The Schrödinger Equation of the Gowdy Model in Reduced Algebraic Quantum Gravity

Thesis · March 2017

DOI: 10.13140/RG.2.2.16811.52002

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The Schrödinger Equation of the Gowdy Model in Reduced Algebraic Quantum Gravity

Master's Thesis in Physics

Presented by

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23. March 2017

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Contents

1 Introduction

At the beginning of the 20th century, it was widely believed that physics came to a halt. The general perception was that physics is completely described by Maxwell's theory of electromagnetism [1], thermodynamics and Newtonian mechanics [2]. These theories have proven to be of great use, since they find their application in engineering as well as in physics, e.g., in classical optics, where Maxwell's theory is used to describe light by electromagnetic waves. Furthermore, Newtonian mechanics can be combined with Newton's law of universal gravity, which specifies gravitation as an attractive force acting between any two massive bodies. As a result of this combination, one can, for instance, describe the motion of the planets around the sun by ellipses. Nevertheless, there were phenomena these established theories could not explain: For example, atoms were known to exist, although they were not widely approved. Explaining the structure of atoms by electrons that orbit the nucleus does not justify their stability, as in Maxwell's theory, the electron should emit Bremsstrahlung and hence fall into the nucleus after a certain time. Moreover, Newtonian gravity could not accurately describe the precession of the perihelion of mercury. Nowadays, these problems are handled using two revolutionary theories, which emerged around the beginning of the 20th century, namely quantum mechanics and general relativity:

- Quantum mechanics gives insights into the microscopic world beyond the atomic scale of $10^{-10}\,m$. Classical trajectories of particles with precisely measurable position and momentum do not exist anymore: We can only make probabilistic statements about the trajectory of a quantum particle, while Heisenberg's uncertainty implies that increasing the measurement precision of the momentum results in a worse resolution of the position and vice versa. Specifically, the stability of atoms is explained within quantum theory, as the electrons energy is bounded from below. One can also quantize fields as described in quantum field theory, which is basically a combination of quantum mechanics with the theory of special relativity. Within special relativity, all inertial systems are equal in the sense that the physical laws of nature have the same form. For the theory of general relativity, this is extended to any coordinate system - a concept called general covariance. The theoretical developments of quantum mechanics and quantum field theory lead to the very important fields of atomic physics, solid state physics, and quantum optics which enabled technological advances such as the invention of lasers, computers etc. Quantum field theory also yields a theoretical description of all particles of the standard model and their interactions. Especially, one can calculate scattering amplitudes that can be tested for instance in the large hadron collider (LHC) at CERN. We refer to [4,5] for books on quantum mechanics and for quantum field theory related literature to [6,7]. The mathematical framework is mainly provided by representation theory of Lie groups and Lie algebras [8] and functional analysis [9,10].

- General relativity is a relativistic theory of gravity and concerned with the macroscopic world, giving us a description of our universe in the context of cosmology. For introductory books on general relativity, see for example [11–14]. General relativity replaces the attractive gravitational force of Newtonian gravity with a purely geometrical picture: The geometry of spacetime is described by the metric tensor g, which is the analogon of the gravitational potential in Newtonian gravity and allows us to calculate distances, areas, and volumes. Then, the gravitational force arises from the backreaction of matter and geometry in the sense that matter curves geometry and geometry determines the movement of matter. This backreaction is encoded in the so-called Einstein equations, introduced in its final form by Albert Einstein in 1915 [15]. On the left hand side of these

equations we have the so-called Einstein tensor G, which contains the metric tensor. On the right-hand side occurs the energy momentum tensor T, which contains information about the matter, e.g., its energy and momentum density. The mathematical language for general relativity is differential geometry, see, e.g., [14, 16–18]. Specifically, general relativity solves the puzzle of the perihelion precession of mercury and also describes several other phenomena, such as the gravitational time dilation, which needs to be taken into account for the global positioning system (GPS). Moreover, in analogy to electromagnetic waves, general relativity predicts gravitational waves and most recently, it was also achieved to directly detect them [19].

Although these theories give much more insight into the structure of our universe and are experimentally verified in a wealth of experiments, they are limited in the following sense which will lead us to quantum gravity:

- The calculation of scattering amplitudes is done perturbatively and in particular, the perturbation series only describes the experimental results correctly up to a certain order and is in general divergent. In addition, there occur so-called ultraviolet (UV) divergences, as loop corrections are included into the perturbation series which come from virtual particles that have in general arbitrary energy. Although renormalization techniques exist, the hope is that a theory of quantum gravity provides a natural UV cutoff and hence a natural regularization.

- One solution of the Einstein equations is the so-called Friedmann-Lemaître-Robertson-Walker (FRLW) metric, which allows to describe our universe by a flat, homogeneous, and isotropic universe that is expanding. Such a universe is experimentally confirmed by the results of WMAP [20] and the redshift measurements of galaxies. An expanding universe, however, suggests that at a certain time, all matter content of the universe was compressed to a single point. This implies an initial singularity with infinite matter density and curvature, where matter means everything on the right hand side of Einstein's equations. Furthermore, general relativity predicts black holes, which are regions of spacetime from which nothing, not even light, can escape, see for instance chapter 6 and 12 of [14] and part VII of [11] for more details. In particular, dying stars may form black holes. In this sense, it is common to say that general relativity predicts its own failure, as the existence of singularities hints at the incompleteness of the theory. In the extreme astrophysical situations of the big bang singularity and black holes, where huge amounts of matter are compressed to tiny regions, it is to be expected that gravitational effects need to be understood on very short scales. This requires a combination of quantum mechanics and general relativity in order to describe physics still appropriately. Length scales relevant for experiments at CERN are much larger and allow to neglect the gravitational interaction at the fundamental level.

Hence, we see that these days, one of the main open problems of physics is to find a quantum theory of gravity. Actually, the first investigations on this topics were already done in the 1930s and thus, quantum gravity has acquired a quite long history of research, see for example appendix B of [21] for an historical overview. Nowadays, among others there are two candidates for quantum gravity theories, namely string theory and loop quantum gravity (LQG). These two theories are fundamentally different: String theory [22–24] is a perturbative approach, which makes use of a background metric and wants to provide a unification of all four fundamental forces. Additionally, string theory postulates a new fundamental picture by perceiving the elementary particles as different excitations of one-dimensional objects,

so-called strings, and also introduces extra dimensions as well as supersymmetry. LQG, however, is a non-perturbative approach, which includes a background-independent quantization of general relativity. Furthermore, LQG does not make any assumptions about possible extra structures. See also [25, 26] for lists of publications and [21, 27] for books and [28, 29] for reviews on LQG. In this thesis, we will be interested in LQG only and will provide more insights in the following paragraph.

Loop quantum gravity requires a canonical quantization of general relativity. Canonical quantization means that one starts from a Hamiltonian formulation of a classical theory, i.e., a formulation in terms of a Hamiltonian and canonically conjugate variables that coordinatize a phase space, and then quantizes this theory. For LQG, the classical starting point is general relativity. In particular, we start from a Lagrangian formulation and perform the Legendre transform to obtain the Hamiltonian formulation. This requires a split of the spacetime manifold into space and time, i.e., a foliation into spatial hypersurfaces. A Hamiltonian formulation of general relativity was first achieved using so-called ADM variables [30]: The components of the $(0,2)$ tensor q, which describes the geometry of the spatial hypersurfaces, are the configuration variables. The momenta are given by the components of the $(2,0)$ tensor P, which is related to the so-called extrinsic curvature K describing how the spatial hypersurfaces are embedded in the spacetime manifold. When performing the Legendre transform, however, there occur constraints, basically due to the fact that not all of the velocities can be solved for their corresponding momenta. Specifically, there appear the four constraints H_a, with $a = 1, 2, 3$, and H, where the former is called spatial diffeomorphism and the latter Hamiltonian constraint. The motivation for this names is that H_a generates diffeomorphisms within the spatial hypersurfaces and H diffeomorphisms orthogonal to the spatial hypersurfaces, i.e., these constraints generate spatial diffeomorphisms and the 'time' evolution, respectively. When quantizing this theory, we have two options: We can solve the constraints at the classical level and quantize the resulting theory, a procedure called reduced phase space quantization. This way, we directly end up with the physical Hilbert space containing all physical states. Alternatively, we can quantize the full phase space but then we also have to quantize the constraints, giving us constraint operators that have to annihilate the physical states. Hence, the physical Hilbert space is not directly determined using Dirac quantization. Both Dirac quantization and reduced phase space quantization are investigated in LQG:

- Concerning Dirac quantization, this was done for ADM variables on a mathematically formal level, which means that a representation for these variables has not been defined, yet. However, the so-called Ashtekar variables [31] provide a formulation of general relativity as a $SU(2)$ Yang-Mills gauge theory and this way make contact with the standard model. The configuration variables are given by the $SU(2)$-connection A and the conjugate momenta are determined by the so-called densitized triad E, which is related to the spatial metric. However, the prize to pay is that there are three additional constraints. These are called Gauß constraints and generate $SU(2)$ gauge transformations. Using Ashtekar's variables, LQG is well-defined at the kinematical level, i.e., when the constraints are not solved. Furthermore, one can define geometric operators such as the area and volume operator at the kinematical level, where they acquire discrete spectra. This is a first hint at a discrete structure of spacetime at the Planck scale, described by the Planck length l_{P}, which is of the order of $10^{-35}\,\mathrm{m}$. Concerning experimental tests at these scales, however, we are in a situation similar to the beginning of the 20th century, where the resolution of the atomic scale was not

possible: Particle collider such as the LHC at CERN produce particles with kinetic energies up to 7 TeV being much smaller than the Planck energy E_{P}, which is of the order of 10^{19} GeV. Nevertheless, there are astrophysical particles which come much closer to the Planck energy and the investigation of so-called γ-ray bursts may be an experimental application of quantum gravity in the future.

- For the reduced phase space quantization, one can use the so-called relational formalism introduced by Rovelli [41–44] and refined by Dittrich in [45, 46]. This formalism introduces so-called reference fields that allow to establish physical time and spatial coordinates with respect to which the dynamics of the remaining geometric and matter degrees of freedom are described. This circumvents the so-called problem of time, which occurs in general relativity: The Hamiltonian of general relativity is constrained to vanish and hence does not generate physical time evolution. Using the relational formalism, however, one can derive a physical Hamiltonian density that indeed generates evolution with respect to physical time. The exact form of the physical Hamiltonian density depends on the specific form of the reference fields, see [47] for an overview of possible choices. Furthermore, one can show that the observables one constructs from the phase space variables coordinatize the physical phase space and satisfy a Poisson algebra that is isomorphic to the original one, for appropriate choices of reference fields. Hence, the kinematical results of LQG can be carried over to the physical phase space and thus, one obtains a direct access to the physical Hilbert space one is finally interested in. In this thesis, we will use the relational formalism to perform a reduced phase space quantization. In particular, we will be interested in so-called Gaussian dust [48], which will hand us a Hamiltonian that is totally equivalent to the geometrical part of the Hamiltonian constraint.

As mentioned before, in this thesis, we will follow the reduced phase space quantization procedure and specifically, we will investigate a simplified version of general relativity, the so-called polarized three-torus Gowdy model. We obtain this model by performing a symmetry reduction of the full theory of general relativity. Afterwards, we carry out the quantization using the techniques of LQG. Moreover, we work in the algebraic quantum gravity (AQG) framework, which is similar to LQG and was developed first in [49–52]. The advantage of considering such toy models is that they are technically much simpler to handle and allow to test the techniques of the full theory. In this way, we hope to gain more insights into the structure of the full theory. Several toy models were investigated up to now, the most striking results were obtained in loop quantum cosmology (LQC), which is reviewed for example in [28, 53]. In particular, within LQC models, the big bang is replaced by a big bounce, a result that is obtained using the discreteness of the spectrum of geometric operators. Nevertheless, it needs to be checked if the results of these toy models also survive in the full theory, for which less results on the quantum dynamics exist. Furthermore, the procedures of symmetry reduction and quantization do not commute in general.

The polarized three-torus Gowdy model falls into a larger class of toy models: It is a special case of Gowdy spacetimes [54], which solve Einstein's equation and possess a two parameter isometry group, i.e., two Killing vector fields. If these Killing vectors are orthogonal, we obtain the polarized Gowdy models and the three-torus part comes from the specialization to spatial hypersurfaces with the topology of a three-torus. Specifically, we will perform a symmetry reduction of the full theory formulated in terms of Ashtekar's variables, where we follow [55, 56]. This will basically result in a $U(1)$ gauge theory on a circle. The loop

quantization of this theory was performed first in [56, 57]. Furthermore, we will work within the framework of AQG which was applied to the Gowdy model in [58, 59]. We also want to point out that for the LQG Gowdy model, the physical Hilbert space could not be constructed so far. This has been achieved by another procedure, where not all degrees of freedom are loop quantized. This so-called hybrid quantization was developed in [60–62] and reviewed in [63]. Within this approach, the classical theory is split into an homogeneous and an inhomogeneous part, where the former is quantized using LQG techniques and the latter via Fock quantization. We will not be concerned with the construction of the physical Hilbert space in this thesis, as we perform a reduced phase space quantization. In particular, the relational formalism together with Gaussian dust gives us a physical Hamiltonian operator $\hat{\mathrm{H}}_{\mathrm{phys}}$ equivalent to the Hamiltonian constraint of [56]. We will then lift this operator to AQG and finally write down the corresponding Schrödinger equation that encodes the quantum dynamics of the Gowdy model. Then, it is the main goal of this thesis to investigate possible solutions of the Schrödinger equation.

This thesis is organized as follows: In section 2, we discuss the Hamiltonian formulation of general relativity, where we proceed chronological by starting with the ADM formalism in section 2.1 and then introducing Ashtekar's variables in section 2.2. Subsequently, we give an introduction to loop quantum gravity in section 3, where we will also discuss the framework of algebraic quantum gravity in section 3.6. Afterwards, in section 4 , we discuss the general procedure to perform reduced phase space quantization and also specialize to Gaussian dust. In section 5, we introduce the polarized three-torus Gowdy model, which includes the symmetry reduction as well as the algebraic loop quantization. After this, we discuss our results in section 6 and finally give a summary and comment on future investigations in section 7.

2 Hamiltonian formulation of general relativity

In the following chapters, we discuss the Hamiltonian formulation of general relativity. The motivation for starting with the Hamiltonian formulation is that we want to quantize the Gowdy model using the techniques of loop quantum gravity (LQG). LQG itself is based on the canonical quantization of general relativity. To perform the canonical quantization, we need a Hamiltonian formulation of general relativity as the classical starting point, which we will discuss in the following sections.

2.1 ADM-formalism

A Hamiltonian formulation of general relativity was first achieved within the so-called ADM-formalism, named after its inventors Richard Arnowitt, Stanley Deser, and Charles W. Misner, see [30] for the original work from 1962 and [64] for the republication from 2008. We will discuss the ADM-framework first, although the classical starting point for the Gowdy model will be the formulation of general relativity in terms of Ashtekar variables, which we will introduce in section 2.2. For Ashtekar variables, the quantization can be rigorously defined, whereas for the ADM variables, this has only been formulated at a rather formal level. It is, however, historically and logically more appropriate to start with the ADM-formalism. Furthermore, we will discuss, with the help of the ADM formalism, how constraints arise and how their Poisson algebra looks like. This will help to interpret the classical theory and emphasize the problems that occur during quantization.

2.1.1 Lagrangian formulation

In this section, we want to set up the Lagrangian formulation of general relativity in the realm of the ADM-formalism. For this, we introduce in analogy to standard classical mechanics configuration variables and velocities. The latter are the time derivatives of the former, therefore, we need to specify a time variable. In particular, this needs to be done without breaking the diffeomorphism invariance imposed by general relativity. We will end up with an action in (2.1.12) depending on the quantities N, N^a, q_{ab}, with $a, b = 1, 2, 3$, which recover the 10 degrees of freedom in the metric tensor $g_{\mu\nu}$, with $\mu, \nu = 0, ..., 3$, we start from. We will mainly follow the ADM-related chapters of [27], [65] and [66]. Further good references are [67] , [68], as well as appendix E of [14].

First, we want to set up the general framework. We start from a $(D+1)$-dimensional, Pseudo-Riemannian manifold (M, g) describing spacetime, where g denotes the symmetric and non-degenerate metric tensor with signature s. In the following, we choose $D = 3$ as well as Lorentzian signature $s = -1$, i.e., a vector field v is called spatial for $g_{\mu\nu}v^\mu v^\nu > 0$, null for $g_{\mu\nu}v^\mu v^\nu = 0$, and timelike for $g_{\mu\nu}v^\mu v^\nu < 0$. Then, the *Einstein-Hilbert action* is given by

$$S_{EH} = \frac{1}{\kappa}\int_M \mathrm{d}^4X\sqrt{|det(g)|}R^{(4)}, \tag{2.1.1}$$

where X^μ, with $\mu = 1, ..., 4$, denote the spacetime coordinates, $\kappa = 16\pi G$ is the coupling constant related to Newton's constant G, and $R^{(4)}$ is the four-dimensional Ricci scalar connected to the four-dimensional Riemann curvature tensor $R^{(4)}_{\mu\nu\rho\sigma}$ via $R^{(4)} = g^{\mu\rho}g^{\nu\sigma}R^{(4)}_{\mu\nu\rho\sigma}$.

In the first step, we perform a split of the spacetime manifold M into space and time without breaking diffeomorphism invariance. We assume that M is globally hyperbolic, i.e., M

can be foliated into spatial hypersurfaces intersecting any causal curve only once. Therefore, the initial value problem is well posed: Given initial data $g, \dot{g}$ on an instant of time, i.e., on a certain spatial hypersurface, one can compute the solution everywhere on the spacetime manifold M. We can now apply a theorem introduced by Geroch [69] and further developed by Sanchez and Bilal [70]: For a globally hyperbolic spacetime there exists a diffeomorphism

$$\varphi : \mathbb{R} \times \sigma \to M, \quad (t, x) \to X = \varphi(t, x), \tag{2.1.2}$$

where σ denotes the three-dimensional spatial manifold with coordinates x^a, where $a = 1, ..., 3$. (2.1.2) establishes $M \cong \mathbb{R} \times \sigma$ and gives us a foliation of the spacetime manifold M into spatial hypersurfaces $\Sigma_t = \varphi(t, \sigma)$ equipped with spatial tangents and timelike normals. The map $\varphi_t : \Sigma_t \to M, x \to \varphi_t(x) := \varphi(x, t)$ provides an embedding of Σ_t into M. Furthermore, we can define a time function τ via $\tau(X) = \tau(\varphi(x, t)) = t$. It seems that the split into space and time corresponds to a choice of coordinate system and therefore breaks diffeomorphism invariance, which, however, is required by general relativity. This is solved by keeping the foliation arbitrary. Then, any diffeomorphism $\Phi \in Diff(M)$ can be written as $\Phi = \varphi \circ \varphi'$, where φ, φ' denote two distinct foliations defined by (2.1.2). Hence, any two foliations can be related by a diffeomorphism via $\varphi = \Phi \circ (\varphi')^{-1}$. This means that the freedom in the choice of foliation is equivalent to the diffeomorphism group Diff(M) and thus diffeomorphism invariance is preserved.

In the second step, we introduce the configuration variables and velocities. We first define a basis for the tangent space $T_p(\mathbb{R} \times \Sigma_t)$ with $p \in \Sigma_t$. For this purpose, we use the embedding and define 3 vector fields tangential to Σ_t adapting the notation of [65]:

$$S_a(X) := (\partial_a)_{\varphi(t,x)=X} = (\varphi^\mu_{,a}(t, x))\partial_\mu \ . \tag{2.1.3}$$

Moreover, we can find a conormal vector field $\bar{n}$ from the condition $\bar{n}(S_a) = \bar{n}_\mu S^\mu_a = 0$. Using the metric g, we obtain a normal vector field n which satisfies $g_{\mu\nu} n^\mu S^\nu_a = 0$ and is normalized according to $g_{\mu\nu} n^\mu n^\nu = -1$. In this way, we have found a basis (n, S_a) of $T_p(\mathbb{R} \times \Sigma_t)$ for every t and every point $p \in \Sigma_t$. Additionally, we can specify the dual basis by $(\bar{n}, S^a)$. If we define $B_A := (n, S_a)$, with $A = 0, ..., 3$ and $B_0 := n, B_1 := S_1$ etc., and accordingly $B^A := (\bar{n}, S^a)$, we can specify the 4-duality relations $B^\mu_A B^A_\nu = \delta^\mu_\nu$ and $B^\mu_A B^C_\mu = \delta^C_A$. Now, we can introduce the configuration variables corresponding to the Riemannian structure defined on each Σ_t by

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{2.1.4}$$

This so-called *intrinsic metric* $q_{\mu\nu}$ is uniquely determined by $q_{\mu\nu} n^\mu = 0$, i.e., $q_{\mu\nu}$ is a spatial tensor, and $q_{\mu\nu} b^\mu = g_{\mu\nu} b^\mu$, i.e., $q_{\mu\nu}$ applied to any vector b^μ tangent to Σ_t describes the same geometry as $g_{\mu\nu}$. To obtain the velocity of $q_{\mu\nu}$, we need to specify the time derivative, which we will do following mainly [27, 65] and chapter 3.2.1.2 of [66]. Up to now, we only defined a time function by $\tau(X) = t = \text{const.}$, but we also need a direction for the time derivative. To this end, we introduce the so-called *deformation* or *time evolution vector field*

$$T(X) := (\partial_t)_{\varphi(t,x)=X} = (\varphi^\mu_{,t}(t, x))_{\varphi(t,x)=X} \partial_\mu, \tag{2.1.5}$$

which basically tells us how points of different spatial hypersurfaces are related for a given foliation. In the basis (n, S_a), we can expand T as

$$T(X) = N(X)n(X) + N^a(X)S_a(X), \tag{2.1.6}$$

where N, the component of T normal to Σ_t, is called *lapse function*, and N^a, the component of T tangential to Σ_t, is called *shift vector field*. N and N^a are completely arbitrary and embody the arbitrariness of the foliation. Note that we consider spacelike embeddings and Lorentzian signature, which implies that T is timelike and the condition $-N^2 + g_{\mu\nu}N^\mu N^\nu < 0$ holds. In addition, we choose N to be positive, i.e., we have a future directed foliation. Using T, we can now define the time derivative of an arbitrary tensor field: The relation $T^\mu \partial_\mu t = 1$ ensures that the integral curves of T intersect each spatial hypersurface only once. Therefore, the Lie derivative along T serves as a time derivative, as it quantifies the change of a tensor field when passing from one spatial hypersurface to another one. In particular, the time derivative of $q_{\mu\nu}$ is given by $\dot{q}_{\mu\nu} := q_\mu^{\mu'} q_\nu^{\nu'} \mathcal{L}_T q_{\mu'\nu'}$, where the spatial projections $q_\mu^{\mu'} q_\nu^{\nu'}$ ensure that we obtain a spatial tensor again. Now, we can introduce the so-called *extrinsic curvature* tensor $K_{\mu\nu}$, which will play the role of the velocity and is defined by

$$K_{\mu\nu} = q_\mu^\rho q_\nu^\sigma \nabla_\rho n_\sigma. \tag{2.1.7}$$

Here, ∇ denotes the covariant derivative being metric-compatible with g, i.e., $\nabla_\rho g_{\mu\nu} = 0$, and torsion-free, i.e., $[\nabla_\mu, \nabla_\nu] f = 0$ for an arbitrary function f. The unique connection associated to the metric-compatible, torsion-free covariant derivative ∇ is called Levi-Civita connection. The components $\Gamma^\rho_{\mu\nu}$ of the Levi-Civita connection are called Christoffel symbols. (2.1.7) defines $K_{\mu\nu}$ as the spatial projection of the parallel transport of the normal n. Hence, $K_{\mu\nu}$ tells us how a spatial hypersurface Σ_t lies in M, justifying the name extrinsic curvature. Furthermore, $K_{\mu\nu}$ is a symmetric tensor and related to $\dot{q}_{\mu\nu}$ via

$$K_{\mu\nu} = \frac{1}{2N} \left(\dot{q}_{\mu\nu} - \mathcal{L}_{\vec{N}} q_{\mu'\nu'} \right). \tag{2.1.8}$$

In the last step, we rewrite (2.1.1) in terms of tensors defined only on σ. The strategy is to pull back spacetime tensors, which are defined on $\Sigma_t = \varphi_t(\sigma)$, via the diffeomorphism φ. Pulling back $q_{\mu\nu}$ to σ gives us

$$q_{ab}(t,x) = (\varphi^* q)_{ab}(t,x) = \frac{\partial \varphi^\mu}{\partial x^a}(t,x) \frac{\partial \varphi^\nu}{\partial x^b}(t,x) q_{\mu\nu}(\varphi(t,x))) = (S_a^\mu S_b^\nu q_{\mu\nu})(X)|_{X=\varphi(t,x)}. \tag{2.1.9}$$

For the pull back of $K_{\mu\nu}$, we obtain

$$K_{ab}(t,x) = (S_a^\mu S_b^\nu \nabla_\mu n_\nu)(X)|_{X=\varphi(t,x)}. \tag{2.1.10}$$

Moreover, the pulled back version of (2.1.8) reads as

$$K_{ab}(t,x) = \frac{1}{2N} (\dot{q}_{ab} - (L_{\vec{N}} q)_{ab})(t,x), \tag{2.1.11}$$

which we will need later on when going over to the Hamiltonian formulation. Note that in (2.1.11), the lapse function and the shift vector field have to be understood as the pulled back quantities $N(x,t) := N(\varphi(x,t))$ and $N^a(x,t) := q^{ab}(x,t)(S_b^\mu g_{\mu\nu} N^\nu)(\varphi(x,t))$. To finalize the 3+1 split of the Einstein Hilbert action, we first need to express the 4-dimensional Ricci scalar $R^{(4)}$ by the three-dimensional one $R^{(3)}$. The latter is related to the 3-dimensional Riemann curvature tensor $R^{(3)}_{\mu\nu\rho\sigma}$ which is defined by $[D_\mu, D_\nu] w_\rho = R^{(3)\sigma}_{\mu\nu\rho} w_\sigma$, where w_σ is an arbitrary one-form and D denotes the covariant derivative that is compatible with $q_{\mu\nu}$, i.e., $D_\mu q_{\nu\rho} = 0$. Furthermore, D should be torsion-free, i.e., $[D_\mu, D_\nu] f = 0$ for f an arbitrary function. D

is connected to the g-compatible, torsion-free covariant derivative ∇, as can be seen by its action $D_\mu w_\nu := q_\mu^{\mu'} q_\nu^{\nu'} \nabla_{\mu'} w_{\nu'}$ on a one-form w_ν. Then, the relation between $R^{(3)}$ and $R^{(4)}$ follows from the so-called Gauss and Codacci equations, see [27] for details. Pulling back the volume form $d^4X\sqrt{|detg|}$, as well as all quantities in the above mentioned Codacci equation, results in the so-called *ADM-action*:

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} \mathrm{d}t \int_\sigma \mathrm{d}^D x \sqrt{|det(q)|} N(R^{(D)} + [K_{ab}K^{ab} - (K_a^a)^2]). \tag{2.1.12}$$

In this equation, we dropped a total derivative term originating from the Codacci equation and resulting in a boundary term that we do not want to discuss here. See [27] for more details on boundary terms. This completes the Lagrangian picture of the ADM-formalism.

2.1.2 Legendre transformation and constraints

In the following, we go over from the Lagrangian to the Hamiltonian formulation. For this, we have to perform the Legendre transformation, i.e., express all velocities by their canonically conjugate momenta. This will hand us certain constraints, where part of them fulfill the so-called Dirac or hypersurface deformation algebra. We will mainly follow [27] and refer to this reference for detailed derivations of the results stated below.

Firstly, we determine the canonically conjugate momenta. For the momenta P^{ab} conjugate to q_{ab}, we find by functionally differentiating the ADM action (2.1.12) with respect to $\dot{q}_{ab}$ that

$$P^{ab} := \frac{\delta S_{EH}}{\delta \dot{q}_{ab}(t,x)} = \frac{1}{\kappa}\sqrt{det(q)}(K^{ab} - q^{ab}K_c^c). \tag{2.1.13}$$

For the momenta conjugate to $\dot{N}$ and $\dot{N}^a$, we get

$$\Pi(t,x) := \frac{\delta S_{EH}}{\delta \dot{N}(t,x)} = 0, \quad \Pi^a(t,x) := \frac{\delta S_{EH}}{\delta \dot{N}^a(t,x)} = 0, \tag{2.1.14}$$

due to the independence of the ADM action on the velocities $\dot{N}$ and $\dot{N}^a$. Then, the phase space is coordinatized by $q_{ab}, P^{ab}, N, \Pi, N^a$, and Π^a. To state the canonical commutation relations satisfied by the phase space variables, we first define the Poisson bracket between two arbitrary tensor fields $F(x)$ and $G(x)$:

$$\{F(x), G(x)\} = \kappa \int_\sigma d^3z \Big[\frac{\delta F(x)}{\delta q_{ab}(z)}\frac{\delta G(x)}{\delta P^{ab}(z)} + \frac{\delta F(x)}{\delta N^a(z)}\frac{\delta G(x)}{\delta \Pi_a(z)} + \frac{\delta F(x)}{\delta N(z)}\frac{\delta G(x)}{\delta \Pi(z)} - F \leftrightarrow G\Big]. \tag{2.1.15}$$

Here, we suppressed any possible index structure of $F(x)$ and $G(x)$. (2.1.15) gives us the following canonical commutation relations:

$$\{N(x), \Pi(y)\} = \kappa\delta^{(3)}(x,y), \ \{N^a(x), \Pi_b(y)\} = \kappa\delta_b^a\delta^{(3)}(x,y), \tag{2.1.16}$$

$$\{q_{ab}(x), P^{cd}(y)\} = \kappa\delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y), \tag{2.1.17}$$

where we left out all Poisson brackets that are zero, namely those between two momenta, two positions, and momenta and positions being not canonically conjugate.

Secondly, we express the velocities by the corresponding momenta. We can solve $\dot{q}_{ab}$ for P^{ab} using (2.1.11) and (2.1.13). However, from (2.1.14) follows that we cannot solve the

other velocities $\dot{N}$ and $\dot{N}^a$ for their corresponding momenta Π and Π_a. This means that we have the case of a singular Lagrangian. For this scenario, we can apply the *Dirac algorithm* developed by Dirac in [71], see also [72] or chapter 24 of [27] for further introductions. The Dirac algorithm involves the imposition of so-called *primary constraints*

$$C(t,x) := \Pi(t,x) = 0,\ C^a(t,x) := \Pi^a(t,x) = 0. \tag{2.1.18}$$

Then, one performs the Legendre transformation for the remaining variables that could be solved for their momenta. After some manipulations, we obtain a so-called *primary Hamiltonian*

$$\mathbf{H} = \int_\sigma d^3x\ [\lambda C + \lambda^a C_a + N^a H_a + NH], \tag{2.1.19}$$

including the primary constraints with Lagrange multipliers λ and λ_a. Furthermore, we have introduced the so-called *spatial diffeomorphism constraint*

$$H_a := -2q_{ac}D_b P^{bc} \tag{2.1.20}$$

and the so-called *Hamiltonian constraint*

$$H := \left[\frac{\kappa}{\sqrt{det(q)}}\left(q_{ac}q_{bd} + \frac{1}{2}q_{ab}q_{cd}\right)P^{ab}P^{cd} + \frac{\sqrt{det(q)}}{\kappa}R\right]. \tag{2.1.21}$$

Note that in the primary Hamiltonian, H and H_a are not yet to be understood as constraints. We denoted them as such because the Dirac algorithm involves a so-called stability analysis resulting in the requirement that H and H_a have to be set to zero. Before discussing the stability analysis, we explain a usual procedure in field theory, which is the introduction of so-called smearing fields: When computing Poisson brackets between fields, we might get a distributional result, as it is the case in (2.1.17). To avoid this, one smears the fields with so-called test functions. In case of (2.1.19), the functions λ, λ_a, N^a, and N naturally arise as smearing functions, indicated by the following notation:

$$\mathbf{H} = \frac{1}{\kappa}\left[C(\lambda) + \vec{C}(\vec{\lambda}) + \vec{H}(\vec{N}) + H(N)\right], \tag{2.1.22}$$

with $C(\lambda) := \int_\sigma d^3x\ \lambda C$ and similar definitions for $\vec{C}(\vec{\lambda}), \vec{H}(\vec{N})$, and $H(N)$. Now, we discuss the so-called stability analysis we have to perform within the Dirac algorithm: The constraint surface determined by the primary constraints should be invariant under the evolution generated by the primary Hamiltonian. For this, we have to compute the Poisson brackets between the Hamiltonian in (2.1.22) and the primary constraints in (2.1.18), resulting in

$$\{\mathbf{H}, C(f)\} = H(f),\ \{\mathbf{H}, \vec{C}(\vec{f})\} = \vec{H}(\vec{f}), \tag{2.1.23}$$

where $f, \vec{f}$ denote arbitrary smearing functions. Then, we demand that the result of the Poisson brackets in (2.1.23) vanishes weakly, i.e., when the constraints are set to zero. This gives us the condition that the quantities H_a and H from (2.1.20) and (2.1.21) have to vanish as so-called *secondary constraints*, justifying the previous denomination of H_a and H as constraints. The same stability analysis has to be performed for the secondary constraints H_a and H, i.e., we again have to compute the Poisson brackets $\{\mathbf{H}, H(f)\}$ and $\{\mathbf{H}, \vec{H}(\vec{f})\}$. Fortunately, it turns out that the resulting expressions are certain combinations of smeared versions of H and H_a and therefore, no further constraints occur. This completes the Dirac

algorithm.

Thirdly, we discuss the equations of motion of the phase space variables and introduce the reduced ADM action. For this purpose, we go to the partially reduced phase space in which the primary constraints $\Pi = 0$ and $\Pi_a = 0$ are satisfied. For the lapse function and the shift vector field, we obtain $\{N, \mathbf{H}\} = \lambda$ and $\{N^a, \mathbf{H}\} = \lambda^a$, respectively. As λ and λ_a are arbitrary Lagrange multipliers, N and N^a are not determined by the dynamics. If we calculate the equations of motion of the canonical pair (q_{ab}, P^{ab}), we see that the terms $C(\lambda)$ and $\vec{C}(\vec{\lambda})$ in $\mathbf{H}$ give no contribution. Hence, concerning the equations of motion of (q_{ab}, P^{ab}), it is legitimate to look only at the *reduced ADM action* given by

$$S = \frac{1}{\kappa} \int_{\mathbb{R}} \mathrm{d}t \int_{\sigma} \mathrm{d}^3x \left[\dot{q_{ab}} P^{ab} - (N^a H_a + N H) \right], \tag{2.1.24}$$

and treat N and N^a as Lagrange multipliers.

Finally, we examine the Poisson algebra that the Hamiltonian and diffeomorphism constraint satisfy. The explicit calculations performed in the Dirac algorithm yield the so-called *Dirac* or *hypersurface deformation algebra* given by

$$\{\vec{H}(\vec{N}), \vec{H}(\vec{N'})\} = -\kappa \vec{H}(\mathcal{L}_{\vec{N}} \vec{N'}), \tag{2.1.25}$$

$$\{\vec{H}(\vec{N}), H(N)\} = -\kappa H(\mathcal{L}_{\vec{N}} N), \tag{2.1.26}$$

$$\{H(N), H(N')\} = -\kappa \vec{H}(q^{-1}(N dN' - N' dN)). \tag{2.1.27}$$

We want to make several comments on the Dirac algebra following [27, 73]:

- From (2.1.25), we see that the diffeomorphism constraints $\vec{H}(\vec{N})$ form a closed Lie subalgebra. As we will justify below, this subalgebra can be identified with $diff(\sigma)$, the Lie algebra of the spatial diffeomorphism group $Diff(\sigma)$.

- From (2.1.26), we can infer that $diff(\sigma)$ is not an ideal of the Dirac algebra, as the Hamiltonian constraint $H(N)$ is not diffeomorphism invariant. An ideal of a Lie algebra L is a Lie subalgebra I of L such that $[L, I] \subset I$.

- From (2.1.27), we see that the Dirac algebra, although being closed, is not a true Lie algebra: The Poisson bracket of two Hamiltonian constraints is again a Hamiltonian constraint, but with a phase space dependent smearing function, as the inverse metric q^{-1} is involved. Therefore, we do not have structure constants but structure functions and thus, the Dirac algebra is a Lie algebroid.

- Additionally, we see from (2.1.25)−(2.1.27) and (2.1.23) that all constraints commute weakly. Such a system of constraints is called *first class*. In accordance with the so-called Dirac conjecture (see again [71] or [72]), we can interpret first class constraints as gauge transformations. We apply this to the case of the reduced ADM framework, cf. (2.1.24). One can show that H_a can be interpreted as the generator of diffeomorphisms that preserve Σ_t, captured by the infinitessimal transformation laws

  $$\{\vec{H}(\vec{N}), q_{ab}\} = (\mathcal{L}_{\vec{N}} q)_{ab}, \ \{\vec{H}(\vec{N}), P^{ab}\} = (\mathcal{L}_{\vec{N}} P)^{ab}. \tag{2.1.28}$$

  Furthermore, H is the generator of diffeomorphisms orthogonal to Σ_t with infinitessimal transformation laws

  $$\{H(N), q_{ab}\} = (\mathcal{L}_{Nn} q)_{ab}, \ \{H(N), P^{ab}\} = (\mathcal{L}_{Nn} P)^{ab}. \tag{2.1.29}$$

However, this interpretation of gauge transformations as spacetime diffeomorphisms is valid only if the vacuum Einstein equations $G_{\mu\nu} = 0$ are satisfied and the constraints are set to zero, i.e., we have a so-called on-shell symmetry. The gauge transformations that are generated off shell are described by the so-called Bergmann Komar group BK(M) being inequivalent to Diff(M). This nonequivalence is not too surprising: On the one hand, Diff(M) is a kinematical symmetry, as there are various ways to construct Diff(M)-invariant Lagrangians, one example being $\mathcal{L} = \sqrt{|det(g)|}R^{(4)\mu\nu\sigma\rho}R^{(4)}_{\mu\nu\sigma\rho}$. On the other hand, BK(M) is a dynamical symmetry, as it depends on the given Lagrangian, because we have to perform the canonical analysis to obtain the constraints generating the BK(M)-transformations off shell.

We want to close the discussion of the ADM framework with the remark that the Hamiltonian $\mathbf{H}$ is a sum of constraints and therefore constrained to vanish. This implies that physical observables do not evolve under the time evolution generated by $\mathbf{H}$: Observables $\mathcal{O}$ are gauge invariant phase space functions, which are called strong Dirac observables if they Poisson commute with all first class constraints. If the observables weakly Poisson commute with all first class constraints, i.e., the Poisson bracket vanishes on the constraint surface, they are called weak Dirac observables. As $\mathbf{H}$ is the sum of all first class constraints, weak/strong Dirac observables have to weakly/strongly Poisson commute with the Hamiltonian $\mathbf{H}$. Hence, if we take the Hamiltonian $\mathbf{H}$ as our physical Hamiltonian generating time evolution, every observable $\mathcal{O}$ would be a constant of motion, since $\dot{\mathcal{O}} = \{\mathcal{O}, \mathbf{H}\} = 0$ on the constraint surface. Therefore, we would have no evolution of any physical observable $\mathcal{O}$, which is clearly in contradiction to experiment. This is known as *the problem of time*. There is, however, a way out, namely by introducing reference fields with respect to which the time evolution of observables can be described. Within this so-called relational framework, we can construct a physical Hamiltonian with the prize to pay that we have to introduce additional matter degrees of freedom, for which we have to find an interpretation. We will discuss this in section 4 in more detail.

2.2 Ashtekar's variables

In this chapter, we will introduce the so-called Ashtekar variables, first developed in 1986 by Abhay Ashtekar [31]. The main motivation behind departing from ADM variables to new variables is that the quantization of the Hamiltonian constraint can be performed using suitably smeared Ashtekar variables. This, however, has not been show for the ADM-variables so far. In particular, the connection to well-known techniques from quantum field theory is much more immediate, as Ashtekar's variables provide an SU(2)-Yang Mills gauge theory formulation of general relativity. We mainly stick to [27] and [29] in our presentation.

2.2.1 The basic variables

In the following paragraphs, we will introduce the basic variables that constitute Ashtekar's variables. For this, we will first introduce the concept of frame fields, handing us an extended phase space being equivalent to the ADM phase space once additional constraints are imposed. We finally arrive at Ashtekar's variables using two canonical transformations.

First, we want to discuss the concept of frame fields following chapter 3.5.1 and appendix AA.4 of [66]. Given a manifold M, we can choose a basis $e_I(p)$, with $I = 0, 1, 2, 3$, for the tangent space T_pM at each point p in M. In our case of four dimensions, the set of vector

fields $e_I(p)$ is called tetrad. We choose e_0 to be timelike and the other basis elements e_1, e_2, and e_3 to be spacelike, captured in the following orthogonality condition:

$$g(e_I, e_J) = g_{\mu\nu} e^\mu_I e^\nu_J = \eta_{IJ} = diag(-1, 1, 1, 1). \tag{2.2.1}$$

A more geometrical picture is obtained by understanding the frame field e as the map $e : M \times \mathbb{R}^4 \to TM$ between the tangent bundle TM and the vector bundle $M \times \mathbb{R}^4$. We refer to [18] for an introduction into the fibre bundle language. The typical fibre $\mathbb{R}^4$ of $M \times \mathbb{R}^4$ is also called internal space, which in this case is just Minkowski space. Therefore, e specifies a pointwise isometry $e(p) : \mathbb{R}^4 \to T_pM$ between Minkowski spacetime, being equipped with a constant metric, and the tangent space T_pM, being equipped with the general metric $g_{\mu\nu}(p)$. Accordingly, we can define the inverse frame field as $e : TM \to M \times \mathbb{R}^4$. Given the usual coordinate basis ∂_μ, with $\mu = 0, .., 3$, as a section of TM and a basis ξ_I, with $I = 0, ..., 3$, as a section of $M \times \mathbb{R}^4$, we can use the frame field to map between the two bases via $e(\xi_I) = e_I = e^\mu_I \partial_\mu$ and $e^{-1}(\partial_\mu) = e^I_\mu \xi_I$. In particular, we can map a vector field v^I in the internal space to a vector field v^μ in TM via $v^\mu = e^\mu_I v_I$. That is why the frame fields are also called soldering forms.

Now, we can apply the frame field formalism to the ADM framework. We can express the spatial ADM metric q_{ab} by coframe fields, which are called *triads* for three dimensions. First, we express the spacetime metric $g_{\mu\nu}$ by cotetrads:

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}. \tag{2.2.2}$$

However, the cotetrads are not uniquely determined by (2.2.2): One can perform a Lorentz transformation $L \in SO^+(1,3)$ of the form $e^I_\mu \to L^I{}_J e^J_\mu$ and as η is invariant under Lorentz transformations, the resulting transformation matrices in (2.2.2) will be canceled. Thus, we have introduced four additional degrees of freedom. To go over to the ADM framework, we recognize that (2.2.2) gives us 10 equations between the components of g and the tetrad e. To solve these equations, we partially fix the $SO^+(1,3)$ gauge freedom in the tetrads by imposing the so-called time gauge $e^0_\mu = n_\mu$, where n_μ denotes the conormal. Furthermore, we use the explicit form of the components of g and n_μ in the ADM frame given by

$$n_a = 0,\ n_t = -N,\ g_{tt} = -N^2 + q_{ab} N^a N^b,\ g_{ta} = q_{ab} N^b,\ g_{ab} = q_{ab}. \tag{2.2.3}$$

Then, we obtain from (2.2.2) that $e^j_t = e^j_a N^a,\ e^0_t = -N,\ e^0_a = 0$ and in particular that

$$q_{ab} = e^i_a e^j_b \delta_{ij}. \tag{2.2.4}$$

This way, we have formulated the ADM framework in terms of triads e^j_a, the lapse function N, and the shift vector field N^a. As for the tetrads, (2.2.4) does not uniquely determine the triads: We have an additional SO(3) freedom, i.e., $e^i_a \to O^j_i e^j_a$ leaves (2.2.4) invariant. To get back the ADM phase space, we will later remove the three additional rotational degrees of freedom by imposing three new constraints. In the following, we consider the triads e^i_a as su(2)-valued one-forms, where we use the isomorphism between the Lie algebras of SO(3) and SU(2). Classically, it makes no difference to choose either su(2) or so(3). At the quantum level, however, we want to be able to couple gravity to fermions, which have half-integer spin. Therefore, su(2) is naturally singled out.

Next, we introduced a phase space that is an extended version of the ADM phase space. This new phase space is coordinatized by to be specified momenta E^a_j and configuration

variables K_a^j. The canonical commutation relations read as

$$\{K_a^i(x), E_j^b(y)\} = \frac{\kappa}{2}\delta_a^b\delta_j^i\delta(x,y),\ \{E_j^a(x), E_k^b(y)\} = \{K_a^j(x), K_b^k(y)\} = 0. \tag{2.2.5}$$

The momenta E_j^a, the so-called *densitized triads*, are defined by

$$E_j^a := \sqrt{det(q)}e_j^a. \tag{2.2.6}$$

The configuration variables K_a^j are related to the extrinsic curvature via $K_{ab} = K_{(a}^j e_{b)}^i\delta_{ij}$, where the symmetrization is necessary since K_{ab} is a symmetric tensor . Thus, the antisymmetric part $K_{[a}^j e_{b]}^i\delta_{ij}$ needs to vanish leading to the so-called rotational constraint $G_{ab} := K_{[a}^j e_{b]}^i\delta_{ij} = 0$. Equivalently, we can consider instead of G_{ab} the constraints

$$G_{jk} := G_{ab}E_j^aE_k^b = K_{a[j}E_{k]}^a = 0, \tag{2.2.7}$$

where in the second step, we made use of the explicit form of G_{ab} and used the relation $E_a^jE_k^a = \delta_k^j$ with the densitized cotriad E_a^j defined by $E_a^j := e_a^j/\sqrt{det(q)}$. In particular, the extended phase space coordinatized by (K_a^j, E_j^a) is equivalent to the ADM phase space for $G_{jk} = 0$.

In the following, we will describe the first of two canonical transformations allowing us to go over from the canonical pairs (K_a^i, E_i^a) to Ashtekar's variables. The first canonical transformation is a rescaling of the basic variables (K_a^i, E_i^a) by an arbitrary complex number, the so-called *Barbero-Immirzi parameter* γ:

$$(K_a^i, E_i^a) \to (\gamma K_a^i, \frac{1}{\gamma}E_i^a) =: ({}^{(\gamma)}K_a^i, {}^{(\gamma)}E_i^a) \tag{2.2.8}$$

For each choice of γ, we get a different set of canonical variables. This ambiguity can for example be fixed within black hole entropy calculations, see for instance [74]. One might choose γ to be complex, requiring certain reality conditions, as otherwise, a doubling of the number of degrees of freedom occurs. The reality conditions, however, are non-trivial to implement at the quantum level. Therefore, one usually assumes γ to be real.

Before we can perform the second canonical transformations to arrive at Ashtekar's variables, we need to introduce the notion of a *spin connection*. For this purpose, we extend the covariant derivative D acting only on tensors with spatial indices a, b, c to a covariant derivative acting on tensors having spatial indices as well as internal indices i, j, k . Specifically, we get for the cotriad that

$$D_a e_b^j = \partial_a e_b^j - \Gamma_{ab}^c e_c^j + \Gamma_{ak}^j e_b^k, \tag{2.2.9}$$

which we impose to be zero, i.e., we extend the metric compatibility $D_a q_{bc} = 0$ to the cotriads. Moreover, we see from $D_a\delta_{jk} = 0$ that the spin connection satisfies $\Gamma_{ajk} = -\Gamma_{akj}$. The condition $D_a e_b^j = 0$ also provides us an expression for the spin connection in terms of triads and cotriads:

$$\Gamma_a^i = \frac{1}{2}\epsilon^{ijk}e_k^b\left(2\partial_{[b}e_{a]}^j + e_j^c e_a^l e_{c,b}^l\right). \tag{2.2.10}$$

Here, we used $\Gamma_{ajk} = \Gamma_a^l\epsilon_{jlk}$ following from $\Gamma_a =: \Gamma_a^l T_l$, with $(T_l)_{jk} = \epsilon_{jlk}$ the generators of $SU(2)$ in the adjoint representation. It turns out that the spin connection is invariant under the canonical transformation of (2.2.8), which we can see by expressing Γ_a^i as a function of

densitized triads and cotriads using (2.2.10) and (2.2.6).

Having introduced the spin connection, we can now perform the second canonical transformation to deduce Ashtekar's variables. To this end, we rewrite the rotational constraint (2.2.7) in the equivalent form

$$G_j = \epsilon_{jkl} \left({}^{(\gamma)}K_a^k\right) \left({}^{(\gamma)}E_l^a\right). \tag{2.2.11}$$

Furthermore, we have the identity

$$0 = D_a \left({}^{(\gamma)}E_a^j\right) = \partial_a \left({}^{(\gamma)}E_j^a\right) + \epsilon_{jkl}\Gamma_a^k \left({}^{(\gamma)}E_l^a\right), \tag{2.2.12}$$

following from $D_a e_b^j = 0$ and the invariance of the $q-$compatible covariant derivative D under the transformation (2.2.8). The latter follows from the invariance of the Christoffelsymbols Γ_{bc}^a. In (2.2.12), we also used that E_j^a has density weight one, which cancels the Christoffel symbol when writing out D_a. Inserting into (2.2.11) a clever zero of the form (2.2.12), we arrive at the so-called *Gauss constraint*

$$G_j =^{(\gamma)} \mathcal{D}_a^{(\gamma)} E_j^a := \partial_a \left({}^{(\gamma)}E_j^a\right) + \epsilon_{jkl} \left({}^{(\gamma)}A_a^j\right). \tag{2.2.13}$$

Here, we introduced the $su(2)$-valued *Ashtekar connection*

$$\left({}^{(\gamma)}A_a^j\right) := \Gamma_a^j + \left({}^{(\gamma)}K_a^j\right) \tag{2.2.14}$$

and the new covariant derivative $\mathcal{D}_a$ acting on spatial indices as $\mathcal{D}_a u_b := D_a u_b$ and on internal indices as $\mathcal{D}_a v_j := \partial_a v_j + \epsilon_{jkl} A_a^k v_l$. One can show that $({}^{(\gamma)}A, {}^{(\gamma)}E)$ constitute a canonically conjugate pair [31, 75]:

$$\{E_j^a(x), A_b^k(y)\} = \frac{\kappa\gamma}{2}\delta_b^a \delta_j^k \delta^{(3)}(x,y), \ \{E_j^a(x), E_k^b(y)\} = \{A_a^j(x), A_k^b(y)\} = 0. \tag{2.2.15}$$

Additionally, the extended phase space described by $({}^{(\gamma)}A, {}^{(\gamma)}E)$ reduces to the ADM phase space for $G_j = 0$, that is, if one performs a symplectic reduction with respect to the Gauß constraints G_j. This way, we have carried out the affine canonical transformation given by

$$({}^{(\gamma)}K_a^i, {}^{(\gamma)}E_i^a) \to^{(\gamma)} A_a^j := \Gamma_a^j +^{(\gamma)} K_a^j. \tag{2.2.16}$$

In the following, we will drop the labels (γ) for a clearer notation.

We want to close this section with some remarks:

- Note that (2.2.13) has exactly the form of a Gauß law known from Yang-Mills gauge theory with gauge group $SU(2)$. Therefore, we can understand GR as a non-abelian $SU(2)$ Yang-Mills gauge theory described by a $SU(2)$ connection $A = A_a^j \, dx^a \otimes \tau_j$, where τ_j denote the generators of $su(2)$, and an electric field $E = E_j^a \, \partial_a \otimes \tau_j$.
- As we have three additional constraints, the Dirac algebra of constraints stated in (2.1.25)$-$(2.1.27) has to be extended. For this, we introduce the smeared Gauß constraint

$$G(\Lambda) := \int_\sigma d^3x \Lambda^j G_j, \tag{2.2.17}$$

with $\Lambda = \Lambda^j \tau_j$ being Lie algebra-valued. Then, we obtain the following relations:

$$\{G(\Lambda), G(\Lambda'\} = G([\Lambda', \Lambda]), \tag{2.2.18}$$

$$\{G(\Lambda), \vec{H}(\vec{N})\} = -G(\mathcal{L}_{\vec{N}}\Lambda), \tag{2.2.19}$$
$$\{H(N), G(\Lambda))\} = 0. \tag{2.2.20}$$

(2.2.18) tells us that $G(\Lambda)$ generates $SO(3)$ or rather $SU(2)$ gauge transformations. (2.2.19) is the expected transformation of a scalar density of weight one under infinitesimal $SU(2)$ gauge transformations. From (2.2.20), we can infer that the Hamiltonian constraint is invariant under SU(2) gauge transformations.

- If we keep the dimension D general, we would have an $SO(D)$ connection A with $D(D-1)/2$ components and a D-bein E with D components. In order to have a canonically conjugate pair, the number of components have to match, which is true only for $D = 3$. However, there exist generalizations to higher dimensions: In [76–79] this was achieved by considering a connection A_{aIJ}. Here, the internal indices I, J correspond to the gauge group $SO(D+1)$ or $SO(D,1)$ in the case of Euclidean or Lorentzian signature for the internal metric, respectively.

2.2.2 New form of the constraints and Thiemann's identities

In this section, we state the Hamiltonian and diffeomorphism constraint in terms of Ashtekar's variables and introduce the so-called Thiemann identities. The latter highlight one of the big advantages of Ashtekar's variables over ADM variables. We mainly follow [27, 29].

In Ashtekar variables, we get the following expressions for the diffeomorphism and Hamiltonian constraint [29]:

$$H_a = F^j_{ab} E^b_j, \tag{2.2.21}$$
$$H = \frac{\gamma}{2} \frac{\epsilon_{jkl} F^j_{ab} E^a_k E^b_l}{\sqrt{det(q)}} - \gamma(\gamma^2+1) \frac{\epsilon_{jmn}\epsilon_{jkl} K^m_a K^n_b E^a_k E^b_l}{\sqrt{det(q)}}, \tag{2.2.22}$$

with the curvature tensor F^j_{ab} defined by

$$F^j_{ab} := 2\partial_{[a} A^j_{b]} + \epsilon_{jkl} A^k_a A^l_b. \tag{2.2.23}$$

Adapting the usual notation, we call the first term of (2.2.22) the Euclidean part H_E and the second term the Lorentzian part H_L. We can also rewrite the reduced ADM action (2.1.24) in terms of Ashtekar variables [27]:

$$S_{EH} = \frac{1}{\kappa} \int_{\mathbb{R}} \mathrm{d}t \int_\sigma \mathrm{d}^3x \left[2\dot{A}^i_a E^a_i - \left(\Lambda^j G_j + N^a H_a + N H \right) \right]. \tag{2.2.24}$$

Note that for $\gamma = \pm i$ the second term in the Hamiltonian constraint would vanish. In this case, however, A would take values in the Lie algebra of the non-compact gauge group $SL(2,\mathbb{C})$. This poses a problem at the quantum level, because the construction of the Hilbert space relies on the harmonic analysis of compact gauge groups, in particular that of $SU(2)$. Furthermore, as noted before, additional reality conditions need to be implemented at the quantum level.

Now we want to introduce the so-called Thiemann identities developed first in [80]. The main motivation behind Thiemann's work is to get rid of the non-polynomial structure of the Hamiltonian constraint due to the $1/\sqrt{det(q)}$ factor. Classically, $\sqrt{det(q)}$ is the integrand of the volume functional of a spatial slice σ:

$$V := \int_\sigma \mathrm{d}^3x \sqrt{|det(q)|}. \tag{2.2.25}$$

In the quantum theory, $\sqrt{det(q)}$ can be quantized using the volume operator that can be defined in the context of LQG. As we need to quantize $1/\sqrt{det(q)}$, we may use the inverse of the volume operator. Proceeding this way, however, we will encounter singularities, as the volume operator has a non-trivial kernel. To solve this problem, one rewrites the Hamiltonian constraint using Thiemann's identities which we present in the following. We have that $det(e) := det(e^j_a) = \frac{1}{6}\epsilon_{jkl}\epsilon^{abc}e^j_a e^k_b e^l_c$, which we can rewrite using $det(e) = sgn(det(e))\sqrt{det(q)}$ and (2.2.6) to get

$$\epsilon^{abc}e^k_c(x) = sgn(det(e))\epsilon^{ijk}\frac{E^a_i E^b_j}{\sqrt{det(q)}}(x). \tag{2.2.26}$$

One can further show that the following identity holds:

$$\frac{4}{\kappa}\epsilon^{abc}\{V, A^k_c(x)\} = 2\epsilon^{abc}\frac{\delta V}{\delta E^c_k(x)} = \epsilon^{abc}e^k_c(x). \tag{2.2.27}$$

Combining (2.2.26) and (2.2.27) gives us the final identity [80]

$$sgn(det(e))\frac{\epsilon^{ijk}E^a_i E^b_j}{\sqrt{det(q)}}(x) = \epsilon^{abc}\{V, A^k_c(x)\}. \tag{2.2.28}$$

This expression allows us to simplify the Euclidean part, which in its smeared version reads as

$$H_E(N) = sgn(det(e))\frac{8}{\kappa}\int_\sigma \mathrm{d}^3x\epsilon^{abc}\mathrm{tr}\left(F_{ab}\{V, A_c(x)\}\right), \tag{2.2.29}$$

where tr denotes the $su(2)-$trace. We see from (2.2.29) that it involves no longer polynomial terms and this was one of the main reasons why in [80], for the first time, a quantization of the Hamiltonian constraint was successfully performed. We will will explain the quantization in more detail in section 3.5. Having simplified the Euclidean part of the Hamiltonian constraint, we now turn to the Lorentzian part. Besides the $1/\sqrt{det(q)}$ factor, we also need to find an expression for K^j_a. For this purpose, we introduce the densitized extrinsic curvature

$$K := \int_\sigma \mathrm{d}^3x K^i_a E^a_i, \tag{2.2.30}$$

satisfying the identities [80]

$$\{V, H_E(1)\} = K, \quad \frac{\delta K}{\delta E^a_j} = K^j_a, \tag{2.2.31}$$

where we defined $H_E(1) := H_E(N = 1)$. This gives us the identity

$$\{\{V, H_E(1)\}, A^j_a\} = K^j_a \tag{2.2.32}$$

helping us to express K^j_a in terms of quantities, for which well defined operators exist. Now, we can rewrite the Lorentzian part of (2.2.22) using (2.2.32) for $K^m_a K^m_b$ and (2.2.28) for $E^a_k E^b_l/\sqrt{det(q)}$. We arrive at the smeared quantity

$$H_L(N) = sgn(det(e))\frac{16}{\kappa^3}\int_\sigma \mathrm{d}^3x\epsilon^{abc}\mathrm{tr}\left(\{\{V, H_E(1)\}, A_a(x)\}\{\{V, H_E(1)\}, A_b(x)\}\{V, A_c(x)\}\right). \tag{2.2.33}$$

As the Euclidean part is contained in the Lorentzian part, it is reasonable to first quantize the Euclidean part and use the result for quantizing the Lorentzian part afterwards. This finishes our discussion of the Hamiltonian formulation of general relativity.

3 Loop quantization of general relativity

In the following sections, we want to review loop quantum gravity (LQG). LQG is obtained by canonically quantizing general relativity. In particular, we start from a Hamiltonian formulation, which we discussed in section 2, and quantize this theory. When quantizing a constrained theory, for which general relativity is an example, we have in general two options:

- **Dirac quantization:** We may quantize the full phase space, i.e., we find representations of the Poisson algebra that our basic phase space variables satisfy. The representation space is called kinematical Hilbert space $\mathcal{H}_{kin}$. On this space, we implement the classical constraints as well-defined constraint operators. The physical states are then annihilated by the constraint operators and comprise the so called physical Hilbert space $\mathcal{H}_{phys}$, with the additional structure of an inner product.
- **Reduced phase space quantization:** We may solve the constraints classically and construct gauge invariant quantities, i.e., a complete set of weak or strong Dirac observables. Then, we compute the algebra of the Dirac observables which we quantize, i.e., we find representations of it. This way, we directly end up with the physical Hilbert space without having to solve any constraint equations.

Both approaches have their advantages and disadvantages: On the one hand, for Dirac quantization, the algebra of the basic variables is quite easily represented by interpreting the configuration variables as multiplication operators and the momenta as derivative operators. To obtain the physical Hilbert space, however, we have to find the joint kernel of all constraint operators. Specifically for general relativity, this is troublesome: The Gauß constraint can be solved, see the discussion at the end of section 3.3.4, and also the diffeomorphism constraint. For the latter, one uses the techniques of refined algebraic quantization and group averaging, see, e.g., section 3.3.2 of [29] as well as [81]. The Hamiltonian constraint operator, however, cannot be solved using the same techniques as for the other constraints and hence renders the Dirac quantization incomplete. On the other hand, for reduced phase space quantization, the advantage is that we directly end up with the physical Hilbert space after we quantized the observable algebra. However, the construction of weak or strong Dirac observables may be a very hard task for certain theories. Especially for general relativity, this has only succeeded in an asymptotically flat scenario where one approaches Minkowski space at spatial infinity and recovers the Poincaré charges as Dirac observables, see section 1.5 of [27]. Furthermore, if we managed to find Dirac observables, the observable algebra may be very complicated and thus, we may not find a representation thereof.

It depends on the theory of interest which approach is more convenient and in this thesis, we want to perform a reduced phase space quantization of the polarized three torus Gowdy model. We avoid the previously mentioned problems of constructing and quantizing observables and their potentially complicated Poisson algebra, respectively, by using the so-called relational formalism, which we will introduce in section 4. We basically add degrees of freedom, allowing us in the end to lift our basic variables to observables satisfying a Poisson algebra that is equivalent to the original one. Hence, finding a representation of this algebra is easy, as we can use the kinematical results established for the Dirac quantization of LQG. We will discuss the kinematics of LQG in more detail in the sections 3.2 and 3.3. Before we come to this, we introduce in the subsequent section the general quantization programme for Dirac quantizing a classical field theory constrained by a set of first class constraints, for which general relativity is a specific example. In particular, we will point out ambiguities we have at various points of the quantization programme.

3.1 The canonical quantization programme

In the following, we introduce the general canonical quantization programme providing us the necessary steps to obtain loop quantum gravity as a quantization of general relativity. This procedure may also be applied to any other classical theory and in particular to field theories. To explain these steps, we mainly follow [65], [73], [27] as well as chapter 13.2 of [82].

The starting point is a phase space modeled by an infinite dimensional symplectic manifold, which is a smooth manifold M equipped with a symplectic form ω. In more detail, ω is a smooth two-form that is non-degenerate, i.e., $\omega(X,Y)=0$ for all $Y\in T(M)$ implies $X=0$, as well as closed, i.e., $d\omega=0$. Then, we can define the Poisson bracket for $f,g\in C^\infty(M)$ as

$$\{f,g\}:=\chi_f(g). \tag{3.1.1}$$

Here, χ_f denotes the Hamiltonian vector field with respect to f that is uniquely determined by the symplectic form via $i_{\chi_f}\omega+df=0$, where i_{χ_f} denotes the interior product. We refer to chapter 19.3 of [27] for more details on symplectic geometry. Furthermore, we assume the symplectic manifold to be constrained by a set of first class constraints C_I with I ranging in some index set $\mathcal{I}$. We remind that a set of constraints is called first class if the constraints weakly Poisson commute, i.e., their Poisson bracket vanishes if the constraints are set to zero. If we also have second class constraints $\tilde{C}_J$, with $J\in\mathcal{J}$, i.e., constraints that do not weakly Poisson commute with all other constraints, we have to replace the Poisson bracket by the corresponding Dirac bracket which is constructed as follows: For two phase space functions f and g we define

$$\{f,g\}^*:=\{f,g\}-\sum_{N,M\in\mathcal{J}}\{f,\tilde{C}_M\}S^{MN}\{\tilde{C}_N,g\}. \tag{3.1.2}$$

Here, S^{MN} denotes the inverse of $S_{MN}:=\{\tilde{C}_M,\tilde{C}_N\}$, i.e., $S^{MK}S_{KN}=\delta^M_{\ N}$.

Now, we list the steps of the quantization programme as presented in [27]:

- Firstly, we choose a classical Poisson algebra $\mathfrak{P}$ which is a set of so-called elementary functions that are globally defined and coordinatize our phase space, i.e., every function on M can be written in terms of them. The corresponding mathematical condition would be that the elements of $\mathfrak{P}$ separate the points of M, i.e., for any $p,p'\in M$ with $p\neq p'$, there exists an $f\in\mathfrak{P}$ such that $f(p)\neq f(p')$. Additionally, we choose $\mathfrak{P}$ to be a Poisson $*$-subalgebra of $C^\infty(M)$: Complex linear combinations of elements of $\mathfrak{P}$ are again in $\mathfrak{P}$ and furthermore, $\mathfrak{P}$ is closed with respect to complex conjugation and the Poisson bracket. This is motived by the fact that we want to quantize the Poisson bracket by $i\hbar\widehat{\{.,.\}}$ and the complex conjugate a^* of $a\in\mathfrak{P}$ as the adjoint operator $\hat{a}^\dagger$.

- Secondly, we construct from the classical Poisson algebra $\mathfrak{P}$ the quantum *-algebra $\mathfrak{A}$ [1]. The general procedure for this would be to first construct the free algebra $\mathcal{J}$ containing finite linear combinations of all words w, which are finite sequences defined by $w:=(a_1...a_n)$, where $a_1,...,a_n\in\mathfrak{P}$ and n is finite[2]. To implement the canonical

[1] An algebra $\mathfrak{A}$ is a vector space over a field $\mathbb{K}$ with a bilinear multiplication map $\circ:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$. In case that $\circ$ is associative, $\mathfrak{A}$ is called an associative algebra. We obtain a *-algebra by equipping $\mathfrak{A}$ with an involution map $*:\mathfrak{A}\to\mathfrak{A},a\mapsto a^*$, which satisfies $(za+z'a')^*=z^*a^*+z'^*a'^*$, $(aa')^*=a'^*a^*$, $(a^*)^*=a$. If $\mathfrak{A}$ also contains a unit element $\mathbb{1}$ satisfying $\mathbb{1}\circ a=a\circ\mathbb{1}=a\ \forall a\in\mathfrak{A}$, we obtain a unital *-algebra.

[2] More precisely, one constructs the tensor algebra $T(\mathfrak{P}):=\oplus_{n=0}^\infty\otimes_{k=1}^n\ \mathfrak{P}=:\mathcal{J}$, where for $n=0$ we get $\mathbb{C}$. Then, a general element of $\mathcal{J}$ is given by $a=(w_0,w_1,...,w_n,...)$, with $w_0\in\mathbb{C}$ and w_n a finite linear combination of elements $w_{1n}\otimes...\otimes w_{nn}$ while $w_{1n},...,w_{nn}\in\mathfrak{P}$.

commutation relations and the involutive structure, we consider the two sided ideal $\mathcal{I}$ generated by elements of $\mathfrak{P}$ that are of the form $aa' - a'a - i\hbar\{a, a'\}$ and $a^* - \bar{a}$, where the bar denotes complex conjugation. Then, we divide the free algebra $\mathcal{J}$ by $\mathcal{I}$ to obtain the quantum *-algebra as

$$\mathfrak{A} := \mathcal{J}/\mathcal{I}. \tag{3.1.3}$$

- Thirdly, we investigate the representations of the quantum *-algebra $\mathfrak{A}$, which involves the construction of the kinematical Hilbert space $\mathcal{H}_{kin}$ on which the elements of $\mathfrak{A}$ are represented as linear operators[3]. To construct a representation from a given algebra, we may use the so-called Gelfand–Naimark–Segal (GNS) construction. This is done in LQG for the so-called Ashtekar-Lewandowski representation, which provides a unique representation under certain assumptions. We will discuss this in more detail in section 3.3.3.

- Finally, we construct the physical Hilbert space. Within Dirac quantization, this requires to find those states that annihilate the constraint operators $\hat{C}_I$, i.e., to solve the equations $\hat{C}_I \Psi_{phys}(q) = 0 \; \forall I \in \mathcal{I}$, where $\Psi_{phys}(q)$ denotes a physical state depending on the configuration variables q. To perform this step, one can apply the group averaging technique under certain assumptions for the constraints, or the more general procedure of direct integral decomposition involving the so-called Master constraint. We refer to chapter 3 of [27] for more details on group averaging and direct integral decomposition, but will discuss the Master constraint programme in section 3.6 as it is important to motivate algebraic quantum gravity.

We now want to comment on the ambiguities occurring in the above quantization programme.

- We begin with the choice of $\mathfrak{P}$: We should avoid any redundancy and hence, we consider the smallest possible subalgebra in the sense that we cannot remove elements of $\mathfrak{P}$ without violating our previously stated conditions on $\mathfrak{P}$. Furthermore, we should adapt the choice of $\mathfrak{P}$ such that the Poisson algebra simplifies and we can find representations of $\mathfrak{P}$ as operators on $\mathcal{H}_{kin}$. Another criterion is that the elements of $\mathfrak{P}$ transform simple under gauge transformations, which facilitates the quantization of Dirac observables. Furthermore, we will define the basic variables in a background independent fashion, which hence prohibits to use the standard Fock quantization procedure that invokes the background structure at various points. Note that if the elements $a \in \mathfrak{P}$ are unbounded, this will lead to unbounded operators which are defined only on dense domains, respectively. These domains, however, do not necessarily have to coincide, which occurs, for example, in standard quantum mechanics for the position and momentum operator. In particular, this leads to problems when calculating the commutator of these operators. In this case, one may consider the so-called Weyl elements, which are bounded and defined by the one-parameter family $U_t(a) := \exp(ita)$. Additionally, note that in case of field theories, one would obtain a distributional Poisson algebra and hence considers smeared quantities. Especially for LQG, we will introduce a specific smearing, see section 3.2.

[3] In more detail, a representation of a *-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, given by the representation Hilbert space $\mathcal{H}$ and the *-homomorphisms $\pi : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ mapping from the $^*-$algebra into the set of linear operators $\mathcal{L}(\mathcal{H})$ defined on $\mathcal{H}$. In particular, the homomorphism properties read as $\pi(aa') = \pi(a)\pi(a')$, $\pi(za + z'a') = z\pi(a) + z'\pi(a')$, while the compatibility with the involution is encoded in $\pi(a^*) = \pi(a)^\dagger$, where $\dagger$ denotes the adjoint. Furthermore, we impose $\pi(1) = \mathbb{1}_{\mathcal{H}}$ for a unital algebra and represent the Poisson brackets via $\pi(\{a, a'\}) = \frac{1}{i\hbar}[\pi(a), \pi(a')]$, where $[.,.]$ denotes the commutator.

- Now, we consider the ambiguities occurring for the quantum *-algebra $\mathfrak{A}$ constructed from $\mathfrak{P}$. Different choices of $\mathfrak{P}$, related by a canonical transformation, will in general lead to different abstract algebras of operators and hence to different quantum theories. Furthermore, if we fix one particular $\mathfrak{P}$, there may be many representations, which also might not be unitary equivalent. This is the generic case in field theories where an uncountably infinite number of unitary inequivalent representations occurs, e.g., already for the Fock quantization of a free scalar field with mass m. In case of quantum mechanics, however, we have the Stone-von Neumann uniqueness theorem, which is proved using the Weyl elements for position and momentum. This theorem tells us that under very mild assumptions, the Schrödinger representation is unique up to unitary transformations. Restricting for simplicity to one dimension, the Schrödinger representation provides a representation of the Heisenberg algebra $\{q,p\} = i\hbar\mathbb{1}_{\mathcal{H}}, \{q,q\} = \{p,p\} = 0$ on the Hilbert space of square integrable functions $\mathcal{H} = L_2(\mathbb{R}, dx)$, with the Lebesgue measure dx on $\mathbb{R}$, via

$$\pi_S(\hat{q}\psi)(x) = x\psi(x), \quad \pi_S(\hat{p}\psi)(x) = -i\hbar\frac{\partial\psi}{\partial x}(x). \tag{3.1.4}$$

  The assumptions going into this uniqueness result are that the representation is irreducible, i.e., there are only trivial invariant subspaces[4], and weakly continous, i.e., the map $t \to < \psi, U_t(a)\phi >$ is continous $\forall\psi, \phi \in \mathcal{H}$, where $U_t(a)$ denote the above introduced Weyl elements. Using the so-called Stone theorem, weak continuity tells us that also the momentum and position operators exist and hence, we can recover the Heisenberg algebra. Weak continuity, however, is not satisfied in LQG and thus, the representation of LQG is not unitarily equivalent to the Schrödinger representation. Nevertheless, as we mentioned before, one can establish a uniqueness result for LQG, see section 3.3.3. Specifically, one uses dynamical input, namely, that the gauge group is represented unitarily.

We conclude this section with the following remarks:

- The Poisson algebra $\mathfrak{P}$ we will be concerned with in LQG corresponds to the case of M having the structure of a cotangent bundle $T^*\mathcal{O}$, where $\mathcal{O}$ denotes the configuration space. Then, $\mathcal{P}$ would be the smallest and closed Lie subalgebra of the product space $Fun(\mathcal{Q}) \times V(\mathcal{Q})$, where $Fun(\mathcal{Q})$ are the functions on $\mathcal{Q}$ and $V(\mathcal{Q})$ the vector fields on $\mathcal{Q}$. The Lie algebra structure is given by

$$\{(f,v),(f',v')\} := (v[f'] - v'[f], [v,v']), \tag{3.1.5}$$

  with the Lie bracket $[.,.]$ defined for vector fields.

- There are also further tasks included in the quantization programme which we did not mention so far. One task would be to investigate if the constraint algebra is represented anomaly-free. This is especially important in LQG when considering the representation of the Poisson bracket between two Hamiltonian constraints that involves structure functions, see our discussion in section 3.6. Furthermore, one has to consider the semiclassical analysis, which we will also discuss in section 3.6. In particular, both of these issues will help us to motivate the introduction of algebraic quantum gravity.

[4] In more detail: Given a representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, a subspace V of $\mathcal{H}$ is called invariant if $\pi(a)V \in \mathcal{H}$, $\forall a \in \mathcal{A}$ and V is called trivial if either $V = \{0\}$ or $V = \mathcal{H}$.

3.2 Holonomy-flux algebra

In this section, we apply the first step of the quantization programme of section 3.1 to general relativity formulated in terms of Ashtekar variables.

3.2.1 Motivation and basic variables

We first observe that the canonical commutation relations of the connection variables in (2.2.15) involve delta distributions. To avoid such distributional results, we have to introduce a smearing of the variables. The main guidance for the smearing will be that the smeared quantities transform nicely under gauge transformations and that they are define in a background independent way.

A first guess may be the smearing [27, 65]

$$E(f) := \int_\sigma \mathrm{d}^3x f_a^j E_j^a, \quad F(A) := \int_\sigma \mathrm{d}^3x F_j^a A_a^j. \tag{3.2.1}$$

However, one can show that the transformation behaviour under gauge transformations, generated by the Gauss constraint, is not nice in the sense that it is not easy to construct gauge invariant quantities. In section 6.1 of [27], further proposals for smearing are discussed, which, however, do not lead to a closed Poisson algebra. Hence, one uses different techniques, in particular inspired from lattice gauge theory as we will present in the next paragraphs.

We will now present the smearing separately for holonomies and fluxes.

Connections: $A = A_a^j \tau_j dx^a$ is a Lie algebra-valued one-form and therefore naturally integrated in one dimension, where naturally means that it can be done without using a background metric. However, it is not convenient to use the connection one-form integrated over some one dimensional submanifold of σ, but instead, one introduces so-called **holonomies**, cf. [29, 65]: We denote a holonomy defined for a curve $c : [0,1] \to \sigma$ by $A(c) := A(c,1) = A(c,t)|_{t=1}$, where $A(c,t) : [0,1] \to SU(2)$ is the unique solution of the parallel transport equation

$$\frac{d}{dt}A(c,t) = A(c,t)A(c(t)), \quad A(c,0) = \mathbb{1}_2, \tag{3.2.2}$$

where $A(c(t)) := A_a^j(c(t))\tau_j/2\dot{c}^a(t)$. This is an ordinary differential equation of first order which is uniquely solved using the initial condition $A(c,0) = \mathbb{1}_2$. Solving (3.2.2) iteratively gives us the following infinite series for $A(c) := A(c,1)$:

$$A(c) = \mathcal{P}\exp\left(\int_e A\right) = \mathbb{1} + \sum_{n=1}^{\infty}\int_0^1 dt_1 \int_{t_1}^1 dt_2 ... \int_{t_{n-1}}^1 dt_n A(c(t_1))...A(c(t_n)). \tag{3.2.3}$$

Here, $\mathcal{P}$ denotes the path ordering symbol and orders the smallest path parameter to the left. Two further important properties are

$$A(c_1 \circ c_2) = A(c_1)A(c_2), \quad A(c^{-1}) = A(c)^{-1}. \tag{3.2.4}$$

The first relation is true for two curves c_1, c_2 whose end and starting point coincide, i.e., $f(c_1) = b(c_2)$, while in the second relation, we defined $c^{-1}(t) := c(1-t)$. Furthermore, $A(c)$ is invariant under reparametrizations. The main motivation for using holonomies is

their simple behaviour under $SU(2)$ gauge transformations: The connection transforms as $A^g = gAg^{-1} - dgg^{-1}$, whereas the holonomy acquires the simple transformation behaviour

$$A^g(c) = g(b(c))A(c)g(f(c))^{-1}), \tag{3.2.5}$$

where the transformation only acts at the starting and end point of the curve along which the connection is smeared. In particular, we can construct the gauge invariant Wilson loop $Tr(\mathcal{P}\exp\left(\int_\alpha A\right)$ with α being a closed loop. To see the gauge invariance, we note that α, being a closed loop, satisfies $b(\alpha) = f(\alpha)$ and hence, when performing a $SU(2)$ gauge transformation, we can use the cyclicity of the $SU(2)$ trace, as well as $g(f(\alpha))^{-1}g(b(\alpha)) = \mathbb{1}_{SU(2)}$ which finishes the argument. The choice of holonomies as elementary configuration variables is the reason why the theory was named **loop quantum gravity**. Furthermore, the connection A transforms as $A^\varphi = \varphi^* A$, for φ being a spatial diffeomorphism. Hence, we obtain for holonomies that

$$A^\varphi(c) = A(\varphi^{-1}(c)). \tag{3.2.6}$$

Densitized triads: So far, we introduced a one-dimensional smearing for the connection. We also have to introduce a smearing for the densitized triads $E = E^a_j \tau_j \partial_a$ in order to avoid a distributional result when calculating the Poisson bracket between E and holonomies. In particular, a two dimensional smearing is needed, as it turns out that for a three dimensional smearing such as $E(f) := \int_\sigma f^j_a E^a_j$, the Poisson algebra does not close. In particular, the Poisson bracket between a matrix element $(A(e))_{mn}$ and $E(f)$ results in an integral over holonomies, see [27] for details. Hence, a two dimensional smearing is necessary. As 2-forms are naturally, i.e., background independently, integrated in two dimensions, we use the Hodge dual $*$ to construct from E^a_j, being a $su(2)$-valued vector density of weight $+1$, the 2-form $(*E)^j_{ab} = E^c_j \epsilon_{abc}$ having density weight zero, as ϵ_{abc} has density weight -1. Smearing the quantity $*E$ with an $su(2)$-valued smearing field n^j and integrating the result over a two-dimensional surface S results in the so called **electric flux**:

$$E(S,n) := \int_S n^j (*E)_j = \int_S n^j \epsilon_{abc} E^a_j dx^b \wedge dx^c. \tag{3.2.7}$$

We obtain the following transformation behaviour under $SU(2)$ gauge transformations and spatial diffeomorphisms, respectively:

$$E^g_n(S) = E_{Ad_{g^{-1}}n}(S), \quad E^\varphi_n(S) = E_{(\varphi^{-1})^* n}(\varphi(S)). \tag{3.2.8}$$

While the spatial diffeomorphisms act in a simple way, the gauge transformations act rather non-local. This, however, is resolved when constructing gauge invariant quantities such as the area operator, cf. section 3.4., where one considers the limit in which the surfaces shrink to points.

We want to remark that the transformations (3.2.5),(3.2.6), and (3.2.8) are implemented as automorphisms of $\mathfrak{P}$, as only the labels are changed under the action of spatial diffeomorphisms and gauge transformations and hence, the algebraic structure remains unchanged.

3.2.2 The Poisson algebra

In the previous section, we have specified the elementary functions of our classical Poisson algebra. Now, we need to compute the Poisson bracket relations. The Poisson bracket $\{A(c'), A(c)\}$ vanishes and we proceed with the calculation of $\{E_n(S), A(c)\}$ in the following. Note, however, that the Poisson bracket we defined in section 2 requires the basic variables to be smeared in 3 dimensions, respectively. The smearing we introduced for the connections and densitized triads, however, is well motivated by background independence, but only one- and two-dimensional, respectively. Hence, one has to introduce a suitable regularization, where one extends the curves to tubes and the surfaces to disks to arrive at a three-dimensional smearing for holonomies and fluxes, respectively. See chapter 6.4 of [27] for more details.

We first consider the simple case of a surface S punctured once by a curve $c : [0, 1] \to \sigma$ in a point p. In particular, we can split up c into $c_1 \circ c_2$ with $f(c_1) = p = b(c_2)$. Then, one can show that [29]

$$\{E_n(S), A(c)\} = \pm\frac{\kappa\gamma}{2} A(c_1)\tau_j n^j(p) A(c_2). \tag{3.2.9}$$

In this equation, we have a positive (negative) sign if the normal to S and the tangent of c are oriented the same (opposite) way. If the edge does not intersect with S, then (3.2.9) is zero.

(3.2.9), however, does not describe the generic case, it may also happen that the edge lies in the surface or intersects it more than once or even infinitely many often. This would result in an infinite number of holonomies on the right hand side of (3.2.9). To avoid this, one restricts to piecewise analytic or rather semianalytic curves and surfaces instead of smooth ones. The transition to analytic structures is justified by the fact that smooth surfaces and paths can intersect infinitely many often and thus, $\{E_n(S), A(c)\}$ would result in infinitely many holonomies. The restriction to piecewise and not entire analytic structure is motivated by locality, because an analytic path specified on an small open neighborhood is determined globally by its analytic extension. Furthermore, a diffeomorphism preserving analytic structures would be non-local which turns out to be undesirable for the proof of the uniqueness result, cf. [27]. Hence, we consider in the following the set of curves $c \in \mathcal{C}$, $c : [0, 1] \to \sigma$, consisting of piecewise semianalytic curves, which are basically a composition of analytic curves meeting in certain points where the curves are at least one time differentiable. Note that the definitions of semianalyticity and piecewise analyticity are similar, with the difference that for example a semianalytic curve has to be at least one time differentiable at points of non-analyticity, while a piecewise analytic curve only has to be continous. Moreover, we restrict to semianalytic surfaces, which are basically unions of analytic surfaces whose closures meet in piecewise analytic curves, see [27] for a precise definition.

We will restrict our choice of curves even further, namely, we define the set of paths $\mathcal{P}$ as the set of equivalence classes of curves $c \in \mathcal{C}$. Two curves are said to be equivalent if 1. they have the same beginning and end point and 2. are identical up to a finite number of retracings and reparametrizations. The latter condition is motivated by the fact that such curves yield the same holonomy due to the properties we described in the context of (3.2.4).

On $\mathcal{P}$. we can define a so-called groupoid structure. For this purpose, we define the composition of two curves c_1, c_2 as

$$(c_1 \circ c_2)(t) := \begin{cases} c_1(2t) & t \in [0, 1/2], \\ c_2(2t-1) & t \in [1/2, 1]. \end{cases} \tag{3.2.10}$$

Furthermore, the inverse of a curve is given by $c^{-1}(t) := c(1-t)$. Then, we can compose and invert paths by applying the respective operations on the representatives:

$$p_{c_1} \circ p_{c_2} := p_{c_1 \circ c_2}, \quad p_c \circ p_c^{-1} = b(p_c). \tag{3.2.11}$$

Here, p_c denotes an equivalence class of curves with representative c. This way, we do not obtain a group but a groupoid structure, as not all elements can be composed. In particular, a path can be broken up into several parts, called edges e_i for $i = 1, ..., n$, of which a path is the composition.

We now introduce so-called *elementary edges* e which lie in a certain way with respect to a given surface S: An edge e is called of type out if $S \cup e = \emptyset$, of type in if $S \cup e = e$, and of type up or down if it intersects S in one point $p = S \cup e$ and lies above or below S. For the latter case, the point p may either be the starting or the end point of e, which we denote as $b(e)$ and $f(e)$, respectively. We illustrated the possible elementary edges in figure 3.2.2, where we chose for e being of type up or down that $S \cup e = b(e)$.

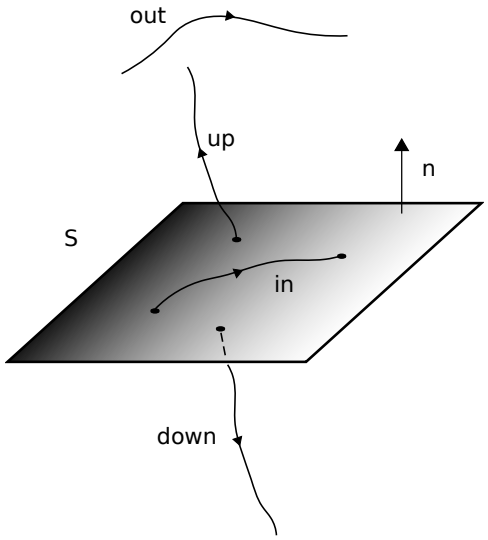

Figure 1: Illustration of the elementary edge of type up, down, out, and in as shown in [65]. Furthermore, n denotes the normal of the surface S.

We can now generalize (3.2.9) and obtain [28]

$$\{E_n(S), A(e)\} = \frac{\kappa\gamma}{2}\frac{\epsilon(S,e)}{2} \times \begin{cases} A(e)\tau_j n^j(p) & \text{if } p = b(e), \\ -\tau_j n_j(p) A(e) & \text{if } p = f(e), \end{cases} \tag{3.2.12}$$

where $\epsilon(S, e)$ is $+1$ for e of type up, -1 for e of type down, and 0 for e of type out.

To extend (3.2.12) to functions of holonomies, we have to introduce the following notions:

- A graph γ is basically a finite collection of edges e_i, with $i = 1, ..., |E(\gamma)|$ where $E(\gamma)$ denotes the set of edges. The edges of γ meet in a certain number of points which are called vertices v_i, with $i = 1, ..., |V(\gamma)|$ where $V(\gamma)$ denotes the set of vertices.
- Denoting the space of smooth connections as $\mathcal{A}$, a function $f : \mathcal{A} \to \mathbb{C}$ is called cylindrical with respect to a graph γ if there exists a smooth function $F_\gamma : SU(2)^{|E(\gamma)|} \to \mathbb{C}$ such that one can write f as

$$f(A) = F_\gamma(A(e_1), ..., A(e_n)). \tag{3.2.13}$$

Here, e_i, with $i = 1, ..., n$, denote the edges of the graph γ.

- To define the space of cylindrical functions, we have to note that given a function f cylindrical with respect to some graph γ, f will also be cylindrical with respect to a larger graph γ'. Identifying such functions as equivalent defines an equivalence relation $\sim$. The space of smooth cylindrical functions is then defined as the quotient space

$$Cyl^\infty := \cup_\gamma Cyl^\infty_\gamma / \sim, \tag{3.2.14}$$

where Cyl^∞_γ contains the smooth functions cylindrical with respect to γ. One can show that Cyl^∞ is a C^*-algebra when equipped with the supremum norm and pointwise addition and multiplication[5].

Now, we can extend (3.2.12) to smooth cylindrical functions $f \in Cyl^\infty$ using the Leibniz rule and obtain

$$\{E_n(S), f\} = \frac{\kappa\gamma}{4} \sum_{e\in E(\gamma)} \epsilon(e,S) n^j(p) \begin{Bmatrix} [A(e)\tau_j]_{AB} & \text{if } p = b(e) \\ -[\tau_j A(e)]_{AB} & \text{if } p = f(e) \end{Bmatrix} \frac{\partial F_\gamma}{\partial A(e)_{AB}}(\{A(e')\}_{e'\in E(\gamma)}). \tag{3.2.15}$$

Note that in this equation, γ is chosen to be adapted to S, i.e., each edge of γ is either of type up, down, in or out. We can achieve this by dividing the edges suitably, which introduces new, so-called virtual vertices.

We now turn to the Poisson bracket between two fluxes $E_n(S)$ and $E_{n'}(S')$. The classical relation $\{E^a_j(x), E^b_k(y)\} = 0$ suggests to set $\{E_n(S), E_{n'}(S')\}$ to zero. However, as discussed in [83], if we write out the Jacobi identity between a cylindrical function and two fluxes $E_n(S), E_{n'}(S')$ and use (3.2.15), we would get a contradiction when assuming that $\{E_n(S), E_{n'}(S')\} = 0$ holds. Hence, we would not have a Lie algebra structure. The way out of this problem is to introduce the derivations $Y_n(S) := \{E_n(S), .\}$, which are the Hamiltonian vector fields of the fluxes and are called flux vector fields. These also satisfy the Jacobi identity for the differential geometric Lie bracket. Using the construction in (3.1.5), we can define a Lie algebra structure. Nevertheless, we have to keep in mind that

$$\{\{E_n(S), E_{n'}(S')\}, f\} = [Y_n(S), Y_{n'}(S')] \cdot f \neq 0, \tag{3.2.16}$$

which implies that at the quantum level, the operators describing the spatial geometry are non-commuting.

We now want to scetch how to rewrite (3.2.15) as the action of the derivation $Y_n(S) := \{E_n(S), .\}$. For this purpose, we introduce left- and right-invariant vector fields defined by [27]

$$(R_j f)(h) := \frac{d}{dt}\Big| f(e^{t\tau_j} h), \quad (L_j f)(h) := \frac{d}{dt}\Big| f(h e^{t\tau_j}), \tag{3.2.17}$$

for h an element of a Lie group G. For matrix Lie groups such as $G = SU(2)$, these vector fields have the coordinate representation $R_j = (\tau_j h)_{AB}\partial/\partial h_{AB}$ and $L_j = (h\tau_j)_{AB}\partial/\partial h_{AB}$, see, e.g., [16] for a proof. This allows us to recast (3.2.15) into the following form:

$$Y_n(S) \cdot f := \{E_n(S), f\} = \frac{\kappa\gamma}{4} \sum_{e\in E(\gamma)} \epsilon(e,S) n^j(p) \begin{Bmatrix} R^j_e & \text{if } p = b(e) \\ -L^j_e & \text{if } p = f(e) \end{Bmatrix} F_\gamma(\{A(e')\}_{e'\in E(\gamma)}). \tag{3.2.18}$$

[5] A C^*-algebra C is an algebra equipped with an involution as well as a norm, such that the induced metric makes the algebra complete, giving us a Banach algebra. If the so-called C^* property $||a^*a|| = ||a^2||$ for all $a \in C$ is satisfied, we obtain a C^*-algebra.

Here, R_e^j and L_e^j only act on the copy of G labeled by e. We see that (3.2.18) is again a cylindrical function and hence, the flux vector fields indeed define derivations on Cyl^∞.

Finally, we can state the definition of the classical Poisson *-subalgebra which is the first step of the quantization programme described in section 3.1: The so called **holonomy flux algebra** $\mathfrak{P}$ is a Lie *-subalgebra of $Cyl^\infty \times V(Cyl^\infty)$, where Cyl^∞ are the smooth cylindrical functions and $V(Cyl^\infty)$ denotes the set of flux vector fields $Y_n(S) := \{E_n(S), .\}$ defined on Cyl^∞. The Lie bracket is defined in (3.1.5). Furthermore, the involution on $\mathfrak{P}$ is given by complex conjugation.

3.3 The Ashtekar-Lewandowski representation

We continue in the quantization programme and construct the quantum *-algebra $\mathfrak{A}$ using the standard construction discussed in the context of (3.1.3). Note that the flux vector fields will correspond to unbounded operators, which causes domain issues. To avoid this, we may consider the corresponding Weyl elements. We will not discuss this in detail at this point and refer to chapter 7 of [27] for a treatment thereof. The next step in the quantization programme is the discussion of representations of the quantum *-algebra $\mathfrak{A}$. We will concentrate on the so-called *Ashtekar-Lewandowski representation*. Note, however, that there are also other representations reported, see, e.g., [84–86].

In standard QM, the classical configuration space is $\mathbb{R}^3$. To find the representation Hilbert space $L_2(\mathbb{R}^3, d^3x)$ of square integrable functions on $\mathbb{R}^3$, one needs to construct a measure on that space, which turns out to be the standard Lebesgue measure. In our case, we have a field theory and the classical configuration space is the infinite dimensional space of smooth connections $\mathcal{A}$. Hence, to obtain the corresponding space of square integrable functions, we need to develop measure theory on infinite dimensional spaces. To handle this, it is a common procedure in the quantization of canonical field theories to extend the classical configuration space to some larger, distributional space, see, e.g., [28, 87] for a discussion. In our case, we need to introduce the space of generalized or distributional connections $\overline{\mathcal{A}}$, which is also called the quantum configuration space. In the following chapter, we want to give a more geometrical explanation of $\overline{\mathcal{A}}$.

3.3.1 The space of generalized connections

In the following, we will establish a more intuitive picture of the space of generalized connections $\overline{\mathcal{A}}$ and also equip it with a topology, such that measure theory can be developed thereon.

In particular, one can show that $\overline{\mathcal{A}}$ is homeomorphic to the set $Hom(\mathcal{P}, G)$ containing *all*, i.e., not necessarily continuous homeomorphisms from the set of paths $\mathcal{P}$ in σ, which we introduced in section 3.2.2, into the gauge group G.

To motivate that $Hom(\mathcal{P}, G)$ is the distributional extension of $\mathcal{A}$, we observe that a connection $A \in \mathcal{A}$ defines an element of $Hom(\mathcal{P}, G)$ due to $A(p_c) = A(c) \in G$. Furthermore, $Hom(\mathcal{P}, G)$ is larger than $\mathcal{A}$, as the latter space actually depends on the principal fibre bundle P and should be written as $\mathcal{A}_P$. In our case, we have P defined via the spatial manifold σ as the base space and $SU(2)$ as the structure group of P. $Hom(\mathcal{P}, G)$, however, does not depend on the principal bundle and hence contains all spaces $\mathcal{A}_P$ at once. Thus, $Hom(\mathcal{P}, G)$ is definitely larger than $\mathcal{A}$. It also contains distributional elements for which specific examples

are discussed in [88].

Now, we will discuss how we can turn $Hom(\mathcal{P}, G)$ into a topological space such that it is compact and Hausdorff[6], which allows us to equip $Hom(\mathcal{P}, G)$ with a measure. We refer to [27, 89] for more details and will only state the main ideas following [65]:

- We first consider the space $X_\gamma := Hom(\gamma, G)$ containing the groupoid homomorphisms from the graph γ, which is a subgroupoid, i.e., $\gamma \subset \mathcal{P}$, into the gauge group $SU(2)$. We introduced the notion of a graph in section 3.2.2. To establish a topology on X_γ, we recognize that each $x_\gamma \in X_\gamma$ is completely determined by the group elements $x_\gamma(e)$ for $e \in E(\gamma)$ and hence, we have a bijection

$$\rho_\gamma : X_\gamma \to G^{|E(\gamma)|}, x_\gamma \mapsto (x_\gamma(e))_{e\in E(\gamma)}. \tag{3.3.1}$$

  Since G is a compact Hausdorff space and $G^{|E(\gamma)|}$ for finite $|E(\gamma)|$, too, we can equip X_γ with a compact Hausdorff topology via the identification (3.3.1).

- Now we proceed to the infinite product

$$X_\infty := \prod_{\gamma\in\Gamma} X_\gamma, \tag{3.3.2}$$

  where Γ denotes the set of all compactly supported and semianalytic graphs. We can define a topology on X_∞ by the so-called product or Tychonov topology such that it becomes compact and Hausdorff.

- We cannot identify $\overline{\mathcal{A}}$ directly with X_∞ as the elements of $\overline{\mathcal{A}}$ have to satisfy an additional compatibility condition: We first define a partial and directed order $\prec$ on Γ[7] by defining $\gamma \prec \gamma'$ if and only if γ is contained in γ', i.e., every $e \in E(\gamma)$ can be written as a finite composition of the edges $e' \in E(\gamma')$ and their inverses. Now, we introduce the projection maps $p_{\gamma'\gamma} : X_{\gamma'} \to X_\gamma$ defined for all $\gamma \prec \gamma'$ as restriction maps. We also have the compatibility condition $p_{\gamma''\gamma} = p_{\gamma'\gamma} \circ p_{\gamma''\gamma'}$, with $\gamma \prec \gamma' \prec \gamma''$. This translates to elements $A \in \overline{\mathcal{A}}$ in the sense that $p_{\gamma'\gamma}(A|_{\gamma'}) = A|_\gamma$. This leads us to consider the set

$$\overline{X} := \{(x_\gamma)_{\gamma\in\Gamma} \mid p_{\gamma'\gamma}(x_{\gamma'}) = x_\gamma,\ \forall \gamma \prec \gamma'\} \tag{3.3.3}$$

  containing those elements of X_∞ that are compatible in the above introduced sense. $\overline{X}$ is called the projective limit of the projective family $(X_\gamma, p_{\gamma\gamma'})_{\gamma\prec\gamma'\in\Gamma}$. To equip $\overline{X}$ with a topology, we use that it is a subspace of X_∞ and take the induced subspace topology, i.e., the open sets of $\overline{X}$ are the sets $U \cap \overline{X}$ with $U \subset X_\infty$ any open set in X_∞. Using this, one can show that $\overline{X}$ is compact and Hausdorff as well.

- The final step is to establish a bijection between $Hom(\mathcal{P}, G)$ and $\overline{X}$ such that the topological properties are inherited from $\overline{X}$ onto $Hom(\mathcal{P}, G)$. See section 6.2.2 of [27] for a proof.

[6] A topological space is called Hausdorff if any two different points of that space have distinct open neighborhoods. It is furthermore called compact if every open cover has a finite subcover, where an open cover is a collection of open subsets, whose union contains the whole topological space.

[7] That the relation $\prec$ on Γ defines a partial order means that it is reflexive, i.e., $\gamma \prec \gamma$, symmetric, i.e., $\gamma \prec \gamma', \gamma' \prec \gamma \Rightarrow \gamma = \gamma'$, and transitive, i.e., $\gamma \prec \gamma', \gamma' \prec \gamma'' \Rightarrow \gamma \prec \gamma''$. Furthermore, directed means that $\forall \gamma, \gamma' \in \Gamma\ \exists \gamma'' : \gamma, \gamma' \prec \gamma''$.

In total, we can identify the space of distributional connections $\overline{\mathcal{A}} := Hom(\mathcal{P}, G)$ with the projective limit such that $\overline{\mathcal{A}}$ becomes a topological and compact Hausdorff space.

We can now extend our definition of cylindrical functions in (3.2.14) to the projective limit $\overline{X}$ and hence to $\overline{\mathcal{A}}$. Denoting the continuous, complex-valued functions on X_l by $C(X_l)$, we define the smooth cylindrical functions on $\overline{X}$ as

$$Cyl^\infty(\overline{X}) = \cup_{\gamma\in\Gamma} C(X_\gamma)/\sim, \tag{3.3.4}$$

with the equivalence relation $\sim$ defined by

$$f \sim f' \Leftrightarrow p^*_{\gamma''\gamma} f = p^*_{\gamma''\gamma'} f' \ \forall \gamma, \gamma' \prec \gamma''. \tag{3.3.5}$$

This is basically the same definition as (3.2.14), i.e., equivalence means that we can find a subgroupoid γ'' that contains γ, γ'. Note that an element of $Cyl(\overline{X})$ is actually an equivalence class of functions on X_γ, i.e., $f \in Cyl(\overline{X})$ might be denoted as $[f_\gamma]_\sim$. One can show, however, that $Cyl(\overline{X})$ can be identified with $C(\overline{X})$, which is the set of continuous functions on the projective limit $\overline{X}$. To show this, one makes use of the so-called Gel'fand-Naimark-Segal theory, of which a review is presented in chapter 27 of [27].

3.3.2 The kinematical Hilbert space

To obtain the kinematical Hilbert space from $\overline{\mathcal{A}}$, we need to define a suitable measure, for which we use projective techniques. This will also hand us an inner product. We again follow [27].

We will first describe the construction of a measure on $\overline{\mathcal{A}}$. The starting point is a projective family $(X_\gamma, p_{\gamma\gamma'})_{\gamma\prec\gamma'\in\Gamma}$ with continuous and surjective projection maps $p_{\gamma'\gamma} : X_{\gamma'} \to X_\gamma$. On this projective family, we can define a family of measures $(\mu_\gamma)_{\gamma\in\Gamma}$ which is said to be consistently defined if

$$\int_{X_{\gamma'}} d\mu_{\gamma'}(X_{\gamma'})\ [p^*_{\gamma'\gamma} f_\gamma](X_{\gamma'}) = \int_{X_\gamma} d\mu_\gamma(X_\gamma)\ f_\gamma(X_\gamma) \quad \forall \gamma \prec \gamma', f_\gamma \in C(X_\gamma). \tag{3.3.6}$$

This condition implies that if we enlarge the subgroupoid γ, with respect to which the sets X_γ are defined, the result of the integral, defined with respect to the corresponding measure μ_γ, does not change. Specializing to the case of $\overline{\mathcal{A}}$, cylindrical functions are defined by

$$f_\gamma(x_\gamma) = F_\gamma(\{x_\gamma(e)\}_{e\in E(\gamma)}) = (\rho^*_\gamma F_\gamma)(x_\gamma), \tag{3.3.7}$$

with $F_\gamma : G^{|E(\gamma)|} \to \mathbb{C}$ being continous and ρ_γ defined in (3.3.1). We see that any cylindrical function over a graph γ can be expressed via a function on $SU(2)^{|E(\gamma)|}$. Hence, if we want to find a measure on $\overline{\mathcal{A}}$, we may take the Haar measure on $SU(2)$ or rather $|E(\gamma)|$ copies of it. This motivates to define the family of cylindrically consistent measures via

$$\mu_{0\gamma}(f_\gamma) = \int_{X_\gamma} d\mu_{0\gamma}(x_\gamma)(\rho^*_\gamma F_\gamma)(x_\gamma) = \int_{G^{|E(\gamma)|}} [d\mu_H(h_e)]\, F_\gamma(\{A(e)\}_{e\in E(\gamma)}). \tag{3.3.8}$$

Here, μ_H denotes the unique Haar probability measure of the compact Lie group $SU(2)$ that is invariant under right and left translations as well as inversions. In particular, (3.3.8) satisfies (3.3.6). The consistency condition (3.3.6) yields us that the linear functional

$$\Lambda(f) := \int_{X_\gamma} d\mu_\gamma(X_\gamma) f_\gamma(x_\gamma), \tag{3.3.9}$$

with $f = [f_\gamma]_\sim \in Cyl(\overline{X})$, is well defined, i.e., it does not depend on the representative f_γ of f. One can show that the linear functional (3.3.9) is even defined on $C(\overline{X})$, the space of continous functions on $\overline{X}$. Together with the fact that $\overline{X}$ is a topological space, which is also compact and Hausdorff, we can apply the so-called Riesz-Markov theorem. This theorem allows us to identify a regular Borel probability measure μ_0 on $\overline{X}$ with the consistently defined family of measures $(\mu_\gamma)_{\gamma\in\Gamma}$, while we have the relation $\mu_0 \circ p_l^{-1} = \mu_\gamma$. μ_0 is also called the *uniform measure* or *Ashtekar-Lewandowski measure*.

We can now define the inner product for two smooth cylindrical functions $f, f' \in Cyl^\infty$ as

$$\int_{SU(2)^n} \prod_{i=1}^{n} d\mu_H(A(e_i))\overline{F_\gamma(A(e_1),...,A(e_n))}F'_\gamma(A(e_1),...,A(e_n)), \tag{3.3.10}$$

where $n := E(\gamma)$. Note that for f, f' being cylindrical with respect to two different graphs γ, γ', we pursue the following strategy: We take a graph γ'' containing γ and γ', e.g., we may choose $\gamma'' = \gamma \cap \gamma'$. Then, by definition, f and f' are still cylindrical with respect to this graph. Furthermore, we assign trivial holonomies to those edges in γ'' not contained in γ, γ'. Cylindrical consistency will guarantee us that the outcome of the inner product does not change.

Finally, we can define the kinematical Hilbert space by taking the completion of Cyl^∞ with respect to the norm induced by the inner product (3.3.10). This gives us the so-called *Ashtekar-Isham-Lewandowski Hilbert space* $\mathcal{H} = L_2(\overline{\mathcal{A}}, \mu_0)$ of functions over $\overline{\mathcal{A}}$ which are square integrable with respect to μ_0.

3.3.3 The representation and uniqueness result

We can now state the so-called *Ashtekar-Lewandowski representation* of the quantum *-algebra $\mathfrak{A}$, which is constructed from the holonomy flux algebra. The representation space is $\mathcal{H} = L_2(\overline{\mathcal{A}}, \mu_0)$ and the cylindrical functions act as multiplication operators

$$(\pi_{AL}(f)\psi)(A) = f(A)\psi(A) \tag{3.3.11}$$

for $\psi \in \mathcal{H}$, while the flux vector fields act as derivative operators

$$(\pi_{AL}(Y_n(S))\psi)(A) = -i\hbar\hat{Y}_n(S)[\psi], \tag{3.3.12}$$

for ψ in the domain of $\hat{Y}_n(s)$. Using (3.2.18), we can also specify the action of $\hat{Y}_n(S)$ on cylindrical functions, which lie dense in $\mathcal{H}$ [29]:

$$\hat{Y}_n(S)\cdot f = \frac{\kappa\gamma}{4}\sum_{v\in V(\gamma)} n^i(v)\left[\sum_{\mathrm{eat}v}\epsilon(S,e)\hat{J}_i^{(v,e)}F_\gamma(\{A(e')\}_{e'\in E(\gamma)})\right], \tag{3.3.13}$$

where we defined

$$\hat{J}_i^{(v,e)} := \mathbb{1}_{\mathcal{H}}\otimes\mathbb{1}_{\mathcal{H}}\otimes...\otimes\begin{Bmatrix} R_e^j & \text{if } p = b(e) \\ -L_e^j & \text{if } p = f(e)\end{Bmatrix}\otimes\mathbb{1}_{\mathcal{H}}\otimes...\otimes\mathbb{1}_{\mathcal{H}}. \tag{3.3.14}$$

In particular, one can show that under certain assumptions, the Ashtekar-Lewandowski representation is unique. This is the content of the so-called LOST theorem [90, 91] imposing the following restrictions on the representation $(\pi, \mathcal{H})$ of the quantum *-algebra $\mathfrak{A}$:

- Firstly, $(\pi, \mathcal{H})$ should be irreducible, i.e., any ψ in a common dense domain $\mathcal{D}$ of all $\pi(a)$ is cyclic. That ψ is cyclic means that $\{\pi(a)\psi | a \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. Specifically, one can show that non-degenerate representations, i.e., those satisfying $\pi(a)\psi = 0 \; \forall a \in \mathfrak{A} \Rightarrow \psi = 0$, are a direct sum of cyclic representations. Hence, it suffices to study cyclic representations, i.e., those admitting a cyclic vector. Now, we can use the so-called Gelfand-Naimark-Segal (GNS) theorem, which establishes a one-to-one correspondence between cyclic representations and positive linear functionals $\omega : \mathfrak{A} \to \mathbb{C}$, where positivity means $\omega(a^* a) \geq 0$. We refer to chapter 29 of [27] for more details. Hence, we can now study GNS states instead of representations.

- Secondly, part of the classical symmetries, i.e., in our case $SU(2)$ gauge transformations and spatial diffeomorphisms, should be implemented as unitary operators. One way to implement this is by requiring the state, corresponding to the cyclic representation, to be invariant under the classical symmetries.

The LOST theorem tells us that there is only one diffeomorphism and gauge invariant state ω, whose so-called GNS data $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ from the GNS theorem coincide with the Ashtekar-Lewandowski representation. Here, π_ω denotes a cyclic representation on the representation Hilbert space $\mathcal{H}_\omega$ and Ω_ω is a normed and cyclic vector contained in $\mathcal{H}_\omega$.

3.3.4 Spin network functions

In this section, we want to introduce a convenient orthonormal basis of the kinematical Hilbert space $\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu_0)$, the so-called spin network functions. We mainly follow [27] and also refer to [92] for more details.

First, we introduce an orthonormal basis for the Hilbert space $L_2(SU(2), d\mu_H)$. Note that the subsequent construction is valid for any compact Lie group. In the following, we fix a representative of the equivalence class of finite-dimensional, unitary, and irreducible representations of $SU(2)$. We denote this representative by $(\pi_j, \mathcal{H}_j)$, labeled by the half-integer spin quantum number $j \in \mathbb{N}/2$. Then, we use the so-called Peter and Weyl theorem, proved for instance in section 31 of [27]. This theorem guarantees that the following functions on $SU(2)$ provide an orthonormal basis of $L_2(SU(2), d\mu_H)$:

$$b_{jmn}(g) := \sqrt{d_j}[\pi_j]_{mn}, \quad m, n = 1, ..., d_j. \tag{3.3.15}$$

Here, $[\pi_j]_{mn}$ denotes a matrix element of the representation π_j labeled by the half-integer spin quantum number j and $d_j = 2j + 1$ is the dimension of the representation space $\mathcal{H}_j$. In particular, we have the orthonormality relation

$$< b_{jmn}, b_{j'm'n'} >= \int_G d\mu_H(g) \overline{[\pi_j]_{mn}} [\pi_{j'}]_{m'n'} = \delta_{jj'}\delta_{mm'}\delta_{nn'}. \tag{3.3.16}$$

We can now use (3.3.15) to define an orthonormal basis of $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu_0)$. To this end, we introduce the following orthonormal functions, which are called spin network functions:

$$T_{\gamma, \vec{j}, \vec{m}, \vec{n}} : \overline{\mathcal{A}} \to \mathbb{C}, \; A \to \prod_{e \in E(\gamma)} b_{j_e m_e n_e}(A(e)). \tag{3.3.17}$$

Here, we defined $\vec{j} := \{j_e\}_{e \in E(\gamma)}$, $\vec{m} := \{m_e\}_{e \in E(\gamma)}$, and $\vec{n} := \{n_e\}_{e \in E(\gamma)}$. These functions provide a basis for $\mathcal{H}_\gamma$, the Hilbert space associated to a graph γ. Now, we introduce so-called

intertwiners, which in general are maps $I : \mathcal{H}_1 \to \mathcal{H}_2$ between representation Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ corresponding to representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ such that $\pi_2(.)I = I\pi_1(.)$. Then, we assign to each vertex an intertwiner and contract them suitably with the spin network functions (3.3.17), giving us the so-called gauge variant spin network functions denoted as $T_{\gamma,\vec{j},\vec{I}}$ with $\vec{I} := (I_v)_{v\in V(\gamma)}$. One can show that these functions provide an orthonormal basis of $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu_0)$.

The concept of intertwiners is especially useful to solve the Gauß constraint. When constructing the Gauß constraint operator explicitly, its action on cylindrical functions is annihilated if at each vertex, the spins associated to the edges meeting in the vertex couple to total spin zero, see, e.g., section 3.3 of [29] and section 9.1 of [27]. This can be achieved by choosing intertwiners that project onto the trivial representation. The resulting gauge invariant spin network functions $T_{\gamma,\vec{j},\vec{I}}$ provide an orthonormal basis for $L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$, where $\overline{\mathcal{A}/\mathcal{G}}$ denotes the distributional extension of the space of connections $\mathcal{A}$ modulo gauge transformation $\mathcal{G}$.

3.4 Geometric operators

We now come to one of the characteristic properties of loop quantum gravity, namely the discreteness of geometry. This is implied by the spectrum of geometrical operators constructed from the classical length, area, and volume functionals.

We start with the area operator, since it is the most easiest geometrical operator to derive. We closely follow [29]. The classical area functional is given by

$$Ar(S) = \int_U d^2u \sqrt{det(X^*q)}(u), \tag{3.4.1}$$

where X^*q denotes the pull back of the spatial metric with respect to the embedding X of the surface defined by $X : U \subset \mathbb{R}^2 \to S$. Using the key classical identity $\sqrt{det(X^*q)} = \sqrt{n_a n_b E^a_j E^b_k \delta^{jk}}$, with n the conormal to S, one can rewrite the area functional in terms of Ashtekar variables. We also need to regularize the area functional, for which we choose a partition of U as $U = \cup_{i=1}^n U_i$, giving us

$$Ar(S) = \sum_{i=1}^n \int_{U_i} d^2u \sqrt{n_a n_b E^a_j E^b_k \delta^{jk}}(X(u)) \approx \sum_{i=1}^n \sqrt{E_j(S_{U_i}) E_k(S_{U_i}) \delta^{jk}}(v). \tag{3.4.2}$$

Here, v denotes a point centered in U_i. Furthermore, we introduced the fluxes $E_j(S_{U_i}) := \int_{U_i} d^2u \; n_a E^a_j(X(u))$, which are defined via the fluxes $E_n(S)$, we introduced in (3.2.7), as $E_j(S_{U_i}) := E_{\tau_j}(S)$, i.e., we choose the $su(2)$ basis elements τ_j as a smearing function. If we consider the limit $n \to \infty$ in (3.4.2), we recover (3.4.1). We can quantize (3.4.2) using (3.3.14), which results in the following action of the area operator on a spin network function:

$$\hat{Ar}(S) T_{\gamma,\vec{j},\vec{I}} = \frac{\gamma l_P^2}{4} \sum_{i=1}^n \sqrt{\left(\sum_{v\in V(\gamma), e at v} \epsilon(S_{U_i}, e) \hat{J}^{(v,e)} \right)^2} T_{\gamma,\vec{j},\vec{I}}. \tag{3.4.3}$$

Here, $l_P^2 := \kappa\hbar$ denotes the Planck length. The spectrum of the area operator can be calculated analytically and turns out to be discrete, see [29] for more details. We also want to

remark that there exists a smallest possible eigenvalue of the are operator. This so-called area gap has the value $l_P^2\gamma\frac{\sqrt{3}}{8}$ and is important for the description of black holes within LQG. Furthermore, note that the area operator is ill-defined in the usual Fock representation motivating the previously introduced, non-standard quantization techniques that allow us to give a meaning to geometrical operators.

Now, we introduce the volume operator. The volume functional of a region R is defined by

$$V(R) := \int_R d^3x\sqrt{det(q)}, \tag{3.4.4}$$

which can be expressed by the densitized triads using the identity

$$det(q) = |det(E_j^a)| = \left|\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}E_i^aE_j^bE_k^c\right|. \tag{3.4.5}$$

To replace the densitized triads by their well defined operator counterparts (3.3.14), we need to introduce a regularization in analogy to our discussion of the area operator. For this, we basically choose a partition of the spatial region R into cubic cells. As derived in detail in [93] and section 13.3 of [27], one arrives at the following expression for the volume operator measuring the volume of the spatial region R for a given graph γ:

$$\hat{V}(R)_\gamma = \int_R d^3x\sqrt{\widehat{det(q)(x)}}_\gamma = \int_R d^3x\hat{V}(x)_\gamma = \left(\frac{l_P}{2}\right)^3\sum_{v\in V(\gamma)}\delta^{(3)}(x,v)\hat{V}_{v,\gamma}, \tag{3.4.6}$$

where

$$\hat{V}_{v,\gamma} := \sqrt{\left|\frac{i}{48}\sum_{e_I\cap e_J\cap e_K)v}\epsilon(e_I,e_J,e_K)\epsilon_{IJK}\hat{Q}_{IJK}\right|}. \tag{3.4.7}$$

Here, we defined $\hat{Q}_{IJK} := \epsilon^{ijk}J_i^{(v,e_I)}J_j^{(v,e_J)}J_k^{(v,e_K)}$ and $\epsilon(e_I,e_J,e_K) = \pm1, 0$ is the sign factor, which vanishes if the three vectors tangent to the edges e_I, e_J, e_K, meeting at the vertex v, are linearly dependent. This coincides with the operator introduced by Ashtekar and Lewandowski in [94].

We now want to discuss some properties of the volume operator (3.4.7). It is defined cylindrically consistent: If we consider the volume operator defined on a graph γ', we can restrict it to functions defined on a smaller graph γ, where it coincides with the volume operator $\hat{V}(R)_\gamma$, i.e., we have $(\hat{V}(R)_{\gamma'})|_\gamma = \hat{V}(R)_\gamma$. Furthermore, the volume operator is essentially self-adjoint and has a pure point spectrum. Concerning the spectral properties, a closed formula for the calculation of matrix elements of the volume operator exists [93]. This is, for instance, used in [95–98] to develop a numerical calculation scheme for matrix elements of the volume operator. These results also suggests the existence of a volume gap which, however, depends on the vertex embedding. Analytic computations of the spectrum are in general hard to perform. Nevertheless, in [51], matrix elements of the volume operator with respect to semiclassical states could be calculated analytically using perturbative methods and in particular, with error control.

Note, however, that (3.4.7), which in the following we denote as $\hat{V}_{AL}$, is not the only volume operator that is reported. There also exists a second one, introduced by Rovelli and

Smolin in [99], which we refer to as $\hat{V}_{RV}$ in the following. Both volume operators use different regularization techniques and are inequivalent. They are also unique up to an arbitrary regularization constant. The main differences between $\hat{V}_{AL}$ and $\hat{V}_{RS}$ are the following: First of all, they are structurally different, while they both contain the operator $\hat{Q}_{IJK}$. However, in $\hat{V}_{AL}$ there occurs the factor $\epsilon(e_I, e_J, e_K)$ which does not appear in $\hat{V}_{RV}$. This implies that $\hat{V}_{AL}$ is only diffeomorphism-covariant, while $\hat{V}_{RV}$ is covariant under homeomorphisms, as it does need any information about the orientation of the edges. Furthermore, also planar edges contribute for $\hat{V}_{RV}$, which is not the case for $\hat{V}_{AL}$. Both operators, however, only act non-trivially at at most three-valent vertices. Note that the operator $\hat{V}_{AL}$ in (3.4.7) is preferred due to the following reasons : On the one hand, we have a kinematical reason provided by the consistency check in [100, 101]. The main results is that $\hat{V}_{RV}$ does not pass this consistency check, while $\hat{V}_{AL}$ does. Furthermore, the regularization constant is fixed in [100, 101], which was already used in (3.4.7). On the other hand, there are also a dynamical reason, namely that the Hamiltonian or Master constraint are not densely defined when using $\hat{V}_{RV}$ instead of $\hat{V}_{AL}$. See also section 13.3 of [27] for a discussion.

We close this chapter with the remark that one can also define a length operator. We will not go into detail about this, but emphasize that the lenght operator turns out to be the most difficult geometric operator. In [102], the Thiemann identities are used to replace the cotriads occurring in the length functional. This procedure, however, results in a length operator that is a function of the volume operator, whose spectrum is difficult to determine which hence carries over to the spectrum of the length operator. Other proposals for length operators can be found for example in [103, 104]. This finishes the discussion of the kinematical setup of LQG.

3.5 The Hamiltonian constraint

Now, we want to discuss the quantization of the Hamiltonian constraint. We already introduced the Thiemann identities in section 2.2.2, which allow us to rewrite the Hamiltonian constraint in the non-polynomial form (2.2.29), (2.2.33). We see that the Euclidean part (2.2.29) consists of the volume, the connection, and the curvature tensor. If we know how to quantize these quantities, we can define the Euclidean constraint operator, which then allows us to quantize the Lorentzian constraint containing the Euclidean part as well as the volume and the connection. Therefore, in the following, we only describe the quantization strategy for the Euclidean part and refer to [27, 32] for details on the quantization of the Lorentzian part.

We first describe the strategy of [32] to express (2.2.29) in terms of holonomies and fluxes. For this, we need to introduce a regularization, i.e., a partition of the spatial manifold into three-dimensional cells. In [32], these cells were chosen to be tetrahedrons $\triangle$, giving us a triangulation $T(\epsilon)$ of σ. ϵ denotes the so-called regulator, whose limit $\epsilon \to 0$ corresponds to shrinking the tetrahedrons to points. In principle, the shape of the regularization cells is arbitrary, they only have to be suitably small such that the Hamiltonian constraint can be well approximated by a Riemann sum, see also section VI, C. of [28] and section 4.1 of [29] for a discussion. To illustrate the regularization, we choose tetrahedrons in the following, for which the Hamiltonian constraint can be written as [40]

$$H(N) = lim_{\epsilon \to 0} \sum_{\triangle \in T(\epsilon)} N(v(\triangle)) H(\triangle). \tag{3.5.1}$$

Here, $v(\triangle)$ is an interior point of the tetrahedron $\triangle$ and $H(\triangle)$ is given by $H(\chi_\triangle)$, where $\chi_\triangle$ denotes the characteristic function of the tetrahedron $\triangle$. The strategy is now to rewrite (3.5.1) in terms of holonomies and fluxes. For this, following [32], we denote the three edges spanning a tetrahedron by e_I, which intersect in the point $v(\triangle)$ with outgoing orientation. Then, we can define the loops $\alpha_{IJ} := e_I \circ a_{IJ} \circ e_J^{-1}$ where a_{IJ} denotes the arc joining the endpoint of the two edges e_I, e_J. Now, we use the following techniques to replace the connection and the curvature in (2.2.29):

- Concerning the connection A, we can express it via holonomies the following way:

$$\{\int_{e_I} A, V(R)\} \approx A(e_I)\{A(e_I)^{-1}, V(R)\} \tag{3.5.2}$$

  which can be shown by approximating the left hand side by $\epsilon \dot{e}_I^a(0)\{A_a^j(e(0))\tau_j/2, V(R)\}$ and expanding the holonomies on the right hand side up to first order, i.e., $A(e_I) \approx 1 + \epsilon A_a^j(e(0))\tau_j/2\dot{e}_I^a(0)$.

- Concerning the curvature F, we will use a technique already known in lattice gauge theory, for which a review can be found in [105]: We consider a loop, e.g., a rectangular one, which is spanned by two edges e_I, e_J of sufficiently small coordinate length ϵ. Denoting this loop as α_{IJ}, one finds that [27]

$$A(\alpha_{IJ}) = \mathbb{1}_{SU(2)} + \epsilon^2 F_{ab}^j \dot{e}_I^a(0)\dot{e}_J^b(0)\tau_j/2 + O(\epsilon^3). \tag{3.5.3}$$

  This allows us to express F_{ab}^j by a holonomy defined with respect to the loop α_{IJ}.

Using the above two points and the explicit form of the Euclidean part (2.2.29), we can rewrite (3.5.1) as

$$H_E(N) = \frac{8}{3\kappa}\, lim_{\epsilon\to 0} \sum_{\triangle\in T(\epsilon)} N(v(\triangle))\epsilon^{IJK}\mathrm{tr}\left(A(\alpha_{IJ})A(e_K)\{A(e_K)^{-1}, V(R_{v(\triangle)})\}\right). \tag{3.5.4}$$

This expression can now be quantized by implementing the holonomies as multiplication operators, using the volume operator (3.4.7), and replacing the Poisson bracket by the commutator times $1/i\hbar$. Afterwards, one considers the limit $\epsilon \to 0$ which is discussed in section 10.4.2 of [27].

In particular, we are interested in triangulations adapted to a graph. This means that every vertex $v \in V(\gamma)$ is contained in a tetrahedron, with $v(\triangle) = v$, and the edges spanning the tetrahedron intersect in a vertex. Then, due to the volume operator (3.4.7) acting only on vertices, the quantum version of (3.5.4) will only give contributions for such tetrahedra, resulting in a finite action. Note that the operator ordering is important. In principle, we could also order $A(\alpha_{IJ})$ to the right such that the Hamiltonian constraint operator would give a contribution for every $v(\triangle) \in \triangle$, which are not necessarily vertices of the graph, hence resulting in a not densely defined operator. Furthermore, if we consider a tetrahedron $\triangle$ containing a vertex of the graph, the edges e_I spanning $\triangle$ are chosen to be segments of the edges of the graph. Then, the arcs a_{IJ} closing a loop α_{IJ} are lying outside of γ. Hence, when acting with the Hamiltonian constraint on a spin network function, it will attach the holonomies defined for the loop α_{IJ} to the spin network function, thereby changing the underlying graph to a new one γ' containing the loop. That is why, the regularization of [27] is called graph-changing. The investigation of the dynamics described by such a Hamiltonian

constraint leads to the subject of quantum spin dynamics, which is discussed in the series of papers [33–40].

However, we are interested in graph-preserving operators due to the following reasons: On the one hand, we later on want to pass to the algebraic quantum gravity formalism, where a graph-preserving quantization is used. On the other hand, we will use the relational formalism applied to Gaussian dust, giving us a physical Hamiltonian that is diffeomorphism invariant at the classical level. When quantizing the physical Hamiltonian, we would like to carry over this classical symmetry, specifically, we construct it as a spatially diffeomorphism invariant operator using the kinematical representation we discussed in section 3.3. However, it was shown in [81] that this representation requires to implement spatially diffeomorphism invariant operators as graph-preserving. To this end, we use a so-called graph-preserving regularization, for which we introduce the notion of minimal loops [106]: A minimal loop α_{IJ} is completely defined by edges of a graph γ. In particular, it starts at a vertex v, where it goes along the edge $e_I \in E(\gamma)$ and ends again at v along e_J^{-1}, the inverse of $e_J \in E(\gamma)$. The loop is furthermore minimal in the sense that there exists no other loop being defined the same way and traversing fewer edges. Using minimal loops to construct the Hamiltonian constraint operator, we do not end up with the aforementioned graph-changing behaviour of the operator introduced in [32].

To construct the graph-preserving Hamiltonian constraint operator, we can again use the approximations (3.5.2) and (3.5.3) to write down a regulated expression for the Euclidean part, analogous to (3.5.4). We arrive at the following operator, acting at a vertex v of γ in graph-preserving fashion, see also section 4.2 of [29]:

$$\hat{H}_{E,v} = \frac{1}{l_{\mathrm{P}}^2 |T_v(\gamma)|} P_\gamma \left[\sum_{(e_1,e_2,e_3)\in T_v(\gamma)} \frac{\epsilon^{IJK}}{|L_{\gamma,v,e_I,e_J}|} \sum_{\alpha_{IJ}\in L_{\gamma,v,e_I,e_J}} \mathrm{tr}\left(\hat{A}(\alpha_{IJ}) \hat{A}(e_K) [\hat{A}(e_K)^{-1}, \hat{V}_{\gamma,v}] \right) \right] P_\gamma \,. \tag{3.5.5}$$

Here, $T_v(\gamma)$ denotes the set of ordered triples at $v \in V(\gamma)$, while the tangent vectors of these edges are linearly independent. Furthermore, L_{γ,v,e_I,e_J} is the set of minimal loops and $\hat{V}_{\gamma,v}$ is the volume operator introduced in (3.4.7). The projection operators P_γ in (3.5.5) ensure that $\hat{H}_{E,v}$ indeed acts graph-preserving: The holonomy operators in (3.5.5) are defined with respect to edges coinciding with the graph and act as multiplication operators. Hence, when considering their action on a spin network function over γ, there occurs a product of representations. To express this product again in terms of spin network functions, one has to perform a Clebsch-Gordan decomposition, in which the trivial representation may occur. Edges with trivial representations, however, are not seen by the action and hence drop out, i.e., the graph underlying the spin network function is changed. To avoid this, the projection operators $P_\gamma : \mathcal{H} \mapsto \mathcal{H}'_\gamma$ are introduced, which project onto the Hilbert space $\mathcal{H}'_\gamma$ of which $\mathcal{H}$ is a direct sum, i.e., $\mathcal{H} = \bigoplus_\gamma \mathcal{H}'_\gamma$. The exact definition of $\mathcal{H}'_\gamma$ can be found for example in [28] and [29]. This way, (3.5.5) acts graph-preserving. Note that for (3.5.5), no limiting procedure is involved, and hence, (3.5.5) corresponds to the so-called effective operator viewpoint discussed in [27]: Here, one drops the regularization parameter ϵ and the continuum limit of infinitely fine triangulations corresponds to considering finer graphs. This way, the classical limit of the operator will just be an approximation of the classical quantity. However, one could see it the other way round, i.e., the classical quantity is only an approximation of the quantum object, which defines a more fundamental theory.

This finishes our discussion of the Hamiltonian constraint operator and we will continue with the framework of algebraic quantum gravity in the next section.

3.6 Algebraic quantum gravity

One of the main open problems of loop quantum gravity is the question if one can obtain general relativity as the semiclassical limit of loop quantum gravity. This is also the main motivation for introducing algebraic quantum gravity (AQG), which was developed first in the series of papers [49–52]. In the following paragraphs, we present the main arguments to motivate AQG, while we mainly follow [49].

The first point is that the infinitessimal generators of diffeomorphisms cannot be implemented in the Ashtekar Lewandowski representation. To see this, we go back to section 3.3.3, where we discussed the LOST theorem which ensures the existence of a unique diffeomorphism invariant state. Now, using the GNS construction, such a state provides us with a unitary action of spatial spatial diffeomorphisms φ as operators $\hat{U}(\varphi)$, see, e.g., section 29.1 of [27]. Explicitly, the action of $\hat{U}(\varphi)$ on spin network function is given by [29]

$$\hat{U}(\varphi)T_{\gamma,\vec{j},\vec{I}} = T_{\varphi(\gamma),\vec{j},\vec{I}}, \tag{3.6.1}$$

i.e., only the graph γ underlying the spin network function is mapped to $\varphi(\gamma)$, while the labels $\vec{j}$ and $\vec{I}$ are left invariant. This action, however, is not weakly continous, which in general means the following for a one-parameter family of unitary operators $\hat{U}(\varphi_t)$:

$$lim_{t\to 0} < T_{\gamma,\vec{j},\vec{I}}, \hat{U}(\varphi_t)T_{\gamma',\vec{j}',\vec{I}'} >=< T_{\gamma,\vec{j},\vec{I}}, T_{\gamma',\vec{j}',\vec{I}'} >, \ \forall T_{\gamma,\vec{j},\vec{I}}, T_{\gamma',\vec{j}',\vec{I}'}. \tag{3.6.2}$$

Since (3.6.2) has to be true for every two spin network functions, we can evaluate it for $T_{\gamma,\vec{j},\vec{I}} = T_{\gamma',\vec{j}',\vec{I}'}$. Then, the right hand side of (3.6.2) will give one, while for the left hand side, we use (3.6.1) to get $< T_s, \hat{U}(\varphi_t)T_s =< T_s, T_{\varphi_t(s)} >$, which vanishes. To see the latter, we take φ_t as the one-parameter family of diffeomorphisms generated by some vector field which is supported on γ. Then, there exists some range for t in which $\varphi_t(\gamma)$ is different from γ, yielding us that the corresponding spin network functions are defined over different graphs and hence orthogonal. Thus, we arrive at $< T_s, T_{\varphi_t(s)} >= 0$ which finishes our argument. In total, we get from (3.6.2), evaluated for the same spin network functions, that $0 = 1$ which is a contradiction and hence, weak continuity is violated. Now, we can apply the so-called Stone theorem which tells us that there exist no infinitessimal generators of diffeomorphisms in the quantum theory. Another way to see this is via the regularization of the diffeomorphism constraint $\vec{C}(\vec{N}) = \int_\sigma \mathrm{d}^3x N^a F^j_{ab} E^b_j(x)$: Approximating the integral as a Riemann sum gives a factor of ϵ^3. For the curvature, we can use (3.5.3) which basically gives us $F \sim h_\alpha/\epsilon^2$ for a loop α. For E^b_j, we introduce the fluxes via $\sim E(S)/\epsilon^2$. In total, we obtain the regularized diffeomorphism constraint $D \sim \epsilon^3 N^a h_\alpha E(S)/\epsilon^4$ which is of the order $1/\epsilon$. Hence, the limit $\epsilon \to 0$ is ill-defined and thus, also the regularization of the diffeomorphism constraint. Therefore, the infinitessimal diffeomorphisms cannot be implemented as operators.

The non-existence of infinitessimal diffeomorphisms in the quantum theory now causes a problem concerning the anomaly-free representation of the Dirac algebra, which contains these operators. For the relations (2.1.25) and (2.1.26) we can define the finite quantum analogues as [27]

$$U(\varphi)U(\varphi')U(\varphi)^{-1} = U(\varphi \circ \varphi' \circ \varphi), \tag{3.6.3}$$

$$U(\varphi)\hat{C}(N)U(\varphi)^{-1} = \hat{C}(N \circ \varphi), \tag{3.6.4}$$

and hence, no problem occurs here. For the relation (2.1.27), however, we have the diffeomorphism constraint involved on the right hand side together with structure functions. Thus, we cannot simply exponentiate this relation as we do not have a true Lie algebra. A way out of this is to require that the commutator of two Hamiltonian constraint annihilates diffeomorphism invariant states, which is exactly what the operator corresponding to the diffeomorphism constraint should do. This was shown to be valid for the operator [32] which, however, is a graph-changing operator, and the property of being graph-changing enters crucially into the result for the commutator.

From the previous paragraph, we see that the anomaly-free representation of the Dirac algebra requires a graph-changing Hamiltonian constraint which now poses a problem concerning the semiclassical analysis: The construction of so-called kinematical coherent states is well-known [107–109]. In particular, the goal is to find coherent states $\Psi_{\gamma,m}$ defined on a given graph γ and peaked at classical phase space point $(A_0, E_0) := m$ such that

$$< \Psi_{\gamma,m}, \hat{H}(N)\Psi_{\gamma,m} >= H(N)^{(m)} + O(\hbar). \tag{3.6.5}$$

Furthermore, the quantum fluctuations should be small. The problem is now that if we use coherent states $\Psi_{\gamma,m}$ defined for one particular graph, so-called cutoff states, the expectation value (3.6.5) will vanish for a graph-changing Hamiltonian constraint: Using that $\Psi_{\gamma,m}$ is a linear combination of spin network functions, the action of $\hat{H}(N)$ will change the underlying graph in the vicinity of its vertices. Hence, $\hat{H}(N)\Psi_{\gamma,m}$ results in a linear combination of spin network functions defined over a graph which is different from γ. As spin network functions defined for different graphs are orthogonal, (3.6.5) vanishes. We may use other coherent states such as the distributional states $\Phi := \sum_\gamma \Psi_{\gamma,m}$, the so-called shadow states, which are basically a sum of cutoff states. However, it turns out that these states are also insufficient, see the discussion on page 380 of [27].

A way out of the previously discussed problems is the so-called *Master constraint programme*, proposed in [106] and tested in various models in [110–114]. The strategy is to replace the infinite number of Hamiltonian and diffeomorphism constraints by one single Master constraint. One can also include the Gauß constraints within the so-called *extended Master constraint programme*, where the Master constraint is defined as

$$M = \int_\sigma d^3x \frac{H^2(x) + H^a H^b q_{ab} + \delta^{ij} G_i G_j}{\sqrt{det(q)}}. \tag{3.6.6}$$

Annihilating the Master constraint (3.6.6) is totally equivalent to annihilating the Gauss, diffeomorphism, and Hamiltonian constraints, respectively. Furthermore, (3.6.6) is also sensitive to Dirac observables $\mathcal{O}$: If we would simply compute $\{\mathcal{O}, \mathrm{M}\}$, this would always vanish weakly, i.e., on the constraint surface, as one can see by direct computation while using the Leibniz rule at various points. This is solved by inspecting $\{\mathcal{O}, \{\mathcal{O}, \mathrm{M}\} \approx 0$ instead, which turns out to be equivalent to $\{\mathcal{O}, G_i\} \approx 0$, $\{\mathcal{O}, C_a\} \approx 0$, and $\{\mathcal{O}, H\} \approx 0$. Now, a big advantage of the Master constraint is that the Dirac algebra trivializes to $\{M, M\} = 0$, which is easily represented in the quantum theory as $[\widehat{M}, \widehat{M}] = 0$. This solves the problem of the anomaly-free representation of the Dirac algebra. However, we need to quantize M, for which we may proceed in two ways [40]: On the one hand, we may quantize M in graph-changing

fashion. As M is spatially diffeomorphism invariant, we must quantize it on the spatially diffeomorphism invariant Hilbert space, cf. [81]. However, we again would encounter a problem in the semiclassical analysis, namely that only kinematical coherent states are well-known. On the other hand, we may quantize M graph non-changing which then becomes an operator on the kinematical Hilbert space. This now allows to do semiclassical analysis as we can use the well-known kinematical coherent states. Hence, the graph non-changing quantization seems to be preferred. However, we still have the problem that the known semiclassical states do not approximate all degrees of freedom of LQG which are encoded in all possible graphs we can embed into σ. Specifically, constructing coherent states by summing over all possible graphs gives an uncountably infinite series which does not result in a normalizable state.

From the previous discussion, we can conclude that the semiclassical tools developed so far are graph-dependent which motivates the introduction of AQG: Within this framework, the graph-dependence is removed by replacing the uncountably infinite amount of finite and embedded graphs γ with a single infinite abstract graph α. The abstract graph is defined by a set of abstract points, which we call vertices in LQG, and has information about how many abstract arrows, which we call edges in LQG, between the abstract points exist. See, e.g., [115] for more details on algebraic graphs. This way, we loose geometrical information such as the topology of σ and the braiding of the edges. However, the algebraic graph can be embedded in all possible ways into σ and only with the embedding we choose the how edges are related. In particular, the embedding of the algebraic graph will only be done within the semiclassical sector. Furthermore, as we can embed α arbitrarily dense, one may say that the continuum limit is built in. Moreover, all diffeomorphism invariant and therefore embedding independent operators can be lifted to AQG. The Master constraint operator is now implemented graph-nonchanging, hence implying that the existing semiclassical tools can be used. To perform the classical limit, information about the spatial manifold together with some initial data thereon as well as an embedding is needed from which one can construct a coherent state using the results of [107–109]. This way, it was shown in [50, 51] that for a cubic and six-valent algebraic graph the semiclassical limit of the Master constraint is correct up to first order and also the first order corrections were shown to be finite.

Now, we want to specify the kinematical structure of AQG which is motivated by LQG. We again follow [49]:

- For the quantum *-algebra, we associate to each edge e of a given abstract graph α an element $A(e)$ of the compact, connected, and semi-simple Lie group G, as well as an element $E(e)$ of the Lie algebra $L(G)$. Furthermore, $A(e)$ and $E(e)$ satisfy the following algebraic relations motivated by the LQG algebra presented in section 3.2.2:

$$[A(e), A(e')] = 0, \tag{3.6.7}$$
$$[E_j(e), A(e')] = i\hbar Q^2 \delta_{e,e'} \tau_j / 2A(e), \tag{3.6.8}$$
$$[E_j(e), E_k(e')] = -i\hbar Q^2 \delta_{e,e'} f_{jkl} E_l(e'). \tag{3.6.9}$$

  Here, Q denotes the coupling constant and we defined $E_j(e) := -2Tr(\tau_j E(e))$, where τ_j are the anti-Hermitian and trace-free generators[8] of the Lie group G normalized such that $Tr(\tau_j \tau_l) = -1/2\delta_{jk}$. Furthermore, an involutive structure is defined by $A(e)^* = [A(e)^{-1}]^T$ and $E_j(e)^* = E_j(e)$.

[8] This is possible as any compact and semi-simple Lie group can be thought of as a subgroup of $SU(2)$.

- For the kinematical Hilbert space, we consider the infinite tensor product (ITP) Hilbert space

$$\mathcal{H}^{\otimes} := \otimes_{e\in\alpha}\mathcal{H}_e, \tag{3.6.10}$$

with $\mathcal{H}_e \simeq L_2(G, d\mu_H)$, where $d\mu_H$ denotes the Haar measure on G. Vectors in $\mathcal{H}^{\otimes}$ are denoted as $\otimes_f := \otimes_e f_e$, with $f_e \in \mathcal{H}_e$. Then, $\mathcal{H}^{\otimes}$ is the closure of the finite linear span of vectors $\otimes_f$. We also can define an inner product by

$$< \otimes_f, \otimes_{f'} >:= \prod_e < f_e, f'_e >_{\mathcal{H}_e}, \tag{3.6.11}$$

which induces a norm. Specifically, we restrict to vectors $\otimes_f$ having finite but nonzero norm. See also [107, 116] for more details on infinite tensor product Hilbert spaces. Choosing the ITP Hilbert space as the representation space is naturally in the sense that we can apply the uniqueness results established in LQG.

- The action of our basic operators $A(e)$ and $E(e)$ on elements of the ITP Hilbert space $\mathcal{H}^{\otimes}$ is given by [49]

$$A(e)\otimes_f := [A(e)f_e] \otimes [\otimes_{e\neq e'} f_{e'}], \tag{3.6.12}$$
$$E(e)\otimes_f := [E(e)f_e] \otimes [\otimes_{e\neq e'} f_{e'}], \tag{3.6.13}$$

with

$$[A(e)f_e](h) := hf_e(h), \quad [E_j(e)f_e](h) := i\hbar Q^2[\frac{d}{dt}]_{t=0} f_e(e^{t\tau_j/2}h), \tag{3.6.14}$$

i.e., $A(e)$ acts as a multiplication operator, which is unitary and bounded, and $E(e)$ as a derivative operator, which is essentially self-adjoint.

We close this section by specifying the most important ingredient for the quantum dynamics, namely the volume operator lifted to AQG, which is given by [49]:

$$\hat{V}_v := l_P^3 \sqrt{\left| \frac{1}{48} \sum_{e_1\cap e_2\cap e_3=v} \epsilon_v(e_1, e_2, e_3)\epsilon^{ijk} E_i(e_1)E_j(e_2)E_k(e_3) \right|}. \tag{3.6.15}$$

Here, e_1, e_2, e_3 denote three different edges meeting at the vertex $v \in \alpha$. The factor $\epsilon_v(e_1, e_2, e_3)$ is totally skew and adapted to the algebraic graph α such that it coincides with the LQG definition in (3.4.7) when choosing an embedding for α.

4 Reduced phase space quantization

In this section, we discuss the necessary ingredients to perform a reduced phase space quantization of general relativity, i.e., to solve the Hamiltonian and diffeomorphism constraint at the classical level and quantize the thus obtained classical theory. For this purpose, we use the so called relational formalism, originally introduced by Rovelli in [41–44] and refined by Dittrich in [45, 46]. This formalism involves the introduction of additional matter degrees of freedom which serve as reference fields, i.e., they provide rulers and clocks with respect to which we describe the time evolution and measure the spatial coordinates. Hence, the problem of time we discussed in section 2 is resolved at the cost of additional matter degrees of freedom which need to be interpreted. Specifically, we will be concerned with Gaussian dust as reference field [47, 48] which provides us a physical Hamiltonian that is equivalent to the Hamiltonian constraint. Furthermore, the observable algebra that we have to quantize is isomorphic to the original Poisson algebra.

4.1 General formalism

First, we discuss the general formalism for the introduction of reference matter and the construction of observables. We follow closely [52] and refer to [117] and [27, 118, 119] for more details.

The starting point is a system of first class constraints C_I, with I taking values in some index set $\mathcal{I}$. Our goal is to construct gauge invariant quantities or rather observables O_f from arbitrary phase space functions f. Then, we want the observables to be weak or, if possible, strong Dirac observables, i.e., $\{O_f, C_I\}$ vanishes weakly or strongly. To this end, we introduce certain reference fields T^I, called clocks, which are basically functions on phase space and parametrize the gauge orbits generated by the first class constraints C_I. We need one clock for each constraint and additionally, we chose the fields T^I such that the matrix $M_I^J := \{C_I, T^K\}$ is invertible. Furthermore, we do not work with C_I directly but consider the equivalent constraints $C'_I := \sum_J [M^{-1}]_I^J C_J$, where the invertibility condition we imposed on M_I^J comes into play. Using the new constraints C'_I, we arrive at $\{C'_I, T_J\} \approx \delta_I^J$, i.e., the clocks and the new constraints form a canonical pair. We can now define the Hamiltonian vector fields corresponding to C'_I by $\chi_I := \{C'_I, .\}$. One can show that the χ_I weakly commute such that later on, when we construct observables, the order in which the χ_I act is not relevant, motivating the introduction of the constraints C'_I. In particular, we consider the linear combination $\chi_\beta := \sum_I \beta^I \chi_I$. Then, we can state the gauge flow generated by these Hamiltonian vector fields χ_β as the following infinite series:

$$\alpha_\beta(f) := \exp(\chi_\beta) \cdot f = \sum_{n=0}^{\infty} \frac{1}{n!} \chi_\beta^n \cdot f. \tag{4.1.1}$$

Using this, we can construct observables O_f from a phase space function f by computing the gauge flow $\alpha_\beta(f)$ and evaluating it for that specific gauge where the clocks T^I assume the values τ^I with respect to the gauge flow, i.e., for $\alpha_\beta(T_I) = \tau_I$ which we shortly denote by $\alpha_\beta(T) = \tau$. In summary, this means that we construct observables via computing

$$O_f(\tau) := [\alpha_\beta(f)]_{\alpha_\beta(T)=\tau}. \tag{4.1.2}$$

Evaluating $\alpha_\beta(T^I)$ and using $\{C'_I, T_J\} \approx \delta_I^J$, one can show that on the constraint surface, we have $\alpha_\beta(T^I) \approx T^I + \beta^I$, which is easy to solve due to the convenient definition of the

constraints C'_I. Moreover, we obtain that (4.1.2) is weakly equivalent to

$$O_f(\tau) := [\alpha_\beta(f)]_{\beta=T-\tau}. \tag{4.1.3}$$

Hence, we see that $O_f(\tau)$ determines the value of f in the gauge $\beta = T - \tau$. Note that in (4.1.3), we first have to compute the action of the gauge flow and then insert $\beta = T - \tau$, as β becomes phase space dependent by the replacement $\beta = T - \tau$ and hence, first inserting and then computing the gauge flow would yield a different result.

In the following, we summarize the key properties of the observables (4.1.3) and the associated multi-parameter family of observable maps $O^\tau : f \to O_f(\tau)$:

- The functions (4.1.3) are weak Dirac observables, i.e. $\{C_I, O_f(\tau)\} \approx 0$.
- $O^\tau : f \to O_f(\tau)$ is a homeomorphism between the commutative algebras of phase space functions and weak Dirac observables, i.e.

$$O_f(\tau) + O_{f'}(\tau) \approx O_{f+f'}(\tau), \quad O_f(\tau)O_{f'}(\tau \approx O_{ff'}(\tau). \tag{4.1.4}$$

- $O^\tau : f \to O_f(\tau)$ is also a Poisson homomorphism in the sense that

$$\{O_f(\tau), O_{f'}(\tau)\} \approx \{O_f(\tau), O_{f'}(\tau)\}^* \approx O_{\{f,f'\}^*}(\tau), \tag{4.1.5}$$

  where $\{.,.\}^*$ denotes the Dirac bracket defined in the following way: In general, one would have a set of gauge fixing conditions G_I forming a second class system with the constraints C_I. Then, we can define the corresponding Dirac bracket by [27]

$$\{f, f'\}^* = \{f, f'\} - \{f, C_\mu\}K^{\mu\nu}\{C_\nu, f'\}, \tag{4.1.6}$$

  where $\mu = (j, I)$, with $j = 1, 2$ and $C_{1I} := C_I$, $C_{2I} := G_I$. Furthermore, $K^{\mu\nu}$ is the inverse matrix of $K_{\mu\nu} := \{C_\mu, C_\nu\}$, i.e., we have $K^{\mu\rho}K_{\rho\nu} = \delta^\mu_\nu$. In our case, we have $\beta_I = G_I = T_I - \tau_I$ and arrive at [52]

$$\{f, f'\}^* = \{f, f'\} - \{f, C_I\}[M^{-1}]^I_J\{T^J, f'\} + \{f', C_I\}[M^{-1}]^I_J\{T^J, f\}. \tag{4.1.7}$$

- Furthermore, for the phase space coordinatized by canonical pairs (q_a, p^a), describing the gravitational part, and (T^I, P_I), describing the reference fields, any function $f(T^I, P_I, q_a, p^a)$ admitting a Taylor expansion has an associated observable defined by

$$O_f(\tau) \approx f(O_{T^I}(\tau), O_{P_I}(\tau), O_{q_a}(\tau), O_{p^a}(\tau)). \tag{4.1.8}$$

  We can show this using (4.1.4) and the explicit form of the Taylor expansion, which in case of a two dimensional phase space is given by $f = \sum_{m,n=0}^\infty f_{mn}q^m p^n$, with suitable coefficients f_{mn}.

There is an important consequence of the Poisson homomorphism property (4.1.5): If we denote the phase space variables describing the geometrical part by (q^a, p_b), we obtain

$$\{O_{q^a}(\tau), O_{p_b}(\tau)\} \approx O_{\{q^a,p_b\}^*}(\tau) = O_{\{q^a,p_b\}}(\tau) = O_{\delta^a_b} = \delta^a_b. \tag{4.1.9}$$

Here, we used (4.1.5) and the fact that the Dirac bracket coincides with the Poisson bracket for phase space functions depending only on (q^a, p_a), since these functions commute with the clocks T^I. From (4.1.9) we observe that on the observable level, (q^a, p_b) are still, at least weakly, canonically conjugate. Thus, using the relational formalism, we can avoid the quantization of a complicated observable algebra, a problem we mentioned at the beginning of section 3.

4.2 Observables and physical Hamiltonian of Gaussian dust

We now want to discuss a specific choice of reference fields T^I, namely Gaussian dust [48]. This will allow us to rewrite the constraints in the form $C_I = P_I + h_I(T, q^a, p_a)$, where the functions h_I depend on the reference field T that we will introduce to describe the time evolution. We will then reduce with respect to the new constraints, for which h_I will be independent of T at least at the gauge-invariant level. This way, we obtain a physical Hamiltonian that is equivalent to the geometrical Hamiltonian constraint.

First, we want to introduce the concept of Gaussian dust. The Lagrangian describing it is given by [47, 48]

$$\mathcal{L}_{GD} = -|det(g)|^{1/2} \left\{ \frac{\rho}{2} \left[g^{\mu\nu} T_{,\mu} T_{,\nu} + 1 \right] + g^{\mu\nu} T_{,\mu} \left[W_j S^j_{,\nu} \right] \right\}, \tag{4.2.1}$$

which contains in total 8 fields T, ρ, W_j, S^j. We will choose the fields T as a clock providing the time coordinate and the fields S^j as rulers providing spatial coordinates. Concerning the interpretation of Gaussian dust, details can be found in [48]. In particular, one can define the vector fields $U^\mu := g^{\mu\nu} T_{,\nu}$ and $V^\mu_j := g^{\mu\nu} S^j_{,\nu}$ for which we can rewrite (4.2.1) and derive the Euler Lagrange equations that give us the following: We obtain $U^\mu U_\mu = -1$, i.e., U defines a congruence of affinely parametrized geodesics. Furthermore, we get $U_\mu V^\mu_j = 0$, i.e., V^μ_j is orthogonal to the geodesics defined by U. We can choose U to be the foliation vector field such that T plays the role of time and labels the spatial hypersurfaces, while S^j, being constant along the geodesics, labels the geodesics and thus plays the role of spatial coordinates. Moreover, the pull back of the metric on the so called dust manifold $\mathcal{S}$, coordinatized by $\sigma^j := S^j(x)$, acquires Gaussian form, i.e., $g_{00} = -1$, $g_{ta} = 0$, with the lapse function being one and the shift vector field vanishing. Furthermore, the energy momentum tensor of Gaussian dust can be calculated via $T_{\mu\nu} = -2/\sqrt{det(g)} \cdot \partial \mathcal{L}_{GD} / \partial g^{\mu\nu}$ which turns out to be the one of a perfect, pressure-less fluid justifying the denomination dust.

Now, we want to derive the new form of the constraints. For this purpose, we add (4.2.1) to the geometrical Lagrangian $\mathcal{L}_{geo} = \frac{1}{\kappa} \sqrt{det(g)} R^{(4)}$. Then, we perform the canonical analysis for $\mathcal{L} = \mathcal{L}_{geo} + \mathcal{L}_{GD}$, i.e., we compute the canonical conjugate momenta giving us the canonical pairs $(T, P), (\rho, \pi_\rho), (W_j, \pi^j), (S^j, P_j)$ and proceed with the steps of the Dirac algorithm, which we introduced in section 2. The detailed analysis is presented in appendix A of [47] and we will only state the result: We obtain the following diffeomorphism and Hamiltonian constraints:

$$c^{tot}_a = P T_{,a} + P_j S^j_{,a} + c_a, \tag{4.2.2}$$

$$c^{tot} = P + c\sqrt{1 + q^{ab} T_{,a} T_{,b}} - q^{ab} T_{,a} c_b. \tag{4.2.3}$$

These constraints contain the dust degrees of freedom as well as the geometrical Hamiltonian and diffeomorphism constraints, which we denoted as c_a and c, respectively. The constraints (4.2.2),(4.2.3) fall into the following theoretical framework: In general, we can split the canonical pairs into two sets (T^I, P_I) and (q^a, p_a), where the former correspond to the clock fields and their conjugate momenta, while the latter correspond to the geometrical phase space variables. Then, we can rewrite the constraints at least locally as

$$C_I = P_I + h_I(T^I, q^a, p_a), \tag{4.2.4}$$

such that one can solve them for P_I. If the functions do not depend on T^I, we would obtain a so called deparametrised theory which provides a further simplification and is discussed

for example in [52] and [117]. For Gaussian dust, however, this is not the case as, e.g., the Hamiltonian constraint (4.2.3) can be written as

$$c^{tot} = P + h(T,q,p), \quad h(T,q,p) := c\sqrt{1+q^{ab}T_{,a}T_{,b}} - q^{ab}T_{,a}c_b, \tag{4.2.5}$$

with $h(T,q,p)$ depending on T. Nevertheless, the constraints (4.2.2) and (4.2.3) strongly Poisson commute. To see this, we inspect (4.2.4) and use an abstract argument as discussed in [52]: The constraints (4.2.4) still form a first class system and hence, they result in a linear combination of first class constraints $b^K_{IJ}C_K$ with b^K_{IJ} some coefficients. Now, we observe that the constraints are linear in P_I and hence, when calculating $\{C_I, C_J\}$, the dependence on P_I drops out. To reach equality with $b^K_{IJ}C_K$, which depends on the P_I, we have to set the coefficients to zero, which finishes the proof. Furthermore, we do not have to introduce new constraints $C'_I := \sum_J [M^{-1}]^J_I C_J$, as done in the previous section, since due to (4.2.4) we have that $M^J_I := \{C_I, T^J\} = \delta^I_J$ and hence $C'_I = C_I$. Following [47], we can now reduce with respect to the constraints (4.2.4) which we will do in the following paragraphs.

To solve (4.2.2) and (4.2.3), i.e., to construct invariants with respect to these constraints, we introduce the Hamiltonian vector fields

$$\chi_\beta = \int_\sigma \mathrm{d}^3x \beta^\mu(x) c^{tot}_\mu(x), \tag{4.2.6}$$

where $\mu = 0, ..., 3$ and $\beta^0 := \tau(x) - T(x)$, $\beta^j = \sigma^j - S^j(x)$, and $c^{tot}_0 := c^{tot}$. As (4.2.2) and (4.2.3) Poisson commute, the Hamiltonian vector fields χ_β Poisson commute as well. Furthermore, $S^j(x)$ commutes with c^{tot} and hence, we can perform the reduction in two steps [117]:

$$O_f(\tau) = [\alpha_{\beta^0}([\alpha_{\vec{\beta}}(f)]_{\vec{\beta}=\vec{\sigma}-\vec{S}})]_{\beta^0=\tau-T} := O^{(2)}_{O^{(1)}_f(\sigma)}(\tau), \tag{4.2.7}$$

i.e., we first reduce with respect to the diffeomorphism constraints to obtain the diffeomorphism invariant quantity $O^{(1)}_f(\sigma)$ and afterwards, we reduce with respect to the Hamiltonian constraint, giving us in total an observable depending on physical time τ and the spatial coordinates $\sigma^j := S^j(x)$ coordinatizing the so called dust manifold $\mathcal{S}$.

Now, we will explicitly perform the two steps contained in (4.2.7):

- Firstly, we reduce with respect to the diffeomorphism constraint. One can show that for a function $f(x)$, which is a scalar on σ, we can perform this reduction by [117]

$$O^{(1)}_f(\sigma) = f(x)_{\vec{S}(x)=\sigma} = f(x_\sigma) = \int_{\mathcal{X}} \mathrm{d}^3x |det(\partial S/\partial x)| \delta(S^k(x), \sigma^k) f(x). \tag{4.2.8}$$

  Here, we denoted the spatial manifold as $\mathcal{X}$ instead of σ, as we did in section 2, to avoid confusion with the dust coordinates σ^j. (4.2.8) means that we construct diffeomorphism invariant scalar functions by evaluating the diffeomorphism variant quantity at that point x_σ, at which $S^j(x)$ has the constant value σ^j [9]. To perform the reduction for more general variables such as the spatial metric q_{ab}, we construct quantities that are scalars on the spatial manifold $\mathcal{X}$. Explicitly, we use the map $S^j : \mathcal{X} \to \mathcal{S}$ or rather its inverse, whose existence we ensure by the condition $det(S) \neq 0$. This way, we end up with the

[9] Note that σ^j =const. and hence, $S^j(x) = \sigma^j(x)$ is solved for x for arbitrary S^j.

duality relations $S^a_j S^j_b = \delta^a_b$ and $S^a_j S^k_a = \delta^j_k$. We can now construct a scalar on $\mathcal{X}$ for instance for the spatial metric by

$$q_{ij} = S^a_i S^b_j q_{ab}. \tag{4.2.9}$$

Evaluating (4.2.9) at x_σ corresponds to the pull back of q_{ab} via the diffeomorphism $S^{-1} : \mathcal{X} \to \mathcal{S}, x \to \sigma$. In particular, the analog to (4.2.8) for the spatial metric is given by

$$\widetilde{q}_{ij}(\sigma) := O^{(1)}_{q_{ij}}(\sigma) = \int_\sigma \mathrm{d}^3x |det(\partial S/\partial x)| \delta(S(x),\sigma) S^a_i S^b_j q_{ab}(x,t). \tag{4.2.10}$$

Similarly, we can construct observables from the scalars P/J, T, and $p^{ij} := p^{ab} S^j_a S^k_b / J$, with $J := det(\partial S/\partial x)$ introduced for the correct density weight. Then, we obtain the partially reduced phase space coordinatized by $(\widetilde{q}_{ij}(\sigma), \widetilde{p}^{ij}(\sigma))$ and $(\widetilde{T}(\sigma), \widetilde{P}(\sigma))$, where the latter pair just equals $(T, P/J)$ evaluated at the point x_σ. For constructing the observables of the Hamiltonian constraint (4.2.3), we can use (4.1.8) to obtain

$$\widetilde{c}^{tot}(\sigma) := O^{(1)}_{c^{tot}}(\sigma) = c^{tot}(O^{(1)}_q, O^{(1)}_p, O^{(1)}_T O^{(1)}_P) = c^{tot}(\widetilde{q}, \widetilde{p}, \widetilde{T}, \widetilde{P}) = \widetilde{P}(\sigma) + \widetilde{h}(\widetilde{T}, \widetilde{q}, \widetilde{p}). \tag{4.2.11}$$

Similarly, we can proceed for the diffeomorphism constraint (4.2.2) to obtain the quantity $\widetilde{c}^{tot}_j(\sigma)$.

- Secondly, we reduce with respect to the Hamiltonian constraint. For a general, diffeomorphism invariant function g, we have to compute

$$O^{(2)}_g(\sigma,\tau) = [\alpha_{\beta^0} \cdot g]_{\beta^0 = \tau - T} = \left[\exp\left(\left\{ \int d^3x \beta^0(x) c_{tot}(x), . \right\} \right) \cdot g \right]_{\beta^0 = \tau - T}. \tag{4.2.12}$$

We now restrict to diffeomorphism invariant functions g that are independent of the dust variables, such that we can evaluate the above expression at $\beta^0(x) = \tau(x) - T(x)$ before computing the Poisson bracket, while we can also replace $c_{tot} = P + h$ by h. Furthermore, choosing $\tau(x) = \tau = \text{const.}$ allows us to rewrite (4.2.12) in terms of the diffeomorphism invariant phase space variables. In total, we can rewrite (4.2.12) as [52]

$$O^{(2)}_g(\sigma,\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \{\widetilde{h}(\tau), g\}_{(n)}, \quad \widetilde{h}(\tau) := \int_{\mathcal{S}} d^3\sigma (\tau - \widetilde{T}(\sigma)) \widetilde{h}(\widetilde{T}, \widetilde{q}, \widetilde{p}). \tag{4.2.13}$$

One can show that this expression strongly Poisson commutes with all constraints. Furthermore, the evolution with respect to τ is given by

$$\frac{d}{d\tau} O^{(2)}_g(\sigma,\tau) = \{O^{(2)}_{\widetilde{h}(1)}(\tau), O^{(2)}_g(\sigma,\tau)\}, \tag{4.2.14}$$

where $\widetilde{h}(1) := \int d^3\sigma \widetilde{h}(\widetilde{T}, \widetilde{q}, \widetilde{p})$. We see that $O^{(2)}_{\widetilde{h}(1)}$ serves as our physical Hamiltonian and using (4.1.8) we can rewrite it as

$$H := O^{(2)}_{\widetilde{h}(1)} = \int d^3\sigma \widetilde{h}(O^{(2)}_{\widetilde{T}}, O^{(2)}_{\widetilde{q}}, O^{(2)}_{\widetilde{p}}). \tag{4.2.15}$$

The dependence on $O^{(2)}_{\widetilde{T}}$ drops out as we will show in the following: If we look at (4.2.3), we see that in the above formula, the dependence on T is only via spatial derivatives of

T, i.e., we only have to compute the observables $O^{(2)}_{\widetilde{T},\sigma_j}(\tau)$. These, however vanish due to

$$O^{(2)}_{\widetilde{T}}(\sigma,\tau) = \tau = \text{const.} \tag{4.2.16}$$

which implies $O^{(2)}_{\widetilde{T},\sigma_j}(\tau) = \partial_{\sigma_j}\tau = 0$. Hence, all terms in (4.2.3) containing spatial derivatives of T simplify or vanish, giving us in the end that

$$\widetilde{h}(O^{(2)}_{\widetilde{T}}, O^{(2)}_{\widetilde{q}}, O^{(2)}_{\widetilde{p}}) = c(O^{(2)}_{\widetilde{q}}, O^{(2)}_{\widetilde{p}}) =: H(\sigma). \tag{4.2.17}$$

Inserting this into (4.2.15) gives us the final form of the physical Hamiltonian:

$$\mathrm{H}_{\mathrm{phys}} := \int_{\mathcal{S}} d^3\sigma\ H(\sigma). \tag{4.2.18}$$

Most importantly, we can conclude from (4.2.17) and (4.2.18) that the physical Hamiltonian is equivalent to the integral over the dust manifold of the geometrical part of the Hamiltonian constraints, which we denoted as c. In particular, $\mathrm{H}_{\mathrm{phys}}$ corresponds to the smeared quantity $c(N=1)$, i.e., the lapse function is one in our formalism. Furthermore, $\mathrm{H}_{\mathrm{phys}}$ generates the time evolution of observables $O_f(\sigma,\tau)$, which are invariant with respect to (4.2.2) and (4.2.3), via

$$\frac{d}{d\tau}O_f(\sigma,\tau) = \{\mathrm{H}_{\mathrm{phys}}, O_f(\sigma,t)\}. \tag{4.2.19}$$

Evaluating this for $O_f(\sigma,\tau) = \mathrm{H}_{\mathrm{phys}}$ shows that the physical Hamiltonian does not explicitly depend on time. Another crucial point concerns the observable algebra of our phase space variables: At the observable level, the canonical pair (T,P) drops out as T is pure gauge and P can be expressed via the constraints (4.2.2) and (4.2.3). Hence, the reduced phase space is coordinatized by

$$Q_{ij}(\sigma,\tau) := O^{(2)}_{\widetilde{q}_{ij}}(\sigma,\tau), \quad P^{ij}(\sigma,\tau) := O^{(2)}_{\widetilde{p}^{ij}}(\sigma,\tau), \tag{4.2.20}$$

which are still canonically conjugate variables, as we can see using (4.1.9) which now holds strongly:

$$\{Q_{ij}(\sigma), P^{kl}(\sigma')\} = O_{\{q_{ab}(x),P^{cd}(y)\}} = O_{\delta^c_{(a}\delta^d_{b)}\delta(x,y)} = \delta^k_{(i}\delta^l_{j)}\delta(\sigma,\sigma'). \tag{4.2.21}$$

Hence, we obtain a Poisson algebra that is isomorphic to the original algebra and thus, the kinematical results of LQG can be carried over to the physical level.

Note that the above discussion was done for ADM variables but analogous steps can be performed for Ashtekar variables, see, e.g., [52].

5 The polarized three-torus Gowdy model

In this thesis, we use Gowdy spacetimes as a toy model, i.e., as a symmetry reduced and hence simplified version of the full theory of general relativity on which we apply the quantization techniques of LQG and AQG. Furthermore, we use the relational formalism introduced in section 4 with Gaussian dust as a reference fluid. This way, we solve the diffeomorphism and the Hamiltonian constraints at the classical level and remain with a physical Hamiltonian density that is equivalent to the geometric contribution of the Hamiltonian constraint.

We first want to define Gowdy spacetimes. They were first studied extensively by Robert Gowdy in [54]. These models comprise globally hyperbolic spacetimes with closed spatial hypersurfaces that solve the vacuum Einstein equations and possess a two parameter isometry group, the torus group $T^2 \simeq U(1) \times U(1)$, i.e., they have two Killing vectors. In particular, Gowdy proved that the two Killing vectors commute and all possible spatial topologies are homeomorphic to a three sphere S^3, a three-handle $S^1 \times S^2$, or a three-torus T^3. It is the latter topology that is technically the most simple and we will concentrate on in this thesis. The Gowdy models we specified so far are called unpolarized Gowdy models but there exists a simpler class of models, the so-called polarized Gowdy models, which we obtain by imposing the two Killing vectors to be orthogonal. The classical solutions of the polarized three-torus Gowdy model are well-known, see for example [120] and [56]. Note that some of these solutions exhibit an initial singularity while others do not. We will work with the polarized Gowdy model as it provides a further simplification of our theory. This was also done in previous work on the loop quantization of the three-torus Gowdy model on which this thesis is based, see [56, 57] and [58, 59].

We will derive the polarized three-torus Gowdy model in section 5.1 by performing a symmetry reduction of the full theory formulated in terms of Ashtekar variables (A_a^j, E_j^b), which we presented in section 2.2. This also includes a partial gauge fixing and two canonical transformations, resulting in the polarized three-torus Gowdy model formulated as a one dimensional field theory. On this model, we then apply the techniques of LQG and AQG, cf. section 5.2. For the LQG formulation, we use the constructions from [56, 57], which are in analogy to the spherical symmetric model discussed in [121, 122]. For the AQG formulation, we rely on the two master theses [58, 59].

Before we continue with the discussion of the symmetry reduction of the three-torus Gowdy model in section 5.1, we discuss in the following paragraphs the role of the Gowdy model compared to other toy models.

To obtain a symmetry reduced quantum theory, there are in principle two scenarios [123]:

- On the one hand, we may first quantize the system and restrict to invariant states, i.e., identify the symmetric sector. For this, we need to have good control over our quantum system. The full theory of loop quantum gravity, however, is not complete yet while the kinematical formulation is well established and further progress was made for example in [52, 124–126]. One can nevertheless use the definition of kinematical invariant states in [55] as distributions with support on invariant connections. We will discuss the classification of invariant connections as done in [55, 127] in section 5.1.1.
- On the other hand, we may first symmetry reduce the classical phase space to an invariant subspace, which we then quantize.

In general, both approaches may lead to different symmetric sectors and hence different results. In this thesis, we choose the toy model approach, i.e., we first symmetry reduce and then quantize the Gowdy model. Specifically, we will use the construction of invariant connections as discussed in section 5.1.1.

The motivation for considering toy models in the first place is that they possess certain symmetries and hence are technically much simpler to handle than the full theory. Thus, using toy models, we can test the full theory for consistency and possibly gain more insights into its structure. In particular, we can investigate open problems of the full theory, for example concerning the dynamics and its semiclassical limit. A problem within the investigation of the dynamics in the full theory is that the Hamiltonian constraint contains the volume operator, whose spectrum, however, is hard to compute analytically while perturbative methods exist, see our discussion in section 3.4. In the polarized three-torus Gowdy model, however, the volume operator acts diagonally and hence we can calculate the spectrum of the Gowdy volume operator more easily. Note that this is a generic feature in toy models, see, e.g., [128] for the construction of volume operators for some cosmological models and [129] for the spherical symmetric model. Another peculiarity of toy models is the avoidance of initial singularities, see for instance [130], which may be a hint that in the full theory this occurs as well, although the results of toy models cannot be directly translated, see for example [131, 132] for a discussion. This can also be investigated in the polarized three-torus Gowdy model.

In general, we can distinguish between midi- and mini-superspace toy models:

- Midi-superspace models involve infinitely many degrees of freedom, i.e., they are symmetry reduced field theories. Especially, they occur in inhomogeneous cosmological models, where one particular example is the Gowdy model. See [63, 133] for reviews on midi-superspace models.

- Mini-superspace models involve only finitely many degrees of freedom, i.e., there are no field degrees of freedom present. These kind of theories occur in homogeneous cosmological models. The quantization of these models using techniques from LQG is performed within loop quantum cosmology (LQC), see [128, 134–136] and for reviews [28, 53]. In particular, one can perform the loop quantization of the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, which has only one geometric degree of freedom described by the scale factor $a(t)$ and hence is one of the simplest possible toy models. The FLRW model is also physically very interesting as experimental data hint at the flatness of the universe. Following [63], we see that within this toy model, the Gauß and the diffeomorphism constraints are trivially satisfied which is the generic case in homogeneous cosmological models. Hence, we are left with only the Hamiltonian constraint. In the Gowdy model, for comparison, the symmetry reduction results in one Gauß, one diffeomorphism and the Hamiltonian constraint. The results of LQC are very peculiar as the big bang singularity is resolved and replaced by a big bounce. To achieve this, the kinematical result of the discreteness of area is adopted from the full theory.

Finally, we want to summarize the motivation for investigating inhomogeneous models such as the Gowdy model following [63]: The quantization of homogeneous models is very well studied and it is the natural next step to investigate models with more degrees of freedom such as the inhomogeneous ones. In particular, the full theory involves field degrees of freedom which can be tested in inhomogeneous but not in homogeneous models. Additionally, as

we mentioned before, singularity avoidance is a generic feature in homogeneous models and it needs to be checked if this happens due to the high symmetry of the model or if it also appears in inhomogeneous theories. Thus, the investigation of such models might give a hint whether singularities will also be avoided in the full theory.

This finishes our general discussion of toy models and we now proceed with the symmetry reduction of the Gowdy model in the next section.

5.1 Symmetry reduction

In the following chapters, we perform the symmetry reduction of the three-torus Gowdy model whose steps we want to summarize in the following: In section 5.1.1, we first explain the general procedure to classify invariant connections for which we introduce the so-called generalized Wang theorem [55, 127]. Afterwards, in section 5.1.2, we apply the generalized Wang theorem to the three-torus Gowdy model. Following [56, 137], we will also perform a partial gauge fixing which sets 4 of the 9 canonical pairs (A_a^j, E_j^a) to zero, thereby solving two Gauß and two diffeomorphism constraints. Additionally, as was done in [56] in analogy to the spherical symmetric model [121, 122], we introduce two canonical transformations giving us new canonical variables that simplify the quantization and especially the volume functional. We also reduce to the polarized Gowdy model, i.e., impose that the two Killing vectors are orthogonal, which eliminates one canonical pair, see again [56]. Additionally, we solve the Gauß constraint at the classical level which eliminates another canonical pair. This was done in [56] but in the follow-up paper [57], the quantization was performed on the gauge variant level. We end up with a $U(1)$ gauge theory constrained by the first class system of one Gauß, one diffeomorphism and the Hamiltonian constraint while we have three canonical pairs, i.e., six basic variables, namely one $U(1)$ connection and two scalar fields on the connection side plus three canonically conjugate fluxes. Furthermore, we have only one physical field degree of freedom remaining.

5.1.1 The generalized Wang theorem

In this section, we first describe the general classification scheme of invariant connections following [53, 55, 66, 138], which use the constructions introduced in [127]. This results in the so-called generalized Wang theorem which we apply to the Gowdy model in section 5.1.2.

The starting point is a principal fibre bundle $P(\sigma, G, \pi)$ whose constituents we explain in the following: σ denotes the base space which is given by the spatial manifold of our globally hyperbolic spacetime manifold $M \simeq \mathbb{R} \times \sigma$. Furthermore, G denotes the structure or gauge group and $\pi : P \to \sigma$ is the projection map from the total space P to the base manifold σ. On this principal bundle, we can define a connection one-form ω. See also, for instance, [18] for more detailed definitions. We now want to classify those connection one forms ω that are invariant with respect to a certain symmetry group S, i.e., those that satisfy

$$s^*\omega = \omega \quad \forall s \in S. \tag{5.1.1}$$

This condition only has to be satisfied up to gauge transformations as the connection ω does not need to be exactly invariant under a symmetry transformation, it will also result in the same physics if it is only invariant up to a gauge transformation.

To classify symmetric connections, the first ingredient is an action of S on the spatial manifold σ, which is such that [121]

$$\sigma \simeq B \times S/F. \tag{5.1.2}$$

Here, $B = \sigma/S$ denotes the reduced base space and S/F are the orbits where F denotes the isotropy group of S. In general, the isotropy group is defined for each point $x \in \sigma$ as the set of elements of S that leave x invariant. To nevertheless arrive at (5.1.2), we assumed that the action of S on σ is such that each orbit is given by S/F, where F denotes a fixed isotropy group. In general, we may use so-called orbit bundles, see for example [55] and [139].

Given (5.1.2), we can classify S-symmetric principal fibre bundles, which are principal fibre bundles that admit an action of the symmetry group as automorphisms on P. In particular, we obtain a classifying pair (λ, Q) which is given by a conjugacy class of homomorphisms $\lambda : F \to G$ and the principal fibre bundle $Q(B, Z_G(\lambda(F)), \pi_Q)$. Concerning the latter, the base space is defined by $B = \sigma/S$, the structure group is given by the centralizer $Z_G(\lambda(F))$, i.e., all those $g \in G$ that commute with all $h \in \lambda(F)$, and the projection map π_Q is defined as projecting from the total space Q down to the base manifold B, i.e., $\pi_Q : Q \to B$.

Given the classifying pair (λ, Q), the so-called generalized Wang theorem allows us to classify symmetric connections. We obtain from this theorem that the invariant connections acquire the local form [121]

$$A = A_B + A_{S/F}, \tag{5.1.3}$$

where A_B is the connection on the reduced bundle Q and $A_{S/F}$ contains Lie algebra-valued scalar fields $\Phi : Q \times L(F)_\perp \to L(G)$ defined on Q. $L(F)_\perp$ denotes the orthogonal complement of $L(F)$ with respect to the Killing form on S, i.e., we have a decomposition of $L(S)$ as $L(F) \oplus L(F)_\perp$ which is true for semi-simple Lie algebras. Furthermore, the scalar fields Φ have to satisfy [138]

$$\mathrm{Ad}_{\lambda(f)}(\Phi(X)) = \Phi(\mathrm{Ad}_f X) \quad \forall f \in F, X \in L(S), \tag{5.1.4}$$

where Ad denotes the adjoint action. Specifically, $A_{S/F}$ is constructed via $\Lambda \circ i^* \theta_{MC}$, where Λ is a map from $L(S)$ to $L(G)$, $i : S/F \hookrightarrow S$ is an embedding map and θ_{MC} denotes the Maurer-Cartan form of S, i.e., the one-form that is invariant with respect to left translations of S and takes values in $L(S)$. We refer to [55,66,127] for proofs and more details on the Wang theorem.

5.1.2 Symmetry reduction of the three-torus Gowdy model

Now, we perform the symmetry reduction of the three-torus Gowdy model for which we first apply the generalized Wang theorem to the Gowdy model, which was done in [59]. For this, we have to specify the following ingredients:

- The constituents of the principal fibre bundle $P(\sigma, G, \pi)$ are given by $\sigma = T^3$ and $G = SU(2)$.
- The symmetry group is the torus group $S = T^2 \simeq U(1) \times U(1)$.
- We can define a left action of S on σ the following way: If we coordinatize a point p in σ by three angular variables $(\theta^1, \theta^2, \theta^3)$ and a group element $s \in S$ by two angular variables (x, y), a left action of the symmetry group $S = \mathbb{T}^2$ is given by

$$l : S \times \sigma \to \sigma, \quad (s, p) \mapsto s \cdot p := (x + \theta^1, y + \theta^2, \theta^3). \tag{5.1.5}$$

This action is free, i.e., we have a trivial isotropy group $F = \{e\}$, where e denotes the identity in S. The Maurer Cartan form invariant with respect to (5.1.5) is just given by $\theta_{MC} = (\theta_{MC})_\rho dx^\rho$ with constant components $(\theta_{MC})_\rho$ for $\rho = 1, 2$ and $dx^1 := dx$, $dx^2 := dy$.

- For the classifying pair (λ, Q) of the S-symmetric principal fibre bundle $P(\sigma, G, \pi)$, we obtain the following: λ is trivial as $F = \{e\}$. Concerning Q, we have that $\sigma/S = S^1$ and from $G = SU(2)$ it follows that $Z_G(\lambda(F)) = SU(2)$, i.e., Q is an $SU(2)$ principal fibre bundle over S^1.

We now have all necessary ingredients to rewrite (5.1.3) in the case of the Gowdy model: A_B is given by the $SU(2)$-valued connection on the reduced bundle Q defined over S^1. In the following, we coordinatize S^1 by the angular variable θ. In particular, the connection components only depend on θ. Concerning $A_{S/F}$, we have Lie algebra-valued scalar fields on the reduced bundle Q defined over S^1 , i.e., they also depend only on θ, and furthermore take values in $SU(2)$. We arrive at the following form of the invariant connection A of the Gowdy model:

$$A = A^i_\theta(\theta)\tau_i \mathrm{d}\theta + A^i_\rho(\theta)\tau_i \mathrm{d}x^\rho, \quad \rho = x, y, \ i = 1, 2, 3, \tag{5.1.6}$$

where we denoted the reduced $SU(2)$ connection components as $A^i_\theta(\theta)$ and the scalar fields as $A^i_\rho(\theta)$. With this notation, we stay close to [56]. Furthermore, in (5.1.6), τ_i denote the generators of $su(2)$ given by the Pauli matrices σ_i via $\tau_i := -i\sigma_i/2$ and x^ρ denote the coordinates of S, i.e., $x^x := x$, $x^y := y$. Note that the gauge group is reduced in the following sense: The connection A is invariant with respect to S only up to gauge transformations. As the connection components depend on θ only, the gauge transformations are reduced in the sense that they are allowed to depend on θ only such that they map invariant connections again to invariant connections. We can also find the conjugate momenta to (5.1.6), which are given by

$$E = E^\theta_i(\theta)\tau_i \partial_\theta + E^\rho_i(\theta)\tau_i \partial_\rho, \quad \rho = x, y, \ i = 1, 2, 3. \tag{5.1.7}$$

We observe that all basic variables only depend on θ. Thus, in the constraints stated in (2.2.13),(2.2.21), and (2.2.22) in terms of Ashtekar variables, all partial derivatives with respect to x and y will drop out. We still have 3 diffeomorphism and 3 Gauß constraints denoted as $\mathrm{G}_I, \mathrm{C}_\rho, \mathrm{G}_3, \mathrm{C}_\theta$ with $I = 1, 2$ and $\rho = x, y$. Specifically, C_θ is the generator of $Diff(S^1)$. As pointed out in [137], the diffeomorphism constraints C_ρ do not have a geometrical interpretation anymore and hence it is not necessary to keep them in the following. In particular, we can introduce a partial gauge fixing, which allows to solve C_ρ as well as G_I which we will do in the following paragraph.

To arrive at the so-called unpolarized Gowdy model, we partially fix the gauge, which solves two diffeomorphism and two Gauß constraints, and furthermore perform two canonical transformations. This way, the quantization becomes technically more easy and especially the form of $det(E)$ simplifies such that the volume becomes a diagonal operator at the quantum level. We describe these two necessary steps, the partial gauge fixing and the two canonical transformations, in the following listing:

- Firstly, we partially fix the gauge following [56, 137]:

$$0 = E^\theta_I = E^\rho_3, \quad 0 = A^I_\theta = A^3_\rho, \tag{5.1.8}$$

where $\rho = x, y$ and $I = 1, 2$. Using this, we can simplify $\det(E)$, the key ingredient to calculate the volume functional:

$$\det(E) = \det\begin{pmatrix} E_1^x & E_2^x & E_3^x \\ E_1^y & E_2^y & E_3^y \\ E_1^\theta & E_2^\theta & E_3^\theta \end{pmatrix} \overset{(5.1.8)}{=} \det\begin{pmatrix} E_1^x & E_2^x & 0 \\ E_1^y & E_2^y & 0 \\ 0 & 0 & E_3^\theta \end{pmatrix} = E_3^\theta (E_1^x E_2^y - E_2^x E_1^y). \quad (5.1.9)$$

Furthermore, one can show that (5.1.8) solves the constraints G_I and C_ρ. We remain with one Gauß, one diffeomorphism and the Hamiltonian constraint, which in smeared versions read as [56]:

$$\mathrm{G} := \mathrm{G}_3 = \frac{1}{\kappa'\gamma}\left[\partial_\theta E_3^\theta + \epsilon_{3J}{}^K A_\rho^J E_K^\rho\right], \quad (5.1.10)$$

$$\mathrm{C}_\theta = \frac{1}{\kappa'\gamma}\left[(\partial_\theta A_\rho^I) E_I^\rho + \epsilon_{3J}{}^K A_\rho^J E_K^\rho A_\theta^3 - \kappa\gamma A_\theta^3 G_3\right], \quad (5.1.11)$$

$$\begin{aligned}\mathrm{C} = \frac{1}{2\kappa'}\frac{1}{\sqrt{|detE|}}\big[&2A_\theta^3 E_3^\theta A_\rho^J E_J^\rho + A_\rho^J E_J^\rho A_\sigma^K E_K^\sigma - A_\rho^K E_J^\rho A_\sigma^J E_K^\sigma - 2\epsilon_{3J}{}^K(\partial_\theta A_\rho^J) E_K^\rho E_3^\theta \\ &- (1+\gamma^2)\left(2K_\theta^3 E_3^\theta K_\rho^J E_J^\rho + K_\rho^J E_J^\rho K_\sigma^K E_K^\sigma - K_\rho^K E_J^\rho K_\sigma^J E_K^\sigma\right)\big]. \end{aligned} \quad (5.1.12)$$

Here, we used that all phase space variables depend on θ only and hence all partial derivatives with respect to x and y drop out which we already remarked after (5.1.7). Furthermore, we defined $\kappa' := \frac{\kappa}{4\pi^2}$, i.e., we absorb a factor of $4\pi^2$. We get this factor when computing the smeared versions of the constraints because these involve integrals over θ, x, and y with the latter two integrals just giving $4\pi^2$ as our basic variables depend on θ only. We will work with the smeared constraints further below, but at this point keep the unsmeared versions, integrate out the x and y integrals, and absorb the factor of $4\pi^2$ into κ. Note that to arrive at (5.1.11), we used, compared to section 2, the diffeomorphism constraint in the form $C_a := H_a - A_a^j G_j$, which generates spatial diffeomorphisms.

- Secondly, we perform two canonical transformations, which are inspired by the spherically symmetric model [121, 122] and were translated to the Gowdy model in [56]:
  - The first canonical transformation we perform is given by

$$\begin{aligned} E_1^x &= E^x \cos\beta, \quad & E_2^x &= E^x \sin\beta, \\ E_1^y &= -E^y \sin\bar{\beta}, \quad & E_2^y &= E^y \cos\bar{\beta} \end{aligned} \quad (5.1.13)$$

    for the electric fields and by

$$\begin{aligned} A_x^1 &= A_x \cos(\alpha+\beta), \quad & A_x^2 &= A_x \sin(\alpha+\beta), \\ A_y^1 &= -A^y \sin(\bar{\alpha}+\bar{\beta}), \quad & A_y^2 &= A_y \cos(\bar{\alpha}+\bar{\beta}) \end{aligned} \quad (5.1.14)$$

    for the connections. This is in analogy to the introduction of polar coordinates with E^x, E^y, A_x, and A_y playing the role of the radius, as for example $E^x = \sqrt{(E_1^x)^2 + (E_2^x)^2}$ and similarly for the other variables. The angular variables in (5.1.14) are introduced in this particular way only for convenience. To complete the first canonical transformation, we define

$$\begin{aligned} X &:= A_x \cos\alpha, \quad & Y &:= A_y \cos\bar{\alpha}, \\ P^\beta &:= -E^x A_x \sin\alpha, \quad & \bar{P}^\beta &:= -E^y A_y \sin\bar{\alpha}. \end{aligned} \quad (5.1.15)$$

This way, we end up with the configuration variables $\mathcal{A} := \gamma^{-1}A^3_\theta, X, Y, \beta, \bar{\beta}$ with conjugate momenta $\mathcal{E} := E^\theta_3, E^x, E^y, P^\beta, \bar{P}^\beta$, where the canonical commutation relations are for example for $\mathcal{A}$ and $\mathcal{E}$ given by

$$\{\mathcal{A}(\theta), \mathcal{E}(\theta')\} = \kappa'\gamma\delta(\theta, \theta'). \tag{5.1.16}$$

In total, we have 10 canonical variables and three first class constraints giving us 2 physical field degrees of freedom and 4 physical degrees of freedom in phase space, respectively.

– The second canonical transformation has the form

$$\begin{aligned} \xi &= \beta - \bar{\beta}, \quad \eta = \beta + \bar{\beta}, \\ P^\xi &= \frac{P^\beta - \bar{P}^\beta}{2}, \quad P^\eta = \frac{P^\beta + \bar{P}^\beta}{2}. \end{aligned} \tag{5.1.17}$$

This way, we replace the canonically conjugate variables $\beta, \bar{\beta}, P^\beta, \bar{P}^\beta$ by η, ξ, P^η, P^ξ.

Using (5.1.13)-(5.1.17), we can rewrite the constraints to complete the formulation of the unpolarized Gowdy model. For this, we refer to [56] as in this thesis, we are only interested in the polarized Gowdy model which we will introduce in the next paragraph. Prior to that, we show explicitly how the two canonical transformations in (5.1.13)-(5.1.17) simplify $\det(E)$ in (5.1.9):

$$\det(E) = \mathcal{E}(E^x \cos\beta \cdot E^y \cos\bar{\beta} + E^x \sin\beta \cdot E^y \sin\bar{\beta}) = \mathcal{E}E^xE^y \cos\xi =: E, \tag{5.1.18}$$

where in the second step, we used the trigonometric identity $\cos\beta\cos\bar{\beta} + \sin\beta\sin\bar{\beta} = \cos(\beta - \bar{\beta}) = \cos\xi$. We see already in the unpolarized Gowdy model that the volume acquires a simple form.

Following [56], now we go over from the unpolarized to the polarized Gowdy model by imposing the two Killing vectors ∂_x, ∂_y, corresponding to the isometry group T^2, to be orthogonal. When we look at the form of the line element of the unpolarized Gowdy model [56]

$$\mathrm{d}s^2 = \cos\xi\frac{E^xE^y}{E^\theta_3}\mathrm{d}\theta^2 + \frac{E^\theta_3}{\cos\xi}\frac{E^y}{E^x}\mathrm{d}x^2 + \frac{E^\theta_3}{\cos\xi}\frac{E^x}{E^y}\mathrm{d}y^2 - 2\frac{E^\theta_3}{\cos\xi}\sin\xi\mathrm{d}x\mathrm{d}y, \tag{5.1.19}$$

we see that for orthogonal Killing vectors ∂_x, ∂_y, the $\mathrm{d}x\mathrm{d}y$-term has to vanish. This is equivalent to the condition $\xi(\theta) = 0$, which we require as an additional constraint. Hence, we have to perform the stability analysis according to the Dirac algorithm, which we introduced in section 2.1.2: We need to compute the Poisson bracket between $\xi(\theta)$ and the primary Hamiltonian and impose the result to be weakly vanishing, i.e., we demand $\dot{\xi}(\theta) \approx 0$. From this, we get the additional constraint

$$\chi(\theta) := 2P^\xi + E^\theta_3\partial_\theta(\ln\frac{E^y}{E^x}) \approx 0, \tag{5.1.20}$$

which together with $\xi(\theta) \approx 0$ forms the so-called *polarization constraints*. These constraints do weakly Poisson commute with the Gauß, the diffeomorphism and the Hamiltonian constraint but not among each other. Hence, they form a system of second class constraints for which we can use the Dirac bracket and implement the constraints strongly. In particular, we eliminate

the variables ξ, P^ξ using the polarization constraints. For completeness, we state the definition of the Dirac bracket for two phase space functions F, G:

$$\begin{aligned}\{F,G\}^* = \{F,G\} - \int \mathrm{d}\theta \int \mathrm{d}\theta' \big[&\{F,\xi(\theta)\}\{\xi(\theta),\chi(\theta')\}^{-1}\{\chi(\theta',G\} \\ &+ \{F,\chi(\theta)\}\{\chi(\theta),\xi(\theta')\}^{-1}\{\xi(\theta'),G\}\big].\end{aligned} \tag{5.1.21}$$

As the polarization constraints weakly Poisson commute with the Gauß, the diffeomorphism and the Hamiltonian constraint, the constraint algebra with respect to Dirac brackets will not differ from the one calculated using the original Poisson brackets. From (5.1.21), we can also see that the equations of motions for all phase space variables remain unchanged using the Dirac bracket, except the ones of ξ and P^ξ. Hence, we can stick to the original Poisson bracket for the polarized model. We end up with the following form of the Gauß, diffeomorphism and Hamiltonian constraints [56, 57]:

$$\mathrm{G} = \frac{1}{\kappa'}\left[\partial_\theta \mathcal{E} + P^\eta\right], \tag{5.1.22}$$

$$\mathrm{C}_\theta = \frac{1}{\kappa'\gamma}\left[(\partial_\theta X)E^x + (\partial_\theta Y)E^y - (\partial_\theta E^\theta_3)A^3_\theta + (\partial_\theta \eta)P^\eta + (\partial_\theta \xi)P^\xi)\right], \tag{5.1.23}$$

$$\begin{aligned}\mathrm{C} = &-\frac{1}{\kappa'}\frac{1}{\sqrt{det(E)}}\Big[\frac{1}{\gamma^2}(XE^xYE^y + \mathcal{A}\mathcal{E}(XE^x + YE^y) + \mathcal{E}\partial_\theta\eta(XE^x + YE^y)) \\ &+ \frac{1}{4}\partial_\theta\mathcal{E} - \frac{1}{4}(\mathcal{E}\partial_\theta ln(\frac{E^y}{E^x}))^2\Big] - \frac{\kappa'}{4}\frac{G^2}{\sqrt{E}} - \frac{\gamma}{2}\partial_\theta\left(\frac{G}{\sqrt{E}}\right),\end{aligned} \tag{5.1.24}$$

where for the Hamiltonian constraint, we eliminated P^η in terms of the Gauß constraint. Furthermore, using $\xi = 0$, we can rewrite the expression for $det(E)$ in (5.1.9) as

$$E := det(E) = \mathcal{E}E^xE^y, \tag{5.1.25}$$

which implies that the volume functional is simply the integrated square root of the product of three electric fluxes. As the electric fluxes correspond to diagonal operators in the quantum theory, cf. section 5.2.2, the volume operator will act diagonally. We see from (5.1.24) that $\sqrt{E}$ enters in every summand and hence the volume operator is a crucial ingredient for quantizing the Hamiltonian constraint. Therefore, the diagonal action of the volume operator provides a significant simplification, especially compared to the full theory.

From the Gauß constraint in (5.1.22), we can also see that $\mathcal{A}$ transforms as a $U(1)$ connection: If we calculate the Poisson bracket between the smeared Gauß constraint $\mathrm{G}(\Lambda) := \int_{S^1} \mathrm{d}\theta\ \Lambda\mathrm{G}$, where $\mathrm{G} := \mathrm{G}_3$ and $\Lambda := \Lambda^3$, and $\mathcal{A}(\theta)$, we get:

$$\{\mathrm{G}(\Lambda), \mathcal{A}(\theta)\} = -\partial_\theta\Lambda(\theta). \tag{5.1.26}$$

This is the infinitesimal version of the transformation law of an $U(1)$ connection $\mathcal{A}$ with respect to finite gauge transformations given by

$$\mathcal{A} \to -dg\ g^{-1} + g\mathcal{A}g^{-1}, \tag{5.1.27}$$

as one can check by inserting into this equation $g = \exp(i\Lambda) \approx 1 + i\Lambda$ for small Λ.

We now solve the Gauß constraint following [56]. For this, we observe from (5.1.22) that all variables except for $\mathcal{A}$ and η are gauge-invariant. Specifically, η is translated under gauge

transformation generated by the Gauß constraint: For an infinitessimal gauge transformation, we get $\{\mathrm{G}(\Lambda), \eta(\theta)\} = \Lambda(\theta)$ and hence arrive for a finite gauge transformation at

$$\exp\left(\{\mathrm{G}(\Lambda), .\}\right) \cdot \eta(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \{\mathrm{G}(\Lambda), \eta(\theta)\}^{(n)} = (\eta + \Lambda)(\theta), \tag{5.1.28}$$

where $\{.,.\}^{(n)}$ denotes the iterated Poisson brackets, cf. section 4. In the second step of (5.1.28), we used $\{\mathrm{G}(\Lambda), \{\mathrm{G}(\Lambda), \eta(\theta)\}\} = 0$. From (5.1.28), it follows that we can assume η to be any constant and in particular, we can impose the constraint $\eta \approx 0$, which forms a second class system with the Gauß constraint while weakly Poisson commuting with all other constraints [56]. Using the associated Dirac bracket, we can strongly set η and G to zero. Furthermore, the Dirac bracket reduces to the original Poisson bracket for phase space functions independent of (η, P^η). We are now left with 6 canonical pairs $(\mathcal{A}, \mathcal{E})$, (X, E^x), (Y, E^y) and can express the constraints in terms of these variables which we do in the next paragraph following [56, 57].

We first consider the Hamiltonian constraint, which we rewrite in the following form:

$$\mathrm{H} := \int_{S^1} \mathrm{d}\theta \; N\mathrm{C} = \int_{S^1} \mathrm{d}\theta \; N \left[\mathrm{C}_{\mathrm{eucl}} + \mathrm{C}_{\mathrm{lor}}\right] := \mathrm{H}_{\mathrm{eucl}} + \mathrm{H}_{\mathrm{lor}}, \tag{5.1.29}$$

where we introduced the so-called Euclidean part

$$\mathrm{H}_{\mathrm{eucl}} := \int_{S^1} \mathrm{d}\theta \; N\mathrm{C}_{\mathrm{eucl}} = -\frac{1}{\kappa'\gamma^2} \int_{S^1} \mathrm{d}\theta N \frac{1}{\sqrt{E}} \left[XE^x YE^y + \mathcal{A}\mathcal{E}(XE^x + YE^y)\right] \tag{5.1.30}$$

and the so-called Lorentzian part

$$\mathrm{H}_{\mathrm{lor}} := \int_{S^1} \mathrm{d}\theta \; N\mathrm{C}_{\mathrm{lor}} = \mathrm{H}^{(1)}_{\mathrm{lor}} + \mathrm{H}^{(2)}_{\mathrm{lor}} + \mathrm{H}^{(3)}_{\mathrm{lor}}, \tag{5.1.31}$$

which we split in three terms given by

$$\mathrm{H}^{(1)}_{\mathrm{lor}} := -\frac{1}{4\kappa'} \int_{S^1} \mathrm{d}\theta N \frac{(\partial_\theta \mathcal{E})^2}{\sqrt{E}}, \tag{5.1.32}$$

$$\mathrm{H}^{(2)}_{\mathrm{lor}} := \frac{1}{4\kappa'} \int_{S^1} \mathrm{d}\theta N \frac{(\mathcal{E})^2}{\sqrt{E}} \left(\frac{\partial_\theta E^x}{E^x} - \frac{\partial_\theta E^y}{E^y}\right)^2, \tag{5.1.33}$$

$$\mathrm{H}^{(3)}_{\mathrm{lor}} := \frac{1}{\kappa'} \int_{S^1} \mathrm{d}\theta N \partial_\theta \left(\frac{\mathcal{E}(\partial_\theta \mathcal{E})}{\sqrt{E}}\right). \tag{5.1.34}$$

For a better comparison with [57], we remark that in [57] the term corresponding to the Euclidean part is named the kinetic term H_K and the term corresponding to the Lorentzian part the potential term H_P. We now want to justify the split (5.1.29) of the Hamiltonian constraint. We see that only in the Euclidean part (5.1.30) the configuration variables $\mathcal{A}, X, Y$ are contained, whereas in the Lorentzian part (5.1.32)-(5.1.34) this is not the case. In particular, the Euclidean part (5.1.30) has a structure that resembles the Euclidean part of the full theory

$$\mathrm{H}_{\mathrm{eucl,full}} = \frac{\epsilon_{jkl} F^j_{ab} E^a_k E^b_l}{\sqrt{E}}, \tag{5.1.35}$$

with the connection components contained in the curvature tensor $F^j_{ab} = \partial_{[a} A^j_{b]} + \epsilon^{jkl} A^k_a A^l_b$. This justifies that we denote (5.1.30) as the Euclidean part. Note, however, that (5.1.30) is

not resulting solely from terms of (5.1.35). We collected the remaining terms of the Gowdy Hamiltonian constraint in another operator which we denoted as the Lorentzian part, cf. (5.1.31). We again split this constraint into three terms in analogy to [57]. Furthermore, the split in (5.1.29) is also justified in the quantum theory where the Euclidean part will correspond to a graph-changing operator in LQG and a label-changing operator in AQG, respectively, while the Lorentzian part acts diagonally in both LQG and AQG.

Besides the Hamiltonian constraint, we also have one diffeomorphism constraint given by [56, 57]

$$\mathrm{C}_\theta = \frac{1}{\kappa'\gamma}\left[E^x\partial_\theta X + E^y\partial_\theta Y - \mathcal{A}\partial_\theta\mathcal{E}\right]. \tag{5.1.36}$$

From this, we see that the basic variables transform as scalars under θ-coordinate transformations on S^1 generated by C_θ. For example, for $X(\theta)$ and the smeared diffeomorphism constraint $\widetilde{\mathrm{C}}(N^\theta) := \int_{S^1} N^\theta \mathrm{C}_\theta$, we obtain

$$\exp\left(\{\widetilde{\mathrm{C}}(N^\theta),.\}\right)\cdot X(\theta) = \sum_{n=0}^{\infty}\frac{1}{n!}\{\widetilde{\mathrm{C}}(N^\theta), X(\theta)\}^{(n)} = X(\theta - N^\theta(\theta)), \tag{5.1.37}$$

where we used (5.1.16) to compute

$$\begin{aligned}\{\widetilde{\mathrm{C}}(N^\theta), X(\theta)\} &= -N^\theta(\theta)\partial_\theta X(\theta),\\ \{\widetilde{\mathrm{C}}(N^\theta), \{\widetilde{\mathrm{C}}(N^\theta), X(\theta)\}\} &= N^\theta(\theta)\partial_\theta\{\widetilde{\mathrm{C}}(N^\theta), X(\theta)\} = (N^\theta(\theta))^2\partial_\theta^2 X(\theta).\end{aligned} \tag{5.1.38}$$

Then, we arrive at $\{\widetilde{\mathrm{C}}(N^\theta), X(\theta)\}^{(n)} = (-N^\theta(\theta))^n\partial_\theta^n X(\theta)$ and finally (5.1.37). For the density weights, we choose the conventions of [56, 57], i.e., $X, Y, \mathcal{E}$ have density weight zero and $E^x, E^y, \mathcal{A}$ have density weight 1. Specifically, we used for $(\mathcal{A}, \mathcal{E})$ that, in one dimension, a tensor density of contravariant rank p, covariant rank q, and density weight w can be seen as a scalar density of weight $w + q - p$, as one can see from the tensor transformation law. Hence, we can assign a density weight of one to the one-form $\mathcal{A}$ which requires $\mathcal{E}$ to have density weight zero.

5.2 Algebraic loop quantization of the polarized three-torus Gowdy model

In the following chapters, we quantize the classical theory of the polarized three-torus Gowdy model. In particular, we will perform a reduced phase space quantization for which we apply the relational formalism in section 5.2.1. This way, we end up with a theory where all constraints are solved at the classical level and that is equivalent to the one described in [56, 57]. The prize we have to pay, however, is that we have introduced additional dust degrees of freedom playing the role of physical observers. Subsequent to section 5.2.1, we proceed in section 5.2.2 with a discussion on the LQG and AQG techniques in the context of the Gowdy model. We apply these to the volume operator, the Hamiltonian constraint and the Gauß constraint in the ensuing sections.

5.2.1 Application of the relational formalism

To perform a reduced phase space quantization of the Gowdy model, we first apply the relational formalism we discussed in section 4: We use Gaussian dust as reference fields which

allows us to rewrite the unsmeared versions $\mathrm{C}_\theta, \mathrm{C}$ of the Hamiltonian and diffeomorphism constraints in (5.1.29) and (5.1.36) in a deparametrised form. Then, we reduce with respect to these new constraints, i.e., we construct observables invariant under the gauge flow of these constraints. Applying this to the basic variables gives us, for instance, the observable $O_X(\tau, \sigma^\theta)$ associated with the configuration variable X and similarly for the remaining variables, where τ denotes the physical time and σ^θ is one of the coordinates $\sigma^j, j = x, y, \theta$, of the dust manifold $\mathcal{S}$ and provides a ruler for the coordinate $\theta \in S^1$. The observables corresponding to the basic variables coordinatize the reduced phase space and in particular, we obtain an observable algebra that is isomorphic to the original Poisson algebra. We also obtain a time-independent physical Hamiltonian

$$\mathrm{H}_{\mathrm{phys}} := \int_{\mathcal{S}} \mathrm{d}^3\sigma \mathrm{C}(\sigma^\theta), \tag{5.2.1}$$

where $\mathrm{C}(\sigma^\theta) := O_{\mathrm{C}}(\sigma^\theta) = \mathrm{C}(O_X, O_Y, O_{\mathcal{A}}, O_{E^x}, O_{E^y}, O_{\mathcal{E}})$ is the observable associated to the unsmeared Hamiltonian constraint C given by (5.1.29)-(5.1.34). This physical Hamiltonian generates the time evolution of an observable via

$$\frac{\partial O_f(\tau,\sigma)}{\partial \tau} = \{O_f(\tau,\sigma), \mathrm{H}_{\mathrm{phys}}\}. \tag{5.2.2}$$

We can now go over to the quantum theory, where (5.2.2) becomes the Heisenberg equation

$$\frac{\partial \hat{O}_f(\tau,\sigma)}{\partial \tau} = i\hbar[\hat{O}_f(\tau,\sigma), \hat{\mathrm{H}}_{\mathrm{phys}}]. \tag{5.2.3}$$

Assuming that $\hat{\mathrm{H}}_{\mathrm{phys}}$ has been implemented as a self-adjoint operator, we can use the unitary transformation $\hat{U}(\tau) = \exp\left(-\frac{i}{\hbar}\tau \hat{\mathrm{H}}_{\mathrm{phys}}\right)$ to go over to the Schrödinger picture and obtain the time-dependent Schrödinger equation

$$i\hbar\partial_\tau \Psi(\tau) = \hat{\mathrm{H}}_{\mathrm{phys}} \Psi(\tau) \tag{5.2.4}$$

for some state Ψ that we will specify at the beginning of section 6.

We conclude by making some important remarks concerning our notation which we adapt to stay close to [56, 57]:

- In the following, in abuse of notation, we will denote the observables corresponding to the basic variables also by $\mathcal{A}, X, Y, \mathcal{E}, E^x, E^y$.
- For quantizing (5.2.1), we can apply the techniques of [57] but just have to replace $\theta \in S^1$ by the dust coordinate σ^θ and the integral over the circle by the integral over the dust manifold. When discussing the quantization in section 5.2.4, however, we will, also in abuse of notation, denote σ^θ still as θ and the dust manifold as $\mathcal{S}^1$ in order to have a better comparison with [57].
- Note that we will not use the quantized version of (5.2.1) as our physical Hamiltonian operator when writing down a Schrödinger equation, as it is not self-adjoint and not even symmetric. Hence, we use a symmetrized version of it, see section 6.

5.2.2 Holonomies, states and fluxes

In the following, our strategy will be to first introduce holonomies, Gowdy states and fluxes in the LQG framework following [57] and then directly translate this to the AQG framework following [58,59]. In particular, we will obtain the kinematical Hilbert space of the Gowdy model.

We begin with the definition of an embedded graph in the polarized three-torus Gowdy model: A graph γ consists of a finite number of non-overlapping arcs or edges e of a circle which meet in vertices v, cf. figure 2. We collect all edges and vertices in the sets $E(\gamma)$ and $V(\gamma)$, respectively.

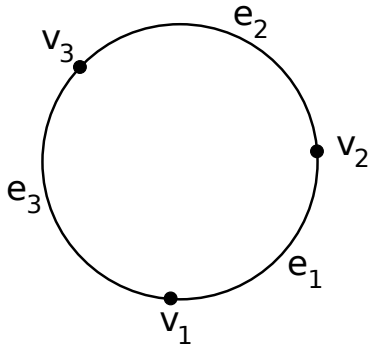

Figure 2: Drawing of an embedded three-valent Gowdy graph γ, where v_1, v_2, v_3 denote the vertices and e_1, e_2, e_3 the edges.

To define a Gowdy state, we need to specify the holonomies, where we follow the notation of [57]:

- $\mathcal{A}$ is a $U(1)$-connection and thus we smear it along an edge $e \in E(\gamma)$ to obtain the $U(1)$-holonomy
$$h_e^{(k)}(\mathcal{A}) := \exp\left(i\frac{k}{2}\int_e \mathcal{A}\right), \tag{5.2.5}$$
with $k \in \mathbb{Z}$ and the factor of $1/2$ introduced for later convenience when we introduce $SU(2)$-valued analogues of the holonomies that are well-defined on the kinematical Hilbert space, cf. (5.2.36)-(5.2.38).

- X and Y are scalar fields, which is a consequence of the symmetry reduction we presented in section 5.1 resulting in (5.1.3). For scalar fields, one introduces so-called point-holonomies, see [27, 55, 121] and [140–142] for details. We apply this procedure to the Gowdy model and define
$$h_v^{(\mu)}(X) := \exp\left(i\frac{\mu}{2}X(v)\right), \quad h_v^{(\nu)}(Y) := \exp\left(i\frac{\nu}{2}Y(v)\right). \tag{5.2.6}$$
We can perceive these holonomies as representations of the Bohr compactification of the real line denoted as $\mathbb{R}_{\text{Bohr}}$, which is the distributional extension of $\mathbb{R}$ and is the analog to the space of generalized connections $\overline{\mathcal{A}}$ in the full theory. It can be understood as the spectrum of the unital C^*-algebra of almost periodic functions[10]. In particular, $\mathbb{R}_{\text{Bohr}}$ is a compact and abelian group, whose representations are labeled by real numbers, which

[10]We introduced the definition of a unital C^*-algebra in the context of (3.2.14) in section 3.2.2. The spectrum of such an algebra $\mathfrak{A}$ is defined as the set of non-zero maps $\chi : \mathfrak{A} \to \mathbb{C}$, the so-called characters,

we denoted as μ and ν respectively in (5.2.6).

The kinematical Hilbert space can be constructed using projective techniques, which we introduced in section 3. We follow the discussion of the spherical symmetric model in [121]: For the $U(1)$-connection, we denote the space of generalized connections as $\overline{\mathcal{A}}_B$, motivated by the form of the invariant connection $A = A_B + A_{S/F}$. $\overline{\mathcal{A}}_B$ is the projective limit with respect to the projective family $(\mathcal{A}_B, p_{\gamma\gamma'})$: The label set is given by closed and oriented graphs γ in S^1, the set $\mathcal{A}_B$ contains elements A_B^γ mapping from the set of edges $E(\gamma)$ to $U(1)^N$, where $|E(\gamma)| = N$, and the projection maps $p_{\gamma\gamma'}$ are defined as the restriction maps $p_{\gamma\gamma'}(A_B^\gamma) = A_B^\gamma|_{\gamma'}$ for $\gamma' \subset \gamma$. Similarly, we proceed for the point holonomies, where we denote the space of generalized connections as $\overline{\mathcal{A}}_{S/F}$, which we obtain as the projective limit of the projective family $(\mathcal{A}_{S/F}, \widetilde{p}_{\gamma\gamma'})$: The label set is given by a set of points in B which we can choose as the set of vertices $V(\gamma)$, the set $\mathcal{A}_{S/F}$ contains maps $A_{S/F}^\gamma$ from $V(\gamma)$ to $\mathbb{R}_{\text{Bohr}}^N$, where $|V(\gamma)| = N$, and the projection maps $\widetilde{p}_{\gamma\gamma'}$ are defined as the restriction maps $\widetilde{p}_{\gamma\gamma'}(A_{S/F}^\gamma) = A_{S/F}^\gamma|_{\gamma'}$ for $\gamma' \subset \gamma$. Now, we can define the space of generalized Gowdy connections $\overline{\mathcal{A}}_{B\times S/F}$ as the projective limit of the tensor product $\mathcal{A}_B \otimes \mathcal{A}_{S/F}$. On $\overline{\mathcal{A}}_{B\times S/F}$, we then can define cylindrical functions and an inner product using the Haar measure of the groups $U(1)$ and $\mathbb{R}_{\text{Bohr}}$: For f_γ, f'_γ two functions cylindrical with respect to a Gowdy graph γ which has N vertices and N edges, we can define the inner product as

$$\begin{aligned} < f_\gamma, f'_\gamma > = & \int_{U(1)^N} \left[\prod_{i=1}^N \mathrm{d}\mu_{U(1)}(h_{e_i}^{(k_i)}(\mathcal{A}))\right] \int_{\mathbb{R}_{\text{Bohr}}^{2N}} \left[\prod_{i=1}^N \mathrm{d}\mu_{\text{Bohr}}(h_{v_i}^{(\mu_i)}(X))\mathrm{d}\mu_{\text{Bohr}}(h_{v_i}^{(\nu_i)}(Y))\right] \times \\ & \times \overline{f_\gamma\left(\{h_{e_i}^{(k_i)}(\mathcal{A}\}_{i=1,\dots,N}, \{h_{v_i}^{(\mu_i)}(X)\}_{i=1,\dots,N}, \{h_{v_i}^{(\nu_i)}(Y)\}_{i=1,\dots,N}\right)} \times \\ & \times f'_\gamma\left(\{h_{e_i}^{(k_i)}(\mathcal{A}\}_{i=1,\dots,N}, \{h_{v_i}^{(\mu_i)}(X)\}_{i=1,\dots,N}, \{h_{v_i}^{(\nu_i)}(Y)\}_{i=1,\dots,N}\right), \end{aligned} \tag{5.2.7}$$

where the Haar measures of $U(1)$ and $\mathbb{R}_{\text{Bohr}}$ are defined by

$$\mu_{U(1)}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\ f(\varphi), \quad \mu_{\text{Bohr}}(g) = \lim_{C\to\infty} \frac{1}{2C}\int_{-C}^{+C} \mathrm{d}x\ g(x), \tag{5.2.8}$$

where f and g are functions on $U(1)$ and $\mathbb{R}_{\text{Bohr}}$, respectively. Taking the completion of the space of cylindrical functions with respect to the inner product (5.2.7) gives us the kinematical Gowdy Hilbert space $\mathcal{H}_{\text{Gowdy}}$. Specifically, at each vertex, we have the Hilbert space $\mathcal{H}_v$, which is given by a tensor product of two copies of $L_2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, the space of functions on $\mathbb{R}_{\text{Bohr}}$ that are square integrable with respect to $d\mu_{\text{Bohr}}$, see also [142], corresponding to the two point holonomies sitting at each vertex, i.e., we have

$$\mathcal{H}_v = L_2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \otimes L_2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}}). \tag{5.2.9}$$

In analogy to the spin network functions in the full theory, we can define an orthonormal basis of the kinematical Hilbert space $\mathcal{H}_{\text{Gowdy}}$ by the so-called *charge network functions* which we shortly call *Gowdy states*:

$$|\gamma, k, \mu, \nu\rangle := \prod_{e\in E(\gamma)} \exp\left(i\frac{k_e}{2}\int_e \mathcal{A}\right) \cdot \prod_{v\in V(\gamma)} \left[\exp\left(i\frac{\mu_v}{2}X(v)\right)\exp\left(i\frac{\nu_v}{2}Y(v)\right)\right]. \tag{5.2.10}$$

which are *-homomorphism, i.e., they satisfy $\chi(ab) = \chi(a)\chi(b)$, $\chi(a+b) = \chi(a) + \chi(b)$, $\chi(a^*) = \chi(a)^*$. Here, we are interested in the algebra of almost periodic functions which is the complex span of functions $T_k := e^{ikx}$, $x, k \in \mathbb{R}$, with period $2\pi/k$. Using pointwise multiplication and addition and the supremum norm on $\mathbb{R}$, this algebra becomes a unital C^*-algebra. See for example chapter 28 of [27] for more details.

More precisely, we define a state of the loop quantized Gowdy model by assigning

- to each edge $e \in E(\gamma)$ an irreducible representation of the $U(1)$-holonomy (5.2.5) labeled by $k_e \in \mathbb{Z}$ and
- to each vertex $v \in V(\gamma)$ the irreducible representations of the point holonomies (5.2.6) labeled by μ_v and $\nu_v \in \mathbb{R}$, respectively.

Multiplying all these irreducible representations of the holonomies at all vertices and edges gives (5.2.10). See the left hand side of figure 3 for an illustration of a Gowdy state. We also refer to chapter 9.1.2 of [138] and [121,122] for a comparison to the spherical symmetric case.

We can now calculate the action of the holonomies on a Gowdy state. For a general $U(1)$-holonomy with charge $k_0 \in \mathbb{Z}$ and the connection $\mathcal{A}$ smeared along an edge $e_i \in E(\gamma)$, we obtain

$$\begin{aligned}\exp\left(i\frac{k_0}{2}\int_{e_i}\mathcal{A}\right)|\gamma,k,\mu,\nu\rangle = &\exp\left(i\frac{k_i+k_0}{2}\int_{e_i}\mathcal{A}\right)\cdot\prod_{e\in E(\gamma)\setminus e_i}\exp\left(i\frac{k_e}{2}\int_e\mathcal{A}\right)\times\\ &\times\prod_{v\in V(\gamma)}\left[\exp\left(i\frac{\mu_v}{2}X(v)\right)\exp\left(i\frac{\nu_v}{2}Y(v)\right)\right] := |\gamma,k_i+k_0,\mu,\nu\rangle\,.\end{aligned} \tag{5.2.11}$$

The notation $|k_i+k_0,\mu,\nu\rangle$ implies that all k-labels remain the same compared to the state $|k,\mu,\nu\rangle$ and only the ith label k_i is shifted by k_0. Similarly, we can proceed for the point holonomies. For example for the X-holonomy defined at a vertex v_i with charge $\mu_0 \in \mathbb{R}$, we get

$$\begin{aligned}\exp\left(i\frac{\mu_0}{2}X(v_i)\right)|\gamma,k,\mu,\nu\rangle = &\prod_{e\in E(\gamma)}\exp\left(i\frac{k_e}{2}\int_e\mathcal{A}\right)\cdot\prod_{v\in V(\gamma)}\exp\left(i\frac{\nu_v}{2}Y(v)\right)\times\\ &\times\exp\left(i\frac{\mu_{v_i}+\mu_0}{2}X(v_i)\right)\cdot\prod_{v\in V(\gamma)\setminus v_i}\exp\left(i\frac{\mu_v}{2}X(v)\right)\\ :=&\,|\gamma,k,\mu_{v_i}+\mu_0,\nu\rangle\,.\end{aligned} \tag{5.2.12}$$

We obtain the action of the point holonomy $\exp\left(i\frac{\nu_0}{2}Y(v_i)\right)$ with $\nu_0 \in \mathbb{R}$ from (5.2.12) by exchanging μ and X with ν and Y.

Now, we define the flux operators. These are implemented as derivative operators, but have to be smeared appropriately according to their density weight. Following our convention introduced below (5.1.38), $\mathcal{E}$ is a scalar and hence we do not have to smear it. The corresponding operator defined for each $\theta \in \sigma/S = S^1$ is given by

$$\hat{\mathcal{E}}(\theta) = -i\gamma l_{\mathrm{P}}^2\frac{\delta}{\delta\mathcal{A}(\theta)} = -i\gamma l_{\mathrm{P}}^2\frac{\delta h_e^{(k)}(\mathcal{A})}{\delta\mathcal{A}(\theta)}\frac{\partial}{\partial h_e^{(k)}(\mathcal{A})}, \tag{5.2.13}$$

with the Planck area l_{P}^2 defined by $l_{\mathrm{P}}^2 = \kappa'\hbar$. This operator has the following action on a Gowdy state [57]:

$$\hat{\mathcal{E}}(\theta)\,|\gamma,k,\mu,\nu\rangle = \frac{\gamma l_{\mathrm{P}}^2}{2}\frac{k_{e^+(\theta)}+k_{e^-(\theta)}}{2}\,|\gamma,k,\mu,\nu\rangle\,. \tag{5.2.14}$$

Here, $e^+(\theta)$ denotes the edge that is outgoing and $e^-(\theta)$ the edge that is ingoing at $\theta \in S^1$. We arrive at (5.2.14) the following way: We consider first the case that θ coincides with a vertex v_i in which two edges e_{i-1} and e_i meet. If we apply $\hat{\mathcal{E}}(\theta)$ to a Gowdy state, there will be two $U(1)$-holonomies in (5.2.10) that depend on v_i. Hence, we have to apply the Leibniz rule and get the sum of two representation labels, namely $k_{e^+(\theta)} = k_{e_i}$ and $k_{e^-(\theta)} = k_{e_{i-1}}$. We also need to take into account that performing the functional differentiation results in an integral over a delta function peaked at the borders of the integral, which gives an additional factor of $1/2$. If θ is not a vertex, it divides the same edge e_i into two pieces and hence $e^+(\theta) = e^-(\theta) = e_i$ resulting in the same label k_{e_i}. In this case, no additional factor of $1/2$ appears from the integral over the delta function. This way, we arrive at (5.2.14) which is valid for a general $\theta \in S^1$. The other two fluxes E^x and E^y are scalar densities of weight one and thus we integrate them over an interval I of the circle. Following [57], we denote the smeared fluxes as

$$\hat{\mathcal{F}}_{x,I} := \int_I \hat{E}^x, \quad \hat{\mathcal{F}}_{y,I} := \int_I \hat{E}^y, \tag{5.2.15}$$

with $\hat{E}^x, \hat{E}^y$ defined in analogy to (5.2.13). Then, we obtain the following action for these flux operators [57]:

$$\hat{\mathcal{F}}_{x,I} \,|\gamma, k, \mu, \nu\rangle = \sum_{v \in V(\gamma) \cap I} \mu_v \,|\gamma, k, \mu, \nu\rangle \,, \tag{5.2.16}$$

$$\hat{\mathcal{F}}_{y,I} \,|\gamma, k, \mu, \nu\rangle = \sum_{v \in V(\gamma) \cap I} \nu_v \,|\gamma, k, \mu, \nu\rangle \,. \tag{5.2.17}$$

Note again that if a vertex coincides with an endpoint of I, we get an additional factor of $1/2$ in the actions of $\hat{\mathcal{F}}_{x,I}$ and $\hat{\mathcal{F}}_{y,I}$, which arises from the integral over a delta function.

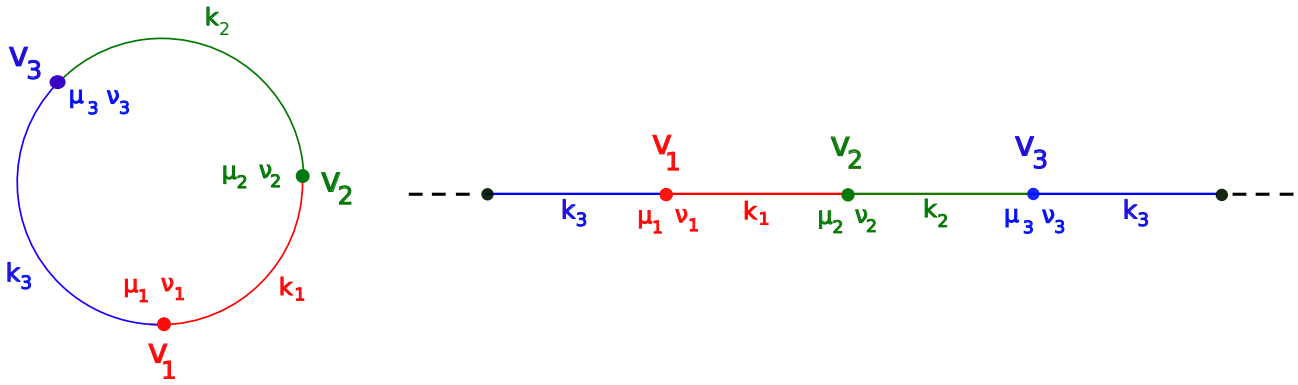

Figure 3: **Left:** Illustration of a Gowdy state (5.2.10) for 3 vertices in LQG. **Right:** Illustration of an abstract Gowdy state (5.2.21) for 3 vertices in AQG.

We now want to go over to the AQG framework but first remark that the big advantage compared to LQG is that the Hamiltonian constraint acts in a graph-preserving way. For LQG, this depends in principle on the chosen regularization, see chapter V of [57] for a discussion. In particular, the action of the LQG Hamiltonian constraint operator of [57] will be more complicated as new vertices are created, see also our discussion at the end of section 5.2.4. Note, however, that the graph-changing action is required such that the Poisson bracket between two Hamiltonian constraints, which results in a diffeomorphism constraint with structure functions, can be implemented in the quantum theory. See also our discussion

in section 3.6.

As introduced in section 3.6, within AQG, we consider one infinite abstract algebraic graph α instead of infinitely many finite graphs γ. An abstract graph for $N = 3$ vertices is illustrated on the right hand side of figure 3. In order to stay close to the LQG graph, we charge only the vertices v_i and edges e_i with $i = 1, ..., N$, $N := |E(\gamma)|$, with the convention that the edge e_i is to the right and the edge e_{i-1} is to the left of the vertex v_i. We also charge the edge e_0, which is to the left of e_1 in figure 3, with the label k_N and charge all other edges and vertices of the infinite graph α trivially such that the circle symmetry of the embedded graph γ is implemented. We can recover γ from the abstract graph α by embedding all vertices v_i, with $i = 1, ..., N+1$, and all edges e_j, with $j = 1, ..., N$, while identifying the vertices v_1 and v_{N+1}. The remaining vertices and edges of α we may map to a common vertex and edge, respectively, giving us accumulation points. However, these edges and vertices are not seen by the Hamiltonian constraint operator as it contains the volume operator, which has eigenvalue zero at trivially charged vertices and edges, cf. (5.2.34).

We now carry over the holonomies and fluxes from LQG to AQG, define so-called abstract Gowdy states, and state the action of the AQG holonomies and fluxes thereon. Following the general procedure presented in section 4, we assign to each edge $e \in E(\alpha)$ Lie group elements $A(e)$ and Lie algebra elements $E(e)$ which correspond to the holonomies and fluxes in the embedded case and satisfy the algebra (3.6.7)-(3.6.9). We construct these variables in the following separately for the holonomies and fluxes:

- **Holonomies:** We define the Lie group element $A(e)$ corresponding to the LQG $U(1)$-holonomy (5.2.5) as $A(e) := \exp\left(i\frac{k_i}{2}\mathcal{A}_{e_i}\right)$, i.e., we replace the integral $\int_e \mathcal{A}$ in (5.2.5) by the expression $\mathcal{A}_e$ that corresponds to an integral over the embedded edge e when choosing an embedding for α. To stay close to the LQG case, we denote $A(e)$ by $h_{e_i}^{(k_i)}(\mathcal{A})$ in the following. Concerning the point holonomies, we observe that all operators that are defined only at vertices straightforwardly carry over to algebraic quantum gravity as they are independent of the embedding. In summary, the AQG holonomies are defined by

$$h_{e_i}^{(k_i)}(\mathcal{A}) := \exp\left(i\frac{k_i}{2}\mathcal{A}_{e_i}\right), \tag{5.2.18}$$

$$h_i^{(\mu_i)}(X_i) := \exp\left(i\frac{\mu_i}{2}X_i\right), \tag{5.2.19}$$

$$h_i^{(\nu_i)}(Y_i) := \exp\left(i\frac{\nu_i}{2}Y_i\right), \tag{5.2.20}$$

  with $X_i := X(v_i), Y_i := Y(v_i)$ and, as discussed above, $\mathcal{A}_e$ equal to $\int_e \mathcal{A}$ in the embedded case, that is for an embedding X we have $A(X(e)) = \exp\left(ik_i/2\int_{X(e)}\mathcal{A}\right)$ where $\mathcal{A}$ is a $U(1)$ connection. Then, we can define a so-called abstract Gowdy state:

$$|k, \mu, \nu\rangle := \prod_{e\in E(\alpha)} \exp\left(i\frac{k_e}{2}\mathcal{A}_e\right) \prod_{i\in V(\alpha)} \exp\left(i\frac{\mu_i}{2}X_i\right) \prod_{i\in V(\alpha)} \exp\left(i\frac{\nu_i}{2}Y_i\right). \tag{5.2.21}$$

  Note that these states do not span the entire infinite tensor product Hilbert space, as only a finite number of edges and vertices of the infinite graph α are charged non-trivially. We obtain the following action of the holonomies in analogy to (5.2.11) and (5.2.12):

$$\exp\left(i\frac{k_0}{2}\mathcal{A}_{e_i}\right)|k, \mu, \nu\rangle = |k_i + k_0, \mu, \nu\rangle\,, \tag{5.2.22}$$

$$\exp\left(i\frac{\mu_0}{2}X_i\right)|k,\mu,\nu\rangle = |k,\mu_i+\mu_0,\nu\rangle\,, \tag{5.2.23}$$

$$\exp\left(i\frac{\nu_0}{2}Y_i\right)|k,\mu,\nu\rangle = |k,\mu,\nu_i+\nu_0\rangle\,. \tag{5.2.24}$$

- **Fluxes:** In LQG, $\hat{\mathcal{E}}(\theta)$ can act on any point of an edge. In AQG, however, an embedded edge does not exist and therefore the action can only be defined on a vertex and is in analogy to (5.2.14) given by

$$\hat{\mathcal{E}}_i\,|\gamma,k,\mu,\nu\rangle = \frac{\gamma l_{\mathrm{P}}^2}{2}\frac{k_i+k_{i-1}}{2}\,|k,\mu,\nu\rangle\,, \tag{5.2.25}$$

  where $\hat{\mathcal{E}}_i := \hat{\mathcal{E}}(v_i)$. Furthermore, we changed our notation for the labels in (5.2.14) slightly by defining $k_i := k_{e^+(\theta)}$ and $k_{i-1} := k_{e^-(\theta)}$ as the k-labels of the edges e_i and e_{i-1} where i denotes the ith vertex of α. The remaining fluxes require a smearing along arcs in LQG, cf. (5.2.15), and hence we have to proceed similar to the definition of the $U(1)$-element $A(e)$: We assign the Lie algebra elements $\hat{\mathcal{F}}_{x,e_i}$ and $\hat{\mathcal{F}}_{y,e_i}$ to each edge $e_i \in E(\alpha)$ and impose them to satisfy the relation (3.6.8) with the corresponding point holonomies (5.2.19), (5.2.20). When we choose an embedding, the flux operators $\hat{\mathcal{F}}_{x,e_i}$ and $\hat{\mathcal{F}}_{y,e_i}$ correspond to the smeared quantities in (5.2.15). This way we obtain for the action

$$\begin{aligned}\hat{\mathcal{F}}_{x,e_i}\,|k,\mu,\nu\rangle &= \frac{\gamma l_{\mathrm{P}}^2}{2}\mu_i\,|k,\mu,\nu\rangle\,,\\ \hat{\mathcal{F}}_{y,e_i}\,|k,\mu,\nu\rangle &= \frac{\gamma l_{\mathrm{P}}^2}{2}\nu_i\,|k,\mu,\nu\rangle\,.\end{aligned} \tag{5.2.26}$$

5.2.3 The volume operator

So far, we quantized the basic variables but now we want to apply these results to more general quantities, where we start in this section with the volume functional and then proceed with the Hamiltonian constraint in the next section. We again first perform the loop quantization of the volume functional following [56] and afterwards translate the result to the AQG framework.

For an arc I, the volume functional is given by

$$V(I) = \int_I \mathrm{d}\theta\sqrt{|\det(E)|} = \int_I \mathrm{d}\theta\sqrt{|\mathcal{E}E^xE^y|}. \tag{5.2.27}$$

We can partition the arc I into N sufficiently small intervals I_i of length ϵ with $i = 1,...,N$. Then, we can rewrite the above volume functional as

$$V(I) \approx \sum_{i=1}^{N}\int_{I_i}\mathrm{d}\theta_i\sqrt{|\mathcal{E}E^xE^y|(\theta_i)} = \sum_{i=1}^{N}\int_{\theta_i}^{\theta_i+\epsilon}\mathrm{d}\tilde{\theta}_i\sqrt{|\mathcal{E}E^xE^y|(\tilde{\theta}_i)}. \tag{5.2.28}$$

We can now approximate the integral and pull the resulting ϵ inside the square roots:

$$V(I) \approx \sum_{i=1}^{N}\sqrt{|\mathcal{E}||\epsilon E^x||\epsilon E^y|(\theta_i)}, \tag{5.2.29}$$

with θ_i a point contained in the interval I_i. Reintroducing the integrals over the fluxes E^x, E^y, we recover the definition of the fluxes $\mathcal{F}_{x,I_i}, \mathcal{F}_{y,I_i}$ smeared along the interval I_i, cf. (5.2.15):

$$V(I) \approx \sum_{i=1}^{N} \sqrt{|\mathcal{E}|(\theta_i)| \int_{\theta_i}^{\theta_i+\epsilon} E^x| | \int_{\theta_i}^{\theta_i+\epsilon} E^y|} = \sum_{i=1}^{N} \sqrt{|\mathcal{E}(\theta_i)||\mathcal{F}_{x,I_i}||\mathcal{F}_{y,I_i}|}. \tag{5.2.30}$$

We can promote this to a well defined operator using (5.2.13) and (5.2.15):

$$\hat{V}(I) := \sum_{i=1}^{N} \sqrt{|\hat{\mathcal{E}}(\theta_i)||\hat{\mathcal{F}}_{x,I_i}||\hat{\mathcal{F}}_{y,I_i}|} =: \sum_{i=1}^{N} \hat{V}(I_i, \theta_i). \tag{5.2.31}$$

Using the action of the fluxes in (5.2.14), (5.2.16), and (5.2.17), we obtain the following action of the volume operator:

$$\hat{V}(I)\,|\gamma, k, \mu, \nu\rangle = \frac{1}{\sqrt{2}} \left(\frac{\gamma l_{\mathrm{P}}^2}{2}\right)^{3/2} \sum_{v \in V(\gamma) \cap I} \sqrt{|\mu_v||\nu_v||k_{e^+(v)} + k_{e^-(v)}|}\,|\gamma, k, \mu, \nu\rangle\,. \tag{5.2.32}$$

As mentioned before, the fluxes give no contribution for intervals that contain no vertex. Hence, the sum in (5.2.31) reduces to contributions from vertices.

Now, we carry over the volume operator $\hat{V}(I)$ defined in LQG for an arc I of S^1 to AQG. Using the AQG equivalents of the fluxes stated in (5.2.25) and (5.2.26), we obtain the volume operator in AQG which consists of a sum over all vertices of the abstract graph, i.e., it is an infinite sum of the following form:

$$\hat{V} := \sum_{i \in V(\alpha)} \hat{V}_i = \sum_{i \in V(\alpha)} \sqrt{|\hat{\mathcal{E}}_i||\hat{\mathcal{F}}_{x,e_i}||\hat{\mathcal{F}}_{y,e_i}|}. \tag{5.2.33}$$

Using the action of the AQG fluxes stated in (5.2.25) and (5.2.26), we obtain the action of the volume operator $\hat{V}_i$ acting at a vertex i of the abstract graph α:

$$\hat{V}_i\,|k, \mu, \nu\rangle = \frac{1}{\sqrt{2}} \left(\frac{\gamma l_{\mathrm{P}}^2}{2}\right)^{3/2} \sqrt{|\mu_i||\nu_i||k_i + k_{i-1}|}\,|k, \mu, \nu\rangle\,. \tag{5.2.34}$$

Having defined the volume operator in this section and holonomies and fluxes in the former, we have now all necessary ingredients to tackle the quantization of the Hamiltonian constraint in the next section.

5.2.4 The Hamiltonian constraint operator

In the following, we explain the quantization procedure for the Hamiltonian constraint. Actually, we are interested in the quantization of the physical Hamiltonian (5.2.1), which is equivalent to the Hamiltonian constraint but depends on the dust coordinate σ^θ and involves an integral over the dust manifold. In the following, as mentioned before in abuse of notation we will denote the dust manifold as $\mathcal{S}^1$ and the physical coordinates as θ. Hence, we start from the classical expression in (5.1.29). We will consider the Euclidean and the Lorentzian part separately. For each parts, we scetch how to derive the LQG-quantized version following [57] and afterwards we go over to the AQG framework as was done before in [58, 59].

Quantization of the Euclidean part: We explain our strategy, following [57], exemplarily for the first term of the Euclidean part (5.1.30). First, we introduce a regularization, i.e., we partition $\mathcal{S}^1$ into N intervals I containing the points θ_i. In particular, we choose the intervals I of sufficiently small length ϵ such that we can rewrite the integral over $\mathcal{S}^1$ as a Riemann sum. This way, we obtain the following expression for the first term of the Euclidean part (5.1.30):

$$-\frac{1}{\gamma^2\kappa'}\int_{\mathcal{S}^1}\mathrm{d}\theta\frac{XE^xYE^y}{\sqrt{|E|}}(\theta)\approx-\frac{1}{\gamma^2\kappa'}\sum_{i=1}^{N}\frac{\epsilon XE^xYE^y}{\sqrt{|E|}}(\theta_i). \tag{5.2.35}$$

Now, for convenience, we introduce the following $SU(2)$-valued analogues of the LQG holonomies defined in (5.2.5) and (5.2.6) :

$$h_\theta(I):=\exp\left(\tau_3k_0\int_I\mathcal{A}\right)=\cos\left(\frac{k_0}{2}\int_I\mathcal{A}\right)+2\tau_3\sin\left(\frac{k_0}{2}\int_I\mathcal{A}\right), \tag{5.2.36}$$

$$h_x(\theta):=\exp\left(\mu_0X(\theta)\tau_1\right)=\cos\left(\frac{\mu_0}{2}X(\theta)\right)+2\tau_1\sin\left(\frac{\mu_0}{2}X(\theta)\right), \tag{5.2.37}$$

$$h_y(\theta):=\exp\left(\mu_0Y(\theta)\tau_2\right)=\cos\left(\frac{\nu_0}{2}Y(\theta)\right)+2\tau_y\sin\left(\frac{\nu_0}{2}Y(\theta)\right), \tag{5.2.38}$$

with $\tau_i, i=1,2,3$, a basis of $su(2)$ that is defined by the Pauli matrices via $\tau_i=-i\sigma_i/2$ and satisfies the following identities:

$$\mathrm{Tr}(\tau_i)=0,\quad \tau_i\tau_j=-\frac{1}{4}\delta_{ij}\mathbb{1}_{SU(2)}+\frac{1}{2}\epsilon_{ijk}\tau_k. \tag{5.2.39}$$

Via (5.2.36)-(5.2.38), we also provide a closer contact to the holonomies of the full theory, which are also $SU(2)$-valued. The action of the $SU(2)$-holonomies on abstract Gowdy states follows from expressing the occurring sines and cosines by complex exponentials $\exp\left(i\frac{k_0}{2}\int_I\mathcal{A}\right)$, $\exp\left(i\frac{\mu_0}{2}X_i\right)$, and $\exp\left(i\frac{\nu_0}{2}Y_i\right)$ which are exactly the holonomies introduced in (5.2.5) and (5.2.6). Hence, the $SU(2)$-holonomies are well defined on the kinematical Hilbert space. This also motivates the factors of $1/2$ we introduced in the definition of the holonomies (5.2.5) and (5.2.6). Furthermore, as stated in (5.2.30), we define the following volume functional for an arc I and a point $\theta\in I$:

$$V(I):=\sqrt{|\mathcal{E}(\theta)|\left|\int_I E^x\right|\left|\int_I E^y\right|}. \tag{5.2.40}$$

Using this and (5.2.36)-(5.2.38), one can show that the following identity holds:

$$\mathrm{Tr}\Big(\left[h_xh_yh_x^{-1}h_y^{-1}-h_yh_xh_y^{-1}h_x^{-1}\right]h_\theta\{h_\theta^{-1},V\}\Big)\approx\frac{\kappa'\gamma}{2}k_0\mu_0\nu_0\frac{\epsilon XYE^xE^y}{\sqrt{E}}(\theta), \tag{5.2.41}$$

where Tr denotes the $SU(2)$-trace and for simplicity, we left out the θ-dependence of the holonomies and fluxes and the I-dependence of the volume functional. We see from (5.2.41) that the right hand side contains exactly the same configuration variables and fluxes as (5.2.35) and thus we found a way to approximate the Euclidean operator by holonomies and the volume. We can prove (5.2.41) in three steps:

- Firstly, we rewrite the expression in square brackets. Plugging in the definitions (5.2.37)

and (5.2.38) of the holonomies h_x and h_y while suppressing any θ- or I-dependence gives

$$\begin{aligned} h_x h_y h_x^{-1} h_y^{-1} - h_y h_x h_y^{-1} h_x^{-1} =& 8 \sin\left(\frac{\mu_0}{2} X\right) \cos\left(\frac{\mu_0}{2} X\right) \left[\sin\left(\frac{\nu_0}{2} Y\right)\right]^2 \tau_1 - \\ &- 8 \sin\left(\frac{\nu_0}{2} Y\right) \cos\left(\frac{\nu_0}{2} Y\right) \left[\sin\left(\frac{\mu_0}{2} X\right)\right]^2 \tau_2 + \\ &+ 8 \sin\left(\frac{\mu_0}{2} X\right) \cos\left(\frac{\mu_0}{2} X\right) \sin\left(\frac{\nu_0}{2} Y\right) \cos\left(\frac{\nu_0}{2} Y\right) \tau_3. \end{aligned} \tag{5.2.42}$$

Then, we evaluate this expression for small X, Y, which allows us to approximate to first order the sine-functions by their argument and the cosine-functions by 1. This gives

$$h_x h_y h_x^{-1} h_y^{-1} - h_y h_x h_y^{-1} h_x^{-1} \approx \mu_0 \nu_0 X Y^2 \tau_1 - \mu_0^2 \nu_0^2 X^2 Y \tau_2 + 2 \mu_0 \nu_0 X Y \tau_3. \tag{5.2.43}$$

- Secondly, we use an identity that is in analogy to the Thiemann identities we introduced in section 2.2.2:

$$h_\theta(I)\{h_\theta^{-1}(I), V(I)\} = -\frac{\kappa' \gamma}{2} \tau_3 k_0 \frac{\int_I E^x \int_I E^y}{V(I)} \approx -\frac{\kappa' \gamma}{2} \epsilon \tau_3 k_0 \frac{E^x E^y}{\sqrt{E}}(\theta), \tag{5.2.44}$$

where θ is a point in the interval I. The first step of this identity can be shown by computing the functional derivatives of h_θ^{-1} with respect to $\mathcal{A}$ and of V with respect to $\mathcal{E}$. In the second step, we approximated the fluxes and the volume by

$$\begin{aligned} &\int_I E^x \approx \epsilon E^x(\theta), \quad \int_I E^y \approx \epsilon E^y(\theta), \\ &V(I) = \sqrt{|\mathcal{E}(\theta)| \left|\int_I E^x\right| \left|\int_I E^y\right|} \approx \sqrt{|\mathcal{E}(\theta)||\epsilon E^x(\theta)||\epsilon E^y(\theta)|} = \epsilon \sqrt{|E(\theta)|}. \end{aligned} \tag{5.2.45}$$

- Thirdly, we multiply (5.2.43) and (5.2.44) and take the $SU(2)$-trace of the resulting expression. Finally, using (5.2.39) when computing the trace, we arrive at (5.2.41).

Similarly, we can proceed for the other two terms of the Euclidean constraint and end up with the following identities, see also [57] and [58, 59]:

$$\begin{aligned} \mathrm{Tr}\Big\{ \left[h_y h_\theta h_{y,\epsilon}^{-1} h_\theta^{-1} - h_\theta h_{y,\epsilon} h_\theta^{-1} h_y^{-1}\right] h_x \{h_x^{-1}, V\} \Big\} &\approx \frac{\kappa' \gamma}{2} k_0 \mu_0 \nu_0 \frac{\epsilon \mathcal{A} \mathcal{E} Y E^y}{\sqrt{E}}, \\ \mathrm{Tr}\Big\{ \left[h_\theta h_{x,\epsilon} h_\theta^{-1} h_x^{-1} - h_x h_\theta h_{x,\epsilon}^{-1} h_\theta^{-1}\right] h_y \{h_y^{-1}, V\} \Big\} &\approx \frac{\kappa' \gamma}{2} k_0 \mu_0 \nu_0 \frac{\epsilon \mathcal{A} \mathcal{E} X E^x}{\sqrt{E}}, \end{aligned} \tag{5.2.46}$$

where $h_{x,\epsilon} := h_x(\theta + \epsilon)$ and similarly for $h_{y,\epsilon}$ and the inverse of these holonomies. We also state the other Thiemann identities necessary for deriving (5.2.46) as we will need them later on for the Lorentzian part:

$$\begin{aligned} h_x(\theta)\{h_x^{-1}(\theta), V(I)\} &\approx -\frac{\kappa' \gamma}{2} \mu_0 \tau_1 \frac{E^y \mathcal{E}}{\sqrt{E}}(\theta), \\ h_y(\theta)\{h_y^{-1}(\theta), V(I)\} &\approx -\frac{\kappa' \gamma}{2} \nu_0 \tau_2 \frac{E^x \mathcal{E}}{\sqrt{E}}(\theta). \end{aligned} \tag{5.2.47}$$

We can now rewrite the Euclidean constraint (5.1.30) using the identities (5.2.41) and (5.2.46). In particular, we can directly state the operator of the Euclidean constraint, as the identities (5.2.41) and (5.2.46) all contain quantities that are well defined operators on the kinematical Hilbert space, namely the volume and the $SU(2)$-holonomies. We also need to replace Poisson brackets by commutators via $\{.,.\} \to \frac{1}{i\hbar}[.,.]$ and finally arrive at the operator

$$\begin{aligned}\hat{\mathrm{H}}_{\mathrm{eucl}} = i\frac{2}{\kappa'\gamma^3\mu_0\nu_0 l_{\mathrm{P}}^2}\sum_{i=1}^{N}\mathrm{Tr}\Big(&\left[\hat{h}_x\hat{h}_y\hat{h}_x^{-1}\hat{h}_y^{-1}-\hat{h}_y\hat{h}_x\hat{h}_y^{-1}\hat{h}_x^{-1}\right]\hat{h}_\theta[\hat{h}_\theta^{-1},\hat{V}_i]+\\&+\left[\hat{h}_y\hat{h}_\theta\hat{h}_{y,\epsilon}^{-1}\hat{h}_\theta^{-1}-\hat{h}_\theta\hat{h}_{y,\epsilon}\hat{h}_\theta^{-1}\hat{h}_y^{-1}\right]\hat{h}_x[\hat{h}_x^{-1},\hat{V}_i]+\\&+\left[\hat{h}_\theta\hat{h}_{x,\epsilon}\hat{h}_\theta^{-1}\hat{h}_x^{-1}-\hat{h}_x\hat{h}_\theta\hat{h}_{x,\epsilon}^{-1}\hat{h}_\theta^{-1}\right]\hat{h}_y[\hat{h}_y^{-1},\hat{V}_i]\Big),\end{aligned} \tag{5.2.48}$$

where we again suppressed the θ- and I_i-dependence of the holonomies and the volume.

In the following, we choose the arbitrary numbers $k_0 \in \mathbb{Z}$, $\mu_0, \nu_0 \in \mathbb{R}$ appearing in the previous formulas to be 1, which lightens our notation a little bit more. This choice will become clearer when seeing the final form of the Hamiltonian constraint operator. We will discuss the choice $k_0 = \mu_0 = \nu_0 = 1$ in more detail at the end of the section.

We can lift (5.2.48) directly to the algebraic framework: The $SU(2)$-holonomies can be expressed in terms of the holonomies in (5.2.5) and (5.2.6) which themselves have well defined equivalents in AQG, cf. (5.2.18)-(5.2.20). Also, the volume operator can be defined in AQG, cf. (5.2.33).

We will, however, not use the AQG version of (5.2.48), as we want to calculate the action of the Euclidean constraint operator for which there exists a more convenient form. To derive this form, we perform the following steps:

- For the holonomies in square brackets, we use the expansion (5.2.42), which is also true at the operator level.
- For the remaining part, we write out the commutator and use the definition of the $SU(2)$-holonomy h_θ in (5.2.36) to arrive at

$$\begin{aligned}\hat{h}_\theta[\hat{h}_\theta^{-1},\hat{V}_i] =&\hat{V}_i-\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)-\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)+\\&+2\tau_3\left[\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)-\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\right].\end{aligned} \tag{5.2.49}$$

  Note that the appearing sines and cosines actually are multiplication operators, i.e., they can be expressed as complex exponentials, which are exactly the holonomies in (5.2.18)-(5.2.20).

- Multiplying (5.2.49) with the expansion (5.2.42) and evaluating the trace using the identities for τ_i in (5.2.39), yields the following expression for the first term of (5.2.48):

$$\begin{aligned}\mathrm{Tr}\left(\left[\hat{h}_x\hat{h}_y\hat{h}_x^{-1}\hat{h}_y^{-1}-\hat{h}_y\hat{h}_x\hat{h}_y^{-1}\hat{h}_x^{-1}\right]\hat{h}_\theta[\hat{h}_\theta^{-1},\hat{V}_i]\right) =& -2\sin(X_i)\sin(Y_i)\times\\&\times\left[\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)-\sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i\cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\right].\end{aligned} \tag{5.2.50}$$

It is now convenient to introduce the operator

$$\hat{O}^{\theta}_{1,i} := \cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i \sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right) - \sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}_i \cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right), \tag{5.2.51}$$

which allows us to write (5.2.50) as

$$-2\sin(X_i)\sin(Y_i)\hat{O}^{\theta}_{1,i}. \tag{5.2.52}$$

Similarly to (5.2.52), we can rewrite the remaining terms of (5.2.48):

$$\begin{aligned}
\mathrm{Tr}\Big(\Big[\hat{h}_y\hat{h}_\theta\hat{h}^{-1}_{y,\epsilon}\hat{h}^{-1}_\theta - \hat{h}_\theta\hat{h}_{y,\epsilon}\hat{h}^{-1}_\theta\hat{h}^{-1}_y\Big]\,\hat{h}_x[\hat{h}^{-1}_x,\hat{V}_i]\Big) &= -4\sin\left(\frac{1}{2}Y_{i+1}\right)\cos\left(\frac{1}{2}Y_i\right)\sin(\mathcal{A}_{e_i})\hat{O}^{x}_{1,i},\\
\mathrm{Tr}\Big(\Big[\hat{h}_\theta\hat{h}_{x,\epsilon}\hat{h}^{-1}_\theta\hat{h}^{-1}_x - \hat{h}_x\hat{h}_\theta\hat{h}^{-1}_{x,\epsilon}\hat{h}^{-1}_\theta\Big]\,\hat{h}_y[\hat{h}^{-1}_y,\hat{V}_i]\Big) &= -4\sin\left(\frac{1}{2}X_{i+1}\right)\cos\left(\frac{1}{2}X_i\right)\sin(\mathcal{A}_{e_i})\hat{O}^{y}_{1,i},
\end{aligned} \tag{5.2.53}$$

where we introduced the following family of operators:

$$\hat{O}^{x}_{\alpha,i} = \cos\left(\frac{1}{2}X_i\right)\hat{V}^{\alpha}_i \sin\left(\frac{1}{2}X_i\right) - \sin\left(\frac{1}{2}X_i\right)\hat{V}^{\alpha}_i \cos\left(\frac{1}{2}X_i\right), \tag{5.2.54}$$

$$\hat{O}^{y}_{\alpha,i} = \cos\left(\frac{1}{2}Y_i\right)\hat{V}^{\alpha}_i \sin\left(\frac{1}{2}Y_i\right) - \sin\left(\frac{1}{2}Y_i\right)\hat{V}^{\alpha}_i \cos\left(\frac{1}{2}Y_i\right), \tag{5.2.55}$$

$$\hat{O}^{\theta}_{\alpha,i} = \cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}^{\alpha}_i \sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right) - \sin\left(\frac{1}{2}\mathcal{A}_{e_i}\right)\hat{V}^{\alpha}_i \cos\left(\frac{1}{2}\mathcal{A}_{e_i}\right), \tag{5.2.56}$$

with α a real number which is 1 for (5.2.52) and (5.2.53). We keep α arbitrary in (5.2.54)-(5.2.56) as we need this later on for the Lorentzian part, see (5.2.64).

Finally, using (5.2.52) and (5.2.53), we can rewrite (5.2.48) in AQG as

$$\begin{aligned}
\hat{\mathrm{H}}_{\mathrm{eucl}} = \sum_{i\in V(\alpha)} \hat{\mathrm{H}}_{\mathrm{eucl,i}} = -i\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\Bigg[&\sin(X_i)\sin(Y_i)\,\hat{O}^{\theta}_{1,i} + 2\sin\left(\frac{1}{2}Y_{i+1}\right)\cos\left(\frac{1}{2}Y_i\right)\sin(\mathcal{A}_{e_i})\,\hat{O}^{x}_{1,i}\\
&+2\sin\left(\frac{1}{2}X_{i+1}\right)\cos\left(\frac{1}{2}X_i\right)\sin(\mathcal{A}_{e_i})\,\hat{O}^{y}_{1,i}\Bigg].
\end{aligned} \tag{5.2.57}$$

The advantage of this form is that we can derive the action of the Euclidean part on an abstract Gowdy state much easier. This finishes our discussion of the quantization of the Euclidean constraint operator in AQG.

Quantization of the Lorentzian part: We consider exemplarily the first term of the Lorentzian part (5.1.32), which we can rewrite as a Riemann sum in analogy to (5.2.35):

$$\mathrm{H}^{(1)}_{\mathrm{lor}} - \frac{1}{\kappa'}\int_{S^1}\mathrm{d}\theta\left[\frac{(\partial_\theta\mathcal{E})^2}{4\sqrt{E}}\right] \approx -\frac{1}{4\kappa'}\sum_{i=1}^{N}\left[\frac{(\epsilon\partial_\theta\mathcal{E}(\theta_i))^2}{\sqrt{\epsilon^2 E(\theta_i)}}\right] = -\frac{1}{4\kappa'}\sum_{i=1}^{N}\left[\frac{(\mathcal{E}(\theta_i+\epsilon)-\mathcal{E}(\theta_i))^2}{V(I_i)}\right], \tag{5.2.58}$$

where in the second step, we used $\mathcal{E}(\theta_i+\epsilon)\approx\mathcal{E}(\theta_i)+\epsilon\partial_\theta\mathcal{E}(\theta_i)$ and (5.2.45) to introduce the volume $V(I_i)$. In (5.2.58), we can quantize the flux $\mathcal{E}$ using (5.2.13), but need to come up with

a strategy to replace the inverse volume: Following [57], we can use the Thiemann identities (5.2.44) and (5.2.47) to show the following classical identity:

$$Z(I) := \epsilon^{abc}\mathrm{Tr}\left[h_a\{h_a^{-1}, V(I)\}h_b\{h_b^{-1}, V(I)\}h_c\{h_c^{-1}, V(I)\}\right] = \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3 \mu_0\nu_0 V(I). \quad (5.2.59)$$

Using this and $h\{h^{-1}, V^\alpha\} = \alpha V^{\alpha-1}h\{h^{-1}, V\}$, with $\alpha \in \mathbb{R}$ arbitrary at this point, we can show another classical identity:

$$\begin{aligned} Z_\alpha(I) := \epsilon^{abc}\mathrm{Tr}\left[h_a\{h_a^{-1}, (V(I))^\alpha\}h_b\{h_b^{-1}, (V(I))^\alpha\}h_c\{h_c^{-1}, (V(I))^\alpha\}\right] = \\ = \frac{3}{2}\left(\frac{\kappa'\gamma}{2}\right)^3 \mu_0\nu_0\alpha^3 \left(V(I)\right)^{3\alpha-2} = \alpha^3 \left(V(I)\right)^{3\alpha-3} Z(I). \end{aligned} \quad (5.2.60)$$

This identity allows us to replace the inverse volume in (5.2.58) by inserting a one of the following form:

$$(1)^l = \left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0}\cdot\frac{Z(I)}{V(I)}\right)^l = \left(\frac{16}{3(\kappa'\gamma)^3\mu_0\nu_0}\right)^l \cdot \left(\frac{Z_\alpha(I)}{\alpha^3(V(I))^{3\alpha-2}}\right)^l, \quad (5.2.61)$$

where $l \in \mathbb{R}$ is arbitrary. In the first step, we used (5.2.59) and in the second step we used (5.2.60) to express $Z(I)$ by $Z_\alpha(I)$ and powers of the volume. To cancel the inverse volume in (5.2.58), we impose $(3\alpha - 2)\cdot l \overset{!}{=} -1$, which is equivalent to the condition

$$\alpha \overset{!}{=} \frac{2}{3} - \frac{1}{3l}. \quad (5.2.62)$$

This way, we have introduced a quantization ambiguity. In summary, we can rewrite the first term of the Lorentzian part in (5.2.58) as

$$-\frac{1}{4\kappa'}\left(\frac{16}{3(\kappa'\gamma\alpha^3)^3}\right)^l \sum_{i=1}^{N}\left[\left(\mathcal{E}(\theta_i+\epsilon) - \mathcal{E}(\theta_i)\right)^2 \left(Z_\alpha(I_i)\right)^l\right]. \quad (5.2.63)$$

We see from this expression that the inverse volume no longer occurs, but is replaced by the quantity $\left(Z_\alpha(I_i)\right)^l$, which motivates the introduction of the classical identities (5.2.59) and (5.2.60). All operators occurring in (5.2.63) can be quantized using the flux and holonomy operators as also the introduced quantities $Z_\alpha(I_i)$ contain only the holonomies and the volume, cf. (5.2.60). We directly go over to the AQG framework, for which we identify in (5.2.63) $\theta_i + \epsilon$ with the next vertex and denote the operators corresponding to $Z_\alpha(I_i)$ as $\hat{Z}_{\alpha,i}$. We can also express these operators in terms of the family of operators in (5.2.56), as was introduced in [57] for LQG:

$$\hat{Z}_{\alpha,i} := \epsilon^{abc}\mathrm{Tr}\left(\hat{h}_a[\hat{h}_a^{-1}, \hat{V}_i]\hat{h}_b[\hat{h}_b^{-1}, \hat{V}_i]\hat{h}_c[\hat{h}_c^{-1}, \hat{V}_i]\right) = -12\hat{O}^x_{\alpha,i}\hat{O}^y_{\alpha,i}\hat{O}^\theta_{\alpha,i} \quad (5.2.64)$$

To prove this identity, we expand the operators $\hat{h}_a$ using (5.2.36)-(5.2.38) and evaluate the trace using (5.2.39). Furthermore, we use that diagonal operators commute, which is the case for the family of operators in (5.2.56), as we show in section 6.1.1. Finally, we can write down the operator corresponding to the first term of the Lorentzian part, which we denote by

$$\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}} = \sum_{i=1}^{N}\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i} := -\frac{1}{4\kappa'}\left(-\frac{i}{\hbar}\right)^{3l}\left(\frac{16}{3(\kappa'\gamma)^3\alpha^3}\right)^l \left(\hat{\mathcal{E}}_{i+1} - \hat{\mathcal{E}}_i\right)\left(\hat{Z}_{\alpha,i}\right)^l. \quad (5.2.65)$$

Similarly, we can proceed for the two remaining terms of the Lorentzian part, yielding the Lorentzian constraint operator

$$\hat{\mathrm{H}}_{\mathrm{lor}} = \sum_{i=1}^{N} \left[\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i} + \hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i} + \hat{\mathrm{H}}^{(3)}_{\mathrm{lor},i} \right], \tag{5.2.66}$$

with $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$ defined in (5.2.65) and the other two parts given by

$$\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i} = \frac{1}{4\kappa'} \left(-\frac{i}{\hbar} \right)^{3l} \left(\frac{16}{3(\kappa'\gamma)^3 \alpha^3} \right)^{l} \left(\hat{\mathcal{E}}_i \right)^4 \left[\hat{\mathcal{F}}_{x,e_i} \hat{\mathcal{F}}_{y,e_{i+1}} - \hat{\mathcal{F}}_{y,e_i} \hat{\mathcal{F}}_{x,e_{i+1}} \right]^2 \left(\hat{Z}_{\beta,i} \right)^{l} \tag{5.2.67}$$

and

$$\hat{\mathrm{H}}^{(3)}_{\mathrm{lor},i} = \frac{1}{\kappa'} \left(-\frac{i}{\hbar} \right)^{3l} \left(\frac{16}{3(\kappa'\gamma)^3 \alpha^3} \right)^{l} \left[\hat{\mathcal{E}}_{i+1} \left(\hat{\mathcal{E}}_{i+2} - \hat{\mathcal{E}}_{i+1} \right) \left(\hat{Z}_{\alpha,i+1} \right)^{l} - \hat{\mathcal{E}}_i \left(\hat{\mathcal{E}}_{i+1} - \hat{\mathcal{E}}_i \right) \left(\hat{Z}_{\alpha,i} \right)^{l} \right]. \tag{5.2.68}$$

Note that for $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$, we deviate from the quantization strategy presented in [57], but proceed as was done in [58, 59]: Looking at the classical expression in (5.1.33), we see that the inverse of E^x and E^y appears in the factor

$$\frac{\partial_\theta E^x}{E^x} - \frac{\partial_\theta E^y}{E^y}. \tag{5.2.69}$$

To quantize this expression, we may introduce inverse fluxes as done on page 17 of [57], or just rewrite the inverse of E^x and E^y as

$$\frac{1}{E^x} = \frac{E^y \mathcal{E}}{E^y \mathcal{E} E^x} = \frac{E^y \mathcal{E}}{(\sqrt{E})^2} \quad \text{and} \quad \frac{1}{E^y} = \frac{E^x \mathcal{E}}{E^x \mathcal{E} E^y} = \frac{E^x \mathcal{E}}{(\sqrt{E})^2}. \tag{5.2.70}$$

This way, we end up with a different power of the inverse volume, which we can eliminate by inserting a one of the form (5.2.61) giving us a parameter β, which is different from α and related to l via $\beta = \frac{2}{3} - \frac{5}{3l}$. Then, the inverse volume in $\mathrm{H}^{(2)}_{\mathrm{lor}}$ can be substituted by the quantity $Z_\beta(I)$, which is defined just like $Z_\alpha(I)$ in (5.2.60) with α replaced by β. We finally arrive at the operator (5.2.67).

Before we continue in the next chapter with a discussion of the Gauß constraint, we want to make two remarks:

- Firstly, we want to comment on the choice $k_0 = \mu_0 = \nu_0 = 1$ in the Euclidean and Lorentzian operators (5.2.57), (5.2.65)-(5.2.68). The main motivation for this is that we want to simplify our notation as the action of the Hamiltonian constraint becomes quite complicated, see section 6.1. Note, however, that in general, $k_0 \in \mathbb{Z}$ and $\mu_0, \nu_0 \in \mathbb{R}$ could be chosen arbitrarily. We may perceive this freedom of choice in analogy to the full theory: When acting with the Hamiltonian constraint of the full theory on a spin network function, the loop holonomy changes the graph underlying the spin network function. The representation of this loop holonomy can in principle be chosen arbitrarily, in particular one chooses it to be the spin 1/2 representation, see [27, 143] for a discussion. This choice is motivated by the results of [144], which indicate that higher spin representations lead to spurious solutions for three spacetime dimensions which hence presumably also holds in four dimensions. Therefore, this ambiguity seems to be fixed in the full theory, while for the Gowdy model and similarly for the LQC models, we do not have such guidance to fix $k_0 \in \mathbb{Z}$ and $\mu_0, \nu_0 \in \mathbb{R}$.

- Secondly, we want to discuss the regularization chosen in [57] and its implications. This will emphasize the simplifications our transition from LQG to AQG provides, especially for calculating the action of the Hamiltonian constraint operator $\hat{\mathrm{H}}$ and investigating the Schrödinger equation (5.2.4). In [57], the regularization is chosen such that each interval I_i of the partition of S^1 with length ϵ contains *at most one vertex*[11]. This has two main consequences:

  - On the one hand, when calculating the action of $\hat{\mathrm{H}}$, for example the operator $\hat{\mathrm{H}}^{(3)}_{\mathrm{lor},i}$ stated in (5.2.68) acts differently in LQG and AQG. The LQG version of (5.2.68) is obtained by replacing all operators defined for vertices and edges of the algebraic graph α by operators defined for points $\theta_i \in I_i$ and intervals I_i of the chosen regularization of the embedded graph γ. When calculating the action, one considers the infinite refinement limit, i.e., one takes the limit $\epsilon \to 0$ which sends the number of intervals I_i to infinity. The regularization of [57] with at most one vertex contained in each regularization interval now implies that if a vertex is contained in the interval I_i, then there will be no vertex in the neighboring intervals. We now have that fluxes act trivially on intervals that contain no vertex, cf. (5.2.16) and (5.2.17). In the LQG version of (5.2.68), the operators $\hat{Z}_{\alpha,i}$ and $\hat{Z}_{\alpha,i+1}$ both contain fluxes and hence the former operator acts non-trivially, while the latter operator acts trivially as it is defined for the interval I_{i+1} where no vertex is contained in the infinite refinement limit. This, however, is not the case in AQG, where we have an abstract graph and do not perform any limit $\epsilon \to 0$ and hence both operators $\hat{Z}_{\alpha,i}$ and $\hat{Z}_{\alpha,i+1}$ in (5.2.68) contribute non-trivially. This change of the action has to be considered but does not cause any trouble.

  - On the other hand, the Hamiltonian constraint operator $\hat{\mathrm{H}}$ will act graph-changing in LQG, see [57], which results in a more involved action compared to AQG. To see this, we consider the graph-changing part of the LQG Hamiltonian constraint, which is called $\hat{H}_K$ in [57]. This operator can be derived from the AQG Euclidean operator (5.2.57) by replacing all operators defined at vertices and edges by the same operators defined at points $\theta_i \in I_i$ and intervals I_i of the regularization. Hence, the LQG operator will contain point holonomies defined for a point $\theta_i \in I_i$, which is in general *not* a vertex, as well as $U(1)$ holonomies defined for intervals I_i, which do *not* coincide with the edges of the graph. Hence, when calculating the action of this operator on a Gowdy state $|\gamma, k, \mu, \nu\rangle$, new vertices and edges are created and hence the underlying graph γ is changed. In AQG, however, the points θ_i and intervals I_i for which the holonomies are defined, respectively, coincide with vertices and edges of the abstract graph α, cf. (5.2.57), and thus the AQG operator acts only label-changing but graph-preserving. We will present the action of the AQG operator in section 6.1.1. The graph-changing action of the LQG operator has in particular consequences when we later want to solve the time-independent Schrödinger equation

    $$\hat{\mathrm{H}}_{\mathrm{phys}}\ \varphi = E\ \varphi, \tag{5.2.71}$$

    where $\hat{\mathrm{H}}_{\mathrm{phys}}$ is a symmetric combination of the Hamiltonian constraint operator, see section 6, and for φ we will choose as an ansatz a linear combination of Gowdy states with coefficients labeled by the representation labels $k_i, \mu_i, \nu_i, i = 1, ..., N$ for N the number of vertices of the embedded graph γ. Furthermore, E plays the

[11] We may also choose different regularizations, see section V of [57] for a discussion.

role of the energy. As $\hat{\mathrm{H}}_{\mathrm{phys}}$ acts graph-changing, the linear combination φ has to include a sum over all graphs γ, which is an uncountably infinite sum and hence results in a non-normalizable state. We may handle this by considering the solution states as distributional in analogy to the discussion in section 6.3.2 below (6.3.46). The problem that remains, however, is that the LQG Hamiltonian constraint acts in a much more complicated way as in AQG. This is undesirable especially when we later on want to derive difference equations from the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$, see section 6.3.1.

5.2.5 The Gauß constraint

In this section, we want to discuss the solution of the Gauß constraint at the quantum level. So far, we solved the Gauß constraint already at the classical level and thus, the configuration variable η does not appear in the Gowdy states and the constraints and neither does the canonically conjugate momentum P^η.

Now, we want to discuss the implications if we do not eliminate the canonical pair (η, P^η) at the classical level. In this case, we introduce a point holonomy for the angular variable $\eta \in \mathbb{R}/\mathbb{Z}$ defined by [57]

$$h_v^\lambda(\eta) = \exp\left(i\lambda\eta(v)\right), \tag{5.2.72}$$

with $\lambda \in \mathbb{Z}$ and hence $h_v^\lambda(\eta) \in U(1)$. As this holonomy is defined at each vertex and hence embedding independent, the generalization to AQG is straightforward, giving the AQG holonomy we may denote by $h_i^\lambda(\eta)$ in analogy to (5.2.19) and (5.2.20). The holonomy corresponding to η also has to be included in the definition of the Gowdy states $|\gamma, k, \mu, \nu\rangle$ in (5.2.10) and abstract Gowdy states $|k, \mu, \nu\rangle$ in (5.2.21), resulting in the states $|\gamma, k, \mu, \nu, \lambda\rangle$ and $|k, \mu, \nu, \lambda\rangle$, respectively. The action of the LQG or rather AQG holonomies $h_v^\lambda(\eta)$ and $h_i^\lambda(\eta)$ can be calculated in analogy to (5.2.11) and (5.2.12) for LQG and (5.2.18)-(5.2.20) for AQG. We also need to define the flux conjugate to η, which is in LQG given by the smeared quantity $\mathcal{F}_{\eta,I_i} := \int_{I_i} P^\eta$, which is implemented as a derivative operator analogous to (5.2.13) and acts as [57]

$$\hat{\mathcal{F}}_{\eta,I_i} |\gamma, k, \mu, \nu, \lambda\rangle = \gamma l_{\mathrm{P}}^2 \sum_{v \in V(\gamma) \cap I} \lambda_v |\gamma, k, \mu, \nu, \lambda\rangle . \tag{5.2.73}$$

To define the AQG equivalent of this flux, we proceed just as for the AQG fluxes $\hat{\mathcal{F}}_{x,e_i}$ and $\hat{\mathcal{F}}_{y,e_i}$ in (5.2.26): We define the quantity $\hat{\mathcal{F}}_{\eta,e_i}$, which corresponds to $\int_{e_i} \widehat{P^\eta}$ when choosing an embedding. The AQG analogue to (5.2.73) is then given by

$$\hat{\mathcal{F}}_{\eta,e_i} |k, \mu, \nu, \lambda\rangle = \gamma l_{\mathrm{P}}^2 \lambda_i |k, \mu, \nu, \lambda\rangle . \tag{5.2.74}$$

Another consequence of keeping (η, P^η) is that for the Euclidean part (5.2.57), we have to do the replacement

$$\sin\left(\mathcal{A}_{e_i}\right) \to \sin\left[\left(\mathcal{A}_{e_i}\right) - \Delta_i\right], \tag{5.2.75}$$

where we introduced the abbreviation $\Delta_i := \eta_i - \eta_{i+1} := \eta(v_i) - \eta(v_{i+1})$ to stay close to [57].

Now, we consider the quantization of the Gauß constraint in LQG following [57]. First, we regularize the Gauß constraint $\mathrm{G} = \int_{S^1} \mathrm{d}\theta \, \left(P^\eta + \partial_\theta \mathcal{E}\right)$ by

$$\mathrm{G} \approx \sum_{i=1}^{N} \left[\mathcal{F}_{\eta,I_i} + \mathcal{E}(\theta_i + \epsilon) - \mathcal{E}(\theta_i)\right] . \tag{5.2.76}$$

This can be quantized using (5.2.13) for $\mathcal{E}(\theta_i)$ and $\mathcal{E}(\theta_i+\epsilon)$ and the flux operator $\hat{\mathcal{F}}_{\eta,I_i} := \widehat{\int_{I_i} P^\eta}$, yielding the constraint operator

$$\hat{\mathrm{G}} \approx \sum_{i=1}^{N} \left[\hat{\mathcal{F}}_{\eta,I_i} + \hat{\mathcal{E}}(\theta_i+\epsilon) - \hat{\mathcal{E}}(\theta_i)\right]. \tag{5.2.77}$$

To solve the LQG Gauß constraint, we have to calculate its action on a Gowdy state $|\gamma, k, \mu, \nu\rangle$. For this, we have to consider the infinite refinement limit. If a vertex is contained in the interval I_i, this will not be the case for the neighboring intervals. In particular, $\theta_i+\epsilon$ and θ_i split up the same edge and hence, suppressing prefactors, $\mathcal{E}(\theta_i+\epsilon)$ results in $k_{e^+}(v)/2$ and $\mathcal{E}(\theta_i+\epsilon)$ in $k_{e^-}(v)/2$. Combining this with the action (5.2.73) gives the following action of $\hat{\mathrm{G}}$ [57]:

$$\hat{\mathrm{G}}\,|\gamma, k, \mu, \nu, \lambda\rangle = \gamma l_{\mathrm{P}}^2 \sum_{v\in V(\gamma)} \left[\lambda_v + \frac{k_{e^+}(v) - k_{e^-}(v)}{2}\right] |\gamma, k, \mu, \nu, \lambda\rangle \tag{5.2.78}$$

In [57], the authors suggest to annihilate the Gauß constraint in this LQG scenario by imposing at each vertex the condition

$$\lambda_v = -\frac{k_{e^+}(v) - k_{e^-}(v)}{2}. \tag{5.2.79}$$

An alternative to this vertex-wise solution would be to annihilate the sum over all vertices in (5.2.78). For this, we recognize that

$$\sum_{v\in V(\gamma)} \frac{k_{e^+}(v) - k_{e^-}(v)}{2} = 0 \tag{5.2.80}$$

due to the periodic boundary conditions on the labels we have for the Gowdy model, i.e., we identify the first and the $(n+1)$th vertex for $n = |V(\gamma)|$. Hence, we can deduce the condition

$$\sum_{v\in V(\gamma)} \left[\lambda_v + \frac{k_{e^+}(v) - k_{e^-}(v)}{2}\right] = \sum_{v\in V(\gamma)} \lambda_v \stackrel{!}{=} 0. \tag{5.2.81}$$

We now want to quantize the Gauß constraint in the AQG-framework. For this, we identify in (5.2.77) θ_i with the vertex i and $\theta_i+\epsilon$ with the vertex $i+1$ of the abstract graph α. Furthermore, we use the AQG operator (5.2.25) for $\mathcal{E}$ and the AQG equivalent of $\hat{\mathcal{F}}_{\eta,I_i}$ given by $\hat{\mathcal{F}}_{\eta,e_i}$. This way, we arrive at the AQG Gauß constraint operator:

$$\hat{\mathrm{G}} \approx \sum_{i=1}^{N} \left[\hat{\mathcal{F}}_{\eta,e_i} + \hat{\mathcal{E}}_{i+1} - \hat{\mathcal{E}}_i\right]. \tag{5.2.82}$$

To solve the AQG Gauß constraint, we consider its action which we obtain using (5.2.74) and (5.2.25):

$$\hat{\mathrm{G}}\,|k, \mu, \nu, \lambda\rangle = \gamma l_{\mathrm{P}}^2 \sum_{i=1}^{N} \left[\lambda_i + \frac{k_{i+1} - k_i}{4} - \frac{k_i + k_{i-1}}{4}\right] |k, \mu, \nu, \lambda\rangle = \left[\lambda_i + \frac{k_{i+1} - k_{i-1}}{4}\right] |k, \mu, \nu, \lambda\rangle. \tag{5.2.83}$$

From this, we can deduce the condition

$$\lambda_i = -\frac{k_{i+1} - k_{i-1}}{4}, \tag{5.2.84}$$

which annihilates the Gauß constraint vertex-wise. The analogue of (5.2.81) in AQG is given by

$$\sum_{i=1}^{N} \lambda_i = 0. \tag{5.2.85}$$

In (5.2.79) and (5.2.84), the impact of the differences in the action of the fluxes in AQG and LQG manifests.

Although the conditions (5.2.79) and (5.2.81) for annihilating the Gauß constrain operator in LQG and (5.2.84),(5.2.85) in AQG are not complicated, we solve the Gauß constraint already at the classical level. One reason is that η is pure gauge, as we discussed in the context of (5.1.28), but the main reason is that due to (5.2.75) holonomies with respect to η are contained in the Euclidean part of the Hamiltonian constraint operator and hence, if we act with this operator on an abstract Gowdy state $|k, \mu, \nu\rangle$, the η degrees of freedom of the Gowdy state will change. This will complicate the action and in particular, we later want to derive difference equations from the Euclidean part in section 6.3 and the inclusion of η degrees of freedom will complicate our analysis. Hence, we solve the Gauß constraint at the classical level and work in the phase space reduced with respect to the Gauß constraint, where the canonical pair (η, P^η) can be dropped.

This ends our discussion of the algebraic loop quantization of the three-torus Gowdy model and we will now proceed with the discussion of the Schrödinger equation in the next section.

6 The Schrödinger equation of the polarized three-torus Gowdy model

In the following sections, we investigate the Schrödinger equation of the polarized three-torus Gowdy model.

First, we want to state the theoretical framework on which the subsequent analysis is based. To write down the Schrödinger equation of the Gowdy model, we use Gaussian dust as a reference field at the classical level to obtain a physical Hamiltonian, see [47,48] and section 4. This classical physical Hamiltonian is equivalent to the Hamiltonian constraint, which we can loop quantize using [57]. As presented in section 5.2, we also apply the techniques of algebraic quantum gravity (AQG) developed in [49–52] to [57]. This way, we can remove the graph dependence in the Gowdy states, which allows to perform a semiclassical analysis of the Gowdy model. This, however, will not be done in this thesis. In particular, using the AQG framework, we obtain a graph-preserving action of the Hamiltonian constraint. This was investigated before in two master theses: In [58], the results of [57] and the AQG framework were applied to the physical Hamiltonian of the Brown-Kuchař dust model [145]. In [59], the same techniques were applied to construct the Master constraint operator of the Gowdy model. Combining [57–59] and using Gaussian dust as a reference field, we arrive at the Schrödinger equation of the Gowdy model, whose constituents we will explain in the following paragraphs.

The time dependent Schrödinger equation reads as

$$i\hbar\partial_\tau \Psi(\tau) = \hat{\mathrm{H}}_{\mathrm{phys}}\Psi(\tau), \tag{6.0.1}$$

with the state $\Psi(\tau)$ depending on the physical time τ and its time evolution with respect to τ being generated by the physical Hamiltonian $\hat{\mathrm{H}}_{\mathrm{phys}}$. We will specify the explicit form of $\Psi(\tau)$ and $\hat{\mathrm{H}}_{\mathrm{phys}}$ in the following two paragraphs.

For the physical Hamiltonian $\hat{\mathrm{H}}_{\mathrm{phys}}$, we could choose the quantization of (5.2.1), which is equivalent to the geometric part of the Hamiltonian constraint $\hat{\mathrm{H}}$ we discussed in section 5.2.4. The Hamiltonian constraint, however, is not self-adjoint and not even symmetric as we will see in section 6.1.2. Since we want the physical Hamiltonian operator to be at least symmetric, we will use the following symmetric combination of $\hat{\mathrm{H}}$:

$$\hat{\mathrm{H}}_{\mathrm{phys}} := \frac{1}{2}\left(\hat{\mathrm{H}} + \hat{\mathrm{H}}^\dagger\right) = \frac{1}{2}\left(\hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}^\dagger_{\mathrm{eucl}} + \hat{\mathrm{H}}_{\mathrm{lor}} + \hat{\mathrm{H}}^\dagger_{\mathrm{lor}}\right) = \frac{1}{2}\left(\hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}\right) + \hat{\mathrm{H}}_{\mathrm{lor}}. \tag{6.0.2}$$

In the last step, we used $\hat{\mathrm{H}}_{\mathrm{lor}} = \hat{\mathrm{H}}^\dagger_{\mathrm{lor}}$ and $\hat{\mathrm{H}}_{\mathrm{eucl}} \neq \hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}$, which we will show in section 6.1.2 and 6.1.3, respectively. This way, we have constructed a symmetric physical Hamiltonian.

For the state $\Psi(\tau)$, we separate the time dependence as $\Psi(\tau) = \chi(\tau)\cdot\varphi$ just like in standard quantum mechanics. $\chi(\tau)$ only depends on the physical time τ and not on any k-, μ- or ν-label. For φ, we consider the following formal infinite linear combination of abstract Gowdy states as an ansatz:

$$\varphi := \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\,, \tag{6.0.3}$$

where N denotes the number of vertices and k,μ,ν are multi-labels defined by $k := (k_1,...,k_N)$, $\mu := (\mu_1,...,\mu_N)$, and $\nu := (\nu_1,...,\nu_N)$. Furthermore, we introduced the arbitrary coefficients

$$C_{k,\mu,\nu} := C_{k_1,...,k_N,\mu_1,...,\mu_N,\nu_1,...,\nu_N} \tag{6.0.4}$$

that depend on all k-, μ- and ν-labels. Moreover, the μ- and ν-labels take values in the sets $m := m_1 \times ... \times m_N$ and $n := n_1 \times ... \times n_N$, respectively, which are infinite subsets of $\mathbb{R}^N$ defined by the sets

$$m_i := \{\widetilde{\mu}_i + p \mid p \in \mathbb{Z}\} \quad \text{and} \quad n_i := \{\widetilde{\nu}_i + p \mid p \in \mathbb{Z}\} \quad \forall i \in \{1, ..., N\}, \tag{6.0.5}$$

where $\widetilde{\mu}_i$ and $\widetilde{\nu}_i$ are arbitrary real numbers defined for each i. We now want to justify the definition (6.0.5), which will also explain our formal ansatz in (6.0.3). For this, we first describe how a general state in the Hilbert space corresponding to the point holonomies is defined. We restrict to a vertex v_i where we have two point holonomies $\exp(\frac{i}{2}\mu_i X_i)$ and $\exp(\frac{i}{2}\nu_i Y_i)$. As discussed below (5.2.6), these point holonomies can be perceived as irreducible representations of the Bohr compactification $\mathbb{R}_{\text{Bohr}}$, labeled by $\mu_i \in \mathbb{R}$ and $\nu_i \in \mathbb{R}$, respectively. Focusing on the X point holonomy, the Hilbert space at the vertex v_i is given by $\mathcal{H}_{v_i}^{X_i} := L_2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{Bohr}})$, which contains functions f over $\mathbb{R}_{\text{Bohr}}$ that are square integrable with respect to the Haar measure $d\mu_{\text{Bohr}}$ of $\mathbb{R}_{\text{Bohr}}$ defined by $\mu_{\text{Bohr}}(f) := \lim_{C\to\infty} \frac{1}{2C} \int_{-C}^{+C} \mathrm{d}x f(x)$. Then, the inner product on $\mathcal{H}_{v_i}^{X_i}$ is defined by

$$< f, g >:= \lim_{C\to\infty} \frac{1}{2C} \int_{-C}^{+C} \mathrm{d}x f(x)^* g(x). \tag{6.0.6}$$

A general basis state of $\mathcal{H}_{v_i}^{X_i}$ is given by $|\mu_i\rangle := \exp(\frac{i}{2}\mu_i x)$ and the inner product for two different basis states $|\mu_i\rangle$ and $|\mu_i'\rangle$ results in

$$\langle \mu_i, \mu_i' \rangle := \lim_{C\to\infty} \frac{1}{2C} \int_{-C}^{+C} \mathrm{d}x \; e^{\frac{i}{2}(\mu_i' - \mu_i)} = \delta_{\mu,\mu'}, \tag{6.0.7}$$

where $\delta_{\mu,\mu'}$ denotes the Kronecker delta. Hence, we have the completeness relation

$$\sum_{\mu_i \in m_i} |\mu_i\rangle \langle \mu_i| = \mathbb{1}_{\mathcal{H}_v^{X_i}}, \tag{6.0.8}$$

with μ_i taking values in a finite subset m_i of $\mathbb{R}$, which we may for example choose as $m_i = \{-1.2, \sqrt{2}, \pi\}$. In particular, an arbitrary state in $\mathcal{H}_{v_i}^{X_i}$ is given by the finite linear combination

$$\phi = \sum_{\mu_i \in m_i} c_{\mu_i} |\mu_i\rangle, \tag{6.0.9}$$

where c_{μ_i} denotes an arbitrary coefficient and m_i is a finite subset of $\mathbb{R}$. The latter implies that the state ϕ has finite norm and hence is an element of $\mathcal{H}_v^{X_i}$. However, our strategy to solve the Schrödinger equation, which we explain in detail in section 6.3, demands to consider formal infinite linear combinations of the type of (6.0.3), where besides the k-labels, with $k \in \mathbb{Z}^N$, also the μ- and ν-labels take values in infinite sets defined by (6.0.5). To justify this, we consider a finite linear combination of the type

$$\varphi := \sum_{k \in K^N} \sum_{\mu \in m'} \sum_{\nu \in n'} C_{k,\mu,\nu} |k, \mu, \nu\rangle, \tag{6.0.10}$$

where K is a finite subset of $\mathbb{Z}$ and m', n' are finite subsets of $\mathbb{R}^N$, e.g., $n' = m' = u \times ... \times u$ with $u := \{-1.2, \sqrt{2}, \pi\}$. We will show, however, that the physical Hamiltonian maps out of the subspace spanned by states of the form of (6.0.10). This is disadvantageous for our strategy to construct special solutions of the time-independent Schrödinger equation $\hat{\mathrm{H}}_{\text{phys}}\varphi = E\varphi$, where E plays the role of the energy. In particular, our strategy includes to derive difference

equations for the coefficients $C_{k,\mu,\nu}$ from the action of the Euclidean part. This action will give us a linear combination of states with shifted labels, see (6.1.8). We then transform the shifts in the states to the coefficients and the ansatz (6.0.10) prohibits to derive difference equations and hence we choose the infinite linear combination (6.0.3). We now want to show that the physical Hamiltonian does not leave the states (6.0.10) invariant. This is caused by the Euclidean part, while the Lorentzian part acts diagonally as we will see in section 6.1.1 and hence does preserve the subspace spanned by the states (6.0.10). For the Euclidean part, we consider exemplarily the first term, explicitly given in (5.2.57) for some arbitrary vertex i. This term contains the operator $\hat{O}^{\theta}_{1,i}$, which acts diagonally, see section 6.1.1, and thus will not map out of the subspace spanned by states of the form of (6.0.10). This, however, will happen for the operators $\sin(Y_i)$ and $\sin(X_i)$ in the first term of (5.2.57). To see this, we rewrite these operators in terms of holonomies:

$$\sin(Y_i) = \frac{1}{2i}\left(e^{i\frac{\nu_i}{2}\cdot 2Y_i} - e^{-i\frac{\nu_i}{2}\cdot 2Y_i}\right), \quad \sin(X_i) = \frac{1}{2i}\left(e^{i\frac{\mu_i}{2}\cdot 2X_i} - e^{-i\frac{\mu_i}{2}\cdot 2X_i}\right). \tag{6.0.11}$$

Now, we have for our ansatz in (6.0.10) that $\nu_i \in u = \{-1.2, \sqrt{2}, \pi\}$ and hence there occurs a state $|k, \mu, \nu\rangle$ in the linear combination of states in (6.0.10) with $\nu_i = \sqrt{2}$, while the other k-, μ- and ν-labels take some value in K and u, respectively. If we act on this state with the holonomy $\exp(i\nu_i/2 \cdot 2Y_i)$ contained in the Euclidean constraint operator via (6.0.11), the resulting state will have the label $\nu_i = \sqrt{2} + 2$ which is not contained in $u = \{-1.2, \sqrt{2}, \pi\}$. Accordingly, this state does not occur in the linear combination in (6.0.10). The same argument holds for any holonomy contained in the Euclidean constraint operator in (5.2.57). Consequently, the Euclidean constraint operator and therefore the physical Hamiltonian maps out of the subspace spanned by states of the form of (6.0.10). We can avoid this problem for the k-labels by choosing K as the whole set of positive and negative integers $\mathbb{Z}$. For the μ- and ν-labels we have to proceed differently: Considering the label ν_i, we can choose an arbitrary value $\widetilde{\nu}_i$ and construct the set n_i in which ν_i takes values as $n_i := \{\widetilde{\nu}_i + p \mid p \in \mathbb{Z}\}$. This way, the action of the holonomy $\exp(i\nu_i/2 \cdot 2Y_i)$ does not map out of the label set n_i. Similarly, we can define all other intervals m_i, n_i and we recover the definitions for the μ- and ν-label sets in (6.0.5). Finally, we arrive at the formal infinite linear combination in (6.0.3).

Now, we rewrite the time-dependent Schrödinger equation in (6.0.1) to arrive at the time-independent Schrödinger equation. First, we use the separation of the time dependence $\Psi(\tau) = \chi(\tau) \cdot \varphi$ to rewrite (6.0.1) as

$$i\hbar(\partial_\tau \chi(\tau))\varphi = (\hat{\mathrm{H}}_{\mathrm{phys}}\varphi)\chi \Leftrightarrow i\hbar\frac{\partial_\tau \chi(\tau)}{\chi} = \frac{\hat{\mathrm{H}}_{\mathrm{phys}}\varphi}{\varphi}. \tag{6.0.12}$$

Note that the assumptions $\chi \neq 0$ and $\varphi \neq 0$ enter here. Both sides of the second equation of (6.0.12) can only be the same if they equal the same constant. We denote this constant as E because it plays the role of the energy. Then, we arrive at the following two equations:

$$\partial_\tau \chi(\tau) = -\frac{i}{\hbar}E\chi(\tau), \tag{6.0.13}$$

$$\hat{\mathrm{H}}_{\mathrm{phys}}\varphi = E\varphi. \tag{6.0.14}$$

The first equation is solved by

$$\chi(\tau) = \chi(0)\exp\left(-\frac{i}{\hbar}E\tau\right). \tag{6.0.15}$$

Thus, we have, in analogy to standard quantum mechanics, reduced the problem of solving the time-dependent Schrödinger equation (6.0.1) to solving the time-independent Schrödinger equation (6.0.14). To investigate (6.0.14), we first have to specify the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}, \hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}_{\mathrm{lor}} = \hat{\mathrm{H}}^{\dagger}_{\mathrm{lor}}$, which we will do in the following chapter 6.1.

6.1 The physical Hamiltonian

In the following sections, we calculate the action of the physical Hamiltonian $\hat{\mathrm{H}}_{\mathrm{phys}}$, which is given by the action of the operators $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}_{\mathrm{lor}}$ as well as their adjoint counterparts $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{lor}}$.

6.1.1 Action of the Euclidean and Lorentzian part

In this chapter, we explain the action of the Euclidean constraint operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and the Lorentzian constraint operator $\hat{\mathrm{H}}_{\mathrm{lor}}$. The action of these operators was first stated in [57] and extended to the AQG framework in [58, 59].

We start with the action of the Euclidean constraint operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$. This operator is the sum of three terms, cf. (5.2.57). In all those terms, the operators $\hat{O}^{a}_{\alpha,i}$, with $a \in \{x, y, \theta\}$, act first. These operators act diagonally, which we show by expressing the sine and cosine operators occurring in the explicit form of $\hat{O}^{a}_{\alpha,i}$ in (5.2.56) by complex exponentials just as we did in (6.0.11). Then, we can rewrite for example $\hat{O}^{x}_{\alpha,i}$ as

$$\hat{O}^{x}_{\alpha,i} = \cos\left(\frac{1}{2}X_i\right)\hat{V}_i^{\alpha}\sin\left(\frac{1}{2}X_i\right) - \sin\left(\frac{1}{2}X_i\right)\hat{V}_i^{\alpha}\cos\left(\frac{1}{2}X_i\right) = \frac{1}{2i}\left[e^{-\frac{i}{2}X_i}\hat{V}_i^{\alpha}e^{\frac{i}{2}X_i} - e^{\frac{i}{2}X_i}\hat{V}_i^{\alpha}e^{-\frac{i}{2}X_i}\right]. \tag{6.1.1}$$

In section 5.2, we introduced the action of the volume operator and the point holonomies $e^{\pm\frac{i}{2}X_i}$ on a basis state $|k, \mu, \nu\rangle$. We use this to specify the action of $\hat{O}^{x}_{\alpha,i}$ on $|k, \mu, \nu\rangle$:

$$\hat{O}^{x}_{\alpha,i}\,|k, \mu, \nu\rangle = o \cdot \left(|\mu_i + 1|^{\frac{\alpha}{2}} - |\mu_i - 1|^{\frac{\alpha}{2}}\right)|\nu_i|^{\frac{\alpha}{2}}|k_i + k_{i-1}|^{\frac{\alpha}{2}}\,|k, \mu, \nu\rangle\,, \tag{6.1.2}$$

where we collected all constant prefactors in the constant

$$o := \frac{1}{2i}\left[\frac{1}{\sqrt{2}}\,\left(\frac{\gamma l_{\mathrm{P}}^2}{2}\right)^{\frac{3}{2}}\right]^{\alpha}. \tag{6.1.3}$$

Similarly, we can compute the action of $\hat{O}^{y}_{\alpha,i}$ and $\hat{O}^{\theta}_{\alpha,i}$:

$$\hat{O}^{y}_{\alpha,i}\,|k, \mu, \nu\rangle = o \cdot \left(|\nu_i + 1|^{\frac{\alpha}{2}} - |\nu_i - 1|^{\frac{\alpha}{2}}\right)|\mu_i|^{\frac{\alpha}{2}}|k_i + k_{i-1}|^{\frac{\alpha}{2}}\,|k, \mu, \nu\rangle\,, \tag{6.1.4}$$

$$\hat{O}^{\theta}_{\alpha,i}\,|k, \mu, \nu\rangle = o \cdot \left(|k_i + k_{i-1} + 1|^{\frac{\alpha}{2}} - |k_i + k_{i-1} - 1|^{\frac{\alpha}{2}}\right)|\mu_i|^{\frac{\alpha}{2}}|\nu_i|^{\frac{\alpha}{2}}\,|k, \mu, \nu\rangle\,. \tag{6.1.5}$$

After the operators $\hat{O}^{a}_{\alpha,i}$ have acted as part of the Euclidean constraint operator, certain sine and cosine operators act. Note that these operators commute as they can be expressed by complex exponentials or rather holonomies which themselves commute. To state the action of the sine and cosine operators, consider for example the first term of the Euclidean constraint operator, where the operators $\sin(X_i)$ and $\sin(Y_i)$ occur, cf. (5.2.57). We obtain the following

action of these operators:

$$\begin{aligned}\sin(X_i)\sin(Y_i)\,|k,\mu,\nu\rangle &= -\frac{1}{4}\left(e^{\frac{i}{2}2X_i}-e^{-\frac{i}{2}2X_i}\right)\left(e^{\frac{i}{2}2Y_i}-e^{-\frac{i}{2}2Y_i}\right)|k,\mu,\nu\rangle = \\ &= -\frac{1}{4}\left(e^{\frac{i}{2}2X_i}-e^{-\frac{i}{2}2X_i}\right)\Big(|k,\mu,\nu_i+2\rangle - |k,\mu,\nu_i-2\rangle\Big) = \\ &= -\frac{1}{4}\Big(|k,\mu_i+2,\nu_i+2\rangle - |k,\mu_i-2,\nu_i+2\rangle \\ &\quad - |k,\mu_i+2,\nu_i-2\rangle + |k,\mu_i-2,\nu_i-2\rangle\Big).\end{aligned} \tag{6.1.6}$$

Similarly, for the second term of the Euclidean constraint operator in (5.2.57), we can compute

$$\begin{aligned}\sin\left(\frac{1}{2}Y_{i+1}\right)\cos\left(\frac{1}{2}Y_i\right)\sin(\mathcal{A}_{e_i})\,|k,\mu,\nu\rangle = -\frac{1}{8}\Big(&|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\ &- |k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle + |k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \\ &- |k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle + |k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\ &+ |k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\Big).\end{aligned} \tag{6.1.7}$$

The action of the product of the operators $\sin\left(\frac{1}{2}X_{i+1}\right)\cos\left(\frac{1}{2}X_i\right)\sin(\mathcal{A}_{e_i})$ contained in the third term of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ follows from (6.1.7) by exchanging the μ- and ν-labels. Using this and (6.1.2)-(6.1.7), we can write down the action of the Euclidean constraint operator on the formal ansatz for φ we stated in (6.0.3):

$$\begin{aligned}\hat{\mathrm{H}}_{\mathrm{eucl}}\,\varphi &= \left(\sum_{i=1}^{N}\hat{\mathrm{H}}_{\mathrm{eucl,i}}\right)\left(\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\right) = \\ &= \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}C_{k,\mu,\nu}\left[\sum_{i=1}^{N}\left(\hat{\mathrm{H}}_{\mathrm{eucl,i}}\,|k,\mu,\nu\rangle\right)\right] = \\ &= A\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}C_{k,\mu,\nu}\sum_{i=1}^{N}\Bigg\{\left[\sqrt{|\mu_i||\nu_i|}\left(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\right)\right]\times \\ &\times\Big[|k,\mu_i+2,\nu_i+2\rangle - |k,\mu_i-2,\nu_i+2\rangle - |k,\mu_i+2,\nu_i-2\rangle + |k,\mu_i-2,\nu_i-2\rangle\Big] \\ &+\left[\sqrt{|k_i+k_{i-1}||\nu_i|}\left(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|}\right)\right]\times \\ &\times\Big[|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle + |k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\ &- |k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle + |k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\ &- |k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle + |k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\Big] \\ &+\left[\sqrt{|k_i+k_{i-1}||\mu_i|}\left(\sqrt{|\nu_i+1|}-\sqrt{|\nu_i-1|}\right)\right]\times \\ &\times\Big[|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle - |k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle + |k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle \\ &- |k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle - |k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle + |k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\ &- |k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle + |k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\Big]\Bigg\}.\end{aligned} \tag{6.1.8}$$

In the above equation, we collected all constant prefactors into the global factor

$$A := \frac{\kappa' \sqrt{\gamma l_{\mathrm{P}}^2}}{8\gamma^2}. \tag{6.1.9}$$

Here, we deviate from [57] by a factor of 2 which might be a typo: In [57], the authors end up with the prefactor $(\kappa' \sqrt{\gamma l_{\mathrm{P}}^2})/(4\gamma^2)$ without carrying out the action of the sine- and cosine-functions, which would lead to an additional factor of $\frac{1}{4}$. Therefore, from [57], we would get $(\kappa' \sqrt{\gamma l_{\mathrm{P}}^2})/(16\gamma^2)$ if we carry out the action of the sine- and cosine-functions, giving the factor of 2 deviation from our calculation.

In this paragraph, we provide more insight into the structure of the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ stated in (6.1.8). We recognize that (6.1.8) basically consists of three summands, which correspond to the three operators of which the operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$ is a sum of , cf. (5.2.57). Each of the three summands in (6.1.8) is a product of two factors that are both parenthesized by square brackets:

- The first factor consists of label-dependent prefactors, e.g., $\sqrt{|\mu_i||\nu_i|}(\sqrt{|k_i + k_{i-1} + 1|} - \sqrt{|k_i + k_{i-1} - 1|})$ for the first summand of (6.1.8). These label-dependent prefactors arise from the action of the operators $\hat{O}^a_{1,i}$, with $a = x, y, \theta$, cf. (6.1.2)-(6.1.5).
- The second factor is the sum of states with certain labels being shifted. This results from the action of the sine and cosine operators. We want to give an overview of the kind of shifted states that are contained in each of the three summands of (6.1.8):
  - The first summand contains in total four shifted states of the form $|k, \mu_i \pm 2, \nu_i \pm 2\rangle$, i.e., states where the ith μ- and the ith ν-label is shifted by ± 2, respectively.
  - The second summand contains in total eight shifted states of the form $|k_i \pm 2, \mu, \nu_i \pm 1, \nu_{i+1} \pm 1\rangle$, i.e., states where the ith k-label is shifted by ± 2, the ith ν-label by ± 1, and the $(i+1)$th ν-label by ± 1.
  - The third summand equals the second summand if we interchange the μ- and the ν-labels.

Now, we explain the action of the Lorentzian part shown in (5.2.65), (5.2.67), and (5.2.68). For this, we first need to discuss the action of the operators $\left(\hat{Z}_{\alpha,i}\right)^l$ and $\left(\hat{Z}_{\beta,i}\right)^l$ introduced in section 5.2.4. The action of the latter follows from the action of the former simply by replacing α with β. Using the explicit expression of $\left(\hat{Z}_{\alpha,i}\right)^l$, we obtain the following result:

$$\begin{aligned}\left(\hat{Z}_{\alpha,i}\right)^l |k, \mu, \nu\rangle = &\left(-12 \hat{O}^x_{\alpha,i} \hat{O}^y_{\alpha,i} \hat{O}^\theta_{\alpha,i}\right)^l |k, \mu, \nu\rangle = \\ = &z \cdot \Big[|\mu_i|^\alpha |\nu_i|^\alpha |k_i + k_{i-1}|^\alpha \left(|\mu_i + 1|^{\frac{\alpha}{2}} - |\mu_i - 1|^{\frac{\alpha}{2}} \right) \times \\ &\times \left(|\nu_i + 1|^{\frac{\alpha}{2}} - |\nu_i - 1|^{\frac{\alpha}{2}} \right) \left(|k_i + k_{i-1} + 1|^{\frac{\alpha}{2}} - |k_i + k_{i-1} - 1|^{\frac{\alpha}{2}} \right) \Big]^l,\end{aligned} \tag{6.1.10}$$

where we collected all constant prefactors in the overall factor

$$z := o \cdot (-12)^l \left(\frac{1}{2i}\right)^{3l} \left[\frac{1}{\sqrt{2}} \left(\frac{\gamma l_{\mathrm{P}}^2}{2}\right)^{\frac{3}{2}} \right]^{3l\alpha}, \tag{6.1.11}$$

with o defined earlier in (6.1.3). We arrive at the result of (6.1.10) the following way: $\hat{Z}_{\alpha,i}$ is a product of the operators $\hat{O}^x_{\alpha,i}$, $\hat{O}^y_{\alpha,i}$ and $\hat{O}^\theta_{\alpha,i}$, as written down in the first step of (6.1.10). Therefore, the action of $\hat{Z}_{\alpha,i}$ is a product of the actions stated in (6.1.2)-(6.1.5). Additionally, $\hat{Z}_{\alpha,i}$ is to the power of l. Putting everything together, we arrive at the result of (6.1.10). Using (6.1.10) and the action of the fluxes stated in section 5.2.2, we can now calculate the action of the Lorentzian part.

We begin with the action of the third term of the Lorentzian constraint operator $\hat{\mathrm{H}}^{(3)}_{\mathrm{lor}}$, cf. (5.2.68), which turns out to vanish if we sum over all N vertices as required for $\hat{\mathrm{H}}_{\mathrm{phys}}$. First, we state its action on an arbitrary basis state $|k,\mu,\nu\rangle$:

$$\begin{aligned}\sum_{i=1}^{N}\hat{\mathrm{H}}^{(3)}_{\mathrm{lor},i}\,|k,\mu,\nu\rangle =\sum_{i=1}^{N}A'\cdot\Big\{&(k_{i+1}+k_i)(k_{i+2}-k_i)\big[(|\mu_{i+1}+1|^{\frac{\alpha}{2}}-|\mu_{i+1}-1|^{\frac{\alpha}{2}})\times\\ &\times(|\nu_{i+1}+1|^{\frac{\alpha}{2}}-|\nu_{i+1}-1|^{\frac{\alpha}{2}})\cdot(|k_{i+1}+k_i+1|^{\frac{\alpha}{2}}-|k_{i+1}+k_i-1|^{\frac{\alpha}{2}})\times\\ &\times|\mu_{i+1}|^{\alpha}|\nu_{i+1}|^{\alpha}|k_{i+1}+k_i|^{\alpha}\big]^l\\ &-(k_i+k_{i-1})(k_{i+1}-k_{i-1})\big[(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}})\times\\ &\times(|\nu_i+1|^{\frac{\alpha}{2}}-|\nu_i-1|^{\frac{\alpha}{2}})\cdot(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})\times\\ &\times|\mu_i|^{\alpha}|\nu_i|^{\alpha}|k_i+k_{i-1}|^{\alpha}\big]^l\Big\}\,|k,\mu,\nu\rangle\,,\end{aligned} \tag{6.1.12}$$

where we defined

$$A':=\frac{\kappa'\sqrt{\gamma l_{\mathrm{P}}^2}}{4}\cdot\left(\frac{1}{2\alpha^3}\right)^l \text{ with } \alpha=\frac{2}{3}-\frac{1}{3l}.$$

Here, we deviate from [57], where a prefactor of $2\kappa'\sqrt{\gamma l_{\mathrm{P}}^2}\cdot\left(\frac{1}{8\alpha^3}\right)^l$ is obtained which might be a typo. To arrive at (6.1.12), we used the action of $\left(\hat{Z}_{\alpha,i}\right)^l$ stated in (6.1.10) and the action of the fluxes stated in section 5.2.2. The latter gives

$$(\hat{\mathcal{E}}_{i+2}-\hat{\mathcal{E}}_{i+1})\,|k,\mu,\nu\rangle\propto(k_{i+2}+k_{i+1}-k_{i+1}-k_i)\,|k,\mu,\nu\rangle=(k_{i+2}-k_i)\,|k,\mu,\nu\rangle\,. \tag{6.1.13}$$

We can do the same calculation for $(\hat{\mathcal{E}}_{i+1}-\hat{\mathcal{E}}_i)$ and end up with (6.1.12). To see why (6.1.12) is zero, we recognize that it is a telescope sum: (6.1.12) is the difference between two terms. Both terms have the same structure and can be obtained from each other by shifting i by ±1. Therefore, (6.1.12) is given by the telescope sum

$$A'\sum_{i=1}^{N}(x_{i+1}-x_i), \tag{6.1.14}$$

with

$$\begin{aligned}x_i:=&(k_i+k_{i-1})(k_{i+1}-k_{i-1})\big[(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}})(|\nu_i+1|^{\frac{\alpha}{2}}-|\nu_i-1|^{\frac{\alpha}{2}})\times\\ &\times(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})|\mu_i|^{\alpha}|\nu_i|^{\alpha}|k_i+k_{i-1}|^{\alpha}\big]^l\end{aligned} \tag{6.1.15}$$

Carrying out the telescope sum in (6.1.14) gives $x_{N+1}-x_1$, which vanishes as $x_{N+1}=x_1$. The latter is true as for Gowdy states we identify the $(N+1)$th with the first label, i.e., $k_{N+1}=k_1$, $\mu_{N+1}=\mu_1$, and $\nu_{N+1}=\nu_1$. Therefore, we have $x_{N+1}=x_N$ and the telescope sum in (6.1.14) vanishes. Consequently, the action of the Lorentzian part reduces to the action of

$\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}}$ and $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$, which we will specify in the following paragraph.

We can calculate the action of the other two terms of the Lorentzian part, $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}}$ and $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$, in analogy to the one of $\hat{\mathrm{H}}^{(3)}_{\mathrm{lor}}$. Hence, we refer to the previous paragraph for more detailed explanations and only state the final result. As the action of $\hat{\mathrm{H}}^{(3)}_{\mathrm{lor}}$ vanishes, we can in particular write down the action of the whole Lorentzian part:

$$\begin{aligned}
\hat{\mathrm{H}}_{\mathrm{lor}}\,\varphi &= \left[\sum_{i=1}^{N}\left(\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}+\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}\right)\right]\left[\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\right] = \\
&= \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\left[\sum_{i=1}^{N}\left(\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}+\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}\right)|k,\mu,\nu\rangle\right] = \\
&= \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\sum_{i=1}^{N}\Bigg\{-B(k_{i+1}-k_{i-1})^2\big[(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}})\times \\
&\quad\times(|\nu_i+1|^{\frac{\alpha}{2}}-|\nu_i-1|^{\frac{\alpha}{2}})\times(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})|\mu_i|^{\alpha}|\nu_i|^{\alpha}|k_i+k_{i-1}|^{\alpha}\big]^l \\
&\quad+B'(k_i+k_{i-1})^4(\mu_i\nu_{i+1}-\mu_{i+1}\nu_i)^2\big[(|\mu_i+1|^{\frac{\beta}{2}}-|\mu_i-1|^{\frac{\beta}{2}})\cdot(|\nu_i+1|^{\frac{\beta}{2}}-|\nu_i-1|^{\frac{\beta}{2}})\times \\
&\quad\times(|k_i+k_{i-1}+1|^{\frac{\beta}{2}}-|k_i+k_{i-1}-1|^{\frac{\beta}{2}})|\mu_i|^{\beta}|\nu_i|^{\beta}|k_i+k_{i-1}|^{\beta}\big]^l\Bigg\}\,|k,\mu,\nu\rangle\,.
\end{aligned} \tag{6.1.16}$$

Here, the first term with the constant prefactor B results from the action of $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}}$ and the second term with the constant prefactor B' results from the action of $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$. We defined the constant prefactors by

$$B := \frac{\sqrt{\gamma l_{\mathrm{P}}^2}}{16\kappa'}\left(\frac{1}{2\alpha^3}\right)^l \text{ with } \alpha=\frac{2}{3}-\frac{1}{3l},\; B'=\frac{\sqrt{\gamma l_{\mathrm{P}}^2}}{16\kappa'}\left(\frac{1}{2\beta^3}\right)^l \text{ with } \beta=\frac{2}{3}-\frac{5}{3l}. \tag{6.1.17}$$

Here, we deviate in B from [57], where a prefactor of $\frac{\sqrt{\gamma l_{\mathrm{P}}^2}}{2\kappa'}\left(\frac{1}{8\alpha^3}\right)^l$ is stated. The prefactor B' cannot be compared to [57], as the authors of [57] use inverse fluxes to quantize $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$, see page 17 of [57] and our discussion in section 5.2.4 in the context of (5.2.70) .

6.1.2 The adjoint of the Euclidean part

In this section, we determine the explicit form of the adjoint of the Euclidean operator, $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, as well as its action. It will turn out that the Euclidean operator is not symmetric and therefore we get a different action for the adjoint operator.

First, we compute $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ by determining the adjoint of the individual parts of $\hat{\mathrm{H}}_{\mathrm{eucl}}$. These are given by sine and cosine operators and the family of operators $\hat{O}^{a}_{\alpha,i}$ with $a\in\{x,y,\theta\}$. All of these operators contain holonomies and fluxes, for which we know how to compute the adjoint operators:

- The fluxes are already symmetric, i.e.:

$$\hat{\mathcal{E}}_i=\hat{\mathcal{E}}_i^{\dagger},\;\hat{\mathcal{F}}_{x,i}=\hat{\mathcal{F}}_{x,i}^{\dagger},\;\hat{\mathcal{F}}_{y,i}=\hat{\mathcal{F}}_{y,i}^{\dagger}. \tag{6.1.18}$$

As the volume operator is the product of all three fluxes, cf. (5.2.33), it is symmetric, too:

$$(\hat{V}_i)^\dagger = \hat{V}_i. \tag{6.1.19}$$

- Holonomies are implemented as unitary operators, which implies that

$$\left(e^{\pm\frac{i}{2}X_i}\right)^\dagger = e^{\mp\frac{i}{2}X_i} \tag{6.1.20}$$

and analogously for $e^{\pm\frac{i}{2}Y_i}$ and $e^{\pm\frac{i}{2}\mathcal{A}_{e_i}}$.

Using these results, we get for the family of operators $\hat{O}^x_{\alpha,i}$ that

$$\left(\hat{O}^x_{\alpha,i}\right)^\dagger = \left[\frac{1}{2i}\left(e^{-\frac{i}{2}X_i}\hat{V}_i^\alpha e^{\frac{i}{2}X_i} - e^{\frac{i}{2}X_i}\hat{V}_i^\alpha e^{-\frac{i}{2}X_i}\right)\right]^\dagger = -\frac{1}{2i}\left(e^{-\frac{i}{2}X_i}\hat{V}_i^\alpha e^{\frac{i}{2}X_i} - e^{\frac{i}{2}X_i}\hat{V}_i^\alpha e^{-\frac{i}{2}X_i}\right) = -\hat{O}^x_{\alpha,i}, \tag{6.1.21}$$

where we used that

$$\left(e^{-\frac{i}{2}X_i}\hat{V}_i^\alpha e^{\frac{i}{2}X_i}\right)^\dagger = \left(e^{\frac{i}{2}X_i}\right)^\dagger\left(\hat{V}_i^\alpha\right)^\dagger\left(e^{-\frac{i}{2}X_i}\right)^\dagger = e^{-\frac{i}{2}X_i}\hat{V}_i^\alpha e^{\frac{i}{2}X_i}$$

due to (6.1.19) and (6.1.20). Similarly, we get

$$\left(e^{+\frac{i}{2}X_i}\hat{V}_i^\alpha e^{-\frac{i}{2}X_i}\right)^\dagger = e^{+\frac{i}{2}X_i}\hat{V}_i^\alpha e^{-\frac{i}{2}X_i}.$$

The minus sign we pick up in (6.1.21) results from the prefactor of $\frac{1}{2i}$. The other ingredient for calculating $\hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}$ is given by the adjoint of the sine- and cosine-operators, such as for example $\sin(X_i)$ and $\cos(X_i)$. We can show that these operators are symmetric using (6.1.20):

$$[\sin(X_i)]^\dagger = \left[\frac{1}{2i}\left(e^{iX_i} - e^{-iX_i}\right)\right]^\dagger = -\frac{1}{2i}\left(e^{-iX_i} - e^{+iX_i}\right) = \sin(X_i), \tag{6.1.22}$$

$$[\cos(X_i)]^\dagger = \left[\frac{1}{2}\left(e^{iX_i} + e^{-iX_i}\right)\right]^\dagger = \frac{1}{2}\left(e^{-iX_i} + e^{+iX_i}\right) = \cos(X_i). \tag{6.1.23}$$

Similarly, we can proceed for all other sine and cosine operators appearing in $\hat{\mathrm{H}}_{\mathrm{eucl}}$. Now, we can calculate the adjoint of the first term of $\hat{\mathrm{H}}_{\mathrm{eucl}}$. In a first step, we obtain

$$\left[i\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\sin(\mu_0 X_i)\sin(\nu_0 Y_i)\,\hat{O}^\theta_{1,i}\right]^\dagger = (-i)\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\left[\hat{O}^\theta_{1,i}\right]^\dagger[\sin(\mu_0 X_i)\sin(\nu_0 Y_i)]^\dagger. \tag{6.1.24}$$

Note that the order of the operators has changed. Using (6.1.21)-(6.1.23), we arrive at

$$\left[i\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\sin(\mu_0 X_i)\sin(\nu_0 Y_i)\,\hat{O}^\theta_{1,i}\right]^\dagger = i\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\hat{O}^\theta_{1,i}\sin(\mu_0 X_i)\sin(\nu_0 Y_i). \tag{6.1.25}$$

Note that the minus sign we get from (6.1.21) is canceled by the overall i-factor contained in $\hat{\mathrm{H}}_{\mathrm{eucl}}$. Furthermore, note that we used in (6.1.25) that all holonomies commute and therefore the change of the order in the sine operators when taking the adjoint does not matter. The analogous steps can be performed for the remaining terms of $\hat{\mathrm{H}}_{\mathrm{eucl}}$, yielding the following expression for the adjoint Euclidean constraint operator:

$$\begin{aligned}\hat{\mathrm{H}}^\dagger_{\mathrm{eucl,i}} = -i\frac{4}{\kappa' l_{\mathrm{P}}^3\gamma^3}\Bigg[&\hat{O}^\theta_{1,i}\sin(X_i)\sin(Y_i) + 2\hat{O}^X_{1,i}\sin\left(\frac{1}{2}Y_{i+1}\right)\cos\left(\frac{}{2}Y_i\right)\sin(\mathcal{A}_{e_i})\\ &+2\hat{O}^Y_{1,i}\sin\left(\frac{\mu_0}{2}X_{i+1}\right)\cos\left(\frac{1}{2}X_i\right)\sin(\mathcal{A}_{e_i})\Bigg].\end{aligned} \tag{6.1.26}$$

We now state the action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$. The action is similar to the one of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ with one substantial difference: The holonomies in $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ act *before* the diagonal operators $\hat{O}^{a}_{\alpha,i}$. For $\hat{\mathrm{H}}_{\mathrm{eucl}}$, it is exactly the other way round. Hence, for $\hat{\mathrm{H}}_{\mathrm{eucl}}$, the diagonal operators $\hat{O}^{a}_{\alpha,i}$ act first and give additional to constant factors also label-dependent prefactors. For $\hat{O}^{x}_{\alpha,i}$, e.g., these label-dependent prefactors are of the form $(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|})\sqrt{|\nu_i||k_i+k_{i-1}|}$. The holonomies then act after the operators $\hat{O}^{a}_{\alpha,i}$ have acted, which results in states with shifted labels, cf. (6.1.6) and (6.1.7). For $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, however, the holonomies act first, resulting in the same shifted states as obtained for $\hat{\mathrm{H}}_{\mathrm{eucl}}$. Afterwards, the fluxes act on each of the shifted states, resulting in label-dependent prefactors different from those of $\hat{\mathrm{H}}_{\mathrm{eucl}}$. Note that compared to $\hat{\mathrm{H}}_{\mathrm{eucl}}$, the constant prefactors stay the same because the same operators are contained in $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, only their order is modified. This way, we arrive at the following action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ on the formal ansatz for φ, which we stated in (6.0.3):

$$
\begin{aligned}
\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}\,\varphi &= \left(\sum_{i=1}^{N}\widehat{\mathrm{H}}^{\dagger}_{\mathrm{eucl,i}}\right)\left(\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\right) = \\
&= \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}\sum_{i=1}^{N}\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl,i}}C_{k,\mu,\nu}\,|k,\mu,\nu\rangle = \\
&= A\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}\sum_{i=1}^{N} C_{k,\mu,\nu}\cdot\Big\{\Big(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\Big)\times \\
&\times\Big[\sqrt{|\mu_i+2||\nu_i+2|}\,|k,\mu_i+2,\nu_i+2\rangle-\sqrt{|\mu_i-2||\nu_i+2|}\,|k,\mu_i-2,\nu_i+2\rangle \\
&-\sqrt{|\mu_i+2||\nu_i-2|}\,|k,\mu_i+2,\nu_i-2\rangle+\sqrt{|\mu_i-2||\nu_i-2|}\,|k,\mu_i-2,\nu_i-2\rangle\Big] \\
&+\Big(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|}\Big)\Big[\ \sqrt{|k_i+k_{i-1}+2|\nu_i+1|}\,|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}+2||\nu_i+1|}\,|k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
&\qquad+\sqrt{|k_i+k_{i-1}+2||\nu_i-1|}\,|k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}+2||\nu_i-1|}\,|k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}-2||\nu_i+1|}\,|k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
&\qquad+\sqrt{|k_i+k_{i-1}-2||\nu_i+1|}\,|k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}-2||\nu_i-1|}\,|k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\
&\qquad+\sqrt{|k_i+k_{i-1}-2||\nu_i-1|}\,|k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\Big] \\
&+\Big(\sqrt{|\nu_i+1|}-\sqrt{|\nu_i-1|}\Big)\Big[\ \sqrt{|k_i+k_{i-1}+2||\mu_i+1|}\,|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}+2||\mu_i+1|}\,|k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
&\qquad+\sqrt{|k_i+k_{i-1}+2||\mu_i-1|}\,|k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}+2||\mu_i-1|}\,|k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}-2||\mu_i+1|}\,|k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
&\qquad+\sqrt{|k_i+k_{i-1}-2||\mu_i+1|}\,|k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
&\qquad-\sqrt{|k_i+k_{i-1}-2||\mu_i-1|}\,|k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle
\end{aligned}
$$

$$+\sqrt{|k_i+k_{i-1}-2||\mu_i-1|}\,|k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\Big]\Big\} \tag{6.1.27}$$

Now, we want to comment on an additional requirement concerning the definition of the abstract Gowdy states we gave in section 5.2.2 in the context of figure 3. For the Euclidean and the Lorentzian operator it is totally fine to excite N vertices and $N+1$ edges only, while charging all other infinitely many edges and vertices trivially, which are then not seen by these operators. To see this, we remark that in both operators, the volume operator is contained and acts first while its eigenvalue (7.0.17) is zero at trivially charged vertices. Note that the volume operator is actually contained in the family of operators $\hat{O}^a_{\alpha,i}$, with $a=x,y,\theta$, cf. (5.2.54)-(5.2.56), such that holonomies act before and after it, but in such a way that we still obtain a zero action at trivially charged vertices and edges. For example, $\hat{O}^x_{\alpha,i}$ will yield us the eigenvalue $\left(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}}\right)|\nu_i|^{\frac{\alpha}{2}}|k_i+k_{i-1}|^{\frac{\alpha}{2}}$, which still vanishes, e.g., for $\mu_i=0$. However, for the adjoint of the Euclidean operator (5.2.57), the order of holonomies and the volume operator changes, in particular the holonomies act first. This has the following consequence: We concentrate on the first term of the adjoint Euclidean operator, cf. (6.1.26), where the point holonomies are involved in $\sin(X_i)\sin(Y_i)$ and the volume operator is contained in $\hat{O}^\theta_{\alpha,i}$. To compute the action on the $N+1$th vertex, we note that this vertex carries the labels $\mu_{N+1}=\nu_{N+1}=0$, while the label of the outgoing edge is $k_{N+1}=0$, but that of the ingoing edge is $k_N\neq 0$. See also figure 3 for an illustration in the case of $N=3$ vertices. The point holonomies in $\sin(X_i)\sin(Y_i)$ will change the labels $\mu_{N+1}=\nu_{N+1}=0$ at the $N+1$th vertex to non-zero values. Hence, when the volume operator acts afterwards, we get a non-zero action, especially due to the fact that the k-labels do not cure this, as the label of the ingoing edge equals k_N which is non-zero in general. A similar thing would happen if we act at the vertex that is preceding the first vertex. For the other parts of the adjoint Euclidean operator, however, this does not occur, as these terms only contain holonomies that change the k-labels together with *one* of the point holonomy charges such that *one* of the μ- or ν-labels always stays zero at trivially charged vertices. Thus, this results in a zero action of the volume operator, as its eigenvalue is proportional to the square root of the point holonomy labels, cf. (7.0.17). We propose the following strategy to avoid the scenario of the adjoint Euclidean operator acting non-trivially at the $N+1$th vertex and the vertex preceding the first vertex: Looking at the eigenvalue of the volume operator (7.0.17), we can observe that the sum of neighboring k-labels occurs and this is actually the reason for the above discussed problem. Instead of setting the label k_{N+1} to zero, we could set it to $-k_N$. Then, the volume operator acting at the $N+1$th vertex measures the sum of the ingoing and outgoing k-labels which now vanishes. We then have to extend this to the whole abstract graph, i.e., we alternately set the k-labels to k_N and $-k_N$, see also figure 4, where we illustrated the situation for $N=2$ vertices. To go back to the embedded case, we may pursue the following strategy: We take the N vertices and edges with charges $\mu_1,...,\mu_N$, $\nu_1,...,\nu_N$, and $k_1,...,k_N$, and map them to a circle where they fill a certain segment. On the remaining part of the circle, we create two new trivially charged vertices v' and v'', giving us two new edges. On one of these, we map the infinitely many edges that carry the charge $+k_N$, and on the other one, we map the infinitely many edges that carry the charge $-k_N$. We do this in such a way that the ordering of the labels is not changed, i.e., at the first vertex, the k label of the ingoing edge is k_N and at the Nth vertex, the k label at the outgoing edge is k_N as well. Hence, the circle symmetry is implemented, see again figure 4. Furthermore, the action of all operators contained in the physical Hamiltonian operator will vanish at the new vertices v' and v''. This discussion shows that the definition of the algebraic graph we introduced in

section 5.2.2 can be extended the following way, at least for the Hamiltonian constraint and the Lorentzian part: While again charging N vertices and $N+1$ edges non-trivially with the circle symmetry being implemented, one may charge the infinitely many other vertices and edges such that the zero volume conditions (7.0.18) are satisfied. Finally, when discussing our results in the sections 6.2-6.7, we always assume suitably defined Gowdy states such that the above described scenario does not occur.

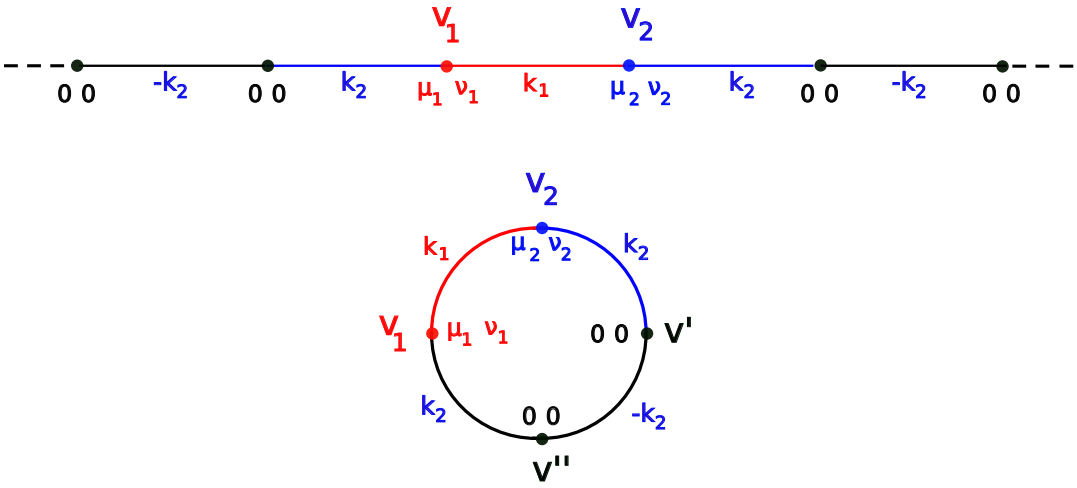

Figure 4: Illustration of the modified definition of an abstract graph (top) and an embedded graph (bottom) for $N = 2$ vertices.

6.1.3 The adjoint of the Lorentzian part

In this section, we determine the adjoint of the Lorentzian part which we denote by $\hat{\mathrm{H}}^{\dagger}_{\mathrm{lor}}$. It will turn out that the Lorentzian part is symmetric and therefore the action of the Lorentzian part and its adjoint is identical.

As shown in section 6.1.1, the first term of the Lorentzian part vanishes when summing up the contributions from all vertices. Accordingly, the Lorentzian part reduces to the expression

$$\hat{\mathrm{H}}_{\mathrm{lor}} = \sum_{i=1}^{N} \left(\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i} + \hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i} \right). \tag{6.1.28}$$

Taking the adjoint of $\hat{\mathrm{H}}_{\mathrm{lor}}$ thus amounts to taking the adjoint of $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$ and $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$.

We start with the calculation of the adjoint of $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$:

$$\begin{aligned} \hat{\mathrm{H}}^{(1)\dagger}_{\mathrm{lor},i} &= \left[-\frac{1}{4\kappa'} \left(-\frac{i}{\hbar} \right)^{3l} \left(\frac{16}{3(\kappa'\gamma)^3\alpha^3} \right)^{l} \left(\hat{\mathcal{E}}_{i+1} - \hat{\mathcal{E}}_i \right) \left(\hat{Z}_{\alpha,i} \right)^{l} \right]^{\dagger} = \\ &= -\frac{1}{4\kappa'} \left(\frac{16}{3(\kappa'\gamma)^3\alpha^3} \right)^{l} \left[\left(-\frac{i}{\hbar} \right)^{3l} \right]^{\dagger} \left[\left(\hat{Z}_{\alpha,i} \right)^{l} \right]^{\dagger} \left(\hat{\mathcal{E}}_{i+1} - \hat{\mathcal{E}}_i \right)^{\dagger}. \end{aligned} \tag{6.1.29}$$

We already know that the fluxes $\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_{i+1}$ are symmetric, cf. (6.1.18), and hence only need two more ingredients to finish this calculation:

- We can write out the factor $(-i/\hbar)^{3l}$ as $(1/\hbar^3)^l \cdot i^l$. Hence, we get a factor of $(-1)^l$ when taking the adjoint of $(-i/\hbar)^{3l}$.
- For the adjoint of $\left(\hat{Z}_{\alpha,i}\right)^l$ we need to compute the adjoint operator $\left(\hat{Z}_{\alpha,i}\right)^\dagger$ for which we get

$$\left(\hat{Z}_{\alpha,i}\right)^\dagger = (-12\hat{O}^x_{\alpha,i}\hat{O}^y_{\alpha,i}\hat{O}^\theta_{\alpha,i})^\dagger = +12\hat{O}^x_{\alpha,i}\hat{O}^y_{\alpha,i}\hat{O}^\theta_{\alpha,i} = -\hat{Z}_{\alpha,i}, \tag{6.1.30}$$

  where we used (6.1.21) in the second step. In consequence, we get a factor of $(-1)^l$ when computing $\left[\left(\hat{Z}_{\alpha,i}\right)^l\right]^\dagger$.

Plugging the results of the above two points into (6.1.29), we see that we get two factors of $(-1)^l$. Due to $(-1)^l \cdot (-1)^l = (-1)^{2l} = +1$ we arrive at

$$\hat{\mathrm{H}}^{(1)\dagger}_{\mathrm{lor},i} = \hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}, \tag{6.1.31}$$

i.e., the first term of the Lorentzian part is symmetric. Note that there appears a change in the order of the fluxes $\hat{\mathcal{E}}$ and the operators $\left(\hat{Z}_{\alpha,i}\right)^l$ when we compute the adjoint in (6.1.29). This, however, does not matter, as both operators act diagonally on any basis state $|k,\mu,\nu\rangle$ and therefore commute.

Now, we discuss the second term of the Lorentzian part $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$. Looking at the explicit form of $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$ in (5.2.67), we see that it has a similar structure as the first term of the Lorentzian part $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$ in (5.2.65): $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$ consists of certain fluxes and the operator $\left(\hat{Z}_{\beta,i}\right)^l$. The latter is the same operator as $\left(\hat{Z}_{\alpha,i}\right)^l$, the operator appearing in in $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$, if we replace α by β. Hence, we see that calculating the adjoint of $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$ is in total analogy to the steps performed for $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i}$: The only parts of $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}$ that will give a change in sign are $(-i/\hbar)^{3l}$ and $\left(\hat{Z}_{\beta,i}\right)^l$. Thus, as explained in the two points below (6.1.29), we pick up two factors of $(-1)^l$ when taking the adjoint and obtain

$$\hat{\mathrm{H}}^{(2)\dagger}_{\mathrm{lor},i} = \hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}, \tag{6.1.32}$$

i.e., also the second term of $\hat{\mathrm{H}}_{\mathrm{lor}}$ is symmetric.

Finally, we combine (6.1.31) and (6.1.32) to arrive at

$$\hat{\mathrm{H}}^\dagger_{\mathrm{lor}} = \left[\sum_{i=1}^{N}\left(\hat{\mathrm{H}}^{(1)}_{\mathrm{lor},i} + \hat{\mathrm{H}}^{(2)}_{\mathrm{lor},i}\right)\right]^\dagger = \hat{\mathrm{H}}_{\mathrm{lor}}, \tag{6.1.33}$$

i.e., $\hat{\mathrm{H}}_{\mathrm{lor}}$ is a symmetric operator. Consequently, also the action of $\hat{\mathrm{H}}_{\mathrm{lor}}$ and $\hat{\mathrm{H}}^\dagger_{\mathrm{lor}}$ is the same.

6.1.4 Action of the physical Hamiltonian

Using (6.1.8), (6.1.27) and (6.1.16), we can state the action of our symmetric, physical Hamiltonian $\hat{\mathrm{H}}_{\mathrm{phys}}$ on the formal ansatz for φ we stated in (6.0.3):

$$\hat{\mathrm{H}}_{\mathrm{phys}}\ \varphi = \left[\sum_{i=1}^{N}\left(\frac{1}{2}\left(\hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}\right) + \hat{\mathrm{H}}_{\mathrm{lor}}\right)\right]\left[\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\right] =$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n}\sum_{i=1}^{N} C_{k,\mu,\nu}\cdot\Bigg\{\frac{1}{2}A\left(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\right)\times\\
&\times\Big[\left(\sqrt{|\mu_i+2||\nu_i+2|}+\sqrt{|\mu_i||\nu_i|}\right)|k,\mu_i+2,\nu_i+2\rangle\\
&-\left(\sqrt{|\mu_i-2||\nu_i+2|}+\sqrt{|\mu_i||\nu_i|}\right)|k,\mu_i-2,\nu_i+2\rangle\\
&-\left(\sqrt{|\mu_i+2||\nu_i-2|}+\sqrt{|\mu_i||\nu_i|}\right)|k,\mu_i+2,\nu_i-2\rangle\\
&+\left(\sqrt{|\mu_i-2||\nu_i-2|}+\sqrt{|\mu_i||\nu_i|}\right)|k,\mu_i-2,\nu_i-2\rangle\Big]\\
&+A\left(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|}\right)\times\\
&\times\Big[\left(\sqrt{|k_i+k_{i-1}+2|\nu_i+1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}+2||\nu_i+1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}+2||\nu_i-1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}+2||\nu_i-1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}-2||\nu_i+1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}-2||\nu_i+1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}-2||\nu_i-1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}-2||\nu_i-1|}+\sqrt{|k_i+k_{i-1}|\nu_i|}\right)|k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\Big]\\
&+A\left(\sqrt{|\nu_i+1|}-\sqrt{|\nu_i-1|}\right)\times\\
&\times\Big[\left(\sqrt{|k_i+k_{i-1}+2||\mu_i+1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}+2||\mu_i+1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}+2||\mu_i-1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}+2||\mu_i-1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}-2||\mu_i+1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}-2||\mu_i+1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle\\
&-\left(\sqrt{|k_i+k_{i-1}-2||\mu_i-1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle\\
&+\left(\sqrt{|k_i+k_{i-1}-2||\mu_i-1|}+\sqrt{|k_i+k_{i-1}|\mu_i|}\right)|k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\Big]\\
&-B(k_{i+1}-k_{i-1})^2\big[(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}})(|\nu_i+1|^{\frac{\alpha}{2}}-|\nu_i-1|^{\frac{\alpha}{2}})\times\\
&\times(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})|\mu_i|^{\alpha}|\nu_i|^{\alpha}|k_i+k_{i-1}|^{\alpha}\big]^l\\
&+B'(k_i+k_{i-1})^4(\mu_i\nu_{i+1}-\mu_{i+1}\nu_i)^2\big[(|\mu_i+1|^{\frac{\beta}{2}}-|\mu_i-1|^{\frac{\beta}{2}})\cdot(|\nu_i+1|^{\frac{\beta}{2}}-|\nu_i-1|^{\frac{\beta}{2}})\times\\
&\times(|k_i+k_{i-1}+1|^{\frac{\beta}{2}}-|k_i+k_{i-1}-1|^{\frac{\beta}{2}})|\mu_i|^{\beta}|\nu_i|^{\beta}|k_i+k_{i-1}|^{\beta}\big]^l\Bigg\}
\end{aligned}
\tag{6.1.34}
$$

Here, we repeated the result for the action of the Lorentzian part stated in (6.1.16). Furthermore, we summarized the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ the following way: As we explained previous to (6.1.27), the label-dependent prefactors make up the difference between the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ and the shifted states of the respective actions in (6.1.8) and (6.1.27) are the same. Therefore, we can collect all identical states when taking the sum of the actions of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$. This way, we obtain the label-dependent prefactors in front of the shifted states in (6.1.34).

6.2 Zero volume states

Before we discuss our strategy to look for special solutions of the Schrödinger equation, we first want to present those formal states φ that are annihilated by the volume operator $\hat{V}_i$ at all vertices $i = 1, .., N$, i.e.:

$$\hat{V}_i\,\varphi = \hat{V}_i \left(\sum_{k\in\mathbb{Z}^N} \sum_{\mu\in m} \sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle \right) = 0 \quad \forall\; i = 1, ...N. \tag{6.2.1}$$

Accordingly, we call the states φ satisfying (6.2.1) zero volume states. As the volume operator is contained in both the Euclidean and the Lorentzian constraint operator, it is advisable to discuss zero volume states in detail.

To specify the zero volume states φ, we first require the action of the volume operator on a single basis state $|k,\mu,\nu\rangle$ to be zero :

$$\hat{V}_i\,|k,\mu,\nu\rangle = \frac{1}{\sqrt{2}} \left(\frac{\gamma l_{\mathrm{P}}^2}{2}\right) \sqrt{|\mu_i||\nu_i||k_i + k_{i-1}|}\,|k,\mu,\nu\rangle \stackrel{!}{=} 0. \tag{6.2.2}$$

Looking at this equation, we can immediately state three possible choices for the sets of labels $k := (k_i)_{i=1,...,N}$, $\mu := (\mu_i)_{i=1,...,N}$, and $\nu := (\nu_i)_{i=1,...,N}$ of a zero volume basis state $|k,\mu,\nu\rangle$:

1. **Possibility**: $\mu_i = 0$ and ν_i, k_i arbitrary $\forall i = 1, .., N$,
2. **Possibility**: $\nu_i = 0$ and μ_i, k_i arbitrary $\forall i = 1, .., N$,
3. **Possibility**: $k_i = 0$ and μ_i, ν_i arbitrary $\forall i = 1, .., N$.

The drawback of these three possibilities is that in each case an entire degree of freedom is eliminated. In particular, the third condition corresponds to the elimination of the U(1)-connection $\mathcal{A}$, which, however, provides a direct link to the gravitational part of the full theory opposed to the scalars X and Y. To circumvent this, we have the following alternative:

4. **Possibility**: Combine possibilities 1-3 by choosing alternately μ_i, ν_i or k_i, k_{i-1} to be zero $\forall i = 1, .., N$

We want to illustrate this possibility with an example: Consider a state $|k,\mu,\nu\rangle$ with $\mu_1 = 0$ and all other labels arbitrary. Then, the volume operator has a zero action when acting at the first vertex, cf. (6.2.2). If als $\nu_2 = 0$ is true for the state $|k,\mu,\nu\rangle$, the volume operator acting at the second vertex has a zero action as well . If we continue setting alternately μ- and ν-labels to zero up to the Nth vertex, we have accomplished that the action of the volume operator is zero at each vertex. In particular, we achieved this without eliminating an entire degree of freedom, as it is the case for either of the possibilities 1-3. In the above example,

one could also substitute the choice of $\nu_2 = 0$ with $k_2 = k_3 = 0$ and similar for the other choices of labels. Thus, we see that using the fourth possibility, we obtain a variety of zero volume states. Note that if we want to eliminate the action of the volume operator at the ith vertex via choosing the k-labels, we need k_i *and* k_{i-1} to be zero, since both labels appear as a sum in the action of the volume operator, cf. (6.2.2).

There is also another possibility which, however, is applicable only for an even number of vertices N:

5. Possibility (*only* for N even): $k_i = -k_{i-1}$ and μ_i, ν_i arbitrary $\forall i = 1, .., N$.

To understand this possibilities, we recognize that in (6.2.2), the sum of two neighboring k-labels k_i and k_{i-1} appears. This sum is zero if the neighboring k-labels have the same modulus but different sign, i.e., if $k_i = -k_{i-1}$ $\forall i = 1, .., N$. For an odd number of vertices, possibility 5 is not applicable, which we will illustrate by an example: Consider a state with three vertices. Choosing $k_1 = -k_2 \neq 0$ annihilates the contribution of the volume operator at the second vertex as $k_1 + k_2 = 0$ and therefore (6.2.2) is satisfied for $i = 2$. If we additionally choose $k_1 = k_3 \neq 0$, the contribution of the volume operator at the third vertex will vanish as well. This, however, is not true at the first vertex as $k_1 + k_3 \neq 0$ due to $k_1 = k_3 \neq 0$. We could, however, also annihilate this contribution by choosing $\mu_1 = 0$ or alternatively $\nu_1 = 0$. We can generalize this to an arbitrary odd number of vertices N:

6. Possibility (*only* for N odd): $k_i = -k_{i-1}$ $\forall i \in \{1, ..., N\} \setminus j$, $k_j = k_{j-1}$ and $\mu_j = 0$ or $\nu_j = 0$; all other μ and ν labels remain arbitrary.

In analogy to possibility 4, we get two additional kind of zero volume states for an even and odd number of vertices, respectively:

7. Possibility (*only* for N even): Combine possibilities 1-3 with 5.

8. Possibility (*only* for N odd): Combine possibilities 1-3 with 6.

We illustrate these possibilities with an example, respectively: For possibility 7, we consider a state with 4 vertices. We can annihilate the contribution of the volume operator at the first vertex by choosing $\mu_1 = 0$, the contribution at the second vertex by selecting $k_2 = -k_1$ and the contribution at the third and fourth vertex by setting $\nu_3 = 0$ and $\mu_4 = 0$. This way, $k_3, k_4, \nu_1, \mu_2, \nu_2, \mu_3$, and ν_4 remain arbitrary. For possibility 8, we consider a state with 3 vertices. We could choose the labels according to $k_1 = -k_2, \mu_2 = 0, \nu_3 = 0$ to annihilate the contributions of the volume operator at each vertex, separately. This way, k_3, μ_1, ν_1, ν_2, and μ_3 remain arbitrary.

Now, we extend the previously stated possibilities 1-8, which specify conditions on a single basis sate $|k, \mu, \nu\rangle$, to a state φ satisfying (6.2.1). In particular, the conditions on the labels k, μ, ν are equivalent to conditions on the coefficients $C_{k,\mu,\nu}$ of a formal state φ: We just set the coefficients to zero if the respective condition of the labels is not satisfied. For the possibilities 1-3, this would give us the conditions:

1. Possibility: $C_{k,\mu,\nu} \neq 0$ if $\mu_i = 0$ $\forall i = 1, ..., N$,

2. Possibility: $C_{k,\mu,\nu} \neq 0$ if $\nu_i = 0$ $\forall i = 1, ..., N$,

3. Possibility: $C_{k,\mu,\nu} \neq 0$ if $k_i = 0$ $\forall i = 1, ..., N$,

which we can combine in analogy to the fourth possibility. Similarly, we can proceed for all other possibilities.

After having classified zero volume states, we make two important remarks:

- If we compute the norm of a formal state φ, this leads us to

$$< \varphi, \varphi > = \sum_{k' \in \mathbb{Z}^N} \sum_{\mu' \in m} \sum_{\nu' \in n} \sum_{k \in \mathbb{Z}^N} \sum_{\mu \in m} \sum_{\nu \in n} C^*_{k',\mu',\nu'} C_{k,\mu,\nu} \langle k', \mu', \nu' | k, \mu, \nu \rangle = \\ = \sum_{k \in \mathbb{Z}^N} \sum_{\mu \in m} \sum_{\nu \in n} |C_{k,\mu,\nu}|^2, \tag{6.2.3}$$

  where in the first step, $*$ denotes complex conjugation. In the second step, we used the orthonormality of the basis states $|k, \mu, \nu\rangle$ to eliminate the sums over k', μ', and ν'. To make φ a zero volume state, we set certain coefficients to zero, according to the above listed possibilities, while imposing the other coefficients to be non-zero. Hence, we can choose the non-zero coefficients arbitrarily and in particular such that the sums over the k-, μ- and ν-labels in (6.2.3) are finite. Therefore, the norm of a zero volume state φ is finite for an appropriate choice of the values of the non-zero coefficients.

- For a zero volume state φ we also have

$$\hat{\mathrm{H}}_{\mathrm{eucl}} \varphi = 0, \quad \hat{\mathrm{H}}_{\mathrm{lor}} \varphi = 0, \tag{6.2.4}$$

  as the volume operator acts first in both operators, which can be seen from the explicit form of the operators given in section 5.2.4. For $\hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}$, however, we have that

$$\hat{\mathrm{H}}^\dagger_{\mathrm{eucl}} \varphi \neq 0. \tag{6.2.5}$$

  To see this, we first note that calculating the action of $\hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}$ on a volume zero state φ reduces to calculating it on a zero volume basis state $|k, \mu, \nu\rangle$. Furthermore, we remind that taking the adjoint of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ changes the order of the operators such that the volume operator acts *after* the holonomies have acted. However, $|k, \mu, \nu\rangle$ is not a zero volume state anymore if a holonomy has acted upon it: Consider for example the zero volume state $|k, \mu, \nu\rangle := |k_1, k_2, \mu_1 = 0, \mu_2, \nu_1, \nu_2 = 0\rangle$ for 2 vertices. After the holonomy $\exp(i/2X_1)$ has acted, we get the state $|k_1, k_2, \mu_1 = 1, \mu_2, \nu_1, \nu_2 = 0\rangle$. This state is not a zero volume state anymore, as the contribution of the volume operator at the first vertex is not annihilated due to $\mu_1 = 1 \neq 0$. We can extend this example to all other kind of zero volume states and any number of vertices. Hence, we arrive at the result from (6.2.5) which basically results from the holonomies acting first in $\hat{\mathrm{H}}^\dagger_{\mathrm{eucl}}$ and therefore changing a zero volume state φ into a state φ' that is not annihilated by the volume operator anymore.

6.3 Annihilation of the action of the Euclidean part

To construct a very special solution of the time-independent Schrödinger equation

$$\hat{\mathrm{H}}_{\mathrm{phys}} \varphi = E \varphi \tag{6.3.1}$$

we pursue the following strategy: Given the expression for the physical Hamiltonian

$$\hat{\mathrm{H}}_{\mathrm{phys}} = \frac{1}{2} \left(\hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}^\dagger_{\mathrm{eucl}} \right) + \hat{\mathrm{H}}_{\mathrm{lor}},$$

we want to exploit the diagonal action of the Lorentzian part on a basis state $|k,\mu,\nu\rangle$ to produce the energy eigenvalue E of (6.3.1). In parallel, we want to annihilate the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$. More precisely, we investigate how to choose the coefficients of φ on the one hand, such that $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ have a zero action on φ, i.e.,

$$\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi = 0,\ \hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}\varphi = 0. \tag{6.3.2}$$

On the other hand, we inspect how to choose the coefficients of φ such that the action of $\hat{\mathrm{H}}_{\mathrm{lor}}$ simplifies to

$$\hat{\mathrm{H}}_{\mathrm{lor}}\ \varphi = \ E\ \varphi. \tag{6.3.3}$$

Note that the ansatz for φ is very special as it contains sums over all possible labels which for general coefficients results in a non-normalizable state, i.e., φ is in general not an element of the Gowdy Hilbert space. We nevertheless work with that ansatz for φ in (6.0.3) because it will allow to derive difference equations from the complicated action of the Euclidean part and its adjoint, cf. (6.1.8) and (6.1.27). This way we gain some insight into the details of the action of our physical Hamiltonian.

In the following sections 6.3-6.5, we first try to accomplish (6.3.2). We will furthermore discuss if the solutions we found this way also allow to obtain (6.3.3). Afterwards, we take the opposite route by starting from (6.3.3) in section 6.6 and discussing if the solutions constructed this way also allow to achieve (6.3.2) or at least simplify the action of the Euclidean operator and its adjoint.

6.3.1 Special separation ansatz and derivation of difference equations

In this section, we impose the condition

$$\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi \stackrel{!}{=} 0, \tag{6.3.4}$$

from which we derive difference equations for the coefficients $C_{k,\mu,\nu}$ of our ansatz φ, which is defined in (6.0.3). For this purpose, we use a special separation ansatz for the coefficients. We recommend [146] and [147] for an introduction into difference equations. In particular, the method of separation of variables for partial difference equations is described in [147] and also in [148, 149].

In the first step, we consider the action of the Euclidean operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$ in (6.1.8) and translate the shifts contained in the states to the coefficients. As explained in section 6.1.1, in the paragraph after (6.1.8), the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ consists of three summands that we will consider separately. For the first summand of (6.1.8), we examine the first state with all sums and prefactors:

$$A\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\sum_{i=1}^{N}\Big\{\sqrt{|\mu_i||\nu_i|}\left(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\right)|k,\mu_i+2,\nu_i+2\rangle\Big\}. \tag{6.3.5}$$

Now, we make the substitutions $\widetilde{\mu}_i = \mu_i + 2$ and $\widetilde{\nu}_i = \nu_i + 2$ which are equivalent to $\mu_i = \widetilde{\mu}_i - 2$ and $\nu_i = \widetilde{\nu}_i - 2$. Plugging this into (6.3.5) gives us

$$A\sum_{k\in\mathbb{Z}^N}\sum_{\widetilde{\mu}\in m}\sum_{\widetilde{\nu}\in n} C_{k,\widetilde{\mu}_i-2,\widetilde{\nu}_i-2}\sum_{i=1}^{N}\Big\{\sqrt{|\widetilde{\mu}_i-2||\widetilde{\nu}_i-2|}\left(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\right)|k,\widetilde{\mu}_i,\widetilde{\nu}_i\rangle\Big\}, \tag{6.3.6}$$

where we introduced the notation

$$C_{k,\widetilde{\mu}_i-2,\widetilde{\nu}_i-2} := C_{k,\mu_1,\ldots,\widetilde{\mu}_i-2,\ldots\mu_N,\nu_1,\ldots\widetilde{\nu}_i-2,\ldots,\nu_N}, \tag{6.3.7}$$

i.e., $C_{k,\widetilde{\mu}_i-2,\widetilde{\nu}_i-2}$ denotes the coefficient for which all labels remain unshifted except for the labels $\widetilde{\mu}_i$ and $\widetilde{\nu}_i$, which are shifted both by -2. Furthermore, looking at the definition of the sets m and n defined in (6.0.5) in which the μ and ν-labels take values, respectively, we see that the sums over the μ and ν labels are invariant under the substitutions $\mu_i = \widetilde{\mu}_i - 2$ and $\nu_i = \widetilde{\nu}_i - 2$. Hence, we can replace the sums in (6.3.5) over μ and ν by sums over $\widetilde{\mu}$ and $\widetilde{\nu}$ in (6.3.6). If we now rename $\widetilde{\mu}_i, \widetilde{\nu}_i$ by μ_i, ν_i, we arrive at

$$A \sum_{k\in\mathbb{Z}^N} \sum_{\mu\in m} \sum_{\nu\in n} C_{k,\mu_i-2,\nu_i-2} \sum_{i=1}^{N} \Big\{ \sqrt{|\mu_i-2||\nu_i-2|} \left(\sqrt{|k_i+k_{i-1}+1|} - \sqrt{|k_i+k_{i-1}-1|} \right) |k,\mu,\nu\rangle \Big\}. \tag{6.3.8}$$

Similarly, we can proceed with all other states of the first summand of (6.1.8). Now, we look at the second summand of the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ in (6.1.8). Again, we consider as an example only the first state with all sums and prefactors:

$$A \sum_{k\in\mathbb{Z}^N} \sum_{\mu\in m} \sum_{\nu\in n} C_{k,\mu,\nu} \sum_{i=1}^{N} \Big\{ \sqrt{|k_i+k_{i-1}||\nu_i|} \left(\sqrt{|\mu_i+1|} - \sqrt{|\mu_i-1|} \right) |k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \Big\}. \tag{6.3.9}$$

In this equation, in analogy to the first summand of (6.1.8), we perform the substitutions $\widetilde{\nu}_i = \nu_i + 1$, $\widetilde{\nu}_{i+1} = \nu_{i+1} + 1$ and $\widetilde{k}_i = k_i + 2$. For the ν-labels, we use again invariance of the sum, see the discussion below (6.3.7). For the k-labels, we sum over all integer values and hence the sum is invariant under the substitution $\widetilde{k}_i = k_i + 2$, too. This would not be the case if we choose the finite linear combination in (6.0.10) which again motivates to pick the one in (6.0.3). After renaming $\widetilde{k}_i, \widetilde{\nu}_i$, and $\widetilde{\nu}_{i+1}$ by k_i, ν_i, and ν_{i+1}, respectively, we obtain the following expression for (6.3.9):

$$A \sum_{k\in\mathbb{Z}^N} \sum_{\mu\in m} \sum_{\nu\in n} C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}-1} \sum_{i=1}^{N} \Big\{ \sqrt{|k_i+k_{i-1}-2||\nu_i-1|} \left(\sqrt{|\mu_i+1|} - \sqrt{|\mu_i-1|} \right) |k,\mu,\nu\rangle \Big\}, \tag{6.3.10}$$

with the notation for the coefficients in analogy to (6.3.7):

$$C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}-1} := C_{k_1,\ldots k_i-2,\ldots k_N,\mu_1,\ldots,\mu_N,\nu_1,\ldots,\nu_i-1,\nu_{i+1}-1,\ldots,\nu_N}. \tag{6.3.11}$$

Similarly, we can proceed for all other states of the second and also the third contribution of (6.1.8), as the second equals the third contribution if we interchange the μ- and ν-labels. Combining the above results, we can rewrite (6.3.4) the following way:

$$\begin{aligned} 0 \stackrel{!}{=} A \sum_{k\in\mathbb{Z}^N} \sum_{\mu\in m} \sum_{\nu\in n} \sum_{i=1}^{N} \Big\{ & \left(\sqrt{|k_i+k_{i-1}+1|} - \sqrt{|k_i+k_{i-1}-1|} \right) \times \\ & \times \Big[\sqrt{|\mu_i-2||\nu_i-2|}\, C_{k,\mu_i-2,\nu_i-2} - \sqrt{|\mu_i+2||\nu_i-2|}\, C_{k,\mu_i+2,\nu_i-2} \\ & - \sqrt{|\mu_i-2||\nu_i+2|}\, C_{k,\mu_i-2,\nu_i+2} + \sqrt{|\mu_i+2||\nu_i+2|}\, C_{k,\mu_i+2,\nu_i+2} \Big] \end{aligned}$$

$$
\begin{aligned}
&+\left(\sqrt{\mu_i+1}-\sqrt{\mu_i-1}\right)\times\\
&\times\Big[\sqrt{|\nu_i-1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}-1}-\sqrt{|\nu_i-1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}+1}\\
&+\sqrt{|\nu_i+1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu,\nu_i+1,\nu_{i+1}-1}-\sqrt{|\nu_i+1||k_i+k_{i-1}-2|}\;C_{k_i-2,\nu_i+1,\mu,\nu_{i+1}+1}\\
&-\sqrt{|\nu_i-1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu,\nu_i-1,\nu_{i+1}-1}+\sqrt{|\nu_i-1||k_i+k_{i-1}+2|}\;C_{k_i+2,\nu_i-1,\mu,\nu_{i+1}+1}\\
&-\sqrt{|\nu_i+1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu,\nu_i+1,\nu_{i+1}-1}+\sqrt{|\nu_i+1||k_i+k_{i-1}+2|}\;C_{k_i+2,\nu_i+1,\mu,\nu_{i+1}+1}\Big]\\
&+\left(\sqrt{\nu_i+1}-\sqrt{\nu_i-1}\right)\times\\
&\times\Big[\sqrt{|\mu_i-1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu_i-1,\mu_{i+1}-1,\nu}-\sqrt{|\mu_i-1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu_i-1,\mu_{i+1}+1,\nu}\\
&+\sqrt{|\mu_i+1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu_i+1,\mu_{i+1}-1,\nu}-\sqrt{|\mu_i+1||k_i+k_{i-1}-2|}\;C_{k_i-2,\mu_i+1,\mu_{i+1}+1,\nu}\\
&-\sqrt{|\mu_i-1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu_i-1,\mu_{i+1}-1,\nu}+\sqrt{|\mu_i-1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu_i-1,\mu_{i+1}+1,\nu}\\
&-\sqrt{|\mu_i+1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu_i+1,\mu_{i+1}-1,\nu}+\sqrt{|\mu_i+1||k_i+k_{i-1}+2|}\;C_{k_i+2,\mu_i+1,\mu_{i+1}+1,\nu}\Big]\Bigg\}\times\\
&\times|k,\mu,\nu\rangle\,.
\end{aligned}
\tag{6.3.12}
$$

In this equation, we can first of all cancel the overall prefactor of A. Second of all, this equation is true iff every summand with respect to the sums over the k-, μ- and ν-labels is zero. The same argumentation holds for the sum over i. Therefore, we will leave out the state $|k,\mu,\nu\rangle$ and all sums in the following. In (6.3.12), we have reached our goal to derive difference equations for the coefficients of a formal state φ. Next, we try to simplify (6.3.12) even further by performing a special separation ansatz for the coefficients.

In the second step we perform the following separation ansatz:

$$
C_{k,\mu,\nu}=C_k\cdot C_{\mu_1}\cdot C_{\mu_2}\cdot\ldots\cdot C_{\mu_N}\cdot C_{\nu_1}\cdot C_{\nu_2}\cdot\ldots\cdot C_{\nu_N},
\tag{6.3.13}
$$

i.e., we keep all the k-labels together in one coefficient but decouple all μ- and ν-labels. We do not separate the k-labels at this point because we see from (6.3.12) that we have prefactors of the form of $\sqrt{k_i+k_{i-1}\pm2}$, which we cannot factorize in a product of two functions $f_1(k_i)$ and $f_2(k_{i-1})$. We now plug in the separation ansatz of (6.3.13) into (6.3.12) which will allow us to write (6.3.12) in a compact way. We explain our strategy with the help of the first summand of (6.3.12): For simplicity, we leave out the k-dependent prefactor and the coefficient C_k giving us

$$
\begin{aligned}
&\Big[\sqrt{|\mu_i-2||\nu_i-2|}\;C_{\mu_i-2}C_{\nu_i-2}-\sqrt{|\mu_i+2||\nu_i-2|}\;C_{\mu_i+2}C_{\nu_i-2}\\
&-\sqrt{|\mu_i-2||\nu_i+2|}\;C_{\mu_i-2}C_{\nu_i+2}+\sqrt{|\mu_i+2||\nu_i+2|}\;C_{\mu_i+2}C_{\nu_i+2}\Big]\,.
\end{aligned}
\tag{6.3.14}
$$

In this equation, we can factor out $\sqrt{|\mu_i-2|}C_{\mu_i-2}$ and $\sqrt{|\mu_i+2|}C_{\mu_i+2}$ to arrive at

$$
\begin{aligned}
&\Big[\sqrt{|\mu_i-2|}C_{\mu_i-2}\left(\sqrt{|\nu_i-2|}C_{\nu_i-2}-\sqrt{|\nu_i+2|}C_{\nu_i+2}\right)\\
&+\sqrt{|\mu_i+2|}C_{\mu_i+2}\left(-\sqrt{|\nu_i-2|}C_{\nu_i-2}+\sqrt{|\nu_i+2|}C_{\nu_i+2}\right)\Big]\,.
\end{aligned}
\tag{6.3.15}
$$

As the terms in round brackets equal up to a minus sign, we can rewrite this equation in a compact form:

$$
\left(\sqrt{|\mu_i+2|}C_{\mu_i+2}-\sqrt{|\mu_i-2|}C_{\mu_i-2}\right)\cdot\left(\sqrt{|\nu_i+2|}C_{\nu_i+2}-\sqrt{|\nu_i-2|}C_{\nu_i-2}\right).
\tag{6.3.16}
$$

We can now proceed analogously for the other two terms of (6.3.12): For the second term of (6.3.12), we use the separation ansatz in (6.3.13) and obtain the following expression whereat we leave out the μ-dependent prefactor and the coefficients $C_{\mu_1}...C_{\mu_N}$:

$$\begin{aligned}\Big[&\sqrt{|\nu_i-1||k_i+k_{i-1}-2|}\; C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}-1} - \sqrt{|\nu_i-1||k_i+k_{i-1}-2|}\; C_{k_i-2,\mu,\nu_i-1,\nu_{i+1}+1}\\ &+\sqrt{|\nu_i+1||k_i+k_{i-1}-2|}\; C_{k_i-2,\mu,\nu_i+1,\nu_{i+1}-1} - \sqrt{|\nu_i+1||k_i+k_{i-1}-2|}\; C_{k_i-2,\nu_i+1,\mu,\nu_{i+1}+1}\\ &-\sqrt{|\nu_i-1||k_i+k_{i-1}+2|}\; C_{k_i+2,\mu,\nu_i-1,\nu_{i+1}-1} + \sqrt{|\nu_i-1||k_i+k_{i-1}+2|}\; C_{k_i+2,\nu_i-1,\mu,\nu_{i+1}+1}\\ &-\sqrt{|\nu_i+1||k_i+k_{i-1}+2|}\; C_{k_i+2,\mu,\nu_i+1,\nu_{i+1}-1} + \sqrt{|\nu_i+1||k_i+k_{i-1}+2|}\; C_{k_i+2,\nu_i+1,\mu,\nu_{i+1}+1}\Big].\end{aligned} \tag{6.3.17}$$

Here, we deviate from our notation introduced in (6.3.7) and include the unshifted label k_{i-1} in $C_{k_i\pm2,k_{i-1}}$. Hereby, we want to indicate that k_i and k_{i-1} are coupled via the prefactors $\sqrt{|k_i+k_{i-1}\pm2|}$. We can now sort (6.3.17) by the factors $\sqrt{|k_i+k_{i-1}\pm2|}C_{k_i\pm2,k_{i-1}}$ which yields

$$\begin{aligned}&\sqrt{|k_i+k_{i-1}-2|}C_{k_i-2,k_{i-1}}\Big(\sqrt{|\nu_i-1|}C_{\nu_i-1}C_{\nu_{i+1}-1} - \sqrt{|\nu_i-1|}C_{\nu_i-1}C_{\nu_{i+1}+1}\\ &\qquad+\sqrt{|\nu_i+1|}C_{\nu_i+1}C_{\nu_{i+1}-1} - \sqrt{|\nu_i+1|}C_{\nu_i+1}C_{\nu_{i+1}+1}\Big)\\ &+\sqrt{|k_i+k_{i-1}+2|}C_{k_i+2,k_{i-1}}\Big(-\sqrt{|\nu_i-1|}C_{\nu_i-1}C_{\nu_{i+1}-1} + \sqrt{|\nu_i-1|}C_{\nu_i-1}C_{\nu_{i+1}+1}\\ &\qquad-\sqrt{|\nu_i+1|}C_{\nu_i+1}C_{\nu_{i+1}-1} + \sqrt{|\nu_i+1|}C_{\nu_i+1}C_{\nu_{i+1}+1}\Big).\end{aligned} \tag{6.3.18}$$

The terms in round brackets equal up to a minus sign and can be factorized themselves: Factoring out $\sqrt{|\nu_i\pm1|}C_{\nu_i\pm1}$ gives us

$$\begin{aligned}&\sqrt{|\nu_i-1|}C_{\nu_i-1}\left(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\right)+\sqrt{|\nu_i+1|}C_{\nu_i+1}\left(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\right)=\\ &\qquad=\left(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\right)\left(\sqrt{|\nu_i-1|}C_{\nu_i-1}+\sqrt{|\nu_i+1|}C_{\nu_i+1}\right),\end{aligned} \tag{6.3.19}$$

where we used that the terms in round brackets in the first equation are the same and can be factored out. Combining (6.3.19) with (6.3.18) gives us the compact equation

$$\begin{aligned}&\left(\sqrt{|k_i+k_{i-1}-2|}C_{k_i-2,k_{i-1}}-\sqrt{|k_i+k_{i-1}+2|}C_{k_i+2,k_{i-1}}\right)\times\\ &\times\left(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\right)\left(\sqrt{|\nu_i-1|}C_{\nu_i-1}+\sqrt{|\nu_i+1|}C_{\nu_i+1}\right).\end{aligned} \tag{6.3.20}$$

We can carry out the same steps for the third term of (6.3.12) which equals the second term of (6.3.12) if we interchange the μ- and ν-labels. Then, we obtain:

$$\begin{aligned}&\left(\sqrt{|k_i+k_{i-1}-2|}C_{k_i-2,k_{i-1}}-\sqrt{|k_i+k_{i-1}+2|}C_{k_i+2,k_{i-1}}\right)\times\\ &\times\left(C_{\mu_{i+1}-1}-C_{\mu_{i+1}+1}\right)\left(\sqrt{|\mu_i-1|}C_{\mu_i-1}+\sqrt{|\mu_i+1|}C_{\mu_i+1}\right)\end{aligned} \tag{6.3.21}$$

We see that (6.3.20) and (6.3.21) share the same difference equation for the k-coefficients. Therefore, when reinserting (6.3.20) and (6.3.21) into (6.3.12), we can factor out the difference equations for the k-coefficients. Using this and (6.3.16), we obtain the following expression for (6.3.12):

$$
\begin{aligned}
0 \overset{!}{=} \Big(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\Big)\, C_k \cdot \Big(\sqrt{|\mu_i+2|}C_{\mu_i+2}-\sqrt{|\mu_i-2|}C_{\mu_i-2}\Big)\times \\
\times\Big(\sqrt{|\nu_i+2|}C_{\nu_i+2}-\sqrt{|\nu_i-2|}C_{\nu_i-2}\Big) \\
+\Big(\sqrt{|k_i+k_{i-1}-2|}\; C_{k_i-2,k_{i-1}}-\sqrt{|k_i+k_{i-1}+2|}\; C_{k_i+2,k_{i-1}}\Big)\times \\
\times\Bigg[\Big(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|}\Big)\, C_\mu\Big(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\Big)\Big(\sqrt{|\nu_i-1|}\; C_{\nu_i-1}+\sqrt{|\nu_i+1|}\; C_{\nu_i+1}\Big) \\
+\Big(\sqrt{|\nu_i+1|}-\sqrt{|\nu_i-1|}\Big)C_\nu\Big(C_{\mu_{i+1}-1}-C_{\mu_{i+1}+1}\Big)\Big(\sqrt{|\mu_i-1|}\; C_{\mu_i-1}+\sqrt{|\mu_i+1|}\; C_{\mu_i+1}\Big)\Bigg].
\end{aligned}
\tag{6.3.22}
$$

Note that this equation has to be satisfied at all vertices i and for all labels $k_i, k_{i-1}, \mu_i, \mu_{i+1}, \nu_i$, and ν_{i+1}. Furthermore, this equation consists of the sum of two terms, each having a nice product form. To make the action of the Euclidean operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$ vanish, we take advantage of the product form of (6.3.22) by setting the individual factors to zero. For this, there are several possibilities for which we have to check if the corresponding difference equations are solvable. We will discuss this in the following two chapters. First, we state some of the most immediate possibilities to satisfy (6.3.22), whereat we have to keep in mind that the corresponding conditions on the coefficients have to be satisfied for *all* i:

- To annihilate the first term of (6.3.22), we could impose the linear, homogeneous difference equations with nonconstant coefficients

$$
\sqrt{|\mu_i+2|}C_{\mu_i+2}-\sqrt{|\mu_i-2|}C_{\mu_i-2}=0 \text{ or } \sqrt{|\nu_i+2|}C_{\nu_i+2}-\sqrt{|\nu_i-2|}C_{\nu_i-2}=0, \tag{6.3.23}
$$

  which are of fourth order in μ_i and ν_i, respectively.

- To annihilate the second term of (6.3.22), we recognize that it consists of two factors: One factor is given by a relation between k-coefficients, which we could impose to be zero, giving us the linear, homogeneous difference equation with nonconstant coefficients

$$
\sqrt{|k_i+k_{i-1}-2|}\; C_{k_i-2,k_{i-1}}-\sqrt{|k_i+k_{i-1}+2|}\; C_{k_i+2,k_{i-1}}=0, \tag{6.3.24}
$$

  which is of fourth order in k_i. The other factor is given by the sum of two terms which both are a product of two relations between the ν- and the μ-coefficients, respectively. Therefore, to make the second term of (6.3.22) vanish, we may, alternatively to (6.3.24), impose the linear, homogeneous difference equations with constant coefficients

$$
C_{\mu_{i-1}}-C_{\mu_{i+1}}=0 \text{ and } C_{\nu_{i-1}}-C_{\nu_{i+1}}=0, \tag{6.3.25}
$$

  which are of second order in μ_i and ν_i, respectively. We could also impose the linear, homogeneous difference equations with nonconstant coefficients

$$
\sqrt{|\mu_i-1|}C_{\mu_i}+\sqrt{|\mu_i+1|}C_{\mu_i}=0 \text{ and } \sqrt{|\mu_i-1|}C_{\nu_i}+\sqrt{|\nu_i+1|}C_{\nu_i}=0, \tag{6.3.26}
$$

  which are of second order in μ_i and ν_i, respectively.

It turns out that any of the above listed equations is solvable $\forall i = 1, ..., N$, the exception being (6.3.24), which cannot be solved at all vertices. The problem in (6.3.24) arises from the square root prefactors, which couple two k-labels from neighboring vertices. We will explain this in more detail in section 6.3.3. In the following two chapters, we discuss the possibilities for combining the relations (6.3.23)-(6.3.26) such that (6.3.22) is satisfied. It appears that we have two main classes of solutions: One class restricts only coefficients with μ- and ν-labels, cf. section 6.3.2, while the other restricts coefficients with all kinds of labels, cf. section 6.3.3. We will also discuss the explicit solutions of (6.3.23)-(6.3.26).

6.3.2 Explicit solutions restricting only μ- and ν-labels

In this chapter, we state explicit solutions of (6.3.22) that only involve conditions on coefficients with μ- and ν-labels, i.e., on the coefficients C_{μ_i} and C_{ν_i} for $i = 1, ..., N$. Thus, for the Gowdy states this corresponds to only the scalar X and Y degrees of freedom being influenced, while we can choose the coefficients C_k freely, i.e., the $U(1)$ degrees of freedom remain arbitrary.

First, we discuss the difference equations we need to impose to satisfy (6.3.22) without restricting the coefficients C_k. These difference equations are given by

$$\sqrt{|\mu_i + 1|}C_{\mu_i+1} + \sqrt{|\mu_i - 1|}C_{\mu_i-1} = 0 \ \forall i \in \{1, ...N\} \quad \text{and} \tag{6.3.27}$$

$$\sqrt{|\nu_i + 1|}C_{\nu_i+1} + \sqrt{|\nu_i - 1|}C_{\nu_i-1} = 0 \quad \text{or} \quad C_{\nu_i+1} - C_{\nu_i-1} = 0 \ \forall i \in \{1, ...N\}. \tag{6.3.28}$$

We now explain why these conditions suffice to satisfy (6.3.22). We see that (6.3.27) is contained in the second term of (6.3.22) but will annihilate only part of it: The term containing shifted ν-coefficients survives. To make this residual term vanish, we observe that it has a product form. Thus we can impose one of the factors to be zero, i.e., we can choose on of the difference equations in (6.3.28). Up to now we only justified how (6.3.27) and (6.3.28) make the second term of (6.3.22) vanish. We claim now that (6.3.27) also annihilates the first term of (6.3.22). In particular, we can show that if (6.3.27) is satisfied then also

$$\sqrt{|\mu_i + 2|}C_{\mu_i+2} - \sqrt{|\mu_i - 2|}C_{\mu_i-2} = 0 \tag{6.3.29}$$

is true which makes the first contribution of (6.3.22) vanish as the relation in (6.3.29) appears there. To show that (6.3.29) follows from (6.3.27), we shift in (6.3.27) μ_i by ± 1 to obtain

$$\sqrt{|\mu_i + 2|}C_{\mu_i+2} + \sqrt{|\mu_i|}C_{\mu_i} = 0 \quad \text{and} \quad \sqrt{|\mu_i|}C_{\mu_i} + \sqrt{|\mu_i - 2|}C_{\mu_i-2} = 0. \tag{6.3.30}$$

Taking the difference between these two equations recovers (6.3.29) which finishes our argument. Note that if we assume that (6.3.29) holds, (6.3.27) is not fulfilled. To see this, we shift (6.3.29) by ± 1 in μ_i to obtain

$$\sqrt{|\mu_i + 3|}C_{\mu_i+3} - \sqrt{|\mu_i - 1|}C_{\mu_i-1} = 0 \quad \text{and} \quad \sqrt{|\mu_i + 1|}C_{\mu_i+1} + \sqrt{|\mu_i - 3|}C_{\mu_i-3} = 0. \tag{6.3.31}$$

Taking the difference between these two equations will not recover (6.3.27). In summary, we have that if (6.3.27) is fulfilled, also (6.3.29) holds, but not vice versa. We conclude that imposing the difference equations (6.3.27) and (6.3.28) solves (6.3.22). We can now proceed to solving these equations explicitly.

In the following, we state the explicit solutions of the difference equations in (6.3.27) and (6.3.28):

- For solving (6.3.27), we recall the definition of the set m_i in which μ_i takes values, cf. (6.0.5):

$$m_i := \{\widetilde{\mu}_i + p \mid p \in \mathbb{Z}\}, \tag{6.3.32}$$

where $\widetilde{\mu}_i$ is an arbitrary real number. We observe that if we choose $\widetilde{\mu}_i$ to be any integer value, the resulting sets m_i will be the same as $\widetilde{\mu}_i$ gets shifted by an integer p in (6.3.32). Hence, also the relations between the coefficients given by (6.3.27) coincide. Analogously we observe that if $\widetilde{\mu}_i$ is not an integer but any other real number, we can restrict to $\widetilde{\mu}_i \in (0,1)$ and also recover all possible sets m_i. In total, we can restrict without loss of generality to the cases of $\widetilde{\mu}_i = 0$ and $\widetilde{\mu}_i \in (0,1)$ and discuss the solutions of (6.3.27) for the corresponding label sets m_i. First, we rewrite (6.3.27) in a form that is more convenient for finding an iterative solution: We first shift μ_i by $+1$ in (6.3.27) and obtain

$$\sqrt{|\mu_i + 2|}C_{\mu_i+2} + \sqrt{|\mu_i|}C_{\mu_i} = 0. \tag{6.3.33}$$

We now isolate one coefficient for which we have two possibilities:

$$C_{\mu_i+2} = -\sqrt{\frac{|\mu_i|}{|\mu_i+2|}}C_{\mu_i} = 0 \tag{6.3.34}$$

$$C_{\mu_i} = -\sqrt{\frac{|\mu_i+2|}{|\mu_i|}}C_{\mu_i+2} = 0. \tag{6.3.35}$$

Using this, we first solve (6.3.27) for $\mu_i \in \mathbb{Z}$ which corresponds to $\widetilde{\mu}_i = 0$. Writing out (6.3.34) and (6.3.35) for some values of μ_i gives us:

μ_i	(6.3.34)
0	$C_2 = 0 \cdot C_0 = 0$
1	$C_3 = -\sqrt{\frac{1}{3}}C_1$
2	$C_4 = -\sqrt{\frac{1}{3}}C_2 = 0$
3	$C_5 = -\sqrt{\frac{3}{5}}C_3 = +\sqrt{\frac{1}{5}}C_1$
4	$C_6 = -\sqrt{\frac{4}{6}}C_4 = 0$
5	$C_7 = -\sqrt{\frac{5}{7}}C_5 = -\sqrt{\frac{1}{7}}C_1$

μ_i	(6.3.35)
-1	$C_{-1} = -C_1$
-2	$C_{-2} = -0 \cdot C_0 = 0$
-3	$C_{-3} = -\sqrt{\frac{1}{3}}C_{-1} = +\sqrt{\frac{1}{3}}C_1$
-4	$C_{-4} = -\sqrt{\frac{2}{4}}C_{-2} = 0$
-5	$C_{-5} = -\sqrt{\frac{3}{5}}C_{-3} = -\sqrt{\frac{1}{5}}C_1$
-6	$C_{-6} = -\sqrt{\frac{4}{6}}C_{-4} = 0$

In these tables, we used, if possible, the expressions for the coefficients from the lines before to either get that the coefficients are zero or related to the coefficient C_1. From the above tables, we can read off the solutions of (6.3.27) for $\mu_i \in \mathbb{Z}$:

$$C_{2\mu_i} = 0 \quad \text{and} \quad C_{2\mu_i+1} = (-1)^{\mu_i}\sqrt{\frac{1}{|2\mu_i+1|}}C_1\ , \tag{6.3.36}$$

i.e., only the odd coefficients are non-zero and C_1 serves as an initial condition. Note that C_0 also remains arbitrary, as can be seen from the first line of the left and the second line of the right above table. Now, we consider the case of non-integer values for $\widetilde{\mu}_i$ which corresponds to $\widetilde{\mu}_i \in (0,1)$. Evaluating (6.3.34) at $\mu_i = \widetilde{\mu}_i$ and considering shifts by positive integers gives us the following equations:

$\mu_i =$	(6.3.34)
$\widetilde{\mu}_i$	$C_{\widetilde{\mu}_i+2} = -\sqrt{\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i+2}\rvert}C_{\widetilde{\mu}_i}$
$\widetilde{\mu}_i+1$	$C_{\widetilde{\mu}_i+3} = -\sqrt{\lvert\frac{\widetilde{\mu}_i+1}{\widetilde{\mu}_i+3}\rvert}C_{\widetilde{\mu}_i+1}$
$\widetilde{\mu}_i+2$	$C_{\widetilde{\mu}_i+4} = -\sqrt{\lvert\frac{\widetilde{\mu}_i+2}{\widetilde{\mu}_i+4}\rvert}C_{\widetilde{\mu}_i+2} = +\sqrt{\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i+4}\rvert}C_{\widetilde{\mu}_i}$
$\widetilde{\mu}_i+3$	$C_{\widetilde{\mu}_i+5} = -\sqrt{\lvert\frac{\widetilde{\mu}_i+3}{\widetilde{\mu}_i+5}\rvert}C_{\widetilde{\mu}_i+3} = +\sqrt{\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i+5}\rvert}C_{\widetilde{\mu}_i+1}$

Evaluating (6.3.35) at $\mu_i = \widetilde{\mu}_i$ and considering shifts by negative integers gives us the following equations:

$\mu_i =$	(6.3.35)
$\widetilde{\mu}_i-1$	$C_{\widetilde{\mu}_i-1} = -\sqrt{\lvert\frac{\widetilde{\mu}_i+1}{\widetilde{\mu}_i-1}\rvert}C_{\widetilde{\mu}_i+1}$
$\widetilde{\mu}_i-2$	$C_{\widetilde{\mu}_i-2} = -\sqrt{\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i-2}\rvert}C_{\widetilde{\mu}_i}$
$\widetilde{\mu}_i-3$	$C_{\widetilde{\mu}_i-3} = -\sqrt{\lvert\frac{\widetilde{\mu}_i-1}{\widetilde{\mu}_i-3}\rvert}C_{\widetilde{\mu}_i-1} = +\sqrt{\lvert\frac{\widetilde{\mu}_i+1}{\widetilde{\mu}_i-3}\rvert}C_{\widetilde{\mu}_i+1}$
$\widetilde{\mu}_i-4$	$C_{\widetilde{\mu}_i-4} = -\sqrt{\lvert\frac{\widetilde{\mu}_i-2}{\widetilde{\mu}_i-4}\rvert}C_{\widetilde{\mu}_i-2} = +\sqrt{\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i-4}\rvert}C_{\widetilde{\mu}_i}$

In the above tables, we used, if possible, the expressions for the coefficients from the lines before. Furthermore, we can read off the solution of (6.3.27) from the above tables:

$$\begin{aligned} C_{\widetilde{\mu}_i+2n} &= (-1)^n\sqrt{\left\lvert\frac{\widetilde{\mu}_i}{\widetilde{\mu}_i+2n}\right\rvert}C_{\widetilde{\mu}_i}, \\ C_{\widetilde{\mu}_i+2n+1} &= (-1)^n\sqrt{\left\lvert\frac{\widetilde{\mu}_i+1}{\widetilde{\mu}_i+2n+1}\right\rvert}C_{\widetilde{\mu}_i+1}, \end{aligned} \tag{6.3.37}$$

with $n \in \mathbb{Z}$. We obtain that all coefficients are in general non-zero and $C_{\widetilde{\mu}_i}, C_{\widetilde{\mu}_i+1}$ serve as an initial condition. We also want to remark that we focused on $\widetilde{\mu}_i = 0$ and $\widetilde{\mu}_i = (0,1)$ because for general $\widetilde{\mu}_i$, we cannot write down a solution of the form of (6.3.37) which is generally valid as we divide by zero for some choices of $\widetilde{\mu}_i$ in (6.3.37). This happens for example in the second equation of (6.3.37) for $\widetilde{\mu}_i = -1$ and $n = 0$.

- Now, we state the explicit solution of (6.3.28). We already solved the first equation of (6.3.28) in the point before. For the other equation, we shift ν_i by $+1$ and obtain

$$C_{\nu_i+2} - C_{\nu_i} = 0. \tag{6.3.38}$$

This difference equations is solved in [147] on page 121: We first determine the characteristic equation of (6.4.19), i.e., we make the ansatz $C_{\nu_i} = r^{\nu_i}$ with r a to be determined

real or complex constant. Plugging this ansatz into (6.3.38) gives us

$$r^2 - 1 = (r+1)(r-1) = 0. \tag{6.3.39}$$

This yields us the roots $r_1 = 1$ and $r_2 = -1$. The fundamental set of solutions of (6.3.38) is given by $C^{(1)}_{\nu_i} = (r_1)^{\nu_i} = 1$ and $C^{(2)}_{\nu_i} = (r_2)^{\nu_i} = (-1)^{\nu_i}$. The general solution is then given by a linear combination of the fundamental solutions:

$$C_{\nu_i} = c_1 + c_2(-1)^{\nu_i}, \tag{6.3.40}$$

where c_1 and c_2 are arbitrary constant coefficients. Using $-1 = e^{i\pi}$ we may rewrite the fundamental solution $C^{(2)}_{\nu_i}$ as $\cos(\pi\nu_i)$ and (6.3.40) as

$$C_{\nu_i} = c_1 + c_2\cos(\pi\nu_i), \tag{6.3.41}$$

where we may also add a function $\sin(\pi\nu_i)$ which, however, is zero for the case $\nu_i \in \mathbb{Z}$.

Note that in (6.3.27) and (6.3.28) we can also exchange all μ- with all ν-labels, giving us another way to solve (6.3.22). The reason for this is that all terms of (6.3.22) are symmetric with respect to exchanging all μ with all ν-labels. Furthermore, we could solve (6.3.27) and (6.3.28) only for certain vertices i and for the remaining vertices, we could solve the versions of (6.3.27) and (6.3.28) with all μ- and ν-labels interchanged.

To conclude this section, we make two remarks:

- Note that (6.3.27) and (6.3.28) do not restrict the coefficients C_k. Therefore, we can define the states φ in (6.0.3) that correspond to the solutions of (6.3.27) and (6.3.28) as

$$\varphi_k := \sum_{\mu\in m}\sum_{\nu\in n} C_k C_{\mu_1}...C_{\mu_N}C_{\nu_1}...C_{\nu_N}\,|k,\mu,\nu\rangle\,, \tag{6.3.42}$$

  i.e., we can omit the sum in (6.0.3) and label the state by $k := (k_1, k_2, ..., k_N)$. Note that we included the coefficient C_k in the above equation, representing the possibility that we can multiply the state by an arbitrary function of the k-labels.

- Now, we comment on the norm of the state given in (6.3.42). We already computed the norm of a general formal state φ in (6.2.3), the only difference now is that we do not have a sum over the k-labels:

$$<\varphi_k, \varphi_k> = \sum_{\mu\in m}\sum_{\nu\in n} |C_k C_{\mu_1}...C_{\mu_N}C_{\nu_1}...C_{\nu_N}|^2. \tag{6.3.43}$$

  The explicit form of the non-zero coefficients C_{μ_i} and C_{ν_i} is for all i either given by a inverse square root function, cf. (6.3.36) and (6.3.37), or a constant or cosine-function, cf. (6.3.41). If we plug these functions into (6.3.43), the sums do not converge, as we either end up with an infinite sum over a constant or cosine-function, or an infinite sum of the form

$$\sum_{\mu_i\in m_i}\frac{1}{\mu_i}, \tag{6.3.44}$$

  where we considered here as an example only the sum over μ_i. This infinite sum can be traced back to the harmonic series which is not converging. Hence, our solutions φ_k are not normalizable. This, however, is a general scenario occurring for an operator having a

continous spectrum. If we consider for example the momentum operator $\hat{\vec{p}} = -i/\hbar\vec{\nabla}$ in three dimensions, it has a continous spectrum and plane waves as eigenfunctions. Plane waves, however, are not square integrable and therefore not contained in the Hilbert space $\mathcal{H} := L_2(\mathbb{R}^3, d^3x)$, which is the space of functions on $\mathbb{R}^3$ being square integrable with respect to the Lebesgue measure d^3x. Nevertheless, one can perceive plane waves as so-called generalized eigenfunctions of $\hat{\vec{p}}$ or rather as distributions. See for example chapter VI of [150] for an introduction into the theory of distributions. More precisely, the eigenfunctions of $\hat{\vec{p}}$ have a meaning as the linear functionals

$$l_{\vec{p}}(.) :=< \vec{p}, . >:= \int \mathrm{d}^3x \; e^{\frac{i}{\hbar}\vec{x}\vec{p}} \quad . \quad , \tag{6.3.45}$$

which are defined on a dense domain of $\mathcal{H}$, namely the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, which is the space of smooth functions on $\mathbb{R}^3$ of rapid decrease. To explain this a little bit more, we consider the general linear functional

$$l(.) := \int \mathrm{d}^3p \; c(\vec{p}) \; < \vec{p}, . >, \tag{6.3.46}$$

with arbitrary coefficients $c(\vec{p})$ and $< \vec{p}, . >$ defined as in (6.3.45). Applying (6.3.46) to an element of $\mathcal{H} := L_2(\mathbb{R}^3, d^3x)$ does not result in a finite complex number for general coefficients $c(\vec{p})$ and hence, (6.3.46) would not define a linear functional on $\mathcal{H}$. However, if we restrict to a dense domain of $\mathcal{H}$, namely the Schwartz space $\mathcal{S}$, (6.3.46) indeed defines a linear functional: Applying l to $f \in \mathcal{S}$ gives us

$$l(f) = \int \mathrm{d}^3p \; c(\vec{p}) \; < \vec{p}, f > . \tag{6.3.47}$$

Now, we can use the Fourier transform of f, given by $f(x) = \int \mathrm{d}^3p' \widetilde{f}(\vec{p}')e^{\frac{i}{\hbar}\vec{x}\vec{p}'}$ where $\widetilde{f}$ is of rapid decrease. Inserting this into (6.3.47) and using $< \vec{p}, \vec{p}' >= \delta(\vec{p} - \vec{p}')$, we obtain

$$l(f) = \int \mathrm{d}^3p \int \mathrm{d}^3p' \; c(\vec{p})\widetilde{f}(p')\delta(\vec{p} - \vec{p}') = \int \mathrm{d}^3p \; c(\vec{p})\widetilde{f}(p), \tag{6.3.48}$$

which is finite as $\widetilde{f}(p)$ is of rapid decrease. Hence, (6.3.46) really defines a linear functional on the dense subset $\mathcal{S} \subset \mathcal{H}$ and therefore also (6.3.45) does. Thus, although the eigenfunctions of the momentum operator are not normalizable, they have a meaning as linear functionals defined on a dense domain of the Hilbert space. Note that the space of linear functionals $\mathcal{S}'$, which is also called the space of tempered distributions, contains the elements (6.3.46) and is also larger than $\mathcal{H}'$ which is the dual of $\mathcal{H}$, giving us the so-called Gelfand triple or rigged Hilbert space $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}'$. Here, $\mathcal{S} \subset \mathcal{H}$ is clear as not every square integrable function is also of rapid decrease[12]. For $\mathcal{H} \subset \mathcal{S}'$ we use that $\mathcal{H}$ being a Hilbert space is reflexive, i.e., it is isomorphic to its dual $\mathcal{H}'$ which is a consequence of the Riesz lemma[13]. Hence, using the identification $\mathcal{H} \simeq \mathcal{H}'$, we can compare $\mathcal{H}$ to the dual space $\mathcal{S}'$. Furthermore, $\mathcal{H}'$ is a subset or more precisely, it can be embedded into $\mathcal{S}'$, which we illustrate using (6.3.46): If we act with (6.3.46) on a function $f \in \mathcal{H}$, we need to restrict the coefficients to obtain a finite complex numbers such that (6.3.46) is in $\mathcal{H}'$. However, if we let the linear functionals (6.3.46) act on

[12] Choose for instance $f(x) = 1/x$ in one dimension.

[13] This theorem ensures that for a continuous linear functional $T : \mathcal{H} \to \mathbb{C}$ on a Hilbert space $\mathcal{H}$ there exists a unique $v \in \mathcal{H}$ such that $T(.) =< v, . >$ and additionally, we have that $||T|| = \sup_{w \neq 0} \frac{|T(w)|}{||w||} = ||v||$.

the dense domain $\mathcal{S}$ we can keep the coefficients $c(\vec{p})$ arbitrary and in particular choose much more singular coefficients, yielding a larger class of linear functionals than $\mathcal{H}'$. Hence, we arrive at $\mathcal{H}' \subset \mathcal{S}'$. Having discussed the case of the momentum operator, now we want to apply this to our case: The states φ_k in (6.3.42), which correspond to the plane waves, are non-normalizable eigenfunctions of the Euclidean operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$, which corresponds to the momentum operator, with eigenvalue zero. Thus, we would have to show that the zero is contained in the continous part of the spectrum of $\hat{\mathrm{H}}_{\mathrm{eucl}}$. For this, we would have to construct the projection-valued measure associated with $\hat{\mathrm{H}}_{\mathrm{eucl}}$, which is a very hard task for this complicated operator and goes beyond the work of this thesis. The fact that the solution states are not normalizable, however, can be handled by constructing the following linear functionals from the solution states φ_k in analogy to (6.3.45):

$$\Phi_k := \sum_{\mu\in m}\sum_{\nu\in n} \left[C_k C_{\mu_1}...C_{\mu_N} C_{\nu_1}...C_{\nu_N}\right]^* < k, \mu, \nu, . >, \tag{6.3.49}$$

where the coefficients C_{μ_i}, C_{ν_i} are those of the solution state φ_k in (6.3.42). We now want to consider the sector of the infinite tensor product Hilbert space spanned by abstract Gowdy states, which we call $\mathcal{H}_{\mathrm{Gowdy}}$, and find a subset thereof, such that when we apply the functional (6.3.49) to states contained in the subset, we obtain a finite complex number. For this, we consider the following general linear functional defined in analogy to (6.3.46):

$$\Phi = \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C'_{k,\mu,\nu} < k, \mu, \nu, . >, \tag{6.3.50}$$

where m, n are subsets of $\mathbb{R}^N$ as defined in (6.0.5) and in particular the same as for φ_k in (6.3.42). Furthermore, $C'_{k,\mu,\nu}$ denote some arbitrary coefficients . We now consider a general element of $\mathcal{H}_{\mathrm{Gowdy}}$ given by

$$\varphi := \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C''_{k,\mu,\nu} \left|k, \mu, \nu\right\rangle, \tag{6.3.51}$$

with the coefficients $C''_{k,\mu,\nu}$ additionally satisfying

$$\sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} |C''_{k,\mu,\nu}|^2 < \infty \tag{6.3.52}$$

such that φ is normalizable. Applying the linear functional Φ to φ and using the orthonormality of the abstract Gowdy states yields

$$\Phi(\varphi) = \sum_{k'\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} \left[C'_{k,\mu,\nu}\right]^* C''_{k,\mu,\nu}, \tag{6.3.53}$$

which for general $C'_{k,\mu,\nu}$ is only finite if the coefficients $C''_{k,\mu,\nu}$ are chosen appropriately. Hence, (6.3.52) and (6.3.53) are two conditions which need to be satisfied for elements $\varphi \in \mathcal{H}_{\mathrm{Gowdy}}$ in (6.3.51) such that Φ defines a linear functional on them. An example of a state fulfilling the conditions (6.3.52) and (6.3.53) is the following state:

$$\widetilde{\varphi} := \sum_{k\in K^N}\sum_{\mu\in m'}\sum_{\nu\in n'} \widetilde{C}_{k,\mu,\nu} \left|k, \mu, \nu\right\rangle, \tag{6.3.54}$$

where $\widetilde{C}_{k,\mu,\nu}$ denote arbitrary, real or complex coefficients, K is a finite subset of $\mathbb{Z}$ and $m' = m'_1 \times ... \times m'_N, n' = n'_1 \times ... \times n'_N$ are finite subsets of $\mathbb{R}^N$ defined in analogy to (6.0.5) by

$$m'_i := \{\widetilde{\mu}_i + p \mid p \in P \subset \mathbb{Z}, |P| < \infty\} \quad \text{and} \quad n'_i := \{\widetilde{\mu}_i + p \mid p \in P \subset \mathbb{Z}, |P| < \infty\}, \tag{6.3.55}$$

where $i \in \{1, ..., N\}$ and $\widetilde{\mu}_i, \widetilde{\nu}_i \in \mathbb{R}$. The norm of the state $\widetilde{\varphi}$ in (6.3.54) is finite, simply because the summation labels take values in finite sets. Therefore, $\widetilde{\varphi}$ really defines an element of the Gowdy Hilbert space. If we apply the linear functional (6.3.50) to such a state $\widetilde{\varphi}$, we can use the orthonormality of the basis states $|k, \mu, \nu\rangle$ to obtain

$$\Phi(\varphi) = \sum_{k \in K^N} \sum_{\mu \in m'} \sum_{\nu \in n'} \left[C'_{k,\mu,\nu}\right]^* \widetilde{C}_{k,\mu,\nu}, \tag{6.3.56}$$

which is a finite, complex number as the sets K^N, m', and n' are finite. In summary, (6.3.49) defines a linear functional on a subset of $\mathcal{H}_{\text{Gowdy}}$, spanned by states (6.3.51) satisfying (6.3.52) and (6.3.53), where (6.3.54) is an explicit example for such states. As we kept the linear functional (6.3.49) general, we can infer that also the special linear functionals Φ_k defined for each k in (6.3.49) are linear functionals on the domain spanned by states (6.3.51) satisfying (6.3.52) and (6.3.53) . Hence, we can conclude that in analogy to the momentum operator, the solution states φ_k in (6.3.42) do not have a meaning as elements of the Hilbert space $\mathcal{H}_{\text{Gowdy}}$, but as linear functionals on a subset of it.

6.3.3 Explicit solutions restricting all labels

In the following, we present explicit solutions of (6.3.22) that involve conditions on coefficients with all kinds of labels, i.e., k-, μ- and ν-labels. We end up with solutions where some of the μ- and ν-coefficients remain arbitrary. Furthermore, in these solutions, the k-coefficients can be separated in a certain way and contain arbitrary functions of certain k-labels. Additionally, in the case of an odd number of vertices, the k-coefficients do not depend on one particular k-label.

First, we show in the following two paragraphs that the difference equation

$$\sqrt{|k_i + k_{i-1} - 2|}\; C_{k_i - 2, k_{i-1}} - \sqrt{|k_i + k_{i-1} + 2|}\; C_{k_i + 2, k_{i-1}} = 0, \tag{6.3.57}$$

which annihilates the second term of (6.3.22), is only solvable for *all* $i = 1, ..., N$ for $N = 2$. Already for $N = 3$, we are forced to the trivial solution $C_{k_1,k_2,k_3} = 0\ \forall k_1, k_2, k_3$ if we want to solve (6.3.57) for all i, that is, at each vertex.

First, we solve (6.3.57) for all $i = 1, ..., N$ for $N = 2$ vertices. For $i = 1$, (6.3.57) reads as

$$\sqrt{|k_1 + k_2 - 2|}\; C_{k_1 - 2, k_2} - \sqrt{|k_1 + k_2 + 2|}\; C_{k_1 + 2, k_2} = 0 \tag{6.3.58}$$

and for $i = 2$ as

$$\sqrt{|k_1 + k_2 - 2|}\; C_{k_1, k_2 - 2} - \sqrt{|k_1 + k_2 + 2|}\; C_{k_1, k_2 + 2} = 0, \tag{6.3.59}$$

where we used in (6.3.58) that for Gowdy states we have $k_0 = k_2$. For solving (6.3.58), we recognize that in this equation the sum of the labels of the coefficients equals the expression

in the square root prefactors. For example, for the first coefficient of (6.3.58), C_{k_1-2,k_2}, the sum of the labels is k_1+k_2-2 which also appears in the square root prefactor. Therefore, one might think that the solution for a general coefficient C_{k_1,k_2} is just given by $1/\sqrt{|k_1+k_2|}$. Plugging this ansatz into (6.3.58) would cancel the square root prefactors in (6.3.58) and result in $1-1=0$, i.e., (6.3.58) is satisfied. However, for $k_1+k_2=0$ we have to find a different solution than $C_{k_1,k_2}=1/\sqrt{|k_1+k_2|}$ as we would divide by zero in this case. In (6.3.58), this scenario corresponds to the case of $k_1+k_2\pm 2=0$. We restrict in the following to the case $k_1+k_2-2=0$, the other one can be treated analogously. For $k_1+k_2-2=0$, (6.3.58) yields

$$0\cdot C_{k_1-2,k_2}-\sqrt{|k_1+k_2+2|}\;C_{k_1+2,k_2}=0, \tag{6.3.60}$$

from which follows that the coefficient C_{k_1+2,k_2} has to be zero whereas the coefficient C_{k_1-2,k_2} remains undetermined. However, (6.3.58) is true for all k_1 and k_2 and thus comprises a whole set of difference equations. Accordingly, the coefficients C_{k_1+2,k_2} and C_{k_1-2,k_2} will show up in another difference equation, respectively. For C_{k_1+2,k_2} this happens for the labels $k_1'=k_1+4, k_2'=k_2$, for which (6.3.58) reads as

$$\begin{aligned}&\sqrt{|k_1'+k_2'-2|}\;C_{k_1'-2,k_2'}-\sqrt{|k_1'+k_2'+2|}\;C_{k_1'+2,k_2'}=\\ =&\sqrt{|k_1+k_2+2|}\;C_{k_1+2,k_2}-\sqrt{|k_1+k_2+6|}\;C_{k_1+6,k_2}=0,\end{aligned}$$

where we resubstituted $k_1'=k_1+4$ and $k_2'=k_2$. As $C_{k_1+2,k_2}=0$, it follows from the above equation that C_{k_1+6,k_2} has to be zero as well. Again, C_{k_1+6,k_2} appears in another relation and will imply $C_{k_1+10,k_2}=0$ etc. Therefore, we obtain in a first step that for labels fulfilling $k_1+k_2-2=0$, we have that $C_{k_1+2+4\cdot n,k_2}=0$, with $n\geq 0$ a positive integer. Similarly, the coefficient C_{k_1-2,k_2} that remains arbitrary in (6.3.60) appears in another relation, namely for $k_1'=k_1-4, k_2'=k_2$, for which (6.3.58) reads as :

$$\sqrt{|k_1+k_2-6|}\;C_{k_1-6,k_2}-\sqrt{|k_1+k_2-2|}\;C_{k_1-2,k_2}=0. \tag{6.3.61}$$

However, C_{k_1-2,k_2} still remains arbitrary, as the square root prefactor is zero due to $k_1+k_2-2=0$, the condition we imposed at the beginning of our argument. Hence, we can infer from (6.3.61) that C_{k_1-6,k_2} has to be zero. Again, C_{k_1-6,k_2} appears in another difference equation resulting in $C_{k_1-10,k_2}=0$ etc. In total, we get that C_{k_1-2,k_2} with $k_1+k_2-2=0$ is arbitrary and all other coefficients resulting from shifting k_1 in C_{k_1-2,k_2} by $4\cdot n$ with $n\in\mathbb{Z}\setminus 0$ are zero. In general, for an arbitrary coefficient C_{k_1,k_2} with labels k_1,k_2 satisfying $k_1+k_2=0$, we get that C_{k_1,k_2} can be chosen arbitrarily and all coefficients $C_{k_1+4\cdot n,k_2}$ with $n\in\mathbb{Z}\setminus 0$ vanish. If we also choose the arbitrary coefficient C_{k_1,k_2} to be zero, we can treat the case $k_1+k_2=0$ in a compact way by just imposing that $C_{k_1,k_2}=0$ if $|k_1+k_2| \bmod 4=0$. This way, we can write down the solution of (6.3.58) as

$$C_{k_1,k_2}=\begin{cases}0 & \text{if} \quad |k_1+k_2| \bmod 4=0,\\ \frac{1}{\sqrt{|k_1+k_2|}} & \text{otherwise}\end{cases}. \tag{6.3.62}$$

This also solves (6.3.59): We see that (6.3.58) and (6.3.59) are symmetric in k_1,k_2. The equations have exactly the same structure, only that either k_1 or k_2 is shifted by ± 2. Thus, to solve (6.3.59), we can follow the same steps we did for (6.3.58). Then, we get for an arbitrary coefficient C_{k_1,k_2} with labels k_1,k_2 satisfying $k_1+k_2=0$, using 6.3.59, that C_{k_1,k_2} can be chosen arbitrarily and all coefficients $C_{k_1,k_2+4\cdot n}$ with $n\in\mathbb{Z}\setminus 0$ vanish. Setting the

arbitrary coefficient to zero, we again can compactly treat this case by imposing $C_{k_1,k_2} = 0$ for $|k_1 + k_2| \bmod 4 = 0$. For $k_1 + k_2 \neq 0$, we can make the ansatz $C_{k_1,k_2} = 1/\sqrt{|k_1 + k_2|}$. Therefore, (6.3.62) also solves (6.3.59). We end up with the result that for 2 vertices, (6.3.57) can be solved for all i and now proceed to the case of 3 vertices.

Now, we try to solve (6.3.57) for all $i = 1, ..., N$ for $N = 3$ vertices. We will show that already when we try to solve two equations of the type of (6.3.57) simultaneously, we get a contradiction that can only be removed if we choose the trivial solution, i.e., all coefficients C_{k_1,k_2,k_3} are zero. First, we rewrite (6.3.57) for convenience as

$$C_{k_i+4,k_{i-1}} = \sqrt{\frac{|k_i + k_{i-1}|}{|k_i + k_{i-1} + 4|}}\, C_{k_i,k_{i-1}}, \tag{6.3.63}$$

which we get by shifting (6.3.57) in k_i by $+2$ and then isolating the coefficient $C_{k_i+4,k_{i-1}}$ on the left hand side. Note that for $|k_i + k_{i-1} + 4| = 0$, we would have divided by zero. For any such labels k_i, k_{i-1}, however, we will choose the corresponding coefficients to be zero and impose $C_{k_1,k_2} = 0$ if $|k_i + k_{i-1}| \bmod 4 = 0$ in analogy to the solution of (6.3.62). Therefore, we implicitly assume in (6.3.63) that the case $|k_i + k_{i-1}| \bmod 4 = 0$ is excluded. For $i = 1, 2$, (6.3.63) reads as

$$C_{k_1+4,k_2,k_3} = \sqrt{\frac{|k_1 + k_3|}{|k_1 + k_3 + 4|}}\, C_{k_1,k_2,k_3}, \quad C_{k_1,k_2+4,k_3} = \sqrt{\frac{|k_1 + k_2|}{|k_1 + k_2 + 4|}}\, C_{k_1,k_2,k_3}. \tag{6.3.64}$$

We now shift k_1 by $+4$ in the second of the above equations, giving us

$$C_{k_1+4,k_2+4,k_3} = \sqrt{\frac{|k_1 + k_2 + 4|}{|k_1 + k_2 + 8|}}\, C_{k_1+4,k_2,k_3} = \sqrt{\frac{|k_1 + k_2 + 4|}{|k_1 + k_2 + 8|}}\sqrt{\frac{|k_1 + k_3|}{|k_1 + k_3 + 4|}}\, C_{k_1,k_2,k_3}, \tag{6.3.65}$$

where in the second step, we used the first equation of (6.3.64) to replace C_{k_1+4,k_2,k_3}. Shifting k_2 by $+4$ in the first equation of (6.3.64) gives us

$$C_{k_1+4,k_2+4,k_3} = \sqrt{\frac{|k_1 + k_3|}{|k_1 + k_3 + 4|}}\, C_{k_1,k_2+4,k_3} = \sqrt{\frac{|k_1 + k_3|}{|k_1 + k_3 + 4|}}\sqrt{\frac{|k_1 + k_2|}{|k_1 + k_2 + 4|}}\, C_{k_1,k_2,k_3}, \tag{6.3.66}$$

where in the second step, we used the second equation of (6.3.64) to replace C_{k_1,k_2+4,k_3}. Now, we recognize that (6.3.65) and (6.3.66) give two relations that involve the same coefficients C_{k_1+4,k_2+4,k_3} and C_{k_1,k_2,k_3}. In order to match these two conditions, we have to impose

$$\sqrt{\frac{|k_1 + k_2|}{|k_1 + k_2 + 4|}} \stackrel{!}{=} \sqrt{\frac{|k_1 + k_2 + 4|}{|k_1 + k_2 + 8|}},$$

which, however, is a contradiction. The only other possibility to match the conditions stated in (6.3.65) and (6.3.66) is that C_{k_1,k_2,k_3} is zero. Now, we chose k_1, k_2, k_3 arbitrary up to the condition $|k_i + k_{i-1}| \bmod 4 \neq 0$ for $i = 1, 2, 3$. For labels fulfilling $|k_i + k_{i-1}| \bmod 4 = 0$ for $i = 1, 2, 3$, we set the corresponding coefficients to zero anyway. Therefore, we can conclude that for all k_1, k_2, k_3 the coefficients have to be zero. This trivial solution, however, is undesirable. Adding the difference equation (6.3.63) for $i = 3$ to (6.3.64) will not improve the situation: We can construct a contradiction for the $i = 2, 3$ contributions of (6.3.63) the same way we did above for the $i = 1, 2$ contributions. This generalizes to an arbitrary number

of vertices and we end up with the result that we cannot solve (6.3.57) for all $i = 1, ..., N$ simultaneously.

It is, however, possible to solve (6.3.57) in the case of 3 vertices for *one* i. In the following, we exemplarily consider (6.3.57) for $i = 2$, which reads as

$$\sqrt{|k_1 + k_2 - 2|}\; C_{k_1,k_2-2,k_3} - \sqrt{|k_1 + k_2 + 2|}\; C_{k_1,k_2+2,k_3} = 0. \tag{6.3.67}$$

This difference equation can be solved in analogy to the solution for 2 vertices stated in (6.3.62):

$$C_{k_1,k_2,k_3} = \begin{cases} 0 & \text{if} \quad |k_1 + k_2| \bmod 4 = 0, \\ \frac{1}{\sqrt{|k_1+k_2|}} \cdot f_1(k_1) \cdot f_2(k_3) & \text{otherwise.} \end{cases} \tag{6.3.68}$$

Here, $f_1(k_1)$ and $f_2(k_3)$ are arbitrary real or complex functions of k_1 or rather k_3. We show that (6.3.68) is indeed a solution of (6.3.67) by plugging (6.3.68) into (6.3.67): That the first case of (6.3.68) solves (6.3.67) was already shown for 2 vertices, see the arguments before (6.3.62). For the second case of (6.3.68), the coefficients appearing in (6.3.67) are given by

$$C_{k_1,k_2-2,k_3} = \frac{f_1(k_1) \cdot f_2(k_3)}{\sqrt{|k_1 + k_2 - 2|}},\; C_{k_1,k_2+2,k_3} = \frac{f_1(k_1) \cdot f_2(k_3)}{\sqrt{|k_1 + k_2 + 2|}}. \tag{6.3.69}$$

If we plug this into (6.3.67), the square root factors will cancel and what remains is $f_1(k_1)f_2(k_3) - f_1(k_1)f_2(k_3) = 0$. Hence, (6.3.68) is indeed a solution of (6.3.67). Furthermore, we see from the product structure of the solution in (6.3.68) for the non-zero case that we can separate the coefficients as $C_{k_1,k_2,k_3} = C_{k_1,k_2}C_{k_3}$, with C_{k_3} arbitrary and C_{k_1,k_2} given by

$$C_{k_1,k_2} = \begin{cases} 0 & \text{if} \quad |k_1 + k_2| \bmod 4 = 0, \\ \frac{1}{\sqrt{|k_1+k_2|}} \cdot f(k_1) & \text{otherwise,} \end{cases} \tag{6.3.70}$$

where again $f(k_1)$ denotes an arbitrary real or complex function, which we can include because (6.3.67) only contains shifts in the k_2 label. We can also solve (6.3.57) for $i = 1$ or $i = 3$ using an ansatz similar to (6.3.70).

In the following, we solve more than one of the equations (6.3.57) simultaneously for graphs with more than 3 vertices. We have to distinguish between an even and an odd number of vertices:

- **N even:** First, we consider the example of 4 vertices. We can perform the separation ansatz $C_{k_1,k_2,k_3,k_4} = C_{k_1,k_2}C_{k_3,k_4}$ with

$$C_{k_1,k_2} = \begin{cases} 0 & \text{if} \quad |k_1 + k_2| \bmod 4 = 0, \\ \frac{1}{\sqrt{|k_1+k_2|}} \cdot f_1(k_1) & \text{otherwise,} \end{cases} \tag{6.3.71}$$

  solving (6.3.57) for $i = 2$, which reads as

$$\sqrt{|k_1 + k_2 - 2|}\; C_{k_1,k_2-2} - \sqrt{|k_1 + k_2 + 2|}\; C_{k_1,k_2+2} = 0. \tag{6.3.72}$$

  Furthermore, we impose

$$C_{k_3,k_4} = \begin{cases} 0 & \text{if} \quad |k_3 + k_4| \bmod 4 = 0, \\ \frac{1}{\sqrt{|k_3+k_4|}} \cdot f_2(k_3) & \text{otherwise,} \end{cases} \tag{6.3.73}$$

to solve (6.3.57) for $i = 4$, which reads as

$$\sqrt{|k_3 + k_4 - 2|}\; C_{k_3,k_4-2} - \sqrt{|k_3 + k_4 + 2|}\; C_{k_3,k_4+2} = 0. \tag{6.3.74}$$

We can generalize this to an arbitrary, even number of vertices $N = 2 \cdot n$, with $n \geq 2$ a positive integer number. We exclude the special case of two vertices, which we treated at the beginning of this section. The first step is to perform the separation ansatz

$$C_{k_1,...,k_N} = C_{k_1,k_2} \cdot C_{k_3,k_4} \cdot ... \cdot C_{k_{N-1},k_N}. \tag{6.3.75}$$

Here, the coefficients C_{k_r,k_s} with $(r, s) = (1, 2), (3, 4), ..., (N - 1, N)$ are determined by

$$C_{k_r,k_s} = \begin{cases} 0 & \text{if} \quad |k_r + k_s| \bmod 4 = 0, \\ \frac{1}{\sqrt{|k_r+k_s|}} \cdot f_I(k_r) & \text{otherwise,} \end{cases} \tag{6.3.76}$$

in analogy to (6.3.71) and (6.3.73). This solves (6.3.57) for $i = 2, 4, ..., N$. Furthermore, in (6.3.76), I ranges from 1 to $N/2$ as we have $N/2$ possible values for the index r. Thus, we end up with $N/2$ arbitrary functions f_I.

- **N odd:** Here, we proceed in analogy to the case of an even number of vertices. First, we consider the example of 5 vertices. We can perform the separation ansatz $C_{k_1,k_2,k_3,k_4,k_5} = C_{k_1,k_2}C_{k_3,k_4}C_{k_5}$, with C_{k_1,k_2} and C_{k_3,k_4} determined by (6.3.71) and (6.3.73). This way, (6.3.57) is solved for $i = 2, 4$. The difference to the case of an even number of vertices is now that we have an arbitrary coefficient C_{k_5}. We can generalize this to an arbitrary number of odd vertices $N = 2 \cdot n + 1$, with $n \geq 1$ a positive integer number. We do not treat the case of one vertex here, for which we would obtain only one equation of the type of (6.3.57) that can be solved iteratively. The separation ansatz for general, odd N is now given by

$$C_{k_1,...,k_N} = C_{k_1,k_2} \cdot C_{k_3,k_4} \cdot ... \cdot C_{k_{N-2},k_{N-1}} \cdot C_{k_N}. \tag{6.3.77}$$

Here, the coefficients C_{k_r,k_s} for $(r, s) = (1, 2), (3, 4), ..., (N - 2, N - 1)$ are determined in analogy to (6.3.76). This way, we solve (6.3.57) for $i = 2, 4, ..., N - 1$. We end up with $(N - 1)/2$ arbitrary functions and one arbitrary coefficient C_{k_N}.

Note that the separation ansätze of (6.3.75) and (6.3.77) are not the only possible ones. For example for an even number of vertices, one could instead of (6.3.75) choose the ansatz

$$C_{k_1,...,k_N} = C_{k_1,k_N} \cdot C_{k_2,k_3} \cdot ... \cdot C_{k_{N-2},k_N-1}, \tag{6.3.78}$$

with the coefficients C_{k_r,k_s} for $(r, s) = (1, N), (2, 3), ..., (N - 2, N - 1)$ satisfying (6.3.76) and solving (6.3.57) for $i = 1, 3, ..., N - 1$. Note that we used in (6.3.78) that only neighboring k-labels are coupled by square root prefactors in (6.3.57) and that the labels k_1 and k_N are adjacent, as we identify k_{N+1} with k_1 and k_0 with k_N for Gowdy states. Similar to (6.3.78), we could proceed in the case of an odd number of vertices.

Finally, we use the results from the previous paragraph to annihilate the action of the Euclidean operator, i.e., we solve (6.3.22). The solutions we state here restrict coefficients with all kind of labels, i.e., k-, μ- and ν-labels.

- For an even number of vertices, we use (6.3.75) and (6.3.76) to annihilate the second term of (6.3.22) for $i = 2, 4, 6, ..., N$. To make also the first contribution of (6.3.22) vanish for $i = 2, 4, 6, ..., N$, we can choose

$$\sqrt{|\mu_i + 2|}C_{\mu_i+2} - \sqrt{|\mu_i - 2|}C_{\mu_i-2} = 0 \text{ or } \sqrt{|\nu_i + 2|}C_{\nu_i+2} - \sqrt{|\nu_i - 2|}C_{\nu_i-2} = 0. \quad (6.3.79)$$

  To solve the remaining contributions of (6.3.22) for $i = 1, 3, 5, ..., N-1$ we can pick

$$\sqrt{|\mu_i + 1|}C_{\mu_i+1} + \sqrt{|\mu_i - 1|}C_{\mu_i-1} = 0 \text{ and } \sqrt{|\nu_i + 1|}C_{\nu_i+1} + \sqrt{|\nu_i - 1|}C_{\nu_i-1} = 0, \quad (6.3.80)$$

  for which we already showed in section 6.3.2 that these conditions suffice to annihilate both terms of (6.3.22). We recognize from (6.3.79) that the coefficients C_{ν_i} or C_{μ_i} for $i = 1, 3, 5, ..., N-1$ remain arbitrary. Furthermore, as stated in the previous paragraph, we have $N/2$ arbitrary functions involved in the solutions for the k-coefficients, cf. (6.3.76).

- For an odd number of vertices, we can proceed in analogy to the case of an even number of vertices. The only difference is that we annihilate the second term of (6.3.22) using (6.3.77) and (6.3.76) not for $i = 2, 4, 6, ..., N$, as it is the case for N even, but for $i = 2, 4, 6, ..., N-1$. For the first term of (6.3.22) for $i = 2, 4, 6, ..., N-1$, we use (6.3.79) and thus, the coefficients C_{ν_i} or C_{μ_i} for $i = 1, 3, 5, ..., N-1$ remain arbitrary. For the remaining contributions of (6.3.22) for $i = 2, 4, 6, ..., N$, we use (6.3.80). Furthermore, we have $(N-1)/2$ arbitrary functions involved in the solutions of the k-coefficients, cf. (6.3.76), and also the coefficient C_{k_N} remains arbitrary.

To complete the solutions for (6.3.22) we presented in this paragraph, we solve (6.3.79) explicitly. The other occurring difference equations in (6.3.80) we already solved in section 6.3.2. We will exemplarily treat the difference equation for the μ_i-coefficient in (6.3.79), where it suffices to consider the cases $\widetilde{\mu}_i = 0$, i.e., $\mu_i \in \mathbb{Z}$, and $\widetilde{\mu}_i = (0, 1)$, see the discussion in section 6.3.2 below (6.3.32). For $\mu_i \in \mathbb{Z}$, we can solve (6.3.79) by

$$C_{\mu_i} = \begin{cases} 0 & \text{if } \mu_i \in 4 \cdot \mathbb{Z}, \\ \frac{1}{\sqrt{|\mu_i|}} & \text{otherwise.} \end{cases} \quad (6.3.81)$$

The second case results from the observation that in (6.3.79), the expressions in the square roots and the labels of the coefficients coincide. Thus, plugging the ansatz $C_{\mu_i} = 1/\sqrt{|\mu_i|}$ into (6.3.79) cancels the square root factors and results in $1 - 1 = 0$. The first condition of (6.3.81) we can deduce by first evaluating (6.3.79) for $\mu_i = 2$ which gives $C_4 = 0$. Then, we look at (6.3.79) for $\mu_i = 2 + 4 \cdot n$ with $n \in \mathbb{Z} \setminus 0$, from which we can iteratively deduce that $C_4 = C_8 = C_{12} = ... = 0$ and $C_{-4} = C_{-8} = C_{-12} = 0$. Note that C_0 remains arbitrary, as it is not fixed by (6.3.79): C_0 appears in (6.3.79) for $\mu_i = 2, -2$ resulting in the equations

$$\sqrt{4}C_4 - 0 \cdot C_0 = 0 \quad \text{and} \quad 0 \cdot C_0 - \sqrt{4}C_{-4} = 0, \quad (6.3.82)$$

in which C_0 remains undetermined. To write down the solution for C_{μ_i} in a compact way, we did set C_0 to zero in (6.3.81). Now, for μ_i taking non-integer values, i.e., $\widetilde{\mu}_i \in (0, 1)$, we can simply make the ansatz

$$C_{\mu_i} = \frac{1}{\sqrt{|\mu_i|}} \quad (6.3.83)$$

because now we do not divide by zero, as μ_i is non-zero.

We close this section with some remarks:

- The solutions we presented in this section allow arbitrary coefficients. Therefore, we could do the same as in the previous section 6.3.2 in (6.3.42), namely we could label the solution states φ by the unfixed labels. For example, for an even number of vertices and C_{μ_i} for $i = 1, 3, 5, ..., N-1$ arbitrary, we could omit the sums in (6.0.3) over μ_i for $i = 1, 3, 5, ..., N-1$ and denote the solution state as $\varphi_{\mu_1,\mu_3,...,\mu_{N-1}}$.

- As we derived for the solution states in section 6.3.2, also here the norm of the solution states is infinite. The μ- or rather ν-coefficients are solved by inverse square root functions, cf. (6.3.81) and (6.3.83). This results in non-converging sums when computing the norm, as we already explained in section 6.3.2 in the context of (6.3.43). For the k-coefficients we will encounter sums of the form

$$\sum_{k_i \in \mathbb{Z}} \frac{1}{|k_i|},$$

  when computing the norm. This series is divergent, as it can be traced back to the divergent harmonic series. Therefore, the solution states we presented in this section are not normalizable.

- We also want to point out the possibility of combining the solutions presented in the current and the previous section with the zero volume conditions in section 6.2. For example, we can satisfy (6.3.22) by choosing for each vertex one of the conditions presented in the sections 6.2, 6.3.2 or 6.3.3.

This finishes our discussion of the annihilation of the Euclidean operator $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and we now turn to the annihilation of its adjoint in the next chapter.

6.4 Annihilation of the action of the adjoint Euclidean part

In this section, we derive conditions on the coefficients $C_{k,\mu,\nu}$ of a formal state φ that make the action of the adjoint Euclidean operator $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ vanish.

First, we proceed in analogy to section 6.3 to derive a difference equation of the form of (6.3.22). We choose a similar separation ansatz as in (6.3.13), but also separate the k-coefficients:

$$C_{k,\mu,\nu} = C_{k_1} \cdot ... \cdot C_{k_N} \cdot C_{\mu_1} \cdot ... \cdot C_{\mu_N} \cdot C_{\nu_1} \cdot ... \cdot C_{\nu_N}. \tag{6.4.1}$$

We will justify further below, why the separation of the k-coefficients is legitimate. The difference between the Euclidean operator and its adjoint is that for $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, the holonomies act before the volume operator whereas for $\hat{\mathrm{H}}_{\mathrm{eucl}}$ it is exactly opposite. This results in a different action and is the main reason why the difference equations we derive in the following for $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ differ from those of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ in (6.3.22). We illustrate this for the first shifted state in first term of the action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ in (6.1.27), while we omit all k-dependent prefactors as well as the sum over the k-labels for simplicity:

$$\sum_{\mu\in m}\sum_{\nu\in n}\sum_{i=1}^{N} C_{\mu_1}...C_{\mu_N}C_{\nu_1}...C_{\nu_N}\sqrt{|\mu_i+2||\nu_i+2|}\,|k,\mu_i+2,\nu_i+2\rangle\,, \tag{6.4.2}$$

where we already performed the separation ansatz (6.4.1). We can now perform the substitutions $\widetilde{\mu_i} = \mu_i + 2$, $\widetilde{\nu_i} = \nu_i + 2$ in analogy to section 6.3.1. Again, we use the translation

invariance of the sums to arrive at the following expression being equivalent to (6.4.2):

$$\sum_{\mu\in m}\sum_{\nu\in n} C_{\mu_1}...C_{\mu_i-2}...C_{\mu_N}C_{\nu_1}...C_{\nu_i-2}...C_{\nu_N}\sqrt{|\mu_i||\nu_i|}\,|k,\mu,\nu\rangle\,. \tag{6.4.3}$$

We see that the label dependent coefficients are now of the form $\sqrt{|\mu_i||\nu_i|}$ after we transferred the shift. For $\hat{\mathrm{H}}_{\mathrm{eucl}}$, the same term would have lead to the factor $\sqrt{|\mu_i-2||\nu_i-2|}$. We can proceed the same way for the other three states contained in the first term of the action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ in (6.1.27). For the other two terms of (6.1.27), we consider exemplarily the first state contained in the second term of (6.1.27), while we omit all μ-dependent parts for simplicity:

$$\sum_{k\in\mathbb{Z}}\sum_{\nu\in n} C_{k_1}...C_{k_N}C_{\nu_1}...C_{\nu_N}\sqrt{|k_i+k_{i-1}+2|\nu_i+1|}\,|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle\,. \tag{6.4.4}$$

We now perform the substitutions $\widetilde{k}_i = k_i+2$, $\widetilde{\nu}_i = \nu_i+1$, and $\widetilde{\nu}_{i+1} = \nu_{i+1}+1$. After renaming $\widetilde{k}_i$, $\widetilde{\nu}_i$, and $\widetilde{\nu}_{i+1}$ as k_i, ν_i, and ν_{i+1}, respectively, we arrive at

$$\sum_{k\in\mathbb{Z}}\sum_{\nu\in n} C_{k_1}...C_{k_i-2}...C_{k_N}C_{\nu_1}...C_{\nu_i-1}C_{\nu_{i+1}-1}...C_{\nu_N}\sqrt{|k_i+k_{i-1}|\nu_i|}\,|k,\mu,\nu\rangle\,. \tag{6.4.5}$$

We can proceed similarly for all other terms of the action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ in (6.1.27) to arrive at

$$\begin{aligned}0 \overset{!}{=} \left(\sqrt{|k_i+k_{i-1}+1|}-\sqrt{|k_i+k_{i-1}-1|}\right)\sqrt{|\mu_i|}\sqrt{|\nu_i|}\Big(C_{\mu_i+2}-C_{\mu_i-2}\Big)\Big(C_{\nu_i+2}-C_{\nu_i-2}\Big)+\\ +\sqrt{|k_i+k_{i-1}|}\,(C_{k_i-2}-C_{k_i+2})\times\\ \times\Bigg[\left(\sqrt{|\mu_i+1|}-\sqrt{|\mu_i-1|}\right)\sqrt{|\nu_i|}\Big(C_{\nu_{i+1}-1}-C_{\nu_{i+1}+1}\Big)\,(C_{\nu_i-1}+C_{\nu_i+1})+\\ +\sqrt{|\mu_i|}\left(\sqrt{|\nu_i+1|}-\sqrt{|\nu_i-1|}\right)\Big(C_{\mu_{i+1}-1}-C_{\mu_{i+1}+1}\Big)\,(C_{\mu_i-1}+C_{\mu_i+1})\Bigg].\end{aligned} \tag{6.4.6}$$

We observe that, compared to the analogous expression for the Euclidean operator in (6.3.22), the label-dependent prefactors can be pulled out of the relations between the coefficients. In particular, k_i and k_{i-1} are not coupled through the prefactor $\sqrt{|k_i+k_{i-1}|}$ anymore, which justifies the separation ansatz we made in (6.4.1).

Now, we state possible solutions of (6.4.6). Just like (6.3.22), (6.4.6) consists of the sum of two terms, where each term has a nice product form. To obtain a zero action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, we can set the individual factors to zero. For this, the possible choices involve, in analogy to section 6.3.2 and 6.3.3, only conditions on μ- and ν-labels or conditions on all labels, while certain coefficients remain arbitrary. We present these possible choices in the following, while we always first state the difference equations that have to be satisfied, then explain why they fulfill (6.4.6) and finally, deduce the explicit form of the coefficients satisfying the respective difference equations:

- The first possibility only restricts coefficients with μ- and ν-labels by the difference equations

$$C_{\mu_i-1}-C_{\mu_i+1}=0 \quad\text{and}\quad C_{\nu_i-1}-C_{\nu_i+1}=0 \quad \forall\ i=1,...,N \tag{6.4.7}$$

or alternatively

$$C_{\mu_i-1} + C_{\mu_i+1} = 0 \quad \text{and} \quad C_{\nu_i-1} + C_{\nu_i+1} = 0 \quad \forall\, i = 1, ..., N. \tag{6.4.8}$$

We see that in both cases, the coefficients C_k remain undetermined. Now, we want to show that these two choices satisfy (6.4.6). We begin with (6.4.7): That these conditions make the second term of (6.4.6) vanish is immediate as both relations appear there. The first term of (6.4.6) also vanishes if (6.4.7) is fulfilled: We can show that (6.4.7) also satisfies

$$C_{\mu_i+2} - C_{\mu_i-2} = 0 \quad \text{or} \quad C_{\nu_i+2} - C_{\nu_i-2} = 0, \tag{6.4.9}$$

which annihilates the first term of (6.4.6). To see this, we first shift μ_i and ν_i by ± 1 in (6.4.7), giving us the four difference equations

$$C_{\mu_i+2} = C_{\mu_i},\ C_{\mu_i-2} = C_{\mu_i},\ C_{\nu_i+2} = C_{\nu_i},\ C_{\nu_i-2} = C_{\nu_i}. \tag{6.4.10}$$

Taking the difference between the first two and the last two equations, respectively, recovers the relations in (6.4.9). Thus, (6.4.7) also satisfies (6.4.9) but not vice versa: Starting from (6.4.9) and shifting both μ_i and ν_i by ± 1 yields

$$C_{\mu_i+3} = C_{\mu_i+1},\ C_{\mu_i-3} = C_{\mu_i-1},\ C_{\nu_i+3} = C_{\nu_i-1},\ C_{\nu_i-3} = C_{\nu_i-1}. \tag{6.4.11}$$

Taking the difference between the first two and the last two equations, respectively, will not recover the relations in (6.4.7). Now, we turn to (6.4.8): These relations appear in the second term of (6.4.6) and thus will make it vanish. Furthermore, (6.4.9) will also be true if (6.4.8) is satisfied and therefore, the first term of (6.4.6) vanishes, too. To see this, we shift μ_i and ν_i by ± 2 in (6.4.8) resulting in the relations

$$C_{\mu_i+2} = -C_{\mu_i},\ C_{\mu_i-2} = -C_{\mu_i},\ C_{\nu_i+2} = -C_{\nu_i},\ C_{\nu_i-2} = -C_{\nu_i}. \tag{6.4.12}$$

If we take the difference between the first two and the last two equations, respectively, we recover the relations in (6.4.9). Thus, (6.4.7) also satisfies (6.4.9) but not vice versa. The argument for the latter statement is in analogy to the one we made for the relations in (6.4.7). In total, (6.4.6) is satisfied if (6.4.7) or (6.4.8) is accomplished. In the following, we discuss the explicit solutions of (6.4.7) and (6.4.8). Concerning (6.4.7), we already derived the explicit solution, which is stated in (6.3.41) exemplarily for C_{ν_i}:

$$C_{\nu_i} = c_1 + c_2 \cos(\pi \nu_i). \tag{6.4.13}$$

Concerning (6.4.8), we consider exemplarily the difference equation for ν_i and proceed in analogy as done for (6.3.41), following [147]: We use the ansatz $C_{\nu_i} = r^{\nu_i}$, where r is an arbitrary real or complex number, and write down the characteristic equation of (6.4.8):

$$r^2 + 1 = (r + i)(r - i) = 0. \tag{6.4.14}$$

From this we obtain the roots $r_1 = i$, $r_2 = -i$ and the fundamental solutions $C^{(1)}_{\nu_i} = i^{\nu_i}$, $C^{(1)}_{\nu_i} = -i^{\nu_i}$. Using $\exp(\pm\frac{\pi}{2} i) = \pm i$ we can rewrite the fundamental solutions as

$$C^{(1)}_{\nu_i} = \cos(\frac{\pi}{2}\nu_i), \quad C^{(2)}_{\nu_i} = \sin(\frac{\pi}{2}\nu_i). \tag{6.4.15}$$

The general solution for C_{ν_i} is then given by

$$C_{\nu_i} = c_1 \cos(\frac{\pi}{2}\nu_i) + c_2 \sin(\frac{\pi}{2}\nu_i), \tag{6.4.16}$$

with c_1 and c_2 arbitrary real or complex constants.

- The second possibility involves conditions on coefficients with all kinds of labels. These conditions are given by

$$C_{\mu_i-2} - C_{\mu_i+2} = 0 \quad \text{or} \quad C_{\nu_i-2} - C_{\nu_i+2} = 0 \quad \forall i = 1, ..., N \tag{6.4.17}$$

$$\text{and simultaneously} \quad C_{k_i-2} - C_{k_i+2} = 0 \quad \forall i = 1, ..., N. \tag{6.4.18}$$

  We deduce from the first equation that either all of the μ- or all of the ν-coefficients remain undetermined. To see that the above difference equations satisfy (6.4.6), we observe that both of the relations in (6.4.17) appear in the first summand of (6.4.6) and therefore make it vanish. (6.4.18) appears in the second summand of (6.4.6) and will annihilate this term. To solve (6.4.17) and (6.4.18) explicitly, we observe that these relations have the same form of the following linear, fourth order, homogeneous difference equation with constant coefficients:

$$C_{n-2} - C_{n+2} = 0, \tag{6.4.19}$$

  with $n \in \{k_i, \mu_i, \nu_i\}$. This equation is solved explicitly in [147] on page 125: First, we determine the characteristic equation of (6.4.19), i.e., we make the ansatz $C_n = r^n$ with r a to be determined real or complex constant. Plugging this ansatz into (6.4.19) gives us

$$r^2 - r^{-2} = 0 \Leftrightarrow r^4 - 1 = 0. \tag{6.4.20}$$

  We can factorize this according to

$$0 = r^4 - 1 = (r^2 - 1)(r^2 + 1) = (r + 1)(r - 1)(r + i)(r - i). \tag{6.4.21}$$

  The roots of this equation are given by $r_{1/2} = \pm i$ and $r_{3/4} = \pm 1$. In general, the fundamental set of solutions of (6.4.18) is given by $C_n^{(j)} = r_j^n$ for $j = 1, .., 4$. Using that $\pm i = e^{\pm i\pi/2}$, we obtain $C_n^{(1,2)} = e^{\pm in\pi/2}$, which is equivalent to $C_n^{(1)} = \cos(\pi n/2)$ and $C_n^{(2)} = \sin(\pi n/2)$. Then, the fundamental set of solutions is given by

$$C_n^{(1)} = \cos\left(\frac{\pi}{2}n\right),\ C_n^{(2)} = \sin\left(\frac{\pi}{2}n\right),\ C_n^{(3)} = 1^n = 1,\ C_n^{(4)} = (-1)^n. \tag{6.4.22}$$

  A general solution of (6.4.18) will be a linear combination of the fundamental solutions:

$$C_n = c_1 \cos\left(\frac{\pi}{2}n\right) + c_2 \sin\left(\frac{\pi}{2}n\right) + c_3 + c_4(-1)^n, \tag{6.4.23}$$

  with arbitrary constants $c_m, m = 1, ..., 4$.

We end this chapter with some remarks that are in analogy to those we made at the end of the sections 6.3.2 and 6.3.3:

- The solution states of this chapter are not normalizable, as the coefficients are either constant or given by sine and cosine-functions, cf. (6.4.13), (6.4.16), and (6.4.23). Plugging these solutions into the computation for the norm, cf. (6.2.3), yields infinity.
- Furthermore, some coefficients remain arbitrary: For the first possibility we presented above, the k-coefficients remain arbitrary. For the second possibility, either the μ- or the ν-coefficients remain arbitrary. Therefore, we could label the solution states by the arbitrary labels, just as we did in (6.3.42).
- Note that the zero volume states do not annihilate the adjoint Euclidean operator, as we already explained at the end of section 6.2. Thus, we cannot combine the solutions of this section with the zero volume states, as we discussed at the end of section 6.3.3 for the Euclidean operator.

6.5 Simultaneous annihilation of the Euclidean part and its adjoint

In the following, we present solutions that annihilate the Euclidean part $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and simultaneously its adjoint $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$.

First, we compare the difference equations we need to impose to annihilate $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ separately, i.e., we examine (6.3.22) and (6.4.6) in contrast. We see that almost all difference equations resulting from $\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi = 0$ contain label dependent prefactors. The difference equations resulting from $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}\varphi = 0$ are , concerning the structure of the coefficients, the same as those of $\hat{\mathrm{H}}_{\mathrm{eucl}}$, with the decisive difference of having *constant* prefactors. The only relations that come up in (6.3.22) as well as in (6.4.6) are given by

$$C_{\mu_{i+1}-1} - C_{\mu_{i+1}+1} \quad \text{and} \quad C_{\nu_{i+1}-1} - C_{\nu_{i+1}+1}. \tag{6.5.1}$$

Now, we discuss solutions that annihilate both the Euclidean operator and its adjoint. If we impose the expressions in (6.5.1) to be zero at all vertices i, $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ will vanish, cf. section 6.4. This way, also the second contribution of the difference equations resulting from $\hat{\mathrm{H}}_{\mathrm{eucl}}$ is annihilated as it contains (6.5.1), cf. (6.3.22). However, the first term of (6.3.22) will *not* vanish, as it contains difference equations with label-dependent prefactors, namely

$$\sqrt{|\mu_i + 2|}C_{\mu_i+2} - \sqrt{|\mu_i - 2|}C_{\mu_i-2} \quad \text{and} \quad \sqrt{|\nu_i + 2|}C_{\nu_i+2} - \sqrt{|\mu_i - 2|}C_{\nu_i-2}. \tag{6.5.2}$$

To show this, we impose the relations in (6.5.1) to be zero and rewrite them as we did in (6.4.10). Plugging this into (6.5.2) gives the relations

$$\left(\sqrt{|\mu_i - 2|} - \sqrt{|\mu_i + 2|}\right) C_{\mu_i} \text{ and } \left(\sqrt{|\nu_i - 2|} - \sqrt{|\nu_i + 2|}\right) C_{\nu_i}, \tag{6.5.3}$$

which are non-vanishing if the coefficients C_{μ_i}, C_{ν_i} are non-zero. Thus, (6.5.2) does not vanish and hence, also the Euclidean operator is not annihilated. We have, however, an alternative to annihilate the Euclidean operator together with its adjoint: We can apply the zero volume states presented in section (6.2) to annihilate $\hat{\mathrm{H}}_{\mathrm{eucl}}$. For example, we may set $C_{\mu_i} = 0$ for $\mu_i \neq 0$, which will erase the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$, as well as that part of the second contribution of (6.4.6) containing the prefactor $(\sqrt{|\mu_i + 1|} - \sqrt{\mu_i - 1})$. For the remaining contributions of (6.4.6), we can impose the fourth order difference equations (6.4.19) for the k- and ν-labels.

We conclude that for our special ansatz, cf. (6.0.3), it seems unavoidable to use zero volume states for annihilating both the Euclidean operator and its adjoint. The Lorentzian part, however, also annihilates zero volume states: In every term of the Lorentzian part, only fluxes and volume operators are contained. As fluxes and the volume operator commute, we can let the volume operator act first. The volume operator, however, vanishes when acting on zero volume states. Therefore, the whole eigenvalue of the Lorentzian part is zero. Hence, if we choose solution states that annihilate both the Euclidean operator and its adjoint, we can only achieve a zero eigenvalue of the Lorentzian part. In particular, this means that our goal in (6.3.3) can only be achieved for $E = 0$. It remains to investigate if the Lorentzian part allows any solutions satisfying (6.6.1) for $E \neq 0$ that simultaneously annihilate or simplify the Euclidean operator and its adjoint. We will examine this in the next chapter.

6.6 Degenerate eigenvalues of the Lorentzian part

Our goal in this section is to find states φ or rather coefficients $C_{k,\mu,\nu}$ satisfying

$$\hat{\mathrm{H}}_{\mathrm{lor}}\ \varphi = \sum_{k\in\mathbb{Z}}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\ \hat{\mathrm{H}}_{\mathrm{lor}}\,|k,\mu,\nu\rangle = \sum_{k\in\mathbb{Z}}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\ E_{k,\mu,\nu}\,|k,\mu,\nu\rangle = E\ \varphi. \quad (6.6.1)$$

Here, we denoted the eigenvalue of $\hat{\mathrm{H}}_{\mathrm{lor}}$ as $E_{k,\mu,\nu}$, which is everything in the curly bracket of (6.1.16), including the sum over i, which is why $E_{k,\mu,\nu}$ depends on all labels of a given state $|k,\mu,\nu\rangle$. To perform the last step of (6.6.1), several options come into mind:

- We may think about deriving difference equations from (6.6.1) as done for the Euclidean part and its adjoint in (6.3.22) and (6.4.6). This, however, is not possible: As no holonomies are contained in the Lorentzian operator, the action of the Lorentzian part contains no shifted states which prohibits to derive any kind of difference equation.

- We can try to combine (6.6.1) with the action of the Euclidean operator and its adjoint to obtain difference equations similar to (6.3.22) and (6.4.6). For this, we inspect the eigenvalue of the Lorentzian part which consists of two terms, cf. (6.1.16): The first one is given by

$$\begin{aligned} &- B(k_{i+1}-k_{i-1})^2\big[(|\mu_i+1|^{\frac{\alpha}{2}}-|\mu_i-1|^{\frac{\alpha}{2}})(|\nu_i+1|^{\frac{\alpha}{2}}-|\nu_i-1|^{\frac{\alpha}{2}})\times \\ &\times(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})|\mu_i|^{\alpha}|\nu_i|^{\alpha}|k_i+k_{i-1}|^{\alpha}\big]^l \end{aligned} \quad (6.6.2)$$

  and results from the action of $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}}$, cf. (5.2.65). The second one is given by

$$\begin{aligned} &B'(k_i+k_{i-1})^4(\mu_i\nu_{i+1}-\mu_{i+1}\nu_i)^2\big[(|\mu_i+1|^{\frac{\beta}{2}}-|\mu_i-1|^{\frac{\beta}{2}})\cdot(|\nu_i+1|^{\frac{\beta}{2}}-|\nu_i-1|^{\frac{\beta}{2}})\times \\ &\times(|k_i+k_{i-1}+1|^{\frac{\beta}{2}}-|k_i+k_{i-1}-1|^{\frac{\beta}{2}})|\mu_i|^{\beta}|\nu_i|^{\beta}|k_i+k_{i-1}|^{\beta}\big]^l \end{aligned} \quad (6.6.3)$$

  and results from the action of $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$, cf. (5.2.67). We observe that the structure of the eigenvalue of the Lorentzian operator prohibits a separation ansatz as we did for the Euclidean operator and its adjoint in section 6.3 and 6.4: In (6.6.2), there appears a factor of $(\mu_i\nu_{i+1}-\mu_{i+1}\nu_i)^2$ which leads to a coupling of neighboring μ- and ν-labels. Therefore, the μ- and ν-coefficients cannot be separated as done in the separation ansätze of (6.3.13) and (6.4.1). Furthermore, (6.6.2), contains the factor $(k_{i+1}-k_{i-1})^2$ which leads to a coupling of k-labels from every second vertex. Additionally, the labels k_i and k_{i-1} are coupled in (6.6.2) which also occurred for the Euclidean operator. This coupling arises in (6.6.2) from the factors $(|k_i+k_{i-1}+1|^{\frac{\alpha}{2}}-|k_i+k_{i-1}-1|^{\frac{\alpha}{2}})$ and $|k_i+k_{i-1}|^{\alpha}$. The same happens in (6.6.3). Thus, in total, the k-labels k_{i-1}, k_i, k_{i+1} are coupled in (6.6.2), prohibiting a separation ansatz for the k-labels as we did for the Euclidean operator in section 6.3.3, in (6.3.75) and (6.3.77), as well as for the adjoint Euclidean operator in section 6.4, in (6.4.1). We conclude that combining the action of the Lorentzian part with that of the Euclidean operator and its adjoint, with the goal to derive difference equations for the coefficients, does not allow any of the previously used separation ansätze. In particular, the complicated structure of the eigenvalue of the Lorentzian part seems to prohibit a separation ansatz at all, which, however, makes it difficult to find solutions.

- We may investigate those sets of labels $(k, \mu, \nu) := (k_1, ..., k_N; \mu_1, ..., \mu_N; \nu_1, ..., \nu_N)$ that result in the same eigenvalue $E_{k,\mu,\nu}$, i.e., we may classify degenerate states $|k, \mu, \nu\rangle$. Then, we apply this to (6.6.1) by setting all coefficients with labels not satisfying the degeneracy conditions to zero. This way, we can pull the eigenvalue $E_{k,\mu,\nu}$ out of the sums in (6.6.1) to finally reach our goal.

As the first two points we listed above seem hard to realize, we explain the third option in more detail. We will state a summary of the results at the end of this paragraph. We first observe that the eigenvalue of the Lorentzian part consists of two terms, cf. (6.6.2) and (6.6.3), that both are symmetric with respect to exchanging all μ- and ν-labels. Thus, given the state $|k, \mu, \nu\rangle$, for which the Lorentzian operator has the eigenvalue $E_{k,\mu,\nu}$, we can perform the following operation on the labels of the state that will not change the value of $E_{k,\mu,\nu}$:

$$\mu_1 \leftrightarrow \nu_1, \ \mu_2 \leftrightarrow \nu_2, ..., \mu_N \leftrightarrow \nu_N. \tag{6.6.4}$$

Furthermore, $E_{k,\mu,\nu}$ is also left invariant with respect to a cyclic permutation of all labels, i.e.:

$$\begin{aligned}&(k_1, k_2, ...k_{N-1}, k_N; \mu_1, \mu_2, ..., \mu_{N-1}, \mu_N; \nu_1, \nu_2, ..., \nu_{N-1}, \nu_N) \rightarrow \\ &\rightarrow (k_N, k_1, ...k_{N-2}, k_{N-1}; \mu_N, \mu_1, ..., \mu_{N-2}, \mu_{N-1}; \nu_N, \nu_1, ..., \nu_{N-2}, \nu_{N-1}) \rightarrow ...\end{aligned} \tag{6.6.5}$$

Here, we basically exploit the circle symmetry of a Gowdy state: At each vertex, we shift the corresponding μ- and ν-label to the next vertex. Simultaneously, at each edge, we shift the corresponding k-label to the next edge. This way, the relation between the labels or rather their overall order is not changed. Hence, the Lorentzian operator acting on a state with certain labels will give the same eigenvalue as for the state with cyclically permuted labels. For the case of an even number of vertices, there arises another possibility. We see that in both terms of the eigenvalue of the Lorentzian operator, cf. (6.6.2) and (6.6.3), the k-labels only appear in form of the sum $k_i + k_{i-1}$ or the difference $k_{i+1} - k_{i-1}$. If we perform the following operation on all k-labels

$$k_1 \rightarrow k_1 + a, \ k_2 \rightarrow k_2 - a, \ ..., \ k_N \rightarrow k_N - a, \tag{6.6.6}$$

with a an arbitrary integer, we see that the sum of neighboring labels, $k_i + k_{i-1}$, as well as the difference between every second vertex, $k_{i+1} - k_{i-1}$, is left invariant. Hence, also the eigenvalue of the Lorentzian operator will not change. In the case of an odd number of vertices, this does not work: For example, for three vertices, we could perform the operation

$$k_1 \rightarrow k_1 + a, \ k_2 \rightarrow k_2 - a, \ k_3 \rightarrow k_3 + a.$$

Taking the sum of the transformed first and third label would result in $k_1 + k_2 + 2a \neq k_1 + k_2$. Thus, (6.6.6) does not apply for an odd number of vertices.

In summary, given an arbitrary state $|k, \mu, \nu\rangle$ with labels $(k, \mu, \nu) := (k_1, ..., k_N; \mu_1, ..., \mu_N; \nu_1, ..., \nu_N)$, we get the same eigenvalue $E_{k,\mu,\nu}$ for the Lorentzian operator if:

1. we exchange all μ- and all ν-labels (cf. (6.6.4)),

2. cyclically permute all labels simultaneously (cf. (6.6.5)),

3. for an even number of vertices, we add to all k-labels some integer whose modulus is the same for all k-labels but whose sign alternates from vertex to vertex (cf. (6.6.6)), or

4. for each step of a cyclic permutation of all labels, we perform 1. or, in the case of an even number of vertices, 2.

Now, we comment on the possibility that the degeneracy conditions also simplify the action of the Euclidean operator and its adjoint. Unfortunately, there occurs no major simplification due to the following reasons:

- If we translate the operations performed in (6.6.4)-(6.6.6) into conditions on the coefficients $C_{k.\mu,\nu}$, we may not be able to separate the coefficients at all. This is for example the case for (6.6.5) which is a condition relating all labels. If we only consider the conditions (6.6.4) and (6.6.6), we may perform the separation ansatz

$$C_{k,\mu,\nu} = C_{k_1,...,k_N} C_{\mu_1,...,\mu_N,\nu_1,...\nu_N}, \tag{6.6.7}$$

  i.e., it is possible to separate the k-coefficients from the μ- and ν-coefficients, as the conditions (6.6.4) and (6.6.6) do not couple the respective labels. The μ- and ν-coefficients itself cannot be separated as (6.6.4) relates all of the respective labels. Furthermore, (6.6.5) couples all of the k-labels and hence, the corresponding coefficients cannot be separated as well. If we use (6.6.7), there occurs another problem: This separation ansatz for the k-coefficients coincides with the ansatz we made for the Euclidean operator in section 6.3. Thus, we could derive the same difference equations for the k-labels:

$$\sqrt{|k_i + k_{i-1} + 2|} C_{k_1,...,k_{i-1},k_i+2,...,k_N} - \sqrt{|k_i + k_{i-1} - 2|} C_{k_1,...,k_{i-1},k_i-2,...,k_N} = 0. \tag{6.6.8}$$

  In this equation, the set of k-labels $(k_1, ..., k_{i-1}, k_i + 2, ..., k_N)$, corresponding to the first coefficient, cannot be generated from the set of k-labels $(k_1, ..., k_{i-1}, k_i - 2, ..., k_N)$ by one of the operations in (6.6.4)-(6.6.6). Thus, if we choose the eigenvalue of the Lorentzian operator E in (6.6.1) to be the one that corresponds to the set $(k_1, ..., k_{i-1}, k_i + 2, ..., k_N)$, we must impose the coefficient $C_{k_1,...,k_{i-1},k_i+2,...,k_N}$ to be non-zero while $C_{k_1,...,k_{i-1},k_i-2,...,k_N}$ has to be zero. Plugging this into (6.6.8), however, yields that also $C_{k_1,...,k_{i-1},k_i+2,...,k_N}$ has to vanish. As we left the k-labels arbitrary, we can infer that all coefficients have to be zero and we obtain the undesirable trivial solution. We end up with the result that imposing (6.6.4)-(6.6.6) either spoils the previously made separation ansätze or results in the trivial solution for the coefficients.

- Furthermore, the subspace spanned by states φ with coefficients satisfying (6.6.4)-(6.6.6) is not preserved by the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$: For both operators, when acting on a basis state $|k, \mu, \nu\rangle$, we obtain states where only two or maximally three labels are shifted, see the discussion in the paragraph after (6.1.8). For example, if we act with the Euclidean operator on a state $|k, \mu, \nu\rangle$, one of the resulting states will be $|k_i + 2, \mu_i + 1, \mu_{i+1} + 1, \nu\rangle$. The eigenvalue of this state, however, is different from that of the original one because none of the operations in (6.6.4)-(6.6.6) can generate the state $|k_i + 2, \mu_i + 1, \mu_{i+1} + 1, \nu\rangle$ out of the state $|k, \mu, \nu\rangle$ and vice versa. Hence, when acting with the Euclidean operator on a state φ with coefficients satisfying one of the conditions in (6.6.4)-(6.6.6), we get back a different state $\widetilde{\varphi}$ with a different eigenvalue $\tilde{E}_{k,\mu,\nu}$ of the Lorentzian operator. Thus, we conclude that the subspace, spanned by states φ for which the Lorentzian operator has some non-zero eigenvalue E, is not left invariant by the Euclidean operator and its adjoint.

The only conditions that drastically simplify the Lorentzian part *and* the Euclidean operator together with its adjoint are zero volume states. For those states, however, the eigenvalue of the Lorentzian operator is zero. Hence, we see that the Euclidean operator and its adjoint are incompatible with the Lorentzian part in the sense that they only have common eigenfunctions with eigenvalue zero. To investigate this incompatibility, we have calculated the commutator between the Lorentzian and the Euclidean operator, cf. appendix A. This computation results in several shifted states with label dependent prefactors coupling all k-, μ-, and ν-labels. Thus, we cannot perform a separation ansatz as done in section 6.3 and 6.4. In particular, the annihilation of the commutator seems to work only for zero volume states, at least when acting with it on a single basis state $|k,\mu,\nu\rangle$ or our special ansatz (6.0.3).

6.7 Further solutions and final discussion

So far, we always tried to annihilate the action of the Euclidean operator and its adjoint. In the following, we want to state a solution that results in a non-zero but simplified action of the Euclidean operator and its adjoint. Choosing

$$C_{\mu_i-1} - C_{\nu_i+1} = 0 \quad \text{and} \quad C_{\nu_i-1} - C_{\nu_i+1} = 0 \quad \forall i = 1, ..., N \tag{6.7.1}$$

annihilates the action of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, cf. section 6.4, and also the second term of $\hat{\mathrm{H}}_{\mathrm{eucl}}$, cf. (6.3.22). For the first first term of (6.3.22), we use that

$$\sqrt{|\mu_i+2|}C_{\mu_i+2} - \sqrt{|\mu_i-2|}C_{\mu_i-2} = \left(\sqrt{|\mu_i+2|} - \sqrt{|\mu_i-2|}\right) C_{\mu_i}, \tag{6.7.2}$$

due to $C_{\mu_i-2} = C_{\mu_i}$ and $C_{\mu_i+2} = C_{\mu_i}$ which follows from (6.7.1) by shifting μ_i by ± 1. Then, the difference equations of the first term of (6.3.22) read as

$$\begin{aligned}\left(\sqrt{|\mu_i+2|}C_{\mu_i+2} - \sqrt{|\mu_i-2|}C_{\mu_i-2}\right)\left(\sqrt{|\nu_i+2|}C_{\nu_i+2} - \sqrt{|\nu_i-2|}C_{\nu_i-2}\right) = \\ \left(\sqrt{|\mu_i+2|} - \sqrt{|\mu_i-2|}\right)\cdot\left(\sqrt{|\nu_i+2|} - \sqrt{|\nu_i-2|}\right) C_{\mu_i}C_{\nu_i}.\end{aligned} \tag{6.7.3}$$

In total, we get for the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ that

$$\hat{\mathrm{H}}_{\mathrm{eucl}}\ \varphi = \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu} A\left(\sqrt{|\mu_i+2|} - \sqrt{|\mu_i-2|}\right)\cdot\left(\sqrt{|\nu_i+2|} - \sqrt{|\nu_i-2|}\right)|k,\mu,\nu\rangle\,, \tag{6.7.4}$$

i.e., we eliminated the shifted states resulting from the action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$. However, we do not get back the original state φ but a different one which we denote as

$$\widetilde{\varphi} := \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} \widetilde{C}_{k,\mu,\nu}\,|k,\mu,\nu\rangle\,, \tag{6.7.5}$$

with the coefficients

$$\widetilde{C}_{k,\mu,\nu} := C_{k,\mu,\nu} A\left(\sqrt{|\mu_i+2|} - \sqrt{|\mu_i-2|}\right)\cdot\left(\sqrt{|\nu_i+2|} - \sqrt{|\nu_i-2|}\right).$$

Then, we can simplify the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$ by combining the action of the Lorentzian part with that of the Euclidean part in (6.7.4) to arrive at

$$\hat{\mathrm{H}}_{\mathrm{phys}}\ \varphi = \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\widetilde{E}_{k,\mu,\nu}\,|k,\mu,\nu\rangle\,, \tag{6.7.6}$$

with

$$\widetilde{E}_{k,\mu,\nu} := E_{k,\mu,\nu} + A\Big(\sqrt{|\mu_i+2|}-\sqrt{|\mu_i-2|}\Big)\Big(\sqrt{|\nu_i+2|}-\sqrt{|\nu_i-2|}\Big). \tag{6.7.7}$$

Here, $E_{k,\mu,\nu}$ denotes the eigenvalue of the Lorentzian part. However, note that (6.7.6) is only true for the special choice of coefficients stated in (6.7.1). In particular, we cannot apply the degeneracy conditions of section 6.6 to pull out $\widetilde{E}_{k,\mu,\nu}$ to obtain $\hat{\mathrm{H}}_{\mathrm{phys}}\,\varphi = E\,\varphi$ for some constant E. This arises from the incompatibility of the conditions for the coefficients we get from the difference equations in (6.7.1) and those from the degeneracy conditions in (6.6.4)-(6.6.6), see also the discussion in the context of (6.6.8).

It is also possible to make the Lorentzian part vanish without using the zero volume states. For this, we impose all k-labels to be equal, as well as all μ- and all ν-labels, i.e.:

$$k_1 = k_2 = ... = k_N,\ \mu_1 = \mu_2 = ... = \mu_N = \nu_1 = \nu_2 = ... = \nu_N. \tag{6.7.8}$$

This annihilates the eigenvalue of the Lorentzian part: In the term resulting from $\hat{\mathrm{H}}^{(1)}_{\mathrm{lor}}$, cf. (6.6.2), the factor $(k_{i+1}-k_{i-1})^2$ shows up, wich is zero if all k-labels are equal. In the term resulting from $\hat{\mathrm{H}}^{(2)}_{\mathrm{lor}}$, cf. (6.6.3), we have a factor of $(\mu_i\nu_{i+1}-\mu_{i+1}\nu_i)^4$, which is zero if all μ- and all ν-labels are equal. However, by imposing (6.7.8), we couple all k-labels with each other and also all μ- with all ν-labels. This prohibits for example a separation ansatz of the kind of (6.3.13), where in particular all μ- and ν-labels are decoupled. We would have to change our separation ansatz to the one in (6.6.7) but then, as discussed in the context of (6.6.7), we obtain the trivial solution for the k-coefficients.

Our final result is that we can achieve

$$\hat{\mathrm{H}}_{\mathrm{phys}}\,\varphi = E\,\varphi \tag{6.7.9}$$

with the ansatz

$$\varphi = \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle \tag{6.7.10}$$

only for $E=0$, i.e., we get a zero action of our physical Hamiltonian. For a non-zero action of our physical Hamiltonian, the best result we can achieve with the specialized ansatz we used in this work is

$$\hat{\mathrm{H}}_{\mathrm{phys}}\,\varphi = \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\widetilde{E}_{k,\mu,\nu}\,|k,\mu,\nu\rangle\,, \tag{6.7.11}$$

for $\widetilde{E}_{k,\mu,\nu}$ defined in (6.7.7). In (6.7.11), we got rid of the shifted states that result from the action of the Euclidean part and its adjoint. However, we cannot pull out the label-dependent factor $E_{k,\mu,\nu}$ and therefore cannot solve the Schrödinger equation with a non-zero energy, at least with the ansatz for φ stated in (6.7.10).

7 Summary and outlook

In the following, we want to summarize this thesis and our results. We will also discuss how our calculations performed in the algebraic quantum gravity context can be applied to other topics such as the Master constraint operator of [59] and the loop quantum gravity version of the Hamiltonian constraint operator.

In this thesis, we performed a reduced phase space quantization of the polarized three-torus Gowdy model, yielding a Schrödinger equation which we investigated. The classical starting point was the Hamiltonian formulation of general relativity in terms of Ashtekar's variables, which we introduced in section 2.2. For the quantization, we used the techniques of loop quantum gravity (LQG) and algebraic quantum gravity (AQG), which we discussed for the full theory in section 3. To perform the reduced phase space quantization, the constraints have to be solved at the classical level. To this end, we introduced the relational formalism and, specifically, Gaussian dust in section 4. We then applied the introduced machinery in section 5 to the Gowdy model using former results of [56,57] and [58,59]: First, we performed the symmetry reduction in section 5.1. This results in a $U(1)$ gauge theory on a circle with 3 canonical pairs $(\mathcal{A},\mathcal{E})$, (X,E^x), (Y,E^y), where $\mathcal{A}$ is a $U(1)$ connection with conjugate momentum $\mathcal{E}$ and X,Y are scalars with conjugate momenta E^x,E^y, respectively. Additionally, we obtained one diffeomorphism and one Hamiltonian constraint. Then, we proceeded with the algebraic loop quantization of the Gowdy model in section 5.2, where we first applied the relational formalism for Gaussian dust in section 5.2.1, which solves the remaining constraints at the classical level. In particular, we constructed the observables associated to our basic variables, which, at the level of the physical phase space, satisfy a Poisson algebra that is isomorphic to the original kinematical Poisson algebra. Furthermore, the physical Hamiltonian density is equivalent to the geometric contribution of the Hamiltonian constraint. This allows us to use the results of [56] for the loop quantization of the Gowdy model, which we did in the subsequent sections, where we also lifted the LQG framework to the AQG level following [58,59]. In section 5.2.5, we also discussed the Gauss constraint and were then finally in the position to write down and discuss a specific ansatz for constructing special solutions of the Schrödinger equation in section 6.

Let us summarize our main results presented in section 6: The starting point is the time-dependent Schrödinger equation

$$i\hbar\partial_\tau\Psi(\tau)=\hat{\mathrm{H}}_{\mathrm{phys}}\Psi(\tau), \tag{7.0.12}$$

where τ denotes the physical time, $\hat{\mathrm{H}}_{\mathrm{phys}}$ the physical Hamiltonian operator, and $\Psi(\tau)$ a time-dependent state. For the state $\Psi(\tau)$, we separate the time-dependence just as in standard quantum mechanics via $\Psi(\tau)=\chi(\tau)\cdot\varphi$ for φ being some time-independent state. This allows us to formulate the time-independent Schrödinger equation

$$\hat{\mathrm{H}}_{\mathrm{phys}}\varphi=E\varphi, \tag{7.0.13}$$

where E plays the role of the energy since classically, the Hamiltonian commutes with itself and hence the system is conservative, which should carry over to the quantum theory. Note that φ is different from the wavefunction $\varphi(x)$ of a quantum particle where x denotes the position, i.e., a single particle interpretation is not possible. Concerning the physical Hamiltonian operator, we want it to be at least symmetric which is not satisfied by the geometric contribution of

the Hamiltonian constraint operator $\hat{\mathrm{H}}$, as we showed in section 6.1.2 and 6.1.3. Hence, we use the following symmetric combination of $\hat{\mathrm{H}}$:

$$\hat{\mathrm{H}}_{\mathrm{phys}} := \frac{1}{2}\left(\hat{\mathrm{H}} + \hat{\mathrm{H}}^{\dagger}\right). \tag{7.0.14}$$

One may also take different symmetric combinations, e.g., $\hat{\mathrm{H}}_{\mathrm{phys}} = \sqrt{\hat{\mathrm{H}}^{\dagger}\hat{\mathrm{H}}}$. Due to the square root, however, we would have to determine the spectrum of $\hat{\mathrm{H}}^{\dagger}\hat{\mathrm{H}}$ and only then we can calculate the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$. Hence, we decided to choose (7.0.14) where no square root appears. Now, if we want to solve (7.0.13), we would have to determine the spectrum of $\hat{\mathrm{H}}_{\mathrm{phys}}$. This amounts to the diagonalization of a complicated and in particular infinite-dimensional matrix. To circumvent this and to learn something about the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$, we choose a specific ansatz for the state φ appearing in (7.0.13). In principle, we also need to show that the eigenvalues E corresponding to the eigenstates φ are indeed contained in the spectrum of $\hat{\mathrm{H}}_{\mathrm{phys}}$, which, however, goes beyond the scope of this work. To analyze (7.0.13) and to construct possible specific solutions, we choose the following ansatz for φ:

$$\varphi := \sum_{k\in\mathbb{Z}^N}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\,|k,\mu,\nu\rangle\,. \tag{7.0.15}$$

The following ingredients are used in this equation:

- The states $|k,\mu,\nu\rangle$ are the so-called abstract Gowdy states, which we defined in (5.2.21). These states provide an orthonormal basis for the Hilbert space obtained in the symmetry reduced Gowdy model and are the analogue of the spin network functions in the full theory.

- Furthermore, the states $|k,\mu,\nu\rangle$ are defined with respect to an abstract graph α, which is basically an infinite spin chain, where only N neighboring vertices and $N+1$ neighboring edges are charged non-trivially, see figure 3 for the case of $N = 3$ vertices. This way, by choosing an embedding and identifying the first and the $N+1$th vertex, we can go back to an embedded graph γ used in LQG, which is basically a circle with a certain number of points, being the vertices, and arcs between those points, being the edges. The infinitely many other trivially charged edges and vertices of α are mapped to accumulation points, which, however, are not seen by the Hamiltonian constraint operator $\hat{\mathrm{H}}$ appearing in (7.0.14). The reason for this is the following: $\hat{\mathrm{H}}$ can be split into the following sum of two operators, cf. section 5.2.4:

  $$\hat{\mathrm{H}} = \hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}_{\mathrm{lor}}. \tag{7.0.16}$$

  Here, $\hat{\mathrm{H}}_{\mathrm{eucl}}$ denotes the Euclidean operator, which acts label-changing. In particular, it can be separated into the sum of three operators that are basically the product of holonomies and the volume operator, respectively, while the volume operator acts first in each term. This is also the case for $\hat{\mathrm{H}}_{\mathrm{lor}}$, the so-called Lorentzian operator, which acts diagonally and is a sum of three operators that are basically a product of the volume operator and the flux operators. In total, in both $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}_{\mathrm{lor}}$ the volume operator acts first, which, however, has eigenvalue zero for trivially charged vertices and edges, cf. section 5.2.3. Thus, $\hat{\mathrm{H}}$ has a zero action on the trivially charged vertices of the abstract graph. Note, however, that this is not true for the adjoint of $\hat{\mathrm{H}}$, which also appears in the physical Hamiltonian operator (7.0.14): While $\hat{\mathrm{H}}_{\mathrm{lor}}$ turns out to be symmetric, cf.

section 6.1.3, and hence also $\hat{\mathrm{H}}^{\dagger}_{\mathrm{lor}}$ annihilates all trivially charged edges, calculating the adjoint of the Euclidean operator changes the operator ordering of holonomies and the volume operator, resulting in $\hat{\mathrm{H}}_{\mathrm{eucl}} \neq \hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$. Therefore, the holonomies act first in $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, thus changing trivial charges to non-trivial ones, which then leads to a nonzero action of the volume operator. Hence, we have to adjust the definition of an abstract Gowdy state. Otherwise, the operator $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ and thus also $\hat{\mathrm{H}}_{\mathrm{phys}}$ would have a non-trivial action on some of the vertices of the abstract graph, for which we actually want the action to be trivial such that the circle symmetry is implemented. We discussed this at the end of section 6.1.2.

- Each vertex v_i, with $i = 1, ..., N$, carries non-trivial labels $\mu_i, \nu_i \in \mathbb{R}$. These are the irreducible representation labels of the Bohr compactification and correspond to the point holonomies (5.2.19) and (5.2.20) constructed from the scalar fields X and Y, respectively. Furthermore, to each edge e_i, with $i = 1, ..., N$, we assign a $U(1)$ representation label k_i, which corresponds to the $U(1)$ holonomy (5.2.18) constructed from the $U(1)$ connection $\mathcal{A}$. Hence, the labels k, μ, ν in (7.0.15) have to be understood as the multilabels $k := (k_1, ..., k_N)$, $\mu := (\mu_1, ..., \mu_N)$, $\nu := (\nu_1, ..., \nu_N)$.

- Finally, in (7.0.15), $C_{k,\mu,\nu} := C_{k_1,...,k_N,\mu_1,...,\mu_N,\nu_1,...,\nu_N}$ denote some arbitrary coefficients and the sets m, n are subsets of $\mathbb{R}^N$, whose exact form we discussed in detail at the beginning of section 6 in (6.0.5).

Note that without any further restrictions on the coefficients, we have infinite series involved in (7.0.15). Hence, this ansatz is rather formal, as it has an infinite norm for generic coefficients. It turns out that for so-called zero volume states, which we discussed in section 6.2, this can be avoided. Zero volume states are those states that are annihilated by the volume operator. If we consider the action of the volume operator $\hat{V}_i$, defined for a vertex v_i of α in (5.2.33), on a single basis state $|k, \mu, \nu\rangle$, we obtain

$$\hat{V}_i \, |k, \mu, \nu\rangle = \frac{1}{\sqrt{2}} \left(\frac{\gamma l_P^2}{2} \right) \sqrt{|\mu_i||\nu_i||k_i + k_{i-1}|} \, |k, \mu, \nu\rangle \, . \tag{7.0.17}$$

This action is zero if the following is satisfied:

$$\mu_i \stackrel{!}{=} 0 \ \vee \ \nu_i \stackrel{!}{=} 0 \ \vee \ k_i \stackrel{!}{=} -k_{i-1}. \tag{7.0.18}$$

We can use the zero volume states to simplify the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$. In particular, the Euclidean and the Lorentzian operator can be annihilated, since for these operators, the volume operator acts first. As we discussed above in the context of (7.0.16), this is not the case for the adjoint Euclidean part and hence, not all of $\hat{\mathrm{H}}_{\mathrm{phys}}$ can be annihilated using zero volume states. To annihilate $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}_{\mathrm{lor}}$, we use that these operators can be written as a sum over all vertices, e.g., $\hat{\mathrm{H}}_{\mathrm{lor}} = \sum_{i=1}^{N} \hat{\mathrm{H}}_{\mathrm{lor},i}$, and therefore they contain $\hat{V}_i$ at all vertices. Thus, we can choose at each vertex one of the conditions of (7.0.18) and obtain a zero action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}_{\mathrm{lor}}$. This results in a rather large class of zero volume states which can also be extended to conditions on the coefficients of our general state (7.0.15): Choosing for example all μ-labels to be zero transfers to the condition that $C_{k,\mu,\nu} \neq 0$ if $\mu = 0$. Note that this solution may sound trivial as it eliminates a whole degree of freedom. To avoid this, we may choose the μ-labels to be nonzero at some vertices, where we choose another zero volume condition (7.0.18) to obtain a zero action, see again section 6.2 for a more detailed classification. Note that this way, the nonzero coefficients have arbitrary values and hence can be chosen such that we obtain a

normalizable state. The zero volume states are useful later on as they will allow us to solve (7.0.13) for $E = 0$. This also means that we have to assume that $E = 0$ is contained in the spectrum of $\hat{\mathrm{H}}_{\mathrm{phys}}$. At the classical level, we can interpret $E = 0$ in the following way: We can construct the observable O_P associated to P, which is the momentum conjugate to the time reference field T, cf. section 4.2. Then, one can show that O_P is determined up to a minus sign by the physical Hamiltonian density. Thus, a zero energy corresponds to the case where the energy density of the dust vanishes and this would be the limit in which back reactions of the dust are neglected. Investigating states with eigenvalue zero is also done in embedded LQG and Dirac quantization or in the context of the Master constraint programme, which we will discuss further below. However, our actual goal was to solve the Schrödinger equation with a non-zero energy. To this end, we pursued the following strategy: Using (7.0.14),(7.0.16), and the fact that $\hat{\mathrm{H}}_{\mathrm{lor}}$ is symmetric while $\hat{\mathrm{H}}_{\mathrm{eucl}}$ is not, we obtain the following form of the physical Hamiltonian operator:

$$\hat{\mathrm{H}}_{\mathrm{phys}} = \frac{1}{2}\left(\hat{\mathrm{H}}_{\mathrm{eucl}} + \hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}\right) + \hat{\mathrm{H}}_{\mathrm{lor}}. \tag{7.0.19}$$

The next step is to calculate the action of $\hat{\mathrm{H}}_{\mathrm{phys}}$, which we derived in detail in section 6.1 using the results of [57–59]. In particular, it turns out that the label-changing operators $\hat{\mathrm{H}}_{\mathrm{eucl}}$ and $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ result in a linear combination of in total 40 states with label-dependent coefficients, respectively, see (6.1.8) and (6.1.27) for the explicit actions. The action of the diagonal operator $\hat{\mathrm{H}}_{\mathrm{lor}}$ we stated in (6.1.16). A new result obtained in this thesis is that one of the terms of the action of $\hat{\mathrm{H}}_{\mathrm{lor}}$ results in a telescope sum for the vertices and vanishes using the circle symmetry of the abstract Gowdy state. See also the discussion in the context of (6.1.12). This did not show up in [58, 59] as the respective authors do not directly work with the Hamiltonian constraint operator, but certain combinations thereof, namely they consider the physical Hamiltonian obtained in the Brown-Kuchař model and the Master constraint, respectively. To analyze (7.0.13), the idea is now to use the diagonal action of $\hat{\mathrm{H}}_{\mathrm{lor}}$ to produce the energy E in (7.0.13) and to annihilate the label-changing operators, i.e., we impose the following:

$$\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi \stackrel{!}{=} 0 \ \wedge \ \hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}\varphi \stackrel{!}{=} 0 \ \wedge \ \hat{\mathrm{H}}_{\mathrm{lor}}\varphi \stackrel{!}{=} E\varphi. \tag{7.0.20}$$

We treated all of this possibilities separately in the sections 6.3-6.6:

- The annihilation of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ was discussed in section 6.3. The action of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ results in 20 shifted states with prefactors that depend on the labels $k_i, k_{i-1}, \mu_i, \nu_i$, cf. (6.1.8). The occurring states are of the form $|k, \mu_i \pm 2, \nu_i \pm 2\rangle$, $|k_i \pm 2, \mu, \nu_i \pm 1, \nu_{i+1} \pm 1\rangle$, and $|k_i \pm 2, \mu_i \pm 1, \mu_{i+1} \pm 1, \nu\rangle$, i.e., either only the μ- and ν-labels at the ith vertex are shifted, or the label k_i together with either ν_i, ν_{i+1} or μ_i, μ_{i+1}. Our strategy is to translate the shifts in the states to the coefficients and derive difference equations, see section 6.3.1. Specifically, we performed the separation ansatz

  $$C_{k,\mu,\nu} = C_k \cdot C_{\mu_1} \cdot C_{\mu_2} \cdot \ldots \cdot C_{\mu_N} \cdot C_{\nu_1} \cdot C_{\nu_2} \cdot \ldots \cdot C_{\nu_N}, \tag{7.0.21}$$

  where we keep all the k-labels together in one coefficient $C_k := C_{k_1,\ldots,k_N}$ but decouple all μ- and ν-labels. This allows us to rewrite $\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi \stackrel{!}{=} 0$ in the convenient product form (6.3.22). Using (6.3.22), we can now derive several difference equations, which can be divided into two categories:

  - The first one we discussed in section 6.3.2. These solutions give restrictions on the coefficients with μ- and ν-labels only. Especially, the difference equations are of

the following form:

$$\sqrt{|n+2|}C_{n+2} - \sqrt{|n-2|}C_{n-2} = 0,\ \sqrt{|n+1|}C_{n+1} + \sqrt{|n-1|}C_{n-1} = 0,$$
$$C_{n+1} - C_{n-1} = 0, \qquad n \in \{\mu_i, \nu_i\},\ i = 1, ..., N. \tag{7.0.22}$$

We have different options to combine these relations such that $\hat{\mathrm{H}}_{\mathrm{eucl}}\varphi \stackrel{!}{=} 0$ is satisfied. Note that the coefficients solving the first two of the above three difference equations are basically inverse square root functions. The last of the three equations is solved by a periodic function with period 2. In particular, as discussed at the end of section 6.3.2, the states φ corresponding to such coefficients are not normalizable. This, however, is a general scenario for operators with continuous spectrum and can be handled by constructing linear functionals from the solution states, in analogy to the momentum operator in standard quantum mechanics. The difficult task that remains, however, is to show that zero lies in the continuous part of the spectrum, which goes beyond the scope of this thesis.

– The second one, we discussed in section 6.3.3. These solutions give restrictions on conditions with all kinds of labels, that is, k-, μ-, and ν-labels. It turns out that the difference equations involving k-labels have the form

$$\sqrt{|k_i + k_{i-1} - 2|}\ C_{k_i-2,k_{i-1}} - \sqrt{|k_i + k_{i-1} + 2|}\ C_{k_i+2,k_{i-1}} = 0. \tag{7.0.23}$$

These relations, however, cannot be solved for all i, while the exceptions are the cases of very simple graphs with one and two vertices. Specifically, we derived a contradiction when trying to solve the above equation for $i = 1$ and $i = 2$ in the case of 3 vertices. This arises basically due to the coupling of neighboring k-labels in the square root prefactors in (7.0.23). Nevertheless, we can solve (7.0.23) at least for one i with the explicit solution of the coefficients basically given by $C_{k_i,k_{i-1}} = 1/\sqrt{|k_i + k_{i-1}|}$. Note that care has to be taken in the case of $k_i = -k_{i-1}$, where we would divide by zero, see section 6.3.3 for more details. We can generalize this to an arbitrary number of vertices; for example for 4 vertices, it is possible to solve (7.0.23), e.g., for $i = 1, 2$. Nevertheless, we cannot solve (7.0.23) at all vertices but for the remaining ones, we can use the solutions of section 6.3.2, i.e., we choose the μ- and ν-coefficients appropriately. Again, the solution states φ will not be normalizable.

- The annihilation of $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$, we discussed in section 6.4. Here, we proceed as for the Euclidean operator and perform a separation ansatz which now has the form

$$C_{k,\mu,\nu} = C_{k_1} \cdot ... \cdot C_{k_N} \cdot C_{\mu_1} \cdot ... \cdot C_{\mu_N} \cdot C_{\nu_1} \cdot ... \cdot C_{\nu_N}, \tag{7.0.24}$$

i.e., we decouple all labels, especially also the k-labels. This is possible due to the following reason: Computing the adjoint of $\hat{\mathrm{H}}_{\mathrm{eucl}}$ changes the operator ordering such that the holonomies act first in $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$. In particular, we get the same shifted states as for the Euclidean operator. Then, the volume part acts on this shifted states resulting in label-dependent prefactors of the shifted states that are similar to but different from those of the Euclidean operator action. Notably, these prefactors are such that if we translate the shift from the states to the coefficients, the label dependent prefactors can just be pulled out and do not manifest themselves in the difference equations.

Then, (7.0.23) looks similar for $\hat{\mathrm{H}}^{\dagger}_{\mathrm{eucl}}$ but without the square root prefactors. This way, the coupling between neighboring k-labels is removed, which allows us to perform the separation ansatz (7.0.24). Also the difference equations involving μ- and ν-coefficients we derive from this ansatz are just like (7.0.22) and (7.0.23), but without any square root prefactors. Specifically, we can again identify two categories of solutions, namely those only restricting μ- and ν-coefficients, and those restricting coefficients with all kinds of labels.

- The next step is to annihilate the Euclidean part and its adjoint simultaneously, which we discussed in section 6.5. Here, it turns out that we need to use the zero volume states of section 6.2. This, however, implies that the eigenvalue of the Lorentzian part vanishes and hence, we can only solve (7.0.13) for a zero energy, at least with the strategy (7.0.20) and the ansatz (7.0.15). Still, it may have been possible that the investigation of $\hat{\mathrm{H}}_{\mathrm{lor}}\varphi \stackrel{!}{=} E\varphi$ gives new insights, which, however, turned out not to be the case, as we will discuss in the next point.

- To extract the energy eigenvalue from the Lorentzian part, we have to solve the following problem:

$$\hat{\mathrm{H}}_{\mathrm{lor}}\ \varphi = \sum_{k\in\mathbb{Z}}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\ \hat{\mathrm{H}}_{\mathrm{lor}}\,|k,\mu,\nu\rangle = \sum_{k\in\mathbb{Z}}\sum_{\mu\in m}\sum_{\nu\in n} C_{k,\mu,\nu}\ E_{k,\mu,\nu}\,|k,\mu,\nu\rangle = E\ \varphi, \tag{7.0.25}$$

  where $E_{k,\mu,\nu}$ denotes the eigenvalue of $\hat{\mathrm{H}}_{\mathrm{lor}}$ when acting on a basis state $|k,\mu,\nu\rangle$, which is given explicitly in (6.1.16). To pull this eigenvalue out in the last step of (7.0.25), we investigated the degeneracy conditions of $\hat{\mathrm{H}}_{\mathrm{lor}}$, i.e., we searched for those sets of labels (k,μ,ν) resulting in the same eigenvalue $E_{k,\mu,\nu}$. The idea is then to choose those coefficients $C_{k,\mu,\nu}$ to zero whose labels (k,μ,ν) do not satisfy the degneracy conditions. We discussed this in section 6.6, which, however, gives only conditions that are in conflict with the Euclidean part and its adjoint, in the sense that the action of these operators is neither annihilated nor simplified.

In total, we end up with the result that we can only solve (7.0.13) for $E = 0$, at least with the formal ansatz (7.0.15) and the strategy (7.0.20). In section 6.7 we also stated some ideas which at least simplify the action of the physical Hamiltonian but do not yield us a solution of (7.0.13). Nevertheless, we gained much more insight into the complicated action of the physical Hamiltonian and found a class of solutions with eigenvalue zero. It would require more advanced techniques to look for general solutions of the Schrödinger equation in the Gowdy model than were applied in this thesis. We hope that our results may be used for example for the Gowdy Master constraint, which was discussed in [59], or even in the framework of Dirac quantization.

Application of the results found in this thesis, in the context of embedded LQG, might go into the following direction: In LQG, the action of the Hamiltonian constraint operator depends on the chosen regularization. For example in [57], every interval of the regularization contains at most one vertex. There are, however, also other regularizations possible, see chapter V of [57] for a discussion. It would be an interesting question for future research how one can compare operators in LQG and AQG in a more general context. In particular, one may fix a certain regularization and then compare the results of LQG and AQG. Furthermore, one may investigate the physical consequences of regularization ambiguities in LQG and the

implications in AQG if one stays close to LQG.

The Master constraint of the Gowdy model is discussed in great detail in [59]. Explicitly, the Master constraint operator is of the form

$$\hat{\mathbf{M}} = \sum_{i=1}^{N} \left(\hat{\tilde{\mathrm{H}}}_i^{\dagger} \hat{\tilde{\mathrm{H}}}_i + \hat{\mathrm{C}}_i^{\dagger} \hat{\mathrm{C}}_i \right). \tag{7.0.26}$$

Here, $\hat{\tilde{\mathrm{H}}}_i$ is the sum of two operators that are constructed in analogy to the Euclidean and Lorentzian operators we stated in (5.2.57) and (5.2.65),(5.2.67),(5.2.68). Note, however that the Master constraint introduces a different density weight, which manifests itself in the action of (7.0.26), as a different power of the volume operator results from this. Nevertheless, the action of $\hat{\tilde{\mathrm{H}}}_i$ and $\hat{\tilde{\mathrm{H}}}_i^{\dagger}$ is quite similar to the one discussed in this thesis. Furthermore, in (7.0.26), the operator $\hat{\mathrm{C}}_i$ arises from the part of the classical Master constraint which contains the diffeomorphism constraint, cf. (3.6.6). Here, also the metric components occur and in particular, we obtain an operator that is new compared to the ones we discussed in this thesis. However, as can be seen in [59], every operator in (7.0.26) contains the volume operator. Specifically, in $\hat{\tilde{\mathrm{H}}}_i$ and $\hat{\mathrm{C}}_i$ the volume operator acts first and hence, the zero volume states we discussed in section 6.2 can be used to annihilate or at least simplify the action of the Master constraint operator. As in (7.0.26) the Euclidean operator acts first, we may also apply our results of section 6.3, where we annihilate the Euclidean operator by deriving and solving difference equations for the coefficients. Note however, that the different density weight has to be taken into account, leading to different label dependent prefactors in the difference equations. How the results of the sections 6.3-6.5 can be applied to the operators $\hat{\mathrm{C}}_i$, $\hat{\mathrm{C}}_i^{\dagger}$ in (7.0.26) remains to be investigated.

Appendices

A Commutator between Euclidean and Lorentzian part

In this appendix, we present the calculation of the commutator between the Euclidean and the Lorentzian operator defined in (5.2.57) and (5.2.65), (5.2.67), (5.2.68), respectively.

Firstly, we write out the commutator acting on an abstract Gowdy state (5.2.21):

$$\left[\widehat{\mathrm{H}}_{\mathrm{eucl}}, \widehat{\mathrm{H}}_{\mathrm{lor}}\right] |k,\mu,\nu\rangle = \widehat{\mathrm{H}}_{\mathrm{eucl}}\left(\widehat{\mathrm{H}}_{\mathrm{lor}}\,|k,\mu,\nu\rangle\right) - \widehat{\mathrm{H}}_{\mathrm{lor}}\left(\widehat{\mathrm{H}}_{\mathrm{eucl}}\,|k,\mu,\nu\rangle\right). \tag{A.0.27}$$

Secondly, we calculate the action of the Lorentzian part in the first term of (A.0.27) and abbreviate the eigenvalue with $H_{lor}(k,\mu,\nu)$, which is stated explicitly in (6.1.16). Furthermore, in the second term of (A.0.27), we compute the action of the Euclidean part using (6.1.8). Then, we get the following expression for (A.0.27):

$$\begin{aligned}
\left[\widehat{\mathrm{H}}_{\mathrm{eucl}}, \widehat{\mathrm{H}}_{\mathrm{lor}}\right] |k,\mu,\nu\rangle = H_{lor}(k,\mu,\nu)(\widehat{\mathrm{H}}_{\mathrm{eucl}}\,|k,\mu,\nu\rangle) - \widehat{\mathrm{H}}_{\mathrm{lor}} \sum_{i=1}^{N} \Bigg\{ a_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k,\mu_i+2,\nu_i+2\rangle \\
- |k,\mu_i-2,\nu_i+2\rangle - |k,\mu_i+2,\nu_i-2\rangle + |k,\mu_i-2,\nu_i-2\rangle \Big] \\
+ b_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
+ |k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
+ |k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle + |k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \Big] \\
+ c_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle - |k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
+ |k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle - |k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle - |k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
+ |k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle - |k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle + |k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle \Big] \Bigg\},
\end{aligned} \tag{A.0.28}$$

where we introduced the short notation

$$a_{k_i,k_{i-1},\mu_i,\nu_i} := A\sqrt{|\mu_i||\nu_i|}\left(\sqrt{|k_i+k_{i-1}+1|} - \sqrt{|k_i+k_{i-1}-1|}\right), \tag{A.0.29}$$

$$b_{k_i,k_{i-1},\mu_i,\nu_i} := A\sqrt{|k_i+k_{i-1}||\nu_i|}\left(\sqrt{|\mu_i+1|} - \sqrt{|\mu_i-1|}\right), \tag{A.0.30}$$

$$c_{k_i,k_{i-1},\mu_i,\nu_i} := A\sqrt{|k_i+k_{i-1}||\mu_i|}\left(\sqrt{|\nu_i+1|} - \sqrt{|\nu_i-1|}\right). \tag{A.0.31}$$

Thirdly, we let the remaining Lorentzian and Euclidean constraint operators act in (A.0.28),

for which we again use (6.1.16) and (6.1.8) resulting in

$$
\begin{aligned}
\left[\widehat{\mathrm{H}}_{\mathrm{eucl}}, \widehat{\mathrm{H}}_{\mathrm{lor}}\right] |k,\mu,\nu\rangle = H_{lor}(k,\mu,\nu) \sum_{i=1}^{N} \Bigg\{ & a_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k,\mu_i+2,\nu_i+2\rangle \\
& - |k,\mu_i-2,\nu_i+2\rangle - |k,\mu_i+2,\nu_i-2\rangle + |k,\mu_i-2,\nu_i-2\rangle \Big] \\
& + b_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
& + |k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle - |k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
& + |k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle - |k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle + |k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \Big] \\
& + c_{k_i,k_{i-1},\mu_i,\nu_i} \Big[|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle - |k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
& + |k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle - |k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle - |k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
& + |k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle - |k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle + |k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle \Big] \Bigg\} \\
& - \sum_{i=1}^{N} \Bigg\{ a_{k_i,k_{i-1},\mu_i,\nu_i} \Big[H_{lor}(k,\mu_i+2,\nu_i+2)\,|k,\mu_i+2,\nu_i+2\rangle \\
& - H_{lor}(k,\mu_i-2,\nu_i+2)\,|k,\mu_i-2,\nu_i+2\rangle - H_{lor}(k,\mu_i+2,\nu_i-2)\,|k,\mu_i+2,\nu_i-2\rangle \\
& + H_{lor}(k,\mu_i-2,\nu_i-2)\,|k,\mu_i-2,\nu_i-2\rangle \Big] \\
& + b_{k_i,k_{i-1},\mu_i,\nu_i} \Big[H_{lor}(k_i+2,\mu,\nu_i+1,\nu_{i+1}+1)\,|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
& - H_{lor}(k_i+2,\mu,\nu_i+1,\nu_{i+1}-1)\,|k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
& + H_{lor}(k_i+2,\mu,\nu_i-1,\nu_{i+1}+1)\,|k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\
& - H_{lor}(k_i+2,\mu,\nu_i-1,\nu_{i+1}-1)\,|k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \\
& - H_{lor}(k_i-2,\mu,\nu_i+1,\nu_{i+1}+1)\,|k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle \\
& + H_{lor}(k_i-2,\mu,\nu_i+1,\nu_{i+1}-1)\,|k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle \\
& - H_{lor}(k_i-2,\mu,\nu_i-1,\nu_{i+1}+1)\,|k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle \\
& + H_{lor}(k_i-2,\mu,\nu_i-1,\nu_{i+1}-1)\,|k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle \Big] \\
& + c_{k_i,k_{i-1},\mu_i,\nu_i} \Big[H_{lor}(k_i+2,\mu_i+1,\mu_{i+1}+1,\nu)\,|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
& - H_{lor}(k_i+2,\mu_i+1,\mu_{i+1}-1,\nu)\,|k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
& + H_{lor}(k_i+2,\mu_i-1,\mu_{i+1}+1,\nu)\,|k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle \\
& - H_{lor}(k_i+2,\mu_i-1,\mu_{i+1}-1,\nu)\,|k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle \\
& - H_{lor}(k_i-2,\mu_i+1,\mu_{i+1}+1,\nu)\,|k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle \\
& + H_{lor}(k_i-2,\mu_i+1,\mu_{i+1}-1,\nu)\,|k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle \\
& - H_{lor}(k_i-2,\mu_i-1,\mu_{i+1}+1,\nu)\,|k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle \\
& + H_{lor}(k_i-2,\mu_i-1,\mu_{i+1}-1,\nu)\,|k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle \Big] \Bigg\}.
\end{aligned}
\tag{A.0.32}
$$

Finally, we can order (A.0.32) by collecting all eigenvalues $H_{lor}(k,\mu,\nu)$ that have the same

state $|k,\mu,\nu\rangle$ giving us

$$\begin{aligned}\left[\widehat{\mathrm{H}}_{\mathrm{eucl}},\widehat{\mathrm{H}}_{\mathrm{lor}}\right]|k,\mu,\nu\rangle = \sum_{i=1}^{N}\Bigg\{ a_{k_i,k_{i-1},\mu_i,\nu_i}\Big[&\left(H_{lor}(k,\mu,\nu)-H_{lor}(k,\mu_i+2,\nu_i+2)\right)|k,\mu_i+2,\nu_i+2\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k,\mu_i-2,\nu_i+2)\right)|k,\mu_i-2,\nu_i+2\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k,\mu_i+2,\nu_i-2)\right)|k,\mu_i+2,\nu_i-2\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k,\mu_i-2,\nu_i-2)\right)|k,\mu_i-2,\nu_i-2\rangle\Big]\\ +b_{k_i,k_{i-1},\mu_i,\nu_i}\Big[&\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu,\nu_i+1,\nu_{i+1}+1)\right)|k_i+2,\mu,\nu_i+1,\nu_{i+1}+1\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu,\nu_i+1,\nu_{i+1}-1)\right)|k_i+2,\mu,\nu_i+1,\nu_{i+1}-1\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu,\nu_i-1,\nu_{i+1}+1)\right)|k_i+2,\mu,\nu_i-1,\nu_{i+1}+1\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu,\nu_i-1,\nu_{i+1}-1)\right)|k_i+2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu,\nu_i+1,\nu_{i+1}+1)\right)|k_i-2,\mu,\nu_i+1,\nu_{i+1}+1\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu,\nu_i+1,\nu_{i+1}-1)\right)|k_i-2,\mu,\nu_i+1,\nu_{i+1}-1\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu,\nu_i-1,\nu_{i+1}+1)\right)|k_i-2,\mu,\nu_i-1,\nu_{i+1}+1\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu,\nu_i-1,\nu_{i+1}-1)\right)|k_i-2,\mu,\nu_i-1,\nu_{i+1}-1\rangle\Big]\\ +c_{k_i,k_{i-1},\mu_i,\nu_i}\Big[&\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu_i+1,\mu_{i+1}+1,\nu)\right)|k_i+2,\mu_i+1,\mu_{i+1}+1,\nu\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu_i+1,\mu_{i+1}-1,\nu)\right)|k_i+2,\mu_i+1,\mu_{i+1}-1,\nu\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu_i-1,\mu_{i+1}+1,\nu)\right)|k_i+2,\mu_i-1,\mu_{i+1}+1,\nu\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i+2,\mu_i-1,\mu_{i+1}-1,\nu)\right)|k_i+2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu_i+1,\mu_{i+1}+1,\nu)\right)|k_i-2,\mu_i+1,\mu_{i+1}+1,\nu\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu_i+1,\mu_{i+1}-1,\nu)\right)|k_i-2,\mu_i+1,\mu_{i+1}-1,\nu\rangle\\ &-\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu_i-1,\mu_{i+1}+1,\nu)\right)|k_i-2,\mu_i-1,\mu_{i+1}+1,\nu\rangle\\ &+\left(H_{lor}(k,\mu,\nu)-H_{lor}(k_i-2,\mu_i-1,\mu_{i+1}-1,\nu)\right)|k_i-2,\mu_i-1,\mu_{i+1}-1,\nu\rangle\Big]\Bigg\}. \end{aligned} \tag{A.0.33}$$

We now discuss the annihilation of the commutator. This is motivated by our results presented in section 6, which imply that the only common eigenstates of the Euclidean and Lorentzian part have eigenvalue zero. Hence, it may happen that the annihilation of the commutator yields new insights or even some states which are diagonal with respect to Euclidean and Lorentzian part with non-vanishing eigenvalue. In particular, we can let the commutator act on our special ansatz (6.0.3). Then, we can use the result of (A.0.33) and transform the shift of the states to the coefficients, as we did for the Euclidean part in section 6.3. However, looking at the complicated result of (A.0.33), we see that the prefactors $H_{lor}(k,\mu,\nu)$ lead to a coupling between the labels prohibiting a separation ansatz as performed in 6.3 or 6.4. This argumentation is similar to that discussed in section 6.6. Nevertheless, we can annihilate the commutator using the zero volume states presented in section 6.2, as both the Lorentzian and Euclidean operator vanish for such states and hence, (A.0.27) will vanish, too. Therefore, we conclude that it seems as our special ansatz (6.0.3) only allows us to annihilate the commutator between the Euclidean and the Lorentzian operator for zero volume states. Also if we let the commutator act on a single basis state $|k,\mu,\nu\rangle$, zero volume states seem to be the only possible solution due to the complicated structure of (A.0.33).

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Danksagung

Zuerst möchte ich mich bei Prof. Kristina Giesel bedanken, die mir diese Arbeit überhaupt erst ermöglicht hat. Insbesondere möchte ich mich für die intensive Betreuung bedanken, es war immer sehr einfach möglich Treffen zu arrangieren, zu diskutieren etc., was wirklich immens hilfreich war mich in die Quantengravitation einzuarbeiten und diese Arbeit zu schreiben.

Des weiteren gilt ein ebenso großer Dank David Winnekens, der mich sehr freundlich aufgenommen hat, mit dem ich einen permanenten Ansprechpartner hatte und mit dem ich in Zusammenarbeit die Ergebnisse dieser Arbeit erreicht habe.

Weiterhin gilt mein Dank auch meinen Bürokollegen Satya Dhandhukiya, Adrian Herzog und David Grüber, sowie Thomas Zilker, Susanne Schander, Klaus Liegener, Thorsten Lang, Max Düll, Andrea Dapor und allen anderen Leuten am Lehrstuhl für die schöne Zeit.

Zu guter Letzt möchte ich meiner Familie danken - Danke für alles.

Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe.

Erlangen, 23. März 2017

Andreas Leitherer