u}\boldsymbol{u}'$ and $\epsilon_1^2\epsilon_2^2\boldsymbol{u}''$ for $n = 6$ *etc.*). All these features have a clear AGT explanation.

4. AGT interpretation

AGT duality is a correspondence relating deformed $\mathcal{N} = 2$ supersymmetric theories and conformal field theories on 2d Riemann surfaces. The topology of this surface is uniquely determined by the flavor and gauge groups of the 4d field theory. Quantitatively, the duality states the equivalence between the instanton partition function of the gauge theory and the conformal block of the correlation function with a suitable choice of the external operator insertions. In doing this, matter field masses and the vacuum expectation value of the scalar field are respectively mapped to the conformal dimensions of the external and intermediate operators in the conformal block, while the deformation parameters determine the central charge.

AGT duality provides an interesting point of view about the chiral ring structure. As is well-known [29], two-dimensional CFTs present a suggestive integrable structure which is proved by the existence of an infinite tower of local integrals of motion. The basic observation in understanding the duality at this level is the strong evidence that the chiral ring of the gauge theory retains the structure of integrals of motion, so chiral trace relations should emerge in a very natural way in this setup. This is clear in particular from the results of [30, 31] for the $SU(2)$ $\mathcal{N} = 2$ theory with $N_f = 4$ fundamentals hypermultiplets. As stated by AGT duality, this theory is related to a Liouville CFT on the 4-punctured sphere, with the instanton partition function being identified with the 4-point conformal block. A convenient way [32] to link the two AGT sides consists in considering the symmetry algebra to a Vir $\oplus$ Heis, with commutation relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [a_m, a_n] &= \frac{m}{2}\delta_{m+n,0}, \qquad [L_m, a_n] = 0. \end{aligned} \tag{13}$$

The first local integrals of motion are in this case

$$\begin{aligned} I_2 &= L_0 - \frac{c}{24} + 2\sum_{k=1}^{\infty} a_{-k}a_k, \\ I_3 &= \sum_{k\in\mathbb{Z}/\{0\}} a_{-k}L_k + 2iQ\sum_{k=1}^{\infty} k a_{-k}a_k + \frac{1}{3}\sum_{i+j+k=0} a_i a_j a_k, \\ I_4 &= 2\sum_{k=1}^{\infty} L_{-k}L_k + L_0^2 - \frac{c+2}{12}L_0 + \ldots, \\ &\ldots \end{aligned} \tag{14}$$

where Q parametrizes the central charge as $c = 1 + 6Q^2$.[2] The authors of [30, 31] found a simple relation between the vacuum expectation values of chiral trace operators and the CFT correlation function with charge insertions:

$$G_k^{N_f=4}(\alpha_i|z) = \langle V_{\alpha_1}(\infty)V_{\alpha_2}(1)I_kV_{\alpha_3}(q)V_{\alpha_4}(0)\rangle, \tag{15}$$

[2] Note that $Q = b + b^{-1}$ where b is related to the deformation parameter as $b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$.

where V_α are primary fields in this extended algebra with conformal dimension $\Delta_\alpha = \alpha(Q-\alpha)$. Guided by these results, in [25] we searched for similar relations in the $SU(2)$ $\mathcal{N}=2^*$ and pure gauge theories. In the former case, AGT correspondence states that the dual quantity in the CFT side is the 1-point conformal block on the torus

$$Z_{\text{inst}} = \left[q^{-\frac{1}{24}}\eta(q)\right]^{2\Delta_m - 1} \text{Tr}_\Delta\left(\mathcal{O}_{\Delta_m} q^{L_0-\Delta}\right), \tag{16}$$

where the trace is taken on the descendants of the Virasoro primary in the intermediate channel and Δ_m is the conformal dimension of the external operator. Starting from Matone's relation [27, 33] and with simple calculations, it is easy to prove that[3]

$$\boldsymbol{u} = (4m^2 - \epsilon_1^2 - \epsilon_2^2)\frac{\text{E}_2 - 1}{24} - \frac{\epsilon_1\epsilon_2}{12} - 2\epsilon_1\epsilon_2 \frac{\text{Tr}_\Delta\left(\mathcal{O}_{\Delta_m}(L_0 - \frac{c}{24})q^{L_0-\frac{c}{24}}\right)}{\text{Tr}_\Delta\left(\mathcal{O}_{\Delta_m} q^{L_0-\frac{c}{24}}\right)}, \tag{17}$$

where the last term is nothing but the insertion of the Virasoro part of the integral of motion I_2 in the instanton partition function. This feature is very similar to the $N_f = 4$ case, so we defined the CFT quantities

$$G_k^{\mathcal{N}=2^*} = \text{Tr}_\Delta\left(\mathcal{O}_{\Delta_m} I_k^{\text{Vir}} q^{L_0-\frac{c}{24}}\right), \tag{18}$$

where I_k^{Vir} is the Virasoro part of integrals (14). We argued that they are related to the chiral trace vacuum expectation values.

The pure theory is something different from the previous cases. In fact, the duality presented in [10] only holds for $\mathcal{N}=2$ superconformal quivers of $SU(2)$. However, asymptotically free theories such as the pure $SU(2)$ case also admit a similar statement. To this aim, Gaiotto introduced in [12] a peculiar state lying in the Verma module of the highest weight of conformal dimension $\Delta = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1\epsilon_2}$. This state is defined as a formal power series in Λ,

$$|\Lambda, \Delta\rangle = \sum_{k=0}^{\infty} \Lambda^{2k} v_k, \tag{19}$$

with each component satisfies the following constraints:

$$L_1 v_k = v_{k-1}, \quad L_2 v_k = 0. \tag{20}$$

The starting point of the sequence is the condition $v_0 = |\Delta\rangle$, which is of course the highest weight state in the Verma module. The AGT statement for the pure $SU(2)$ theory is that the norm of such a state exactly reproduces the Nekrasov instanton partition function, so Λ^4 acquires the meaning of instanton counting parameter:

$$Z_{\text{inst}} = \langle\Lambda, \Delta|\Lambda, \Delta\rangle = \sum_{k=0}^{\infty} \Lambda^{4k} ||v_k||^2. \tag{21}$$

[3] Here, E_2 is the Eisenstein quasi-modular form

$$\text{E}_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

with $\sigma_1(n)$ is the sum of divisors on n.

With the same approach as before, we can built new quantities in the CFT side by inserting the local integrals of motion in the brackets:

$$G_k^{\text{pure}} = \langle \Lambda, \Delta | I_k^{\text{Vir}} | \Lambda, \Delta \rangle, \quad G^{\text{pure}} = \langle \Lambda, \Delta | \Lambda, \Delta \rangle. \tag{22}$$

Again, we claim that these quantities are related to the chiral trace vacuum expectation values in the gauge theory. However, a systematical check of this conjecture requires a basis choice in the algebra of integrals of motion and a precise matching between the two sides, which unfortunately is still missing in general.

4.1. Leading terms and universality

Despite the absence of the dictionary between the chiral ring elements and correlation functions with the insertion of integrals of motion, it is possible to extract some predictions. The basic observation is that in all cases chiral trace relations share some common terms:

$$\begin{aligned}
\mathbf{t}_4 &= \frac{1}{2}\boldsymbol{u}^2 - \epsilon_1\epsilon_2\boldsymbol{u}' + \dots, \\
\mathbf{t}_6 &= \frac{1}{4}\boldsymbol{u}^3 - \frac{3}{2}\epsilon_1\epsilon_2\boldsymbol{u}\boldsymbol{u}' + \epsilon_1^2\epsilon_2^2\boldsymbol{u}'' + \dots, \\
\mathbf{t}_8 &= \frac{1}{8}\boldsymbol{u}^4 - \frac{3}{2}\epsilon_1\epsilon_2\boldsymbol{u}^2\boldsymbol{u}' + \frac{3}{2}\epsilon_1^2\epsilon_2^2(\boldsymbol{u}')^2 + 2\epsilon_1^2\epsilon_2^2\boldsymbol{u}\boldsymbol{u}'' - \epsilon_1^3\epsilon_2^3\boldsymbol{u}''' + \dots, \\
&\dots
\end{aligned} \tag{23}$$

These leading terms are special monomials of $\boldsymbol{u}$ and its derivatives with coefficients depending only the deformation parameters ϵ_i. They are indeed universal, as they not depend on the particular model. In order to understand their origin, the key point is that even integrals of motion present powers of the Virasoro generators L_0^n, which act in CFT correlation function through the differential operators $(q\partial_q)^n$. It is therefore natural to guess that leading terms are associated to such special terms. In mathematical terms, this is nothing but the requirement that

$$G_{2n} \underset{\text{leading}}{\sim} (q\partial_q)^n G. \tag{24}$$

We can therefore write

$$\mathbf{t}_{2n} \underset{\text{leading}}{\sim} 2\frac{(-\epsilon_1\epsilon_2 q\partial_q)^n G}{G}. \tag{25}$$

From Matone's relation

$$\boldsymbol{u} = -2\epsilon_1\epsilon_2 q\partial_q \log Z, \tag{26}$$

where Z now is the total partition function $Z = Z_{\text{cl}} Z_{\text{1-loop}} Z_{\text{inst}}$. The classical part is $Z_{\text{cl}} = \exp\left(-(\epsilon_1\epsilon_2)^{-1}a^2\log q\right)$, while the 1-loop contribution is q-independent. Then (apart for contributions which are not relevant in this discussion) we can identify G with the whole partition function Z, or in other words

$$G = \exp\left(-\frac{U}{2\epsilon_1\epsilon_2}\right), \tag{27}$$

where $\boldsymbol{u} = q\partial_q U$. Using this representation and (25), we can easily determine the terms in (23), even without mentioning which theory we are considering. Indeed, the results reported in this section depends only on the structure of the local integrals of motion of the conformal field theory. Eq. (24) holds both for G_k defined through (18) in the $\mathcal{N} = 2^*$ model and through (22) for the pure theory. Therefore, we conclude the leading terms are universal, in perfect agreement with the explicit form of chiral trace relations.

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