RIGOROUS PROPERTIES OF AMPLITUDES AT HIGH ENERGIES

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I. INTRODUCTION

The purpose of this summer school is to survey the present theories of strong interactions at high energies. My task in these lectures is limited specifically to a review of the rigorous bounds and other asymptotic properties of scattering amplitudes at high energies. The word 'rigorous' sometimes evokes strong reactions from many down-to-earth theorists and it would perhaps be wise to state in a few words in what context we are using this term. By rigorous I mean generally results that follow without any further assumptions from the Wightman axioms. One recalls that these axioms in a direct and mathematically precise way embody the general physical principles of relativistic invariance, causality, positivity of energy, and certain technical assumptions which are general features of most local field theories.

It is important to keep in mind that these axioms are not God given. In fact some of the general properties or inequalities derived from them might well turn out to disagree with experiment. If that should happen in the future it would consist of a very important contribution. For example, if the forward dispersion relation is violated at high energies, we would have to face at least as fundamental a change as that involved in CP violation. On a much less ambitious level one can say that it is useful to follow this approach since it tells us that certain results are very much model independent.

The logical (but not the historical) structure of a complete chain of argument would be:

<u>Wightman Axioms</u>
↓ Hepp, Haage, Ruelle
<u>LSZ Formalism</u>
↓ Bogoliubov, Lehmann, Hepp, Martin, Bros, Epstein, Glaser ---
<u>Analyticity Properties, Crossing, Unitarity</u>
↓
<u>Bounds, Inequalities, Asymptotic Relations</u>

In these lectures we shall only concentrate on the last arrow in the diagram. We shall only state the known analyticity and crossing results without going into their proofs. From there we proceed to derive bounds, inequalities, and asymptotic relations for scattering amplitudes. In both the work related to this last arrow and the crucial analyticity results needed for this work, much of the credit goes to A. Martin.

II. KINEMATICS AND DEFINITIONS

In the general results derived in these lectures, spin presents us only with a technical problem. So unless we specify otherwise we will always be dealing with two body collisions involving equal mass, zero spin, and neutral particles shown schematically below.

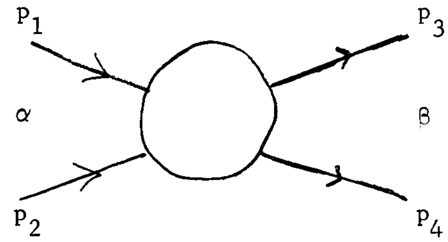


Fig. 1

Here we set $p_i^2 = m^2$ and define the Mandelstam variables in the usual way,

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t &= (p_1 - p_3)^2, \\ u &= (p_1 - p_4)^2. \end{aligned} \tag{2.1}$$

The center of mass momentum k and the scattering angle θ are given by

$$\begin{aligned} s &= 4m^2 + 4k^2, \\ t &= -2k^2(1-\cos\theta), \\ u &= -2k^2(1+\cos\theta) \end{aligned} \tag{2.2}$$

The S-matrix element for the transition shown in Fig. 1 is defined as

$$S_{\beta\alpha} = \langle p_3 p_4 \text{ out } | p_1 p_2 \text{ in} \rangle = \langle p_3 p_4 \text{ in } | S | p_1 p_2 \text{ in} \rangle \tag{2.3}$$

Its relation to the T matrix is

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) T_{\beta\alpha}. \tag{2.4}$$

We now define our elastic scattering amplitude as

$$F(s,t) = \langle p_3 p_4 \text{ out } | T | p_2 p_1 \text{ in} \rangle \ . \tag{2.5}$$

With this normalization the relation between F and the elastic differential cross section is

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s-4m^2)} |F|^2, \tag{2.6}$$

and the total elastic cross section is

$$\sigma_{el} = \frac{1}{16\pi} \int_{4m^2-s}^{0} dt \, \frac{|F|^2}{sk^2} . \tag{2.7}$$

It is sometimes convenient to work with an amplitude normalized in a slightly different way and which we denote by f(s,t). This amplitude is chosen so that

$$\frac{d\sigma}{d\Omega} = |f|^2, \tag{2.8}$$

and clearly we have

$$f(s,t) = F(s,t)/8\pi(s)^{\frac{1}{2}}. \tag{2.9}$$

The optical theorem for our amplitudes has the form

$$\begin{aligned} \mathrm{Im}F(s,0) &= 2k(s)^{\frac{1}{2}} \sigma_{tot.}, \\ \mathrm{Im}f(s,0) &= \frac{k}{4\pi} \sigma_{tot}. \end{aligned} \tag{2.10}$$

Finally, we write down the partial wave expansion for f(s,t):

$$\begin{aligned} f(s,t) &= \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \, f_\ell(s) P_\ell(\cos\theta) \\ &= F(s,t)/8\pi(s)^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

The convergence of this expansion even for nonphysical values of θ is a consequence of the analyticity of the amplitude in an ellipse in the $\cos\theta$-plane.

The partial wave amplitudes $f_\ell(s)$ in the elastic region have as a consequence of unitarity the simple form

$$f_\ell(s) = \frac{e^{2i\delta_\ell}-1}{2i} = e^{i\delta_\ell} \sin\delta_\ell, \tag{2.12}$$

with real δ_ℓ's. In the inelastic region δ_ℓ becomes complex but unitarity still gives us the restriction that

$$|S_\ell^{el.}| = |\, 1+2if_\ell(s)| \le 1 . \tag{2.13}$$

From this it is trivial to show that

$$0 \le |f_\ell(s)|^2 \le \mathrm{Im}f_\ell(s) \le 1. \tag{2.14}$$

Thus, both in the elastic and inelastic regions, $\mathrm{Im}f_\ell$ is <u>positive</u> and bounded above by unity. This simple consequence of unitarity will turn out to be the most crucial and important input for most of the bounds that we shall derive.

III. RIGOROUS ANALYTICITY PROPERTIES

We briefly review the main analyticity properties of $F(s,t)$ that follow rigorously from the Wightman axioms. No effort is made here to state the best possible results but only to give what is needed and relevant for the derivation of bounds.

A. Lehmann Ellipse[1]

We will write $F(s,t)$ and $F(s,\cos\theta)$ interchangeably and set $z=\cos\theta$. For fixed physical s, $F(s,z)$ is analytic in z in an ellipse with foci at $z=\pm 1$ and a semi-major axis $z_0(s)$ given by

$$z_0(s) = \left[1 + \frac{(m^2 - M^2)^2}{k^2 s}\right]^{\frac{1}{2}}. \qquad (3.1)$$

Here M is the smallest mass for which $\langle 0|j(x)|M\rangle \neq 0$ and $j(x)$ is the source current for our colliding particles, $(\Box + m^2)\phi(x) = j(x)$. Thus $M = 2m$ for scalar bosons and $M = 3m$ for pions. For $\mathrm{Im}F(s,z)$ one has analyticity in a larger ellipse. However, the crucial property of both ellipses is the rate at which they shrink to the line $-1 \leq z \leq +1$ as $s \to \infty$. In both cases we have, for large s, $[z_0(s) - 1] \sim s^{-2}$. In the t-plane, $t = 2k^2(z-1)$, the foci of the ellipse are at $t=0$ and $t=-4k^2$ and the distance between the focus at $t=0$ and the turning point shrinks like s^{-1} for large s, (see Fig. 2 below).

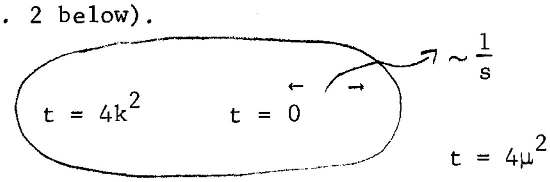


Fig. 2

On the other hand, perturbation theory suggests that the nearest a singularity could be in the t-plane is at $t = 4\mu^2$ for any value of s. The existence of a finite region of analyticity near the origin in the t-plane with minimum size independent of s was established by Martin[2] in 1965.

B. Martin Analyticity[2]

The input for Martin's work consisted of i) dispersion relations at fixed t, ii) positivity and iii) the analyticity properties obtained by Bros, Epstein, and Glaser[3].

Martin proved that for fixed s near a physical point, $F(s,t)$ is analytic in t for $|t| < R$, where R is independent of s. For pion-pion scattering $R=4m^2$.

This result also holds for $\mathrm{Im}F(s,t)$. We also know that $\mathrm{Im}F(s,t)$ is analytic in the Lehmann ellipse with foci at $t=-4k^2$ and $t=0$ and expandable in Legendre polynomials of $\cos\theta$. The Legendre expansion of ImF has positive coefficients as a consequence of (2.14). Therefore this expansion must have a singularity at the extreme right of the largest ellipse of convergence. But ImF(s,t) has no singularities for $0 < t < 4\mu^2$ and consequently ImF(s,t) must be analytic in an ellipse $E_1(s)$

$$E_1(s): \quad \text{foci} \quad t=0 \;, \quad t = 4m^2 - s \qquad \text{extremities} \quad t=4m^2 \;, \quad t = -s. \tag{3.2}$$

This region of analyticity can be further enlarged as is shown in Ref. 2., but the results stated so far are sufficient for our purposes which are limited here to the derivation of upper and lower bounds.

C. Dispersion Relations and Polynomial Boundedness[4]

For fixed t, $-t_1 \le t \le 0$, F(s,t) satisfies a dispersion relation in s with a fixed number of subtractions.

The crucial fact is that for $-t_1 \le t \le 0$, and all $|s| > 1$ we have a bound

$$|F(s,t)| \le c|s|^N. \tag{3.3}$$

Martin[2] showed that this was true for all t such that $|t| \le R$ and that $N \le 2$. We shall derive these results again below. Polynomial boundedness is to some extent technically built into the Wightman axioms and related to the fact that the Wightman functions are taken to be tempered distributions. However, both Jaffe's axioms and theories based on local observables which do not require temperedness lead to on-shell scattering amplitudes which are polynomially bounded.[5]

IV. FROISSART AND GREENBERG-LOW BOUNDS

A. Intuitive Argument

Suppose we consider scattering by a potential $V(r) = g(E)\, e^{-\kappa r}/r$. A particle with impact parameter 'a' has an interaction of order $g\, e^{-\kappa a}$. Thus the interaction is negligible when $|g\, e^{-\kappa a}| \ll 1$. An estimate of the cross-section is given by the value of a for which $g\, e^{-\kappa a} \simeq 1$, or

$$a \sim \frac{1}{\kappa} \log|g|, \tag{4.1}$$

and

$$\sigma_t \sim \frac{1}{\kappa^2} \log^2|g|. \tag{4.2}$$

Now if g(E) is polynomially bounded in E for large E, $|g(E)| \le E^N$, then

$$\sigma_t \le \frac{c}{\kappa^2} \log^2 E. \tag{4.3}$$

If κ_{min} is independent of energy this is just the Froissart bound. The existence of a finite analyticity region in t near t=0 independent of energy is of course closely related in the language of potential scattering to the fact that the range, κ_{min}, is independent of E.

B. Derivation of the Froissart and Greenberg-Low Bounds[6,7]

From (2.11) we have

$$\mathrm{Im} f(s,z) \equiv A(s,z) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell(s) P_\ell(z) \ , \tag{4.4}$$

where we have set $\mathrm{Im} f_\ell = a_\ell$. From the results summarized in Chapters II and III we know that

i) $0 \leq |f_\ell|^2 \leq a_\ell \leq 1$.

ii) $|A(s,z_1)| < s^N$ for large s and z_1 inside the Martin or Lehmann ellipse.

iii) A(s,z) is analytic in z in the Lehmann ellipse or the Martin ellipse. Both ellipses have foci at $z = \pm 1$. The semi-major axis is given by

$$z_0(s) = 1 + \frac{t_0(s)}{2k^2} \quad , \tag{4.5}$$

with the following property for large s

$$t_0(s) \sim \frac{c}{s} \quad \text{(Lehmann)},$$

$$t_0(s) \sim c \quad \text{(Martin)} . \tag{4.6}$$

We now pick a $z_1(s)$ of the form $z_1 = 1+(t_1(s)/2k^2)$, such that $0 < t_1(s) < t_0(s)$. Using the fact that $P_\ell(x) \geq 0$ for $x > 1$ and the positivity property i) we easily get a bound on $a_\ell(s)$,

$$s^N \geq \sum_{\ell=0}^{\infty} (2\ell+1) \frac{a_\ell}{k} P_\ell\left(1+\frac{t(s)}{2k^2}\right) \geq (2\ell+1) \frac{a_\ell(s)}{k} P_\ell\left(1+\frac{t(s)}{2k^2}\right) \quad , \tag{4.7}$$

and hence for all ℓ

$$0 \leq a_\ell(s) \leq \frac{s^N \cdot k}{(2\ell+1) P_\ell\left(1+\frac{t(s)}{2k^2}\right)} \quad . \tag{4.8}$$

At the end of this chapter we shall prove the following inequality for $x > 1$ (see also reference 7):

$$P_\ell(x) \geq \frac{c}{(2\ell+1)^{\frac{1}{2}}} \left(x+(x^2-1)^{\frac{1}{2}}\right)^\ell > \frac{c}{(2\ell+1)^{\frac{1}{2}}} \left(1+(2(x-1))^{\frac{1}{2}}\right)^\ell . \tag{4.9}$$

Thus the bound (4.8) with (4.9) gives us

$$a_\ell(s) \leq \frac{C\ s^N k}{(2\ell+1)^{\frac{1}{2}} \left(1+\frac{(t(s))^{\frac{1}{2}}}{k}\right)^\ell} \quad . \tag{4.10}$$

As $s \to \infty$, $(t(s)/k) \to 0$, and we get

$$a_\ell(s) \lesssim C\, s^N \cdot k\, e^{-\ell \frac{(t(s))^{\frac{1}{2}}}{k}} , \qquad s > s_0 \quad . \qquad (4.11)$$

We can now consider two distinct cases depending on the choice of $t(s)$.

Case (i): (Greenberg and Low)

Here one uses the analyticity in the Lehmann ellipse only and $t(s)$ must decrease like s^{-1} as $s \to \infty$. We choose $t(s)$ such that

$$t(s) = \frac{c}{s} . \qquad (4.12)$$

Hence for $s > s_0$,

$$a_\ell(s) \lesssim C\, s^{N+\frac{1}{2}} e^{-\frac{c'\ell}{s}} . \qquad (4.13)$$

We define $L(s)$ to be the value of ℓ for which the right hand side of (4.13) is of order unity. Then we have

$$L(s) \cong C\, s \ln s \; ; \qquad (C = C(N)) \quad . \qquad (4.14)$$

For fixed large s and $\ell < L(s)$ the unitarity bound, $a_\ell(s) \lesssim 1$, is a better bound than (4.13). However, for $\ell > L(s)$, (4.13) gives a smaller upper bound and in fact $a_\ell(s)$ is negligible for $\ell \gtrsim L(s)+1$. Thus $L(s)$ gives us the number of partial waves that effectively contribute to the amplitude at large s.

Let $\hat{L}$ be the smallest integer greater than $L(s)$, then we can write

$$A(s,z) \lesssim \frac{1}{k} \sum_{\ell=0}^{\hat{L}} (2\ell+1) |P_\ell(\cos\theta)| + \frac{1}{k} \sum_{\ell=\hat{L}+1}^{\infty} (2\ell+1) C\, s^{N+\frac{1}{2}} e^{-\frac{c'\ell}{s}} |P_\ell(\cos\theta)| \quad . \qquad (4.15)$$

In the first sum above we have used the unitarity bound $a_\ell(s) \lesssim 1$. In the second sum we have used (4.13).

For the forward case, $\theta = 0$, $|P_\ell(\cos\theta)| = 1$, and one obtains

$$A(s,1) \lesssim \frac{1}{k} \sum_{\ell=0}^{\hat{L}} (2\ell+1) + \text{Remainder}. \qquad (4.16)$$

The remainder term can be easily shown to be negligible compared to the first term, and we get

$$A(s,1) \lesssim \frac{(\hat{L}+1)^2}{k} (1+\epsilon) \quad . \qquad (4.17)$$

Substituting our estimate for $\hat{L}(s)$ we finally have

$$A(s,1) \le \frac{C\, s^2 \ln^2 s}{k} \quad . \tag{4.18}$$

Since unitarity gives us also the bound $|f_\ell| \le (a_\ell)^{\frac{1}{2}}$ we can easily follow the same method as above to get a bound similar to (4.18) for the full amplitude

$$|F(s,1)| \le \frac{C\, s^2 \ln^2 s}{k} \; . \tag{4.19}$$

The Greenberg-Low bound for σ_{tot} follows from (4.18),

$$\sigma_{tot} = \frac{4\pi\, A(s,1)}{k} \le \text{Const. } s \ln^2 s \quad . \tag{4.20}$$

For the non-forward case, $\theta \ne 0$ and $\theta \ne \pi$, one has

$$|P_\ell(\cos\theta)| \le \left(\frac{2}{\pi \ell \sin\theta}\right)^{\frac{1}{2}} \; . \tag{4.21}$$

Using this bound, and the inequality $|f_\ell| < (a_\ell)^{\frac{1}{2}}$ and proceeding as before, we get

$$\begin{aligned} |A(s,\cos\theta)| &\le |f(s,\cos\theta| \\ &\le \frac{c}{k(\sin\theta)^{\frac{1}{2}}} \sum_{\ell=0}^{\hat{L}} \frac{(2\ell+1)}{(\ell)^{\frac{1}{2}}} + \text{negligible terms} \; . \end{aligned} \tag{4.22}$$

For large s this gives the bound

$$|A(s,\cos\theta)| \le |f(s,\cos\theta)| \le \text{Const.} \frac{s\, \ell n^{3/2} s}{(\sin\theta)^{\frac{1}{2}}} \quad . \tag{4.23}$$

Case (ii): (Froissart bound)

For this case we have analyticity in the Martin ellipse defined in (3.2) and one can choose t(s) to be a constant independent of s,

$$t(s) = t_0 \quad . \tag{4.24}$$

Proceeding as in case (i) we get

$$|f_\ell|^2 \le a_\ell(s) \le C\, s^N \cdot k\, e^{-c\ell/(s)^{\frac{1}{2}}} \quad . \tag{4.25}$$

Hence instead of (4.14) we now have

$$L(s) \cong C\, (s)^{\frac{1}{2}} \ell n s \; . \tag{4.26}$$

With this improved estimate of the number of significant partial waves we get following the same steps as in case (i) for $s > s_0$

$$A(s,1) \leq |f(s,1)| \leq \frac{\text{Const. } s\ln^2 s}{k} \quad , \qquad (4.27)$$

$$\sigma_{tot} \leq \text{Const. } \ln^2 s \quad , \qquad (4.28)$$

and

$$|f(s,\cos\theta)| \leq \text{Const.} \frac{s^{3/4}(\ln s)^{3/2}}{k(\sin\theta)^{\frac{1}{2}}} \quad , \ \theta \neq 0,\pi \ . \qquad (4.29)$$

These bounds can be easily translated to bounds for $|F(s,t)|$ for fixed physical negative t. We use $\cos\theta = 1 + t/2k^2$ and $F = 8\pi(s)^{\frac{1}{2}}f$ and obtain from (4.29)

$$|F(s,t)|_{t \text{ fixed, physical}} \leq \text{Const. } s\ln^{3/2}s, \quad t \neq 0. \quad (4.30)$$

By a Phragmen-Lindelof type of argument one can show that the bound (4.20) holds in all directions in the s-plane. Thus the maximum number of subtractions needed for fixed t dispersion relations is 2 as long as t is negative.

We close our discussion of the Froissart bounds by making two related remarks.

1) The non-forward bound (4.29) cannot be reached for a finite interval $\theta_1 < \theta < \theta_2$. For recalling that $d\sigma/d\Omega = |f|^2$, we would then have

$$\frac{d\sigma}{d\Omega} \cong \frac{s^{\frac{1}{2}}\ln^3 s}{\sin\theta} \ , \qquad \theta_1 < \theta < \theta_2 \qquad .$$

This would give a $\sigma_{el} \gg \sigma_{tot}$ for large s and an obvious contradiction. So the bound can only be reached at isolated peaks in θ. At this point one immediately is led to the suspicion that (4.29) could and should be improved.

2) Kinoshita, Loeffel and Martin[8] considered the problem of improving the Froissart bounds by assuming analyticity in a domain larger than the Martin ellipse. They assumed:

(i) Unitarity, $0 \leq a_\ell \leq 1$

(ii) Analyticity in the cut z-plane

(iii) Polynomial boundedness for z inside analyticity domain.

Their main results were:

(a) In the forward case <u>no</u> improvement is possible. An explicit counter-example satisfying (i), (ii), and (iii) was constructed.

(b) For $\theta \neq 0$ or π an improvement of (4.29) is possible and they obtained

$$|F(s,\cos\theta)| \leq \frac{\text{Const.}(\ln s)^{3/2}}{\sin^2\theta} \quad . \qquad (4.31)$$

This bound gives us a more acceptable bound on the differential cross-section, which vanishes as $s \to \infty$,

$$\frac{d\sigma}{d\Omega} \leq \frac{C(\ell n s)^3}{s\ \sin^4\theta} \tag{4.32}$$

However, translating (4.31) back to a fixed t bound does not lead to a change in (4.30).

Actually, as is clear from ref. 8, it is not necessary to assume full cut plane analyticity in the z-plane to obtain (4.31). For more detail on this point the reader should read ref. 8.

Before proceeding to the next chapter we still have the task of proving the inequality (4.9) for $P_\ell(x)$, with $x > 1$. The second inequality in (4.9) follows trivially from the first. To prove the first we introduce the variable $y \equiv x+(x^2-1)^{\frac{1}{2}}$ for which $y+y^{-1} = 2x$, or $x = \cos\theta$, $y = e^{\pm i\theta}$. Then we can write

$$P_\ell(x) = \sum_{-\ell}^{+\ell} C_n^{(\ell)} y^n \ . \tag{4.33}$$

The coefficients $C_n^{(\ell)}$ have two properties: (a) $C_n^{(\ell)} \geq 0$; (b) $C_\ell^{(\ell)} = \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}$. The property (b) follows immediately from the fact that $C_\ell^{(\ell)}$ is $2^{-\ell}$ times the coefficient of x^ℓ in the expansion of $P_\ell(x)$. From (a) and (b) we have

$$P_\ell(x) \geq C_\ell^{(\ell)} y^\ell > \frac{c}{(2\ell+1)^{\frac{1}{2}}}\,[x+(x^2-1)^{\frac{1}{2}}]^\ell \ . \tag{4.34}$$

A simple proof of property (a) due to Martin follows from some elementary properties of rotation group. One can write

$$\begin{aligned} P_J(\cos\theta) &= C < J,J_z=0|e^{iJ_y\theta}|J,J_z=0 > \\ &= C < J,J_z=0|\cos J_y\theta|J,J_z=0 > \quad . \end{aligned} \tag{4.35}$$

Introducing a complete set of states in (4.35), we get

$$\begin{aligned} P_J(\cos\theta) &= C \sum_{M=-J}^{J} < J,J_z=0|J,J_y=M > < J,J_y=M|\cos(J_y\theta)|J,J_z=0 > , \\ &= C \sum_{M=-J}^{J} \cos M\theta |< J,J_z=0|J,J_y=M >|^2 \quad . \end{aligned} \tag{4.36}$$

But $\cos M\theta = \frac{1}{2}(e^{iM\theta}+e^{-iM\theta}) = \frac{1}{2}[y^M+y^{-M}]$ and that completes the proof of (b).

V. BOUNDS FOR UNPHYSICAL VALUES OF t AND MAXIMUM NUMBER OF SUBTRACTIONS IN DISPERSION RELATIONS

For negative (physical) values of t we have the bound (4.30),

$$|F(s,t)| \leq C\ s\ell n^{3/2} s \quad . \tag{5.1}$$

In this chapter we shall derive a fixed t bound for unphysical values of t. More specifically we shall be interested in the region $0 < t < 4\mu^2$ and the region $|t| < 4\mu^2$. The dispersion relations are valid for any t in the Martin ellipse and this unphysical bound will give us the maximum number of subtractions needed in the fixed t dispersion relations. It will also give us a determination of the constant that appears on the right in the Froissart bound (4.20). We follow the method of Jin and Martin,[9] although many of the results of this paper can be also obtained as a byproduct of the proof in reference 2.

Let us, to be slightly more general consider an unequal mass collision $a+b \to a+b$, and allow the following kinematics: (i) $s+t+u = 2m_a^2+2m_b^2$; (ii) 2μ is the lowest mass in the crossed t channel; (iii) no states in the s or u channels with mass less than (m_a+m_b); (iv) $\mu \leq m_a$; $\mu \leq m_b$.

The amplitude, $F(s,t)$, for fixed t, $|t| < 4\mu^2$, is analytic in the cut s-plane. The right hand cut starts at $s = (m_a+m_b)^2$ and the left hand cut at $s = (m_a-m_b)^2-t$. We further assume that as $|s| \to \infty$, $|t| < 4\mu^2$, F is polynomially bounded,

$$|F(s,t)| \leq |s|^N \quad . \tag{5.2}$$

We shall prove by making use of the Froissart bounds that, in fact

$$\lim_{|s| \to \infty} \left|\frac{F(s,t)}{s^2}\right| = 0 \ , \quad |t| < 4\mu^2 \quad . \tag{5.3}$$

From the analyticity of F and the bound (5.2) we can write an N+1 subtracted dispersion relation

$$F(s,t) = \sum_{n=0}^{N} C_n(t)s^n + \frac{s^{N+1}}{\pi}\int_{(m_a+m_b)^2}^{\infty} ds' \frac{A_s(s',t)}{s'^{N+1}(s'-s)}$$

$$+ \frac{u^{N+1}}{\pi}\int_{(m_a+m_b)^2}^{\infty} du' \frac{A_u(u',t)}{u'^{N+1}(u'-u)} \quad . \tag{5.4}$$

It is easy to see from the partial wave expansion that

$$A_s(s',t) \geq 0, \quad 0 \leq t \leq 4\mu^2 \quad . \tag{5.5}$$

In the s-channel, $A_s \equiv \mathrm{Im}F$. Similarly, A_u is the absorptive part for the u-channel reaction $a+\bar{b} \to a+\bar{b}$ and by unitarity

$$A_u(u',t) \geq 0; \quad 0 \leq t \leq 4\mu^2 \quad . \tag{5.6}$$

For complex s and Res large enough the integrals in (5.4) are uniformly convergent for any t such that $|t| < 4\mu^2$. Therefore both integrals define functions analytic in $|t| < 4\mu^2$. The function F(s,t) is also analytic in the disc, $|t| < 4\mu^2$. Hence $C_n(t)$ are analytic in t in the same disc, since $\sum_{n=0}^{N} C_n(t)s^n$ is analytic for at least (N+1) distinct values of s.

From the Froissart bound we have,

$$|F(s,0)| \leq C\, s\ell n^2 s \quad .$$

At the end of this chapter we shall show that using the techniques of Chapter IV, we can always choose an $\epsilon > 0$ and find a t_0, $0 < t_0 < 4\mu^2$, such that

$$|F(s,t)| \leq cs^{1+\epsilon}, \qquad 0 \leq t \leq t_0 \quad . \tag{5.7}$$

We choose $\epsilon < 1$, and thus for $0 \leq t \leq t_0$ we need only two subtractions in our dispersion relations. We obtain

$$F(s,t) = a(t)+b(t)s+\frac{s^2}{\pi}\int\frac{A_s(s',t)}{s'^2(s'-s)}\,ds'$$

$$+\frac{u^2}{\pi}\int\frac{A_u(u',t)}{u'^2(u'-u)}\,du' \;; \qquad 0 \leq t \leq t_0 \;. \tag{5.8}$$

Then using the identity,

$$\frac{s^{N+1}}{s'^{N+1}(s'-s)} = \frac{s^2}{s'^2(s'-s)} - \frac{s^2}{s'^3} - \ldots - \frac{s^{N-1}}{s'^N} - \frac{s^N}{s'^{N+1}} \quad , \tag{5.9}$$

and comparing (5.8) with (5.4) we get

$$\sum_{n=0}^{N} C_n(t)s^n = a(t)+b(t)s+\sum_{n=2}^{N}\left[\frac{s^n}{\pi}\int ds'\,\frac{A_s(s',t)}{s'^{n+1}} + \frac{u^n}{\pi}\int du'\,\frac{A_u(u',t)}{u'^{n+1}}\right] \;. \tag{5.10}$$

This last equation is at this stage only valid for $0 \leq t \leq t_0$. Two cases have to be treated separately depending on whether N is even or odd.

(i) N even > 2.

Comparing the coefficients of s^N for large s, (5.10) gives us

$$C_N(t) = \frac{1}{\pi}\int ds'\,\frac{A_s(s',t)}{s'^{N+1}} + \frac{1}{\pi}\int du'\,\frac{A_u(u',t)}{u'^{N+1}} \quad . \tag{5.11}$$

This equation is again only true at this stage for $0 \leq t \leq t_0$. From unitarity we know that

$$\left.\frac{d^n}{dt^n}A_s(s',t)\right|_{t=0} \geq 0 \quad ,$$

$$\left.\frac{d^n}{dt^n} A_u(u',t)\right|_{t=0} \geq 0 \quad . \tag{5.12}$$

These positivity requirements follow from the partial wave expansion and the fact that

$$\left.\frac{d^n P_\ell(\cos\theta)}{d(\cos\theta)^n}\right|_{\cos\theta=1} \geq 0 \quad . \tag{5.13}$$

To prove this last inequality easily one can use the expansion (4.36) and the positivity of the coefficients in that expansion.

Returning to (5.11) we can expand both A_s and A_u in power series

$$A_s(s',t) = \sum_{n=0}^{\infty} A_s^{(n)}(s')t^n \quad ,$$

$$A_u(u',t) = \sum_{n=0}^{\infty} A_u^{(n)}(u')t^n \quad . \tag{5.14}$$

The coefficients in both series are positive. Now, for $0 \leq t \leq t_0$, we can exchange the summation and integration in (5.11) and get

$$C_N(t) = \sum_{n=0}^{\infty} t^n \left[\int \frac{A_s^{(n)}(s')}{s'^{N+1}} ds' + \int \frac{A_{u'}^{(n)}(u')}{u'^{N+1}} du' \right] \quad . \tag{5.15}$$

The function $C_N(t)$ is analytic for $|t| < 4\mu^2$. The right hand side of (5.15) is a power series expansion and therefore it must also converge for all $|t| \leq 4\mu^2$. Then reversing the order again in (5.15) we conclude that

$$\int \frac{A_s(s',t)}{s'^{N+1}} ds' \quad \text{and} \quad \int \frac{A_u(u',t)}{u'^{N+1}} du'$$

are finite for all t such that $|t| \leq 4\mu^2$. It is easy to show using (5.4) that if these integrals are finite, then

$$\lim_{|s| \to \infty} \left| \frac{F(s,t)}{s^N} \right| = 0 \quad , \tag{5.16}$$

and we can undo one subtraction in (5.4).

(ii) N odd ≥ 3

In this case an analogous argument shows that we can reduce the number of subtractions by two.

Repeating the argument as many times as necessary we prove that

$$\int \frac{A_s(s',t)}{s'^3}\, ds' \quad \text{and} \quad \int \frac{A_u(u',t)}{u'^3} du'$$

converge absolutely for all $|t| < 4\mu^2$. Hence it is easy to show that (5.3) follows.

In closing we still have to prove the inequality (5.7). Namely, we have to show from the estimates of the previous chapter for $|f_\ell(s)|$, that for any ϵ there exists a t_0 such that for $0 \leq t \leq t_0$; $|F(s,t)| \leq C|s|^{1+\epsilon}$.

From the partial wave expansion we have

$$|F(s,t)| \leq C \sum_{0}^{\hat{L}} (2\ell+1)\left|P_\ell\left(1+\frac{t}{2k^2}\right)\right| + C \sum_{\hat{L}+1}^{\infty} (2\ell+1)s^N e^{-\frac{2\mu\ell}{(s)^{\frac{1}{2}}}} \left|P_\ell\left(1+\frac{t}{2k^2}\right)\right| \quad . \qquad (5.17)$$

Here in the first sum we used the unitarity bound for $|f_\ell|$ and in the second sum we used (4.11) with $t(s) \cong 4\mu^2-\epsilon$. In (5.17) we have convergence for $0 \leq t < 4\mu^2$. We choose t_0 such that $(t_0)^{\frac{1}{2}} \ll \mu$, then in the second sum in (5.17) the decreasing exponential will damp the increasing exponential behavior of $P_\ell(1+ t/2k^2)$, for $0 \leq t \leq t_0$. In fact, one has the bound

$$|P_\ell(\cosh\xi)| \leq C \frac{e^{(\ell+\frac{1}{2})\xi}}{(2\ell+1)^{\frac{1}{2}}(\sinh\xi)^{\frac{1}{2}}} \; ; \quad \xi \neq 0 \quad , \qquad (5.18)$$

where

$$\cosh\xi = 1 + t/2k^2 \quad ,$$

or

$$\xi = \ell n\left[1 + \frac{t}{2k^2} + \left(\left(1+\frac{t}{2k^2}\right)^2 - 1\right)^{\frac{1}{2}}\right] \quad . \qquad (5.19)$$

For large s

$$\xi \cong \frac{2(t)^{\frac{1}{2}}}{(s)^{\frac{1}{2}}} \quad .$$

We then get for large s

$$\left|P_\ell\left(1+\frac{t}{2k^2}\right)\right| \leq \frac{C}{(2\ell+1)^{\frac{1}{2}}} \frac{s^{\frac{1}{4}}}{t^{\frac{1}{4}}} e^{\ell \frac{2(t)^{\frac{1}{2}}}{(s)^{\frac{1}{2}}}} \quad . \qquad (5.20)$$

Now as long as $0 < t \leq t_0 \ll \mu$, the second sum in (5.17) is negligible compared to the first. Hence, we get

$$|F(s,t)| \leq C \sum_{\ell=0}^{\hat{L}} (2\ell+1) \frac{s^{\frac{1}{4}}}{t^{\frac{1}{4}}} \frac{e^{2\ell(t)^{\frac{1}{2}}/(s)^{\frac{1}{2}}}}{(\ell)^{\frac{1}{2}}}$$

$$\leq C \, e^{\bar{c} \frac{(s)^{\frac{1}{2}} \ell ns (t)^{\frac{1}{2}}}{(s)^{\frac{1}{2}}}} \frac{s^{\frac{1}{4}}}{t^{\frac{1}{4}}} \sum_{\ell=0}^{\hat{L}} \frac{(2\ell+1)}{(\ell)^{\frac{1}{2}}}$$

$$\leq \frac{C'}{t^{\frac{1}{4}}} s^{1+\bar{c}(t)^{\frac{1}{2}}} (\ell ns)^{3/2}; \qquad t \neq 0 \quad . \tag{5.21}$$

Here we have used the fact that $\hat{L}(s) \cong (s)^{\frac{1}{2}} \ell ns$. Thus by choosing t_0 small enough (5.7) will follow from (5.21). It is interesting to note the vague similarity between (5.21) and Regge like terms.

The knowledge that in the region $0 \leq t \leq 4\mu^2$ we have no more than two subtractions in the dispersion relation enables us to determine the constant in the Froissart bound. We have derived the bound $\sigma_t \leq \text{Const.}\, \ell n^2(s/s_0)$; we shall calculate an upper bound for the constant. We have

$$\sigma_t = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell(s) \quad . \tag{5.22}$$

From (4.10) we get

$$a_\ell(s) \leq \frac{C\, s^{N'} \cdot k}{\left(1+\frac{(t_0)^{\frac{1}{2}}}{k}\right)^{\ell}} \quad . \tag{5.23}$$

From the results of this chapter we know that $|F| \leq C(s/s_0)^2$ and hence $|f| \leq C(s/s_0)^{2-\frac{1}{2}}$, and $N' = 3/2$. This gives us

$$a_\ell(s) \leq C\left(\frac{s}{s_0}\right)^2 e^{-\frac{2\mu\ell}{k}} \quad , \tag{5.24}$$

and

$$L(s) \cong \frac{(s)^{\frac{1}{2}}}{2\mu} \ell n(s/s_0) \quad . \tag{5.25}$$

From (5.22) we get the bound

$$\sigma_t \leq (1+\epsilon) \frac{4\pi}{k^2} L^2 \quad ,$$

$$\leq (1+\epsilon) \left(\frac{4\pi}{\mu^2}\right) \ell n^2 (s/s_0) \quad , \tag{5.26}$$

where ϵ can be as small as we like.

VI. UNITARITY BOUNDS

In this chapter we shall derive some very simple bounds which follow mainly from unitarity. The additional input we use is that only L(s) partial waves are significant.

A. Lower Bounds on $\sigma_{elastic}$[10]:

By definition we have

$$\sigma_{el} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) |f_\ell|^2 \quad . \tag{6.1}$$

Hence we write

$$\sigma_{el} \geq \frac{4\pi}{k^2} \sum_{\ell=0}^{L} (2\ell+1) a_\ell^2 \quad . \tag{6.2}$$

Using the Cauchy-Schwartz inequality we obtain:

$$\left[\sum_{\ell=0}^{L} ((2\ell+1)^{\frac{1}{2}} a_\ell)^2\right] \left[\sum_{\ell=0}^{L} (2\ell+1)\right] \geq \left[\sum_{\ell=0}^{L} (2\ell+1) a_\ell\right]^2 . \tag{6.3}$$

Hence

$$\sigma_{el} \geq \frac{\frac{4\pi}{k^2}\left[\sum_{\ell=0}^{L} (2\ell+1) a_\ell\right]^2}{\sum_{0}^{L} (2\ell+1)} \quad . \tag{6.4}$$

If $\sigma_{tot} \geq C\, s^{-N}$, then (6.4) gives us

$$\sigma_{el} \geq \frac{k^2}{4\pi} \frac{\sigma_t^2}{(L+1)^2} \geq \text{Const.} \frac{\sigma_t^2}{\ell n^2 s} \quad . \tag{6.5}$$

We recall that for $L \cong C(s)^{\frac{1}{2}}\ell ns$,

$$\sum_{\ell=0}^{L} (2\ell+1)a_\ell = \sum_{\ell=0}^{\infty} (2\ell+1)a_\ell + 0(s^{-M})$$

$$= \frac{k^2}{4\pi}\sigma_t + 0(s^{-M}) \quad . \tag{6.6}$$

B. Lower Bound on Width of Diffraction Peak

We define the "width" of the diffraction peak, $\triangle$, as

$$\triangle \equiv \frac{F(s,0)}{\left|\left.\frac{dF(s,t)}{dt}\right|_{t=0}\right|} \quad . \tag{6.7}$$

In Regge theory for large s

$$\triangle \cong \frac{1}{c\ell ns} \; . \tag{6.8}$$

From the partial wave expansion we have

$$\left.\frac{dF(s,t)}{dt}\right|_{t=0} = \frac{8\pi(s)^{\frac{1}{2}}}{k} \sum_{\ell=0}^{\infty} (2\ell+1) \frac{f_\ell(s)}{2k^2} \frac{\ell(\ell+1)}{2} \quad . \tag{6.9}$$

Therefore,

$$\left|\frac{dF(s,t)}{dt}\right|_{t=0} \leq \frac{C(s)^{\frac{1}{2}}}{k^3} \sum_{\ell=0}^{L} (2\ell+1) \frac{\ell(\ell+1)}{2}$$

$$\leq C\, s\ell n^4 s \quad . \tag{6.10}$$

Since $|F(s,0)| \geq |\mathrm{Im}F(s,0)| \cong cs\sigma_t$, one concludes that

$$\triangle \geq \frac{C\sigma_t}{(\ell ns)^4} \; . \tag{6.11}$$

Thus if $\sigma_t \rightarrow$ const. the diffraction peak width cannot be rapidly decreasing.

C. Bound on Regge Trajectories

The Froissart bound gives us immediately a bound on Regge trajectories at $t=0$, $\alpha(0) \leq 1$. However, if one assumes that the asymptotic behavior of $F(s,t)$ is dominated by a Regge pole it is possible to prove directly from unitarity, and without using the analyticity input necessary for the Froissart bound, that $\alpha(0) \leq 1$. We go through this proof which is due to Leader [11] just to demonstrate how powerful a restriction unitarity gives us.

We start by writing down the expression for σ_{el},

$$\sigma_{el} = \frac{1}{32\pi} \int_{-4k^2}^{0} dt \, \frac{|F(s,t)|^2}{2k^2 s} \ . \tag{6.12}$$

From this we have the obvious inequality

$$\sigma_t \geq \sigma_e \geq \frac{1}{32\pi s} \int_{-T}^{0} dt \, \frac{|F|^2}{2k^2} \ , \tag{6.13}$$

where T is a constant such that $T \ll 4k^2$. Let F have Regge asymptotic behavior

$$F(s,t) \sim \beta(t) s^{\alpha(t)} \ ; \tag{6.14}$$

where we assume that $\beta(t)$ and $\alpha(t)$ are continuous in t for $-t_1 \leq t \leq 0$. Given any $\epsilon > 0$ we can choose $T(\epsilon)$ small enough so that $\alpha(t)$, for $-T(\epsilon) \leq t \leq 0$, is greater than $\alpha(0)-\epsilon$, i.e.,

$$\alpha(t) > \alpha(0)-\epsilon \ , \qquad -T(\epsilon) \leq t \leq 0 \ . \tag{6.15}$$

The optical theorem for large s gives us $\mathrm{Im}F \sim Cs\sigma_t$. Substituting this on the left in (6.13) and using (6.15) and continuity on the right we get

$$Cs^{\alpha(0)-1} \geq \frac{C'}{k^2 s} s^{2\alpha(0)-2\epsilon} \ . \tag{6.16}$$

As $s \to \infty$ we obtain

$$\alpha(0)-1 \geq 2\alpha(0)-2-2\epsilon \ ,$$

or

$$\alpha(0) \leq 1+2\epsilon \ . \tag{6.17}$$

But ϵ can be made arbitrarily small, hence

$$\alpha(0) \leq 1 \ . \tag{6.18}$$

This result follows only from unitarity, continuity, and Regge behavior - no analyticity need be assumed.

D. Unitarity Bound on the Diffraction Peak

MacDowell and Martin[12] derived a lower bound for the logarithmic derivative of $\mathrm{Im}F(s,t)$ at $t=0$. Again only unitarity is used. This bound also has the advantage that it can be compared with experiment even at non-asymptotic energies.

Let us write

$$A(s,t) \equiv \mathrm{Im}F(s,t) \ , \tag{6.19}$$

and

$$A(s,t) = \frac{8\pi(s)^{\frac{1}{2}}}{k} \sum (2\ell+1) a_\ell(s) P_\ell(1+\frac{t}{2k^2}) \ .$$

We also write the definitions

$$\sigma_t = \frac{4\pi}{k^2} \sum (2\ell+1) a_\ell \quad , \tag{6.20}$$

$$\sigma_{el} = \frac{4\pi}{k^2} \sum (2\ell+1) |f_\ell|^2 \; , \tag{6.21}$$

$$\sigma_{el.im} = \frac{4\pi}{k^2} \sum (2\ell+1) a_\ell{}^2 \quad , \tag{6.22}$$

and

$$\left.\frac{dA(s,t)}{dt}\right|_{t=0} = \frac{8\pi(s)^{\frac{1}{2}}}{2k^3} \sum (2\ell+1) \, \frac{\ell(\ell+1)}{2} a_\ell(s) \qquad . \tag{6.23}$$

The problem is to find a minimum for this last series for a given fixed σ_t and $\sigma_{el.im}$.

Using the method of Lagrange multipliers we get from (6.20), (6.22), and (6.23)

$$\sum_{\ell=0}^{\infty} \left[\frac{(s)^{\frac{1}{2}}}{k} \frac{\ell(\ell+1)}{2} - p - 2qa_\ell \right] (2\ell+1) \delta a_\ell = 0 \qquad . \tag{6.24}$$

The Lagrange multipliers p and q are picked to make the bracket zero for $\ell = 0,1$ and the a_ℓ's for $\ell = 2,3,\ldots,$ are taken to be independent variables. Hence an extremum is reached for

$$a_\ell = \alpha - \beta\ell(\ell+1) \; , \tag{6.25}$$

where α and β are functions of s. It is clear from the case $\ell=0$ that α has to be positive. On the other hand taking β negative in (6.25) will obviously not give the minimum we are seeking and $\beta \geq 0$.

In addition to (6.25) unitarity gives us the restriction

$$0 \leq a_\ell \leq 1 \qquad . \tag{6.26}$$

There are two cases to consider $\alpha > 1$ and $\alpha < 1$.

a) $\alpha > 1$

The minimum is reached when

$$a_\ell = 1 \quad ; \qquad \ell < L_0 \quad ;$$

$$a_\ell = \alpha - \beta\ell(\ell+1) \quad ; \qquad L_0 \leq \ell < L_1 \quad ;$$

$$a_\ell = 0 \quad ; \qquad \ell > L_1 \qquad ; \tag{6.27}$$

where L_0 is the smallest integer such that $\alpha - \beta L_0(L_0+1) \leq 1$ and L_1 is the largest integer for which $\alpha - \beta L_1(L_1+1) > 0$.

Using (6.27) in (6.20) and (6.22) we get σ_t and $\sigma_{e.i}$ as functions of α and β. Inverting these functions we can write $\alpha = \alpha(\sigma_t, \sigma_{e.i})$ and $\beta = \beta(\sigma_t, \sigma_{e.i})$. With these values of α and β in (6.27) we calculate the minimum of (6.23). The algebra is tremendously simplified by converting all sums to integrals and errors are only of order $1/k^2$. The result is

$$\frac{d}{dt}\, \ell n A(s,t)\Big|_{t=0} \geq \frac{1}{8}\frac{\sigma_t}{4\pi}\left[1+3\left(1-\frac{\sigma_{e.i}}{\sigma_t}\right)^2\right] + 0\left(\frac{1}{k^2}\right) . \qquad (6.28)$$

However, the case $\alpha > 1$ corresponds to small inelasticity and is unphysical. One can check that for $\alpha > 1$ we have

$$\sigma_{el} > \sigma_{el.im} \geq \frac{2}{3}\sigma_t \quad , \qquad (6.29)$$

which is in disagreement with present data.

b) $\alpha < 1$

In this case the minimum is reached when

$$a_\ell = \alpha - \beta\ell(\ell+1) \qquad \ell \leq L_1 \quad ,$$
$$a_\ell = 0 \quad , \qquad \ell > L_1 \quad . \qquad (6.30)$$

Using this expression in (6.20), (6.22), and (6.23) one gets, after some algebra,

$$\frac{d}{dt}\, \ell n A(s,t)\Big|_{t=0} \geq \frac{1}{9}\left[\left(\frac{\sigma_t}{4\pi}\right)\frac{\sigma_t+(\sigma_t{}^2+12\pi\, \sigma_{e.i}/k^2)^{\frac{1}{2}}}{2\sigma_{e.i}} - \frac{3}{2k^2}\right] . \qquad (6.31)$$

The bracket has a close lower bound, $[\frac{\sigma_t}{4\pi}(\frac{\sigma_t}{\sigma_{e.i}}) - \frac{1}{k^2}]$. We have then

$$\frac{d}{dt}\, \ell n A(s,t)\Big|_{t=0} > \frac{1}{9}\left(\frac{\sigma_t}{4\pi}\frac{\sigma_t}{\sigma_{e.i}} - \frac{1}{k^2}\right) . \qquad (6.32)$$

The algebra in going from (6.30) to (6.31) is quite involved. To get a quick derivation correct to order $1/k^2$ one takes leading orders in ℓ and converts all sums to integrals. In that case (6.30) becomes

$$a_\ell \cong \alpha - \beta\ell^2 \quad , \qquad \ell \leq L_1 \quad ,$$
$$a_\ell \cong 0 \quad , \qquad \ell > L_1 \quad ; \qquad (6.33)$$

where now $\alpha - \beta L_1{}^2 = 0$ and $L_1 = (\alpha/\beta)^{\frac{1}{2}}$. From (6.20) we obtain

$$\sigma_t = \frac{2\pi}{k^2}\frac{\alpha^2}{\beta} \quad , \qquad (6.34)$$

and from (6.22)

$$\sigma_{e.i} = \left(\frac{2\pi}{k^2}\right) \cdot \frac{2}{3}\frac{\alpha^3}{\beta} \quad . \qquad (6.35)$$

Note now that for $\alpha < 1$, $\sigma_{e.i} < 2/3\ \sigma_t$. From (6.23) we obtain

$$\left.\frac{dA}{dt}\right|_{t=0} \geq \frac{\pi(s)^{\frac{1}{2}}}{3k^3} \cdot \frac{\alpha^3}{\beta^2} \quad . \tag{6.36}$$

Hence finally one gets

$$\left.\frac{d}{dt} \ell nA(s,t)\right|_{t=0} \geq \frac{1}{12k^2} \frac{\alpha}{\beta} = \frac{\sigma_t^{\ 2}}{36\pi\sigma_{e.i}} \quad , \tag{6.37}$$

which is the same as (6.32) when one ignores terms $0(1/k^2)$.

It is interesting to note how close (6.37), with $\sigma_{e.i}$ replaced by σ_{el}, is to the experimental situation. From Dr. White's lectures we learned that a good fit to the pp data is obtained with the form $\frac{d\sigma}{dt} \cong A\exp(bt)$, $b \cong 9(\mathrm{GeV})^2$. One also learns that, ignoring spin, Ref and Imf have approximately the same t-dependence. One usually computes σ_{el} as

$$\sigma_{el} \cong \int_{-\infty}^{0} A\ e^{+bt} dt \cong \frac{A}{b} \quad , \tag{6.38}$$

and

$$A = |f(0)|^2 \ \frac{2\pi}{2k^2} \quad . \tag{6.39}$$

From the fact that $|\mathrm{Ref}|$ and $|\mathrm{Imf}|$ have the same t dependence we get

$$\left.\frac{d\ell nf}{dt}\right|_{t=0} = \frac{b}{2} \quad . \tag{6.40}$$

Substituting this in (6.37) we obtain

$$\frac{b}{2} \geq \frac{1}{9}\left[\frac{4\pi(\mathrm{Imf}(0))^2}{k^2} \cdot \frac{b}{A}\right] \quad , \tag{6.41}$$

where we have used the optical theorem $\sigma_t = 4\pi\mathrm{Imf}(0)/k$. Finally (6.39) gives us

$$\frac{1}{2} \geq \frac{4}{9}\left(\frac{\mathrm{Imf}(0)}{|f(0)|}\right)^2 \quad . \tag{6.42}$$

The bound is rather close when Ref/Imf is small.

VII. LOWER BOUND ON THE FORWARD AMPLITUDE

The results on lower bounds are so far much weaker than one would like. In this chapter we prove the Jin-Martin[13] lower bound of the form

$$|F(s,0)| \geq Cs^{-2} \quad . \qquad (7.1)$$

More precisely we only show that there exist a sequence $\{s_n\}$, $s_n \to \infty$ as $n \to \infty$, such that

$$|F(s_n,0)| \geq Cs_n^{-2} \quad . \qquad (7.2)$$

Although a more direct proof[14] of (7.2) for either large positive <u>or</u> large negative s, but not both, is possible, we shall here follow the proof of Jin and Martin which uses Herglotz functions. We do this for pedagogical reasons since Herglotz functions are quite useful in scattering theory and they will come up again in later lectures.

Let us start then by giving the definition of Herglotz functions and listing a few of their properties.

<u>Definition</u>: a function H(z) is called a Herglotz function if H(z) is analytic in the upper half plane and if $\mathrm{Im}H(z) > 0$ for $\mathrm{Im}z > 0$.

Any Herglotz function has the representation

$$H(z) = A+Bz + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{Im}H(x)\,(1+zx)}{(1+x^2)(x-z)}\, dx \quad , \qquad (7.3)$$

with $B \geq 0$, A real, and $\mathrm{Im}H(x) \geq 0$. Also we have the convergence condition

$$\int_{-\infty}^{+\infty} \frac{\mathrm{Im}H(x)}{(1+x^2)}\, dx < \infty \quad . \qquad (7.4)$$

Another property that is easy to verify is that if H(z) is a Herglotz function then

$$F(z) \equiv -\frac{1}{H(z)} \quad , \qquad (7.5)$$

is also in the Herglotz class. From the defining representation (7.3) one can easily show that, for all z such that $\epsilon < \arg z < \pi-\epsilon$, the following bounds hold:

$$C|z|^{-1} \leq |H(z)| \leq C|z| \quad . \qquad (7.6)$$

For the physical amplitudes we want lower bounds on $|H(x)|$ for real x and (7.6) is not directly applicable. However one has the following lemma due to Martin.[15]

<u>Lemma 1:</u>

Given a Herglotz function, H(z), satisfying (7.6), and such that H(z) is analytic in the cut z-plane with right hand cut from z=a to $z=+\infty$ and left hand cut from z=-b to $z=-\infty$, and any positive number C' larger than C; one can find an infinite real sequence $\{x_n\}$, $x_n \to +\infty$, such that

$$|H(x_n)| \leq C'|x_n| . \qquad (7.7)$$

Similarly, one can find a sequence $\{x_n\}$, $x_n \to -\infty$, along the negative real axis such that (7.7) holds.

Since $(-1/H(z))$ is also Herglotz, it follows from (7.8) that there also exists a sequence $\{x_n\}$, $x_n \to \pm\infty$, such that

$$|H(x_n)| \geq C''|x_n|^{-1} \qquad (7.8).$$

Proof of Lemma 1:

Let F(z) be H(z) minus the contribution of the left hand cut in (7.3), i.e.,

$$F(z) \equiv A+Bz + \frac{1}{\pi}\int_a^\infty \frac{\mathrm{Im}H(x)(1+zx)}{(1+x^2)(x-z)}\,dx \quad . \qquad (7.9)$$

F(z) is regular in the cut z-plane with a right hand cut only. It is bounded by $C|z|$ in any complex direction including the negative real axis. Let us now assume that on the right hand cut for large x

$$|F(x)| \geq C'x, \quad C' > C \quad . \qquad (7.10)$$

We show that (7.10) leads to a contradiction. This is obtained by first applying the Phragmen-Lindelof theorem to the function $1/F(x)$. Since $(1/F(z))$ is analytic outside the right hand cut and a finite circle around the origin we get for large $|z|$

$$\left|\frac{1}{F(z)}\right| \leq \frac{1}{C'}|z|^{-1} \qquad (7.11)$$

in all directions. Hence we get for all directions

$$|F(z)| \geq C'|z| \quad , \qquad (7.12)$$

which contradicts the bound given below (7.9) for $C' > C$. Thus for any $C' > C$ there must exist a sequence $\{x_n\}$, $x_n \to \infty$ on the positive real axis such that

$$|F(x_n)| < C'x_n \quad . \qquad (7.13)$$

The integral over the left hand cut in (7.3) when divided by $|x|$ tends to zero as $x_n \to +\infty$. This gives us for the same sequence $\{x_n\}$ as in (7.13), as $x_n \to +\infty$

$$|H(x_n)| < C'x_n \quad . \qquad (7.14)$$

An analogous result is obtained by interchanging the roles of the left and right hand cuts. This completes the proof of the lemma 1.

In order to use the lemma 1 and (7.8) to obtain lower bounds we have first to get Herglotz functions from the scattering amplitudes. In general, scattering amplitudes are not Herglotz. We shall need the following second lemma.

Lemma 2:

Let F(z) be analytic in the cut z-plane with a right hand cut extending from $z=a$ to $z=+\infty$ and a left hand cut from $z=-\infty$ to $z=-b$. Also let F(z) have the following properties:

(i) $|F(z)| \leq \mathrm{Const.}|z|^N$, $\qquad |z| > z_0$.

(ii) $\text{Im}F(x+i0) > 0$.

(iii) $F(z) = F^*(z^*)$. (reality in gap).

Then $F(z)$ has only a finite number of zero.

Proof of Lemma 2:

Complex zeros occur in pairs z_i, z_i^*, and real zero x_i occurs in the gap $-b \le x_i \le a$. If the number of zeros is infinite we can take p-pairs of complex zeros and 2q real zeros and define a new function, $G(z)$:

$$G(z) = \frac{F(z)}{\prod_{i=1}^{p} (z-z_i)(z-z_i^*) \prod_{j=1}^{2q} (z-x_i)} , \tag{7.15}$$

where we take $2p+2q > N+2$. Hence from (i) we have for large $|z|$

$$G(z) \le \frac{1}{|z|^2} . \tag{7.16}$$

By construction $\text{Im}G(z) > 0$ on the real axis, and we write the a an unsubtracted dispersion relation for G,

$$G(z) = \frac{1}{\pi} \int_{-\infty}^{-b} dx \frac{\text{Im}G(z)}{x-z} + \frac{1}{\pi} \int_{a}^{\infty} dx \frac{\text{Im}G(x)}{x-z} . \tag{7.17}$$

But the positivity of $\text{Im}G(x)$ implies that $|G(z)|$ cannot decrease faster than $|z|^{-1}$ which contradicts (7.16). Hence the number of zeros must be finite and one can easily show that

$$\text{number of zeros} \le N+2 . \tag{7.18}$$

This ends the proof of lemma 2.

Next we want to construct a Herglotz function from an $F(z)$ satisfying the conditions of lemma 2. Here two cases occur:

(a) Number of Real Zeros Even:

In this case we remove all the zeros and define a function Φ as

$$\Phi(z) = \frac{F(z)}{\prod_{i=1}^{p} (z-z_i^*)(z-z_i) \prod_{j=1}^{2q} (z-x_j)} . \tag{7.19}$$

Now $\Phi(z)$ has no zeros. Furthermore on the cuts $\text{Im}\Phi(x+i0) > 0$, and for large $|z|$

$$\Phi(z) \le \text{Const.}|z|^{N-2p-2q} . \tag{7.20}$$

In lemma 3 below we show that such a function is a Herglotz function. Hence there exist a sequence $\{x_n\}$, $x_n \to \infty$, such that

$$|\Phi(x_n)| \ge \bar{C}\, x_n^{-1} . \tag{7.21}$$

This is a consequence of lemma 1. From (7.21) and (7.19) we now get

$$|F(x_n)| \geq \bar{C}\, x_n^{2p+2q-1} \geq \bar{C}\, x_n^{-1} , \tag{7.22}$$

for the same sequence.

(b) Number of Real Zeros Odd:

We remove all complex zeros and all real zeros except one, writing

$$\psi(z) = \frac{F(z)}{\prod_{i=1}^{p} (z-z_i^*)(z-z_i) \prod_{j=1}^{2q} (z-x_j)} . \tag{7.23}$$

In this case $\psi(z)$ still has only one real zero at $z = x_0$ but it has no complex zeros. The treatment of the remaining zero depends on the sign of the derivative of F at the remaining zero. A simple modification of the previous argument (see ref. 13) will still give a sequence $\{x_n\}$, $x_n \to \infty$, such that

$$|\psi(x_n)| \geq \bar{\bar{C}}\, x_n^{-1} , \tag{7.24}$$

which gives again

$$|F(x_n)| \geq \bar{\bar{C}}\, x_n^{-1} . \tag{7.25}$$

To apply (7.22) and (7.25) to scattering amplitudes we still have to consider the situation where ImF(x+i0) changes sign ν-times from $-\infty$ to $+\infty$. If this happens then we write

$$G(z) = \prod_{k=1}^{\nu} (z-\tilde{x}_k) F(z) , \tag{7.26}$$

where the $\tilde{x}_k$'s are chosen so that ImG(x+i0) > 0 on both cuts. Proceeding in the same way for G as we did for F we get a sequence $\{x_n\}$, $x_n \to +\infty$ such that

$$|G(x_n)| \geq \text{Const.} x_n^{-1} , \tag{7.27}$$

and hence

$$|F(x_n)| \geq \text{Const.} x_n^{-1-\nu} , \tag{7.28}$$

where ν is the number of times ImF(x+i0) changes sign in $-\infty < x < +\infty$.

If we now consider the scattering process $a+b \to a+b$, $\text{Im}F(s,0) > 0$ for $S > (M_a+M_b)^2$, and $\text{Im}F(s,0) < 0$ for $-\infty < s < (M_a-M_b)^2$. Hence for an actual scattering process all the conditions on F are satisfied with $\nu = 1$. We conclude that there exist a sequence $\{s_n\}$, $s_n \to +\infty$, and a constant C such that

$$|F(s_n,0)| \geq C s_n^{-2} . \tag{7.29}$$

In closing we state and prove lemma 3 which we have used to assert that $\Phi(z)$ was Herglotz.

Lemma 3:

Given a function $\Phi(z)$ regular in the upper half plane and satisfying the following properties:

(a) $|\Phi(z)| < C|z|^N, \quad |z| > z_0 \quad ,$

(b) $\Phi(z)$ has no zeros for $\text{Im}z \geq 0 \quad ,$

(c) $\text{Im}\Phi(x+i0) > 0$ on the real axis, (i.e., $0 \leq \arg\Phi(x+i0) \leq \pi$).

Then $\Phi(z)$ is a Herglotz function.

Proof of Lemma 3:

Take an integer $N' > N$ with $N' \geq 2$. The function $(\Phi)^{1/N'}$ will also be analytic in the upper half plane because of (b). Furthermore it is clear that $\text{Im}[\Phi^{1/N'}]$ will also be positive on the real axis as a consequence of (c). We consider the function

$$W(z) = \exp[i\Phi^{1/N'}] \quad . \tag{7.30}$$

This function is regular for $\text{Im}z > 0$. On the real axis $|W(x+i0)| < 1$ holds. Now the Phragmen-Lindelof principle tells us that only one of the following two possibilities holds

(i) $|W(z)| < 1$ everywhere in the upper half plane

OR (ii) There exists a sequence of points $\{z_n\}$, $|z_n| \to \infty$ along some direction in the upper half plane such that

$$|W(z_n)| \geq e^{\alpha|z_n|}, \quad \alpha > 0 \quad . \tag{7.31}$$

Alternative (ii) implies that along some direction $(\text{Im}\Phi^{1/N'})$ blows up at least linearly to $-\infty$. Hence we get

$$|(\Phi(z_n))^{1/N'}| \geq \alpha|z_n| \quad ,$$

and

$$|\Phi(z_n)| \geq \alpha|z_n|^{N'} \quad . \tag{7.32}$$

This contradicts assumption (a) of our lemma. Only alternative (i) can hold, and that implies

$$\text{Im}(\Phi)^{1/N'} > 0 \text{ for } \text{Im}z > 0. \tag{7.33}$$

Let $\ell n\Phi^{1/N'} \equiv u+iv$. The function $v(x,y)$ is a harmonic function and $v \equiv \arg\Phi^{1/N'}$. From (7.33) we know that $0 \leq v \leq \pi$ for all points in the upper half plane and therefore $v(x,y)$ is bounded for $y \geq 0$. However, from our original construction we know that on the x axis

$$0 \leq \arg(\Phi(x+i0))^{1/N'} \leq \frac{\pi}{N'} \quad ,$$

or

$$0 \leq v(x,0) \leq \frac{\pi}{N'} \quad . \tag{7.34}$$

Since $v(x,y)$ is harmonic and bounded for $y > 0$ then it follows from (7.34) that for all $y \geq 0$,

$$0 \leq v(x,y) \leq \frac{\pi}{N'} \quad , \tag{7.35}$$

or

$$0 \leq \arg(\Phi(z)^{1/N'}) \leq \frac{\pi}{N'} \quad , \tag{7.36}$$

and

$$0 \le \arg\Phi(z) \le \pi \quad , \quad \mathrm{Im}z \ge 0; \tag{7.37}$$

we conclude that $\Phi(z)$ is Herglotz. Q.E.D.

The lower bound obtained above is as we mentioned earlier rather weak. The question of improving it is an immediate challenge.

It is clear from (7.22) that we can immediately improve (7.2) by one power of s if we know the amplitude has a real zero and by two powers if we know the amplitude has a pair of complex zeros. However, to guarantee the existence of such zeros requires more detailed information about the signs of the scattering lengths and the relative magnitudes of subtraction terms as compared to dispersion integrals over the low energy region. Such conditions are given in Ref. 13.

Another possibility for improvement occurs in the case of a symmetric amplitude, i.e. $\bar{b}=b$, if one knows in advance that the scattering length is negative. To show how this happens it is convenient to use the variable E, the laboratory energy, instead of s, where for $\pi^o K \to \pi^o K$ for example

$$s = \mu^2 + M_K^2 + 2M_K E \quad . \tag{7.38}$$

The amplitude F(E) is even in E and satisfies the dispersion relation

$$F(E) = F(0) + \frac{2E^2}{\pi}\int_{\mu}^{\infty} \frac{\mathrm{Im}F(E')}{E'(E'^2-E^2)}\, dE' \quad . \tag{7.39}$$

We define a new variable $z=E^2$, write $F(E)=G(z)$, and get

$$G(z) = G(0) + \frac{z}{\pi}\int_{\mu^2}^{\infty} \frac{\mathrm{Im}G(z')}{z'(z'-z)}\, dz' \quad . \tag{7.40}$$

First note that G(z) is a Herglotz function of z, $(\mathrm{Im}F(E') > 0)$. Next it is evident from (7.40) that G(x) is real and monotonically increasing in the interval $-\infty < x \le \mu^2$; $dG/dz > 0$ for $z < \mu^2$. If the scattering length is negative,

$$G(\mu^2) \equiv F(\mu) < 0 \quad , \tag{7.41}$$

then for $-\infty < x < \mu^2$,

$$G(x) \le G(\mu^2) < 0 \ . \tag{7.42}$$

We can therefore construct another Herglotz function $H(z)=G(z)/z$. This is trivial to check from (7.40). Hence we have a real sequence $\{z_n\}$ such that,

$$|H(z_n)| \ge C|z_n|^{-1} \ ,$$

$$|G(z_n)| \ge C \qquad ,$$

$$|F(E_n)| \ge C \qquad . \tag{7.43}$$

This improves (7.2) by two powers of s.

So far we have only derived lower bounds for the full amplitude. The question of lower bounds for ImF or σ_t is also relevant. We need such a lower bound to complete the proof of (6.5).

We show that

$$\mathrm{Im}F(s_n,0) \geq C\ s_n^{-5}(\ell n s_n)^{-2} \quad . \tag{7.44}$$

From Chapter IV we write

$$|F(s,0)|^2 = \left|\sum_0^{L(s)} (2\ell+1)\ f_\ell(s)\ \frac{8\pi\sqrt{s}}{k} + O(s^{-N})\right|^2 \quad . \tag{7.45}$$

Using the Cauchy-Schwartz inequality we get,

$$|F(s,0)|^2 \leq \frac{s}{k^2}(8\pi)^2\left(\sum_0^L(2\ell+1)\right)\left(\sum_0^L(2\ell+1)\ |f_\ell|^2\right) \quad ,$$

$$\leq C^2\ s\ell n^2 s\ \mathrm{Im}F(s,0) \quad . \tag{7.46}$$

This with (7.2) gives us (7.44).

VIII. POMERANCHUK THEOREMS

A. Historical Remarks

The Pomeranchuk theorem[16] in its original form states that asymptotically the total cross section for the scattering of a particle on a target is equal to the total cross section for the scattering of the antiparticle on the same target, i.e. $\lim_{E\to\infty} \sigma_+(E) = \lim_{E\to\infty} \sigma_-(E)$. The total cross sections were assumed to reach a constant limit.

This theorem is not purely a rigorous consequence of analyticity and polynomial boundedness alone. Other assumptions had to be made. These assumptions have been considerably weakened by succeeding authors as shown below. The only extra input now needed concerns the relative magnitude of ReF to ImF.

We consider the processes

$$a + T \to a + T \quad ,$$
$$\bar{a} + T \to \bar{a} + T \quad , \tag{8.1}$$

with forward amplitudes, $F_\pm(E)$, respectively and E = lab. energy. We set $m_a=\mu$ and $m_T=M$.

Pomeranchuk[16] assumed that:

i) $F_\pm(E)$ are such that:

$$\lim_{E\to\infty} \frac{\mathrm{Im}F_\pm(E)}{\sqrt{E^2-\mu^2}} = \text{const.} \tag{8.2}$$

and for large E,

$$\left|\frac{F_\pm(E)}{\sqrt{E^2-\mu^2}}\right| \leq \text{const.} \tag{8.3}$$

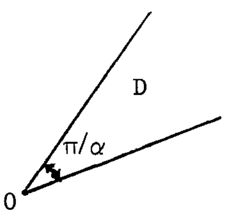
ii) $F_\pm(E)/(E^2-\mu^2)$ satisfies an unsubtracted dispersion relation.

Weinberg[17] weakened the condition $\sigma_\pm\to$const. as $E\to\infty$ and replaced it by the condition that for large enough E, $\Delta\sigma \equiv \sigma_+-\sigma_-$ should have one sign. He also showed that if one assumes that for large E, $\sigma_{\pm} \sim C_{\pm}(\ell nE)^m$, $0 < m < 1$, then $C_+ = C_-$. In that case one has $\lim(\sigma_+/\sigma_-) = 1$.

Amati, Fierz, and Glazer[18] proved that $\int^{\infty} \frac{\Delta\sigma}{E}\, dE$ converges.

A simple and elegant proof of the Pomeranchuk theorem in a slightly weakened form was given by Meiman.[19] We shall follow it here in section C. Before we do that we have to discuss some mathematical preliminaries related to the Phragmén-Lindelöf principle. We have used this principle several times in these lectures and for the sake of completeness should state it somewhere.

B. Mathematical Preliminaries: Phragmén-Lindelöf



(a) Phragmén-Lindelöf Theorem: (Classical Form)

Let f(z) be analytic in z regular in a sector, D, defined by two straight lines intersecting at 0 with an angle π/α. Further let f(z) be continuous in $\bar{D}$. As $r \to \infty$, let

$$\max_{\theta\epsilon \text{ sector}} |f(re^{i\theta})| = O(e^{r^\beta}), \ \beta < \alpha \quad . \tag{8.4}$$

If now $|f(z)| \leq M$ on the straight lines, then $|f(z)| \leq M$ for all $z\epsilon D$.

We specialize this theorem to the case of the half-plane which is of interest to us and we will use in this chapter only the following version of (a).

Theorem (a'):

If f(z) is analytic in z, regular in Imz > 0, continuous in Imz ≥ 0, and $|f(x)| \leq M$ on the real axis, then:

either $|f(z)| \leq M$ for all z, Imz ≥ 0,

or There exists a constant $\alpha > 0$ such that

$$M(R_n) \geq e^{\alpha R_n} \quad ,$$

for an infinite sequence $\{R_n\}$, $R_n \to \infty$ as $n \to \infty$, where

$$M(R) = \max_{0<\theta<\pi} |f(Re^{i\theta})| \quad .$$

A corollary of this theorem states that if f(z) satisfies the conditions of theorem (a') and has a bound sufficient to exclude exponential growth in any direction, and if

$$\lim_{x\to+\infty} f(x) = L_1 \quad ,$$

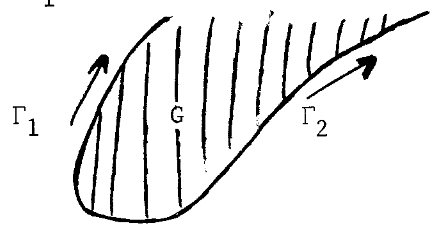
$$\lim_{x\to-\infty} f(x) = L_2 \quad , \tag{8.5}$$

then $L_1 = L_2$ and $f(z) \to L_1$ uniformly in all directions.

Another more general form of the Phragmén-Lindelöf principle which we shall use below is contained in the following theorem.

Theorem (b):

We are given a domain G bounded by two curves Γ_1 and Γ_2 which extend to infinity in such a way that in going to infinity along Γ_2 G is to the left, and as one moves to infinity along Γ_1 G lies to the right.



Consider a function f(z) analytic in G and continuous and bounded on boundary. Let $\mathcal{E}_1$ be the manifold of limit values of f(z) as $z \to \infty$ along Γ_1 and $\mathcal{E}_2$ the manifold of limit values as $z \to \infty$ along Γ_2. ($\mathcal{E}_1$ and $\mathcal{E}_2$ each consist of either one point or a continuum.) If $\mathcal{E}_1$ and $\mathcal{E}_2$ have no points in common and if one of them does not surround the other, then f(z) cannot be bounded in G.

If G is the half plane, then it follows from theorem (a') that if f(z) is unbounded it increases faster than $\exp(\alpha|z|)$ in some direction. <u>We note also that theorem (a') is valid for a region obtained from the half plane by deforming a finite part of the real axis</u>. We do not need analyticity in the full upper half plane to apply these theorems and we can easily exclude a region that lies inside a semicircle centered at the origin.

C. <u>Meiman's Proof of Pomeranchuk's Theorem</u>

We start with functions $F_{\pm}(E)$ with the following properties:

i) $F_+(E+io) = F_-(-E-io)$.

ii) Analytic in cut plane with cuts $E=\mu$ to $E=\infty$ and $E=-\infty$ to $E=-\mu$.

iii) $F_{\pm}(E+io) = F_{\pm}^*(E-io)$.

iv) For an arbitrary $\epsilon > 0$, and $|E| > E_o(\epsilon)$

$$|F(E)| \le \exp(\epsilon|E|), \text{ in all directions.} \quad (8.6)$$

v) $F_{\pm}(E)/\sqrt{E^2-\mu^2}$ bounded on real axis as $E \to \pm\infty$.

Properties i)-iv) are consequences of local field theory. In fact iv) can be replaced by polynomial boundedness in local field theory. However, v) has not been proved rigorously, and although one can only make the original Pomeranchuk statement about cross sections that do tend to constants, still as we shall see below there is no guarantee that the real parts are bounded as in v).

We now prove that i)-v) imply that there exists a sequence, $\{E_n\}$, $E_n \to +\infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty}[\sigma_+(E_n)-\sigma_-(E_n)] = 0. \tag{8.7}$$

To do this we define, g(E), as

$$g(E) = \frac{F_+(E)-F_-(E)}{\sqrt{E^2-\mu^2}} \quad . \tag{8.8}$$

The optical theorem gives us

$$\operatorname{Im}g(E) = 2M[\sigma_+(E)-\sigma_-(E)] \quad . \tag{8.9}$$

From the crossing condition i) and the reality condition iii) we obtain

$$g(-E+i0) = g^*(E+i0) \quad , \tag{8.10}$$

and

$$g(-E-i0) = -g(E+i0) \quad . \tag{8.11}$$

Using theorem (b) of section B with G corresponding to the upper half plane, Γ_2 is the positive real axis and Γ_1 is the negative real axis. Let $\mathcal{E}_2$ be the manifold of limit values as $E \to +\infty+i0$ and $\mathcal{E}_1$ the manifold of limit values of g(E) as $E \to -\infty+i0$. From (8.10) it is clear that $\mathcal{E}_2$ and $\mathcal{E}_1$ should be symmetric with respect to the real-g axis in the g-plane. These manifolds must intersect the real axis, for if they do not one of them would lie in the upper half plane and one in the lower half plane and they will have no points in common. If they have no points in common g(E) is unbounded by theorem (b), and hence by theorem (a'), g(E) increases faster than some exponential ($\exp\alpha|E|$) along some sequence of points in the upper half plane. This is excluded by the bound (8.6). Therefore there must exist at least one real point, u_o, which is in both $\mathcal{E}_2$ and $\mathcal{E}_1$, and thus there exists a sequence $\{E_n\}$, $E_n \to +\infty$ as $n \to +\infty$, such that $g(E_n) \to u_o$ as $n \to \infty$, or

$$\lim_{n\to\infty}\{\operatorname{Im}g(E_n)\} = 0 \quad ,$$

and

$$\lim_{n\to\infty}[\sigma_+(E_n)-\sigma_-(E_n)] = 0 \quad . \tag{8.12}$$

This is a weaker statement than the original Pomeranchuk theorem, but an experimentally meaningful statement.

However, if the limit of $\Delta\sigma(E)$ as $E \to \infty$ exists then it immediately follows from (8.12) that

$$\lim_{E\to\infty} \Delta\sigma(E) = 0 \quad . \tag{8.13}$$

Meiman's method can also be used to study the case where the cross-sections do not tend to a constant but increase or decrease like some power of logE or loglogE.

For example let us assume that for large E along the cut,

$$\Delta\sigma(E) \cong C|\varphi(E)| \quad . \tag{8.14}$$

The function $\varphi(E)$ is assumed analytic in the upper half plane and satisfies the conditions: 1) for any $\epsilon > 0$ and large $|E| > E_o(\epsilon)$ we have $|E|^{-\epsilon} < |\varphi(E)| < |E|^{\epsilon}$; 2) along the real axis $(\text{Im}\varphi(E)/|\varphi|) \to 0$ as $E \to \infty$, and $\frac{\varphi(-E)}{\varphi(E)} \to 1$ as $E \to \infty$. An example of such functions is $\varphi(E) \equiv (\log E)^p (\log\log E)^q$.

If now in addition to (8.14) we assume that $g(E)/\varphi(E)$ is <u>bounded</u> as $E \to +\infty$ on the real axis, then by applying theorem (b) to this ratio we get

$$\lim_{n\to\infty} \left(\frac{\Delta\sigma(E_n)}{|\varphi(E_n)|}\right) = 0 \quad , \tag{8.15}$$

hence C=0 in (8.14). More precisely, if $\sigma_+(E) \sim C_+|\varphi(E)|$ and $\sigma_-(E) \sim C_-|\varphi(E)|$ as $E \to \infty$, then $C_+=C_-$ and

$$\lim_{E\to\infty}\left[\frac{\sigma_+(E)}{\sigma_-(E)}\right] = 1 \quad . \tag{8.16}$$

The only input going into the Pomeranchuk theorem which is not a direct consequence of local field theory is, as we mentioned earlier, condition v). Thus even when the total cross sections tend to constants at infinity, there is no guarantee that the ratio $\text{ReF}_\pm/E$ is bounded as $E \to \infty$ along the real axis. Martin[20] was able to weaken v) and replace it by

$$\lim_{E\to\infty} \left|\frac{F_\pm(E)}{E \log E}\right| = 0 \quad . \tag{8.17}$$

With this and the assumption that $\Delta\sigma(E)$ has a limit as $E \to \infty$ he showed that $\Delta\sigma(\infty) = 0$.

Thus the main gap remains to find a bound on the growth of the real parts. To close this gap Eden[21] and Kinoshita[22] have shown that using unitarity and analyticity in the Martin ellipse in $\cos\theta$ one gets the inequality

$$\frac{|\text{ReF}(E)|}{\text{ImF}(E)} \le \frac{C\sqrt{E}\,\ell n\, E}{[\text{ImF}(E)]^{\frac{1}{2}}} \quad , \tag{8.18}$$

for sufficiently large E. One can easily check that for any process for which $\sigma(E) \to \infty$ as $E \to \infty$, (8.18) implies

$$\lim_{E\to\infty} \frac{|\text{ReF}(E)|}{(\text{ImF}(E))\,\ell n\, E} = 0 \quad . \tag{8.19}$$

Unfortunately, (8.19) does not help us in the most interesting case, namely when $\sigma_\pm(E)$ tend to constants. In that case we are still faced with the counter example,

$$F_+(E) = -AE\,\ell n(\mu-E) + BE\,\ell n(\mu+E) \quad , \tag{8.20}$$

where A and B are real and positive $A \neq B$.

For this example we have

$$|\sigma_+(\infty)-\sigma_-(\infty)| = \frac{1}{2M}|A-B| \neq 0 \quad , \tag{8.21}$$

also

$$\lim_{E\to\infty} \frac{\sigma_+(E)}{\sigma_-(E)} = \frac{A}{B} \neq 1 \quad . \tag{8.22}$$

The real part of course grows like $E\ell nE$, a behavior which seems in contradiction with the measurements of the Lindenbaum group.

Finally, using the remark at the end of section B, we stress the fact that neither the Meiman proof nor the Martin proof need full analyticity in the upper half E-plane. It is clear that all the results of this chapter are true even if $F_{\pm}(E)$ is only analytic outside some large but finite semicircle centered at the origin. In Ref. 20 Martin points out that this minimal analyticity and the crossing property used are known to hold for all collisions of the type $a+b \to a+b$ as a result of the work of Bros, Epstein, and Glaser, Ref. 3.

IX. ASYMPTOTIC RELATIONS BETWEEN PHASES AND TOTAL CROSS-SECTIONS

A. Motivations and Examples

In deriving upper bounds we have so far barely used the analyticity in energy. In general it is not yet clear how to fully incorporate this information into the derivation of upper bounds. In this chapter we shall show that there is a strict correlation between the behavior of the ratio ReF/ImF as $E \to \infty$ and that of $\sigma(E)$ as $E \to \infty$. Of course, the dispersion relations in principle determine ReF from σ_{tot}. But because of the principal value integrals in the dispersion relations, it is easier to use either geometric methods or the phase representation to determine what certain assumptions on (ReF/ImF) imply about the asymptotic behavior of $\sigma(E)$.

In this chapter we restrict ourselves to studying the symmetric amplitude F(E):

$$F(E) = \frac{1}{2}[F_+(E)+F_-(E)] - \text{pole terms}. \tag{9.1}$$

Let us first write down a few simple examples to give an idea of the kind of relation we are seeking.

(a) ReF/ImF $\to$ const.

An example in this case would be

$$F(E) = \frac{C}{\sin\pi\alpha}\,[(E-\mu)^{\alpha}+(-E-\mu)^{\alpha}] \;;\; 0 < \alpha \leq 1. \tag{9.2}$$

For this example

$$\lim_{E\to\infty} \frac{\mathrm{Re}F(E)}{\mathrm{Im}F(E)} = \cot\frac{\pi\alpha}{2} \quad , \tag{9.3}$$

and for large E,

$$\sigma(E) \sim E^{\alpha-1} \quad . \tag{9.4}$$

Thus one feature of this example is that if ReF/ImF → constant as $E \to \infty$, then $\sigma(E)$ goes to zero like an inverse power and this power is determined by the constant in (9.3). We shall see below that this feature is quite general and can be made more precise.

(b) $\underline{\mathrm{Re}F/\mathrm{Im}F \sim C(\ell nE)^{-1}}$

Here we take

$$F(E) = iCE[\ell nE - i\tfrac{\pi}{2}]^{\gamma} \quad . \tag{9.5}$$

This gives us

$$\sigma(E) \sim (\ell nE)^{\gamma}$$

$$\frac{\mathrm{Re}F(E)}{\mathrm{Im}F(E)} \sim \frac{C}{\ell nE} \quad . \tag{9.6}$$

(c) $\underline{\frac{F(E)}{E} \to \text{const.}}$

If $(F(E)/E) \to C$ as $E \to +\infty$, it is easy to show that $\mathrm{Re}F(E)/E \to 0$. For by Phragmén-Lindelöf $(F(E)/E) \to C$ also as $E \to -\infty$, but $F(E)/E$ has an odd real part and hence C must be purely imaginary.

To make some of the features of these examples more general, we follow the techniques used by Khuri and Kinoshita.[23] Similar results can be obtained by using the phase representation method.[24]

B. Mathematical Preliminaries

We first state an inequality due to Meiman.[19]

Let g(E) be analytic in ImE > 0 and continuous in ImE ≥ 0. Furthermore, we assume that g(E) has the symmetry property

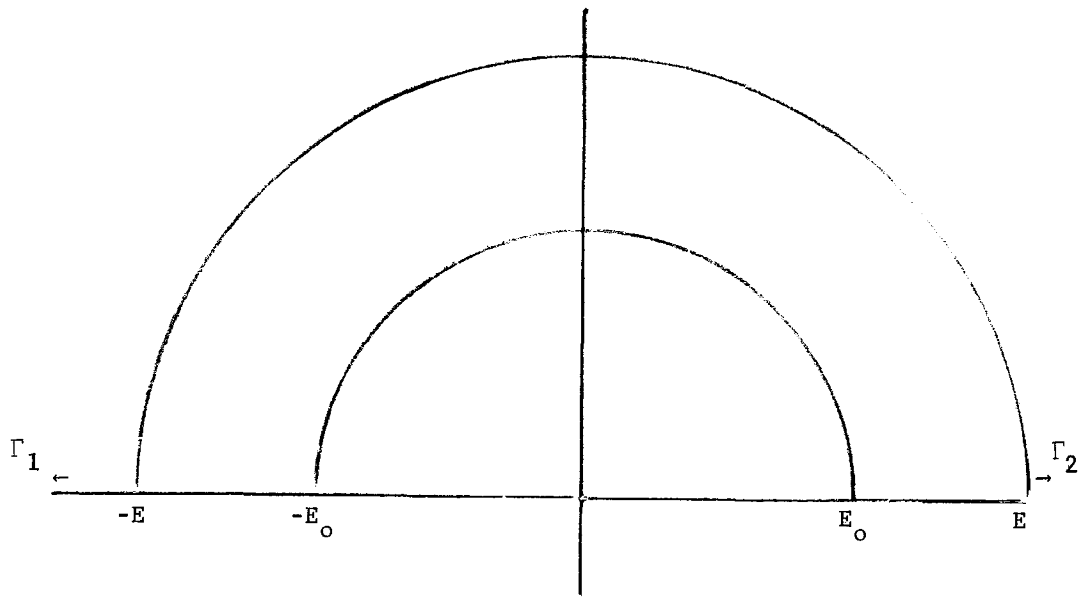
$$g(-E + i0) = g^{*}(E + i0). \tag{9.7}$$

Also we assume that

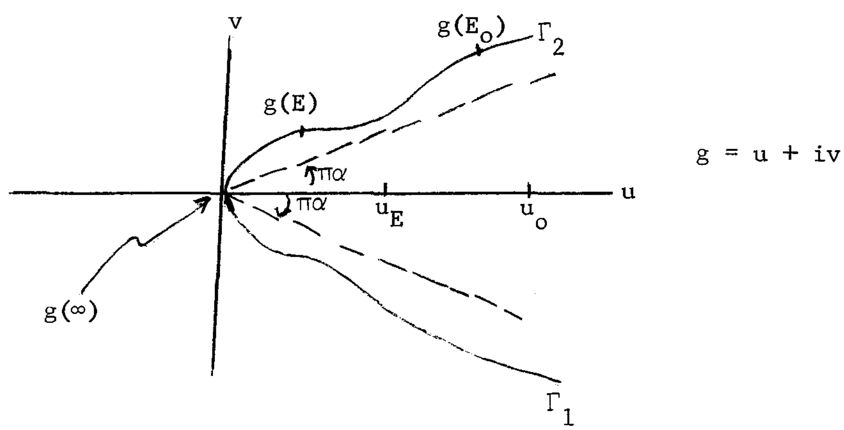
$$\lim_{|E|\to\infty} g(E) = 0 \tag{9.8}$$

The function g(E) maps the upper half E-plane into a certain domain of the g-plane. In particular the upper edges of the semi real axis $(0,-\infty)$ and $(0,+\infty)$ are mapped onto two curves Γ_1 and Γ_2 symmetrically located with respect to the real g-axis. In this mapping a sufficiently distant upper-half neighborhood of the point $E = \infty$ is mapped onto a certain neighborhood (perhaps many sheeted) of the point g=0. To be more specific we take two large real values E and E_o, such that $E, E_o \gg E_1$;

E_1 being a large positive constant. We consider a domain in the E-plane bounded by two semi-circles centered at the origin with radii E and E_o as shown below.



Let us assume that for $|E| > E_1$, Γ_1 and Γ_2 have no points in common. The image of the above region in the g-plane is shown below



We let u_o be the nearest point of intersection of the map of the smaller semi-circle with the positive u-axis. Similarly, we let u_E be the farthest point of intersection of the map of the large semi-circle with the positive u-axis. Meiman's inequality can now be stated as

$$\int_{u_E}^{u_o} \frac{du}{\rho(u)} \geq \frac{1}{4} \ln \frac{E}{E_o} \quad , \qquad (9.9)$$

where $E \gg E_o$ and $\rho(u)$ is the shortest distance from the point u to the curve $\Gamma(E,E_o)$. For a proof of this inequality, see Ref. 23 or for more detail

Nevannlina's book.

Using this inequality we state and prove the following two theorems due to Meiman. For simplicity we ignore violent oscillations in the functions g(E).

Theorem I:

Let g(E) satisfy (9.7) and (9.8) and be analytic in ImE > 0. Furthermore, let g(E) have no zeros for $|E| > E_1$. If in addition for large real $E > E_1$, we have the inequality

$$\left|\frac{\mathrm{Im} g}{\mathrm{Re} g}\right| \geq \tan \pi\alpha \ , \quad 0 < \alpha \leq \tfrac{1}{2} \ , \qquad (9.10)$$

then starting with some $E_o \gg E_1$, we have for $E \gg E_o$,

$$|g(E)/g(E_o)| \leq C\left(\frac{E_o}{E}\right)^{\alpha/2} \ , \qquad (9.11)$$

(actually when we have oscillations, (9.11) holds only on a sequence of intervals extending to infinity).

Proof:

In this case if we draw straight lines through the origin (in the previous figure) with slope tan $\pi\alpha$ and -tan $\pi\alpha$, then Γ_2 will lie above one line and Γ_1 below the other as shown.

It suffices to consider the case $\alpha = \frac{1}{2}$ since other cases can be reduced to it by considering the function $g' = g^{\frac{1}{2\alpha}}$.

For $\alpha = \frac{1}{2}$ we have

$$\rho(u) \geq (u^2 + |g(E)|^2)^{\frac{1}{2}} \ , \qquad (9.12)$$

with $u_E \leq u \leq u_o$. Thus we have

$$\int_{u_E}^{u_o} \frac{du}{\rho(u)} \leq \int_{o}^{u_o} \frac{du}{(u^2+|g(E)|^2)^{\frac{1}{2}}} = \ln\left[\frac{u_o+(u_o^2+|g(E)|^2)^{\frac{1}{2}}}{|g(E)|}\right] \qquad (9.13)$$

Combining this with (9.9) we get

$$\frac{|g(E)|}{u_o + (u_o^2+|g(E)|^2)^{\frac{1}{2}}} \leq C\left(\frac{E_o}{E}\right)^{\frac{1}{4}} \ , \qquad (9.14)$$

and hence

$$|g(E)| \leq C\left(\frac{E_o}{E}\right)^{\frac{1}{4}} . \quad \text{Q.E.D.} \qquad (9.15)$$

Remark:

Even though we always choose $E \gg E_o$ and E_o large, it is still possible that for some real E', $E_o \ll E' < E$, we have

$$|g(E')| \leq |g(E)| \ ; \qquad (9.16)$$

i.e. E' is the point at which $|g(E)|$ takes its minimum value. In this case (9.12) reads $\rho(u) \geq (u^2+|g(E')|^2)^{\frac{1}{2}}$. One then gets instead of (9.15) the inequality

$$|g(E')| \leq C\left(\frac{E_o}{E}\right)^{\frac{1}{4}} , \tag{9.17}$$

hence since E' < E, we get

$$|g(E')| \leq C\left(\frac{E_o}{E'}\right)^{\frac{1}{4}} . \tag{9.18}$$

We see here that if $|g(E)|$ oscillates (9.15) holds only on a sequence of points.

Theorem II:

If g(E) satisfies the same conditions as in theorem I but instead of (9.10) we have,

$$|\text{Im } g(E)| \geq C|\text{Re } g(E)|^{\nu}, \quad C > 0, \ \nu > 1, \tag{9.19}$$

for large real E; then starting with some E_o we have,

$$|\text{Im } g(E)/\text{Im } g(E_o)| \leq \left(1 + \tfrac{1}{4} C'(\nu-1)\ell n \frac{E}{E_o}\right)^{\frac{-\nu}{\nu-1}}, \tag{9.20}$$

where C' is a constant.

For a proof see the Appendix of Ref. 23.

C. Applications to the Even Amplitude

We now show how the two theorems just stated can be used to derive phase relations for the even amplitude F(E) defined in (9.1).

First we recall the rigorous properties of F(E).

(i) Analyticity in cut E-plane and continuity on boundary.

(ii) $F(E+i0) = F^*(E-i0)$; (reality).

(iii) $F(-E-i0) = F(E+i0)$; (crossing).

(iv) $|F(E)| \leq C\ |E|\ \ln^2 |E|$; (Froissart Bound).

Theorem A:

If F(E) satisfies i) - iv) and if for large real E

$$\left|\frac{\text{Re}F}{\text{Im}F}\right| \geq \tan \pi\alpha \ ; \quad 0 < \alpha \leq \tfrac{1}{2} \ ; \tag{9.21}$$

then for large enough $|E|$,

$$|F(E)| \leq C\ |E|^{1-\alpha/2}\ (\ell n|E|)^2. \tag{9.22}$$

Proof:

We define the function w(E) as

$$w(E) = \frac{F(E)}{iE\left[\ell nE - \frac{i\pi}{2}\right]^{\gamma}} \quad , \quad \gamma > 2 \ ; \tag{9.23}$$

and $0 \leq \arg E \leq \pi$.

By construction w(E) is analytic in the upper half E-plane excluding the unit half disc around the origin. Furthermore, $w(E) \to 0$ as $|E| \to \infty$ in all directions. It also follows from iv) and the dispersion relations for F(E) that F(E)/E has no zeros for $|E|$ greater than some constant E_1. Thus w(E) has no zeros for sufficiently large $|E|$ in the upper half plane and also $\mathrm{Re}\, w \geq 0$ in that region. From unitarity we have $\mathrm{Im}F(E) > 0$ for $E \geq \mu$; hence along the positive real axis Re w(E) is positive for large E.

The reality condition (ii) and the crossing relation (iii) give us

$$\begin{aligned} \mathrm{Re}\, w(E+i0) &= \mathrm{Re}\, w(-E+i0) , \\ \mathrm{Im}\, w(E+i0) &= -\mathrm{Im}\, w(-E+i0) . \end{aligned} \tag{9.24}$$

For large E we have,

$$\frac{\mathrm{Im}\, w(E)}{\mathrm{Re}\, w(E)} \cong - \frac{\mathrm{Re}F}{\mathrm{Im}F} , \tag{9.25}$$

and hence

$$\left|\frac{\mathrm{Im}\, w(E)}{\mathrm{Re}\, w(E)}\right| \geq \tan \pi\alpha. \tag{9.26}$$

Thus w(E) satisfies all the conditions of theorem I of Meiman, and we get for large enough real E

$$\left|\frac{w(E)}{w(E_o)}\right| \leq C \left(\frac{E_o}{E}\right)^{\alpha/2} . \tag{9.27}$$

This gives us

$$|F(E)| \leq C\, |E|^{1-\frac{\alpha}{2}} (\ell n|E|)^{\gamma};\ \gamma > 2. \tag{9.28}$$

For the total cross section we have

$$\sigma_{tot}(E) \leq C\, E^{-\alpha'/2} \ ; \quad \alpha' < \alpha. \tag{9.29}$$

This completes the proof of theorem A. If the cross section tends to a nonvanishing value or decreases more slowly than any negative power of E as $E \to \infty$, then $|\mathrm{Re}F/\mathrm{Im}F|$ must tend to zero in that limit.

Again for the mathematically afflicted we stress that (9.28) and (9.29) hold only on a sequence of points in general.

The cases where $|\mathrm{Re}F/\mathrm{Im}F| \to 0$ as $E \to \infty$ have been treated in detail in Ref. 23. We state here one more theorem to give a flavor of the results.

Theorem B:

If F(E) satisfies (i) - (iv) and if for large real E

$$|\mathrm{Re}F/\mathrm{Im}F| \geq \frac{C}{(\ell n E)^a} , \quad 0 < a < 1, \tag{9.30}$$

then for large $|E|$

$$|F(E)| \leq C\, |E|\, (\ell n|E|)^{-\lambda}, \quad \lambda > 0. \qquad (9.31)$$

Here λ can be chosen arbitrarily large.

In this case the total cross section decreases faster than any inverse power of ℓnE. The proof of this theorem depends on using theorem II of Meiman. The details are in Ref. 23, where also more theorems of this type are given.

X. ASYMPTOTIC PHASE RELATIONS AND UNIVALENT FUNCTIONS

The theorems and inequalities proved in the previous chapter could be improved and strengthened if the symmetric amplitude, F(E) is also univalent. However, one can easily show that this is not true in general, namely F(E) is not necessarily univalent. On the other hand as we shall see below, we can easily construct from F(E) functions that are univalent in the upper half E-plane. We then use the strong restrictions on univalent functions to derive some asymptotic phase relations for these new functions. The details of this chapter can be found in Ref. 25.

A. Mathematical Preliminaries

A function, f(z), analytic in a domain D is called univalent in D if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$ for any z_1, z_2 in D. Locally any analytic function f(z) is univalent in some neighborhood of a point z_0 if $f'(z_0) \neq 0$. A theorem on univalent functions states that if f(z) is regular for z in a simple domain D, and $f'(z) \neq 0$ for all z in D, and if the simple boundary curve of D is mapped by f(z) into a simple boundary curve (i.e., with no double points) of the image domain in the f-plane, then f(z) is univalent in D.

We now wish to construct a univalent function from the function F(E) defined by Eq. (9.1) and satisfying the properties i)...iv) given above theorem A of Chapter IX.

We write a twice subtracted dispersion relation for F(E),

$$F(E)-F(0) = \frac{2E^2}{\pi} \int_{\mu}^{\infty} dE' \, \frac{\mathrm{Im}F(E')}{E'(E'^2-E^2)} \quad . \tag{10.1}$$

We define the new function H(E) as,

$$H(E) = \frac{F(E)-F(0)}{E} \quad . \tag{10.2}$$

This function H(E) has the following properties:

a) H(E) is regular for $\mathrm{Im}E > 0$ and continuous for $\mathrm{Im}E \geq 0$.

b) H(E) is a Herglotz function, i.e., $\mathrm{Im}H(E) > 0$ when $\mathrm{Im}E > 0$.

c) $H(i\lambda)$, $\lambda > 0$ is purely imaginary and positive.

d) $\mathrm{Re}H(E+i0) = -\mathrm{Re}H(-E+i0)$, and $\mathrm{Im}H(E+i0) = \mathrm{Im}H(-E+i0)$. Thus the function H(E) maps the upper half E-plane into a domain in the upper half H-plane. This mapping is of course not necessarily one to one.

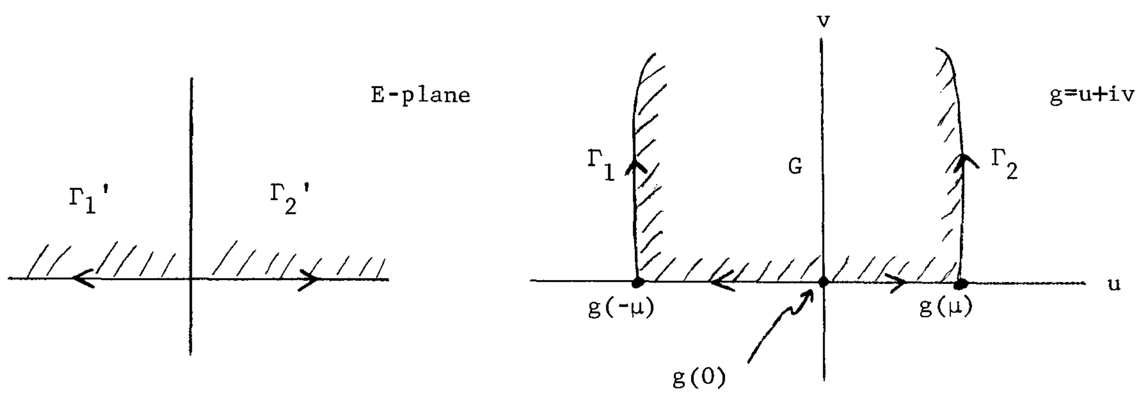
To get a univalent function we write

$$g(E) = \int_0^E \frac{H(E')}{E'} \, dE' \; ; \qquad \mathrm{Im}E \geq 0 \quad ; \tag{10.3}$$

where the integration contour is taken to be in the upper half E-plane. The integral is convergent at E'=0 since crossing tells us that $F'(0) = 0$.

One can easily check that g(E) has the following properties: (a) g(E) is regular in ImE $>$ 0 and continuous in ImE $\geq$ 0, (b) Img(E) $>$ 0 if ImE $>$ 0, (c) dg/dE $\neq$ 0 for all E such that ImE $>$ 0, (d) Img(-E+i0) = Img(E+i0) and Reg(E+i0) = -Reg(-E+i0), (e) for E $\geq \mu$, Img(E+i0) is non-negative and increases monotonically with E as E increases along the positive real axis, (f) Reg(E+i0) is non-negative and increases monotonically in the interval $0 \leq E \leq \mu$.

As is seen from (b), g(E) maps the upper half E-plane into a domain G located in the upper half g-plane. We know from (c) that this mapping is locally one-to-one everywhere in the upper half E-plane. The mapping will be globally univalent if the boundary curve of G does not have double points. Let us denote by Γ_1 and Γ_2 the images of the negative and positive real E-axes respectively. We know from (f) that the part of Γ_2 corresponding to $0 \leq E \leq \mu$, does not intersect with itself and lies on the positive real g-axis as shown in the figure below.



For E $> \mu$, g(E) becomes complex and the corresponding part of Γ_2 goes away monotonically from the real g-axis according to (e). Thus Γ_2 cannot have any double points. The same holds for Γ_1. Hence the only remaining possibility is that Γ_1 and Γ_2 have some common points. Because of the monotonicity and the symmetry the only possible common points of Γ_1 and Γ_2 can be found only on the imaginary g-axis. One can easily show from (10.1) that Reg(E+i0) $>$ 0 for E $> \mu$. Hence the only possible common point is $g(\infty)$. If $g(\infty)$ is finite Γ_1 and Γ_2 meet on the v-axis. If $g(\infty)$ is infinite they continue going up. Thus the boundary curve of G has no double point which proves the univalence of g(E) in the upper half E-plane. (Actually, to handle the behavior at infinity properly requires a few more detailed arguments. Those interested should consult Ref. 25).

One may ask what does one gain by going from F(E) to the univalent function g(E)? We give a few examples of the sort of restrictions on the possible behavior of g(E) which are imposed by some theorems on univalent functions.

For this purpose we introduce the new variable z defined by

$$z = \frac{E-i\lambda}{E+i\lambda} \quad ; \quad \lambda > 0 \quad . \tag{10.4}$$

This function maps the upper half E-plane onto a unit circular disc, $|z| < 1$. We also define, for fixed λ, the function

$$\varphi(z) = \frac{g(E)-g(i\lambda)}{2i\lambda g'(i\lambda)} \quad . \tag{10.5}$$

By construction $g'(i\lambda)$ is never zero for $\lambda > 0$. Thus $\varphi(z)$ is regular and univalent in the unit disc, $|z| < 1$, and its power series has the normalized form

$$\varphi(z) = z+a_2z^2+a_3z^3+ \ldots \quad . \tag{10.6}$$

Furthermore, one can easily check that $\varphi(z)$ is real if and only if z is real, $|z| < 1$. For such univalent functions one can prove that the coefficients a_n in (10.6) satisfy the inequality

$$|a_n| \leq n \quad ; \quad n = 2,3,4,\ldots \quad . \tag{10.7}$$

This puts upper bounds on all derivatives of g(E) at $E = i\lambda$ which depend only on λ and $g'(i\lambda)$. Although it is unlikely that the bounds (10.7) are of direct practical use, they might be useful in some theoretical considerations.

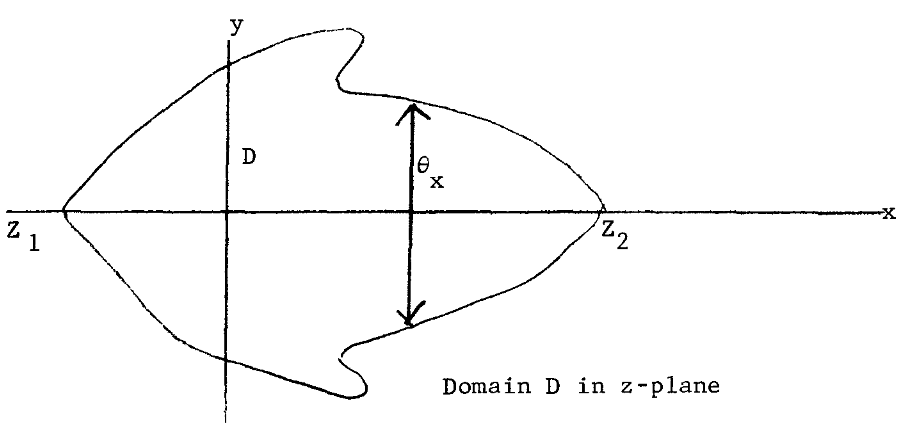
Another possibly more practical inequality follows from a theorem of Koebe. It gives us

$$|\varphi(e^{i\theta})| \geq \frac{1}{4} \quad , \quad 0 \leq \theta \leq 2\pi \quad . \tag{10.8}$$

B. Asymptotic Phase Theorems to g(E)

The fact that g(E) is univalent allows us to use much more powerful inequalities than that of Meiman given in the previous chapter. The main results of Ref. 25 follow from using a theorem due to Ahlfors. We start by stating that theorem.

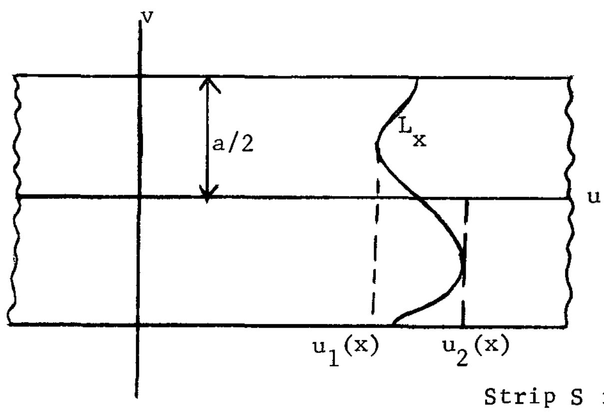
We consider a simple domain D in the z-plane ($z = x+iy$) which is simply connected and symmetric with respect to the x-axis. Let $Z_1 = Y_1+iY_1$ and $Z_2 = X_2+iY_2$ be the points on the boundary curve of D with the smallest and largest real part, respectively.



Domain D in z-plane

For any x, $X_1 < x < X_2$, the vertical line Rez = x will have one or more intersections with D, each of which bisects D into two disconnected parts. Under our assumptions on D, there is one intersection which crosses the x-axis. This line segment we denote by θ_x and its length by $\theta(x)$ as shown above. The line segment θ_x divides D into two disconnected parts in such a way that Z_1 and Z_2 belong to different parts. We require that $\theta(x)$ is a continuous function of x for $x_1 < x < x_2$ except at some isolated points.

Let $w = u+iv = w(z)$ be a function which is regular and univalent in D and maps D conformally onto a strip S defined by $|v| < \frac{1}{2}a$, $a > 0$, in such a way that $u(Z_1) \to -\infty$ and $u(Z_2) \to +\infty$



Strip S in w-plane

In this mapping the line segment θ_x will be mapped onto a continuous curve L_x which connects the two boundary lines $v = \pm\frac{1}{2}a$. The largest and smallest values of u on L_x are denoted by $u_2(x)$ and $u_1(x)$ respectively, as shown above. The theorem of Ahlfors now states that

$$u_1(x_2)-u_2(x_1) \geq a\int_{x_1}^{x_2}\frac{dx}{\theta(x)} - 4a \qquad (10.9)$$

holds for any pair $x_2 > x_1$ such that

$$\int_{x_1}^{x_2}\frac{dx}{\theta(x)} > 2 \quad . \qquad (10.10)$$

We now use the Ahlfor's theorem to prove the following results.

Theorem 1:

If g(E) for large real E satisfies

$$\frac{\text{Reg}(E)}{\text{Img}(E)} \geq \tan\pi\alpha\ , \quad 0 < \alpha < \frac{1}{2} \quad , \qquad (10.11)$$

then g(E) has the lower bound for $|E| > E_0$

$$|g(E)| \geq C(\frac{E}{E_0})^{2\alpha} \quad , \qquad 0 < \alpha < \frac{1}{2} \quad . \tag{10.12}$$

Proof:

We define z and w(z) by

$$z = \ln(\frac{E}{C}) - i\pi/2 \ , \qquad C > 0 \quad ,$$

$$w(z) = \ln g(E) - i\pi/2 \quad . \tag{10.13}$$

We apply Ahlfor's theorem to the mapping $z \to w$. The lines Reg/Img = $\pm \tan\pi\alpha$ correspond to the two straight lines in the w plane which are parallel to the u-axis ($w = u+iv$) and separated by the distance $a = 2\alpha\pi$. We choose D to be the domain whose boundary curve consists of two vertical line segments, $\text{Re}z = x_1 = \ln(\frac{E_2}{C})$ and $\text{Re}z = x_2 = \ln(\frac{E_2}{C})$, and two Jordan curves which are the inverse maps of the two parallel lines in the w-plane mentioned above, by the inverse transformation $z = z(w)$. It is obvious to check in this case that by definition

$$\theta(x) \leq \pi \ . \tag{10.14}$$

Thus from the Ahlfor's inequality we obtain

$$\begin{aligned} u_1(x_2)-u_2(x_1) &\geq 2\alpha\pi(\frac{x_2-x_1}{\pi} - 4) \\ &= 2\alpha \ln \frac{E_2}{E_1} - 8\pi\alpha \quad . \end{aligned} \tag{10.15}$$

From the definition of $u_1(x_2)$ we know that

$$\ln|g(|E_2|e^{i\varphi})| \geq u_1(x_2), \qquad \epsilon \leq \varphi \leq \pi-\epsilon \quad . \tag{10.16}$$

If we choose $\varphi = \pi/2$ we get

$$|g(i|E_2|)| \geq C(\frac{E_2}{E_1})^{2\alpha} \quad , \tag{10.17}$$

for $E_2 \gg E_1$. In Appendix C of Ref. 25 one show that

$$|g(E)| > \frac{1}{(2)^{\frac{1}{2}}} |\text{Im}g(iE)| \quad , \tag{10.18}$$

for real positive E. This with (10.17) completes the proof.

Theorem 2:

If g(E) satisfies

$$\frac{\text{Re}g(E)}{\text{Im}g(E)} \leq \tan\pi\alpha' \quad , \qquad 0 < \alpha' < \frac{1}{2} \quad , \tag{10.19}$$

for $E > E_0$ then g(E) has the upper bound

$$|g(E)| \leq C(\frac{E}{E_0})^{2\alpha'} \quad . \tag{10.20}$$

The proof of this theorem is given in Ref. 25 and is very similar to that of Theorem 1 except that one reverses the definitions of z and w(z).

Several other theorems follow from the Ahlfor's inequality and the reader is referred to Ref. 25 for details. We close by mentioning one such result.

If for sufficiently large real E we have

$$\text{Re}g(E) \leq b \ , \qquad b > 0 \quad , \tag{10.21}$$

where b is a positive constant, then g(E) has the following upper bound

$$|g(E)| \leq C \ \ell n|E| \quad . \tag{10.22}$$

This essentially means that the total cross-section is bounded by a constant for large energies. Now if, for example, ReF(E) is negative for all $E \geq E_0$ where E_0 is some fixed large energy, then clearly (10.21) will be satisfied for all $E > E_0$ with $b = \text{Re}g(E_0)$. Thus if the forward amplitude is repulsive at high energies as $E \to \infty$ then the total cross section cannot go to infinity or more precisely, $\int_\mu^E \sigma(E')/E'dE'$ does not diverge faster than ℓnE as $E \to \infty$.

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