QUANTIZATION OF $g\ell(N,\mathbb{C})/U(1)$ AT ROOTS OF UNITY AND PARAFERMIONS

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ABSTRACT

The equivalence between the quantum group $(g\ell(N,\mathbb{C};L)/U(1))_{q,s} \equiv g\ell(N;L)_{q,s}$ at $q^2=-1$ over a non-Grassmannian field and $s\ell(L|N-L)_s$ over a Grassmannian field is discussed. The equivalence extends to $g\ell(N;L)_{q,1}(q^2=-1) \sim s\ell(L|N-L)$. This suggests a generalization of $s\ell(L|N-L)$ to Z_m-grading via $g\ell(N;L)_{q,s}$ at $q^2=m^{th}$ root of unity, m>2. Specifically, representations $g\ell(2;1)_{q,s}$ at m^{th} root of unity are shown, via their fusion and braiding properties, to transform as s-deformed parafermions, or spin-1/m anyons. They contrast sharply with corresponding representations of $s\ell(2)_s$.

Recently representations of quantum groups, especially $s\ell(N,\mathbb{C})_q$ (henceforth $s\ell(N)_q$) at roots of unity have attracted a great deal of attention.[1] Here we discuss representations at roots of unity of another quantum group $(g\ell(N,\mathbb{C};L)/U(1))_{q,s}$ (henceforth $g\ell(N;L)_{q,s}$ and called twisted quantum group of A_{N-1} in [2,3]). Some of the especially interesting properties of these representations are already known: *(a)* For N=2, s generic and $q^2=-1$, the representation gives the Alexander-Conway link polynomial, whose counterpart is the Jones polynomial derivable from the fundamental representation of $s\ell(2)_q$, q generic.[2] *(b)* The state model associated with the Alexander-Conway polynomial is the free fermion model.[3,4] *(c)* There is a hierarchy of Alexander-Conway link polynomials corresponding N=2, s generic and $q^2=m^{th}$ root of unity.[2,5] *(d)* The representations of $g\ell(N;L)_{q,s}$ at $q^2=-1$ coincide with those of $s\ell(L|N-L)_s$, whose associated link polynomials are just Witten's Wilson-lines for the 3D topologocal Chern-Simons theory with gauge group SU(L|N-L).[3,6]

In this report (where $\omega_m \equiv \exp(2\pi i/m)$, $q_m \equiv \omega_m^{-1/2}$) we give a summary of properties of the representations of $g\ell(2;1)_{q_m,s}$, s generic ($g\ell(2;L)_{q,s}$ reduces to $s\ell(2)_s$ unless L=1). They are parafermionic and unlike the representations of $s\ell(2)_s$, s generic, which have a one-to-one correspondence to the representations of $s\ell(2)$. We show that $g\ell(2;1)_{q_m,s}$ provides a generalization of the Z_2-grading of $s\ell(2)$ to Z_m-grading.

The generators of the Hopf algebra[2] of $g\ell(2;1)_{q,s}$, denoted by $\mathcal{A}$, are I, H and $X^{\pm}$, where in the classical limit I generates the U(1) factor in $g\ell(2) \sim s\ell(2)\times U(1)$ and the other three generate $s\ell(2)$. In the quantized case, I is still central to $\mathcal{A}$, and $[H,X^{\pm}]= \pm 2X^{\pm}$ and $[X^+,X^-]=(k^2-k^{-2})/(q-q^{-1})$ as in $s\ell(2)_q$, except that

$$k = q^{(H-I)/2}\, s^{I/2} \tag{1}$$

instead of $k=q^{H/2}$ in $s\ell(2)_q$. It is possible to absorb the effect of I on k in (1) into H by a redefinition of the latter, which will no longer be traceless.[7] For reasons that will become transparent we use the expression (1) in which the role of I and that of the second parameter s is made explicit from the outset (in which case the respective numbers of generators in the Cartan subalgebra and deformation parameters still match). For convenience we write $p\equiv s/q$. First note the trivial special case of (1) at $p^2=1$, whence the U(1) factor in $g\ell(N;L)_{q,s}$ is modded out and $\mathcal{A}$ is reduced to $s\ell(2)_q$, whose properties are well known.

We consider only the nontrivial case $p^2\neq 1$. Then $\mathcal{A}$ has a finite representation over the vector field V only when q^2 is a root of unity:

$$q^2=\omega^{-1}=e^{-2\pi i/m}, \quad m = \text{positive integer} \tag{2}$$

The same result obtains when one chooses, instead of (2), $q^2=\omega^{m'}$, provided m' is prime to m. Given (2), the elements $(X^{\pm})^m$ are central in $\mathcal{A}$, and a fundamental m-dimensional matrix representation π: $\mathcal{A}\rightarrow End(V)$ is obtained when the relations

$$\pi((X^{\pm})^m) = 0 \tag{3}$$

are imposed. In what follows, it will be understood that all expressions given for elements in $\mathcal{A}$ are those under the homomorphism π, and that $[\rho]=\{|i\rangle;\ i=1$ to $m\}$ is a basis for V, with the highest (lowest) state with respect to X^+ being $|1\rangle$ ($|m\rangle$). Then $|m\rangle$ ($|1\rangle$) are the highest (lowest) state with respect to X^-. With the aid of the derived relation[8] (meant to hold when acted on a state $\in \mathrm{Ker}X^+\backslash \mathrm{Im}(X^+)^{m-1}$)

$$[(X^+)^u,(X^-)^v] =$$

$$(X^-)^{v-u}\ \frac{[v]_q!}{[v-u]_q!}\ \prod_{j=1}^{v}\ \frac{k^2q^{-u+j}-k^{-2}q^{u-j}}{q-q^{-1}} \tag{4}$$

one obtains from standard methods:

$$I|i\rangle = (m-1)|i\rangle, \qquad H|i\rangle = (m+1-2i)|i\rangle \tag{5}$$

$$X^-|i\rangle = \left([i]_q(s^{m-1}q^{-l+1}-s^{-m+1}q^{l-1})/(q-q^{-1})\right)^{1/2}|i+1\rangle,$$

$$\langle i|X^+|i+1\rangle = \langle i+1|X^-|i\rangle \tag{6}$$

The R-matrix may be calculated from the method either of Drinfeld[9] or of [2]. Here we only give its m^2 eigenvalues, whose degeneracies determine the *fusion rule* of the direct product $[\rho]\otimes[\rho]$ and whose values characterize the *braiding* of the irreducible representations in the direct product, as expressed in the following two relations

$$[\rho]\otimes[\rho]=\bigoplus_{j=1}^{m}[\sigma_j],\ \text{(dimensionality of } [\sigma_j]=n_j) \tag{7}$$

$$R[\sigma_j] = r_j[\sigma_j] \tag{8}$$

That is, the degeneracy of r_j is n_j, and $\Sigma_j n_j=m^2$. For the R-matrix under study, r_j and n_j are given by

$$\{r_j,\ n_j;\ j=1 \text{ to } m\} =$$
$$\{-(-1)^j\omega^{(1-j)(j-2)/2}s^{(m-1)(3-2j)},\ m;\ j=1 \text{ to } m\} \tag{9}$$

There are m distinct eigenvalues, all with degeneracy m. This contrasts sharply with the R-matrix, denoted by R′, of the m-dimensional representation $[\rho']$ of $s\ell(2)_s$, whose eigenvalues r'_j and degeneracies n'_j for generic s are given by

$$\{r'_j,\ n'_j;\ j=1 \text{ to } m\} =$$
$$\{-(-1)^j s^{(m-1)[1-2m+(2m+1)j-j^2]},\ 2m-2j+1;\ j=1 \text{ to } m\} \tag{10}$$

For m=2, the link polynomials corresponding to $[\rho]$ and $[\rho']$ are respectively just the Alexander-Conway and Jones polynomials[2,5]. It follows from the fact $\mathcal{A}$ coincides with the Hopf algebra $\mathcal{A}'$ of $s\ell(2)_s$ in the limit $s^2=q^2=\omega^{-1}$ that $R(s^2=\omega^{-1})=R'(s^2=\omega^{-1})$. On the other hand (9) and (10) are discretely distinct. Therefore at least one of the relations cannot be continuous in that limit. It turns out that both are not; for a detailed discussion see [10].

To have a better understanding of the difference between (9) and (10) we return to $\mathcal{A}$ (instead of the homomorphism π) and consider, instead of $X^\pm$, the generators

$$Y^+ = q^{H/2}X^+, \qquad Y^- = X^-q^{H/2} \tag{11}$$

Define an x-commutator to be $[A,B]_x\equiv AB-xBA$. Then, instead of having a commutation relation like $X^\pm$ do, $Y^\pm$ satisfy

$$[Y^+,Y^-]_\omega = A\omega^{H/2}(\omega^{(I-H)/2}s^I - \omega^{-(I-H)/2}s^{-I}) \tag{12}$$

where $\omega=q^{-2}$ and A is a nonessential normalization constant so long as $q^2\neq1$. The coproduct on $Y^\pm$ now has a nonstandard appearance[2]: $\Delta(Y^\pm) = Y^\pm\otimes q^Hp^{I/2} + p^{-I/2}\otimes Y^\pm$.

The left-hand side of (12) is an ω-commutator. In particular, when $\omega=-1$, it is an *anti-commutator*. In this case, under the homomorphism π of (5) for m=2, the right-hand side of (12) is proportional to (s^I-s^{-I}), which vanishes in the limit $s\to1$. If one replaces the normalization constant A by $(s-s^{-1})^{-1}$, then (12) is exactly the commutation relation satistfied by the raising and lowering generators of $s\ell(1|1)_s$ (note that the fundamental representation of H in $s\ell(1|1)_s$ is proportional to the unit matrix, just as that of I is). In this sense $g\ell(2;1)_{q2,s}$ is equivalent to $s\ell(1|1)_s$.

To understand this notion further, consider (9) and (10) for the case m=2, and write the two states $|1\rangle$ and $|2\rangle$ as $|+\rangle$ and $|-\rangle$, the representations $[\sigma_j]$ for j=1 and 2 (see (7)) as [b] and [f], and $[\sigma'_j]$ as [s] and [a], respectively. For reason that will be clear presently, b, f, s and a stand for boson, fermion, symmetric and antisymmetric, respectively. We have

$$R[b]=s[b],\quad R[f]=-s^{-1}[f]\quad (\text{for } g\ell(2;1)_{q2,s}) \tag{13}$$

$$R'[s]=s[s],\quad R'[a]=-s^{-1}[a]\quad (\text{for } s\ell(2)_s) \tag{14}$$

The two sets of equations appear identical, but they carry quite different meanings. It suffices to point out that whereas both the symmetric states $|+\rangle|+\rangle$ and $|-\rangle|-\rangle$ lie in the three dimensional [s] in the case of $s\ell(2)_s$, in the case of $g\ell(2;1)_{q2}$ $|+>|+>$ lies in the two dimensional [b] while $|->|->$ lies in the two dimensional [f]. Thus, in the limit $s\to 1$, [f] changes sign under braiding not because it is antisymmetric, like [a] is, but because its constituents are fermionic.

It is important to distinguish how [f] is given a fermionic exchange property (here, because $(Y^{\pm})^2$ are central, there is no difference between braiding and transposition) in (the unquantized) $s\ell(1|1)$ and in $g\ell(2;1)_{q2,1}$. In the former, which has a trivial coproduct, the task is achieved by making the vector space explicitly contain a Grassmann variable, namely the state $|->$. In the latter the fermionic property of $|->$ is encoded in the braiding property of R in a Hopf algebra with a nontrivial coproduct, while the vector space is *nonGrassmannian*.

The analysis above can be transplanted onto $g\ell(N;L)_{q2,s}$ [3] to demonstrate its equivalence to $s\ell(L|N-L)_s$. This explains why, for the fundamental representations of the two quantum groups, the link polynomials, which are actually eigenvalues of invariants of the quantum group, are identical, as are their associated graded vector models, and why the latter are nonquasiclassical.[3] The equivalence carries over to the limit $s\to 1$ to establish the equivalence between the Hopf algebra $g\ell(N;L)_{q2,1}$ and the graded Lie algebra $s\ell(L|N-L)$. For $g\ell(N;L)_{q2,s}$ the formula (13) still applies, except that the dimensionality of [b] is N(N-1)/2+L and that of [f] is N(N+1)/2-L. These are to be contrasted with the dimensionalities of [s] and [a] in $s\ell(N)_s$, being respectively N(N+1)/2 and N(N-1)/2.

The Z_2-grading of $s\ell(2)$ into $s\ell(1|1)$ does not lend itself to a direct generalization to higher gradings. However, the discussion above shows that a Z_m-grading can be achieved by way of the Hopf algebra of $g\ell(2;1)_{q,s}$ at $q^2 = \omega_m^{-1}$, which in the following we call $\mathcal{A}_m$. Recall that the configuration space for a system of states having the property of higher than Z_2 grading is nonsimply connected, so that, instead of transposition, one must speak of braiding of two states. This explains why a quantum group is necessary for higher gradings. That $\mathcal{A}_m$ has the property of a Z_m-graded algebra is already clear from (9) and (12), especially when the latter is recast into the form

$$[Y^+,Y^-]_{\omega_m} = \alpha(s)(P_m-\beta(s)) \qquad (m>2) \qquad (15)$$

where P_m is idempotent of order m, and α and β are central elements depending on s and I. The right-hand side of (15) does not vanish in the limit $s\to 1$ for m>2, so it is not necessary to have a factor $(s-s^{-1})^{-1}$.

From (9), the fusion states $[\sigma_j]$ defined in (8) for $\mathcal{A}_m$ at s=1 braid as

$$R[\sigma_j] = (-1)^{j+1}\omega^{v_j}[\sigma_j];$$
$$v_j = -(j-1)(j-2)/2 \quad (\mathrm{mod}\ m) \qquad (16)$$

In particular $[\sigma_1]$=[b] is bosonic, $[\sigma_2]$=[f] is fermionic, while the other states are such that $R^m[\sigma_j]=\pm[\sigma_j]$. These latter states may be interpreted as anyonic states with "spin" 1/m; they are direct generalizations of a fermionic state, which has spin 1/2. The dimensionality of $[\sigma_j]$ is m, independent of j. Thus the representation $[\rho]$ of $\mathcal{A}_m$ is parafermionic. (Since the link polynomial for $\mathcal{A}_2$, is just the Wilson line for the supersymmetric Chern-Simons theory with SU(1|1) gauge symmetry,[3,6] one is intrigued with the possibility of the link polynomials for $\mathcal{A}_m$, m>2, being related to the Wilson lines for fractionally supersymmetric[11] Chern-Simons theories.) In comparison, for $s\ell(2)$, the corresponding fusion states $[\sigma'_j]$ are just normal spin m-j states: they have respective dimensionalities 2(m-j)+1 and are either symmetric (j odd) or antisymmetric (j even) under R'. Since $s\ell(2)_s$ is a continuous deformation of $s\ell(2)$, the eigenstates of R' for generic s cannot be anyonic even as they have unusual braiding properties. They are just normal spin states deformed. For a discussion of the situation at $s^2=\omega_m^{-1}$, when $\mathcal{A}_m$ coincides with $s\ell(2)_s$, see [10].

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DISCUSSION

Q. **A. LeClair** *(Cornell Univ.)*: Why do you call your symmetries fractional supersymmetries if you don't have the Poincaré generators in the algebra? I don't think the name is justified.

A. **H. C. Lee**: The representations are those for $1/m$— statistics anyons. I mention fractional supersymmetry because I think the representations are characteristic of those of fractional supersymmetric systems, plus the fact that the link invariants for $(sl(n/n)\times U(1))_{q^2=-1,s}$ are exactly the link invariants of Wilson lines in the three-dimensional supersymmetric topological field theory with $SU(n/n)$ gauge group.