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Article

Classification of Petrov Homogeneous Spaces

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Abstract: In this paper, the final stage of the Petrov classification is carried out. As it is known, the Killing vector fields specify infinitesimal transformations of the group of motions of space V_4. In the case where the group of motions G_3 acts in a simply transitive way in the homogeneous space V_4, the geometry of the non-isotropic hypersurface is determined by the geometry of the transitivity space V_3 of the group G_3. In this case, the metric tensor of the space V_3 can be given by a nonholonomic reper consisting of three independent vectors $\ell^{\alpha}_{(a)}$, which define the generators of the group G_3 of finite transformations in the space V_3. The representation of the metric tensor of V_4 spaces by means of vector fields $\ell^{\alpha}_{(a)}$ has a great physical meaning and makes it possible to substantially simplify the equations of mathematical physics in such spaces. Therefore, the Petrov classification should be complemented by the classification of vector fields $\ell^{\alpha}_{(a)}$ connected to Killing vector fields. For homogeneous spaces, this problem has been largely solved. A complete solution of this problem is presented in the present paper, where I refine the Petrov classification for homogeneous spaces in which the group G_3, which belongs to type $VIII$ according to the Petrov classification, acts simply transitively. In addition, this paper provides the complete classification of vector fields $\ell^{\alpha}_{(a)}$ for space V_4 in which the group G_3 acts simply transitivity on isotropic hypersurfaces.

Keywords: algebra of symmetry operators; linear partial differential equations

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1. Introduction

Space-time manifolds V_4 admitting groups of motions occupy a special place in the theory of gravitation, since the vector Killing fields defining generators of these groups serve to construct the conserved physical quantities connected with the properties of the gravitational field. Therefore, these fields play an important role in describing the geometry of curved spacetime and physical phenomena occurring within it.

A large number of papers have been devoted to this role of the Killing vector fields (see, for example, [1–4]). The question of group motions in Riemannian spaces was first considered in the works of Bianchi [5,6], Fubini and Rimini [7,8] back in the late 19th and early 20th centuries.

These spaces are of particular interest in cosmology, which, as a modern science, in fact, began to develop after the appearance of works [9,10], in which special cases of homogeneous spaces were used for the first time. These spaces are still the foundation of cosmological models, including in alternative theories of gravitation (see [11,12]), as well as in the early stages of the Universe's development.

The classification of space-time manifolds with groups of motions is based on the theory of continuous groups of transformations (see [13,14]).

A complete classification of four-dimensional pseudo-Riemannian spaces founded on the canonical real structures of the motion groups acting in the spaces was realized by A.Z. Petrov in [15–17]. Petrov constructed non-equivalent sets of operators for all groups of motions of four-dimensional pseudo-Riemannian spaces, integrated the Killing equations and found the metrics of all these spaces. When classifying isotropic spaces, Petrov used the approaches described in the papers by Kruchkovich [18,19].

The obtained results of this classification have found application in the theory of gravitation, first of all, when considering the problem of exact solutions of the field gravitational equations (see [17,20,21]). In more recent works, various aspects of using spaces with groups of motions were considered. Thus, the authors of [22–24] studied the intersections of homogeneous and Stackel spaces, as well as physical processes in homogeneous spaces. In [25,26], the problems of constructing integrals of motion for classical and quantum scalar charged particles moving in spaces with groups of motions and in admissible electromagnetic fields were considered. The classification of the exact solutions of Maxwell equations in homogeneous spaces in the presence of admissible electromagnetic fields was carried out in [27–29]. A special direction of application of homogeneous spaces is the construction of the theory of noncommutative integration of the equations of motion of charged quantum particles in external fields [30–33].

In this paper, the final stage of the Petrov classification is carried out. As it is known, the Killing vector fields specify infinitesimal transformations of the group of motions of space V_4. In the case where the group of motions G_3 acts in a simply transitive way in the homogeneous space V_4, the geometry of the non-isotropic hypersurface is determined by the geometry of the transitivity space V_3 of the group G_3. In this case, the metric tensor of the space V_3 can be given by a nonholonomic reper consisting of three independent vectors $\ell^{\alpha}_{(a)}$, which define the generators of the group G_3 of finite transformations in the space V_3. The representation of the metric tensor of V_4 spaces by means of vector fields $\ell^{\alpha}_{(a)}$ has a great physical meaning and makes it possible to substantially simplify the equations of mathematical physics in such spaces. Therefore, the Petrov classification should be complemented by the classification of vector fields $\ell^{\alpha}_{(a)}$ connected to Killing vector fields. For homogeneous spaces, this problem has been largely solved (see, e.g., [34–36]). A complete solution of this problem is presented in the present paper, where I refine the Petrov classification for homogeneous spaces in which the group G_3, which belongs to type $VIII$ according to the Petrov classification, acts simply transitively. In addition, this paper provides the complete classification of vector fields $\ell^{\alpha}_{(a)}$ for spaces V_4 in which the group G_3 acts simply transitively on isotropic hypersurfaces.

2. Homogeneous Petrov Spaces

Consider a Riemannian space V_4 with signature $(-1,1,1,1)$, on the hypersurface V_3 of which the motion groups $G_3(N)$ act simply transitively (N corresponds to the number of the group G_3 in the Bianchi classification). In the case of a null hypersurface of transitivity $V_3^*(N)$, the space $V_4^*(N)$ will be called a null homogeneous Petrov space of type N. If the hypersurface of transitivity $V_3(N)$ is non-zero, then $V_4(N)$ will be called a homogeneous non-isotropic Petrov space of type N.

The geometry of a homogeneous Petrov space is connected with the geometry of the three-dimensional space of transitivity $\tilde{V}_3(N)$ of the group $G_3(N)$. For a non-null homogeneous Petrov space, this connection is direct, since in this case a hypersurface of transitivity $V_3(N)$ is in fact the space $\tilde{V}_3(N)$. For a null Petrov space, this connection is less obvious, but also exists, since the Killing vector fields of the space $V_4^*(N)$ are generally a linear combination of the Killing vector fields of the space $\tilde{V}_3(N)$ with coefficients depending on the wave variable u^0.

The contravariant components of the metric tensor in a non-null semi-geodesic coordinate system can be represented in the following form:

$$g^{ij} = \begin{pmatrix} g^{\alpha\beta} & (u^i) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} \quad (\varepsilon \pm 1). \tag{1}$$

The admissible transformations of the coordinates u^i, which do not violate the form of (1), are of the following form:

$$\tilde{u}^\alpha = \tilde{u}^\alpha(u^\beta), \quad \tilde{u}^0 = u^0. \tag{2}$$

It follows from the Killing equations that

$$\xi^\alpha_{(a),0} = 0, \quad \xi^0_{(a)} = 0. \tag{3}$$

The following index designations are used here and hereafter:

$$i, j, r, \ell \div 1, 2, 3, 0. \quad \alpha, \beta, \gamma \div 1, 2, 3. \quad a, b, c \div 1, 2, 3.$$

Zero indices number the fourth row and column. Greek letters denote the coordinate indices of the semi-geodetic coordinate system u^i. The small Latin letters a, b, and c denote the indices of the nonholonomic coordinate system associated with the group $G_3(N)$.

The classification of null homogeneous Petrov spaces was carried out in the isotropic semi-geodesic coordinate system (see [15], p. 158). In this coordinate system, the contravariant components of the metric tensor g^{ij} have the form

$$g^{ij} = \begin{pmatrix} g^{\alpha\beta}(u^i) & & & A(u^0, u^p) \\ & & & 0 \\ & & & 0 \\ A(u^0, u^p) & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

There and further, the indices denoted by p, q change within $\div 1, 2$. In the present paper, the canonical form of the isotropic semi-geodesic coordinate system, in which

$$A = 1, \tag{5}$$

is used:

$$g^{ij} = \begin{pmatrix} g^{\alpha\beta}(u^i) & & & 1 \\ & & & 0 \\ & & & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

It follows from the Killing equations that in the canonical coordinate system the components of the Killing vector fields of the group $G_3(N)$ satisfy the following conditions (see [17]):

$$\xi^\alpha_{(a),1} = \xi^0_{(a)} = 0. \tag{7}$$

The admissible coordinate transformations $\{u^i\}$, which do not violate the conditions (6), (7), are of the following form:

$$\tilde{u}^0 = \varphi_0(u^0), \quad \tilde{u}^1 = u^1/\varphi_{0,0} + \varphi^1(u^0, u^p), \quad \tilde{u}^p = \varphi^p(u^0, u^q) \tag{8}$$

The homogeneous space $V_3(N)$ has two equivalent definitions. According to the first one, $V_3(N)$ is the space of transitivity for the group $G_3(N)$, whose operators are constructed with Killing vector fields $\xi^\alpha_{(a)}$:

$$\mathrm{X}_a = \xi^\alpha_{(a)} \mathrm{p}_\alpha, \tag{9}$$

According to the second definition, in the homogeneous space $V_3(N)$ the group $G_3(N)$ of non-infinitesimal shifts acts. The group operators have the form

$$\mathrm{Y}_a = \ell^\alpha_{(a)} \mathrm{p}_\alpha. \tag{10}$$

According to the first definition, two infinitely close points of space $V_3(N)$ can be combined using the motion group of space $G_3(N)$. According to the second definition, this can be done for any two points of the space. As is known (see, for example, [34]) p. 483), the metric tensor of the homogeneous space $V_3(N)$ is expressed through the vector fields $\ell^{\alpha}_{(a)}$ as follows:

$$g^{\alpha\beta} = \ell^{\alpha}_{(a)}\ell^{\beta}_{(a)}\eta^{ab}. \tag{11}$$

$\alpha, \beta, \gamma \div 1,2,3$; $a, b, c, \div 1,2,3$; $\eta^{ab} = const$ (in the case where η^{ab} is considered part of the metric tensor g^{ij}, η^{ab} are functions of the variable u^0). The sets of operators $\mathrm{X}_a, \mathrm{Y}_b$ obey the same structure equations:

$$[\mathrm{X}_a, \mathrm{X}_b] = C^c_{ab}\mathrm{X}_c, \tag{12}$$

$$[\mathrm{Y}_a, \mathrm{Y}_b] = C^c_{ab}\mathrm{Y}_c. \tag{13}$$

Hence, the sets of vector fields $\{\xi^{\alpha}_{(a)}\}$ and $\{\ell^{\alpha}_{(a)}\}$ are equivalent to each other with respect to coordinate transformations. Therefore, both fields can be simultaneously considered as Killing vector fields and as nonholonomic reper vector fields (see (11)) in the same space $V_3(N)$ but in different coordinate systems connected by admissible transformations of the form (2) or (8). In one coordinate system, these fields are connected by relations (see [34], p. 484):

$$\ell^{\alpha}_{(a),\beta} = \xi^{(b)}_{\beta}\xi^{\alpha}_{(b),\gamma}\ell^{\gamma}_{(a)} \Rightarrow \xi^{\alpha}_{(a),\beta} = \ell^{(b)}_{\beta}\ell^{\alpha}_{(b),\gamma}\xi^{\gamma}_{(a)}, \tag{14}$$

where

$$\xi^{(b)}_{\beta}\xi^{\alpha}_{(b)} = \ell^{(b)}_{\beta}\ell^{\alpha}_{(b)} = \delta^{\alpha}_{\beta}. \tag{15}$$

These considerations apply to non-null homogeneous Petrov spaces. In the case of null Petrov spaces, they cannot be used directly because the covariant metric tensor of the space $V_3(N)$ is not the metric tensor of the isotropic hypersurface of the space V^*_4. Therefore, the proof of the (4) relations based on the above second definition of homogeneous space for null spaces is inapplicable. Nevertheless, the representation of (11), where the vector fields $\ell^{\alpha}_{(a),\beta}$ satisfy the system of Equation (14), holds for null homogeneous Petrov spaces as well. It can be shown that if

$$det|\zeta^{\alpha}_{a}| \neq 0 \Rightarrow \zeta^{\alpha}_{a}\zeta^{\beta}_{a} = \delta^{\beta}_{\alpha}, \tag{16}$$

vector fields satisfy a system of equations of the form (14), then ζ^{α}_{a} are Killing vector fields for a space with metric tensor (11). Indeed, substituting the expressions

$$\zeta^{\alpha}_{(a),\beta} = \ell^{(b)}_{\beta}\ell^{\alpha}_{(b),\gamma}\zeta^{\gamma}_{(a)} \tag{17}$$

and (11) into the Killing equations, we obtain the identity. As Equation (17) has three independent solutions, the vector fields $\zeta^{\alpha}_{(a),\beta}$ (also as ℓ^{α}_{a}) form the group G_3.

The classification has been carried out by Petrov in three stages.

1. Selection of a semi-geodesic coordinate system.

2. Solving structural Equation (4). In the second stage, all sets of Killing vector fields that are non-equivalent with respect to admissible transformations of variables have been listed. In this way, the choice of semi-geodesic coordinate systems in which the solutions of the systems of structural equations can be represented in elementary functions is also realized.

3. Solving the Killing equations. In the third step, using the solutions found in the second step, the Killing equations have been integrated:

$$g^{\alpha\beta}_{,\gamma}\zeta^{\gamma}_{a} = g^{\alpha\gamma}\zeta^{\beta}_{a,\gamma} + g^{\beta\gamma}\zeta^{\alpha}_{a,\gamma}, \tag{18}$$

and the metric tensors of all the appropriate homogeneous Petrov spaces have been found.

Representation of the metric tensor of homogeneous space in the form (11) has a deep physical meaning, and it is demanded when solving problems of mathematical physics in the theory of gravitation. Therefore, in this paper, the classification of homogeneous Petrov spaces has been supplemented by the fourth stage.

4. Solving the systems of Equation (14). In the fourth stage, all non-equivalent solutions of the systems of Equation (14) have been found. When the components of the Killing vector fields do not depend on the wave variable u^0, the components of the metric $g^{\alpha\beta}$ are found directly from the (16) relations without directly using the Killing equations. Let us consider the peculiarities of the implementation of this step in the presence of such dependence.

Each of the groups $G_3(II), G_3(III), G_3(V)$ has several independent sets of group operators. One of these sets contains a linear combination of group operators of the following form (see [17]):

$$\tilde{\mathrm{X}}_{a'} = \mathrm{X}_{a'} + \ell_0(u^0)\mathrm{X}_1, \quad \mathrm{X}_1 = \mathrm{p}_2, \quad \mathrm{X}_{a,0} = (\xi^\alpha_{(a)}\mathrm{p}_\alpha)_{,0} = 0, \quad [\mathrm{X}_a, \mathrm{X}_b] = C^c_{ab}\mathrm{X}_c.$$

Using the solutions of the system of Equation (15), one can find the contravariant components (7) of the metric tensor $g^{\alpha\beta}$ of the space V_4^* admitting Killing vector fields $\xi^\alpha_{(a)}$. Then the contravariant components of the metric tensor $\tilde{g}^{\alpha\beta}$ can be represented as

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} + \dot{\ell}_0 \Delta g^{\alpha\beta}. \tag{19}$$

The function $\Delta g^{\alpha\beta}$ is found from the Killing equations:

$$\tilde{g}^{\alpha\beta}_{,\gamma}\tilde{\xi}^\gamma_a = \tilde{g}^{\alpha\gamma}\tilde{\xi}^\beta_{a,\gamma} + \tilde{g}^{\beta\gamma}\tilde{\xi}^\alpha_{a,\gamma}.$$

Here and hereafter,

$$\dot{\ell}_0 = \ell_{0,0}.$$

3. The Diagonalization of Group Vectors

When making the classification of homogeneous Petrov spaces, one can restrict oneself to the null case only, since the group of admissible coordinate transformations (8) is a subgroup of the group of admissible coordinate transformations (2). Therefore, all sets of the group operators $G_3(N)$, which act in the space $V_4^*(N)$, form equivalence classes with respect to the transformation group (2) for the set of operators of the same group, which acts in the space $V_4(N)$. One feature must be borne in mind when constructing the metric tensor $g^{\alpha\beta}$ according to the formula (11). In the non-null space $V_4(N)$, the number of independent functions $\eta^{ab}(u^0)$ cannot be reduced by admissible transformations of form (2).

The first step of the classification is to choose a coordinate system, in which one of the Killing fields has to be represented in the form

$$\xi^\alpha_{(1)} = \delta^\alpha_1 \mathrm{A}_1(u^0, u^p). \tag{20}$$

Using admissible coordinate transformations (2), it can always be made for any vector. Moreover, it is possible to make the function A equal to unity. However, using admissible transformations (8) this can be carried out only for the vector that is already in this shape. Indeed, under the transformation (8), the operator X_a is transformed as follows:

$$\tilde{\mathrm{X}}_a = \left(\mathrm{A}_a + \mathrm{B}_a\Phi^1_2 + R_a\Phi^1_{,3}\right)\tilde{\mathrm{p}}_1 + \left(\mathrm{B}_a\Phi^2_2 + R_a\Phi^3_{,2}\right)\tilde{\mathrm{p}}_2 + \left(\mathrm{B}_a\Phi^2_3 + R_a\Phi^3_{,3}\right)\tilde{\mathrm{p}}_3. \tag{21}$$

The following notations are used here and hereafter:

$$\mathrm{X}_a = \mathrm{A}_a\mathrm{p}_1 + \mathrm{B}_a\mathrm{p}_2 + \mathrm{R}_a\mathrm{p}_3 \quad \tilde{u}^1 = u^1 + \Phi^1, \quad \tilde{u}^p = \Phi^p.$$

The letters $\Phi^\alpha, \mathrm{A}_a, \mathrm{B}_a$, and R_a denote functions depending on the variables u^0 and u^p. Functions φ^a, a_a, b_a, r_a depend on u^0 and one of the variables u^p. Since the Jacobian of the transformation (8) does not equal to zero, it is possible to represent $\tilde{X}_1$ in the form (20) only if $B_1 = R_1 = 0$. In this case, any of the remaining operators, for example, X_2, can be diagonalized using admissible transformations (8). Indeed, without restriction of generality, let us require that in a new coordinate system $\{\tilde{u}\}$ the operator X_2 has the form

$$X_2 = p_2.$$

For this purpose, the following condition must be fulfilled:

$$\begin{cases} A_2 + B_2\Phi^1_{,2} + R_2\Phi^1_{,3} = 0 \\ B_2\Phi^2_{,2} + R_2\Phi^3_{,2} = 1 \\ B_2\Phi^2_{,3} + R_2\Phi^3_{,3} = 0 \end{cases} \quad (22)$$

The system (22) reduces to the form of the Cauchy–Kovalevskaya system and is consistent. Thus, operator X_2 can be presented in the following form:

$$X_2 = p_2. \quad (23)$$

The first option is received.
1. First version:

$$X_1 = a_1(u^0, u^3)p_1, \quad X_2 = p_2, \quad X_3 = A_3p_1 + B_3p_2 + R_3p_3.$$

Obviously, this version is valid only for solvable groups. Hereinafter, operators X_1, X_2 are chosen as operators of the abelian subgroups of the groups $G_3(I) - G_3(VII)$. Therefore, from $A_{,2} = 0$ it follows that $A_1 = a_1(u^0, u^3)$.
If $|B_1| + |R_1| \neq 0$, from (22) it follows that X_1 can be presented in the form

$$X_1 = p_2. \quad (24)$$

The admissible coordinate transformations that do not violate the condition (24) are as follows:

$$\tilde{u}^0 = \varphi_0, \quad \tilde{u}^1 = \frac{u^1}{\varphi_{0,0}} + \varphi^1\left(u^0, u^3\right), \quad \tilde{u}^p = \varphi^p\left(u^0, u^3\right). \quad (25)$$

Operator X_2 commutes with operator X_1. Therefore, after the admissible transformations (25) it takes the following form:

$$X_2 = \left(a_2 + r_2\varphi^3_{,3}\right)p_1 + \left(b_2 + r_2\varphi^2_{,3}\right)p_2 + r_2\varphi^3_{,3}p_3. \quad (26)$$

Obviously, only the following options are possible:

1. $r_2 \neq 0 \Rightarrow \quad X_2 = p_3;$
2. $r_2 = 0 \Rightarrow X_2 = a_2p_1 + b_2p_2 \quad (a_{2,2} = b_{2,2} = 0).$

In the last case, one can use the transformations (25) to simplify operator X_3. Thus, these options together with versions for unsolvable groups have to be considered separately when making the classification.
2. Second version:

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = A_3p_1 + B_3p_2 + R_3p_3. \quad (27)$$

3. Third version:

$$X_1 = p_2, \quad X_2 = a_2p_1 + b_2p_2, \quad X_3 = A_3p_1 + B_3p_2 + R_3p_3. \quad (28)$$

When solving structural equations, admissible transformations (25) can be used to simplify the form of the operator X_3.

For insoluble groups $G_3(VIII), G_3(IX)$ there is following version:

4. Fourth version:

$$X_1 = p_2, \quad X_p = A_p p_1 + B_p p_2 + R_p p_3. \tag{29}$$

In this case, the admissible transformations have the form (25). They can be used to simplify the form for one of the operators X_p.

This paper is constructed as follows.

1. Since the method used in this paper for classifying non-equivalent sets of motion group operators differs somewhat from the method used in [17] (in particular, admissible coordinate transformations of the form (28) are used), some details of the computations in solving the structural equations are given below. Some of the results obtained have a simpler form than those given in Petrov's book. In addition, I managed to complete the Petrov classification with two new homogeneous non-isotropic spaces of type $V_4(VII)$.
2. For each non-equivalent set of group operators constructed using Killing vector fields, the classification of the reper vectors $\ell^\alpha_{(a)}$ is given.
3. Using formulas (11) and (19), the components of the contravariant metric tensor are constructed for each non-equivalent set of group operators.
4. In the final section, all non-equivalent sets of the group operators X_2 and Y_2 are listed.

4. Solvable Groups

4.1. Group$G_3(I)$

Since three mutually commuting vector fields can always be diagonalized, even by using coordinate transformations of the form (8) [14], the metric tensor of an isotropic space of type I according to the Bianchi classification can be transformed to the following form ($\xi^\alpha_a = \delta^\alpha_a$):

$$g^{ij} = \begin{pmatrix} \eta^{\alpha\beta} & (u^0) & & 1 \\ & & & 0 \\ & & & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $\eta^{\alpha\beta} = a_{\alpha\beta}(u^0)$. It contains 6 independent functions of u^0. The number of these functions can be reduced by an admissible transformation of variables:

$$\tilde{u}^\alpha = u^\alpha + \varphi^\alpha\left(u^0\right) \tag{30}$$

As a result, the functions $a_{1\alpha}$ can be inverted to zero and the metric tensor has the form

$$g^{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{31}$$

4.2. Group$G_3(II)$

The structural equations have the following form:

$$[X_1 X_2] = 0, \quad [X_2, X_3] = X_1, \quad [X_1 X_3] = 0. \tag{32}$$

Let us consider all versions presented above.

First version. The operators X_a are of the following form:

$$X_1 = a_1 p_1, \quad X_2 = p_2, \quad X_3 = A_3 p_1 + B_3 p_2 + R_3 p_3 \tag{33}$$

From the third equation of system (32), it follows that $a_1 = 1$. Using admissible transformation (25), one can find solution of the second equation from the system (32) in the form

$$\mathrm{X}_1 = \mathrm{p}_1, \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = u^2 \mathrm{p}_1 + \mathrm{p}_3. \tag{34}$$

Let us construct the matrices:

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u^2 & 0 & 1 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u^2 & 0 & 1 \end{pmatrix}, \quad \xi^{(b)}_{\beta} \xi^{\alpha}_{(b),\gamma} = \delta^2_{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{35}$$

Then, the set of Equation (15) takes the form

$$\ell^1_{(a),1} = \ell^p_{(a),\beta} = 0, \quad \ell^1_{(a),3} = \ell^2_{(a)}. \tag{36}$$

From (36), it follows that

$$\ell^{\alpha}_{(a)} = \delta^{\alpha}_1 \left(\delta^1_a + u^3 \delta^3_a \right) + \delta^{\alpha}_2 \delta^2_a + \delta^{\alpha}_3 \delta^3_a \tag{37}$$

From (37), one can obtain the matrix $g^{\alpha\beta} = \ell^{\alpha}_{(a)} \ell^{\beta}_{(b)} \eta^{ab}$:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} + 2a_{13}u^3 + a_{33}u_3^2 & a_{12} + u^3 a_{23} & a_{13} + u^3 a_{33} \\ a_{12} + u^3 a_{23} & a_{22} & a_{23} \\ a_{13} + u^3 a_{33} & a_{23} & a_{33} \end{pmatrix} \tag{38}$$

The number of independent functions in the metric tensor (38) can be reduced to three by admissible coordinate transformations.
Second version. The operators X_a are of the following form:

$$\mathrm{X}_1 = p_2, \quad \mathrm{X}_2 = p_3, \quad \mathrm{X}_3 = \mathrm{A}_3 p_1 + \mathrm{B} p_2 + \mathrm{R} p_3.$$

From structure Equation (32), it follows that

$$\mathrm{X}_1 = \mathrm{p}_2, \quad \mathrm{X}_2 = \mathrm{p}_3, \quad \tilde{\mathrm{X}}_3 = \mathrm{X}_3 + 2\ell_0 \mathrm{p}_3, \quad \mathrm{X}_3 = \mathrm{p}_1 + u^3 \mathrm{p}_2 \tag{39}$$

One can find the vectors ℓ^{α}_a using the following operators:

$$\mathrm{X}_1 = \mathrm{p}_2, \quad \mathrm{X}_2 = \mathrm{p}_3, \quad \mathrm{X}_3 = \mathrm{p}_1 + u^3 \mathrm{p}_2$$

Let us construct the matrices:

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} 1 & u^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} 1 & -u^3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{(b)}_{\beta} \xi^{\alpha}_{(b),\gamma} = \delta^3_{\gamma} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

Then, the set of Equation (14) takes the following form:

$$\ell^{\alpha}_{(a),p} = \ell^1_{(a),\beta} = \ell^3_{(a),\beta} = 0, \quad \ell^1_{(a),2} = \ell^2_{(a)}. \tag{41}$$

From (41), it follows that

$$\ell^{\alpha}_{(a)} = \delta^{\alpha}_1 \delta^1_a + \delta^{\alpha}_2 \left(\delta^2_a + u^1 \delta^3_a \right) + \delta^{\alpha}_3 \delta^3_a \tag{42}$$

Let us find the matrix

$$g^{\alpha\beta} = \ell^{\alpha}_{(a)} \eta^{ab} \ell^{\beta}_{(b)} = \begin{pmatrix} a_{11} & a_{12} + u^1 a_{13} & a_{13} \\ a_{12} + a_{13} u^1 & a_{22} + 2u^1 a_{23} u_1 + u_1^2 a_{33} & a_{23} + u^1 a_{33} \\ a_{13} & a_{23} + u^1 a_{33} & a_{33} \end{pmatrix}$$

From the Killing equations, we obtain

$$G^{\alpha\beta} = \dot{\ell}_0 u_1 \Big(2(\delta_1^\alpha \delta_3^\beta + \delta_3^\alpha \delta_1^\beta) + u_1(\delta_1^\alpha \delta_2^\beta + \delta_2^\alpha \delta_1^\beta)\Big)$$

Finally, we obtain the following:

$$\tilde{g}^{\alpha\beta} = G^{\alpha\beta} + g^{\alpha\beta} = \begin{pmatrix} a_{11} & a_{12} + u^1 a_{13} + \dot{\ell}_0 u^{1\,2} & a_{13} + 2\dot{\ell}_0 u^1 \\ a_{12} + a_{13} u^1 + \dot{\ell}_0 u^{1\,2} & a_{22} + u^1 a_{23} + u_1^2 a_{33} & a_{23} + u^1 a_{33} \\ a_{13} + 2\dot{\ell}_0 u^1 & a_{23} + u^1 a_{33} & a_{33} \end{pmatrix}. \quad (43)$$

Using the admissible transformations of coordinates, one can show that $g^{\alpha\beta}$ contains four independent functions of u^0.

4.3. Group$G_3(III)$

Let us write the structural equations in the following form:

$$[\mathrm{X}_1 \mathrm{X}_2] = [\mathrm{X}_1 \mathrm{X}_3] = 0, \quad [\mathrm{X}_2 \mathrm{X}_3] = \mathrm{X}_2. \quad (44)$$

First version. The operators X_a are of the following form:

$$\mathrm{X}_1 = a_3 \mathrm{p}_1, \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = \mathrm{A}_3 \mathrm{p}_1 + \mathrm{B}_3 \mathrm{p}_2 + \mathrm{R}_3 \mathrm{p}_3.$$

From the equations of the system (44), it follows that $a_3 = 1, \mathrm{A}_3 = 0, B_3 = u^2, \mathrm{R}_3 = 1$. Thus, in this case, the set of group operators has the following form:

$$\mathrm{X}_1 = \mathrm{p}_1, \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = \mathrm{p}_3 + \mathrm{u}^2 \mathrm{p}_2. \quad (45)$$

Using the following matrices,

$$\xi^\alpha_{(a)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u^2 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -u^2 & 1 \end{pmatrix}, \quad \xi^{(b)}_\beta \xi^\alpha_{(b),\gamma} = \delta^2_\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

one can find the system of Equation (15) in the following form:

$$\ell^\alpha_{(a),\beta} = \delta^\alpha_2 \delta^3_\beta \ell^2_{(a)}. \quad (46)$$

The solution of system (46) is

$$\ell^\alpha_{(a)} = \delta^\alpha_1 \delta^1_a + \delta^\alpha_2 \delta^2_a \exp(-u_3) + \delta^\alpha_3 \delta^3_a \quad (47)$$

Using (47), one can obtain the matrix $g^{\alpha\beta}$:

$$g^{\alpha\beta} = \ell^\alpha_{(a)} \ell^\beta_{(b)} \eta^{ab} = \begin{pmatrix} a_{11} & a_{12}\exp(-u^3) & a_{13} \\ a_{12}\exp(-u^3) & a_{22}\exp(-2u^3) & a_{23}\exp(-u^3) \\ a_{13} & a_{23}\exp(-u^3) & a_{33} \end{pmatrix} \quad (48)$$

Second version. The operators X_a have the following form:

$$\mathrm{X}_1 = p_2, \quad \mathrm{X}_2 = p_3, \quad \mathrm{X}_3 = \mathrm{A}_3 p_1 + \mathrm{B}_3 p_2 + \mathrm{R}_3 p_3.$$

From the equations of the system (44), after performing an admissible transformation, it follows that

$$\mathrm{A}_3 = 1, \quad \mathrm{B}_3 = \ell_0\big(u^0\big), \quad \mathrm{R}_3 = 1.$$

Therefore, the set of group operators in this case is of the following form:

$$\tilde{X}_3 = X_3 + \Delta X_3, \quad X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^3 p_3, \quad \Delta X_3 = \ell_0 p_2. \tag{49}$$

Using the operators X_a, one can find the vector fields $\ell^\alpha_{(a)}$ and the matrix $g^{\alpha\beta} = \ell^\alpha_{(a)} \ell^\beta_{(b)} \eta^{ab}$. To achieve this, the following matrices must be used:

$$\xi^\alpha_{(a)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & u^3 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} 0 & -u^3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi^\alpha_{(a),\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta^3_\gamma \end{pmatrix}.$$

The solution of system (14),

$$\ell^\alpha_{(a),\beta} = \delta^\alpha_3 \delta^1_\beta \ell^3_{(a)},$$

can be written as follows:

$$\ell^\alpha_{(a)} = \delta^\alpha_1 \delta^1_a + \delta^\alpha_2 \delta^2_a + \exp u^1 \delta^\alpha_3 \delta^2_a \tag{50}$$

Therefore, the matrix $g^{\alpha\beta}$ has the following form:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \exp u^1 \\ a_{12} & a_{22} & a_{23} \exp u^1 \\ a_{13} \exp u^1 & a_{23} \exp u^1 & a_{33} \exp 2u^1 \end{pmatrix} \tag{51}$$

Let us represent the metric tensor $\tilde{g}^{\alpha\beta}$ as

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} + G^{\alpha\beta}$$

From the the Killing equations, it follows that

$$G^{\alpha\beta} = \dot{\ell}_0 u^1 (\delta^\alpha_1 \delta^\beta_2 + \delta^\alpha_2 \delta^\beta_1)$$

Then,

$$\tilde{g}^{ij} = \begin{pmatrix} a_{11} & a_{12} & \dot{\ell}_0 u^1 + a_{13} \exp u^3 \\ a_{12} & a_{22} & a_{23} \exp u^3 \\ \dot{\ell}_0 u^1 + \exp u^3 a_{13} & a_{23} \exp u^3 & a_{33} \exp 2u^3 \end{pmatrix} \tag{52}$$

Third version. The operators X_a are of the form $X_1 = p_2, X_2 = a_2 p_1 + b_2 p_2$, $X_3 = a_3 p_1 + b_3 p_2 + p_3$. After admissible transformations, the functions a_3, b_3 are converted to zero. From the equation $[X_2 X_3] = X_2$, it follows that

$$X_2 = p_1 \exp -u^3$$

Then, the set of group operators has the following form:

$$X_1 = p_2, \quad X_2 = p_1 \exp -u^3, \quad X_3 = p_3. \tag{53}$$

Using the matrices

$$\xi^\alpha_{(a)} = \begin{pmatrix} \exp(-u^3) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} \exp u^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\xi^\alpha_{(a),\gamma} = -\delta^3_\gamma \exp\left(-u^3\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

let us construct the system of Equation (14):

$$\ell^1_{(a),1} = -\ell^3_a, \quad \ell^p_{(a),\alpha} = 0 \tag{54}$$

and find the vector fields $\ell^\alpha_{(a)}$:

$$\ell^\alpha_{(a)} = \delta^\alpha_1(\delta^1_a - u^1\delta^3_a) + \delta^\alpha_2(\delta^2_a + \delta^\alpha_3\delta^3_a). \tag{55}$$

The matrix of tensor $g^{\alpha\beta} = \ell^\alpha_{(a)}\ell^\beta_{(b)}\eta^{ab}$ has the following form:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} - 2u_1a_{13} + u_1^2 & a_{12} - u^1a_{23} & a_{13} - a_{23}u^1 \\ a_{12} - u^1a_{23} & a_{22} & a_{23} \\ a_{13} - u^1a_{33} & a_{23} & a_{33} \end{pmatrix} \tag{56}$$

4.4. GroupG$_3(IV)$

The structural equations have the following form:

$$[\mathrm{X}_1\mathrm{X}_2] = 0, \quad [\mathrm{X}_1\mathrm{X}_3] = \mathrm{X}_1, \quad [\mathrm{X}_2\mathrm{X}_3] = \mathrm{X}_1 + \mathrm{X}_2. \tag{57}$$

First version. The group operators have the following form:

$$\mathrm{X}_1 = a_1\mathrm{p}_1, \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = a_3\mathrm{p}_1 + b_3\mathrm{p}_2 + r_3\mathrm{p}_3. \tag{58}$$

From the second equation of the system (57), it follows that $\mathrm{X}_1 = \mathrm{p}_1\exp(-u_3)$. Let us substitute this in the last equation of the system (57) and perform an admissible transformation of coordinates. As a result, we obtain the following:

$$\mathrm{X}_1 = \mathrm{p}_1\exp\left(-u^3\right), \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = \mathrm{p}_3 + u^2\left(\mathrm{p}_1\exp\left(-u^3\right) + \mathrm{p}_2\right) \tag{59}$$

The matrices $\xi^\alpha_{(a)}, \xi^{(a)}_\alpha, \xi^\beta_{\alpha,\gamma}$ have the following form:

$$\xi^\alpha_{(a)} = \begin{pmatrix} \exp(-u^3) & 0 & 0 \\ 0 & 1 & 0 \\ u_2\exp(-u^3) & u^2 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} expu^3 & 0 & 0 \\ 0 & 1 & 0 \\ -u^2 & -u^2 & 1 \end{pmatrix},$$

$$\xi^\alpha_{(a),\gamma} = \delta^2_\gamma\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \exp(-u^3) & 1 & 0 \end{pmatrix} - \delta^3_\gamma\exp\left(-u^3\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ u^2 & 0 & 0 \end{pmatrix}$$

Using them, let us obtain the set of Equation (15):

$$\ell^1_{(a),1} = -\ell^3_{(a)}, \quad \ell^2_{(a),3} = \ell^2_{(a)}, \quad \ell^1_{(a),3} = \ell^2_{(a)}\exp(-\mathrm{u}^3), \quad \ell^3_{(a),\alpha} = 0.$$

The solution can be written in the following form:

$$\ell^\alpha_a = \delta^\alpha_1\left(\delta^1_a - u^1\delta^3_a + u^3\delta^2_a\right) + \exp u^3\delta^\alpha_2\delta^2_a + \delta^\alpha_3\delta^3_a. \tag{60}$$

Let us find matrix $g^{\alpha\beta} = \ell^\alpha_{(a)}\ell^\beta_{(b)}\eta^{ab}$:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} + 2(u^3a_{12} - u^1a_{13}) + \left(a_{33}{u^1}^2 + a_{22}{u^3}^2 - 2u^1a_{23}\right) & g^{12} & g^{13} \\ (a_{12} - u^1a_{23} + u^3a_{22})\exp u^3 & a_{22}\exp 2u^3 & g^{23} \\ a_{13} - u^1a_{33} + u^3a_{23} & a_{23}\exp u^3 & a_{33} \end{pmatrix} \tag{61}$$

Second version. The operators X_a have the following form:

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = A_3 p_1 + B_3 p_2 + R_3 p_3.$$

From structural Equation (57), it follows that

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + (u^2 + u^3) p_2 + u^3 p_3. \tag{62}$$

To obtain Equation (15), we use the following matrices:

$$\xi^\alpha_{(a)} = \begin{pmatrix} 1 & (u^2+u^3) & u^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} 1 & -(u^2+u^3) & -u^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^\alpha_{(a),\gamma} = \begin{pmatrix} 0 & (\delta^2_\gamma + \delta^3_\gamma) & \delta^3_\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Equations (15) have the following form:

$$\ell^\alpha_{(a),\beta} = \begin{pmatrix} 0 & \ell^2_{(a)} + \ell^3_{(a)} & \ell^3_{(a)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \ell^\alpha_{(a)} = \delta^\alpha_1 \delta^1_a + (\delta^\alpha_2 (\delta^2_a + u^1 \delta^3_a) + \delta^\alpha_3 \delta^3_a) \exp u^1 \tag{63}$$

Let us find matrix $g^{\alpha\beta} = \ell^\alpha_{(a)} \ell^\beta_{(b)} \eta^{ab}$:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} & (a_{12} + a_{13}u^1) \exp u^1 & a_{13} \exp u^1 \\ (a_{12} + a_{13}u^1) \exp u^1 & (a_{22} + 2a_{23}u^1 + a_{33}{u^1}^2) \exp 2u^1 & (a_{23} + a_{33}) \exp 2u^3 \\ a_{13} \exp u^1 & (a_{23} + a_{33}) \exp 2u^1 & a_{33} \exp 2u^1 \end{pmatrix} \tag{64}$$

Third version. The operators X_a have the following form:

$$X_1 = p_2, \quad X_2 = a_2 p_1 + b_2 p_2, \quad X_3 = A_3 p_1 + B_{3,} p_2 + R_3 p_3$$

Let us substitute this in equation $[X_1 X_3] = X_1 + X_2$. As a result, we obtain the following: $B_3 = u^2, R_3 = 1$. From the last equation of system (57), it follows that

$$X_1 = p_2, \quad X_2 = p_1 \exp(-u^3) - p_2 u^3, \quad X_3 = p_3 + p_2 u^2. \tag{65}$$

Using the following matrices,

$$\xi^\alpha_{(a)} = \begin{pmatrix} \exp(-u^3) & -u_3 & 0 \\ 0 & 1 & 0 \\ 0 & u^2 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} \exp u^3 & u^3 \exp u^3 & 0 \\ 0 & 1 & 0 \\ 0 & -u^2 & 1 \end{pmatrix},$$

$$\xi^\alpha_{(a),\gamma} = \begin{pmatrix} -\delta^3_\gamma \exp(-u_3) & -\delta^3_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & \delta^2_\gamma & 0 \end{pmatrix}.$$

one can obtain the system of Equation (14):

$$\ell^1_{(a),1} = -\ell^3_{(a)}, \quad \ell^2_{(a),1} = -\ell^3_{(a)} \exp\left(u^3\right), \quad \ell^2_{(a),3} = \ell^2_{(a)}, \quad \ell^\alpha_{(a),2} = 0, \quad \ell^1_{(a),3} = 0.$$

The solution has the following form:

$$\ell^\alpha_{(a)} = \delta^\alpha_1 \left(\delta^1_a + u^1 \delta^3_a\right) + \delta^\alpha_2 \left(\delta^2_a + \delta^3_a u^1\right) \exp u^3 - \delta^\alpha_3 \delta^3_a, \tag{66}$$

Let us find the matrix of the metric tensor $g^{\alpha\beta}$:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} - 2u_1 a_{13} + {u^1}^2 a_{33} & a_{12} - u^1(a_{33} + a_{23}) + {u^1}^2 a_{33} & a_{13} - u^1 a_{33} \\ a_{12} - u^1(a_{33} + a_{23}) + {u^1}^2 a_{33} & a_{22}^2 - 2u^1 a_{23} + {u^1}^2 a_{33} & a_{23} - u^1 a_{33} \\ a_{13} - u^1 a_{33} & a_{23} - u^1 a_{33} & a_{33} \end{pmatrix} \quad (67)$$

4.5. GroupG$_3$(V)

The structural equations have the form

$$[X_1 X_2] = 0, \quad [X_1 X_3] = X_1, \quad [X_2 X_3] = X_2. \quad (68)$$

First version. From structural Equation (68), it follows immediately that

$$X_1 = p_1 \exp u_3, \quad X_2 = p_2, \quad X_3 = -p_3 + u^2 p_2$$

This version is equivalent to Petrov's result ([17], (f. 25.24), p. 163) (after variable transformation: $\tilde{u}^1 = u^1 \exp u_3$). On the other hand, it is a partial case of the third variant, which is considered below. Therefore, it should be excluded from the classification.

Second version.(O'k) $X_1 = p_2, X_2 = p_3$. From the structural equations, it follows (after using the admissible transformations of variables) that

$$X_3 = p_1 + u^2 p_2 + u^3 p_3.$$

Using the following matrices,

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & u^2 & u^3 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} -u^2 & -u^3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi^{\alpha}_{(a),\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta^2_\gamma & \delta^2_\gamma \end{pmatrix},$$

one can obtain the system of Equation (14):

$$\ell^1_{(a),\alpha} = 0, \quad \ell^2_{(a), \quad \alpha} = \delta^1_\alpha \ell^2_{(a)}, \quad \ell^3_{(a),\alpha} = \delta^1_\alpha \ell^3_{(a)}$$

The solutions have the following form:

$$\ell^{\alpha}_{(a)} = \delta^1_a \delta^{\alpha}_1 + \exp u^1 (\delta^a_2 \delta^2_a + \delta^{\alpha}_3 \delta^3_a)$$

Let us find components of the matrix $g^{\alpha\beta} = \ell^{\alpha}_{(a)} \ell^{\beta}_{(b)}$:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} & a_{12} \exp u^1 & a_{13} \exp u^1 \\ a_{12} \exp u^1 & a_{22} \exp 2u^1 & a_{23} \exp 2u^1 \\ a_{13} \exp u^1 & a_{23} \exp 2u^1 & a_{33} \exp 2u^1 \end{pmatrix}, \quad (69)$$

Third version $X_1 = p_2, X_2 = a_2 p_1 + b_2 p_2$. From structural Equation (68), it follows that

$$X_2 = p_1 \exp u^3 + \ell_0 p_2, \quad X_3 = p_3 + u^2 p_2.$$

Using the following matrices,

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} \exp u^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u^2 & 1 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} \exp(-u^3) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u^2 & 1 \end{pmatrix},$$

$$\xi^{\alpha}_{(a),\gamma} = \begin{pmatrix} \delta^3_\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \delta^2_\gamma & 0 \end{pmatrix},$$

one can obtain the system of Equation (14):

$$\ell^1_{(a),1} = \ell^3_{(a)}, \qquad \ell^2_{(a),3} = \ell^2_{(a)}, \quad \ell^\alpha_{(a),2} = 0,$$

$$\ell^3_{(a),\alpha} = 0, \quad \ell^\alpha_{(a),3} = 0, \quad \ell^1_{(a),p} = 0, \qquad \ell^2_{a,1} = 0.$$

The solution can be represented in the following form:

$$\ell^\alpha_{(a)} = \delta^\alpha_1 \left(\delta^1_a + u^1 \delta^3_a\right) + \delta^\alpha_2 \delta^2_a \exp u^3 + \delta^\alpha_3 \delta^3_a \tag{70}$$

Then, the components of matrix $g^{\alpha\beta} = \ell^\alpha_{(a)} \ell^\beta_{(b)} \eta^{\alpha\beta}$ will be as follows:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} + 2u^1 a_{13} + a_{33} {u^1}^2 & (a_{12} + u^1 a_{13}) \exp u^3 & a_{13} + u^1 a_{33} \\ (a_{12} + u^1 a_{13}) \exp u^3 & a_{22} \exp 2u^3 & a_{23} \exp u^3 \\ a_{13} + u^1 a_{33} & a_{23} \exp u^3 & a_{33} \end{pmatrix}. \tag{71}$$

The tensor $\tilde{g}^{\alpha\beta}$ can be represented as follows (Killing equations have been used):

$$\tilde{g}^{\alpha\beta} = g^{\alpha\beta} + \ell_{0,0} u^1 \left(\delta^\alpha_1 \delta^\beta_2 + \delta^\beta_1 \delta^\alpha_2\right) \exp u^3$$

4.6. *Group$G_3(VI)$*

The structural equations have the following form:

$$[\mathrm{X}_1 \mathrm{X}_2] = 0, \quad [\mathrm{X}_2 \mathrm{X}_3] \quad = q\mathrm{X}_2, \quad [\mathrm{X}_1 \mathrm{X}_3] = \mathrm{X}_1. \tag{72}$$

First version. $\mathrm{X}_1 = p_1 \exp(-u^3), \mathrm{X}_2 = p_2$. From the structural equation, it follows that

$$\mathrm{X}_1 = p_1 \exp\left(-u^3\right), \quad \mathrm{X}_2 = \mathrm{p}_2, \quad \mathrm{X}_3 = \mathrm{p}_3 + q u^2 \mathrm{p}_2. \tag{73}$$

The set corresponds to the Petrov set ([17], (f. 25.24), p. 163, when substituting $\widetilde{u}^1 = u^1 \exp u^3$). Let us find the vectors $\ell^\alpha_{(a)}$ using the following matrices:

$$\xi^\alpha_{(a)} = \begin{pmatrix} \exp(-u^3) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & qu^2 & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} \exp u^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -qu^2 & 1 \end{pmatrix}, \quad \xi^\alpha_{(a),\gamma} = \begin{pmatrix} -\exp(-u^3)\delta^3_\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q\delta^2_\gamma & 0 \end{pmatrix}.$$

The equations of (14) take the form

$$\ell^1_{(a),1} = -\ell^3_{(a)}, \quad \ell^2_{(a),3} = q\ell^2_{(a)}, \quad \ell^\alpha_{(a),2} = 0, \quad \ell^3_{(a),\alpha} = 0, \quad \ell^2_{(a),1} = 0.$$

The solution can be written through the operators $Y_a = \ell^\alpha_{(a)} p_\alpha$ in the following form:

$$Y_1 = p_1, \quad Y_2 = p_2 \exp\left(qu^3\right), \quad Y_3 = p_3 - u^1 p_1 \Rightarrow \ell^\alpha_{(a)} = \delta^\alpha_1 \delta^1_a + \delta^\alpha_2 \delta^2_a + \exp u^3 \delta^\alpha_3 \delta^2_a \tag{74}$$

$$\Rightarrow \ell^\alpha_{(a)} = \delta^\alpha_1 \delta^1_a + \delta^\alpha_2 \delta^2_a + \exp u^3 \delta^\alpha_3 \delta^2_a.$$

Let us provide the elements of the matrix $g^{\alpha\beta}$:

$$g^{11} = \begin{pmatrix} a_{11} + 2u^1 a_{13} + {u^1}^2 a_{33} & (a_{12} + u^1 a_{23}) \exp qu^3 & a_{13} + u^1 a_{33} \\ (a_{12} + u^1 a_{23}) \exp qu^3 & a_{22} \exp 2qu^3 & a_{23} \exp qu^3 \\ a_{13} + u^1 a_{33} & a_{23} \exp qu^3 & a_{33} \end{pmatrix}.$$

Second version. $X_1 = p_2, X_2 = p_3$. From the structural equations we immediately obtain

$$X_3 = p_1 + u^2 p_2 + qu^3 p_3.$$

Let us find the vector fields $\ell^{\alpha}_{(a)}$. Using the following matrices,

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} 1 & u^2 & qu^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} 1 & -u_2 & -qu^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \xi^{\alpha}_{(a),\gamma} = \begin{pmatrix} 0 & \delta^2_{\gamma} & q\delta^3_{\gamma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

one can obtain Equation (14) in the following form:

$$\ell^1_{(a,\alpha)} = \ell^{\alpha}_{(a,2)} = \ell^{\alpha}_{(a,3)} = 0, \quad \ell^2_{(q,1)} = \ell^2_{(q)}, \quad \ell^3_{(a,1)} = q\ell^3_{(a)} \tag{75}$$

Let us find solution of system (75):

$$Y_1 = p_1, \quad Y_2 = p_2 \mathrm{exp} u^1, \quad Y_3 = p_3 \mathrm{exp}(qu^1) \Rightarrow \tag{76}$$

$$\ell^1_{(a)} = \delta^1_a, \quad \ell^2_{(a)} = \delta^2_a \mathrm{exp} u^1, \quad \ell^3_{(a)} = \delta^3_a \mathrm{expq} u^1$$

Matrix $g^{\alpha\beta} = \ell^{\alpha}_{(a)} \ell^{\beta}_{(b)} \eta^{\alpha\beta}$ has the following form:

$$g^{\alpha\beta} = \begin{pmatrix} a_{11} & a_{12}\mathrm{exp} u^1 & a_{13}\mathrm{exp} qu^1 \\ a_{12} exp u^1 & a_{22} exp 2u^1 & a_{23} exp (q+1) u^1 \\ a_{13} exp qu^1 & a_{23} exp (q+1) u^1 & a_{33} exp 2qu^1 \end{pmatrix} \tag{77}$$

Third version. The operators X_a have the following form:

$$X_1 = \mathrm{p}_2, \quad X_2 = a_2 \mathrm{p}_1 + b_2 \mathrm{p}_2, \quad X_3 = A_3 \mathrm{p}_1 + B_3 \mathrm{p}_2 + R_3 \mathrm{p}_3.$$

From the structural equations, it follows that

$$A_{3,2}\mathrm{p}_1 + B_{3,2}\mathrm{p}_2 + R_{3,2}\mathrm{p}_3 = \mathrm{p}_2.$$

Hence,

$$A_3 = a_3, \quad B_3 = u^2 + b_3, \quad R_3 = r_3.$$

Since $r_3 \neq 0$, the function r_3 can be be turned to (-1). Then, the functions a_3, b_3 are reversed to zero by admissible coordinate transformations. The operator X_3 will take the form $X_3 = -p_3 + u^2 p_2$. Let us substitute X_2 and X_3 into the last structural equation. The result is as follows:

$$b_2 p_2 + a_{2,3} p_1 + b_{2,3} = q(a_2 p_1 + b_2 p_2)$$

Hence, $a_2 = a_0 \exp(qu^3), b_2 = b_0 \exp((q+1)u^3)$. By using the admissible coordinate transformation, one can turn a_0, b_0 to unity. Finally, the set has the following form:

$$X_1 = \mathrm{p}_2, \quad X_2 = p_1 \exp(qu^3) + \mathrm{p}_2 \exp((q+1)u^3), \quad X_3 = -\mathrm{p}_3 + u^2 \mathrm{p}_2. \tag{78}$$

The set differs from Petrov's set ([17], (f. 25.32), p. 165) by the transformation of the coordinate u^1.

Using the following matrices,

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} \exp(u^3 q) & \exp(q+1)u^3 & 0 \\ 0 & 1 & 0 \\ 0 & u^2 & -1 \end{pmatrix}, \quad \xi^{(a)}_{\alpha} = \begin{pmatrix} \exp(-qu^3) & -\mathrm{exp} u^3 & 0 \\ 0 & 1 & 0 \\ 0 & u_2 & -1 \end{pmatrix},$$

$$\xi^{\alpha}_{(a),\gamma} = \begin{pmatrix} \delta^3_\gamma q exp\left(u^3 q\right) & \delta^3_\gamma (q+1) exp\left(u^3(q+1)\right) & 0 \\ 0 & 0 & 0 \\ 0 & \delta^2_\gamma & 0 \end{pmatrix}$$

we obtain the system of Equation (14) in the following form:

$$\ell^1_{(a),1} = q\ell^3_{(a)}, \quad \ell^2_{(a),1} = (q+1)\ell^3_{(a)}, \quad \ell^2_{(a)},$$

$$\ell^3_{(a)} = c^3_a = const, \quad \ell^\alpha_{(a),2} = \ell^1_{(a),3} = 0.$$

The solution can be written in the following form:

$$\mathrm{Y}_1 = p_1, \quad \mathrm{Y}_2 = \mathrm{p}_2 \mathrm{exp} u^3, \quad \mathrm{Y}_3 = p_3 + u^1(\mathrm{p}_2(1-q)\mathrm{exp} u^3 - q\mathrm{p}_1) \Rightarrow \tag{79}$$

$$\ell^\alpha_{(a)} = \delta^\alpha_1(\delta^1_a - qu^1\delta^3_a) + \mathrm{exp} u^3 \delta^\alpha_2(\delta^2_a + u^1(1-q)\delta^3_a) + \delta^\alpha_3\delta^3_a$$

Using vectors $\ell^\alpha_{(a)}$, let us find elements of the matrix $g^{\alpha\beta} = \ell^\alpha_{(a)}\ell^\beta_{(b)}\eta^{\alpha\beta} =$

$$\begin{pmatrix} a_{11} - 2qu^1 a_{13} + q^2 {u^1}^2 a_{33} & \exp u_3(a_{12} - u^1(a_{13}(q-1) + qa_{23}) - q(q-1){u^1}^2 a_{33}) & g^{13} \\ g^{12} & \exp 2u^3(a_{22} + 2(q-1)a_{23}u^1 + (q-1)^2 {u^1}^2 a_{33}) & g^{23} \\ a_{13} - qu^1 a_{33} & \exp u^3(a_{23} + (q-1)u^1 a_{33}) & a_{33}. \end{pmatrix} \tag{80}$$

4.7. $Group G_3(VII)$

The structural equations have the following form:

$$[\mathrm{X}_1\mathrm{X}_2] = 0 \quad [\mathrm{X}_1\mathrm{X}_3] = \mathrm{X}_2 \quad [\mathrm{X}_2\mathrm{X}_3] = 2\cos\alpha \mathrm{X}_2 - \mathrm{X}_1 \ . \tag{81}$$

Here, I denote $q = 2\cos\alpha = const$.

First version. Obviously, the first variant cannot be realized, since the first two equations of the structure imply $\mathrm{X}_1 = \mathrm{p}_1$. In this case, X_1 commutes with X_p, which is impossible.

Second version. $\mathrm{X}_1 = p_2, \mathrm{X}_2 = p_3, \mathrm{X}_3 = \mathrm{A}p_1 + \mathrm{B}p_2 + Rp_3$. From structural Equation (81), it follows that

$$\mathrm{X}_1 = p_2, \quad \mathrm{X}_2 = p_3, \quad \mathrm{X}_3 = p_1 + u^3.(2\cos\alpha p_3 - p_2) + u^2 p_3 \tag{82}$$

Let us find the vector fields $\ell^\alpha_{(a)}$. Using the following matrices,

$$\xi^\alpha_{(a)} = \begin{pmatrix} 1 & -u^3 & 2u^3\cos\alpha + u^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \xi^{(a)}_\alpha = \begin{pmatrix} 1 & u^3 & -\left(u^2 + 2u^3\cos\alpha\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\xi^\alpha_{(a),\gamma} = \begin{pmatrix} 0 & -\delta^3_\gamma & \delta^2_\gamma + 2\delta^3_\gamma\cos\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

one can obtain the system of Equation (14) in the following form:

$$\ell^1_{(a),1} = 0, \quad \ell^2_{(a),1} = -\ell^3_{(a)}, \quad \ell^3_{(a),1} = \ell^2_{(a)} + 2\ell^3_a\cos\alpha, \quad \ell^\alpha_{(a),2} = \ell^\alpha_{(a),3} = 0. \tag{83}$$

Let us represent the solutions in the form $Y_a = \ell^\alpha_a p_\alpha$:

$$Y_1 = p_1, \quad Y_2 = (p_2\sin p - p_3\sin(p+\alpha))\exp q, \quad Y_3 = (p_2\cos p + p_3\cos(p+\alpha))\exp q \Rightarrow \tag{84}$$

$$\ell^\alpha_{(a)} = \delta^\alpha_1\delta^1_a + \exp q(\delta^\alpha_2(\delta^2_a\sin p + \delta^3_a\cos p) + \delta^\alpha_3(\delta^2_a\sin(p+\alpha) + \delta^3_a\cos(p+\alpha))), \tag{85}$$

where $q = u^1\cos\alpha, p = u^1\sin\alpha$. Using $\ell^\alpha_{(a)}$ from (85), one can find the elements of the matrix $g^{\alpha\beta} = \ell^\alpha_{(a)}\ell^\beta_{(b)}\eta^{ab}$:

$$\begin{cases} g^{11} = a_{11}, g^{12} = \exp q(a_{12}\sin p + a_{13}\cos p), g^{13} = -\exp q(a_{12}\sin(p+\alpha) + a_{13}\cos(p+\alpha)) \\ g^{23} = -\exp 2q\left(\frac{(a_{22}+a_{33})}{2}\cos\alpha + \frac{(a_{33}-a_{22})}{2}\cos(2p+\alpha) + a_{23}\sin(2p+\alpha)\right) \\ g^{22} = \exp 2q\left(\frac{a_{22}+a_{33}}{2} + \frac{(a_{33}-a_{22})}{2}\cos 2p + a_{23}\sin 2p\right), \\ g^{33} = -\exp 2q\left(\frac{a_{22}+a_{33}}{2} + \frac{(a_{33}-a_{22})}{2}\cos 2(p+\alpha) + a_{23}\sin 2(p+\alpha)\right) \end{cases} \tag{86}$$

Third version. The operators X_a have the following form:

$$X_1 = p_2, \quad X_2 = a_2 p_1 + b_2 p_2, \qquad X_3 = A p_1 + B p_2 + R p_3$$

From equation, after performing admissible transformations $[X_1 X_3] = X_1$, it follows that $X_3 = p_3 + u^2 p_2$. Let us substitute it into the equation $[X_2 X_3] = 2\cos\alpha X_2 - X_1$. As a result, we obtain a condition on the functions a_2, b_2:

$$a_{2,3} = a_2(b_2 - 2\cos\alpha), \quad b_{2,3} = (b_2 - \cos\alpha)^2 + \sin^2\alpha.$$

From here, it follows that

$$a_2 = \frac{\exp(-q)}{\cos p}, \qquad b_2 = \cos\alpha + \sin\alpha\frac{\sin p}{\cos p}.$$

Here, $p = u^3\sin\alpha, q = u^3\cos\alpha$. Thus, the set of operators X_a is reduced to the following form:

$$X_1 = p_2, \quad X_3 = p_3 + u^2 X_2, \quad X_2 = p_1\frac{\exp -q}{\cos p} + p_2\frac{\cos(p-\alpha)}{\cos p}. \tag{87}$$

Let us find the vector fields $\ell^\alpha_{(a)}$ using the following matrices:

$$\xi^\alpha_{(a)} = \begin{pmatrix} 0 & 1 & 0 \\ \exp(q) & \frac{\cos(p-\alpha)}{\cos p} & 0 \\ u_2\frac{\exp(q)}{\cos p} & u_2\frac{\cos(p-\alpha)}{\cos p} & 1 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} -\exp(-q)\cos(p-\alpha) & \cos p\exp(-q) & 0 \\ 1 & 0 & 0 \\ 0 & -u_2 & 1 \end{pmatrix},$$

$$\xi^\alpha_{(a),\gamma} = \delta^2_\gamma\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\exp q}{corp} & \frac{\cos(p-\alpha)}{\cos p} & 0 \end{pmatrix} + \delta^3_\gamma\begin{pmatrix} 0 & 0 & 0 \\ -\frac{\exp q\cos(p+\alpha)}{\cos^2 p} & \frac{\sin^2\alpha}{\cos^2 p} & 0 \\ -u_2\frac{\exp q\cos(p+\alpha)}{\cos^2 p} & \frac{u_2\sin^2\alpha}{\cos^2 p} & 0 \end{pmatrix}$$

The system of Equation (14) has the following form:

$$\ell^1_{(a),1} = \frac{\cos(p+\alpha)}{\cos p}\ell^3_{(a)}, \quad \ell^2_{(a),1} = \frac{\sin^2\alpha\exp(-q)}{\cos p}\ell^3_a, \quad \ell^3_{(a),\alpha} = 0, \quad \ell^\alpha_{(a),2}, = 0 \tag{88}$$

$$\ell^1_{(a),3} = \frac{\exp q}{\cos p}, \quad \ell^2_{(a),3} = \ell^2_{(a)}\frac{\cos(p-\alpha)}{\cos p}.$$

The solution can be represented in the following form:

$$Y_1 = p_2, \quad Y_2 = p_1\sin\alpha\frac{\sin p}{\cos p} + p_2\sin^2(\alpha)\frac{\exp q}{\cos p}, \quad Y_3 = p_3 + u^1(Y_2 - p_1\cos\alpha). \tag{89}$$

$$\ell^\alpha_{(a)} = \delta^\alpha_1(\delta^1_a + \exp q(\delta^2_a + u^2\delta^3_a)) + \delta^\alpha_2\cos(p-\alpha)(\delta^2_a + \delta^2_a u^2) + \delta^\alpha_3\delta^3_a. \tag{90}$$

Here are the components of the tensor $g^{\alpha\beta} = \ell^\alpha_{(a)}\ell^\beta_{(b)}\eta^{ab}$:

$$\begin{cases} g^{11} = a_{11} + \frac{\cos^2(p+\alpha)}{\cos^2 p}\left(a_{22} + 2u^1 a_{23} + {u^1}^2 a_{33}\right) - \frac{2\cos(p+\alpha)}{\cos p}\left(a_{12} + u^1 a_{13}\right) \\ g^{22} = \frac{\sin^4\alpha \exp(2q)}{\cos^2 p}\left(a_{22} + 2u^1 a_{23} + {u^1}^2 a_{33}\right), \qquad g^{33} = a_{33} \\ g^{12} = \frac{\sin^2\alpha \exp q}{\cos p}\left(a_{12} + u^1 a_{13} - \frac{\cos(p+\alpha)}{\cos p}\left(a_{22} + 2u^1 a_{23} + {u^1}^2 a_{33}\right)\right) \\ g^{13} = a_{13}\frac{\cos(p+\alpha)}{\cos p}\left(a_{23} + u^1 a_{33}\right), \qquad g^{23} = \frac{\sin^2\alpha \exp q}{\cos p}\left(a_{23} + u^1 a_{33}\right) \end{cases} \tag{91}$$

5. Unsolvable Groups

For unsolvable groups $G_3(VIII), G_3(IX)$, there is only the fourth version:

$$X_1 = p_2.$$

For group $G_3(IX)$, it does not matter which of the operators X_a have to be diagonalized. In the case of group $G_3(VIII)$, following Petrov, we choose the operator X_1 as the diagonalized operator. For both groups, the sets of Killing vector fields are the same for isotropic and non-isotropic Petrov spaces, so in the case of group $G_3(IX)$ acting on an isotropic Petrov space, the same local coordinate system $\{u^a\}$ is chosen as for the non-isotropic case (Petrov [17] (f. 25.5) p. 157). As to the case of group $G_3(VIII)$ acting on an isotropic Petrov space, this cannot be carried out because two spaces were omitted when classifying non-isotropic Petrov spaces of type $V_3(VIII)$ (see Petrov [17] (f. 25.4) p. 157). Therefore, we accept the coordinate systems used by Petrov when classifying the spaces of type $V_4^*(VIII)$ ([17], formulas (25.35)–(25.37) p. 166).

5.1. *Group* $G_3(VIII)$

The structural equations are of the following form:

$$[X_1 X_2] = X_1, \quad [X_2 X_1] = X_3, \quad [X_1 X_3] = 2X_2. \tag{92}$$

As already noted, the case $\xi^{\alpha}_{(2)} = A_2 p_1$ must be omitted, since from the structural equations it follows that $A_2 = 1 \Rightarrow X_1$ commutes with operators X_1, X_3, which is impossible. From the structural equations of (92), it follows that

$$X_1 = \tilde{X}_1 \exp(-u^2), \quad X_2 = \tilde{X}_2 \exp(u^2),$$

where $\tilde{X}_1 = a_1 p_1 + b_1 p_2 + r_1 p_3, \tilde{X}_3 = a_3 p_1 + b_3 p_2 + r_3 p_3, \tilde{X}_{1,3} = \tilde{X}_{3,3} = 0$. Without loss of generality, one can assume $r_1 \neq 0$. Using the admissible transformations of variables, it is possible to reduce the operators X_a to the following form:

$$X_1 = p_3 \exp\left(-u^2\right), \quad X_2 = p_2, \quad X_3 = \left(p_1 - 2u^3 p_2 + \left({u^3}^2 - \varepsilon\right)\right)\exp u^2 \quad (\varepsilon = 0, \pm 1). \tag{93}$$

These operators of group $G_3(VIII)$ for the case of null Petrov spaces were found by Petrov in [17] (p. 165). It is impossible to turn the parameter ε to zero, even by coordinate transformations of the general form (2). Since the types of solutions of the Killing equations and equations (14) depend essentially on the values of the parameter ε, the relations (93) represent three non-equivalent sets of operators X_a, each of which corresponds to a non-equivalent set of operators of the group $G_3(VIII)$, which acts in the non-isotropic Petrov space $V_4(VIII)$. Thus, the list of non-null homogeneous Petrov spaces of type $V_4(VIII)$ (see [17], f. 25.5) should be supplemented. Using the following matrices,

$$\xi^{\alpha}_{(a)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & u^2 & 1 \\ -\exp u^3 & \left(u^{2^2} + \varepsilon \exp 2u^3\right) & 2u^2 \end{pmatrix},$$

$$\xi^{(a)}_{\alpha} = \begin{pmatrix} \varepsilon \exp u^3 - u^{2^2} \exp(-u^3) & 2u^2 \exp(-u^3) & -\exp(-u^3) \\ 1 & 0 & 0 \\ -u^2 & 1 & 0 \end{pmatrix},$$

$$\xi^{\alpha}_{(a),\gamma} = \delta^2_{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2u^2 & 2 \end{pmatrix} + \delta^3_{\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\exp u^3 & 2\varepsilon \exp 2u^3 & 0 \end{pmatrix},$$

one can obtain the system of Equation (14) in the following form:

$$\ell^1_{(a),1} = \ell^3_{(a)}, \quad \ell^2_{(a),1} = -2\varepsilon \ell^3_{(a)} \exp u^3, \quad \ell^3_{(a),1} = -2\ell^2_{(a)} \exp\left(-u^3\right), \tag{94}$$

$$\ell^1_{(a),3} = \ell^3_{(a),3} = 0, \quad \ell^2_{(a),3} = \ell^2_{(a)}, \quad \ell^{\alpha}_{(a),2} = 0. \tag{95}$$

From the set of Equation (94), it follows that

$$\ell^1_{(a)} = f^1_a(u^1), \quad \ell^2_{(a)} = f^2_a(u^1) \exp u^3, \quad \ell^3_{(a)} = f^3_a(u^1). \tag{96}$$

Let us substitute (96) to (95). As a result, we obtain the following:

$$\dot{f}^1_{(a)} = f^3_{(a)} \quad \dot{f}^2_{(a)} = -2\varepsilon f^3_{(a)} \quad \dot{f}^3_{(a)} = -2f^2_{(a)} \tag{97}$$

The dot denotes the derivative of u^1. Depending on the values of the parameter ε, one can obtain following solutions of the set of Equation (97):

1. $\varepsilon = 0 \Rightarrow f^2_{(a)} = -C^3_{(a)}, f^1_{(a)} = C^3_a u^{1^2} + C^2_a u^1 + C^1_a, f^3_{(a)} = 2C^3_a u^1 + C^2_a \quad (C^{\alpha}_a = const)$. Hence the operators Y_a can be represented as follows:

$$\mathrm{Y}_1 = \mathrm{p}_1 \quad \mathrm{Y}_2 = \mathrm{p}_3 + u^1 \mathrm{p}_1, \quad \mathrm{Y}_3 = u^{1^2} \mathrm{p}_1 + 2u^1 \mathrm{p}_3 - \mathrm{p}_2 \exp u^3 \Rightarrow \tag{98}$$

$$\ell^{\alpha}_{(a)} = \delta^{\alpha}_1 (\delta^1_a + u^{1^2} \delta^2_a + u^1 \delta^3_a) - \exp(u^3) \delta^{\alpha}_2 \delta^2_a + \delta^{\alpha}_3 (2u^1 \delta^2_a + \delta^3_a). \tag{99}$$

Matrix $g^{\alpha\beta} = \ell^{\alpha}_{(a)} \ell^{\beta}_{(b)} \eta^{ab}$ has the following form:

$$\begin{cases} g^{11} = \left(a_{11} + 2u^1 a_{13} + u^{1^2}(a_{33} + 2a_{12}) + 2u^{1^3} a_{23} + a_{22} u^{1^4}\right) \\ g^{12} = -\exp u^3 \left(a_{12} + a_{23} u^1 + a_{22} u^{1^2}\right), \quad g^{22} = a_{22} \exp 2u^3 \\ g^{33} = a_{33} + 4u^1 a_{23} + 4u^{1^2} a_{22}, \quad g^{23} = -\exp u^3 (a_{23} + 2u^1 a_{22}) \\ g^{13} = a_{13} + u^1(a_{33} + 2a_{12}) + 3u^{1^2} a_{23} + 2a_{22} u^{1^3} \end{cases} \tag{100}$$

2. $\varepsilon = \pm 1$. Let us introduce the functions $st(u_1), ct(u^1)$, which, depending on the value of the parameter ε, have the following form:

$$st(u) = \frac{\exp(\sqrt{\varepsilon} u) - \exp(-\sqrt{\varepsilon} u)}{2\sqrt{\varepsilon}}, ct(u) = \frac{\exp(\sqrt{\varepsilon} u) + \exp(-\sqrt{\varepsilon} u)}{2}.$$

Then, solutions of Equations (95)–(97) can be represented in the following form:

$$\ell^{\alpha}_{(a)} = \delta^{\alpha}_1 (\delta^1_a + st(2u^1) \delta^2_a + \varepsilon ct(2u^1) \delta^3_a) + 2\delta^{\alpha}_2 (\delta^2_a st(2u^1) + \varepsilon \delta^3_a ct(2u^1)) \exp u^3 + 2\delta^{\alpha}_3 (ct(2u^1) \delta^2_a + st(2u^1) \delta^3_a). \tag{101}$$

The components of the matrix $g^{\alpha\beta}$ are as follows:

$$\begin{cases} g^{11} = \left[a_{11} + \frac{(a_{33}-\varepsilon a_{22})}{2}\right] + ct(4u^1)\left(\frac{a_{33}+\varepsilon a_{22}}{2}\right) + \varepsilon a_{23}st(4u^1) + 2\left(a_{12}st(2u^1) + \varepsilon a_{13}ct(2u^1)\right) \\ g^{12} = \left[\frac{(a_{33}-\varepsilon a_{22})}{2} + ct(4u^1)\left(\frac{a_{33}+\varepsilon a_{22}}{2}\right) + 2\varepsilon a_{23}st(4u^1) + 2a_{12}st(2u^1) + 2a_{13}ct(2u^1)\right]\exp u^3 \\ g^{22} = 2\exp 2u^3\left[(a_{33} - \varepsilon a_{22}) + (a_{33} + \varepsilon a_{22})ct4u^1 + 2\varepsilon a_{23}st(4u^1)\right] \\ g^{33} = 2\left[(a_{22} - \varepsilon a_{33}) + (a_{22} + \varepsilon a_{33})ct(4u^1) + 2a_{23}st(4u^1)\right] \\ g^{13} = 2a_{12}ct(2u^1) + 2a_{13}st(2u^1) + (a_{22} + \varepsilon a_{33})st(4u^1) + 2a_{23}ct(4u^1) \\ g^{23} = \left[2(a_{22} + \varepsilon a_{33})st(4u^1) + 4\varepsilon a_{23}ct(4u^1)\right]\exp u^3 \end{cases} \quad (102)$$

The difference between the null space case and the non-null space case is that in the case of the space $V_3^*(VIII)$ the function a_{11} can be converted to zero (as is actually done in formulas (25.35)–(25.37) on p. 166 in [17]). In the case of non-null space, this cannot be carried out.

5.2. $GroupG_3(IX)$

The structural equations have the following form:

$$[X_1X_2] = X_3, \quad [X_2X_3] = X_1, \quad [X_3X_1] = X_2 \ . \quad (103)$$

A vector $\xi^\alpha_{(1)}$ will be chosen to diagonalize. Obviously, the first variant leads to degeneracy of the set. Therefore, without restriction of generality, one can assume $X_1 = p_2$. From the first and third equations of the structure of (103), it follows that

$$X_{2,2} = X_3, \quad X_{3,2} = -X_2 \Rightarrow X_{2,22} + X_2 = 0 \quad (104)$$

The solution can be presented in the following form: $X_2 = \tilde{X}_2 \sin u^2 + \tilde{X}_3 \cos u^2, X_3 = \tilde{X}_2 \cos u^2 - \tilde{X}_3 \sin u^2$, where the operators $\tilde{X}_p$ have the following form:

$$\tilde{X}_p = a_p p_1 + b_p p_2 + r_p p_3 \quad (105)$$

Without loss of generality, one can believe that $r_3 = 1$. Then, by admissible transformations of variables that conserve the form of the operator X_1, the functions a_3, b_3 can be converted to zero. Thus, the operators $\tilde{X}_p$ have the following form:

$$\tilde{X}_3 = p_3, \quad \tilde{X}_2 = a_2p_2 + b_2p_2 + r_2p_3. \quad (106)$$

From the second equation of system (103), it follows that

$$X_1 = p_2, \quad X_2 = p_3 \cos u_2 + \frac{\sin u^2}{\cos u_3}\left(p_1 + p_2 \sin u^3\right), \quad X_3 = X_{2,2} \ . \quad (107)$$

This form corresponds to Petrov [17] (f. (25.6), p. 157). Let us find the solution of the system of Equation (14). To do this, we use the following matrices:

$$\xi^\alpha_{(a)} = \begin{pmatrix} \frac{\sin u^2}{\cos u^3} & \sin u^2 \frac{\sin u^3}{\cos u^3} & \cos u^2 \\ 0 & 1 & 0 \\ \frac{\cos u^2}{\cos u^3} & \cos u^2 \frac{\sin u^3}{\cos u^3} & -\sin u^2 \end{pmatrix}, \quad \xi^{(a)}_\alpha = \begin{pmatrix} \sin u^2 \cos u^3 & -\sin u^3 & \cos u^2 \cos u^3 \\ 0 & 1 & 0 \\ \cos u^2 & 0 & -\sin u^2 \end{pmatrix}, \quad (108)$$

$$\xi^{(b)}_\beta \xi^\alpha_{(b),\gamma} = \delta^2_\gamma \begin{pmatrix} 0 & 0 & -\cos u^3 \\ 0 & 0 & 0 \\ \frac{1}{\cos u^3} & \frac{\sin u^3}{\cos u^3} & 0 \end{pmatrix} + \delta^3_\gamma \begin{pmatrix} \frac{\sin u^3}{\cos u^3} & \frac{1}{\cos u^3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The system of Equation (14) will take the following form:

$$\begin{cases} \ell^1_{(a),3} = \frac{\ell^2_{(a)}}{\cos u^3} & \ell^2_{(a),3} = \ell^2_{(a)} \frac{\sin u^3}{\cos u^3} & \ell^3_{(a),3} = 0 \\ \ell^1_{(a),1} = \ell^3_{(a)} \frac{\sin u^3}{\cos u^3} & \ell^2_{(a),1} = \frac{\ell^3_{(a)}}{\cos u^3} & \ell^3_{(a),1} = -\cos u^3 \ell^2_{(a)} \end{cases} \quad (109)$$

The system (109) has a solution:

$$Y_1 = p_1, \quad Y_2 = \frac{\sin u^3}{\cos u^3}(p_2 + p_1 \sin u^3) + p_3 \cos u^1, \quad Y_3 = Y_{2,1} \Rightarrow$$

$$\ell^{\alpha}_{(a)} = \delta^{\alpha}_1(\delta^1_a + \frac{\sin u^3}{\cos u^3}(\delta^2_a \sin u^1 + \delta^3_a \cos u^1)) + \frac{\delta^{\alpha}_2}{\cos u^3}(\delta^2_a \sin u^1 + \delta^3_a \cos u^1) + $$
$$+ \delta^{\alpha}_3(\delta^2_a \cos u^1 - \delta^3_a \sin u^1).$$

The components of the matrix $g^{\alpha\beta}$ are as follows:

$$\begin{cases} g^{11} = a_{11} - 2\frac{\cos u^3}{\sin u^3}(a_{12}\sin u^1 + a_{13}\cos u^1) + (\frac{\cos u^3}{\sin u^3})^2(\frac{a_{33}+a_{22}}{2} + (\frac{a_{33}-a_{22}}{2})\cos 2u^1 + a_{23}\sin 2u^1) \\ g^{12} = \frac{1}{\sin u^3}(a_{12}\sin u^1 + a_{13}\cos u^1 - \frac{\cos u^3}{\sin u^3}(\frac{a_{33}+a_{22}}{2} + \frac{a_{33}-a_{22}}{2}\cos 2u^1 + a_{23}\sin 2u^1)) \\ g^{22} = \frac{1}{\sin^2 u^3}(\frac{a_{33}+a_{22}}{2} + \frac{a_{33}-a_{22}}{2}\cos 2u^1 + a_{23}\sin 2u^1) \\ g^{33} = \frac{1}{\sin u^3}(\frac{a_{33}+a_{22}}{2} - \frac{a_{33}-a_{22}}{2}\cos 2u^1 - a_{23}\sin 2u^1) \\ g^{13} = a_{12}\cos u^1 - a_{13}\sin u^1 - \frac{\cos u^3}{\sin u^3}(\frac{a_{22}-a_{33}}{2}\sin 2u^1 + a_{23}\cos 2u^1) \\ g^{23} = \frac{a_{33}+a_{22}}{2} + \frac{a_{22}-a_{33}}{2}\cos 2u^1 - a_{23}\sin 2u^1 \end{cases} \tag{110}$$

6. List of Obtained Results

In this section, all non-equivalent sets of operators of groups $G_3(N)$ acting simply transitively in homogeneous Petrov spaces $V_4(N)$ and $V_4^*(N)$ are given. The generators of infinitesimal transformations are given by Killing vector fields: $X_a = \xi^i_a p_i$. The generators of non-infinitesimal transformations are given by the vector fields of the reper: $Y_a = \ell^i_a p_i$. All sets of operators X_a, except for those acting in Petrov spaces $V_4(VIII)$, are equivalent to those in Petrov's book [17] (formulas (25.1)–(25.6) and (25.24)–(25.38)). In contrast to Petrov's book, all solutions for spaces $V_4^*(VIII)$ are given in the canonical semi-geodesic coordinate system (6). The contravariant components of the metric tensor of the spaces are given in Sections 5 and 6. All sets of operators of each group that are non-equivalent with respect to the admissible coordinate transformations of the form (8) are equivalent with respect to admissible coordinate transformations of the form (2). The exception is the group $G_3(N)$. Each of the three sets of operators of the group $G_3(VIII)$ is non-equivalent to the other two sets with respect to both coordinate transformations (2) and (8).

1. Group $G_3(II)$

First version

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^2 p_1 + p_3. \tag{111}$$

$$Y_1 = p_1, \quad Y_2 = u^3 p_1 + p_2, \quad Y_3 = p_3. \tag{112}$$

Second version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^3 p_2, \quad \tilde{X}_3 = X_3 + \ell_0(u^0)p_3. \tag{113}$$

$$Y_1 = p_1, \quad Y_2 = p_2, \quad Y_3 = p_1 + u^3 p_2. \tag{114}$$

2. Group $G_3(III)$

First version

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = p_3 + u^2 p_2, \tag{115}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^3, \quad Y_3 = p_3. \tag{116}$$

Second version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^3 p_3, \quad \tilde{X}_3 = X_3 + \ell_0(u^0)p_2. \tag{117}$$

$$Y_1 = p_1, \quad Y_2 = p_2, \quad Y_3 = p_3 \exp u^1. \tag{118}$$

Third version

$$X_1 = p_2, \quad X_2 = p_1 \exp -u^3, \quad X_3 = p_3. \tag{119}$$

$$Y_1 = p_1, \quad Y_2 = p_2, \quad Y_3 = p_3 - u^1 p_1. \tag{120}$$

3. Group $G_3(IV)$
First version

$$X_1 = p_1 \exp\left(-u^3\right), \quad X_2 = p_2, \quad X_3 = p_3 + u^2(p_1 \exp(-u_3) + p_2) \tag{121}$$

$$Y_1 = p_1, \quad Y_2 = p_1 u^3 + p_2 \exp u_3, \quad Y_3 = p_3 - p_1 u^3, \tag{122}$$

Second version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + (u^2 + u^3) p_2 + u^3 p_3. \tag{123}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^1, \quad Y_3 = (p_3 + p_1 u^1) \exp u^1, \tag{124}$$

Third version

$$X_1 = p_2, \quad X_2 = p_1 \exp(-u^3) - p_2 u^3, \quad X_3 = p_3 + p_2 u^2. \tag{125}$$

$$Y_1 = p_2 \exp u^3, \quad Y_2 = p_1, \quad Y_3 = p_3 + u^1(p_1 + p_2 \exp u^3), \tag{126}$$

4. Group $G_3(V)$
First version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^2 p_2 + u^3 p_3 \tag{127}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^1, \quad Y_2 = p_2 \exp u^1. \tag{128}$$

Second version

$$X_1 = p_2, \quad X_2 = p_1 \exp u_3, \quad X_3 = -p_3 + u^2 p_2, \quad \tilde{X}_2 = X_2 + \ell_0 p_2. \tag{129}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^3, \quad Y_2 = p_2 + u^1 p_1. \tag{130}$$

5. Group $G_3(VI)$
First version

$$X_1 = p_1 \exp\left(-u^3\right), \quad X_2 = p_2, \quad X_3 = p_3 + q u^2 p_2. \tag{131}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp(q u^3), \quad Y_3 = p_3 - u^1 p_1. \tag{132}$$

Second version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^2 p_2 + q u^3 p_3. \tag{133}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^1, \quad Y_3 = p_3 \exp(q u^1). \tag{134}$$

Third version

$$X_1 = p_2, \quad X_2 = p_1 \exp(-q u^3) + p_2 \exp((1-q) u^3), \quad X_3 = p_3 + u^2 p_2. \tag{135}$$

$$Y_1 = p_1, \quad Y_2 = p_2 \exp u^3, \quad Y_3 = p_3 + u^1(p_2(1-q) \exp u^3 - q p_1). \tag{136}$$

Group $G_3(VII)$
First version

$$X_1 = p_2, \quad X_2 = p_3, \quad X_3 = p_1 + u^3(2 \cos\alpha p_3 - p_2) + u^2 p_3. \tag{137}$$

$$Y_1 = p_1, \quad Y_2 = (p_2 \sin p - p_3 \sin(p + \alpha)) \exp q, \quad Y_3 = (p_2 \cos p - p_3 \cos(p + \alpha)) \exp q \tag{138}$$

$(p = u^3 \sin\alpha, \quad q = -u^3 \cos\alpha)$.
Second version

$$X_1 = p_2, \quad X_3 = p_3 + u^2 X_2, \quad X_2 = p_1 \frac{\exp -q}{\cos p} + p_2 \frac{\cos(p - \alpha)}{\cos p}. \tag{139}$$

$$Y_1 = p_2, \quad Y_2 = p_1 \sin\alpha \frac{\sin p}{\cos p} + p_2 \sin^2(\alpha) \frac{\exp q}{\cos p}, \quad Y_3 = p_3 + u^1 (Y_2 - p_1 \cos\alpha). \tag{140}$$

Group $G_3(VIII)$

$$X_1 = p_3 \exp\left(-u^2\right), \quad X_2 = p_2, \quad X_3 = \left(p_1 - 2u^3 p_2 + \left({u^3}^2 - \varepsilon\right)\right) \exp u^2 \quad (\varepsilon = 0, \pm 1). \tag{141}$$

First version $\varepsilon = 0$

$$Y_1 = p_1 \quad Y_2 = p_3 + u^1 p_1, \quad Y_3 = {u^1}^2 p_1 + 2u^1 p_3 - p_2 \exp u^3. \tag{142}$$

Second version $\varepsilon^2 = 1$

$$Y_1 = p_1, \quad Y_2 = p_1 st(2u^1) + 2\exp u^3 st(2u^1) p_2 + 2ct(2u^1) p_3, \quad Y_3 = \varepsilon Y_{2,1}, \tag{143}$$

where the following functions are introduced:

$$st(u) = \frac{\exp(\sqrt{\varepsilon}u) - \exp(-\sqrt{\varepsilon}u)}{2\sqrt{\varepsilon}}, \quad ct(u) = \frac{\exp(\sqrt{\varepsilon}u) + \exp(-\sqrt{\varepsilon}u)}{2}.$$

Group $G_3(IX)$

$$X_1 = p_2, \quad X_2 = p_3 \cos u^2 + \tfrac{\sin u^2}{\cos u^3}\left(p_1 + p_2 \sin u^3\right), \quad X_3 = X_{2,2} \ . \tag{144}$$

$$Y_1 = p_1, \quad Y_2 = \frac{\sin u^3}{\cos u^3}(p_2 + p_1 \sin u^3) + p_3 \cos u^1, \quad Y_3 = Y_{2,1}.$$

7. Conclusions

In the theory of gravitation, a special place is occupied by Riemannian manifolds, on the spacelike hypersurfaces of which three-parameter groups of motions act simply transitively. These manifolds, which are usually called homogeneous spaces of type N by Bianchi, generalize the homogeneous isotropic model of the Universe (see [34]). They are models of a homogeneous but non-isotropic Universe and may be of interest, for example, in the early stages of the Universe's life. Obviously, the traditional models of the Universe are special cases of these spaces. Homogeneous Petrov spaces, the classification of which is completed in the present paper, generalize these homogeneous spaces in the following way. First, in homogeneous Petrov spaces, non-null hypersurfaces of transitivity can be time-like. Second, these hypersurfaces can be isotropic. Homogeneous Petrov spaces with null hypersurfaces of transitivity can serve as models of plane-wave metrics (examples of the study of such spaces can be found in [22–24]). The geometry of homogeneous Petrov spaces in both cases is determined by the geometry of transitivity spaces $V_3(N)$ (in the null case under the condition that the Killing vector fields are independent of u^0).

Note that homogeneous Petrov spaces belong to the type of spaces admitting three Killing fields (vector or tensor). Besides them, such a feature is also possessed by Stackel spaces (see, for example, [21] and the bibliography therein). For spaces of this type there are methods of exact integration of the equations of motion of test bodies. Gravitational equations in these spaces can be reduced to systems of ordinary differential equations. The classification carried out in this paper facilitates the transition to these systems. Thus, the classification of all Riemannian spaces with Lorentz signature admitting a triple of Killing fields is completed (see also [35,36]).

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