



Probing the minimal geometric deformation with trace and Weyl anomalies

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Abstract

The method of minimal geometric deformation (MGD) is used to derive static, strongly gravitating, spherically symmetric, compact stellar distributions. The trace and Weyl anomalies are then employed to probe the MGD in the holographic setup, as a realistic model, playing a prominent role in AdS/CFT.

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1. Introduction

The minimal geometric deformation, MGD, and the MGD-decoupling methods consist of well-succeeded procedures that can construct analytical solutions of the brane Einstein's effective field equations, in AdS/CFT and its membrane paradigm [1–5]. Our universe, the codimension-1 brane with intrinsic tension, is assumed to be embedded in a bulk [6]. The minimal anisotropic procedure onto the brane has been thrivaly employed to engender exact inner solutions to Einstein's field equations for static and nonuniform stellar configurations, containing local and nonlocal bulk terms. The MGD is a formal approach that generates holographic and realistic varieties of not only any solution in General Relativistic (GR) but extended ones, on fluid branes [7]. Weyl functions into stellar distributions can create effective physical signatures of the bulk.

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Gravitational field equations, in GR, take into account the regime of a rigid brane, where the tension is infinite. However, to match recent observational data, the brane tension, that emulates the vacuum energy itself, must be finite. The current range of the brane tension, σ , is given by $\sigma \gtrsim 2.813 \times 10^{-6}$ GeV [8]. The brane tension controls the MGD of the Schwarzschild space-time, that is a solution of the brane Einstein equations, and also drives other deformed solutions that include tidal charge [9–13]. The brane tension is proportional to the universe temperature, shown by WMAP regarding the CMB anisotropy [14,15]. The MGD was employed to construct compact stellar distributions on the brane [16–18]. Refs. [19,20] introduced the bridge between braneworld models and the holographic AdS/CFT setup [19–21].

The MGD and extended MGD solutions were studied in Refs. [22–24] under different phenomenological setups. Analog MGD models of gravity, using acoustics in a moving fluid, were proposed [25]. Refs. [26–35] have paved a robust way to construct solutions of the brane Einstein effective field equations, using MGD methods. Moreover, MGD-decoupling procedure has been employed to engender anisotropic solutions that describe anisotropic stellar configurations [36–42]. The observation of gravitational lensing effects to detect signatures of MGD stellar configurations was proposed in [43], and MGD stars composed of glueball condensates have been also discussed [44,45]. MGD Dirac stars were introduced and scrutinized in Ref. [46].

The MGD-decoupling procedure iteratively constructs, upon a given isotropic source of gravitational field, anisotropic compact sources of gravity, that are weakly coupled. One starts with a perfect fluid, then coupling it to more elaborated stress-energy-momentum tensors that underlie realistic compact configurations [47–52]. Ref. [53] demonstrated that for positive anisotropy, when the radial pressure is smaller than the tangential pressure, the stellar distribution exerts a force that is repulsive and compensates for the pressure. In this way, anisotropic compact stars are more plausible stellar configurations to happen, as proposed and studied in Refs. [54–65]. Current observations of anisotropy in compact stellar distributions, via gravitational waves, have contributed for MGD-decoupling to occupy a successful role, as a framework that describe high density, anisotropic, astrophysical entities, that comprise X-ray sources, X-ray pulsars and X-ray bursters as well [66,67]. The MGD-decoupling has been also employed to study strange stellar configurations, as the astrophysical object SAX J1808.4-3658 [53], as the brightest X-ray burst ever observed, as spotted by the Neutron Star inner Composition Explorer (NICER). In addition, anisotropic neutron compact stellar configurations were used to describe the compact astrophysical objects 4U 1820.30 and 1728.34, RX J185635.3754, PSR J0348+0432 and 0943+10, for instance, [68,69]. Strange quark stellar configurations were also explored in Ref. [70]. The extension of isotropic compact solutions to anisotropic ones, through the MGD-decoupling, was also proposed in Refs. [71,72].

The so-called Weyl anomaly represents the fact that conformal invariance under Weyl rescaling displayed by gravity-interacting classical fields, is no longer present after quantization. This anomalous behaviour was first pointed out by Ref. [73], back in 1973, and since then has been applied to many areas of theoretical physics (for a review see [74]). Of particular interest for this work is its application to AdS/CFT correspondence, where this anomaly is related to black holes on braneworlds, and, in this context, it is called holographic Weyl anomaly [75]. Other relevant aspects of trace anomalies were comprehensively developed in the seminal Refs. [76–79].

Comparison of the holographic Weyl anomaly to the trace anomaly of the energy-momentum tensor from 4D field theory leads to a coefficient [80] that measures the back-reaction of the brane on the bulk geometry. Therefore it is a good way to understand how accurate the AdS/CFT description of the on-brane boundary theory is, given a particular solution of the codimension-1 bulk.

This paper is organized as follows: Sect. 2 is dedicated to reviewing the MGD derivation as a complete method to deform the Schwarzschild solution and to describe realistic stellar distributions on finite tension branes. In Sect. 3, the trace anomalies are computed for MGD solutions, from the point of view of 4D QFT, and compared to that predicted by the AdS-CFT correspondence. Sect. 4 is devoted to conclusions and important perspectives.

2. MGD: framework and metric

The MGD method provides bulk corrections to well-known solutions of GR, including high energy and nonlocal corrections [3,12]. Underlying the MGD procedure, fluid branes are endowed with an intrinsic tension, mimicking the vacuum energy [14,81].

For $\mu, \nu = 0, 1, 2, 3$, the brane effective Einstein field equations are given by

$$G_{\mu\nu} = -\Lambda_{\text{brane}} g_{\mu\nu} + \frac{8\pi G}{c^4} \mathfrak{T}_{\mu\nu}, \quad (1)$$

where $g_{\mu\nu}$ is the brane metric, G stands for the brane Newton coupling constant and $G_{\mu\nu}$ is the well known Einstein tensor; Λ_{brane} is the cosmological constant on the brane, that is set to zero by fine tuning [82]. The stress-energy-momentum tensor, appearing in Eq. (1), can be decomposed as [83]

$$\mathfrak{T}_{\mu\nu} = T_{\mu\nu} - E_{\mu\nu} + \sigma^{-1} \kappa_5^4 \Pi_{\mu\nu} + \mathcal{K}_{\mu\nu} + M_{\mu\nu}, \quad (2)$$

where $\kappa_5^2 = 48\pi G$. The $T_{\mu\nu}$ term represents the brane stress-energy-momentum tensor, that encodes the brane energy (including dark energy) and brane matter (including dark matter) content. Given the bulk Weyl tensor $C_{\mu\nu\rho\sigma}$, its projection onto the brane, $E_{\mu\nu} \equiv C_{\mu\nu\rho\sigma} n^\rho n^\sigma$, where n^σ is a unitary vector field out of the brane, brings nonlocal ingredients to the brane effective Einstein field equations (1) and, in general, is σ^{-1} -dependent. When the brane is infinitely rigid, corresponding to the GR $\sigma \rightarrow \infty$ case, the tensor $E_{\mu\nu}$ is equal to zero. It is worth to mention that the brane Weyl tensor contains the so-called Weyl functions, namely, the Weyl scalar, \mathcal{U} , and the anisotropy, \mathcal{P} , encoded into any stellar configuration that is solution of (1), being both proportional to the stellar configuration compactness. The Weyl functions arise from the MGD undertaken by the g_{rr} component of the metric, due to AdS bulk effects. In addition, generalized models, modifying the pressure by bulk effects, also encompass nonlocal terms encoding the bulk Weyl curvature [9]. The $\Pi_{\mu\nu}$ component of the stress-energy-momentum tensor encrypts quadratic terms involving the stress-energy tensor, arising from the extrinsic curvature terms in the Einstein tensor projection onto the brane. In fact, the brane matching conditions, applied to the extrinsic curvature tensor, $K_{\mu\nu}$, makes it to be expressed as $K_{\mu\nu} = -\kappa_5^2 (\Pi_{\mu\nu} - \Pi g_{\mu\nu}/3)$, where $\Pi = \Pi_\mu^\mu$ [83]. Also denoting $T = T_\mu^\mu$, one can explicitly write

$$\Pi_{\mu\nu} = \left(\frac{1}{8} T^{\rho\sigma} T_{\rho\sigma} - \frac{1}{4!} T^2 \right) g_{\mu\nu} + \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\rho} T_\nu^\rho. \quad (3)$$

The tensor $\mathcal{K}_{\mu\nu}$ in Eq. (2) describes an eventual asymmetric embedding to the brane into the AdS bulk and the $M_{\mu\nu}$ tensor includes bulk gravitons and moduli fields [81,82].

The solution considered in this work has $\mathcal{K}_{\mu\nu} = M_{\mu\nu} = 0$ in Eq. (2), and the corresponding energy-momentum tensor on the brane, $T_{\mu\nu}$, is that of a perfect fluid for the inner region of the star, $r < R$, and vanishes for $r > R$, as is discussed below.

Compact stellar configurations represent analytical solutions of the gravitational field equations (1), with metric

$$ds^2 = -A(r)dt^2 + B(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{4}$$

where $d\Omega^2$ denotes the solid angle element. One denotes $A(r) = e^{\nu(r)}$ and $B(r) = e^{\xi(r)}$, for the sake of conciseness.

We define the integral

$$\mathcal{I}(r) = \int_0^r \frac{2\tau^2 \nu''(\tau) + (\tau \nu'(\tau) + 2)^2}{\tau^2 \nu'(\tau) + 4\tau} d\tau, \tag{5}$$

denoting by a prime the derivative with respect to the radial coordinate.

The MGD asserts that the $B(r)^{-1}$ radial component of the metric (4) can be deformed as [9]

$$e^{-\xi(r)} = \mu(r) + \kappa(r, \sigma^{-1}), \tag{6}$$

where [7]

$$\kappa(r, \sigma) = e^{-\mathcal{I}(r)} [\mathfrak{b}(r, \sigma) + \mathcal{J}(r)], \tag{7}$$

for

$$\mathcal{J}(r) = \int_0^r \frac{2\tau e^{\mathcal{I}(\tau)}}{\tau \nu'(\tau) + 4} \left[L(\tau) + \frac{\rho(\tau)}{G^2 \sigma} (\rho(\tau) + 3p(\tau)) \right] d\tau. \tag{8}$$

In Eq. (8), $p(\tau)$ denotes the pressure and $\rho(\tau)$ the density of the compact star. In the limit of an infinitely rigid brane, $\sigma \rightarrow \infty$ case, the deformation reads $\kappa(r) \rightarrow 0$, recovering General Relativity. Moreover, the function appearing in Eq. (6) reads

$$\mu(r) = \begin{cases} 1 - 2GM_{\text{Schw}}/c^2 r, & r > R \\ 1 - \frac{1}{G^2 r} \int_0^r \rho(\tau) \tau^2 d\tau, & r \leq R, \end{cases} \tag{9}$$

where R denotes the star surface radius and M_{Schw} is the Schwarzschild star GR mass. The function $L(r) = L(\nu(r), p(r), \rho(r))$ encodes bulk-induced anisotropy. The function $\mathfrak{b} = \mathfrak{b}(r, \sigma)$ in Eq. (7) will be derived soon. The MGD $\kappa(r)$, in vacuum, where $p(r) = \rho(r) = 0$, will be hereon represented by $\tilde{h}(r)$ [2]:

$$\tilde{h}(r, \sigma) = \mathfrak{b}(r, \sigma) e^{-\mathcal{I}(r)}. \tag{10}$$

The MGD can be split into terms that are factors of σ^{-1} , encompassing high energy terms, and terms that encode nonlocal effects of the Weyl fluid. Junction conditions match the inner MGD metric, meaning that $r < R$, for $\tilde{\kappa}(r, \sigma)$ given in Eq. (6), making $L(r) = 0$, in the exterior region, $r > R$.

The Weyl fluid, that moistens the brane, can be represented by Weyl functions. The on-brane Weyl tensor, being inversely proportional to the brane tension, can be split off as

$$E_{\mu\nu} = -2\sigma^{-1} \left[\mathfrak{Q}_{(\mu} u_{\nu)} + \mathcal{U} \left(u_{\mu} u_{\nu} + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} \right], \tag{11}$$

where the vector field u^μ is the velocity that describes the flow of the Weyl fluid and the tensor $h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu}$ projects quantities orthogonally to the flow direction. In addition, the σ -dependent Weyl scalar, explicitly given by $\mathcal{U} = -\frac{1}{2} \sigma E_{\mu\nu} u^\mu u^\nu$ represents the energy density,

whereas $\mathcal{P}_{\mu\nu} = -\frac{1}{2}\sigma \left(h^\rho_\mu h^\sigma_\nu - \frac{1}{3}h^{\rho\sigma} h_{\mu\nu} \right) E_{\rho\sigma}$ denotes the (nonlocal) anisotropic energy-stress-momentum tensor. Besides, the brane nonlocal flux of energy is given by $\Omega_\alpha = -\frac{1}{2}\sigma h^\rho_\alpha E_{\rho\beta} u^\beta$.

The trace of the anisotropic energy-stress-momentum tensor and the Weyl scalar in the outer region are, respectively, given by

$$\mathcal{P}_+(r) = -\frac{\mathfrak{b}(\sigma) \left(1 - \frac{4GM}{3c^2 r} \right)}{9G^2 r^3 \left(1 - \frac{3GM}{2c^2 r} \right)^2 \sigma}, \tag{12}$$

$$\mathcal{U}_+(r) = \frac{\mathfrak{b}(\sigma) M}{12Gc^2 r^4 \left(1 - \frac{3GM}{2c^2 r} \right)^2 \sigma}. \tag{13}$$

In the vacuum, $p(r) = \rho(r) = 0$ in the $r > R$ outer sector. Therefore, the outer metric can be written as [2,18]

$$ds^2 = -e^{\mathfrak{v}^+(r)} dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r} + \tilde{h}(r)} + r^2 d\Omega^2. \tag{14}$$

Junction conditions that match the outer to the inner stellar sector, at the stellar configuration surface $r = R$, imply that [2]

$$\mathfrak{v}^\pm(R) = \ln \left(1 - \frac{2GM}{c^2 R} \right), \tag{15}$$

$$M - M_{\text{Schw}} = \frac{R}{2G} \left(\tilde{h}(R, \sigma) - \tilde{\kappa}(R, \sigma) \right). \tag{16}$$

Regarding Eq. (16), the ADM mass reads $M = M_{\text{Schw}} + \mathcal{O}(\sigma^{-1})$. Owing to the current brane tension lower bound $\sigma \approx 2.813 \times 10^{-6}$ GeV [8], all terms involving $\mathcal{O}(\sigma^{-2})$ are disregarded. The junction conditions at the star surface $r = R$ yield

$$[g_{\mu\nu} a^\nu] = 0, \tag{17}$$

where a^α denotes a radial vector field and one denotes the matching function by

$$[\kappa] \equiv \lim_{r \rightarrow R_-} \kappa(r) - \lim_{r \rightarrow R_+} \kappa(r), \tag{18}$$

averaging any quantity on the star surface by its values in both the inner and the outer surface neighbourhoods.

Thus, Eq. (17) implies $[T^{\text{T}\mu\nu} a_\nu] = 0$, yielding [10]

$$\left[2[\sigma + \rho(r)] p(r) + \rho^2(r) + 4G^2 [\mathcal{U}(r) + 2G^2 \mathcal{P}(r)] \right] = 0. \tag{19}$$

The Schwarzschild-type coefficient $e^{\mathfrak{v}_{\text{Schw}}(r)} = e^{-\xi_{\text{Schw}}(r)} = 1 - \frac{2GM}{c^2 r}$ can be reinstated in Eq. (10), implying that

$$\tilde{h}(r) = -\frac{2\left(1 - \frac{2GM}{c^2 r}\right)\mathfrak{b}(\sigma)}{\left(r - \frac{3GM}{2c^2}\right)r}. \tag{20}$$

To derive the function $\mathfrak{b}(\sigma)$, firstly Eq. (19) must be rewritten as [2],

$$R^2 p(R) - G^2 \tilde{\kappa}(R) (R\mathfrak{v}'(R) + 1) = -\tilde{h}(R). \tag{21}$$

It means that the function $\tilde{h}(R)$ – evaluated at the star surface – attains negative values. Therefore, the MGD coordinate singularity, $r_{\text{MGD}} = 2GM/c^2$, is spotted near the centre of the MGD stellar configuration, when compared to the Schwarzschild coordinate singularity $r_{\text{Schw}} = 2GM_{\text{Schw}}/c^2$. In fact, recall that $M = M_{\text{Schw}} + \mathcal{O}(\sigma^{-1})$. Therefore, Weyl fluid effects on the brane induce a weaker gravitational force, when compared with standard Schwarzschild solutions [2,27].

Eqs. (20), (21) imply that [2]

$$b(\sigma) = \frac{\frac{3GM}{2} - c^2R}{2GM - c^2R} \left[R^3 p(R) - R(Rv'(R) + 1) G\tilde{\kappa}(R) \right] \equiv \frac{\partial_0}{\sigma}, \tag{22}$$

where ∂_0 is given by the awkward expression in Eq. (31) in Ref. [9].

Ref. [23] derived the corresponding experimental and observational signatures of a bulk Weyl fluid, obtained from the Solar system classical tests, encompassing the perihelion precession of Mercury, the deflection of light by the Sun, and the radar echo delay. The bound $\left| \frac{\partial_0}{\sigma} \right| \lesssim 2.8 \times 10^{-11}$ has been obtained in Ref. [23].

The MGD metric can be then written as [2]

$$A(r) = e^{v_{\text{Schw}}(r)} = 1 - \frac{2GM}{c^2r}, \tag{23a}$$

$$B(r) = A(r) \left(\frac{\ell}{r - \frac{3GM}{2c^2}} + 1 \right), \tag{23b}$$

where

$$\ell = R \left(R - \frac{3GM}{2c^2} \right) \left(R - \frac{2GM}{c^2} \right)^{-1} \frac{\partial_0}{\sigma}. \tag{24}$$

In the GR, rigid brane limit $\sigma \rightarrow \infty$, the MGD metric is clearly lead to the Schwarzschild metric, as $M = M_{\text{Schw}} + \mathcal{O}(\sigma^{-1})$. Hence, using the classical tests of GR and replacing the bound $\left| \frac{\partial_0}{\sigma} \right| \lesssim 2.8 \times 10^{-11}$ into Eq. (24), with units $c = 2.998 \times 10^8$ m/s, $M_{\odot} = 1.989 \times 10^{30}$ kg, $R_{\odot} = 6.955 \times 10^8$ m, yields the fundamental gravitational length in the MGD metric (23b) to be

$$|\ell| \lesssim 6.259 \times 10^{-4} \text{ m}. \tag{25}$$

This universal bound holds for MGD compact stellar configurations of any mass. In fact, varying the mass in Eq. (24) also makes the star surface radius, R , and hence the functions in Eq. (22) to be modified, accordingly.

3. MGD anomalies

As stated briefly in the introduction, our goal is to compare the trace anomalies from the field theory side against the one found in the CFT. This was introduced in Ref. [80] and consists in defining the coefficient

$$\Upsilon_{\text{CFT}} = \left| \frac{\langle T \rangle_{4\text{D}} - \langle T \rangle_{\text{CFT}}}{\langle T \rangle_{\text{CFT}}} \right|, \tag{26}$$

where¹

$$\langle T \rangle_{4D} = \frac{1}{2880\pi^2} \left(\mathring{K} - R^{\mu\nu} R_{\mu\nu} - \square R \right), \tag{27}$$

$$\langle T \rangle_{\text{CFT}} = \frac{1}{4\ell_p^2 \sigma^2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right), \tag{28}$$

are the trace and holographic Weyl anomalies, both normalized respectively to the number of bosons in the boundary and the number of stack branes. As the form of the Weyl anomaly is different for different types of matter fields, one has to sum over all massless fields of spin less than two. They also arise by momentum-space dimensional regularization. A counterterm that must be added to eliminate divergences in the contributions of the field to the graviton self-energy [84,85]. The quantity $\langle T \rangle_{4D}$ is obtained using only field theoretic methods in curved spacetimes [85], and does not vanish in general for curved backgrounds (even for Ricci flat ones), as it depends on the Kretschmann scalar, \mathring{K} . The holographic Weyl anomaly appears when one evaluates the effective action of a CFT, via the AdS/CFT procedure. When computing the effective action of the boundary theory on the brane one is forced to choose one amongst the equivalence class of metrics forming the conformal structure of the boundary, therefore explicitly breaking the conformal symmetry to obtain a finite value. This anomaly is perceived as an UV effect, since it is present in the boundary theory, but arises from a divergence whose origin is on the infrared (IR) scale. Such a divergence is present in the bulk [75].

The explicit form of this anomaly depends on the dimension of the spacetime where the CFT boundary is placed. For odd dimensions, the anomaly always vanishes, whereas for even dimensionality the expressions get more intricate as the number of dimensions increase [75]. We only present the 4D case, which is the case we are interested in, the anomaly reads

$$\mathcal{A} = -\frac{N^2}{\pi^2} \left(E_{[4]} + I_{[4]} \right), \tag{29}$$

considering a stack of N branes, where $E_{[4]}$ is the Euler density and $I_{[4]}$ is the conformal invariant. This result holds in the context of 5D asymptotic AdS backgrounds [75] and here we consider the possibility of emulating this result to the MGD metric.

In 4D, there is only one conformal invariant, which is the Weyl tensor contracted with itself. The explicit expressions for the invariants are

$$\begin{aligned} E_{[4]} &= \frac{1}{64} \left(K - 4R^{\mu\nu} R_{\mu\nu} + R^2 \right), \\ I_{[4]} &= -\frac{1}{64} \left(K - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \right). \end{aligned} \tag{30}$$

It is clear from Eq. (30) that $E_{[4]} + I_{[4]} = -\frac{1}{32} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right)$. Eq. (28) is then obtained by using the AdS/CFT dictionary on braneworlds, that relates N degrees of freedom to the Planck length and brane tension [20,80].

The developments in the beginning of Sect. 2 can be now equivalently implemented in the context of AdS/CFT. Firstly, the brane Einstein equations can be expressed as [75,86–89]:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \frac{4}{a| -g|^{1/2}} \frac{\delta}{\delta g_{\mu\nu}} \left(\Gamma_{\text{CFT}} + \frac{1}{32} \mathcal{S}_o \right), \tag{31}$$

¹ Here $R_{\mu\nu}$ is the Ricci tensor and R the scalar curvature.

for $a = 4K^{-1}$, where $K = K^\mu_\mu$ is the extrinsic curvature trace. In addition, the quantity Γ_{CFT} carries the CFT action on the boundary. Its trace anomaly reads [75,88]:

$$g_{\mu\nu} \frac{\delta \Gamma_{\text{CFT}}}{\delta g^{\mu\nu}} = \frac{a^3}{16} | -g |^{1/2} \left(R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right). \tag{32}$$

The term S_o has R^2 counterterms that yield a finite action, whereas the trace of the term $\delta S_o / \delta g_{\mu\nu}$ equals zero, for

$$\frac{\delta S_o}{\delta g^{\mu\nu}} = \frac{a^3}{2} \left[\square R_{\mu\nu} - \frac{1}{3} \nabla_\mu \nabla_\nu R - 2R^{\alpha\beta} R_{\mu\alpha\nu\beta} + \frac{2}{3} R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\frac{1}{3} (\square R + R^2) - \frac{1}{4} R^{\alpha\beta} R_{\alpha\beta} \right) \right], \tag{33}$$

where $\alpha, \beta = 0, 1, 2, 3$. Now, taking the trace of all terms in Eq. (31) yields $R = -\frac{8\pi G}{c^4} T + \frac{a^2}{4} \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right)$. Therefore, up to linear order, the CFT stress-energy-momentum tensor emulates the brane Weyl tensor part, that can be read off as [86,88,89]

$$E_{\mu\nu} = -\frac{K}{| -g |^{1/2}} \frac{\delta \Gamma_{\text{CFT}}}{\delta g^{\mu\nu}}. \tag{34}$$

Hence, it appropriately locates the on-brane Weyl tensor (11), and its consequences to the Einstein effective brane equations (1), in the AdS/CFT setup.

The coefficient (26) measures the reliability of results obtained through the AdS/CFT correspondence when a given spacetime metric is under investigation in the following sense: by measuring the trace anomalies, one can check how the back-reaction of bulk perturbations affect results on the brane or vice-versa, that is if the presence of the brane has any effect on the bulk geometry. The value of Υ_{CFT} ranges from 0 to infinity, and the predictions from AdS/CFT become less reliable the higher the value of the coefficient is [80].

If one considers the Schwarzschild spacetime [80], setting $\ell = 0$ in Eq. (23b), the immediate results $\langle T \rangle_{4\text{D}} \propto K$ and $\langle T \rangle_{\text{CFT}} = 0$ are found. The second identity leads to the conclusion that $\Upsilon_{\text{CFT}} \rightarrow \infty$, and therefore results from the AdS/CFT correspondence are, at best, questionable.

For the MGD metric (4) with functions $A(r)$ and $B(r)$ given by Eqs. (23a) and (23b), respectively, one can thoroughly compute the anomalies, as well as the Υ_{CFT} coefficient, as

$$\langle T \rangle_{4\text{D}}^{\text{MGD}} = \frac{1}{360\pi^2 r^6 (3cGM - 2c^2r)^2} \left[486G^6 M^6 + 9c^{12} \ell^2 r^4 - 648c^2 G^5 M^5 (\ell + 2r) - 16c^{10} G \ell M r^3 (5\ell + 3r) + 216c^4 G^4 (\ell + r) (\ell + 6r) - 36c^6 G^3 M^3 r (12\ell^2 + 35\ell r + 16r^2) + 6c^8 G^2 M^2 r^2 (49\ell^2 + 72\ell r + 16r^2) \right] \tag{35}$$

$$\langle T \rangle_{\text{CFT}}^{\text{MGD}} = \frac{\ell^2}{\sigma^2 \ell_p^2} \frac{2c^2 (6G^2 M^2 - 8c^2 G M r + 3c^4 r^2)}{r^4 (3GM - 2c^2r)^4} \tag{36}$$

$$\Upsilon_{\text{CFT}}^{\text{MGD}} = \left| 1 - \frac{\sigma^2 \ell_p^2}{720\pi^2 c^8 \ell^2 r^2 (3c^4 r^2 - 8c^2 G M r + 6G^2 M^2)} \left[9c^{12} \ell^2 r^4 - 16c^{10} G \ell M r^3 (5\ell + 3r) + 6c^8 G^2 M^2 r^2 (49\ell^2 + 72\ell r + 16r^2) \right] \right|$$

$$\begin{aligned}
 & -36c^6G^3M^3r(12\ell^2 + 35\ell r + 16r^2) + 216c^4G^4M^4(\ell^2 + 7\ell r + 6r^2) \\
 & -648c^2G^5M^5(\ell + 2r) + 486G^6M^6 \Big], \tag{37}
 \end{aligned}$$

where $\ell_p = \sqrt{\hbar G/c^3}$ denotes the Planck length. In the large- r limit, Eq. (35) reads

$$\langle T \rangle_{4D}^{\text{MGD}} = \left(1 - \frac{Gc^2M}{120}\ell + \frac{c^4}{640}\ell^2 \right) \langle T \rangle_{4D}^{\text{Schwarzschild}} \tag{38}$$

that is, it constitutes corrections to the trace anomaly for the Schwarzschild black hole solution. This motivates to analyze Eq. (37) on the same regime, where we obtain

$$\Upsilon_{\text{CFT}}^{\text{MGD}(r \rightarrow \infty)} = \left| \frac{\sigma^2 \ell_p^2 (3c^4 \ell^2 - 16c^2 G \ell M + 32G^2 M^2)}{720\pi^2 c^4 \ell^2} - 1 \right| \tag{39}$$

Eq. (39) allows us to draw some conclusions about the reliability of bulk/boundary correspondence, as long as one bears in mind that such conclusions apply for $r \gg 1$.

We proceed to extract numerical values for Υ_{CFT} for two known values of the ℓ parameter, that appears in the MGD metric radial component (23b), respectively for two regimes of the ADM mass M [23,43]. In the following calculations, we will use SI units for all quantities.² Ref. [9] has shown that $\ell\sigma = -0.0042572$. This is particularly interesting because we can then eliminate the brane tension in Eq. (39) and write it only in terms of known quantities:

$$\Upsilon_{\text{CFT}}^{r \rightarrow \infty} = \left| \frac{10^{-6} \ell_p^2 (3c^4 \ell^2 - 16c^2 G \ell M + 32G^2 M^2)}{4\pi^2 c^4 \ell^4} - 1 \right|. \tag{40}$$

The first set of data to be considered is the already upper bound (25) [23], the other relevant values are given above that equation. Besides, in Ref. [43] the MGD solution was applied to modelling gravitational lensing effects, where the Sagittarius A* black hole of mass $M = (4.154 \pm 0.014) \times 10^6 M_\odot$ was considered, therefore $R = 2M$ is the event horizon radius. For this case, the observational value of $\ell = 0.06373$ m was obtained [43]. For both sets, $|\ell| \ll 1$ leading Eq. (40) to imply that

$$\Upsilon_{\text{CFT}}^{\text{MGD}} \approx 1. \tag{41}$$

When $\ell \rightarrow 0$, all the results regarding the Schwarzschild solution are recovered, however the current range of the brane tension, $\sigma \gtrsim 2.813 \times 10^{-6}$ GeV [8], together with the equality $\ell\sigma = -0.0042572$, makes ℓ not to attain a null value. Therefore, it places the MGD, in the AdS/CFT formulation on the brane, into a trustworthy position to be a realistic model to describe, in this context, stellar distributions that are compatible with AdS/CFT. Although the most precise lower bound that the brane tension attains $\sigma \gtrsim 2.813 \times 10^{-6}$ GeV [8] can be used, it is worth to mention the physically feasible and realistic value $\sigma \sim 7.3 \times 10^{19}$ kg.m²/s² = 0.3 GeV was reported in Refs. [90,91]. This value was also found in the membrane paradigm describing neutron stars, which although stronger than the big-bang nucleosynthesis constraint probed by the Cosmic Background Explorer [92], is still weaker than the bound obtained by Newton’s law at submillimetric scale constraints. Therefore this value can be realistically employed for our purposes here.

² $\ell_p = 1.616 \times 10^{-35}$ m, $G = 6.674 \times 10^{-11}$ m³ kg⁻¹ s⁻².

Using strong gravitational lensing effects of anisotropic stellar configurations, in particular the catalogued PSR B0943+10, 4U 1820.30, 4U 1728.34, PSR J0348+0432, RX J185635.3754, PSR J0348+0432, PSR 0943+10, SAX J1808.4-3658, Her X-1, Cen X-3, XTE J1739-285, RX J1856, XTE J1739-285, Sagittarius A* [43,53,66,68,69], including also the possibility that our sun may be described by the MGD as posed in Ref. [23], we can then engender the more general relation

$$\ell(M) = 1.95243 \times 10^{-75} M^2 - 7.84876 \times 10^{-39} M + 0.06373, \tag{42}$$

with mass M units in kg. Applying this expression to (37) and taking the limit $r \rightarrow 2M$ yields

$$\begin{aligned} & \lim_{r \rightarrow 2M} \Upsilon_{\text{CFT}} \\ &= \left| \frac{-3.812 \times 10^{-150} M^4 + 3.065 \times 10^{-113} M^3 - 3.105 \times 10^{-76} M^2 + 1.001 \times 10^{-39} M - 4.001 \times 10^{-3}}{(6.373 \times 10^{-2} - 7.849 \times 10^{-39} M + 1.952 \times 10^{-75} M^2)^2} \right|. \end{aligned} \tag{43}$$

One can check that $\lim_{r \rightarrow 2M} \Upsilon_{\text{CFT}} \approx 1$ in Eq. (43), for all mass values in the range $0 \leq M \lesssim 10^{70}$ kg. Some studies have suggested that the maximum mass that a black hole can reach is $\sim 10^{12} M_{\odot} \approx 10^{42}$ kg [93,94], and that the mass of the entire observable universe is often quoted as $\sim 10^{53}$ kg, regarding the ordinary matter that includes the interstellar and the intergalactic media, up to $\sim 10^{55}$ kg if one includes dark matter and dark energy as well. Therefore, we are safe to assert that Eq. (43) will be effectively equal to 1, for all values of stellar configuration masses in the range from zero to 17 orders of magnitude more massive than the observable universe. According to Ref. [80], it provides a good agreement for the AdS/CFT correspondence in the IR regime.

To summarize, one can safely assume that Υ_{CFT} is effectively unity in both the infinity and near the event horizon, unless one considers either exceptionally high masses that are not physically attainable – since one should consider stellar objects 17 orders of magnitude more massive than the entire observable universe – or models where the brane tension is very high, approaching a rigid brane. The latter case goes towards the GR limit and such behaviour is expected.

4. Conclusions

The MGD, usually employed to obtain static, strongly gravitating spherically symmetric and compact stellar distributions, was here explored with the tools of trace and Weyl anomalies. Contrary to the Schwarzschild solution, for which the implausible result $\Upsilon_{\text{CFT}} \rightarrow \infty$ in Eq. (26) yields the AdS/CFT correspondence to be difficult to be implemented in this context, the MGD is a reliable attempt to describe realistic models, in the AdS/CFT setup. The parameter in Eq. (26) quantifies how safe AdS/CFT is when bulk/brane back-reaction effects are taken into account. Since the value of the Υ_{CFT} coefficient, for the MGD case, was shown to be near unity, it means that the MGD solutions may occupy a privileged place and can play a prominent role on emulating AdS/CFT on braneworld scenarios.

Similarly to AdS/QCD models, where the extra dimension is interpreted as an energy scale in QCD, in the setup here established, phenomena regarding CFT coupled to gravity can be exclusively interpreted from the point of view of the brane. Bulk gravitons that propagate in the bulk correspond to 4D gauge bosons on the boundary. The difference between these two countenances, in the AdS/CFT setup, cannot be identifiable, from any phenomenological point of view.

CRedit authorship contribution statement

P. Meert: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **R. da Rocha:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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