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Article

STABILITY CONDITIONS IN GEOMETRIC INVARIANT THEORY

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ABSTRACT

We explain how structures analogous to those appearing in the theory of stability conditions on abelian and triangulated categories arise in geometric invariant theory. This leads to an axiomatic notion of a central charge on a scheme with a group action and ultimately to a notion of a stability condition on a stack analogous to that on an abelian category. In the appendix by Ibáñez Núñez, it is explained how central charges can be viewed through the graded points of a stack. We use these ideas to introduce an axiomatic notion of a stability condition for polarized schemes, defined in such a way that K-stability is a special case. In the setting of axiomatic geometric invariant theory on a smooth projective variety, we produce an analytic counterpart to stability and explain the role of the Kempf–Ness theorem. This clarifies many of the structures involved in the study of deformed Hermitian Yang–Mills connections, Z-critical connections and Z-critical Kähler metrics.

1. INTRODUCTION

Mumford's geometric invariant theory (GIT) gives a method for constructing quotient spaces in algebraic geometry, with many important applications to the construction of moduli spaces [**39**]. These quotients parameterize *polystable* orbits—the unstable orbits are discarded to ensure a separated quotient.

Perhaps the most powerful outcome of Mumford's work was not GIT itself, but rather the introduction of the notion of *stability*, which has been fundamental to an enormous amount of further work. We mention two examples. The first is the notion of *slope stability* of a coherent sheaf, which led to Rudakov's abstraction to stability on general *abelian categories* (where there is no possible interpretation via GIT) [**40**] and ultimately to Bridgeland's very general theory of stability conditions on *triangulated categories* [**8**], building on ideas of Douglas motivated by string theory [**19**]. Here GIT is used more as a motivational philosophy rather than as a direct tool.

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The second example we give is Mumford's construction of the moduli space of (stable) curves [38], and its higher-dimensional (and largely conjectural) analogue of *K-stability* of a polarized variety [46, 18]. Here again while Mumford's theory can successfully be applied in the special case of curves, in higher dimensions K-stability is only *modelled on* GIT, which is similarly used as a motivational philosophy.

Bridgeland's notion of a stability condition is the appropriate one in the presence of some linearity, essentially due to the necessity of the presence of abelian categories, but many interesting problems in algebraic geometry have no such linearity; this is notably the case for the theory of stability of polarized varieties. Roughly speaking, in Bridgeland's theory stability is only defined for certain abelian subcategories $\mathcal{A}$ of the triangulated category (satisfying various hypotheses), where stability involves choosing a *central charge* $Z : \mathcal{A} \to \mathbb{C}$ and demanding that for subobjects $S \subset E$ the *phase inequality*

$$\arg Z(E) > \arg Z(S)$$

holds (with arg denoting the argument of a complex number); this type of inequality is precisely of a form analogous to those arising in the traditional theory of slope stability of coherent sheaves. It is then important to assume that central charges are complex valued, as this is a basic step in Bridgeland's proof that the space of stability conditions on a triangulated category forms a complex manifold. Just as Bridgeland's theory gives a general way of understanding stability in the presence of a sort of linearity, it is natural to ask if one can extend the theory beyond the linear setting.

This note revisits some of the foundational ideas of GIT, with the aim of developing a general parallel of Bridgeland's work. We accomplish roughly the easier half of this: motivated by a new notion of a central charge associated with a group action on a projective scheme, we introduce a notion of a central charge on a general stack, motivated by Halpern–Leistner's approach to GIT on stacks [25]. When this stack is the stack of coherent sheaves, we then explain how this relates to the classical notion of a central charge. Our real interest, however, is in the stack of polarized schemes: here we use this to define an *axiomatic notion of stability for polarized schemes*. As we explain, K-stability is then a special case of *Z-stability*, with Z being a central charge. An *ad hoc* notion of stability for polarized varieties was introduced in [14], and our motivation here is to give a more axiomatic approach to essentially the same notion.

In the appendix Ibáñez Núñez, the relationship with Halpern–Leistner's notion of the stack of graded points of a stack is discussed and explained in detail, and a thorough stack-theoretic treatment of the notion of a central charge is given. These results emphasize that central charges are natural objects associated with stacks. In addition, it is observed there that the space of central charges naturally has the structure of a complex vector space, proving a basic (loosely, abelian) counterpart to the complex-manifold structure of the space of stability conditions on a triangulated category.

We emphasize that this only solves half of the problem, really giving a stacky analogue of Rudakov's notion of stability on an abelian category. To generalize Bridgeland's theory is considerably more challenging—the closest analogy is the requirement to extend from coherent sheaves to *complexes* of coherent sheaves, and it is not clear what the right analogue of a complex should be for more general stacks—especially the stack of polarised schemes. The author expects that the right categorical generalization of K-stability—parallel to stability conditions on triangulated categories—should involve a larger overlying categorical structure, with stability defined then for appropriate substacks of this larger structure. On these substacks, stability should precisely require choosing a central charge on the stack, with stability then meaning what we introduce in the present work.

While this more categorical generalization of K-stability is speculative, general notions of stability are of interest even for a fixed polarised variety, where the lack of 'global' structures analogous to Bridgeland stability conditions should be less problematic. The reason for this interest is in links with geometric partial differential equations and moment maps, as we explain in more detail further. Developing the theory purely algebraically appears to be very challenging, and we leave this for future work; for example—away from the important special case of Fano varieties [47]—basic questions such as Zariski openness of the stable locus are completely open even for K-stability of polarized varieties.

1.1. Moment maps

Beyond the introduction of the general notion of stability in algebraic geometry, the most powerful outcomes of Mumford's GIT arose through resulting links between algebraic and differential geometry. These arise through the *Kempf–Ness theorem* [33, 34], which states that *polystability* of an orbit can be characterized through the existence of zeroes of *moment maps*, as used in *symplectic* geometry. This characterization then descends to a homeomorphism of the algebraic and symplectic quotients by Kirwan [34]. The notions of slope stability of holomorphic vector bundles and K-stability of smooth polarised varieties—which can be thought of as infinite-dimensional analogues of stability in GIT—then have analytic counterparts through the existence of *Hermitian Yang–Mills connections* and *constant scalar curvature Kähler metrics*, which can each be viewed as zeroes of infinite-dimensional moment maps [3, 17, 21]. Thus these moment map equations give differential-geometric approaches to moduli problems in algebraic geometry.

The second goal of this note is to give—in light of the axiomatic notion of a central charge in GIT—a differential-geometric counterpart to central charges and resulting stability conditions, motivated by (and using) the Kempf–Ness theorem on a smooth projective variety. For a central charge Z we define a *complex moment map* and use it to define *Z-critical points*, and under a strong 'subsolution' hypothesis we prove that the existence of Z-critical points is equivalent to Z-polystability—the axiomatic notion of stability defined using the central charge.

Infinite-dimensional counterparts of this idea have become increasingly prominent in recent years, notably for equations that arose in string theory and mirror symmetry. The most well-understood of these is the *deformed Hermitian Yang–Mills equation*, which is the mirror of the *special Lagrangian equation* under SYZ mirror symmetry [35, 37]. The deformed Hermitian Yang–Mills equation roughly corresponds to stability of coherent sheaves with respect to a certain central charge, and for more general central charges one obtains the condition for a connection on a holomorphic vector bundle to be a *Z-critical connection*, as introduced by McCarthy and Sektnan [16]. A parallel, more challenging analogue for smooth polarized varieties was introduced in [14, 15], through the theory of *Z-critical Kähler metrics*, linked with the notion of *Z-stability* of the polarized variety [14]. While—with the notable exception of deformed Hermitian Yang–Mills connections on line bundles [9, 10, 31]—it remains an open problem to characterize the existence of solutions of these partial differential equations in terms of the associated notions of stability, the present work gives a finite-dimensional analogue of these results which does provide evidence for generalizations of the Kempf–Ness theorem to the aforementioned infinite-dimensional settings. In joint work with Hallam providing a sequel to the present note, a geometric recipe is given which canonically associates complex moment maps with central charges in the categories of polarized manifolds and holomorphic vector bundles, building on some of the ideas presented here [15].

In both the variety and coherent sheaf settings, the Z-critical equation can be associated with very general classes of central charges, but basic properties—such as ellipticity of the partial differential equation—cannot hold in complete generality. This is best understood in the (smooth) coherent sheaf setting, where the *Z-subsolution* condition governs ellipticity of the equation along with various other geometric properties of the equation [10, 16, 44]. This subsolution condition—which as we explain here has a finite dimensional analogue—is a positivity condition, and it seems plausible to suggest that these notions are related to the various axioms of Bridgeland's stability conditions, which themselves can be thought of as positivity conditions (as an explicit example, the existence of Harder–Narasihman filtrations assumed in the definition of a Bridgeland stability condition is a consequence of a kind of convexity, which is in turn a consequence of a kind of positivity).

The analytic state-of-the-art in both the variety and coherent sheave cases is closer to stability on an abelian category than a triangulated one, and it remains an important problem to extend the notion of a Z-critical connection to more general complexes. The manifold setting is more mysterious again, but the perspective presented here, along with the parallels between the coherent sheaf and variety theories, provides analytic motivation for the appearance of an analogue of the derived category of coherent sheaves in the setting of polarized schemes. That is, given that one should expect the need to consider the triangulated setting and categorical obstructions to fully understand the existence of Z-critical connections in relation to Bridgeland stability, to understand the existence of Z-critical Kähler metrics one should also need to understand deeper categorical obstructions. Optimistically one may expect to be able to actually *obtain* the right categorical extensions from a deep understanding of the analysis underlying the Z-critical Kähler condition, much as one would hope to be able to recover the notion of Bridgeland stability through understanding the analysis underlying the Z-critical connection condition. We also remark that there is evidence that Bridgeland stability may not be exactly the right condition to be equivalent to the existence of deformed Hermitian Yang–Mills connections [**11**, **30**], with this philosophy then suggesting that one may be able to obtain a similar algebro-geometric stability condition through the analysis underlying the deformed Hermitian Yang–Mills condition: the stability condition should be the one that is equivalent to existence of solutions of this equation.

In any case, our work here gives a finite-dimensional motivation for these geometric partial differential equations and, in particular, gives a dictionary for passing from standard moment map constructions—such as geometric flows—to interesting infinite-dimensional ones. We thus emphasize that the main purpose of this note is to use GIT to understand the right general axiomatic notion of stability for polarized schemes—and for general stacks—and to highlight how this viewpoint can be used to motivate the appearance of differential-geometric counterparts to abstract stability conditions. There are relatively few proofs, with the focus instead being on providing definitions which we hope will motivate further work. We also hope that these ideas clarify many of the basic structures appearing in recent work on general relationships between abstract stability conditions and geometric partial differential equations.

1.2. Remark

Independent forthcoming work of Haiden–Katzarkov–Kontsevich–Pandit aims to develop an analytic counterpart to Bridgeland stability for general triangulated categories; they title their programme 'categorical Kähler geometry' (see [**23**, **24**] for precursors). Their programme has a substantial overlap with the ideas developed here, and indeed the Z-critical condition used here bears some similarity to equations that appear in their work. The author thanks Haiden and Pandit for discussions on these ideas.

1.3. Outline

We discuss stability in GIT in Section 2.1, before turning to the axiomatic notion of Z-stability for group actions on schemes in Section 2.2 and on stacks in Section 2.3. Section 2.4 then explains the special cases of stability conditions on the stacks of coherent sheaves and polarized schemes. The analytic counterpart to Z-stability is described in Section 3 through the theory of complex moment maps, and we end with a discussion of various natural analytic structures in complex moment map theory in Section 3.2.

2. STABILITY CONDITIONS IN GIT

2.1. The classical theory

Let (X, L) be a polarized projective scheme, and let G be a reductive group acting on (X, L). We briefly recall some of the basic ideas of GIT, which produces a quotient $X /\!\!/ G$ of X by G. The original reference is Mumford's book [**39**], and good surveys are given by Hashimoto [**27**] and Thomas [**45**]. Rather than parameterizing all orbits, the quotient represents only *polystable* orbits, which we now define.

Let $\lambda : \mathbb{C}^* \hookrightarrow G$ be a one-parameter subgroup. For a given $x \in X$, we call

$$y = \lim_{t\to 0} \lambda(t).x$$

the *specialization* of x under λ; we also say that x *degenerates* to y under λ. As y is fixed by the $\mathbb{C}^*$-action, there is a $\mathbb{C}^*$-action on the one-dimensional complex vector space L_y; this action is multiplication by

$t^{\nu(y,\lambda)}$ for an integer $\nu(y,\lambda) \in \mathbb{Z}$ called the *weight*. We think of the weight as an assignment

$$(y,\lambda) \to \nu(y,\lambda) \in \mathbb{Z},$$

where $\lambda : \mathbb{C}^* \hookrightarrow G_y$ is a one-parameter subgroup of the stabilizer G_y of y.

DEFINITION 2.1 We say that $x \in X$ is

(1) *semistable* if for all one-parameter subgroups λ of G, we have $\nu(y,\lambda) \geq 0$;
(2) *polystable* if for all one-parameter subgroups λ of G, we have $\nu(y,\lambda) \geq 0$, with equality if and only if $\lim_{t\to 0} \lambda(t).x \in G.x$;
(3) *stable* if for all non-trivial one-parameter subgroups λ of G, we have $\nu(y,\lambda) > 0$;
(4) *unstable* otherwise.

Here in each case y is the specialization of x under λ. These are conditions on the orbit $G \cdot x$, so we also have that the orbit $G \cdot x$ is semistable, polystable or stable, respectively.

The general machinery of GIT produces a quotient space $X /\!\!/ G$ which parameterises *polystable orbits*. The first important step in this process is that both the stable locus X^s and the semistable locus X^{ss} are Zariski open. The next step is to prove the existence of a surjective morphism $X^{ss} \to X /\!\!/ G$ such that distinct polystable orbits are mapped to distinct points in $X /\!\!/ G$. This map is well-defined is equivalent to the fact that the closure of each semistable orbit contains a *unique* polystable orbit.

REMARK 2.2 This is not how stability in GIT is usually presented; rather, it is an equivalent characterization provided by the Hilbert–Mumford criterion. More typically GIT is presented instead using invariant global sections of L^k, from which constructing the GIT quotient is almost tautological via the Proj construction. The Hilbert–Mumford criterion then gives a way of geometrically interpreting polystable orbits [39, Section 2].

It will be useful to relate one-parameter subgroups to associated elements of the Lie algebra $\mathfrak{g}$ of G. First consider rst a maximal (complex) torus $T^{\mathbb{C}} \subset G$ with Lie algebra $\mathfrak{t}^{\mathbb{C}}$. The *cocharacter lattice* has points consisting of the kernel of the exponential map $\exp : \mathfrak{t}^{\mathbb{C}} \to T^{\mathbb{C}}$; these are in bijection with the one-parameter subgroups $\mathbb{C}^* \to T^{\mathbb{C}}$. Splitting $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$, the $\mathbb{R}$-span of the cocharacter lattice is the real Lie algebra $\mathfrak{t}$ of the torus T, and the $\mathbb{C}$-span is the Lie algebra $\mathfrak{t}^{\mathbb{C}}$ itself. We denote by $\mathfrak{t}_{\mathbb{Q}} \subset \mathfrak{t}$ the $\mathbb{Q}$-span of the cocharacter lattice, so that the $\mathbb{Q}$-vector space $\mathfrak{t}_{\mathbb{Q}} \oplus i\mathfrak{t}_{\mathbb{Q}} \subset \mathfrak{t}^{\mathbb{C}}$ is a dense subset. We call the spaces $\mathfrak{t}_{\mathbb{Q}}$ and $\mathfrak{t}_{\mathbb{Q}} \oplus i\mathfrak{t}_{\mathbb{Q}}$ the collection of *rational points* of $\mathfrak{t}$ and $\mathfrak{t}^{\mathbb{C}}$, respectively.

We next turn to the Lie algebra $\mathfrak{g}$, where we say that $u \in \mathfrak{g}$ is *rational* if there exists a maximal torus $T^{\mathbb{C}}$ of G such that $u \in \mathfrak{t}^{\mathbb{C}} \subset \mathfrak{g}$ is a rational point. We denote by $\mathfrak{g}_{\mathbb{Q}}$ the set of rational points of $\mathfrak{g}$, and note that while this is not a $\mathbb{Q}$-vector space in general, $\mathfrak{g}_{\mathbb{Q}}$ is nevertheless a dense subset of $\mathfrak{g}$. Similarly, writing G as the complexification of a maximal compact subgroup $K \subset G$, the Lie algebra $\mathfrak{g}$ splits as

$$\mathfrak{g} \cong \mathfrak{k} \oplus i\mathfrak{k},$$

where $\mathfrak{k} = \mathrm{Lie}K$. With an analogous definition of $\mathfrak{k}_{\mathbb{Q}}$, the $\mathbb{R}$-span of $\mathfrak{k}_{\mathbb{Q}}$ is $\mathfrak{k}$ itself, meaning that $\mathfrak{k}_{\mathbb{Q}} \subset \mathfrak{k}$ is again dense.

As the stabilizer G_x of $x \in X$ is not reductive in general, it is not the case that $\mathfrak{g}_{x,\mathbb{Q}}$ as defined above is dense in $\mathfrak{g}_x$ for all $x \in X$. Instead we fix a maximal compact subgroup $K_x \subset G_x$, and let $K_x^{\mathbb{C}} \subset G_x$ denote its complexification. We then repeat the above discussion for the reductive group $K_x^{\mathbb{C}}$ and denote by $\mathfrak{k}_{x,\mathbb{Q}}$ and $\mathfrak{k}_{x,\mathbb{Q}}^{\mathbb{C}}$ the rational points of $\mathfrak{k}_x$ and $\mathfrak{k}_x^{\mathbb{C}}$, respectively.

LEMMA 2.3 (Additivity) *The weight function extends to a Lie algebra character*

$$\nu(x,\cdot):\mathfrak{k}_x \to \mathbb{R}.$$

Similarly the weight function extends to a *complex*-valued Lie algebra character $\nu(x,\cdot): 1\mathfrak{k}_x^{\mathbb{C}} \to \mathbb{C}$. That the weight is additive on the cocharacter lattice implies that it extends linearly to a linear function $\mathfrak{k}_x \to \mathbb{R}$. Denoting by

$$g\cdot\lambda = g\circ\lambda\circ g^{-1}$$

the adjoint action of G on one-parameter subgroups, associated with a one-parameter subgroup $\lambda \hookrightarrow G_x$, there is then an induced one-parameter subgroup $g\cdot\lambda \hookrightarrow G_{g(x)}$. The conjugation-invariance property

$$\nu(g(x),\lambda) = \nu(x, g\cdot\lambda)$$

implies from this that ν is actually a Lie algebra character, namely, that $\nu(x,[\cdot,\cdot]) = 0$.

We also recall an equivariance property that is more global in nature. Consider a connected subscheme $B \subset X$ such that λ is a one-parameter subgroup contained in the stabilizer G_x of x for all (closed) points $x \in B$.

LEMMA 2.4 (Equivariant constancy) *The value $\nu(x,\lambda)$ is independent of $x \in B$.*

This property can be seen by equivariantly trivializing the restriction of L to an affine chart of B.

EXAMPLE 2.5 Fix a point $x \in X$, and consider two commuting one-parameter subgroups λ,γ of G. These induce a $(\mathbb{C}^*)^2$-equivariant morphism $\mathbb{C}^2 \to X$, where $\mathbb{C}^2$ is given the natural $(\mathbb{C}^*)^2$-action, such that the image of $(\mathbb{C}^*)^2$ is contained in $G.x$. The image of $\mathbb{C}^2$ is the affine toric variety produced by taking the partial closure of the $(\mathbb{C}^*)^2$-orbit $(\mathbb{C}^*)^2.x \subset X$. The closure of the $(\mathbb{C}^*)^2$-orbit thus contains fixed points of γ and λ, which we denote by $y_{\lambda,s}, y_{\gamma,t}$, respectively, where we have parameterized using coordinates s,t on $\mathbb{C}^2$. These families intersect at the image of the origin in $\mathbb{C}^2$: $y_{\gamma,0} = y_{\lambda,0}$, as follows from commutativity of λ,γ. Equivariant constancy then implies that $\nu(y_{\lambda,s},\lambda) = \nu(y_{\lambda,0},\lambda)$ and likewise for γ. This example is related to forthcoming work of Kirwan–Nanda on representations of 2-quivers.

2.2. Axiomatic stability conditions

We would like to axiomatize the key structures of GIT. Thus consider a projective scheme X given the action of a reductive group G. Denote by

$$\mathcal{S} = \{(y,\lambda) \mid y \in X \text{ and } \lambda \textit{ is a one}-\textit{parameter subgroup of } G_y\}/\sim,$$

where $(y,\lambda) \sim (z,\gamma)$ if there is $g \in G$ with $g(y) = z$ and $g\cdot\lambda = \gamma$. Thus $\mathcal{S}$ is an upgrade of the space of orbits in X by also remembering a one-parameter subgroup fixing the point in the orbit. There is a canonical injective map

$$X \to \mathcal{S}, \qquad x \to (x, \mathrm{Id}),$$

where Id denotes the trivial one-parameter subgroup of G.

The set $\mathcal{S}$ admits the structure of a directed graph, with vertices given by elements of $\mathcal{S}$ and arrows given by equivariant specialization. That is, we write

$$(x,\zeta) \rightsquigarrow (y,\lambda)$$

if λ commutes with ζ and y is the specialization of x under λ. We write $x \rightsquigarrow (y,\lambda)$ to mean that $(x,\mathrm{Id}) \rightsquigarrow (y,\lambda)$.

Lemma 2.6 *Equivariant specialization is well-defined on orbits.*

Proof. Let $(x,\zeta) \sim (z,\gamma)$, so that there is $g \in G$ with $(g(x), g\cdot\zeta) = (z,\gamma)$. If $(x,\zeta) \rightsquigarrow (y,\lambda)$, then

$$(z,\gamma) = (g(x), g\cdot\zeta) \rightsquigarrow (g(y), g\cdot\lambda) \sim (y,\lambda),$$

as required. □

Fixing a maximal compact subgroup K_x of G_x with Lie algebra $\mathfrak{k}_x$, as in Section 2.1, it is then useful to consider $\mathcal{S}$ as consisting of pairs (x,v), where $v \in \mathfrak{k}_x$ satisfies $\exp(v) = \mathrm{Id}$, meaning that v corresponds to a one-parameter subgroup of G_x.

Definition 2.7 A *central charge* is a function $Z : \mathcal{S} \to \mathbb{C}$ such that

(1) (Additivity) For a fixed $x \in X$, Z induces a function

$$Z(x,\cdot) : \mathfrak{k}_x \to \mathbb{C},$$

which is a Lie algebra character;

(2) (Equivariant constancy) Suppose $B \subset X$ is a connected subscheme such that λ is a one-parameter subgroup of G_x for each $x \in X$. Then $Z(x,\lambda)$ is independent of $x \in X$.

Additivity in particular asks that $Z(x,\cdot)$ is additive under compositions of commuting one-parameter subgroups, so that it canonically extends to a linear function on $\mathfrak{t}_x = \mathrm{Lie}\, T_x$ for any maximal torus $T_x \subset K_x$. That Z is well-defined on $\mathcal{S}$ implies that $Z(x,v) = Z(x, g\cdot v)$ for $v \in \mathfrak{k}_x$ and $g \in G_x$, so that its extension to $\mathfrak{t}_x$ induces an extension to $\mathfrak{k}_x$ canonically, since maximal tori are always conjugate.

We record a motivating property for the definitions given here, for which we take ζ, λ commuting one-parameter subgroups. Their composition $\zeta \circ \lambda$ is then well-defined and corresponds to exponentiating the sum of the associated elements of $\mathfrak{k}$. In the following we view Z as taking values on pairs of points and one-parameter subgroups.

Lemma 2.8 *Suppose* $(x,\zeta) \rightsquigarrow (y,\lambda)$. *Then*

$$Z(x,\zeta) + Z(y,\lambda) = Z(y, \zeta\circ\lambda).$$

Proof. Since

$$Z(y,\zeta\circ\lambda) = Z(y,\zeta) + Z(y,\lambda),$$

the claim follows from the claim that

$$Z(x,\zeta) = Z(y,\zeta),$$

which in turn follows from equivariant constancy. □

This notion of a central charge allows us to define stability, for which we must in addition fix a *phase* $\varphi \in (-\pi,\pi)$.

Definition 2.9 We say that $x \in X$ is

(1) *Z-semistable* if for all $x \rightsquigarrow (y,\lambda)$ we have

$$\mathrm{Im}(e^{-i\varphi}Z(y,\lambda)) \geq 0;$$

(2) *Z-polystable* if for all $x \rightsquigarrow (y,\lambda)$ we have $\mathrm{Im}(e^{-i\varphi}Z(y,\lambda)) \geq 0$ with equality if and only if $G.x = G.y$;
(3) *Z-stable* if x is polystable and in addition G_x is finite;
(4) *Z-unstable* otherwise.

Thus stability is measured relative to the chosen phase φ. Z-semistability and polystability can be defined similarly for points $(x,\zeta) \in \mathcal{S}$, and in this way Z-semistability of (x,ζ) asks for 'equivariant Z-semistability' of x, namely Z-semistability with respect to one-parameter subgroups commuting with ζ. In standard GIT, equivariant semistability is equivalent to semistability, and so we choose not to emphasize this slightly more general situation.

Remark 2.10 Under an additional hypothesis on φ and $Z(y,\lambda)$, we can rephrase the numerical inequality governing stability in a way more reminiscent of Bridgeland stability [8]. Let $\arg : \mathbb{C}\backslash\mathbb{R}_{\leq 0} \to (-\pi,\pi)$ denote the principal branch of the argument function and denote by

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0 \textit{ and } z \notin \mathbb{R}_{<0}\}.$$

Provided $Z(y,\lambda) \in \mathbb{H}$ we define the *phase* of $Z(x,\zeta)$ to be

$$\varphi(y,\lambda) = \arg((y,\lambda)) \in (-\pi,\pi).$$

Suppose further that $e^{-i\varphi}$ and $Z(y,\lambda)$ both lie in $\mathbb{H}$ (so that $\varphi \in [0,\pi)$). Then the condition

$$\mathrm{Im}(e^{-i\varphi}Z(y,\lambda)) \geq 0$$

holds if and only if the phase inequality

$$\varphi \geq \varphi(y,\lambda)$$

holds and similarly if one demands strict inequalities. In fact what is important is to fix a given half-plane in $\mathbb{C}$; the specific choice $\mathbb{H}$ is not essential, as stability is independent of simultaneously rotating the phase φ and the central charge Z by the same angle.

Remark 2.11 To obtain a working theory with standard properties—such as Zariski openness of the stable locus—one certainly needs further hypotheses on the central charge. What we have given here seems to be the minimum required to give a definition analogous to the usual definition of a central charge on an abelian category, as we discuss in Section 'Coherent sheaves', and to obtain a link with moment maps and differential geometry, as we discuss in Section 3.

2.3. Stability conditions on stacks

In Section 2.2 we considered a scheme X with the action of a reductive group G. Viewing this as producing a global quotient stack $[X/G]$, we next make analogous definitions for arbitrary stacks. What will be important is that—just as with $[X/G]$—we can speak of a point along with a one-parameter subgroup of its stabilizer. Our discussion will be rather formal, but to match with the hypotheses of the appendix—where a more thorough stack-theoretic treatment is provided—we assume that $\mathcal{X}$ is a quasi-separated algebraic stack, which is locally of finite type over $\mathbb{C}$ and has affine stabilizers.

Thus let $\mathcal{X}$ be such a stack. For $y \in \mathcal{X}$ we denote by $\mathrm{Aut}(y)$ the automorphism group (or stabiliser) of y. We then define

$$\mathcal{S} = \{(y,\lambda) \mid y \in \mathcal{X} \; and \lambda \; is \; a \; one-parameter \; subgroup \; of \; \mathrm{Aut}(y)\}/\sim,$$

where we say $(x,\lambda) \sim (y,\gamma)$ if there is an isomorphism $g : x \cong y$ such that $g \cdot \lambda = \gamma$, where $g \cdot \lambda$ denotes the one-parameter subgroup of the stabilizer G_y of y under the isomorphism $g : x \cong y$.

We endow $\mathcal{S}$ with the structure of a directed graph as follows. We begin with the simplest case, namely when we consider (x, Id), with Id denoting the trivial (identity) one-parameter subgroup. Here we say that

$$(x, \mathrm{Id}) \rightsquigarrow (y,\lambda)$$

if there is a morphism $\Psi : [\mathbb{C}/\mathbb{C}^*] \to \mathcal{X}$ such that $\Psi(1) = x$ and

$$\Psi(0,\mathbb{C}^*) = (y,\lambda);$$

that is, we consider $\mathbb{C}^* \hookrightarrow \mathrm{Aut}(0)$ for $0 \in \mathbb{C}$ and ask that this one-parameter subgroup is mapped to $\lambda \hookrightarrow \mathrm{Aut}(y)$.

To define what it means for there to exist a degeneration

$$(x,\zeta) \rightsquigarrow (y,\lambda),$$

we consider the quotient stack $[\mathbb{C}^2/(\mathbb{C}^*)^2]$ given the natural action. We choose coordinates (s,t) on $\mathbb{C}^2$ and $\mathbb{C}^*$-actions $\hat{\lambda}, \hat{\zeta}$, so that for $s = 1$ the point $(0,1)$ has stabilizer ξ and similarly for $t = 0$ the point $(1,0)$ has stabilizer $\hat{\zeta}$. The notation is suggestive: we will ask that $\hat{\lambda}, \hat{\zeta}$ map to λ, ζ, respectively. We then say that $(x,\zeta) \rightsquigarrow (y,\lambda)$ if there is a morphism $\Psi : [\mathbb{C}^2/(\mathbb{C}^*)^2] \to \mathcal{X}$ satisfying the following two conditions. Firstly, on points we ask that $\Psi(t,1) = x$ for all $t \in \mathbb{C}$ and $\Psi(t,0) = y$. Secondly, on stabilizers we ask that the $\mathbb{C}^*$-stabilizer $\hat{\zeta}$ of $(0,1)$ is mapped to $\zeta \subset \mathrm{Aut}(\Psi(0,1)) = \mathrm{Aut}(x)$, and we in addition ask that the stabilizer $\hat{\lambda}$ of $(t,0)$ is mapped to $\lambda \subset \mathrm{Aut}(\Psi(t,0)) = \mathrm{Aut}(y)$. The consideration of $\mathbb{C}^2$ is not essential, and one can instead consider the quotient stack $[\mathbb{C}/(\mathbb{C}^*)^2]$, where one copy of $\mathbb{C}^*$ lies in the stabilizer of every point of $\mathbb{C}$.

Definition 2.12 We call a morphism $\Psi : [\mathbb{C}/\mathbb{C}^*] \to \mathcal{X}$ with $\Psi(1) = x$ being a *test configuration* for x.

Remark 2.13 Morphisms $[\mathbb{C}/\mathbb{C}^*] \to \mathcal{X}$ are called 'filtrations' of x by Halpern-Leistner [**25**], motivated by the case when $\mathcal{X}$ is the stack of coherent sheaves on a scheme, but since this word has various other meanings in the theory of K-stability, we instead use Donaldson's terminology.

Remark 2.14 With this terminology, maps $[\mathbb{C}^2/(\mathbb{C}^*)^2] \to \mathcal{X}$ (as considered in our definition of a degeneration $(x,\zeta) \rightsquigarrow (y,\lambda)$) are essentially 'test configurations of test configurations'; we give a geometric example of this condition for polarized schemes in Remark 2.18.

We can now define a central charge analogously to before. Two pieces of notation will be useful. Firstly we let $\mathfrak{k}_x \subset \mathrm{LieAut}(x)$ be the Lie subalgebra associated with a maximal compact subgroup $K_x \subset G_x$.

Definition 2.15 A *central charge* is a function $Z : \mathcal{S} \to \mathbb{C}$ such that

(1) (Additivity) For each $x \in \mathcal{X}$, Z extends to a Lie algebra character

$$Z(x,\cdot) : \mathfrak{k}_x \to \mathbb{C}.$$

(2) (Equivariant constancy) Suppose $\Psi : [B/\mathbb{C}^*] \to \mathcal{X}$ is a morphism where $\mathbb{C}^*$ fixes each point $b \in B$ of a connected finite-type scheme B, and denote by $\lambda_{\Psi(b)}$ the associated one-parameter subgroup of $\mathrm{Aut}(\Psi(b))$. Then $Z(\Psi(b), \lambda_{\Psi(b)})$ is independent of $b \in B$.

REMARK 2.16 A central charge as defined here is a variant of Halpern-Leistner's notion of a 'numerical invariant' [**25**], and in particular the equivariant constancy used here is motivated by a property he demands.

A central charge induces a notion of stability just as before: fixing a phase $\varphi \in (-\pi, \pi)$, we say that x is x*Z-semistable* if for all $(x, \mathrm{Id}) \rightsquigarrow (y, \lambda)$ the inequality

$$\mathrm{Im}(e^{-i\varphi} Z(y, \lambda)) \geq 0$$

holds, and *Z-stability*, *Z-polystability* and *Z-instability* are defined analogously.

2.4. Examples

We give two examples of central charges for particular stacks: the stack of *coherent sheaves* and the stack of *polarized schemes*.

Coherent sheaves

Let $\mathcal{C}$ denote the stack of coherent sheaves over a projective scheme X, with $\mathcal{S}$ parameterizing sheaves along E along with a one-parameter subgroup of $\mathrm{Aut}(E)$. There is a classical notion of a central charge on $\mathcal{C}$: this associates with each coherent sheaf E on X a complex number $Z(E)$ which is deformation invariant (that is, constant in flat families of sheaves; this is a consequence of central charges being assumed to factor through the numerical Grothendieck group of coherent sheaves and constancy of numerical invariants in flat families) and additive in short exact sequences. Here we explain how this canonically induces a central charge in the sense of Section 2.3, so that our notion can be seen as a generalization of the classical notion.

In the stack of coherent sheaves, test configurations $[\mathbb{C}/\mathbb{C}^*] \to \mathcal{C}$ for E correspond to filtrations of E labelled by integers (this is standard; see, for example, [**25**, Example 0.0.2]). For example, any subsheaf $S \subset E$ induces the specialization

$$(E, \mathrm{Id}) \rightsquigarrow (S \oplus E/S, (\mathrm{Id}, \exp(t)).$$

Suppose that we are given for each coherent sheaf E a complex number $Z(E)$ which is constant in flat families and additive in short exact sequences (such as from a central charge in the classical sense). For an element

$$(E_1 \oplus \ldots \oplus E_k, (\mathrm{Id}, \ldots, \exp(t), \ldots, \mathrm{Id})) \in \mathcal{S},$$

with $\exp(t)$ in the j^{th} spot, first set

$$Z(E_1 \oplus \ldots \oplus E_k, (\mathrm{Id}, \ldots, \exp(t), \ldots, \mathrm{Id})) = Z(E_j).$$

Then setting

$$Z(E_1 \oplus \ldots \oplus E_k, (\exp(a_1 t), \ldots, \exp(a_k t))) = \sum_{j=1}^{k} a_j Z(E_j)$$

induces a central charge: additivity follows by definition, while equivariant constancy is a consequence of $Z(E)$ being constant in flat families of sheaves.

The classical example of a central charge on the stack of coherent sheaves over a polarized scheme (X, L) is given by

$$Z(E) = \mathrm{rk}E - i\deg E,$$

where rk$E$ denotes the rank and $\deg E = c_1(E)\cdot L^{n-1}$ denotes the degree. With this choice Z-semistability recovers *slope semistability*, while if one restricts to test configurations induced by *saturated* subsheaves of E then Z-polystability recovers slope polystability (for a survey explaining slope stability and its relation to Bridgeland stability see [**4**]).

Polarized schemes

Denote by $\mathcal{X}$ the stack of $\mathbb{Q}$-polarized schemes with fixed Hilbert polynomial (that is, we consider schemes together with an ample $\mathbb{Q}$-line bundle). The main differences between the polarized scheme theory and the coherent sheaf theory is that test configurations no longer correspond directly to *subobjects*, so it is more natural to consider polarized schemes with fixed Hilbert polynomial rather than considering all polarized schemes at the same time.

Unravelling the definition, a test configuration for (X, L) in $\mathcal{X}$ corresponds to a flat family $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}}) \to \mathbb{C}$ of polarized schemes along with a $\mathbb{C}^*$-action covering the natural one on $\mathbb{C}$, such that the fibres satisfy $(\mathcal{Y}_t, \mathcal{L}_{\mathcal{Y},t}) \cong (X, L)$ for all $t \neq 0$. This agrees with the usual definition of a test configuration due to Donaldson [**18**], generalizing Tian's prior work [**46**]. Given a test configuration with associated $\mathbb{C}^*$-action λ, we write $(X, L) \rightsquigarrow (\mathcal{Y}_0, \mathcal{L}_{\mathcal{Y}_0}, \lambda)$.

The set $\mathcal{S}$ consists of triples (X, L, ζ) where (X, L) is a polarized scheme and ζ is a one-parameter subgroup of $\mathrm{Aut}(X, L)$. Denote by $\mathfrak{k}_{(X,L)}$ the Lie algebra of a maximal compact subgroup $K \subset \mathrm{Aut}(X, L)$. The notion of a central charge for polarized schemes is the following.

Definition 2.17 A central charge is a function $Z : \mathcal{S} \to \mathbb{C}$ which satisfies the following:

(1) (Additivity) For a fixed polarized scheme, Z induces a Lie algebra character

$$Z((X, L), \cdot) : \mathfrak{k}_{(X,L)} \to \mathbb{C}.$$

(2) (Equivariant constancy) Suppose that $\pi : (\mathcal{Y}, \mathcal{L}_{\mathcal{Y}}) \to B$ is a flat family of polarised schemes, and suppose that there is a $\mathbb{C}^*$-action λ on $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ such that $\pi \circ \lambda(t) = \pi$ for all t. Then $Z(\mathcal{Y}_b, \mathcal{L}_{\mathcal{Y}_b}, \lambda)$ is independent of $b \in B$, where $(\mathcal{Y}_b, \mathcal{L}_{\mathcal{Y}_b})$ denotes the fibre of π over b.

Given a central charge, one can then ask for a polarized scheme (X, L) to be *Z-semistable*, *Z-stable*, *Z-polystable* or *Z-unstable* in the natural way. For example, fixing a phase $\varphi \in (-\pi, \pi)$ for (X, L) to be Z-semistable that for all $(X, L) \rightsquigarrow (Y, L_Y, \lambda)$ we have

$$\mathrm{Im}\left(e^{-i\varphi} Z(Y, L_Y, \lambda)\right) \geq 0.$$

For a test configuration $(\mathcal{Y}, \mathcal{L})$, if we define $Z(\mathcal{Y}, \mathcal{L}_Y)$ to be $Z(\mathcal{Y}_0, \mathcal{L}_{\mathcal{Y}_0}, \lambda)$, we may think of a central charge as associating a complex number with each test configuration, in an additive and equivariantly constant manner. With this notation Z-semistability then for all test configurations $(\mathcal{Y},\mathcal{L})$ for (X,L) we have

$$\operatorname{Im}\left(e^{-i\varphi}Z(\mathcal{Y},\mathcal{L})\right)\geq 0.$$

Remark 2.18 The analogue for polarized schemes of the condition that $(x,\zeta)\rightsquigarrow(y,\lambda)$ is the following. Suppose that (X,L) is a polarized scheme, with ζ being a $\mathbb{C}^*$-action on (X,L). We can then ask for a test configuration $(\mathcal{Y},\mathcal{L}_{\mathcal{Y}})$ to be *ζ-equivariant*, in the sense that there is a $\mathbb{C}^*$-action on $\mathcal{Y}$ acting fibrewise (so preserving the map to $\mathbb{C}$), extending the action of ζ on the general fibre (X,L) and commuting with the ξ-action on $\mathcal{Y}$ coming from the definition of a test configuration. Given such a ζ-equivariant test configuration, we obtain a family $\mathcal{Y}\times\mathbb{C}\to\mathbb{C}^2$ with a $(\mathbb{C}^*)^2$-action induced from ξ on the first factor and (the extension of) ζ on the second. There are two $\mathbb{C}^*$-actions on $(\mathcal{Y}_0,\mathcal{L}_{\mathcal{Y}_0})$ induced by ξ and ζ, and the condition that $(X,L,\zeta)\rightsquigarrow(\mathcal{Y}_0,\mathcal{L}_{\mathcal{Y}_0},\lambda)$ asks that there is a ζ-equivariant test configuration as described such that in addition $\xi\circ\zeta=\zeta\circ\xi=\lambda$ as subgroups of $\operatorname{Aut}(\mathcal{Y}_0,\mathcal{L}_{\mathcal{Y}_0})$. Note that this implies λ and ζ also commute in $\operatorname{Aut}(\mathcal{Y}_0,\mathcal{L}_{\mathcal{Y}_0})$. As before, it is not essential to consider families over $\mathbb{C}^2$, as one can instead consider families over $\mathbb{C}$ at the expense of working with the ineffective quotient $[\mathbb{C}/(\mathbb{C}^*)^2]$.

Example 2.19 (K-stability) Suppose that $(\mathcal{Y},\mathcal{L})$ is a test configuration for an n-dimensional scheme (Y,L), inducing a $\mathbb{C}^*$-action λ_0 on $H^0(\mathcal{Y}_0,\mathcal{L}_0^k)$ for all k. The dimension of $H^0(\mathcal{Y}_0,\mathcal{L}_0^k)$ and the total weight of the action on $H^0(\mathcal{Y}_0,\mathcal{L}_0^k)$ are polynomials for $k\gg 0$ which we may write

$$\begin{aligned} h(k) &= a_0k^n+a_1k^{n-1}+O(k^{n-2}),\\ w(k) &= b_0k^{n+1}+b_1k^n+O(k^{n-1}),\end{aligned}$$

respectively. Setting

$$Z((\mathcal{Y}_0,\mathcal{L}_0),\lambda_0)=-ib_0+b_1,\qquad Z((Y,L),\mathrm{Id})=ia_0-a_1,$$

produces a central charge, and the notion of Z-semistability recovers Donaldson's notion of K-semistability [18] (extending Tian's analytic definition in the Fano case [46]) and similarly for Z-stability and Z-polystability.

After Donaldson's original work, a subtlety in the definition of K-stability (rather than K-semistability) was realized: for normal varieties one must exclude certain 'almost trivial' test configurations (test configurations whose total space normalizes to the trivial test configuration) to have a sensible theory [7, 13, 36, 42]; it is not clear what role these degenerate test configurations play in the theory for more general central charges. Excluding almost trivial test configurations is analogous to the restriction to considering saturated subsheaves in the definition of slope stability of torsion-free coherent sheaves; by contrast torsion sheaves play a central role in Bridgeland stability.

Example 2.20 The notion of a central charge is intended to axiomatize the notion of a central charge introduced in [14]. There a specific smooth polarized variety (X,L) was fixed, and a central charge was defined explicitly through a choice of topological information on (X,L). This choice canonically induces a phase $\varphi=\arg Z(X,L)$, which is then independent of (X,L) itself, provided that it varies in a flat manner. For each test configuration $(\mathcal{Y},\mathcal{L})$ with smooth total space, a number $Z(\mathcal{Y},\mathcal{L})\in\mathbb{C}$ was then defined via intersection theory on a natural compactification of the total space $\mathcal{Y}$. Thus the definition relies on $(\mathcal{Y}_0,\mathcal{L}_0)$ being the central fibre of a test configuration with reasonable total space. It would be interesting to define these quantities intrinsically on $(\mathcal{Y}_0, \mathcal{L}_0)$, more in line with the perspective of this note.

3. *Z*-CRITICAL POINTS AND COMPLEX MOMENT MAPS

We next describe the analytic counterpart to Z-stability, through what we call *complex* moment maps. In the traditional theory of moment maps, one can either consider maps to the Lie algebra or its dual; the former requires a choice of an inner product. This is mostly an aesthetic choice, and we choose to fix an inner product so that the links with the motivating *infinite*-dimensional problems are most transparent, as this is one of the main goals of our work. We refer to Kirwan [34], Georgoulas–Robbin–Salamon [22] and Hashimoto [27] for comprehensive accounts of the relationship between moment maps and GIT.

To define the inner product, we proceed as follows. Firstly we fix a faithful representation of G on a complex vector space V, giving an embedding

$$\mathfrak{g} \subset \operatorname{End} V.$$

Thus we may *multiply* elements of $\mathfrak{g}$. The most important example to keep in mind is when $X \subset \mathbb{P}^n = \mathbb{P}(V)$ is a subvariety of projective space and G acts faithfully and linearly on projective space, meaning that there is a natural G-action on V. We assume that G is reductive and is hence the complexification of a maximal compact subgroup $K \subset G$. We also fix a K-invariant Hermitian inner product on V, which induces one on $\operatorname{End} V$ and hence induces an isomorphism

$$\mathfrak{g} \cong \mathfrak{g}^*.$$

In this way, for $u \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$ we have

$$\langle u, \alpha \rangle = \operatorname{tr}(u^* \alpha^\vee),$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing, $\alpha^\vee \in \mathfrak{g}$ is the dual element of α and u^* is the conjugate transpose.

We now return to a smooth projective variety X, which we endow with a closed K-invariant *complex* $(1,1)$-form ω_Z. For the moment we do *not* assume any positivity hypotheses on ω_Z.

Let G be a reductive linear algebraic group acting holomorphically on X, and fix a representation of G and a Hermitian inner product as earlier. We write K for the maximal compact subgroup of G, and let $\mathfrak{k}$ denote the Lie algebra of K. We will—slightly abusively—identify an element $v \in \mathfrak{g}$ with its induced vector field on X.

Definition 3.1 We say that a smooth map

$$\tilde{Z} : X \to \mathfrak{g}$$

is a *complex moment map* if for all $u \in \mathfrak{k}$ we have

$$d\operatorname{tr}\left(u^* \tilde{Z}\right) = -\iota_u \omega_Z$$

and $\tilde{Z}$ is K-equivariant with respect to the adjoint action on $\mathfrak{k}$.

Remark 3.2 Similar structures to complex moment maps arise in Bérczi–Kirwan's recent work providing a moment map interpretation of non-reductive GIT [6], and it would be interesting to understand the relationship between their work and what we consider here.

We now assume that $\tilde{Z}$ is a complex moment map. To link with the definition of a central charge, it is useful to upgrade $\tilde{Z}$ to a smooth function

$$\tilde{Z} : X \times \mathfrak{k} \to \mathfrak{g}$$

by defining

$$\tilde{Z}(x,v) = v^*\tilde{Z}(x),$$

where the second term is interpreted as multiplication of elements of $\mathfrak{g} \subset \operatorname{End} V$. In the following we view $\mathcal{S}$ as consisting of pairs (x,u) such that $x \in X$ and $u \in \mathfrak{k}_x$. We fix a central charge Z on $\mathbb{S}$ and a phase $\varphi \in (-\pi,\pi)$.

Definition 3.3 We say that $\tilde{Z}$ is *compatible with a central charge* $Z : \mathcal{S} \to \mathbb{C}$ if for all $(x,v) \in \mathcal{S}$ we have

$$\operatorname{tr}(\tilde{Z}(x,v)) = Z(x,v)$$

and

$$\arg \operatorname{tr}(\tilde{Z}(x)) = \varphi.$$

Remark 3.4 Compatibility is a key point of the definitions: if one thinks of Z as analogous to a choice of topological classes—as will be the case in the examples given below—compatibility is analogous to asking that $\tilde{Z}$ produces Chern–Weil and equivariant Chern–Weil representatives of these topological classes. Being able to phrase the compatibility condition is the main advantage of choosing a representation of G on the vector space V.

From here we fix a central charge Z compatible with the complex moment map $\tilde{Z}$. We thus turn to linking complex moment maps with Z-stability, and in particular we will exclusively be interested in understanding $\tilde{Z}$ on points, rather than general pairs (x,u). We fix a phase $\varphi \in (-\pi,\pi)$.

Definition 3.5 We say that a point x is *Z-critical* if

$$\operatorname{Im}(e^{-i\varphi}\tilde{Z}(x)) = 0,$$

where Im refers to the skew Hermitian part of an element of $\mathfrak{g}$ with respect to the Hermitian inner product.

As we will explain, this is the key condition related to Z-polystability. A basic observation shows that compatibility is actually automatic at Z-critical points:

Lemma 3.6 *Suppose x is a Z-critical point. Then*

$$\arg \operatorname{tr}(\tilde{Z}(x)) = \varphi.$$

Proof. Write

$$\tilde{Z}(x) = M_h(x) + M_s(x),$$

where $e^{-i\varphi}M_h(x)$ is Hermitian and $e^{-i\varphi}M_s(x)$ is skew-Hermitian. Then we see that

$$\operatorname{Im}(e^{-i\varphi}\tilde{Z}(x)) = e^{-i\varphi}M_s(x),$$

so

$$\begin{aligned}\mathrm{tr}\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x)) &= \mathrm{tr}e^{-i\varphi}M_s(x),\\ &= e^{-i\varphi}\mathrm{tr}M_s(x),\\ &= \mathrm{Im}(e^{-i\varphi}\mathrm{tr}\tilde{Z}(x)),\end{aligned}$$

where in the final step we used the similar fact that $\mathrm{tr}e^{-i\varphi}M_h(x)$ is real. Thus if $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x)) = 0$, we must have $\mathrm{Im}(e^{-i\varphi}\mathrm{tr}\tilde{Z}(x)) = 0$, which implies

$$\arg\mathrm{tr}(\tilde{Z}(x)) = \varphi,$$

concluding the proof. □

While we actually assume compatibility throughout, this result nevertheless makes clear that it is a natural condition. We next relate complex moment maps to usual moment maps for compact group actions.

DEFINITION 3.7 We call a map $\mu : X \to \mathfrak{k}^*$ a *formal moment map* with respect to a closed $(1,1)$-form η if μ is K-equivariant and

$$d\langle \mu, v\rangle = -\iota_v\eta.$$

Thus if η is positive—hence defining a symplectic form—μ is a moment map in the usual sense. In the language of equivariant cohomology, the condition asks that the (complex) equivariant differential form $\eta + \mu$ is equivariantly closed.

PROPOSITION 3.8 *Suppose that $\tilde{Z}$ is a complex moment map with respect to ω_Z. Then*

$$i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^\vee : X \to \mathfrak{k}^*$$

is a formal moment map with respect to the (1, 1)-form $\mathrm{Re}(e^{-i\varphi}\omega_Z)$.

Here the notation $i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^\vee$ means the composition of $i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot)) \to \mathfrak{k}$ with the isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$, while $\mathrm{Im}(e^{-i\varphi}\omega_Z)$ denotes the imaginary part of the complex $(1,1)$-form $e^{-i\varphi}\omega_Z$. Note that $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^\vee$ itself has image in $i\mathfrak{k}$, meaning that $i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^\vee$ takes values in Hermitian matrices. This extra factor of i is compensated for in the $(1,1)$-form as

$$\mathrm{Re}(e^{-i\varphi}\omega_Z) = \mathrm{Im}(e^{-i\varphi}i\omega_Z).$$

Proof. K-equivariance of $\tilde{Z} : X \to \mathfrak{g}$ implies that $i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot)) : X \to \mathfrak{k}$ is K-equivariant, while the K-equivariance of the Hermitian inner product on V implies that the isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$ is K-equivariant.

To prove the moment map equation, it is equivalent to show that

$$d\langle u, \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))^\vee\rangle = -\iota_u\mathrm{Im}(e^{-i\varphi}\omega_Z).$$

Since $u \in \mathfrak{k}$ corresponds to a real vector field on X, we have

$$\iota_u\mathrm{Im}(e^{-i\varphi}\omega_Z) = \mathrm{Im}(e^{-i\varphi}\iota_u\omega_Z),$$

which by the complex moment map identity gives

$$-\iota_u \mathrm{Im}(e^{-i\varphi}\omega_Z) = \mathrm{Im}\left(e^{-i\varphi} d\mathrm{tr}\left(u^* \tilde{Z}\right)\right).$$

Using a similar linear algebra argument as Lemma 3.6 along with the fact that u viewed as an element of End$V$ corresponds to a Hermitian matrix, we see

$$\mathrm{Im}\left(e^{-i\varphi} d\mathrm{tr}\left(u^* \tilde{Z}\right)\right) = d\mathrm{tr}\left(u^* \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))\right).$$

Since the isomorphism $\mathfrak{k} \cong \mathfrak{k}^*$ arises from the Hermitian inner product on V, it follows that

$$\mathrm{tr}\left(u^* \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))\right) = \langle u, \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))^\vee \rangle$$

and hence

$$d\langle u, \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))^\vee \rangle = -\iota_u \mathrm{Im}(e^{-i\varphi}\omega_Z),$$

proving the result. □

Positivity is, of course, crucial to the theory of moment maps. While many aspects of the theory require *global* positivity, others rely only on *local* positivity; for example, to obtain a symplectic quotient, one only needs positivity in an open neighbourhood of the zero set of the moment map.

Definition 3.9 We say that a point $x \in X$ is a *Z-subsolution* if the form

$$\mathrm{Re}(e^{-i\varphi}\omega_Z)$$

is positive on T_xX, in the sense that

$$\mathrm{Re}(e^{-i\varphi}\omega_Z)(u, J_x u) > 0$$

for all $u \neq 0$, with $J_x : T_xX \to T_xX$ being the almost complex structure. We further say that $\tilde{Z}$ satisfies the *global subsolution hypothesis* if every point $x \in X$ is a subsolution.

The global subsolution hypothesis is strong: analogues fail in infinite dimensions, as discussed in Remark 3.13. As mentioned there, in the better-understood infinite-dimensional problems, what is expected to be true is that every solution of the equation (that is, being an analogue of a Z-critical point) *is* also a subsolution. This is often enough to obtain geometric consequences:

Theorem 3.10 *Suppose that every Z-critical point is a Z-subsolution. Then the symplectic quotient*

$$X/_Z K := \mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^{-1}(0)/K$$

admits the structure of a Kähler space.

Proof. This is classical under the global subsolution hypothesis [29, 28], but the proofs only require a Kähler metric in a *neighbourhood* of the zero set of the moment map. Thus since every Z-critical point is a subsolution, and the subsolution condition is *open* in x (as it is a positivity condition on an inner product on the tangent space), the form $\mathrm{Re}(e^{-i\varphi}\omega_Z)$ is indeed a Kähler metric in a neighbourhood of $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^{-1}(0)$. □

More explicitly, in this generality the Kähler metric on the quotient is produced as follows. Let $\psi_Z \in C^\infty(X, \mathbb{C})$ be a local potential for the complex $(1,1)$-form ω_Z in a neighbourhood of a Z-critical point x, in the sense that near x

$$\omega_Z = i\partial\bar{\partial}\psi_Z.$$

Restricting ψ_Z to (a neighbourhood of x intersected with) $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^{-1}(0)$, as ψ_Z is a K-invariant function, it descends to a continuous function $\tilde{\psi}_Z$ on the quotient $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))^{-1}(0)/K$. The function

$$\mathrm{Re}(e^{-i\varphi}\tilde{\psi}_Z) \in C^0(X/_Z K, \mathbb{R})$$

is then a weak Kähler potential for the induced form on the complex space $X/_Z K$ in the sense used by Heinzner–Huckleberry–Loose [28].

Our main result explains how to relate the existence of Z-critical points to complex moment maps. The proof reduces to a version of the classical Kempf–Ness theorem (due to Kempf–Ness being in the affine setting [33] and to Kirwan being in the projective setting [34]).

Theorem 3.11 *Suppose that $\tilde{Z}$ satisfies the global subsolution hypothesis. Then the following are equivalent:*

(1) there is a point $y \in G.x$ such that $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(y)) = 0$;
(2) x is Z-polystable.

Proof. By the global subsolution hypothesis, the form $\mathrm{Im}(e^{-i\varphi}\omega_Z)$ is a Kähler metric on X and $\mathrm{Im}(e^{-i\varphi}\tilde{Z}(\cdot))$ is a moment map. Although X is a smooth projective variety, the form $\mathrm{Re}(e^{-i\varphi}\omega_Z)$ may not lie in an integral class, meaning that we cannot apply the classical Kempf–Ness theorem. Instead we apply the Kempf–Ness theorem for *Kähler* manifolds (see, for example, the survey [22, Section 12]), which implies that the existence of a Z-critical point in the orbit of x is equivalent to the condition that for all $x \rightsquigarrow (y,u)$ we have

$$\langle i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(y))^\vee, u\rangle \leq 0,$$

with equality if and only if $y = x$. Here our slightly extended notation means that

$$y = \lim_{t\to\infty} \exp(-itu).x.$$

What this essentially means is that, in the Kähler setting one must also include 'irrational' vector fields to obtain the existence of a zero of the moment map, rather than merely rational ones inducing one-parameter subgroups of G.

The rest of the proof will compare this numerical condition to the one governing Z-polystability and will then explain that in fact it *is* enough to merely consider 'rational' vector fields (equivalently one-parameter subgroups) in our situation.

The definition of the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ gives

$$\begin{aligned}\langle i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(y)^\vee), u\rangle &= \mathrm{tr}((i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(y)))^* u),\\ &= \mathrm{tr}(i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(y)u),\\ &= -\mathrm{Im}(e^{-i\varphi}Z(y,u)).\end{aligned}$$

In slightly more detail, the final step follows from the fact that for a (complex) matrix A and a Hermitian matrix B we have

$$\mathrm{tr}(i(\mathrm{Im}A)B) = -\mathrm{Im}\,\mathrm{tr}(AB),$$

and also from the compatibility condition $\mathrm{tr}\left(u^*\tilde{Z}(y)\right) = \mathrm{tr}\left(\tilde{Z}(y)u\right) = Z(y,u)$ again using that u is Hermitian.

To conclude we must show that it is enough to check stability with respect to *rational* elements of $\mathfrak{k}$, generating one-parameter subgroups. If x is Z-unstable, it is standard to produce rational destabilizing elements of $\mathfrak{k}$ given the existence of an irrational one by an approximation argument, and thus we may assume that x is Z-semistable; here we note that Z-semistability with respect to rational vector fields implies Z-semistability also with respect to irrational ones. Then from the 'semistable' case of the Kempf–Ness theorem there is a point $z \in \overline{G.x}$ which is itself Z-critical. We can then take a slice of the G-action in a neighbourhood of the Z-critical point z (essentially by construction of the quotient in the complex setting [29]), so that the action is modelled on the linear action on T_zX, where it is clear that one can find a one-parameter subgroup taking x to z. □

REMARK 3.12 Rather than the global subsolution hypothesis, the proof only requires the weaker condition that $\mathrm{Re}(e^{-i\varphi}\omega_Z)$ be positive in a neighbourhood of $\overline{G.x}$.

In the classical projective case a consequence of this sort of result is a homeomorphism between the symplectic and algebraic quotients (this is due to Kirwan [34]). As we have appealed to a Kähler version of the Kempf–Ness theorem, there is no purely algebraic definition of the quotient. So while—under the global subsolution hypothesis—one still obtains a quotient $X/_K K$ which is a complex space endowed with a Kähler metric by Theorem 3.10, there is no direct algebraic construction to compare it with.

3.1. Examples in infinite dimensions

We next briefly explain the link between the categorical notions of stability for coherent sheaves and polarized schemes and moment maps.

Vector bundles

Associated with a class of central charges on $\mathrm{Coh}(X)$—which in particular take the form of Section 'Coherent sheaves'—is a partial differential equation on Hermitian metrics on holomorphic vector bundles on X, solutions of which are called *Z-critical connections* [16]. Briefly, these central charges involve a choice of Kähler class on X, a choice of (products of) Chern classes of the sheaf and a choice of topological classes on X (as motivated by Bayer's polynomial stability conditions [5]). To a Kähler metric on X, a Hermitian metric on E producing a Chern connection A and a closed differential form on X representing the topological class is then associated with an $\mathrm{End}E$-valued (n,n) form $\tilde{Z}(E,A)$ which satisfies

$$\int_X \tilde{Z}(A) = Z(E),$$

which is analogous to the compatibility of the central charge as given in Definition 3.3. The inner product corresponding to the trace used there is the L^2-inner product defined with respect to the volume form associated with the Kähler metric.

The Z-critical equation then asks that

$$\mathrm{Im}(e^{-i\varphi(E)}\tilde{Z}(h)) = 0,$$

where this denotes the skew-Hermitian part of the $\mathrm{End}E$ component as defined through the Hermitian metric h. The various sign conventions used in the present work are chosen to match with the Z-critical connection and the deformed Hermitian Yang–Mills literature. An especially noteworthy example is given by the deformed Hermitian Yang-Mills equation [35, 37] (which appeared long before the more general notion of a Z-critical connection), which corresponds to the central charge

$$Z(E) = \int_X e^{-i[\omega]} \cdot \mathrm{ch}(E);$$

other equations relevant to string theory and mirror symmetry involve including Chern classes of X itself [**16**, Example 2.8].

REMARK 3.13 On line bundles, an almost complete theory of deformed Hermitian Yang–Mills connections exists, especially due to the studies by Chen, Collins, Jacob and Yau [**9**, **10**, **12**, **31**], and their theory emphasizes many of the structures that one should expect to be relatively general. For example, the existence of a solution to the deformed Hermitian Yang–Mills equation *implies* that the associated Hermitian metric is a subsolution [**10**] (in the so-called 'supercritical phase range'). An important aspect of the theory is that this same statement ('solution implies subsolution') is true along a continuity method one can use to solve the equation under a stability hypothesis [**9**], so one always has positivity along the path designed to solve the equation.

In fact, the *converse* of this statement also holds on line bundles: the existence of deformed Hermitian Yang–Mills connections on a line bundle is equivalent to the existence of a subsolution (again in the appropriate phase range); this is due to the study by Collins–Jacob–Yau [**10**]. In higher rank, for appropriate classes of central charge it should still be the case that the existence of a solution implies the existence of a subsolution, but the converse cannot hold. For example, the Hermitian Yang–Mills condition is a special case of the Z-critical condition, and here the subsolution condition is automatic (as it asks that ω^{n-1} is a positive $(n-1, n-1)$-form where ω is the Kähler metric), but nevertheless obstructions to the existence of solutions appear from saturated subsheaves.

Polarized varieties

The theory for smooth polarized varieties is analogous, but with additional complications on the analytic side [**14**, **15**]. Here the equation is for a Kähler metric $\omega \in c_1(L)$ on a smooth projective variety X, and one makes analogous choices—namely a choice of topological classes on X and products of Chern characters of X. The equation is only explicitly available in the case of powers of the *first* Chern class of X (along with arbitrary auxiliary differential forms on X and powers of the ample line bundle), where one associates with ω a complex valued function

$$\tilde{Z}(\omega) : X \to \mathbb{C}.$$

To tighten the parallel with the bundle theory, equivalently by multiplying by ω^n one can consider $\tilde{Z}(\omega)$ as a complex valued (n, n)-form. This complex (n, n)-form satisfies the 'compatibility' condition

$$\int_X \tilde{Z}(\omega) = Z(X, L),$$

and the Z-critical equation asks

$$\mathrm{Im}(e^{-i\varphi(X,L)}\tilde{Z}(\omega)) = 0.$$

The actual construction of $\tilde{Z}(\omega)$, however, is more subtle than its bundle analogue. The reason is that its construction involves not only various Chern–Weil representatives but also higher-order terms essential for a link with algebraic geometry. A good understanding of the Z-subsolution condition—along with various other foundational structures—remains to be achieved.

3.2. Structures in complex moment map theory

We now briefly explain the appearance of several standard structures in classical moment map theory in our setup: the norm-squared of the moment map, the moment map flow, the log-norm functional and the log-norm functional as a Kähler potential. Many of these have appeared in the infinite-dimensional theories discussed earlier, and our new perspective gives some finite-dimensional motivation for their appearance. All these structures are discussed at great length in the survey [**22**].

For any $x \in X$, the norm-squared of the moment map is simply the value

$$\|\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))\|^2 = \mathrm{tr}(\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x)) \cdot \mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))).$$

This is the functional whose Euler–Lagrange equation produces both Z-critical points and Z-*extremal points*: points which satisfy

$$\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x)) \in i\mathfrak{k}_x.$$

To define the moment map flow, note that for $x \in X$ the value $i\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x) \in \mathfrak{k}$ can be thought of as an element of T_xX through the infinitesimal action. Thus for any x_0 we may define a flow by $x(0) = x_0$ and

$$\frac{d}{dt}x(t) = -\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x(t))).$$

This is the downward gradient flow of the norm-squared of the moment map

$$x \to \|\mathrm{Im}(e^{-i\varphi}\tilde{Z}(x))\|^2,$$

and we call this flow the Z-*flow*. The asymptotics of this flow are related to 'optimal destabilizing one-parameter subgroups', which are in turn analogous to Harder–Narasihman-type filtrations in the coherent sheaf setting. In the deformed Hermitian Yang–Mills setting this flow corresponds to the *tangent Lagrangian phase flow* of Takahashi [43], and we note that in that setting there is also the *line bundle mean curvature flow*, introduced by Jacob–Yau [31], which is instead motivated by the *Lagrangian mean curvature flow* in the study of special Lagrangians.

The log-norm functional is a functional on a fixed orbit, which is defined through its variation. Fixing a reference point $x \in X$, any other point is of the the form $g.x$ for some $g \in G$. We first define a one-form dE_Z on G (the notation will be justified by this one-form being exact) by setting

$$\langle dE_Z, u\rangle_g = \mathrm{tr}(u^*\tilde{Z}(g(t).x)).$$

This is then K-invariant and hence descends to a one-form on the symmetric space G/K. A standard calculation, identical to the usual one in moment map theory, shows that this one form is closed and is hence exact. In particular it is well-defined, independent of the choice of path. Thus we obtain a functional

$$E_Z : G/K \to \mathbb{C},$$

and we define the Z-*energy* to be

$$\mathrm{Re}(e^{-i\varphi}E_Z) : G/K \to \mathbb{R}.$$

This is the analogue of the log-norm functional; this is convex along geodesics in the symmetric space G/K in the locus of Z-subsolutions, and it is strictly decreasing along the Z-flow (again in the locus of Z-subsolutions). In the deformed Hermitian Yang–Mills setting this corresponds to what Collins–Yau call the *Calabi–Yau functional* [12, Definition 2.13] and in the setting of Z-critical Kähler metrics corresponds to the Z-*energy* [14, Definition 3.7].

We lastly turn to the potential for the form ω_Z. A G-orbit $G.x \subset X$ is affine, hence on this locus $\omega_Z = i\partial\bar{\partial}\psi_Z$ for some complex-valued function ψ_Z. We can consider E_Z as a function $G.x \to \mathbb{R}$ by defining the Z-energy relative to the base point x. Then on this locus a calculation shows that

$$i\partial\bar{\partial}E_Z = \omega_Z,$$

so that we can view the Z-energy as a potential for the form ω_Z. In particular on this locus we have

$$i\partial\overline{\partial}(\mathrm{Re}(e^{-i\varphi}E_Z)) = \mathrm{Re}(e^{-i\varphi}\omega_Z).$$

Thus the Z-subsolution condition forces the complex Hessian $i\partial\overline{\partial}(\mathrm{Re}(e^{-i\varphi}E_Z))$ to be positive at the point x.

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A. BY ANDRÉS IBÁÑEZ NÚÑEZ

In this appendix we explain how the notion of a central charge on an algebraic stack $\mathcal{X}$ can be formulated using the formalism of graded points on $\mathcal{X}$, in the spirit of Halpern-Leistner's definition of numerical invariant [**25**]. In the present section we will call these *complex linear forms* on $\mathcal{X}$, to make the statements of our results (especially the equivalence with Definition 2.7) transparent.

We denote $\mathbb{G}_m = \mathrm{Spec}\mathbb{C}[t,t^{-1}]$ the multiplicative group scheme over $\mathbb{C}$, whose $\mathbb{C}$-points are $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$. A crucial role will be played by the classifying stack $B\mathbb{G}_m$ of $\mathbb{G}_m$. While $B\mathbb{G}_m$ is determined by the fact that, for any algebraic stack $\mathcal{X}$ over $\mathbb{C}$, the groupoid $\mathrm{Hom}(\mathcal{X},B\mathbb{G}_m)$ of maps $\mathcal{X} \to B\mathbb{G}_m$ is equivalent to that of line bundles on $\mathcal{X}$, here we will rather be interested in maps *from* $B\mathbb{G}_m$ into other algebraic stacks.

We fix an algebraic stack $\mathcal{X}$, quasi-separated and locally of finite type over $\mathbb{C}$, with affine stabilizers. Examples of stacks satisfying these assumptions are moduli stacks of polarized projective schemes over $\mathbb{C}$ [**41**, Tag 0DPS], [**32**, Section 2.1], moduli stacks of objects in suitable $\mathbb{C}$-abelian categories [**2**, Section 7] and stacks of G-bundles on a proper scheme X over $\mathbb{C}$ for G a linear algebraic group over $\mathbb{C}$ [**26**, Tag 00BK].

Definition A.1 The *stack of graded points* $\mathrm{Grad}(\mathcal{X})$ of $\mathcal{X}$ is the stack over $\mathbb{C}$ defined by setting, for any scheme T over $\mathbb{C}$,

$$\mathrm{Hom}(T,\mathrm{Grad}(\mathcal{X})) = \mathrm{Hom}(B\mathbb{G}_m \times T,\mathcal{X}).$$

In other words, $\mathrm{Grad}(\mathcal{X})$ is the mapping stack $\underline{\mathrm{Maps}}(B\mathbb{G}_m,\mathcal{X})$. Thus a map $T \to \mathrm{Grad}(\mathcal{X})$ is the same data as a map $B\mathbb{G}_m \times T \to \mathcal{X}$. It is a nontrivial result [**1**, Theorem 5.10] that $\mathrm{Grad}(\mathcal{X})$ is an algebraic stack locally of finite type over $\mathbb{C}$. We denote $|\mathrm{Grad}(\mathcal{X})|$ as its underlying topological space.

We will use the notation $\Gamma^{\mathbb{Z}}(\mathbb{G}_m^n) = \mathrm{Hom}(\mathbb{G}_m,\mathbb{G}_m^n)$ for the group of cocharacters of $\mathbb{G}_m^n$, which is isomorphic to $\mathbb{Z}^n$. More generally, for a linear algebraic group G over $\mathbb{C}$ we denote with $\Gamma^{\mathbb{Z}}(G)$ the set of cocharacters of G and with $\Gamma_{\mathbb{Z}}(G)$ the abelian group of characters of G.

The main definition in this appendix is as follows:

Definition A.2 A *complex linear form* Z on $\mathcal{X}$ is a locally constant map

$$Z\colon\ |\mathrm{Grad}(\mathcal{X})| \to \mathbb{C}$$

such that, for any $n \in \mathbb{Z}_{>0}$ and any morphism $g\colon B\mathbb{G}_m^n \to \mathcal{X}$, the map

$$Z_g\colon \Gamma^{\mathbb{Z}}(\mathbb{G}_m^n) \to \mathbb{C}$$

induced by g and Z is $\mathbb{Z}$-linear, where the definition of Z_g is as follows: if $\alpha\colon \mathbb{G}_m \to \mathbb{G}_m^n$ is a cocharacter, the composition

$$B\mathbb{G}_m \xrightarrow{B\alpha} B\mathbb{G}_m^n \xrightarrow{g} \mathcal{X}$$

defines a point p of $\mathrm{Grad}(\mathcal{X})$, and we set $Z_g(\alpha) = Z(p)$.

We denote with $\mathrm{PStab}(\mathcal{X})$ the set of complex linear forms on $\mathcal{X}$, which is naturally a $\mathbb{C}$-vector space. The notation is intended to signify that we think of a complex linear form as a 'pre-stability condition', where the eventual full structure of a stability condition should in addition require a positivity property. The relationship with central charges and stability in the sense of Sections 2.3 and 2.2 is explained by Remarks A.7 and A.8, respectively.

Remark A.3 Definition A.2 makes sense whenever $\mathrm{Grad}(\mathcal{X})$ is an algebraic stack, which holds under very general assumptions on $\mathcal{X}$ (see [1, Theorem 5.10]). In this generality, the linearity condition in the definition should be imposed for all $g\colon B\mathbb{G}_{m,k}^n \to \mathcal{X}$, where k is an arbitrary algebraically closed field.

Complex linear forms on the classifying stack of a group have a transparent description.

Lemma A.4 *Let G be an affine algebraic group over $\mathbb{C}$. Then there is a canonical isomorphism*

$$\mathrm{PStab}(BG) \cong \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G)$$

between the vector space of complex linear forms on BG and that of complex characters of G.

Proof. Let T be a maximal torus of G, and let $W = N_G(T)/C_G(T)$ be the associated Weyl group. Let C be a complete set of representatives of Weyl orbits in $\Gamma^{\mathbb{Z}}(T)$. Then by [25, Theorem 1.4.8] there is a canonical isomorphism

$$\mathrm{Grad}(BG) \cong \bigsqcup_{\lambda \in C} BL(\lambda),$$

where $L(\lambda)$ is the centralizer of λ in G. Therefore $\pi_0(\mathrm{Grad}(\mathcal{X})) = \Gamma^{\mathbb{Z}}(T)/W$, and a complex linear form on BG is given by a map $Z\colon \Gamma^{\mathbb{Z}}(T)/W \to \mathbb{C}$. The linearity condition in Definition A.1 amounts to the composition

$$\Gamma^{\mathbb{Z}}(T) \longrightarrow \Gamma^{\mathbb{Z}}(T)/W \xrightarrow{Z} \mathbb{C}$$

being a homomorphism. Thus we have an isomorphism

$$\mathrm{PStab}(BG) = \mathrm{Hom}(\Gamma^{\mathbb{Z}}(T), \mathbb{C})^W = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(T)^W.$$

The result follows from the fact that the natural map $\mathbb{Q} \otimes \Gamma_{\mathbb{Z}}(G) \to \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(T)_{\mathbb{Z}}^W$ is an isomorphism [20]. □

Let now denote $\mathcal{P} = \mathrm{Grad}(\mathcal{X})(\mathbb{C})$ the groupoid of $\mathbb{C}$-points of $\mathrm{Grad}(\mathcal{X})$, and $\mathcal{S}$ the set of equivalence classes of $\mathcal{P}$, namely,

$$\mathcal{S} = \pi_0(\mathcal{P}).$$

Thus $\mathcal{P}$ has objects (x, λ), where $x\colon \operatorname{Spec}\mathbb{C} \to \mathcal{X}$ is a point and $\lambda\colon \mathbb{G}_m \to G_x$ is a cocharacter of the stabilizer group G_x of x. A map $(x, \lambda) \to (y, \mu)$ in $\mathcal{P}$ is an isomorphism $g\colon x \to y$ such that $\mu = \lambda^g$, where $(-)^g\colon G_x \to G_y$ denotes the isomorphism that g induces on automorphism groups by conjugation. From this we see that the set $\mathcal{S}$ defined here coincides with that in Section 2.3. We now compare Definition 2.15 with Definition A.2.

LEMMA A.5 *The data of a locally constant map $|\operatorname{Grad}(\mathcal{X})| \to \mathbb{C}$ are equivalent to those of a map $Z\colon \mathcal{S} \to \mathbb{C}$ satisfying that, for every connected finite type scheme T over $\mathbb{C}$ and every map $B\mathbb{G}_m \times Z \to \mathcal{X}$, the composition*

$$T(\mathbb{C}) \longrightarrow \mathcal{S} \xrightarrow{Z} \mathbb{C}$$

is constant.

This is precisely the equivariant constancy condition of Definition 2.15.

Proof. Under the natural injection $\mathcal{S} \to |\operatorname{Grad}(\mathcal{X})|$, $\mathcal{S}$ is the set of points of $|\operatorname{Grad}(\mathcal{X})|$ that can be realized by a map $\operatorname{Spec}\mathbb{C} \to \operatorname{Grad}(\mathcal{X})$, and thus $\mathcal{S}$ inherits a topology from $|\operatorname{Grad}(\mathcal{X})|$.

For any closed subset R of $|\operatorname{Grad}(\mathcal{X})|$, the intersection $\mathcal{S} \cap R$ is dense in R, so $\mathcal{S}$ and $|\operatorname{Grad}(\mathcal{X})|$ have the same connected components. For any morphism $T \to \operatorname{Grad}(\mathcal{X})$, where T is a scheme over $\mathbb{C}$, the induced map $T(\mathbb{C}) \to \mathcal{S}$ is continuous. Moreover, if $Y \to \operatorname{Grad}(\mathcal{X})$ is a smooth atlas, then the induced map $Y(\mathbb{C}) \to \mathcal{S}$ is a submersion. Therefore, giving a locally constant map $|\operatorname{Grad}(\mathcal{X})| \to \mathbb{C}$ is equivalent to giving a locally constant map $\mathcal{S} \to \mathbb{C}$, which is in turn equivalent to giving a map $\mathcal{S} \to \mathbb{C}$ such that, for every morphism $T \to \operatorname{Grad}(\mathcal{X})$ with T a scheme of finite type over $\mathbb{C}$ and connected, the composition $T(\mathbb{C}) \to \mathcal{S} \to \mathbb{C}$ is constant. □

PROPOSITION A.6 *Let $Z\colon |\operatorname{Grad}(\mathcal{X})| \to \mathbb{C}$ be a locally constant map. For a point $x \in \mathcal{X}(\mathbb{C})$, with stabilizer group G_x, we denote $\psi_x\colon \Gamma^{\mathbb{Z}}(G_x) \to \mathbb{C}$ as the map induced by Z and x. Then the following conditions are equivalent:*

(1) The map Z defines a complex linear form on $\mathcal{X}$.
(2) For every $x \in \mathcal{X}(\mathbb{C})$, the map ψ_x is induced by a (uniquely determined) complex character $\chi \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G_x)$ of G_x.
(3) For every $x \in \mathcal{X}(\mathbb{C})$, if K_x is a maximal compact subgroup of G_x and $\mathfrak{k}_x$ is the Lie algebra of K_x, then the map ψ_x is induced by a (uniquely determined) complex Lie algebra character $\mathfrak{k}_x \to \mathbb{C}$.
(4) For every $x \in \mathcal{X}(\mathbb{C})$ and for all commuting cocharacters $\lambda, \lambda' \in \Gamma^{\mathbb{Z}}(G_x)$ we have $\psi_x(\lambda\lambda') = \psi_x(\lambda) + \psi_x(\lambda')$.

Proof. If Z is a complex linear form on $\mathcal{X}$ and $x \in \mathcal{X}(\mathbb{C})$, then there is an induced monomorphism $\iota\colon BG_x \to \mathcal{X}$ where G_x is the stabilizer group of x. The pullback $\iota^* Z$, that is, the composition

$$|\operatorname{Grad}(BG_x)| \xrightarrow{|\operatorname{Grad}(\iota)|} |\operatorname{Grad}(\mathcal{X})| \xrightarrow{Z} \mathbb{C},$$

is a complex linear form on BG_x. It follows from Lemma A.4 that ψ_x is given by a complex character $\chi \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G_x)$ that is uniquely determined. Therefore (1) implies (2).

Now fix $x \in \mathcal{X}(\mathbb{C})$. Let U be the unipotent radical of G_x and $L = G_x/U$, which is reductive. Let $\mathfrak{l}$ be the Lie algebra of L. Then for any maximal compact subgroup K_x of G_x, the composition $K_x \to G_x \to L$ exhibits L as the complexification of K_x. Therefore, the Lie algebra $\mathfrak{l}$ of L equals $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}_x$, and thus a homomorphism $\mathfrak{k}_x \to \mathbb{C}$ of real Lie algebras is the same data as a homomorphism $\mathfrak{l} \to \mathbb{C}$ of complex Lie algebras. Any character $G_x \to \mathbb{G}_m$

factors through $G_x \to L$ and, taking the differential of the induced $L \to \mathbb{G}_m$, it gives a Lie algebra character $\mathfrak{l} \to \mathbb{C}$. This gives a homomorphism $\Gamma_{\mathbb{Z}}(G_x) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{l}, \mathbb{C})$ and, by extending scalars, a map $r\colon \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G_x) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{l}, \mathbb{C})$.

An element $\chi \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G_x)$ gives a pairing map $\langle -, \chi \rangle\colon \Gamma^{\mathbb{Z}}(G_x) \to \mathbb{C}$. Similarly, a Lie algebra character $\alpha \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{l}, \mathbb{C})$ gives a map $\langle -, \alpha \rangle\colon \Gamma^{\mathbb{Z}}(G_x) \to \mathbb{C}$ as follows. If $\lambda\colon \mathbb{G}_m \to G_x$ is a one-parameter subgroup, then $\langle \lambda, \alpha \rangle$ is the composition of $\mathrm{Lie}(\mathbb{G}_m \xrightarrow{\lambda} G_x \to L)$ and α, which is a linear map $\mathbb{C} \to \mathbb{C}$ and thus identified with a complex number. Both pairings are compatible in the sense that $\langle -, \chi \rangle = \langle -, r(\chi) \rangle$ for all $\chi \in \mathbb{C} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{Z}}(G_x)$. Therefore, if $\psi_x = \langle -, \chi \rangle$ for some χ, then it also equals $\psi_x = \langle -, r(\chi) \rangle$ and it is thus induced by a Lie algebra character $r(\chi)\colon \mathfrak{l} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}_x \to \mathbb{C}$, which is uniquely determined because the image of the map $\Gamma^{\mathbb{Z}}(L) \to \mathfrak{l}\colon \lambda \to \mathrm{Lie}(\lambda)(1)$ spans $\mathfrak{l}$ by reductivity of L. This shows that (2) implies (3).

Any Lie algebra character respects addition, so it is clear that (3) implies (4).

If $g\colon B\mathbb{G}_m^n \to \mathcal{X}$ is a map, then there is a point $x \in \mathcal{X}(\mathbb{C})$ such that g factors as $B\mathbb{G}_m^n \to BG_x \to \mathcal{X}$. Thus the map $Z_g\colon \Gamma^{\mathbb{Z}}(\mathbb{G}_m^n) \to \mathbb{C}$ induced by g and Z factors through the map $\psi_x\colon \Gamma^{\mathbb{Z}}(G_x) \to \mathbb{C}$ induced by Z and x. Therefore Z_g is additive for all g if (4) is satisfied for all x, and thus (4) implies (1). □

Remark A.7 Together, Lemma A.5 and Proposition A.6 establish that Definitions 2.15 and A.2 are equivalent.

Remark A.8 If $\mathcal{X} = X/G$ is a quotient stack, then we can describe

$$\mathcal{S} = \{(x, \lambda)\colon x \in X(\mathbb{C}), \quad \lambda\colon \mathbb{G}_m \to G_x\}/\sim,$$

where $(x, \lambda) \sim (y, \mu)$ if there is $g \in G(\mathbb{C})$ such that $y = gx$ and $\mu = \lambda^g$.

Therefore, Definition 2.7 of a central charge for the G-scheme X is equivalent to Definition A.2 of a complex linear form on X/G.

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