Z_3-graded colour Dirac equations for quarks, confinement and generalized Lorentz symmetries

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ABSTRACT

We propose a modification of standard QCD description of the colour triplet of quarks by introducing a 12-component colour generalization of Dirac spinor, with built-in Z_3 grading playing an important algebraic role in quark confinement. In "colour Dirac equations" the $SU(3)$ colour symmetry is entangled with the Z_3-graded generalization of Lorentz symmetry, containing three 6-parameter sectors related by Z_3 maps. The generalized Lorentz covariance requires simultaneous presence of 12 colour Dirac multiplets, which lead to the description of all internal symmetries of quarks: besides $SU(3) \times SU(2) \times U(1)$, the flavour symmetries and three quark families.

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1. Introduction

It is well known that colour symmetries play a double role - they describe $SU(3)$ gauge symmetry group in QCD and are linked with quark confinement, which is obtained usually as dynamical consequence of strong forces between quarks growing linearly with their spatial separation (see e.g. [1], [2]).

In the present paper we would like to propose an alternative algebraic approach to the confining aspect of colour symmetries. For that purpose we replace the usual tensor product of Lorentz and colour $SU(3)$ group actions by the entanglement of space-time and colour symmetries generated by the Z_3 symmetry which plays also an important role in the appearance of fractional electric and baryonic charges of quarks.

We shall show that such an entanglement appears naturally when we generalize the derivation of the 4-component Dirac equation as given by particular $Z_2 \times Z_2$ symmetric coupling of a pair of 2-component Pauli spinors ([3]), to the $Z_3 \times Z_2 \times Z_2$ symmetry which unifies in a specific manner (see (13)) the system of six linear equations for six Pauli spinors. In such a way we obtain a new description of quarks endowed with colour as irreducible

12-component analogs of Dirac spinors, with internal (colour) and space-time degrees of freedom entangled in a non-trivial manner.

By studying the solutions of free colour Dirac equation one arrives at possible algebraic explanation of quark confinement phenomenon: all exponential solutions of this system, which are wave-like, depend on complex wave vectors with imaginary parts damping the free propagation of coloured quarks; however, certain cubic products of such solutions result in the cancellation of imaginary parts and produce propagating waves corresponding to the freely moving composite particle states (see e.g. [4], [5], [6]).

The plan of our paper is as follows:

In Section 2 we present how to obtain the 12-component colour Dirac equation which describes the dynamics of quark and anti-quark endowed with colour and spin by introducing 12-component colour Dirac field equations displaying the $Z_3 \times Z_2 \times Z_2$ symmetry.

In Section 3 we consider the extension of relativistic symmetry by incorporating in 18-parameter generalized Lorentz symmetry the standard 6-parameter Lorentz subgroup and the additional pair of complex-conjugated 6-parameter Lorentz-like sectors. Such Z_3-generalization $\mathcal{L}$ of the Lorentz algebra can be decomposed into the following Z_3-graded sum of three sectors:

$$\mathcal{L} = L^{(0)} \oplus L^{(1)} \oplus L^{(2)}, \quad \left[L^{(k)}, L^{(m)} \right] \in L^{(k+m)}, \tag{1}$$

where $k, m = 0, 1, 2$ and $(k+m)$ is mod 3, $L^{(0)}$ describes the standard Lorentz sector, while adding $L^{(1)}$ and $L^{(2)} = \left(L^{(1)} \right)^\dagger$ († denotes

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Hermitian conjugate) extends it to a Z_3-graded generalized Lorentz algebra.

To obtain the representation of generalized Lorentz algebra $\mathcal{L}$ one should introduce the set of twelve 12×12 generalized Dirac matrices $\Gamma_F^\mu = (\Gamma_F^0, \Gamma_F^k)$ ($F = 1, 2, ..12$). In order to show it we introduce 12×12 matrix $\mathcal{S}$ describing the spinor representation of the generalized Lorentz algebra $\mathcal{L}$, which contains a 6-parameter subgroup $\mathcal{S}^{(0)} \subset \mathcal{S}$ representing the standard Lorentz group.

We shall study the transformations of colour Dirac matrices $\Gamma_{(F)}^\mu$ under the 18-parameter spinor transformation $\Psi' = \mathcal{S}\Psi$, where Ψ is a 12-component colour Dirac spinor, and includes its standard Lorentz subgroup $\Psi' = \mathcal{S}^{(0)}\Psi$. We obtain that the covariance under spinor Lorentz transformations $\mathcal{S}^{(0)}\Gamma_{(F)}^\mu[\mathcal{S}^{(0)}]^{-1}$ requires the introduction of Lorentz doublets of colour-Dirac matrices, while the closure of the map $\mathcal{S}\Gamma_{(F)}^\mu\mathcal{S}^{-1}$ leads to the appearance of 12 different Lorentz doublets of matrices $\Gamma_{(F)}^\mu$. One can further argue that the lowest-dimensional spinor space on which act the generalized Lorentz transformations in a closed and faithful way describes six different types of coloured quarks. The standard Lorentz covariance and the presence of Lorentz doublets of colour Dirac fields leads to an additional Z_2-factor in front of $Z_3 \otimes Z_2 \otimes Z_2$, which may be responsible for the appearance of weak isospin doublets of quarks. In this way extending Z_3 to $Z_2 \times Z_3$ permits to introduce the Standard Model's internal symmetries $SU(3) \times SU(2) \times U(1)$, which by gauging generate the octet of gluons, three vector mesons B_μ^a and the electromagnetic field A_μ. Finally, the generalized Lorentz covariance implies the six-fold enlargement of the representation space of standard Lorentz symmetries and allows to accomodate quark flavour doublets and the triplet of quark families.

In Section 4 we display the complete basis of solutions of the generalized "colour" Dirac equation, all of which are represented by exponential with non-vanishing damping factors. Further, we illustrate the confinement by showing that certain ternary products of such damped solutions can propagate freely, and can asymptotically represent the composite free baryon state.

The colour Dirac spinor components satisfy sixth order homogeneous field equation, which in the four-momentum space factorizes into three mass shells: one with real mass and two with conjugate complex masses, related respectively with the Z_3-graded sectors $L^{(0)}$, $L^{(1)}$ and $L^{(2)}$ of $\mathcal{L}$. We point out that such a triplet of masses can coincide with the mass spectrum of a particular Z_3-covariant perturbative Lee-Wick QFT [8].

We conclude that recent results in the description of renormalizability and unitarity of the Lee-Wick perturbative QFT [9] justify the conjecture that our model of coloured quarks may be also renormalizable and unitary.

2. From Dirac to coloured Dirac equation

The Dirac equation for the electron (or any spin $\frac{1}{2}$ particle with non-zero mass m) ([7]) can be written in a compact way as follows:

$$\gamma^\mu p_\mu \psi = mc \psi \quad \text{with} \quad \psi = (\psi_+, \psi_-)^T, \tag{2}$$

where $p_\mu = -i\hbar\partial_\mu$, $\psi_\pm$ are two complex 2-component Pauli spinors, and as Dirac matrices γ^μ one can choose

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2, \ \gamma^k = (i\sigma_2) \otimes \sigma^k, \tag{3}$$

where $\sigma_0 = \mathbb{1}_2$, and σ^k ($k = 1, 2, 3$) are Pauli matrices. The Dirac matrices realize the 4-dimensional Clifford algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4, \ \eta^{\mu\nu} = \text{diag}(+, -, -, -). \tag{4}$$

Under the Lorentz transformation

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu \tag{5}$$

the spinor field $\psi = \psi^A$ ($A = 1, 2, 3, 4$) in (2) transforms as follows:

$$\psi'(x^{\rho'}) = \psi'(\Lambda^{\rho'}_\mu x^\mu) = S\psi(x^\mu). \tag{6}$$

In order to ensure the standard Lorentz covariance, the condition relating the vectorial and spinorial realizations of the Lorentz group $O(3, 1) \simeq SL(2, \mathbf{C})$ is:

$$S\gamma^{\mu'}S^{-1} = \Lambda^{\mu'}_\nu\gamma^\nu. \tag{7}$$

The spinorial representation S is given by the formula

$$S = exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{8}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the corresponding infinitesimal vectorial representation is given by the formula

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \ \text{where} \ \omega_{\mu\nu} = \eta_{\mu\lambda}\omega^\lambda_\nu = -\omega_{\nu\mu}, \tag{9}$$

with three independent Lorentz boosts ($\omega_{0k} = -\omega_{k0}$) and three independent spatial rotations ($\omega_{ij} = -\omega_{ji}$)).

The generalized Dirac equation incorporating colour degrees of freedom in a Z_3-symmetric way was proposed in [4], [5], [6] after introducing three pairs of independent Pauli spinors

$$\varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \ \varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \ \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix},$$
$$\chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \ \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \ \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}, \tag{10}$$

with Pauli sigma-matrices acting on them in a natural way. These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".

The cyclic group Z_3 is represented on the complex plane by multiplicative group of three complex numbers, generated by powers of $j = e^{\frac{2\pi i}{3}}$, namely:

$$j = e^{\frac{2\pi i}{3}}, \ j^2 = e^{\frac{4\pi i}{3}}, \ j^3 = 1, \ 1 + j + j^2 = 0. \tag{11}$$

The $Z_2 \times Z_2$ symmetry of the Dirac equation can be made explicit if we multiply (2) by γ^0 and get a system of two equations entangling two Pauli spinors:

$$\begin{aligned} E\psi_+ &= mc^2\psi_+ + c\boldsymbol{\sigma}\mathbf{p}\psi_-, \\ E\psi_- &= -mc^2\psi_- + c\boldsymbol{\sigma}\mathbf{p}\psi_+. \end{aligned} \tag{12}$$

The system (12) displays two discrete Z_2 symmetries: the space reflection simultaneously changing directions of spin and momentum, $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, $\mathbf{p} \to -\mathbf{p}$, and the particle-antiparticle symmetry which is realized in (12) by the transformation $m \to -m$, $\psi_+ \to \psi_-$, $\psi_- \to \psi_+$.

In what follows, we extend the $Z_2 \times Z_2$ symmetry by Z_3 group, so that the system will mix not only the two spin $\frac{1}{2}$ states and particles with anti-particles, but the three colours as well. The standard Dirac equation (12) is extended in the following way in terms of six entangled Pauli spinors:

$$E \, \varphi_+ = mc^2 \, \varphi_+ + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \chi_-,$$
$$E \, \varphi_- = -mc^2 \, \varphi_- + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \chi_+$$
$$E \, \chi_+ = j \, mc^2 \, \chi_+ + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi_-,$$
$$E \, \chi_- = -j \, mc^2 \, \chi_- + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi_+ \qquad (13)$$
$$E \, \psi_+ = j^2 \, mc^2 \, \psi_+ + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \varphi_-,$$
$$E \, \psi_- = -j^2 \, mc^2 \, \psi_- + c \, \boldsymbol{\sigma} \cdot \mathbf{p} . \varphi_+$$

The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-colour symmetry is realized by multiplication of mass m by j each time the colour changes, i.e. more explicitly, Z_3 symmetry is realized as follows:

$$m \to jm, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm, \qquad (14)$$
$$m \to jm, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm. \qquad (15)$$

The energy operator is diagonal; the mass operator is diagonal as well, but its elements are described by the powers of the sixth root of unity $q = e^{\frac{2\pi i}{6}}$: $q^6 = 1$, $q = -j^2$, $q^2 = j$, $q^3 = -1$, $q^4 = j^2$, $q^5 = -j$. Choosing a particular basis in the space of "coloured spinors" (10), such that $\Psi^T = [\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$, we rewrite (13) in compact form as a 12-component equation:

$$E \, \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \, \Psi = \Big[Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p} + mc^2 B \otimes \sigma_3 \otimes \mathbb{1}_2 \Big] \Psi. \qquad (16)$$

$$\text{where} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (17)$$

Note that $B^2 = B^\dagger$, with $B^\dagger$ the Hermitian conjugate of B, $B^3 = \mathbb{1}_3$, $Q_3^2 = Q_3^\dagger$ and $Q_3^3 = \mathbb{1}_3$. Eq. (16) can be presented in a way recalling much better the structure of the original Dirac equation (2) if we multiply the equation (16) from the left by $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$. We get

$$\Big[E \, B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c \boldsymbol{\sigma} \cdot \mathbf{p} \Big] \Psi = mc^2 \, \mathbb{1}_{12} \Psi. \qquad (18)$$

Here $B^\dagger = \text{diag}[1, j^2, j]$, so that $BB^\dagger = B^\dagger B = \mathbb{1}_3$ and $Q_2 = BQ_3$. The full set of matrices Q_A and $Q_B^\dagger$, $A, B = 1, 2, 3$, together with two diagonal traceless matrices B and $B^\dagger$ generated by B and Q_3 form a special basis of the $SU(3)$ algebra [10]. They can be obtained by iteration, using the following multiplication table:

$$B Q_A = j^2 Q_A B = Q_{A+1}, \quad B^\dagger Q_A = j Q_A B^\dagger = Q_{A-1},$$
$$Q^\dagger B = j^2 B Q_A^\dagger = Q_{A-1}^\dagger, \quad Q_A^\dagger B^\dagger = B^\dagger Q_A^\dagger = Q_{A+1}^\dagger,$$
$$Q_A Q_{A-1} = j Q_{A+1}^\dagger, \quad Q_{A-1}^\dagger Q_A^\dagger = j^2 Q_{A+1}, \qquad (19)$$
$$Q_A Q_{A+1}^\dagger = B^\dagger, \quad Q_A Q_{A-1}^\dagger = B,$$
$$Q_A^\dagger Q_{A-1} = j B^\dagger, \quad Q_A^\dagger Q_{A+1} = j^2 B,$$

and of course $Q_A Q_A^\dagger = Q_A^\dagger Q_A = \mathbb{1}_3$, where the indices A, $A + 1$, $A - 1$ are always taken modulo 3, so that e.g. $3 + 1 \,|_{modulo\ 3} = 4 \,|_{modulo\ 3} = 1$, etc., and the cube of each of the eight matrices in (19) is the unit 3×3 matrix.

Let us introduce the four 12×12 matrices Γ^0, Γ^i given by the following formula (20).

$$\Gamma^0 = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k, \qquad (20)$$

Now the system (18) can be written in the 12-dimensional Dirac-like form:

$$\Gamma^\mu \, p_\mu \, \Psi = mc \, \Psi, \quad \text{with} \quad p_0 = E/c. \qquad (21)$$

It can be calculated [5] that the "colour Dirac operator" on the left-hand side of Eq. (21) has the following important algebraic properties:

$$\left(\Gamma^\mu \, p_\mu \right)^6 = (p_0^6 - |\mathbf{p}|^6) \, \mathbb{1}_{12}, \quad \det(\Gamma^\mu \, p_\mu) = (p_0^6 - |\mathbf{p}|^6)^2,$$
$$p_0^6 - |\mathbf{p}|^6 = (p_0^2 - |\mathbf{p}|^2)(p_0^2 - j \, |\mathbf{p}|^2)(p_0^2 - j^2 \, |\mathbf{p}|^2). \qquad (22)$$

3. Implementing standard and generalized Lorentz covariance

It should be stressed that the 12×12 matrices Γ^μ appearing in the coloured Dirac equation (21) do not span 4-dimensional Clifford algebra. In fact, the $Z_3 \otimes Z_2$ structure of Γ^μ-matrices implies that only their sixth powers are proportional to the unit matrix $\mathbb{1}_{12}$ (see also (22)). Thus, in order to obtain the realization of $D = 4$ Lorentz algebra generators one can not use just two standard commutators

$$J_i = \frac{i}{2} \epsilon_{ijk} \left[\Gamma^j, \Gamma^k \right], \quad K_l = \frac{1}{2} \, [\Gamma_l, \Gamma_0] \,. \qquad (23)$$

However, the generators $\left(J_i^{(0)}, K_l^{(0)} \right)$ satisfying the standard Lorentz algebra relations (see also (28) for $r = 0, s = 0$) can be defined by triple commutators:

$$[J_i, [J_j, J_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \, J_l^{(0)},$$
$$[K_i, [K_j, K_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \, K_l^{(0)} \,. \qquad (24)$$

Indeed, substituting in (24) the explicit form of Γ^μ given in (20), we get

$$J_i = -\frac{i}{2} Q_2^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l = -\frac{1}{2} Q_1 \otimes \sigma_1 \otimes \sigma_l,$$
$$J_i^{(0)} = -\frac{i}{2} \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l^{(0)} = -\frac{1}{2} \mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_l. \qquad (25)$$

In order to close the generalized Lorentz algebra (1) where $L^{(0)} = (J_i^{(0)}, K_j^{(0)})$, $L^{(1)} = (J_i^{(1)}, K_j^{(1)})$, $L^{(2)} = (J_i^{(2)}, K_j^{(2)})$, one should supplement (24) by two missing triple commutators:

$$[J_i, [J_j, K_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \, K_l^{(2)},$$
$$[K_i, [K_j, J_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \, J_l^{(1)}, \qquad (26)$$

where using the representation (25) we get

$$J_l^{(1)} = -\frac{i}{2} Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(2)} = -\frac{1}{2} Q_3^\dagger \otimes \sigma_1 \otimes \sigma_i. \qquad (27)$$

The full set of Z_3-graded relations defining the algebra (1) (r, s, $r + s$ are modulo 3):

$$\left[J_i^{(r)}, J_k^{(s)} \right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)} \right] = \epsilon_{ikl} K_l^{(r+s)},$$
$$\left[K_i^{(r)}, K_k^{(s)} \right] = -\epsilon_{ikl} J_l^{(r+s)}. \qquad (28)$$

We see that from commutators $[K_i^{(1)}, K_m^{(1)}] \simeq J^{(2)}$ and $[J^{(1)}, J^{(1)}] \simeq J^{(2)}$ one gets the remaining generators of $\mathcal{L}$:

$$J_i^{(2)} = -\frac{i}{2} Q_3^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_m^{(1)} = -\frac{1}{2} Q_3 \otimes \sigma_1 \otimes \sigma_m. \qquad (29)$$

The formulae (25), (27) and (29) describe the realization of $\mathcal{L}$ which follows from the choice (20) of matrices Γ^μ.

Before considering standard and generalized Lorentz covariance we shall introduce the following notation:

$$\Gamma^\mu_{(A;\alpha)} = I_A \otimes \sigma_\alpha \otimes \sigma^\mu, \ A = 0, 1, .., 8; \ \alpha = 2, 3; \ \mu = 0, 1, 2, 3 .$$

(30)

Let the 3×3 "colour matrices" I_A appearing as the first factor in (30) be defined as follows: $I_0 = \mathbb{1}_3$, $I_r = Q_r$, $I_{r+3} = Q_r^\dagger$, $I_7 = B$, $I_8 = B^\dagger$. Then the original Γ-matrices given by (20) are encoded as $\Gamma^0_{(8,3)} = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$ and $\Gamma^i_{(2;2)} = Q_2 \otimes (i\sigma_2) \otimes \sigma^i$. The eight matrices with $A = 1, 2, ...8$ with the multiplication rules given in (19) span the ternary basis, generated by the cyclic Z_3-automorphism of the $SU(3)$ algebra ([10], Sect. 8).

In order to get the closed formula for the action $\mathcal{S}^{(0)}\Gamma^\mu[\mathcal{S}^{(0)}]^{-1}$ of classical spinorial Lorentz symmetries generated by $L^{(0)}$, we should introduce the pairs of Γ^μ-matrices $\Gamma^\mu = (\Gamma^i_{(A;2)}, \ \Gamma^0_{(B;3)})$ and $\tilde{\Gamma}^\mu = (\Gamma^i_{(B;2)}, \ \Gamma^0_{(A;3)})$, $A \ne B$. For any choice of Γ^μ's in (30) we get:

$$\left[J^{(0)}_i, \Gamma^j_{(A;\alpha)} \right] = \epsilon_{ijk}\Gamma^k_{(A;\alpha)}, \quad \left[J^{(0)}_i, \Gamma^0_{(A;\alpha)} \right] = 0,$$

(31)

and the boosts $K^{(0)}_i$ act covariantly on doublets $\left(\Gamma^\mu, \tilde{\Gamma}^\mu \right)$ as follows:

$$\left[K^{(0)}_i, \Gamma^j_{(A;2)} \right] = \delta^j_i \, \Gamma^0_{(A;3)}, \quad \left[K^{(0)}_i, \Gamma^0_{(B;3)} \right] = \Gamma^i_{(B;2)},$$
$$\left[K^{(0)}_i, \Gamma^j_{(B;2)} \right] = \delta^j_i \, \Gamma^0_{(B;3)}, \quad \left[K^{(0)}_i, \Gamma^0_{(A;3)} \right] = \Gamma^i_{(A;2)},$$

(32)

(with $A \ne B$), i.e. the standard Lorentz covariance requires the *doublet* of coloured Dirac spinors; In particular, the Γ^μ matrices (20) should be supplemented by:

$$\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes (\sigma_3) \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^k_{(8;2)} = B^\dagger \otimes i\sigma_2 \otimes \sigma^k. \ (33)$$

One can conjecture that the pairs of Γ-matrices generated by the standard Lorentz covariance requirement can be used for the introduction of weak isospin doublets of the $SU(2) \times U(1)$ electroweak symmetry. In such a way one can conclude that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of Standard Model follow from the imposition of standard Lorentz covariance on colour Dirac multiplets.

Our next goal is to study the generalized Lorentz covariance of coloured Dirac equations, by generalization of standard invariance condition (7) and incorporating the standard Γ^μ-matrices (20) into an irreducible representation of $\mathcal{L}$. For this purpose, we should study the 18-parameter symmetry transformation $\Gamma^\mu \to \mathcal{S}\Gamma^\mu\mathcal{S}^{-1}$, where

$$\mathcal{S} = \prod_{r=0}^2 e^{i\left[\alpha^k_{(r)} J^{(r)}_k + \beta^m_{(r)} K^{(r)}_m \right]},$$

(34)

with $\alpha^k_{(0)}$, $\beta^k_{(0)}$ real, $(\alpha^k_{(1)})^* = \alpha^k_{(2)}$, $(\beta^k_{(1)})^* = \beta^k_{(2)}$, $J^{\dagger(1)}_k = J^{(2)}_k$ and $K^{\dagger(1)}_m = K^{(2)}_m$. It follows that in order to obtain the closure of the faithful action of generators $(J^{(s)}_k, \ K^{(s)}_m)$ $(s = 0, 1, 2)$ on matrices Γ^μ_k, one should introduce two sets Γ^μ_k, $\Gamma^\mu_{\dot{a}} = (\Gamma^\mu_{(a)})^\dagger$ $(a = 1, 2, ...6)$ of coloured 12×12 Dirac matrices supplemented by Lorentz doublet partners $(\tilde{\Gamma}^\mu_{(a)}, \tilde{\Gamma}^\mu_{(\dot{a})})$. If we choose $(J^{(1)}_k, K^{(1)}_m)$ as given by Eqs. (27), (29), and assume that $\Gamma^\mu_{(1)}$ is described by the formula (20), by calculating the multicommutators of $\left(J^{(1)}_i, K^{(1)}_l \right)$ with the set $\Gamma^\mu_{(a)}$, $(a = 1, 2...6)$, we get the following sextet of Γ-matrices closed under the action of $L^{(1)}$:

$$\Gamma^\mu_{(1)} = \left(\Gamma^0_{(8;3)}, \ \Gamma^i_{(2;2)} \right); \ \Gamma^\mu_{(2)} = \left(\Gamma^0_{(2;2)}, \ \Gamma^i_{(4;3)} \right);$$
$$\Gamma^\mu_{(3)} = \left(\Gamma^0_{(4;3)}, \ \Gamma^i_{(8;2)} \right); \ \Gamma^\mu_{(4)} = \left(\Gamma^0_{(8;2)}, \ \Gamma^i_{(2;3)} \right);$$
$$\Gamma^\mu_{(5)} = \left(\Gamma^0_{(2;3)}, \ \Gamma^i_{(4;2)} \right); \ \Gamma^\mu_{(6)} = \left(\Gamma^0_{(4;2)}, \ \Gamma^i_{(8;3)} \right).$$

(35)

The realization of $L^{(2)}$ sector is obtained by introducing the Hermitean-conjugate sextet $\Gamma^\mu_{(\dot{a})} = (\Gamma^\mu_{(a)})^\dagger$; further one should add $\tilde{\Gamma}^\mu_{(\dot{a})} = (\tilde{\Gamma}^\mu_{(a)})^\dagger$ due to standard Lorentz covariance. The generalized Lorentz transformations of 24 matrices $\Gamma^\mu_{(F)} = (\Gamma^\mu_{(a)}, \Gamma^\mu_{(\dot{a})}; \ \tilde{\Gamma}^\mu_{(a)}, \tilde{\Gamma}^\mu_{(\dot{a})})$ will be expressed by the following generalization of the formula (7)

$$\mathcal{S}\Gamma^\mu_{(F)}\mathcal{S}^{-1} = \Lambda^{\mu \ (G)}_{\nu \ (F)} \, \Gamma^\nu_{(G)}, \quad \mu, \nu = 0, 1, 2, 3; \ \ F, G = 1, 2, ..., 24,$$

(36)

where with the help of the Baker-Campbell-Hausdorff type formula ([11]) the matrix $\Lambda^{\mu \ (G)}_{\nu \ (F)}$ can be calculated explicitly if the multi-commutators of $\Gamma^\mu_{(F)}$ with the generators of $\mathcal{L}$ are known up to the sixth order.

In order to describe in compact way the action of generalized Lorentz algebra on coloured Dirac matrices, we can introduce the 144×144 "master" $\mathbf{\Gamma}^\mu$ matrices built up in a suitable manner as a 12×12 matrix with its entries being the 12×12 coloured $\Gamma^\mu_{(a)}$ defined in (35), their Hermitean conjugates $\Gamma^\mu_{(\dot{a})}$ and their Lorentz doublet partners. In such a way one can obtain the "master" colour Dirac equation for 144-component master spinor field describing six known relativistic quarks in three flavour doublets (u, d), (c, s), (t, b). In such a scheme the sextet (35) defining six colour multiplets introduces an additional $Z_2 \times Z_3$ grading with the discrete degrees of freedom, related with flavour doublets (Z_2 grading) and the three quark families, called also "generations" (Z_3 grading). At the present stage we assume that this second Z_3 grading, related with quarks' families, contrary to the colour Z_3 grading, does not imply any entanglement of symmetries.

4. Solutions, ternary products and confinement

Let us consider the solutions of the coloured Dirac equation (21) in the exponential form $e^{k_\mu x^\mu}$. The characteristic equation of the 12×12 operator (22) yields the dispersion relation in the 4-momentum space of Fourier transforms:

$$k_0^6 - | \mathbf{k} |^6 = M^6.$$

(37)

The general solution is therefore $e^{k_0 x^0 - \mathbf{k} \cdot \mathbf{r}}$, provided the above relation (37) is satisfied, what for the choice of real k_0 means that it is given by

$$ck_0 = c\sqrt[6]{| \mathbf{k} |^6 + M^6} = \Omega(\mathbf{k}).$$

(38)

Any sixth-order root of real number provides six different values, two real ones, ($\pm\Omega$, $\mathbf{k}$), $\mathbf{k}$ real), and four other ones obtained by multiplying $\pm\Omega(\mathbf{k})$ by j and j^2, yielding full set of six solutions with ck_0 given by $\pm\Omega, \pm j\Omega, \pm j^2\Omega$. Further, if we have one solution with given k_0 and $\mathbf{k}$ satisfying (38), we get other solutions of the same form, with $\mathbf{k}$ replaced by $j\mathbf{k}$ or $j^2\mathbf{k}$ (the change $\mathbf{k} \to -\mathbf{k}$ does not introduce independent solutions, because a three-vector $\mathbf{k}$ covers the entire sphere S^2; $k_0 \to -k_0$ does matter as it distinguishes positive and negative energy states).

Combining all these possibilities we arrive at 18 different exponentials, 9 with positive k_0 and 9 with negative k_0. They can be organized in the following two sets of solutions

$$\Psi^+_{(r,s)}(t, \mathbf{r}) = e^{j^r \Omega t + j^s \mathbf{k} \cdot \mathbf{r}}, \quad \Psi^-_{(r,s)}(t, \mathbf{r}) = e^{-j^r \Omega t + j^s \mathbf{k} \cdot \mathbf{r}},$$

(39)

where $s, r = 0, 1, 2$ and Ω is given by (38).

The colour Dirac equation (21) as a system of 12 differential equations of first order should display only 12 independent solutions, six with positive and six with negative frequencies. We can

choose the six off-diagonal entries in (39), with $r \neq s$, which can be displayed in the following matrix:

$$
\begin{pmatrix}
0 & e^{\Omega t + j\mathbf{k}\cdot\mathbf{r}} & e^{\Omega t + j^2\mathbf{k}\cdot\mathbf{r}} \\
e^{j\Omega t + \mathbf{k}\cdot\mathbf{r}} & 0 & e^{j\Omega t + j^2\mathbf{k}\cdot\mathbf{r}} \\
e^{j^2\Omega t + \mathbf{k}\cdot\mathbf{r}} & e^{j^2\Omega t + j\mathbf{k}\cdot\mathbf{r}} & 0
\end{pmatrix},
\tag{40}
$$

and similarly for the negative energy values ($\Omega \to -\Omega$); one can check that the determinant of the matrix (40) displaying the six independent solutions is equal to 1.

All these twelve functions, describing propagation of coloured quarks, do not represent free waves due to the presence of damping factors. However, observing that there are only two ways of obtaining imaginary units as linear combinations of the Z_3 roots 1, j, j^2, namely

$$
\frac{1}{\sqrt{3}}(1 + 2j) = i, \quad \frac{1}{\sqrt{3}}(1 + 2j^2) = -i
\tag{41}
$$

one can produce propagating free wave-like solutions by forming two independent cubic products with positive Ω, and two ones with negative Ω. Following (40), (41) we choose the first pair of solutions as

$$
\Psi^{+}_{(B)}(t, \mathbf{r}) = \Psi^{+}_{(2,0)}\Psi^{+}_{(0,1)}\Psi^{+}_{(2,1)} = e^{-i\sqrt{3}(\Omega t + \mathbf{kr})},
\tag{42}
$$

$$
\bar{\Psi}^{+}_{(B)}(t, \mathbf{r}) = \Psi^{+}_{(1,0)}\Psi^{+}_{(0,2)}\Psi^{+}_{(1,2)} = e^{i\sqrt{3}(\Omega t + \mathbf{kr})},
\tag{43}
$$

and two ones with negative Ω. With two additional solutions obtained by replacing Ω by $-\Omega$ we get just the right number of four plane wave solutions needed to describe a massive spin $\frac{1}{2}$ particle - a composite three-quark free baryon wave function. Similarly, due to the relation $(j - j^2)/\sqrt{3} = i$, the quark-antiquark pairs of solutions with positive and negative frequencies will provide the particle and anti-particle spin-0 meson plane waves.

The 4-vector $[\frac{\Omega}{c}, \mathbf{k}]$ in baryonic wave functions (42), (43) does not satisfy the usual quadratic dispersion relation $\omega^2 = c^2\mathbf{k}^2 + m^2$, where m is the baryonic mass, but the relation (37), i.e. $\Omega^6 = c^6|\mathbf{k}|^6 + M^6$. One can argue however that because for $|\mathbf{k}| \gg M$ we have

$$
\Omega^2 = \sqrt[3]{c^6(\mathbf{k}^2)^3 + M^6} = c^2\mathbf{k}^2\sqrt[3]{1 + \frac{M^6}{|\mathbf{k}|^6}} \simeq c^2\mathbf{k}^2 + m^2(\mathbf{k}), \tag{44}
$$

where

$$
m^2(\mathbf{k}) = \frac{1}{3}M^2\left[\frac{M^4}{c^4\mathbf{k}^4} + \mathcal{O}\left(\frac{M^2}{c^2\mathbf{k}^2}\right)^5\right],
\tag{45}
$$

the baryonic wave functions (42), (43) satisfy the d'Alembert equation with source term which quickly converges on the solutions (42), (43) to zero in the high energy limit $\mathbf{k} \to \infty$.

An important future task is to construct a QCD framework with colour Dirac spinors. The presented ideas are preliminary; in principle it should be possible to introduce the generalized Dirac action incorporating the "master" colour Dirac matrices which could describe all phenomenologically known quarks.

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