



Binary Encoded Recursive Generation of Quantum Space-Times

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Abstract. Real geometric algebras distinguish between space and time; complex ones do not. Space-times can be classified in terms of number n of dimensions and metric signature s (number of spatial dimensions *minus* number of temporal dimensions). Real geometric algebras are periodic in s , but recursive in n . Recursion starts from the basis vectors of either the Euclidean plane or the Minkowskian plane. Although the two planes have different geometries, they have the same real geometric algebra. The direct product of the two planes yields Hestenes' space-time algebra. Dimensions can be either open (for space-time) or closed (for the electroweak force). Their product yields the eight-fold way of the strong force. After eight dimensions, the pattern of real geometric algebras repeats. This yields a spontaneously expanding space-time lattice with the physics of the Standard Model at each node. Physics being the same at each node implies conservation laws by Noether's theorem. Conservation laws are not pre-existent; rather, they are consequences of the uniformity of space-time, whose uniformity is a consequence of its recursive generation.

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1. Introduction

An n -dimensional space-time has various geometric elements: 0-dimensional points, 1-dimensional line segments (vectors), 2 -dimensional plane segments

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(bi-vectors), etc., up to the n -dimensional volume element. These geometric elements form the bases of a geometric algebra, in which geometric elements can be represented by square matrices of rank K .

The rank of a tensor \mathbf{T} of type $\binom{I}{J}$ with I contra-variant indices and J co-variant indices is $I+J$. A matrix is a 2nd rank tensor of type $\binom{1}{1}$. Write matrix \mathbf{A} (index-free notation) with K rows by L columns (use K/L , not $K \times L$ for “ K by L ”) in abstract-index notation ([16], p. 240) as A_L^K , whose elements are a_{ℓ}^k , where $k = 0, 1, \dots, K-1$ and $\ell = 0, 1, \dots, L-1$. K is the row rank of the matrix; L is the column rank. For square matrices, $K = L$. Starting the count at 0, rather than at 1, simplifies certain formulas in the binary index notation introduced in [13].

In geometric algebra, the dot product (symmetric product) of vectors represented by square matrices \mathbf{A} and \mathbf{B} is given by

$$\mathbf{A} \cdot \mathbf{B} = \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) = \frac{1}{2}\{\mathbf{A}, \mathbf{B}\} \quad (1)$$

and the wedge product (antisymmetric product) is given by

$$\mathbf{A} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{AB} - \mathbf{BA}) = \frac{1}{2}[\mathbf{A}, \mathbf{B}] \quad (2)$$

([12], p.11). The dot product leads to a Clifford algebra; the wedge product leads to a Grassmann algebra; a geometric algebra is both a Clifford algebra and a Grassmann algebra [4]. The left-hand equalities of equations (1) and (2) give a geometrical interpretation useful for general relativity, while the right-hand equalities give commutators and anti-commutators useful for quantum mechanics. Geometric algebra has been suggested as a unifying language for all of physics [2, 6, 7, 10].

A dot product implies a metric, characterized by the number p of spatial dimensions, the number q of temporal dimensions, and the number n_0 of null dimensions. For non-degenerate metric spaces, $n_0 = 0$. Geometric algebras for non-degenerate space-times have previously been classified in terms of p and q [18]. That classification can be simplified by expressing it in terms of the number of dimensions $n = p + q$ and metric signature $s = p - q$. This paper uses a semicolon rather than a comma to distinguish variables $(n; s)$ from (p, q) . Section 2 reviews real geometric algebras in terms of n and s . Section 3 presents the construction of a compound set of basis vectors as the direct product of two sets of basis vectors, a geometric interpretation of the construction, and a formulation of the recursive relation as a matrix of matrices. Section 4 gives applications of the process of building geometric algebras from the basis vectors of either the Euclidean plane or the Minkowskian plane. It includes complex numbers, dyreal numbers, qubits, quaternions, Pauli spin matrices, Hestenes’ Space-Time Algebra, Dirac gamma matrices, and Kaluza-Klein construction of the Standard Model of physics. It shows how the recursion of real geometric algebras leads to an expanding 4-dimensional space-time lattice with the physics of the Standard Model at each node of the lattice. Finally, it suggests understanding the generation and expansion of the universe in terms of the recursive generation of real geometric algebras. Section 5 recapitulates the recursion in terms of n

and s , discusses the different time scales of the 4-d and 8-d recursion, and touts the unified framework for physics provided by recursive generation.

2. Geometric Algebra

A fundamental concept in geometry is the position vector $\mathbf{r}_{n;s}$, where $n \in (0, 1, 2, \dots)$ is the number of dimensions and $s \in (-n, -n+2, \dots, n-2, n)$ is the metric signature.

2.1. Basis Vectors

If the position is parametrized, at least locally, by coordinates x^i , where $i = 0, 1, \dots, n-1$, then basis vectors are

$$\boldsymbol{\gamma}_{i|n;s} = \frac{\partial \mathbf{r}_{n;s}}{\partial x^i}. \quad (3)$$

The label $n; s$ may be dropped if the context is clear. The dot products of the basis vectors are the metric tensor $g_{ij|n;s}$:

$$\boldsymbol{\gamma}_{i|n;s} \cdot \boldsymbol{\gamma}_{j|n;s} = g_{ij|n;s} \mathbf{I}_{n;s}, \quad (4)$$

where $\mathbf{I}_{n;s}$ is the identity matrix appropriate for dimension n and metric signature s , as discussed below.

Let's use the Einstein summation convention that repeated indices, one sub-scripted and one super-scripted, are to be summed over the n indices. The inverse metric tensor g^{ki} is related to g_{ij} by the definition

$$g^{ki} g_{ij} = \delta_j^k. \quad (5)$$

As usual, the contra-variant form of the basis vectors is generated by using the inverse metric tensor to raise indices:

$$\boldsymbol{\gamma}^k = g^{ki} \boldsymbol{\gamma}_i. \quad (6)$$

These $\boldsymbol{\gamma}^k$ are called “reciprocal vectors.” The $\boldsymbol{\gamma}^k$ space is dual to the $\boldsymbol{\gamma}_k$ space. Multiplying equation (4) by g^{ki} yields

$$\boldsymbol{\gamma}_{n;s}^k \cdot \boldsymbol{\gamma}_{j|n;s} = \delta_j^k \mathbf{I}_{n;s}. \quad (7)$$

Hence

$$\boldsymbol{\gamma}^k \cdot \boldsymbol{\gamma}_j = 0 \text{ for } k \neq j \quad (8)$$

and

$$\boldsymbol{\gamma}^k \cdot \boldsymbol{\gamma}_k = \mathbf{I}_{n;s} \text{ (no sum on } k\text{).} \quad (9)$$

By Eq. (8), $\boldsymbol{\gamma}^k$ is perpendicular to all $\boldsymbol{\gamma}_j$ other than $\boldsymbol{\gamma}_k$, but is not necessarily parallel to $\boldsymbol{\gamma}_k$. However, if the $\boldsymbol{\gamma}_i$ are mutually perpendicular, then $\boldsymbol{\gamma}^k$ is parallel to $\boldsymbol{\gamma}_k$, in which case, $\boldsymbol{\gamma}^k$ and $\boldsymbol{\gamma}_k$ commute. Then

$$\boldsymbol{\gamma}^k \cdot \boldsymbol{\gamma}_k = \boldsymbol{\gamma}^k \boldsymbol{\gamma}_k = \mathbf{I}_{n;s} \text{ (no sum on } k\text{),} \quad (10)$$

so contra-variant vector $\boldsymbol{\gamma}^k$ is the matrix inverse of the co-variant form $\boldsymbol{\gamma}_k$:

$$\boldsymbol{\gamma}^k = (\boldsymbol{\gamma}_k)^{-1} = \frac{\boldsymbol{\gamma}_k}{\boldsymbol{\gamma}_k \cdot \boldsymbol{\gamma}_k} \text{ (no sum on } k\text{)} \quad (11)$$

for mutually orthogonal basis vectors.

Using the Gram-Schmidt process, one can construct, at least locally, orthonormal basis vectors, designated by \mathbf{e} instead of γ , that satisfy

$$\mathbf{e}_{i|n;s} \cdot \mathbf{e}_{j|n;s} = \eta_{ij|n;s} \mathbf{I}_{n;s}, \quad (12)$$

where

$$\eta_{ij} = \begin{cases} +1 & \text{for } i = j = 0, 1, \dots, p-1 \\ -1 & \text{for } i = j = p, p+1, \dots, p+q-1 \\ 0 & \text{for } i \neq j. \end{cases} \quad (13)$$

The \mathbf{e}_i are n orthogonal (since $\eta_{ij} = 0$ for $i \neq j$) anti-commuting matrices, normalized in the sense that the first p of them square to $+\mathbf{I}$ (p space-like basis vectors) and the remaining q of them square to $-\mathbf{I}$ (q space-like basis vectors). In 4 dimensions, the \mathbf{e}_i form a *vierbein*; in n dimensions, a *vielbein*. In Cartesian coordinates in the tangent space,

$$\mathbf{r}_{n;s} = x^i \mathbf{e}_{i|n;s}. \quad (14)$$

Then

$$(\mathbf{r}_{n;s})^2 = ((x^0)^2 + \dots + (x^{p-1})^2 - (ct^0)^2 - \dots - (ct^{q-1})^2) \mathbf{I}_{n;s}. \quad (15)$$

Sums and products of the orthonormal basis vectors $\mathbf{e}_{i|n;s}$ over the real/complex numbers generate a real/complex geometric algebra $\mathbf{R}_{n;s} / \mathbf{C}_n$. Since multiplying a basis vector \mathbf{e}_i by the imaginary unit i makes space-like basis vectors time-like, and vice versa, complex geometric algebras do not distinguish between space and time. Therefore let's restrict our attention to real geometric algebras.

2.2. Isomorphisms

Real geometric algebras for non-degenerate space-times have previously been classified in terms of the number of spatial dimensions p and the number of temporal dimensions q ([18], p.133). That classification can be simplified by expressing it in terms of the number of dimensions $n = p + q$ and the metric signature $s = p - q$, instead of p and q . In terms of n and s , the periodic table for real geometric algebras $\mathbf{R}_{n;s}$ ([12], p. 217) is shown in Table 1. The sign σ_s of the square of the n -volume is defined in Eq. (28).

TABLE 1. Periodic table for real geometric algebras $\mathbf{R}_{n;s}$
(n -volume is space-like for $\sigma_s = +1$, time-like for $\sigma_s = -1$)

4	$\mathbf{H}(2)$	$\mathbf{H}(2)$	$\mathbf{R}(4)$	$\mathbf{R}(4)$	$\mathbf{H}(2)$
3	${}^2\mathbf{H}$	$\mathbf{C}(2)$	${}^2\mathbf{R}(2)$	$\mathbf{C}(2)$	
2	\mathbf{H}	$\mathbf{R}(2)$	$\mathbf{R}(2)$		
1	\mathbf{C}	${}^2\mathbf{R}$			
$n = 0$					\mathbf{R}
$s =$	-4	-3	-2	-1	0
$\sigma_s =$	+1	+1	-1	-1	+1

$\mathbf{R}(K)$, $\mathbf{C}(K)$, and $\mathbf{H}(K)$ are K by K matrices with real, complex, and quaternionic entries, respectively, and the double fields ${}^2\mathbf{K}(K) = \mathbf{K}(K) \oplus \mathbf{K}(K)$ can be represented by 2 by 2 matrices with K by K matrices of kind \mathbf{K} along the diagonal

$$\begin{pmatrix} \mathbf{K}(K) & 0 \\ 0 & \mathbf{K}(K) \end{pmatrix}. \quad (16)$$

Elements of ${}^2\mathbf{H} \cong {}^2\mathbf{R} \otimes \mathbf{H}$ are often called *split-biquaternions*, but Rosenfeld calls them *dyquarterions*, to be distinguished from *biquaternions*, $\mathbf{C} \otimes \mathbf{H} \cong \mathbf{C}(2)$ ([20], p. 48). In the same way, elements of ${}^2\mathbf{R}$ have been given a variety of names, but are better called *dyreals*.

Euclidean geometries have only space-like dimensions and no time-like dimensions ($q = 0$), and so are to be found on the rightmost diagonal ($s = n$) of Table 1; Minkowskian geometries have one time-like dimension ($q = 1$), and so are to be found on the diagonal ($s = n - 2$) immediately to the left of the rightmost one. Universes with more than one time-like dimension cannot have coherent histories and so can be excluded as unphysical.

Real geometric algebras $\mathbf{R}_{n;s}$ are isomorphic to matrix algebras, with the kind of matrix depending only on the metric signature s and the rank of that kind of matrix depending only on the number of dimensions n . The dependence of matrix type on s is $|(s+3)_{mod} 8 - 4|$, not $|(s-1)_{mod} 8|$ found in ([22], pp. 300). Explicitly, $\mathbf{R}_{n;s}$ is isomorphic to algebras of dyreal ${}^2\mathbf{R}(2^{\frac{n-1}{2}})$, real $\mathbf{R}(2^{\frac{n}{2}})$, complex $\mathbf{C}(2^{\frac{n-1}{2}})$, quaternionic $\mathbf{H}(2^{\frac{n-2}{2}})$, or dyquaternionic ${}^2\mathbf{H}(2^{\frac{n-3}{2}})$ matrices, for $|(s+3)_{mod} 8 - 4| = 0, 1, 2, 3$, or 4, respectively. Because of this separation of variables with matrix kind depending only on s and matrix rank depending only on n , $\mathbf{R}_{n;s}$ is a more useful form than $\mathbf{R}_{p,q}$. The isomorphisms of real geometric algebras $\mathbf{R}_{n;s}$ are shown in Table 2.

The periodic table of geometric algebras can be extended by a number of theorems, usually expressed in terms of p and q , but better expressed in terms of n and s . The periodic table is extended horizontally in s by the theorem

$$\mathbf{R}_{p,q} \cong \mathbf{R}_{p+4k, q-4k} \quad (17)$$

TABLE 2. Isomorphisms of real geometric algebras $\mathbf{R}_{n;s}$

$ (s+3)_{mod} 8 - 4 $	Matrix
0	${}^2\mathbf{R}(2^{\frac{n-1}{2}})$
1	$\mathbf{R}(2^{\frac{n}{2}})$
2	$\mathbf{C}(2^{\frac{n-1}{2}})$
3	$\mathbf{H}(2^{\frac{n-2}{2}})$
4	${}^2\mathbf{H}(2^{\frac{n-3}{2}})$

for any integer k such that $p + 4k \geq 0$ and $q - 4k \geq 0$ ([22], p. 296), which can be rewritten as

$$\boxed{\text{Periodicity in } s : \mathbf{R}_{n;s} \cong \mathbf{R}_{n;s+8k}} \quad (18)$$

for any integer k such that $|s + 8k| \leq n$. This shows that real geometric algebras are periodic in metric signature s , not in dimension n . This 8-fold periodicity, called Bott periodicity, was proven for Clifford algebras in [1].

The periodic table can be extended vertically in n by adding two more dimensions, one space-like and one time-like, so as to preserve the signature.

$$\boxed{\text{Recursion in } n : \mathbf{R}_{n;s} \otimes \mathbf{R}_{2;0} = \mathbf{R}_{n+2;s}.} \quad (19)$$

For each two additional dimensions, the rank of the matrix representation doubles. This forms the foundation of recursive generation of larger geometric algebras discussed below. $\mathbf{R}_{n;s}$ is periodic in s , but recursive in n , which is further reason to prefer $\mathbf{R}_{n;s}$ over $\mathbf{R}_{p,q}$. Porteous's "periodicity theorem" in q ([18] p. 133) conflates periodicity in s and recursion in n .

2.3. Binary Index Notation

A bit is a unit real number: $+1$ or -1 . The binary index b of $\text{bit}_b \in (0, 1)$ is such that

$$\text{bit}_b = (-1)^b. \quad (20)$$

The elements $\mathbf{E}_{b|n;s}$ of a geometric algebra $\mathbf{R}_{n;s}$ are generated by all possible products of the basis vectors $\mathbf{e}_{i|n;s}$. By repeated application of equation (12), the basis vectors in a product can be arranged with the indices increasing:

$$\mathbf{E}_{b|n;s} = \mathbf{E}_{b_0 \dots b_i \dots b_{n-1}|n;s} = (\mathbf{e}_{0|n;s})^{b_0} \dots (\mathbf{e}_{i|n;s})^{b_i} \dots (\mathbf{e}_{(n-1)|n;s})^{b_{n-1}}. \quad (21)$$

By Eq. (11), the inverse $(\mathbf{E}_{b|n;s})^{-1}$ is

$${}^{b|n;s} \mathbf{E} = {}^{b_{n-1} \dots b_i \dots b_0|n;s} \mathbf{E} = (\mathbf{e}^{(n-1)|n;s})^{b_{n-1}} \dots (\mathbf{e}^{i|n;s})^{b_i} \dots (\mathbf{e}^{0|n;s})^{b_0}, \quad (22)$$

where, as usual, superscripts/subscripts designate contra-/co-variance and, following [13], prescripts/postscripts designate descending/ascending order. Since $\mathbf{e}_i^2 = \pm \mathbf{I}$, each of the n \mathbf{e}_i occurs in the ordered product \mathbf{E}_b either once or not at all; hence each b_i is either 1 or 0. These possible values of b_i led [13] to treat b_i as the bits of a binary number b :

$$b = \sum_{i=0}^{n-1} b_i 2^i \equiv b_i 2^i, \quad (23)$$

where the Einstein summation convention is extended to exponents as well as to superscripts. The binary label b of the geometric element \mathbf{E}_b for the basis vector \mathbf{e}_i is a power of two: $\mathbf{E}_{2^i} = \mathbf{e}_i$. For binary labels that are not powers of two, the geometric element is the unique product of those basis vectors whose labels sum to the label of the element: for example, $\mathbf{E}_5 = \mathbf{E}_1 \mathbf{E}_4 = \mathbf{e}_0 \mathbf{e}_2$.

The grade g_b of element $\mathbf{E}_{b|n;s}$ is the number of basis vectors it contains:

$$g_b = \sum_{i=0}^{n-1} b_i \equiv b_i 1^i. \quad (24)$$

In the geometric algebra for n dimensions with signature s , there are 2^n geometric elements, labeled by the binary index b , which runs from 0 to $2^n - 1$. The zeroth element is

$$\mathbf{E}_{0|n;s} = \mathbf{E}_{0\dots 0\dots 0} = (\mathbf{e}_{0|n;s})^0 \cdots (\mathbf{e}_{i|n;s})^0 \cdots (\mathbf{e}_{(n-1)|n;s})^0 = \mathbf{I}_{n;s}, \quad (25)$$

where $\mathbf{I}_{n;s}$ is the identity matrix of the rank and kind appropriate for that number of dimensions and metric signature, as given in the previous section. (Equation (1.10) of [13] erroneously identified $\mathbf{I}_{n;s}$ as \mathbf{I}_n , the $n \times n$ identity matrix. $\mathbf{I}_{n;s}$ happens to be \mathbf{I}_n , in the cases $n = 2$ or 4 , $s = 0$ or 2 , but not in general.) The last element of the geometric algebra is the n -dimensional volume element

$$\mathbf{E}_{2^n-1} = \mathbf{E}_{1\dots 1\dots 1} = (\mathbf{e}_{0|n;s})^1 \cdots (\mathbf{e}_{i|n;s})^1 \cdots (\mathbf{e}_{(n-1)|n;s})^1 = \mathbf{V}_{n;s}. \quad (26)$$

By repeated application of equation (13),

$$(\mathbf{V}_{n;s})^2 = (-1)^{\frac{n(n-1)}{2}} (-\mathbf{I}_{n;s})^q (\mathbf{I}_{n;s})^p = (-1)^{\frac{s(s-1)}{2}} \mathbf{I}_{n;s}, \quad (27)$$

where the factor $(-1)^{\frac{n(n-1)}{2}}$ arises from reversing the order of the n elements of $\mathbf{V}_{n;s}$. Introducing the sign

$$\sigma_s = (-1)^{\frac{s(s-1)}{2}} \quad (28)$$

gives

$$(\mathbf{V}_{n;s})^2 = \sigma_s \mathbf{I}_{n;s}. \quad (29)$$

Thus the volume element $\mathbf{V}_{n;s}$ is space-like for $s_{mod\ 4} = 0$ or 1 and time-like for $s_{mod\ 4} = 2$ or 3 . The inverse volume element is

$$(\mathbf{V}_{n;s})^{-1} = {}^{n;s}\mathbf{V} = \sigma_s \mathbf{V}_{n;s}. \quad (30)$$

Likewise, by reversion,

$${}_{n;s}\mathbf{V} = \sigma_n \mathbf{V}_{n;s} \quad (31)$$

and

$$\mathbf{V}^{n;s} = \sigma_n \sigma_s \mathbf{V}_{n;s} = (-1)^{(n-s)/2} \mathbf{V}_{n;s} = (-1)^q \mathbf{V}_{n;s}, \quad (32)$$

as expected from the q time-like basis vectors.

The Hodge dual is usually defined as $\mathbf{E}_{b|n;s} \mathbf{V}_{n;s}$ ([12], pp 38-39). A better definition is

$${}^*\mathbf{E}_{b|n;s} = {}^{n;s}\mathbf{V} \mathbf{E}_{b|n;s} = h_b \mathbf{E}^{\bar{b}|n;s}, \quad (33)$$

where \bar{b} is the bit inverse of b and

$$h_b = \prod_{i=0}^{n-1} (-1)^{b_i(n-i-1)}. \quad (34)$$

The Hodge dual is an element of the geometric algebra of the cotangent space.

For odd/even strictly positive n , the volume element $\mathbf{V}_{n;s}$ commutes/anti-commutes with each basis vector:

$$\mathbf{e}_{i|n;s} \mathbf{V}_{n;s} = (-1)^{n-1} \mathbf{V}_{n;s} \mathbf{e}_{i|n;s} \text{ for } n \geq 1. \quad (35)$$

The commutativity/anti-commutativity of the n -volume is used below in the construction of larger geometric algebras as direct products of smaller geometric algebras. See [3] and [13] for additional applications of binary notation.

3. Construction of Larger Geometric Algebras

Let's build larger geometric algebras from smaller ones by using the direct product. Consider two real geometric algebras, $\mathbf{R}_{n;s}$ with n orthonormal basis vectors \mathbf{e}_i with signature s , and $\mathbf{R}_{n';s'}$ with n' orthonormal basis vectors $\mathbf{e}_{i'}$ with signature s' . Construct basis vectors:

$$\mathbf{e}_{i|n'';s''} = \mathbf{e}_{i|n;s} \otimes \mathbf{I}_{n';s'} \text{ for } i = 0, 1, \dots, n-1; \quad (36)$$

$$\mathbf{e}_{n+i'|n'';s''} = \mathbf{V}_{n;s} \otimes \mathbf{e}_{i'|n';s'} \text{ for } i' = 0, 1, \dots, n'-1. \quad (37)$$

The n basis vectors of equation (36) are orthonormal because of the orthonormality of the original \mathbf{e}_i . The n' basis vectors of equation (37) are orthonormal because of the orthonormality of the original $\mathbf{e}_{i'}$; they are orthogonal to the n basis vectors of equation (36) by Eq. (35), provided n is a strictly positive even integer. For strictly positive even n , the $(n + n')$ basis vectors $\mathbf{e}_{i''|n'';s''}$ form an orthogonal set of basis vectors. While this construction is not unique, it does have a simple geometrical interpretation: the n basis vectors of equation (36) are the n basis vectors $\mathbf{e}_{i|n;s}$ extruded into the n' additional dimensions without change, while the n' basis vectors of equation (37) are the n' basis vectors $\mathbf{e}_{i'|n';s'}$ appended onto n -volume element $\mathbf{V}_{n;s}$.

The space-like or time-like nature of each basis vector can be found by squaring each one. Since $(\mathbf{A} \otimes \mathbf{B})^2 = \mathbf{A}^2 \otimes \mathbf{B}^2$,

$$(\mathbf{e}_{i|n'';s''})^2 = (\mathbf{e}_{i|n;s})^2 \otimes (\mathbf{I}_{n';s'})^2 \text{ for } i = 0, 1, \dots, n-1; \quad (38)$$

$$(\mathbf{e}_{n+i'|n'';s''})^2 = (\mathbf{V}_{n;s})^2 \otimes (\mathbf{e}_{i'|n';s'})^2 \text{ for } i' = 0, 1, \dots, n'-1. \quad (39)$$

Each of the first n basis vectors $\mathbf{e}_{i|n;s} \otimes \mathbf{I}_{n';s'}$ in the product space has the same space-like or time-like nature as the corresponding basis vector $\mathbf{e}_{i|n;s}$. The space-like or time-like nature of the last n' basis vectors $\mathbf{V}_{n;s} \otimes \mathbf{e}_{i'|n';s'}$ depends on the sign of $(\mathbf{V}_{n;s})^2$. If the n -volume is space-like, $(\mathbf{V}_{n;s})^2 = +\mathbf{I}_{n;s}$, each of the n' basis vectors $\mathbf{V}_{n;s} \otimes \mathbf{e}_{i'|n';s'}$ has the same space-like or time-like nature as the corresponding $\mathbf{e}_{i'|n';s'}$. If the n -volume is time-like, $(\mathbf{V}_{n;s})^2 = -\mathbf{I}_{n;s}$, each of the n' basis vectors $\mathbf{V}_{n;s} \otimes \mathbf{e}_{i'|n';s'}$ changes from space-like to time-like and vice versa. Thus for strictly positive even n , the direct product of $\mathbf{R}_{n;s}$ with n orthonormal basis vectors e_i with signature s , and $\mathbf{R}_{n';s'}$ with n' orthonormal basis vectors $e_{i'}$ with signature s' , is isomorphic to $\mathbf{R}_{n+n';s+s'\sigma_s}$ with $n + n'$ orthonormal basis vectors $e_{i''}$ with signature $s + s'\sigma_s$:

$$\mathbf{R}_{n;s} \otimes \mathbf{R}_{n';s'} = \mathbf{R}_{n+n';s+s'\sigma_s} \text{ for } n = 2m > 0. \quad (40)$$

Similarly,

$$\mathbf{e}_{i'|n'';s''} = \mathbf{I}_{n;s} \otimes \mathbf{e}_{i'|n';s'} \text{ for } i' = 0, 1, \dots, n'-1; \quad (41)$$

$$\mathbf{e}_{n'+i|n'';s''} = \mathbf{e}_{i|n;s} \otimes \mathbf{V}_{n';s'} \text{ for } i = 0, 1, \dots, n-1. \quad (42)$$

In that case,

$$\mathbf{R}_{n;s} \otimes \mathbf{R}_{n';s'} = \mathbf{R}_{n'+n;s+s'\sigma_s} \text{ for } n' = 2m' > 0. \quad (43)$$

For $n' = 2, s' = 0$, Eq. (43) becomes

$$\mathbf{R}_{n;s} \otimes \mathbf{R}_{2;0} = \mathbf{R}_{n+2;s}. \quad (44)$$

This proves the recursion relation (Eq. 19) by construction. Since direct products can be written as matrices of matrices, the recursion relation can be written as

$$\mathbf{r}_{n+2;s} = \begin{pmatrix} \mathbf{r}_{n;s} & (x^p + ct^q)\mathbf{V}_{n;s} \\ (x^p - ct^q)\boldsymbol{\sigma}_s \mathbf{V}_{n;s} & (-1)^n \mathbf{r}_{n;s} \end{pmatrix}. \quad (45)$$

Each geometric algebra can be constructed from the one directly below it in the periodic table of geometric algebras. The matrix representation doubles in rank for each increase of n by 2 dimensions, keeping s constant. That is to say, adding the next space-like dimension x^p and the next time-like dimension t^q doubles the rank of the matrix representation.

4. Applications

4.1. The Minkowskian Plane

$\mathbf{R}_{2;0}$ has one space-like basis vector, $\mathbf{e}_{0|2;0}$, and one time-like basis vector, $\mathbf{e}_{1|2;0}$. By Eq. (21), the geometric elements of $\mathbf{R}_{2;0}$ are

$$\mathbf{E}_{0|2;0} = \mathbf{E}_{00|2;0} = (\mathbf{e}_{0|2;0})^0 (\mathbf{e}_{1|2;0})^0 = \mathbf{I}_{2;0}; \quad (46)$$

$$\mathbf{E}_{1|2;0} = \mathbf{E}_{10|2;0} = (\mathbf{e}_{0|2;0})^1 (\mathbf{e}_{1|2;0})^0 = \mathbf{e}_{0|2;0}; \quad (47)$$

$$\mathbf{E}_{2|2;0} = \mathbf{E}_{01|2;0} = (\mathbf{e}_{0|2;0})^0 (\mathbf{e}_{1|2;0})^1 = \mathbf{e}_{1|2;0}; \quad (48)$$

$$\mathbf{E}_{3|2;0} = \mathbf{E}_{11|2;0} = (\mathbf{e}_{0|2;0})^1 (\mathbf{e}_{1|2;0})^1 = \mathbf{e}_{0|2;0} \mathbf{e}_{1|2;0} = \mathbf{V}_{2;0}. \quad (49)$$

$\mathbf{V}_{2;0}$, the element of area in the Minkowskian plane, is space-like.

4.2. The Euclidean Plane

$\mathbf{R}_{2;2}$ has two space-like basis vectors, $\mathbf{e}_{0|2;2}$ and $\mathbf{e}_{1|2;2}$. By equation (21), the geometric elements of $\mathbf{R}_{2;2}$ are

$$\mathbf{E}_{0|2;2} = \mathbf{E}_{00|2;2} = (\mathbf{e}_{0|2;2})^0 (\mathbf{e}_{1|2;2})^0 = \mathbf{I}_{2;2}; \quad (50)$$

$$\mathbf{E}_{1|2;2} = \mathbf{E}_{10|2;2} = (\mathbf{e}_{0|2;2})^1 (\mathbf{e}_{1|2;2})^0 = \mathbf{e}_{0|2;2}; \quad (51)$$

$$\mathbf{E}_{2|2;2} = \mathbf{E}_{01|2;2} = (\mathbf{e}_{0|2;2})^0 (\mathbf{e}_{1|2;2})^1 = \mathbf{e}_{1|2;2}; \quad (52)$$

$$\mathbf{E}_{3|2;2} = \mathbf{E}_{11|2;2} = (\mathbf{e}_{0|2;2})^1 (\mathbf{e}_{1|2;2})^1 = \mathbf{e}_{0|2;2} \mathbf{e}_{1|2;2} = \mathbf{V}_{2;2}. \quad (53)$$

$\mathbf{V}_{2;2}$, the element of area in the Euclidean plane, is time-like.

4.3. Different Geometries; Same Geometric Algebra

Both $\mathbf{R}_{2;0}$, generated by basis vectors in the Minkowskian plane, and $\mathbf{R}_{2;2}$, generated by basis vectors in the Euclidean plane, are isomorphic to the matrix algebra $\mathbf{R}(2)$ of 2×2 matrices of real numbers. Indeed, identify

$$\mathbf{I}_{2;0} = \mathbf{I}_{2;2} = \mathbf{I}_2; \quad (54)$$

$$\mathbf{e}_{0|2;0} = \mathbf{e}_{0|2;2} = \mathbf{X}_2; \quad (55)$$

$$\mathbf{e}_{1|2;0} = \mathbf{V}_{2;2} = \mathbf{T}_2; \quad (56)$$

$$\mathbf{e}_{1|2;2} = \mathbf{V}_{2;0} = \mathbf{Y}_2. \quad (57)$$

The consistency of these assignments is established by noting that $\mathbf{V}_{2;0} = \mathbf{X}_2 \mathbf{T}_2 = \mathbf{Y}_2$ is consistent with $\mathbf{V}_{2;2} = \mathbf{X}_2 \mathbf{Y}_2 = \mathbf{X}_2 (\mathbf{X}_2 \mathbf{T}_2) = \mathbf{T}_2$.

Although the Euclidean plane and the Minkowskian plane do not have the same geometry, they have the same geometric algebra, $\mathbf{R}(2)$ ([11]). Since the basis elements of $\mathbf{R}(2)$, namely \mathbf{I}_2 , \mathbf{T}_2 , \mathbf{X}_2 , and \mathbf{Y}_2 , can be used via Eq. (40) to construct the elements of all geometric algebras, $\mathbf{R}(2)$ can be taken as foundational. $\mathbf{R}(2)$ itself can be generated by any two of the three basis vectors: \mathbf{T}_2 , \mathbf{X}_2 , and \mathbf{Y}_2 . There is no unique set of elements out of which $\mathbf{R}(2)$ is generated: the basis vectors of either the Euclidean plane or the Minkowskian plane will do, so neither can be taken as more fundamental. $\mathbf{R}(2)$ is indeed foundational – it can be used to construct all larger space-times, but it itself is not composed of something more fundamental.

Any element \mathbf{R}_2 of $\mathbf{R}(2)$ can be decomposed into a trace, an antisymmetric matrix, and a symmetric trace-free matrix:

$$\mathbf{R}_2 = \frac{1}{2} \text{Tr}(\mathbf{R}_2) \mathbf{I}_2 + \frac{1}{2} (\mathbf{R}_2 - \mathbf{R}_2^\top) + \frac{1}{2} (\mathbf{R}_2 + \mathbf{R}_2^\top - \text{Tr}(\mathbf{R}_2) \mathbf{I}_2), \quad (58)$$

where \mathbf{A}^\top is the transpose of matrix \mathbf{A} . The trace of a matrix is a scalar, with no directionality. Entities with direction, such as vectors and bi-vectors, are trace-free. Antisymmetric matrices are necessarily trace-free. Real antisymmetric 2×2 matrices \mathbf{A}_2 are of the form:

$$\mathbf{A}_2 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}. \quad (59)$$

Now $\mathbf{A}_2^2 = -a^2 \mathbf{I}_2$, so \mathbf{A}_2 is necessarily time-like. The unit antisymmetric real 2×2 matrix has $a = \pm 1$, so a is on the 0-dimensional unit sphere $\mathbf{S}^0 = \{x_0 \in \mathbf{R} | x_0^2 = 1\}$. In the Euclidean plane, the two signs correspond to the two sides of the ordered element of area $\mathbf{V}_{2;2}$. In the Minkowskian plane, the two signs correspond to particles and antiparticles moving forward and backward, respectively, through time.

Real symmetric trace-free 2×2 matrices \mathbf{S}_2 are of the form:

$$\mathbf{S}_2 = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}. \quad (60)$$

Now $\mathbf{S}_2^2 = (b^2 + c^2) \mathbf{I}_2$, so \mathbf{S}_2 is necessarily space-like. Thus the decomposition in equation (58) separates \mathbf{R}_2 into scalar, time-like, and space-like parts. The unit symmetric real 2×2 matrix has (b, c) on the 1-dimensional unit sphere, i.e. a circle, $\mathbf{S}^1 = \{x_i \in \mathbf{R}^2 | x_0^2 + x_1^2 = 1\}$. So space has an additional degree of freedom, θ , corresponding to rotation in the Euclidean plane:

$$\mathbf{S}_2 = \begin{pmatrix} \cos \theta & \pm \sin \theta \\ \pm \sin \theta & -\cos \theta \end{pmatrix}, \quad (61)$$

with the \pm sign corresponding to clockwise or counterclockwise rotation. Without loss of generality, one can choose θ as the angle of \mathbf{S}_2 from the \mathbf{X}_2 axis going towards the \mathbf{Y}_2 axis. Then

$$\mathbf{S}_2 = \mathbf{X}_2 \cos \theta + \mathbf{Y}_2 \sin \theta. \quad (62)$$

A matrix representation of a vector in the Euclidean plane is thus

$$\mathbf{r}_{2;2} = r \begin{pmatrix} \cos \theta \sin \theta \\ \sin \theta - \cos \theta \end{pmatrix} = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} = x\mathbf{X}_2 + y\mathbf{Y}_2. \quad (63)$$

With that choice of θ ,

$$\mathbf{T}_2 = \mathbf{X}_2 \mathbf{Y}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (64)$$

In the Euclidean plane, $\mathbf{A} \mathbf{r}_{2;2} \mathbf{A}^{-1}$ reflects vector $\mathbf{r}_{2;2}$ over vector \mathbf{A} . Consequently, $(\mathbf{B}\mathbf{A})\mathbf{r}_{2;2}(\mathbf{B}\mathbf{A})^{-1}$ reflects vector $\mathbf{r}_{2;2}$ over vector \mathbf{A} and then over vector \mathbf{B} ; that is, it rotates vector $\mathbf{r}_{2;2}$ in the direction from \mathbf{A} to \mathbf{B} by twice the angle between \mathbf{A} and \mathbf{B} ([12], p. 13-14). In $\mathbf{R}_{2;2}$, the geometric algebra of the Euclidean plane, bi-vectors are generators of rotations. Euler's relation becomes

$$e^{i\theta} \mapsto e^{\mathbf{T}_2\theta} = e^{\mathbf{X}_2 \mathbf{Y}_2 \theta} = \mathbf{I} \cos \theta + \mathbf{X}_2 \mathbf{Y}_2 \sin \theta. \quad (65)$$

$\mathbf{X}_2 \mathbf{Y}_2$ is a real matrix representation of the imaginary number i in Euler's relation. The operation

$$\mathbf{r}'_{2;2} = e^{\mathbf{Y}_2 \mathbf{X}_2 \theta / 2} \mathbf{r}_{2;2} e^{\mathbf{X}_2 \mathbf{Y}_2 \theta / 2} \quad (66)$$

actively rotates vector $\mathbf{r}_{2;2}$ in the direction from \mathbf{X}_2 towards \mathbf{Y}_2 by angle θ .

A matrix representation of a vector in the Minkowskian plane is

$$\mathbf{r}_{2;0} = x \mathbf{X}_2 + y \mathbf{T}_2 = \begin{pmatrix} x & y \\ -y & -x \end{pmatrix}, \quad (67)$$

where $y = ct$. Note the difference in signs between (63) and (67).

In $\mathbf{R}_{2;2}$, the geometric algebra of the Euclidean plane, bi-vectors are generators of rotations. In $\mathbf{R}_{2;0}$, the geometric algebra of the Minkowskian plane, bi-vectors are generators of boosts. There Euler's relation is

$$e^{\mathbf{X}_2 \mathbf{T}_2 \alpha} = \mathbf{I}_2 \cosh \alpha + \mathbf{X}_2 \mathbf{T}_2 \sinh \alpha. \quad (68)$$

The operation

$$\mathbf{r}'_{2;0} = e^{\mathbf{T}_2 \mathbf{X}_2 \alpha / 2} \mathbf{r}_{2;0} e^{\mathbf{X}_2 \mathbf{T}_2 \alpha / 2} \quad (69)$$

actively boosts vector $\mathbf{r}_{2;0}$ by rapidity α , where $\tanh \alpha = \beta = v/c$.

4.4. Complex Numbers

A matrix representation of complex numbers in polar coordinates $C = r e^{i\theta}$ becomes

$$\mathbf{C}_2 = r e^{\mathbf{T}_2 \theta} = r \cos \theta \mathbf{I}_2 + r \sin \theta \mathbf{T}_2 = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \quad (70)$$

Note the difference in signs among (63), (67), and (70). Geometrically, the so-called "complex plane" is not a plane since \mathbf{I}_2 is a scalar, not a vector. \mathbf{T}_2 , which squares to $-\mathbf{I}_2$, is a real matrix representation of the imaginary number i . By Eq. (64), \mathbf{T}_2 , the i of ict , is the same as $\mathbf{X}_2 \mathbf{Y}_2$, the i of $e^{i\theta}$. However, as we shall see below, it differs from the i of $\sigma_x \sigma_y \sigma_z$ or of $i\hbar$.

4.5. Dyreal Numbers

A matrix representation of dyreal numbers is

$$\mathbf{D}_2 = x\mathbf{I}_2 + y\mathbf{Y}_2 = \begin{pmatrix} x & y \\ y & x \end{pmatrix}. \quad (71)$$

Again, note the differences in signs among (63), (67), (70), and (71). This exhausts the four choices of signs (either same or opposite on either major or minor axis). Like complex numbers, dyreal numbers do not form a plane.

4.6. Qubits

Qubits are usually described by complex amplitudes of two orthogonal states:

$$|\mathbf{Q}\rangle = C_0 |\mathbf{0}\rangle + C_1 |\mathbf{1}\rangle, \quad (72)$$

where $C_0^*C_0 + C_1^*C_1 = 1$. In geometric algebra, the two orthogonal states can be taken as \mathbf{X}_2 and \mathbf{Y}_2 . Then

$$\mathbf{Q} = \begin{pmatrix} C_0 & C_1 \\ C_1 & -C_0 \end{pmatrix}, \quad (73)$$

where $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}_2$. This \mathbf{Q} is an element of $\mathbf{C}(2)$ and so describes 3 real dimensions – the three axes of the Bloch sphere. It is really a *qutrit*, not a qubit. In geometric algebra, a qubit is

$$\mathbf{Q} = \cos \theta \mathbf{X}_2 + \sin \theta \mathbf{Y}_2 = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \quad (74)$$

where $\mathbf{Q}\mathbf{Q} = \mathbf{I}_2$. This is a unit vector in the Euclidean plane, not a complex number. [19] recently asserted that quantum theory based on real numbers can be experimentally falsified. Unfortunately, [19] chose as the real density matrix $\bar{\rho}$ corresponding to the complex density matrix ρ

$$\bar{\rho} = \Re(\rho) \otimes \frac{1}{2}\mathbf{I}_2 + \Im(\rho) \otimes \frac{1}{2}\mathbf{X}_2\mathbf{Y}_2, \quad (75)$$

instead of one constructed with anti-commuting basis vectors

$$\bar{\rho} = \Re(\rho) \otimes \mathbf{X}_2 + \Im(\rho) \otimes \mathbf{Y}_2. \quad (76)$$

\mathbf{I}_2 and $\mathbf{X}_2\mathbf{Y}_2$ commute, but \mathbf{X}_2 and \mathbf{Y}_2 anti-commute, thereby guaranteeing that if complex ρ is orthonormal, so is real $\bar{\rho}$. Using anti-commuting basis vectors enables real quantum mechanics to reproduce the results of complex quantum mechanics, including Bell correlations. Thus we can use Euclidean spaces, which have anti-commuting basis vectors over the real numbers, in place of Hilbert spaces over complex numbers for all of quantum mechanics.

4.7. Quaternions

By Eq. (40), $\mathbf{R}_{2;0} \otimes \mathbf{R}_{2;0} = \mathbf{R}_{4;0}$. By equations (36) and (37), the basis vectors of $\mathbf{R}_{4;0}$ are

$$\begin{aligned} \mathbf{e}_{0|4;0} &= \mathbf{X}_2 \otimes \mathbf{I}_2, \text{ which is space-like;} \\ \mathbf{e}_{1|4;0} &= \mathbf{T}_2 \otimes \mathbf{I}_2, \text{ which is time-like;} \\ \mathbf{e}_{2|4;0} &= \mathbf{Y}_2 \otimes \mathbf{X}_2, \text{ which is space-like;} \\ \mathbf{e}_{3|4;0} &= \mathbf{Y}_2 \otimes \mathbf{T}_2, \text{ which is time-like.} \end{aligned} \quad (77)$$

The time-like basis vectors can be chosen as the first two quaternions:

$$\begin{aligned}\mathbf{i} &= \mathbf{T}_2 \otimes \mathbf{I}_2; \\ \mathbf{j} &= \mathbf{Y}_2 \otimes \mathbf{T}_2.\end{aligned}\quad (78)$$

Then

$$\mathbf{k} = \mathbf{ij} = \mathbf{X}_2 \otimes \mathbf{T}_2. \quad (79)$$

Thus the quaternions can be expressed in terms of $\mathbf{R}(2)$.

4.8. Pauli Algebra

By Eq. (40), $\mathbf{R}_{2;2} \otimes \mathbf{R}_{1;-1} = \mathbf{R}_{3;3}$ or $\mathbf{R}(2) \otimes \mathbf{C} = \mathbf{C}(2)$. By equations (36) and (37), the basis vectors of $\mathbf{R}_{3;3}$ are

$$\begin{aligned}\mathbf{e}_{0|3;3} &= \mathbf{X}_2 \otimes \mathbf{I}_2, \text{ which is space-like - call it } \mathbf{X}_3; \\ \mathbf{e}_{1|3;3} &= \mathbf{Y}_2 \otimes \mathbf{I}_2, \text{ which is time-like - call it } \mathbf{Y}_3; \\ \mathbf{e}_{2|3;3} &= \mathbf{T}_2 \otimes \mathbf{T}_2, \text{ which is space-like - call it } \mathbf{Z}_3;\end{aligned}\quad (80)$$

This is 3-dimensional Euclidean space. The volume element $\mathbf{V}_{3;3} = \mathbf{X}_3 \mathbf{Y}_3 \mathbf{Z}_3 = -\mathbf{I}_2 \otimes \mathbf{T}_2$ squares to $-\mathbf{I}_4$ and can be taken as the imaginary i_3 . One usually uses complex variables and simply writes i_3 as i , but the use of real variables shows that $i_3 = \mathbf{V}_{3;3}$, a time-like 3-volume, is geometrically different from $i_1 = \mathbf{T}_2$, the time-like vector of ict , and $i_2 = \mathbf{X}_2 \mathbf{Y}_2$, the time-like bivector of $e^{i\theta}$. By Eq. (64), i_1 can be equated to i_2 , since both are elements of $\mathbf{R}(2)$, but i_3 is an element of $\mathbf{R}(4)$. From Eq. (45),

$$\mathbf{r}_{3;3} = \begin{pmatrix} \mathbf{r}_{2;2} & z\mathbf{V}_{2;2} \\ z\boldsymbol{\sigma}_2 \mathbf{V}_{2;2} & (-1)^2 \mathbf{r}_{2;2} \end{pmatrix} = \begin{pmatrix} x & y & 0 & z \\ y & -x & -z & 0 \\ 0 & -z & x & y \\ z & 0 & y & -x \end{pmatrix} = x\mathbf{X}_3 + y\mathbf{Y}_3 + z\mathbf{Z}_3. \quad (81)$$

This shows how 3-dimensional Euclidean space, represented by 4×4 real matrices, is generated by the geometrical elements of 2-dimensional Euclidean space, represented by 2×2 real matrices. The Pauli representation of $\mathbf{r}_{3;3}$ in terms of 2×2 complex matrices $\mathbf{C}(2)$,

$$\mathbf{r}_{3;3} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = z\mathbf{X}_2 + x\mathbf{Y}_2 - iy\mathbf{V}_{2;2}, \quad (82)$$

obscures its 2-dimensional roots. In quantum information theory, \mathbf{X}_2 and \mathbf{Y}_2 are called Pauli-Z and Pauli-X, respectively. $\mathbf{C}(2)$ also leads to loop quantum gravity [21], which can be reformulated geometrically in terms of $\mathbf{R}(2)$.

4.9. Space-Time Algebra

By Eq. (40), $\mathbf{R}_{2;2} \otimes \mathbf{R}_{2;0} = \mathbf{R}_{4;2}$. By Eq. (45),

$$\begin{aligned}\mathbf{r}_{4;2} &= \begin{pmatrix} \mathbf{r}_{2;2} & (z+ct)\mathbf{V}_{2;2} \\ (z-ct)\boldsymbol{\sigma}_2 \mathbf{V}_{2;2} & (-1)^2 \mathbf{r}_{2;2} \end{pmatrix} \\ &= \begin{pmatrix} x & y & 0 & z+ct \\ y & -x & -z-ct & 0 \\ 0 & -z+ct & x & y \\ z-ct & 0 & y & -x \end{pmatrix}.\end{aligned}\quad (83)$$

Then

$$(\mathbf{r}_{4;2})^2 = (x^2 + y^2 + z^2 - c^2 t^2) \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix} = (x^2 + y^2 + z^2 - c^2 t^2) \mathbf{I}_4. \quad (84)$$

This is 4-dimensional Minkowskian space-time. Its geometric algebra forms the elements of Hestenes' space-time algebra [9]: scalar \mathbf{I}_4 ; space-time basis vectors: $\mathbf{X}_4, \mathbf{Y}_4, \mathbf{Z}_4, \mathbf{T}_4$; spin-area bi-vectors: $\sigma_x = \mathbf{T}_4 \mathbf{X}_4, \sigma_y = \mathbf{T}_4 \mathbf{Y}_4, \sigma_z = \mathbf{T}_4 \mathbf{Z}_4$; Area_x = $\mathbf{Y}_4 \mathbf{Z}_4$, Area_y = $\mathbf{Z}_4 \mathbf{X}_4$, Area_z = $\mathbf{X}_4 \mathbf{Y}_4$; energy-momentum trivectors: $E/c = -\mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4, \hat{p}_x = \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4, \hat{p}_y = -\mathbf{X}_4 \mathbf{Z}_4 \mathbf{T}_4, \hat{p}_z = \mathbf{X}_4 \mathbf{Y}_4 \mathbf{T}_4$; and the action 4-volume $\mathbf{V}_{4;2} = \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4$. Now $\mathbf{V}_{4;2}$ squares to $-\mathbf{I}_4$ and can be taken as the imaginary i_4 , which is the i of $i\hbar$.

In standard quantum mechanics, one treats momentum as a differential operator

$$\hat{p}_\mu = -i\hbar \frac{\partial}{\partial x^\mu}. \quad (85)$$

In geometric algebra, one recognizes that this i is the 4-volume, $\mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4$, so

$$\hat{p}_\mu = -\mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \hbar \mathbf{X}_\mu = (-\mathbf{T}_4 \mathbf{Z}_4 \mathbf{Y}_4 \mathbf{X}_4) \hbar \mathbf{X}_\mu = \hbar^* \mathbf{X}_\mu. \quad (86)$$

Thus

$$\mathbf{p}_{4;2} = \begin{pmatrix} p_x & -p_y & -p_z + E/c & 0 \\ -p_y & -p_x & 0 & -p_z + E/c \\ -p_z - E/c & 0 & -p_x & p_y \\ 0 & -p_z - E/c & p_y & p_x \end{pmatrix}. \quad (87)$$

Then

$$(\mathbf{p}_{4;2})^2 = (p_x^2 + p_y^2 + p_z^2 - E^2/c^2) \mathbf{I}_4. \quad (88)$$

$$\mathbf{r} \cdot \mathbf{p} = \frac{1}{2} (\mathbf{r}_{4;2} \mathbf{p}_{4;2} - \mathbf{p}_{4;2} \mathbf{r}_{4;2}) = (xp_x + yp_y + zp_z + Et) \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4. \quad (89)$$

$$\begin{aligned} \mathbf{r} \wedge \mathbf{p} &= \frac{1}{2} (\mathbf{r}_{4;2} \mathbf{p}_{4;2} + \mathbf{p}_{4;2} \mathbf{r}_{4;2}) \\ &= -[(xp_y - yp_x) \mathbf{Z}_4 + (yp_z - zp_y) \mathbf{X}_4 + (zp_x - xp_z) \mathbf{Y}_4] \mathbf{T}_4 \\ &\quad - (xE/c + ctp_x) \mathbf{Y}_4 \mathbf{Z}_4 - (yE/c + ctp_y) \mathbf{Z}_4 \mathbf{X}_4 \\ &\quad - (zE/c + ctp_z) \mathbf{X}_4 \mathbf{Y}_4. \end{aligned} \quad (90)$$

The trivector for p_x commutes with the basis vectors $\mathbf{Y}_4, \mathbf{Z}_4$, and \mathbf{T}_4 that comprise it, but anti-commutes with \mathbf{X}_4 , and similarly for the other components of the four-momentum. In the momentum representation, the four-momentum is the vector and the position is the trivector. (A way to visualize this is to think of position in terms of a three-cornered reflector.)

The relationship between space-time vectors and energy-momentum trivectors is an example of Hodge duality between the $\binom{n}{g}$ elements of grade g and the same number of elements $\binom{n}{n-g}$ of grade $n-g$. Each element \mathbf{E}_b of a geometric algebra commutes or anti-commutes with its Hodge dual according to

$$\mathbf{E}_b \cdot^* \mathbf{E}_b = (-1)^{g_b(n-g_b)} \cdot^* \mathbf{E}_b \mathbf{E}_b. \quad (91)$$

In $n = 4$ dimensions, vectors ($g_b = 1$) and their Hodge dual momenta anti-commute, as a consequence of the anti-commutativity of the basis vectors. The Heisenberg anti-commutation relations between position and conjugate momentum are consequences of the anti-commutativity of the basis vectors, without the need of introducing an additional postulate about momenta. The postulates of quantum mechanics are consequences of the geometric algebra.

4.10. Dirac Gamma Matrices

By Eq. (40), $\mathbf{R}_{4;2} \otimes \mathbf{R}_{1;-1} = \mathbf{R}_{5;3} \cong \mathbf{C}(4) \cong \mathbf{R}(4) \otimes \mathbf{C}$. By equations (36) and (37), the basis vectors of $\mathbf{R}_{5;3}$ can be taken as

$$\begin{aligned} \mathbf{e}_{0|5;3} &= \mathbf{X}_4 \otimes \mathbf{I}_2, \text{ which is space-like - identify it as } \Gamma_1; \\ \mathbf{e}_{1|5;3} &= \mathbf{Y}_4 \otimes \mathbf{I}_2, \text{ which is space-like - identify it as } \Gamma_2; \\ \mathbf{e}_{2|5;3} &= \mathbf{Z}_4 \otimes \mathbf{I}_2, \text{ which is space-like - identify it as } \Gamma_3; \\ \mathbf{e}_{3|5;3} &= \mathbf{T}_4 \otimes \mathbf{I}_2, \text{ which is time-like - identify it as } \Gamma_0; \\ \mathbf{e}_{4|5;3} &= \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \otimes \mathbf{T}_2, \text{ which is space-like - call it } \Gamma_4. \end{aligned} \quad (92)$$

This is a matrix representation of the Dirac gamma matrices $\in \mathbf{C}(4)$, which naturally describe a 5-dimensional space-time with signature $-,+,+,+,+.$

By Eq. (40), $\mathbf{R}_{2;0} \otimes \mathbf{R}_{3;3} = \mathbf{R}_{5;3}$. Various choices of the matrix representation of $\mathbf{R}_{2;0}$ lead to various representations of Dirac matrices as 2×2 matrices of Pauli matrices.

4.11. Kaluza-Klein Theory

$\mathbf{R}(4) \otimes \mathbf{C}$ is the direct product of 4-dimensional space time and something with symmetry $\mathbf{C} \cong U(1) \cong SO(2)$, taken to be electromagnetism. To describe the electroweak force with symmetry $U(1) \otimes SU(2) \cong SO(2) \otimes SO(3)$, one needs 1 dimension of time and 3 of space, namely another copy of $\mathbf{R}_{4;2}$, only with compactified dimensions. By Eq. (40), $\mathbf{R}_{4;2} \otimes \mathbf{R}_{4;2} = \mathbf{R}_{8;0}$.

By Eqs. (36) and (37), the basis vectors of $\mathbf{R}_{8;0}$ can be taken as

$$\begin{aligned} \mathbf{e}_{0|8;0} &= \mathbf{X}_4 \otimes \mathbf{I}_4, \text{ which is space-like - call it } \mathbf{X}_8; \\ \mathbf{e}_{1|8;0} &= \mathbf{Y}_4 \otimes \mathbf{I}_4, \text{ which is space-like - call it } \mathbf{Y}_8; \\ \mathbf{e}_{2|8;0} &= \mathbf{Z}_4 \otimes \mathbf{I}_4, \text{ which is space-like - call it } \mathbf{Z}_8; \\ \mathbf{e}_{3|8;0} &= \mathbf{T}_4 \otimes \mathbf{I}_4, \text{ which is time-like - call it } \mathbf{T}_8; \\ \mathbf{e}_{4|8;0} &= \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \otimes \mathbf{X}_4, \text{ which is time-like - call it } \mathbf{U}_8; \\ \mathbf{e}_{5|8;0} &= \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \otimes \mathbf{Y}_4, \text{ which is time-like - call it } \mathbf{V}_8; \\ \mathbf{e}_{6|8;0} &= \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \otimes \mathbf{Z}_4, \text{ which is time-like - call it } \mathbf{W}_8; \\ \mathbf{e}_{7|8;0} &= \mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4 \otimes \mathbf{T}_4, \text{ which is space-like - call it } \mathbf{S}_8. \end{aligned} \quad (93)$$

However, $\mathbf{R}_{8;0} \cong \mathbf{R}(16)$ has neutral metric signature ($s = 0$). What is needed for quantum mechanics is the Euclidean space $\mathbf{R}_{8;8}$, which by Bott periodicity is also isomorphic to $\mathbf{R}(16)$. Its basis vectors can be taken as the four space-like basis vectors of equation (93), together with the four space-like trivectors composed of three time-like vectors:

$$\begin{aligned} \bar{\mathbf{T}}_8 &= \mathbf{U}_8 \mathbf{V}_8 \mathbf{W}_8; \\ \bar{\mathbf{U}}_8 &= \mathbf{V}_8 \mathbf{W}_8 \mathbf{T}_8; \\ \bar{\mathbf{V}}_8 &= \mathbf{W}_8 \mathbf{T}_8 \mathbf{U}_8; \\ \bar{\mathbf{W}}_8 &= \mathbf{T}_8 \mathbf{U}_8 \mathbf{V}_8. \end{aligned} \quad (94)$$

$\mathbf{R}_{8;8} \cong \mathbf{R}(16)$ expresses the “eightfold way” of the strong force [8]. In turn, $\mathbf{R}(16) \otimes \mathbf{R}(16) \cong \mathbf{R}(32)$ leads to the $SO(32)$ formulation of M-Theory.

4.12. Recursion

After $\mathbf{R}_{8;8}$, the pattern of geometric algebras repeats. Repeated application of equation (43) gives

$$\mathbf{R}_{n+8k;s+8k} \cong \mathbf{R}_{n;s} \otimes \underbrace{\mathbf{R}_{8;8} \otimes \cdots \otimes \mathbf{R}_{8;8}}_{k \text{ times}} \cong \mathbf{R}_{n;s} \otimes \underbrace{\mathbf{R}(16) \otimes \cdots \otimes \mathbf{R}(16)}_{k \text{ times}}. \quad (95)$$

Suppose that the open basis vectors are similarly recursive, but doubling in size at each recursion k :

$$\mathbf{e}_{m+8k|n+8k;s+8k} = 2^k \mathbf{e}_{m|n;s}, \quad (96)$$

for $m = 0, 1, 2, 3$; but that the closed basis vectors do not double in size:

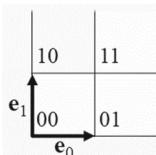
$$\mathbf{e}_{m+8k|n+8k;s+8k} = \mathbf{e}_{m|n;s}, \quad (97)$$

for $m = 4, 5, 6, 7$. This forms a 4-dimensional recursive integer lattice of the open space-time dimensions with 4 closed dimensions, encoding the physics of the Standard Model, at each node of the space-time lattice. Specifically, the 8 dimensions are 1 open time dimension, 3 open space dimensions, 1 closed time-like dimension for electromagnetism, and 3 closed space-like dimensions for the weak force, which collectively form 8 dimensions for the strong force. The open dimensions double by Eq. (96), while by Eq. (97), the closed dimensions do not, thus resolving the conundrum: if everything in the universe expands at the same rate, how could you tell? The answer is that space-time grows, but atomic clocks and rulers do not ([14], p. 719).

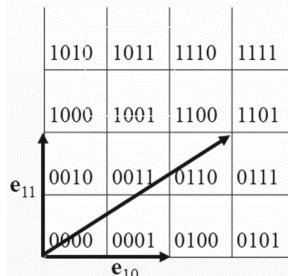
It is easier to visualize a 2-d Euclidean recursive integer lattice:

$$\mathbf{e}_{m+2k|2+2k;2+2k} = 2^k \mathbf{e}_{m|2;2}, \text{ with } m = 0, 1. \quad (98)$$

Then



doubles to



,

where $\mathbf{r} = x^k \mathbf{e}_k$ with $x^k \in \{0, 1\}$ and $k = 0, 1, 2, 3$; and where each binary number labels the node to its lower left, for example, $1101\mathbf{r} = 1\mathbf{e}_3 + 1\mathbf{e}_2 + 0\mathbf{e}_1 + 1\mathbf{e}_0$.

The line element for the basis vectors of equation (96) is

$$ds^2 = 2^{2k} (dx^2 + dy^2 + dz^2 - d(ct)^2) = 2^{2k(t)} (dx^2 + dy^2 + dz^2) - d(ct)^2. \quad (99)$$

This is of the form of the Robertson-Walker line element for a flat spatially isotropic universe ([14], p. 722):

$$ds^2 = a(t) (dx^2 + dy^2 + dz^2) - d(ct)^2. \quad (100)$$

For a Λ -dominated universe,

$$a(t) \propto \exp^{ct\sqrt{\Lambda/3}} = 2^{2k(t)}, \quad (101)$$

which implies a doubling time of

$$\tau_2 = \frac{\ln 2}{c} \sqrt{\frac{3}{\Lambda}} = \frac{\ln 2}{\sqrt{\Omega_\Lambda}} T_0 \approx 0.835 T_0 \quad (102)$$

[17]. From this perspective, the various terms of the Einstein field equation

$$\mathbf{R} - \frac{1}{2} R \mathbf{I} + \Lambda \mathbf{I} = \frac{8\pi G}{c^4} \mathbf{T} \quad (103)$$

have different origins: the \mathbf{R} terms are the Einstein curvature over dimensions 0, 1, 2, 3; the Λ term is over dimensions $(0, 1, 2, 3)_{mod\ 8}$ (Eq. (96)); and the \mathbf{T} term is over dimensions 4, 5, 6, 7 (Eq. (97)). Einstein is said to have complained that the field equation is like a mansion with two wings, one of which is made from fine marble and the other is made from cheap wood [23]. Perhaps the whorls of Einstein's wood are the compact dimensions curling back on themselves.

Noether's theorem [15] states that, if the action is invariant under continuous displacement in coordinate q^i , then the momentum p_i conjugate to q^i is conserved. Noether's theorem is also valid for discrete displacements [5]. By Eq. (96), space-time coordinates form an integer lattice for dimensions 0, 1, 2, 3, with the same physics over dimensions 4, 5, 6, 7, by Eq. (97) at each node of the space-time lattice. The lattice being invariant under discrete displacement of dimensions 0, 1, 2, 3 implies conservation of energy-momentum by Noether's theorem. Conservation laws are not pre-existent; rather, they are consequences of the uniformity of space-time, whose uniformity is a consequence of its recursive generation.

4.13. Cosmogenesis

Consider the basis vectors of the Euclidean plane \mathbf{X}_2 and \mathbf{Y}_2 as possible actions: "going a step in some direction" and "going a step in some other direction." They generate a geometric algebra, $\mathbf{R}_{2,2}$, which also includes \mathbf{I}_2 , "going neither some way nor some other way," and $\mathbf{X}_2 \mathbf{Y}_2$, "going some way while going some other way," which introduces an element of time, \mathbf{T}_2 . The direct product of a unit of the Euclidean plane ($\mathbf{X}_2, \mathbf{Y}_2$) and a unit of the Minkowskian plane ($\mathbf{X}_2, \mathbf{T}_2$) produces an element of 4-d Minkowskian space-time ($\mathbf{X}_4, \mathbf{Y}_4, \mathbf{Z}_4, \mathbf{T}_4$) if the coordinates don't close back on themselves and a

quantum of the electroweak force if they do. The geometric algebra $\mathbf{R}_{4;2}$ of 4-d space-time automatically satisfies the Heisenberg commutation relations. The direct product of the two 4-d algebras is an 8-d algebra, $\mathbf{R}_{8;0} \cong \mathbf{R}_{8;8} \cong \mathbf{R}(16)$, that describes the Standard Model of physics. After eight dimensions, the pattern of geometric algebras repeats itself, generating recursively a spontaneously expanding space-time lattice with the physics of the Standard Model at each node of the lattice. The total energy needed to generate the universe is zero, within the limits of observation ($\Omega_K = 0.0005^{+0.0038}_{-0.0040}$ [17]), so it can arise spontaneously without the need for an initiating agent, either inflaton or deity.

5. Discussion

Real geometric algebras have previously been classified in terms of the number of spatial dimensions p and the number of temporal dimensions q [18], but are better classified in terms of the number of dimensions $n = p + q$ and metric signature $s = p - q$. $\mathbf{R}_{n;s}$ is periodic in s (equation 18):

$$\mathbf{R}_{n;s} \cong \mathbf{R}_{n;s+8k} \quad (104)$$

for any integer k such that $|s + 8k| \leq n$; but recursive in n (Eq. 19):

$$\mathbf{R}_{n;s} \otimes \mathbf{R}_{2;0} = \mathbf{R}_{n+2;s}, \quad (105)$$

which can be written as a matrix of matrices:

$$\mathbf{r}_{n+2;s} = \begin{pmatrix} \mathbf{r}_{n;s} & (x^p + ct^q)\mathbf{V}_{n;s} \\ (x^p - ct^q)\boldsymbol{\sigma}_s \mathbf{V}_{n;s} & (-1)^n \mathbf{r}_{n;s} \end{pmatrix}, \quad (106)$$

where $\mathbf{r}_{n;s}$ is an n -dimensional position vector with signature s , x^p is the next dimension of space, t^q is the next dimension of time, and $\mathbf{V}_{n;s}$ is the n -volume, whose signature σ_s is space-like or time-like depending only on s (Eq. 29):

$$\sigma_s = (-1)^{\frac{s(s-1)}{2}} = \begin{cases} +1 & \text{for } s_{mod} 4 = 0, 1 - \text{space-like n-volumes;} \\ -1 & \text{for } s_{mod} 4 = 2, 3 - \text{time-like n-volumes.} \end{cases} \quad (107)$$

These equations provide a way of constructing larger real geometric algebras as direct products of smaller ones.

Real geometric algebras provide a unified framework for various aspects of physics. Geometric algebras can be generated recursively, which shows a path whereby the physical processes described mathematically by real geometric algebras may have arisen. One normally thinks of mathematical operations as occurring instantaneously. However, the mathematical operations in this paper describe physical processes that take time. The formation of area $\mathbf{X}_2 \mathbf{Y}_2 = \mathbf{T}_2$, which is the i of ict , cannot happen faster than the speed of light. The scale of the unit 4-volume of Minkowskian space-time, $\mathbf{X}_4 \mathbf{Y}_4 \mathbf{Z}_4 \mathbf{T}_4$, which is the i of $i\hbar$, sets the unit of action. The scale of $\mathbf{R}_{4;2}$ with closed dimensions sets the Planck length. The rate of recursion of $\mathbf{R}_{8;8}$ sets the cosmological constant. Recursion of $\mathbf{R}_{8;8}$ is a different physical process from the scale of the closed copy of $\mathbf{R}_{4;2}$, so it's not surprising that the time scale for the cosmological constant is very different from the Planck time.

Despite the wide-ranging scales of the physical processes, they can all be expressed mathematically in terms of real geometric algebras, all of which can be generated recursively from the set of real 2×2 matrices, $\mathbf{R}(2)$, the geometric algebra generated by basis vectors of either the Euclidean plane or the Minkowskian plane. Recursive generation provides sets of basis vectors for space-times of any number of dimensions n and metric signature s . It also provides matrix representations for complex numbers, dyreal numbers, qubits, quaternions, Pauli spin matrices, 4-dimensional space-time, Dirac gamma matrices, the electroweak and strong forces, and the cosmological constant. Constructing all these entities from the same two basic geometric elements shows the deep relationships among them. It also simplifies computerized symbolic manipulation of them. The recursive generation of real geometric elements raises the possibility that the universe generates itself recursively from two basic geometric elements. Growing recursively, they generate an exponentially expanding space-time lattice with the Standard Model of physics at each node of the lattice. The uniformity of space-time and the consistency of physics across the universe are consequences of its recursive generation. Conservation of energy-momentum is a consequence of the uniformity of space-time. Conservation laws are not antecedent to the existence of the universe; rather they are consequences of how the universe grows, recursively.

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