FUNDAMENTAL THEORETICAL QUESTIONS

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I will use my Rapporteur's prerogative to single out for discussion a small subset of the papers submitted; the complete list will be found elsewhere in these Proceedings. I will also comment on some relevant progress made, since the XIIIth Conference on High-Energy Physics, in papers not submitted here.

Of course, the grand unsolved problem of this session is to find a suitable framework on which to hang theories of elementary particles. As you well know, there are contending proposals for such a framework that centre around two main sets of ideas: S-matrix theory and field theory. I will spend about equal time on each.

Let me begin with S-matrix theory and for the moment exclude infra-red problems; I will come back to them later.

It is a standard feature of S-matrix theory that analyticity, crossing symmetry, and unitarity are together enormously restrictive. It is also notorious that their consequences are not trivial to work out. For example, it was nearly nine years after Mandelstam proposed his representation for two-particle scattering amplitudes that it was finally shown by Martin that the analyticity arising directly from causality, elastic unitarity, and crossing cannot be used to prove the Mandelstam representation in the equal mass case[1]. Martin's work left open the question whether the larger analyticity domain implied by the Mandelstam representation is even compatible with unitarity and crossing. This kind of question is one which also appears in bootstrap dynamics customarily coupled there with special approximations.

In an interesting paper[2] (see also Ref. 3 for details), Atkinson has settled this question for the scattering amplitudes of charged and neutral pions. Since the method involved is relatively new and likely to be fruitful in the future, let me describe it in some detail. The idea is to use fixed-point theorems of non-linear functional analysis to prove the existence of solutions of the unitarity and crossing relations. These theorems provide one of the few highly developed techniques of functional analysis useful in proving the existence and determining the properties of the solutions of non-linear equations. Their use has been eloquently advocated in S-matrix theory by Lovelace[4] and earlier, in field theory, by J.G. Taylor[5]. The simplest result of this kind is the so-called contraction mapping theorem. It says that if T is a mapping of a complete normed space into itself, satisfying

$$\|T(x) - T(y)\| \leq k\,\|x - y\| \tag{1}$$

with $k < 1$, then T possesses a unique fixed point x_0. In fact, pick any point z of the space and consider the sequence $z, T(z), T^2(z), \ldots$. It converges to x_0. [Proof: For any two points x and y, and any integers $k, n \geq 0$, $\|T^{k+n}(x) - T^n(y)\| \leq k^n \|T^k(x) - y\| \to 0$ as $n \to \infty$.] There is a rich variety of other theorems and devices for telling whether a fixed point is isolated or part of a continuous family.

The general strategy of the application to the present problem for the case of neutral pions is as follows. The scattering amplitude is expressed in terms of the spectral weight ρ by

$$A(s,t) = \frac{1}{\pi^2} \int_4^\infty ds' \int_{\sigma(s')}^\infty dt'\, \rho(s',t')\left[(s'-s)^{-1}(t'-t)^{-1} + (s'-s)^{-1}(t'-4+s+t)^{-1} + (s'-4+s+t)^{-1}(t'-t)^{-1}\right], \tag{2}$$

where $\sigma(s) = \min\,(4s/s-16,\ 16s/s-4)$.

Given ρ one can compute also

$$D(s,t) = \frac{1}{\pi} \int_{\sigma(t)}^{\infty} ds' \frac{\rho(s',t)}{s'-s} + \frac{1}{\pi} \int_{\sigma(t)}^{\infty} ds' \frac{\rho(s',t)}{s'-4+s+t}, \tag{3}$$

the t-channel absorptive part of A, and

$$\rho^{el}(s,z) = \frac{1}{\pi}\,\theta(z-2z_0+1)\left(\frac{s-4}{s}\right)^{\frac{1}{2}} \int_{z_0}^{h(z_1 z_0)} dz \int_{z_0}^{h(z,z_1)} dz_2\, k^{-\frac{1}{2}}(z,z_1,z_2)\, D^*(s,z_1)\, D(s,z_2), \tag{4}$$

where $k(z,z_1,z_2) = z^2 + z_1^2 + z_2^2 - 2z\, z_1 z_2 - 1$, $z_0 = 1 + 2t/(s-4)$, and $h(z,z_0) = z\, z_0 - (z^2-1)^{1/2}$. Then finally one can compute a new ρ' by writing

$$\rho'(s,t) = \rho^{el}(s,t) + \rho^{el}(t,s) + v(s,t), \tag{5}$$

where v is some symmetric function. For each fixed v, Eqs. (3), (4), and (5) can be written symbolically as

$$\rho' = T(\rho). \tag{6}$$

Straightforward arguments show that if v(s,t) is chosen to vanish for

$$s \le \frac{16t}{t-16}\left[1 + \frac{8t}{(t-16)(t-4)}\right],$$

then a ρ for which ρ' = ρ, i.e. a fixed point of T, is a solution to the problem. It remains to show that for a suitable choice of admissable v and ρ, T maps admissable ρ into themselves in such a manner that a fixed-point theorem is applicable. I spare you the technical details[6], which are long and ingenious, but I want to emphasize the significance of the result. v is a kind of measure of the contribution to ρ from inelastic processes. For each v from a rich class, one gets an admissible scattering amplitude. Thus, crudely speaking, the π-π scattering amplitude is no more limited by Mandelstam analyticity, crossing symmetry, and unitarity than is the scattering amplitude of a potential scattering problem. In each case the possible amplitudes are parametrized by a real function, here v; in the potential scattering case, by the potential. Of course, the significance of the two real functions is completely different.

The above deals with the case of neutral pions. Atkinson has obtained similar results for charged and neutral pions taken together, again assuming no subtraction in the Mandelstam representation.

Whilst this work settles a long-standing question of principle, it leaves open another of considerable practical importance. Can the results be extended to the case in which subtractions are necessary? Here Atkinson reported that he has been able to carry the argument through, except for the proof that there are solutions satisfying inelastic unitarity. If this difficulty could be overcome, the theory would become a practical tool for the description of pions as we see them in Nature.

In three interesting papers[7-9], Wanders and coworkers have shown that known exact information on the structure of the π-π scattering amplitude has significant practical consequences. In Ref. 7, Wanders derives four sum rules for the amplitudes $A^I(s,t)$ under four assumptions as to their high-energy behaviour. For example, from

$$\int^{\infty} ds\, \frac{1}{s}\, A^0(s,t) < \infty \qquad 0 \le t \le t_0 \tag{7}$$

he gets

$$\int_4^{\infty} dx \left[\frac{2x-4}{x(x-4)} \frac{\partial}{\partial t} A^0(x,0) - \frac{1}{x^2} A^0(x,0) - \frac{2}{(x-4)^2} A^1(x,0) \right] = 0. \tag{8}$$

Expansion in partial waves then yields a sum rule containing only observables

$$\sum^{\infty} \sum F_\ell^I = 4 \int_4^{\infty} ds \frac{A_0^0(s)}{(s-4)^{\frac{1}{2}} s^{\frac{3}{2}}} = F_0^0, \tag{9}$$

with

$$F_\ell^I = \int_4^{\infty} ds\, f_\ell^I(s)\, A_\ell^I(s), \tag{10}$$

where the f_ℓ^I are known positive functions. When one inserts the observed ρ, f, and G resonances and a reasonable estimate for A_0^0, one gets a violation of this sum rule.

Conclusion: the fixed t-dispersion relation for the isospin 0 T-matrix element $T^{(0)}(s,t,u)$ requires a subtraction. The remaining three sum rules are more difficult to check and, on first investigation, gave no clear-cut discrepancy with experiment.

In Ref. 9, the objective is somewhat different. There the authors develop various parametrizations of S-wave π-π scattering amplitudes and ask whether the imposition of known exact inequalities yields practical restrictions on the parameters. The answer is that they do. Reference 8 is devoted to the problem of parametrizing partial wave π-π scattering amplitudes satisfying analyticity, unitarity, and a simple mathematical assumption about the form of the discontinuity on the left-hand cut. The problem is reduced to the solutions of a set of difference equations for the N and D functions of the N/D method; these are solved for some simple cases. The variety of solutions reflects the presence of CDD poles.

Several papers on the structure of form factors were submitted to this session[10-12]. Of these, I will discuss first that of Nagel[10] which describes generalizations of results of Balachandran and Loeffel[13]. Nagel considers a function F analytic in the complex t-plane with the exception of the cut $0 < b < t < \infty$, and satisfying f real for $-\infty < t < b$ and for each $\varepsilon > 0$

$$|F(t)| \le \left[1 + (\Delta t)^{-m}\right] M_\varepsilon e^{\varepsilon |t|^{\frac{1}{2}}} , \qquad (11)$$

where Δt is the distance from t to the cut. Various assumptions are made about how F(t) approaches zero as $t \to -\infty$ along the real axis. In all cases, F has a unique boundary value on the cut in the sense of distribution theory, and satisfies a generalized unsubtracted dispersion relation

$$F(t) = \lim_{\delta \to +0} \frac{1}{\pi} \int_{6}^{\infty} \frac{c(\tau,\delta)\ \mathrm{Im}\ F(\tau)\ d\tau}{t - \tau} , \qquad (12)$$

where

$$c(\tau,\delta) = \left[1 + \delta\tau\right]^{-1} \exp -\delta\, t^{\frac{1}{2}} .$$

If

$$\lim_{t \to -\infty} \left[t^N F(t) - K\right] = 0$$

for some integer $N \ge 1$, then

$$\lim_{\delta \to 0} \int_{6}^{\infty} c(\tau,\delta)\tau^{N-1}\ \mathrm{Im}\ F(\tau)\ d\tau = -\pi K . \qquad (13)$$

If $N \ge 2$, then also

$$\lim_{\delta \to 0} \int_{6}^{\infty} c(\tau,\delta)\tau^{n}\ \mathrm{Im}\ F(\tau)\ d\tau = 0 \qquad n = 0, 1, \ldots, N-2 . \qquad (14)$$

These are generalized superconvergence relations.

In addition to these very general and precise results on form factors, there were several others submitted to the Conference where the purpose is to derive relations suitable for parametrizing the data on the pion form factor. An example is the work of Truong and Vinh-Mau[12]. Here the existence of experimental data on the cross-section $e^+ + e^- \to \pi^+ + \pi^-$ suggests an attempt to write a dispersion relation which contains $|F(t)|$. The trick is to consider $\ln F(t)/\sqrt{t - 4\mu^2}$. If F is analytic in the plane cut from $4\mu^2$ to $+\infty$ and sufficiently bounded, one gets, for $t < 0$,

$$\frac{\ln F(t)}{\sqrt{4\mu^2 - t}} = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\ln|F(t')|\, dt'}{\sqrt{t' - 4\mu^2}\,(t' - t)} + \text{terms arising from the zeros of F.} \qquad (15)$$

The integral in Eq. (15) is directly computable from experiment. Experimental information can also be used to estimate the remaining terms, and then Eq. (15) can be used to obtain upper and lower bounds on the form factor for $t < 0$, and for the pion electromagnetic radius which is given by $1/6\ \langle r^2 \rangle = F'(0)/F(0)$.

Before leaving the subject of S-matrix theory, I would like to list some outstanding developments of the last year and comment on their significance.

1. Hepp's proof[14] that the collision states are complete in non-relativistic n-body Schrödinger theory.

Here is something taken completely for granted in relativistic S-matrix theory, but which for years resisted all attempts at proof in non-relativistic Schrödinger theory. The key was the n-body generalization by Yakubovsky[15] of the Faddeev equations.

2. The proof of superconvergence relations by Mahoux and Martin[16]. The practical importance of such relations is notorious. What is remarkable is the generality of the conditions under which they were proved.

3. The proof by Iagolnitzer and Stapp[17] of the analyticity of collision amplitudes from assumptions on their dependence on impact parameters.

It is good to have phenomenological S-matrix conditions which guarantee some of the properties normally derived from field theory.

4. Martin's counter-example[18] showing the impossibility of passing to the second sheet of a partial wave amplitude.

Here is a puzzle found by Martin. He asks what general principle is there to prevent a partial wave amplitude from having a set of zeros at the points $(p/N) + (i/N)$, $p = 1, \dots, N$; $N = 1, 2, \dots$? These zeros cluster in such a way as to produce a natural boundary on the elastic cut $[0,1]$. The boundary value of such an amplitude is a rather nasty beast since it has a dense set of discontinuities. Of course, Freund and Karplus contemplated the possibility of such a natural boundary some time ago, but it lay inside the second sheet. The practical importance of the phenomenon described here is that it would block Martin's use of unitarity to enlarge the analyticity domain of the full scattering amplitude. The only general principle I know of, that Martin's counter-example would violate, is Goldberger's Principle: it is absurd that Nature could be so unkind.

5. The work of H. Epstein and V. Glaser on the five-point function[19].

The authors start from the observation that, for the inelastic amplitude (2 particles → 3 particles), the mass shell does not, in general, lie inside the analyticity domain of any single generalized retarded function. To obtain the existence of scattering amplitudes as the restriction of such a retarded function to the mass shell, they propose a decomposition (non-unique and only valid locally) of the p-space analytic functions into functions with better analytic properties. The existence of such decompositions has been established for the five-point function, and appears to be provable for the n-point function.

Now I turn to the second general area of discussion: model quantum field theories. Before reviewing the papers submitted to the Conference, let me comment on the state of the problem of proving the existence of solutions in model field theories. It was reported at Berkeley two years ago that it had been possible to prove the existence of Green's functions for two cases: the neutral scalar field with quartic self-coupling[20]; and the theory of a spinor field coupled to a scalar field by a Yukawa interaction[21]. In both cases the theory was butchered by the introduction of an ultra-violet cut-off and a box cut-off. The ultra-violet cut-off was more drastic in the ϕ^4 theory since only a finite number of modes were allowed to be coupled. It has turned out that these results can be generalized in two ways. First, one can now couple an infinite number of Bose modes; and secondly, one can handle all derivative-free couplings of a set of Bose fields to a set of Fermi fields, provided there are present formally positive Bose self-couplings which dominate the rest of the couplings in which bosons appear[22] (see also Ref. 23).

$$\left[\text{Example: } \mathscr{H}_I(x) = g\,\bar{\psi}\psi\phi + \lambda\phi^3 + \lambda_1\phi^4 \text{ with } \lambda_1 > 0\right].$$

Thus, one has available a large class of cut-off models, and attention has turned to the problem of removing the cut-offs. Here very interesting progress has been made.

Let me recall that the reason for the box cut-off is that one wants to start by giving meaning to the Hamiltonian of a model by using the ϕOK representation of the commutation and anticommutation relations. When the box is removed so that the theory becomes Euclidian-invariant, strange representations of the CCR and CAR have to be used (Haag's theorem), and we find it difficult to specify which ones. On the other hand, the reason for the introduction of the ultra-violet cut-off is ultra-violet divergences, and the traditional remedy for that is renormalization. The customary first step is therefore to try to define a renormalized Hamiltonian in a box. From it one computes the unitary operator describing the time development of the theory; and from it, in turn, one computes the vacuum and vacuum expectation values; the hope then is that these will have a limit as the box becomes large. They will then define a theory that is free from cut-offs. Let me try to summarize the situation, as it now stands, in Table 1.

TABLE 1

Status of existence theorems for model field theories.

		2-dim. space-time		3-dim. space-time	
		ϕ^4	Yukawa	ϕ^4	Yukawa
Box, No UV cut-off	Existence of H_{ren}	yes (24)	yes (25)	yes (29)	yes (23)(24)
	$H_{ren} \geq -B$	yes (24)	yes (26)		
	Existence of vacuum	yes (30)			
UV cut-off, No box	Existence of rep. of CCR and CAR	yes (27)		yes (27)	
	Existence of fields	yes (28)			
No cut-offs	Existence of Haag fields	yes (30)			
	Existence of good vacuum	( ) (31)			
	Existence of vacuum expectation values	( ) (31)			

H_{ren} is the renormalized Hamiltonian
CCR ≡ Canonical Commutation Relations
CAR ≡ Canonical Anticommutation Relations
Haag Fields ≡ exp i $\phi(f,t)$ and exp i $\pi(f,t)$, where ϕ and π are the ordinary canonical fields smeared in space with test functions f and g, respectively. Good vacuum means it is the limit of the cut-off ground states and is the ground state.

The first point to be made is that all the results are for super-renormalizable theories. The step to the renormalizable case looks difficult with present techniques, so the attempt of the authors to push the super-renormalizable case through to the end before worrying about the renormalizable but not super-normalizable theories seems sensible. The second point is that to define H_{ren} as a bilinear form is not enough to show that it is an operator. However, if one can show $H_{ren} \geq B$, one can use Friedrich's extension to define an operator and a unitary propagator exp i H_{ren} t. The trouble with this procedure is that one does not have very much control over the spectrum of H_{ren}. It is much more satisfactory when one can show that H_{ren} is essentially self-adjoint on a suitable domain. That has so far been done only for the ϕ^4 theory in two-dimensional space-time. What has been done in the other cases is to prove the existence of a dressing transformation T_∞, which maps the domain of the unperturbed Hamiltonian on the domain of the renormalized Hamiltonian. Here there arises a significant technical distinction between those cases in which the cut-off Hamiltonian H_K converges strongly to H_{ren} as the cut-off $K \to \infty$:

$$\lim_{K\to\infty} H_K T_K \Phi = H_{ren} T_\infty \Phi \tag{16}$$

for Φ in a dense domain, and those for which the convergence is weak:

$$\lim_{K\to\infty} (T_K \Phi, H_K T_K \Psi) = (T_\infty \Phi, H_{ren} T_\infty \Psi). \tag{17}$$

Strong convergence has been established for the Yukawa interaction in two-dimensional space-time, and even for the interaction $\bar{\psi}\psi\phi^2$ (ψ a fermion field and ϕ a boson field)[32], but it does not hold for the ϕ^4 theory in three-dimensional space-time. In fact, in this latter theory the range of the dressing transformation is not in ϕOK space, but rather in a Hilbert space with an altered scalar product.

It is out of the question to attempt to summarize the methods by which these results have been achieved, but it is probably worth while to note that one basic tool is just the kind of canonical transformation used by Schwinger in 1946-1947 to display the self-energy divergences. The book of Friedrichs[33] based on his Boulder Lectures of 1960 has also been a source of inspiration. Whatever one may regard as the historical sources of inspiration, the theory includes a new and deep singular perturbation theory directly adapted to the problems of field theory[34].

The significance of the ( ) in the last two rows of the table is this. In Paper I, Glimm and Jaffe (Ref. 34) have announced that their Paper II will deal with the existence of the vacuum and the vacuum expectation values. They have written Paper II, but are still checking details and do not wish to make a statement at this time. There is a reasonable expectation that we will soon have what we have been itching to get our hands on for two decades: a non-trivial relativistically invariant theory on which to test the heuristic core of our subject: perturbation theory as a guide to the structure of the S-matrix .

In my opinion these developments are exceedingly promising. You can try to catch up with them by working through the Varenna Courses of Jaffe and Glimm (August 1968) which should be published shortly.

I will now turn to papers in this field submitted to this meeting.

I have emphasized that because of Haag's theorem the ϕOK representation of the CCR and CAR cannot be used in Euclidian-invariant theories with vacuum polarization. Last year saw the emergence of a proof that a large class of strange representations, the so-called product representations, are not usuable either[35]. Since product representations are at present the largest, easily accessible, class of strange representations, it is of interest to know what kinds of Hamiltonians can be made meaningful by using them. Reed has shown[36] that this is the case for the Hamiltonian

$$H = \sum_{k=1}^{\infty} \omega_k \left[p_k^2 + q_k^2 - \tau_k\right] + \sum_{k,\ell,m,n=1}^{\infty} d_{k\ell mn} \; q_k \, q_\ell \, q_m \, q_n \,. \quad (18)$$

Here it is assumed that $\omega_k > 0$, $d_k \equiv d_{kkkk} > 0$, and the off-diagonal d_{klmn} are small in the sense that they satisfy

$$\sum{}'' \left| d_{k\ell mn} \right| \left| d_{rstu} \right| \left[1 + \left(\frac{\omega_k}{d_k}\right) + \cdots \left(\frac{\omega_u}{d_u}\right)\right] < \infty$$

$$\frac{1}{8}\sum{}'' \left| d_{k\ell mn} \right| \left| d_{rstu} \right| \left(\frac{1}{d_k^2} + \frac{1}{d_\ell^2} + \cdots \frac{1}{d_n^2}\right) \le a < 1. \quad (19)$$

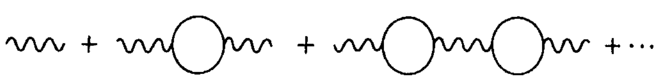
(The double prime means omit diagonal terms in both summations.) One has to choose the τ_k's so as to subtract the vacuum energy of the Hamiltonian omitting off-diagonal terms. H then has a pure discrete spectrum with a non-degenerate ground state (vacuum). The techniques used in proving this result (Kato perturbation theory) are closely related to those in Ref. 22 although the two investigations were carried out independently. The conditions [Eq. (19)] are not satisfied in the usual ϕ^4 theory.

Simon[37] has taken a half-step forward in a perennial unsolved problem of field theory: the convergence of renormalized perturbation expansions for Green's functions in a theory of a spinor and scalar field interacting through the Yukawa interactions. There have been questions raised whether the behaviour of the renormalized series might be completely different from that of the regularized unrenormalized series. [See Guerra and Marinaro[38] where that seems to be the case for a butchered model.] Simon does not answer that question for the theory he considers, but rather shows that the regularized renormalized series has a non-zero radius of convergence. The theory is that of the Yukawa interaction in two-dimensional space-time in which the only primitively divergent diagrams are

Now the usual expression for the n^{th} order contribution to a Green's function is

$$\frac{e^n}{n!}\int\cdots\int \begin{pmatrix} 1,\ldots,n & x_1,\ldots,x_r \\ 1,\ldots,n & y_1,\ldots,y_n \end{pmatrix} \times$$
$$\times \left[1,\ldots,n \;\; z_1,\ldots,z_\ell\right] d_1,\ldots,d_n, \quad (20)$$

where the first factor of the integrand is a determinant of fermion propagators and the second is a hafnian of boson propagators. Simon rearranges this series to obtain a sum in which the n^{th} term is the same, except that the determinant is replaced by a <u>bubblessian</u> and the bosons propagators in the hafnian must become aerated propagators which are the renormalized sums of bubbles



A bubblessian is a determinant where the terms in its expansion that contain two cycles are deleted. Remarkably enough, the bubblessian satisfies an analogue of the Hadamard inequality for determinants, so that if the propagators are now regularized and the integrations are taken over a finite space-time box, the standard proof of convergence of the series works. This proves absolutely nothing about the renormalized series itself.

Now I turn to the paper submitted by Ruijgrok[39]. which treats a nice old problem: the fixed-source charge-symmetric scalar meson theory. Here the Hamiltonian is

$$H = \sum_{i=1}^{3} a_i^* a_i + g \sum_{i=1}^{3} \tau^i a_i \;,$$

where the a_i and a_i^* are annihilation and creation operators for S-wave mesons of isospin index i, and τ^i are the Pauli matrices acting on the isospin of a fixed nucleon. By a shrewd choice of basis, Ruijgrok reduces the problem of calculating the bound-state energies numerically to a convenient form. Numerical results are then compared with weak and strong coupling expansions. It is very instructive for those who like to see how strong, intermediate, and weak couplings compare in a concrete non-trivial case.

The third and last part of my report is devoted to general field theory. From among the numerous contributions I have chosen four, not only because what they do is interesting, but because the questions they raise are likely to be interesting in the future.

Mandelstam's communication[40], which is based on two thick preprints, solves the problem: derive the Feynman-de Witt rules for the evaluation of the perturbation series of Green's functions of the quantum theory of gravitation and extend them to the Yang-Mills theory. Mandelstam's derivation uses a gauge-invariant formalism which he developed some years ago for the treatment of quantum electrodynamics and gravitation. The basic field variables in this formalism depend not only on a point x but on a path to x. The procedure is to define path-dependent Green's functions in terms of these fields, to define path-independent Green's functions in terms of the path-dependent ones, and finally to develop a perturbation theory for the path-independent Green's functions. In this process the mysterious closed loops of auxiliary scalar particles, introduced by Feynman and de Witt to save unitarity, automatically appear.

Mandelstam's work raises a number of problems of principle which, in my opinion, display some of the most serious gaps in the existing general theory of fields:

1. How is operator-gauge invariance to be formulated in a precise mathematical way?

2. How does one express the fact that some fields are local functions of others?

3. How can the unique role of mass-zero fields be explained in this operator language?

4. What is the substitute for the S-matrix in the presence of infra-red problems?

5. Does the occurrence of such expressions as $\exp\left[i\ e \int^x d\xi^\mu A_\mu(\xi)\right] \psi(x)$ in operator gauge transformations mean that the Green's functions of the theory will in general be non-tempered?

If we are ever to reach the stage in which we can appreciate what Weinberg was saying yesterday about chiral invariance as a dynamical symmetry, some answers will be needed to the above questions.

There was one paper submitted to this Conference which goes in the right direction to increase our comprehension of the above situation. That is the paper of Łopuszański[41]. In it he shows that for a field theory of a single kind of massive particle, there exists no one parameter symmetry group whose generator is the integral of the fourth component of a conserved vector field. Whilst the proof offered is incomplete, the theorem deserves careful study.

Now I come to the work of Efimov[42] (and earlier papers). To explain it, let me recall that in the definition of a field the requirements on the test function space are not unequivocal. The use of test functions in $\mathcal{S}$ (the infinitely differentiable functions of fast decrease) leads to tempered fields. On the other hand, using the spaces of test functions introduced by Jaffe, one can admit worse than polynomial boundedness in momentum space and still formulate a notion of strict localizability. The first important idea in Efimov's work is to introduce a definition of the notion of support of a functional which works when the test functions are entire functions: a linear functional F defined on a set of entire functions has a region G as support if it is continuous in the test functions when the test functions converge uniformly on G. (Since a sequence of entire functions can converge uniformly on G whilst going wild outside G, it is clear that if F has G as support it cannot depend on what is going on outside G.)

Using this notion of support, it is possible to define local commutativity of fields: A and B are relatively local if the functional

$$(\Phi, [A(f), B(g)]\Psi)$$

has support in or on the light cone in the difference variable $x_A - x_B$. Much remains to be done to work out the properties of this interesting definition. It is a candidate for the formalism to deal with Green's functions which increase exponentially or worse in momentum space.

In the work of Efimov, cleverly chosen entire functions are also used in quite a different way. They appear as form factors in a non-local interaction Lagrangian. Efimov has found a set of rules for evaluating the perturbation series in such a theory, and finds it compatible with the usual unitarity relativistic invariance and analyticity requirements. This is a remarkable result, and the problem deserves careful study to be sure that no hidden troubles have been overlooked.

My last remark in this report directs your attention to the work of Oksak and Todorov[43], who have given a free field transforming according to an infinite-dimensional representation of SL(2,C) satisfying the spectral condition and violating the PCT theorem. This simple example makes clear the desirability of further study of the relationships between local observables, local fields, and symmetries.

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DISCUSSION

WEINBERG: I have a comment and a question.

Does the failure of Wanders' sum rule depend on the use only of the ρ, f, and g resonances in the numerical analysis? With only these resonances contributing, all other sum rules also fail.

In the case where a subtraction appears in the Mandelstam representation, can the contraction-mapping theorem be used for all values of the subtraction constants, or only for sufficiently small values?

WIGHTMAN: I will let Wanders answer to the first point, but it is characteristic of the applications of the fixed-point theorem that you form your space of admissible solutions with some restrictions, so it would be unlikely if there were restrictions on the subtraction constant. You are looking for a fixed point in some neighbourhood, and if you move away you will be in another neighbourhood in which you can carry on the discussion. If somebody would give you a suggestion of what the amplitude would look like, you would adapt your norms to that and repeat the proof for it. It is not a local theory but a kind of semi-local theory. Typically you would have neighbourhoods and do different things in different neighbourhoods.

ATKINSON: My point is that the fixed-point theorem works without subtraction constant or if the subtraction constant is sufficiently small. It is a possibility that you can go away to some other region but not using the contraction-mapping but perhaps other methods.

WANDERS: I think Professor Weinberg misunderstood our result. We assume dispersion relations without subtractions for $\pi\pi$ scattering and derive a sum rule from this assumption. This sum rule is violated, and we conclude that subtractions are needed. Of course this is not surprising because everyone guesses that one needs subtractions, because everybody believes that one should have constant cross-sections at infinity.

CHEW: It was suggested by Professor Wightman that the Atkinson result seems to undermine the bootstrap idea. However, it has been recognized for some time by bootstrappers that with ordinary methods of analysis, one needs a postulate usually described as "second-degree analyticity". This assumptions sometimes is expressed in terms of asymptotic behaviour and sometimes in terms of Regge poles. In either case, second-degree analyticity is a condition not required by Atkinson.

One reason that axiomatists have not considered second-degree analyticity is that it still lacks a precise definition. I should like to emphasize the possible relevance here of the type of integral equation described yesterday in Frazer's report--an idea discovered by Amati, Stanghellini and Fubini, and recently generalized by Low and Goldberger. This type of equation gives an unambiguous meaning to Regge asymptotic behaviour, and rests only on the existence of recursive relations between physical region multiparticle unitarity integrals. As emphasized by Low, it is conceivable that this type of equation can be given a rigorous basis. If so, it may then be unnecessary to add a postulate of second-degree analyticity--the latter being a consequence of multi-particle unitarity.

Note that Atkinson included in his considerations only a small part of inelastic unitarity. He ignored the detailed connection between production amplitudes and the two-particle imaginary part that constitutes the basis for Regge behaviour in the above-described equation.

FEINBERG: Does the work of Oksok and Todorov on the possibility of non-TCP invariance for infinite component fields imply that if the infinite component fields interact with finite component fields, then TCP invariance will also not be satisfied for the particles described by the finite component fields?

WIGHTMAN: I do not think that has been worked out, but I suppose it would be true. It might be that you could not make them interact at all. I do not know. Of course, the paper in question is academic in that it treats free fields and does not ask what happens to interactions.

LEE: Referring to the work of Glimm and Jaffe, does "no cut-off" imply relativistic invariance automatically?

WIGHTMAN: It is not yet proved. The construction goes at a fixed time, and therefore in a given Lorentz frame, constructing the temporal development in that frame, and then a vacuum and Green's functions. That part guarantees to you that space-time translations are represented by unitary operators, but does not gurantee that there is a unitary operator for Lorentz transformations. However, it seems very unlikely that Lorentz invariance would fail, because you prove local commutativity and therefore one has the light cone. I am also surprised that it has not been proved yet.

KÄLLEN: Concerning the results of Jaffe et al., what is the difference with the old field-theoretical ideas of the fifties, say, the work of Kristensen? They gave up because of problems of convergence, I believe. What is the new ingredient that makes things work now?

WIGHTMAN: I think that the difference is that now we do not impose anything on the point limit that enforced conditions on the form factors. I think these form factors would violate that condition. This is an impression; I have not looked at the matter carefully.