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**ASYMPTOTIC STRUCTURE OF SPACE-TIME WITH A  
POSITIVE COSMOLOGICAL CONSTANT**

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by  
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# Abstract

In general relativity a satisfactory framework for describing isolated systems exists when the cosmological constant  $\Lambda$  is zero. The detailed analysis of the asymptotic structure of the gravitational field, which constitutes the framework of asymptotic flatness, lays the foundation for research in diverse areas in gravitational science. However, the framework is incomplete in two respects. First, asymptotic flatness provides well-defined expressions for physical observables such as energy and momentum as ‘charges’ of asymptotic symmetries at null infinity,  $\mathcal{I}^+$ . But the asymptotic symmetry group, called the Bondi-Metzner-Sachs group is infinite-dimensional and a tensorial expression for the ‘charge’ integral of an arbitrary BMS element is missing. We address this issue by providing a charge formula which is a 2-sphere integral over fields local to the 2-sphere and refers to no extraneous structure.

The second, and more significant shortcoming is that observations have established that  $\Lambda$  is not zero but positive in our universe. Can the framework describing isolated systems and their gravitational radiation be extended to incorporate this fact? In this dissertation we show that, unfortunately, the standard framework does not extend from the  $\Lambda = 0$  case to the  $\Lambda > 0$  case in a physically useful manner. In particular, we do not have an invariant notion of gravitational waves in the non-linear regime, nor an analog of the Bondi ‘news tensor’, nor positive energy theorems. In addition, we argue that the stronger boundary condition of conformal flatness of intrinsic metric on  $\mathcal{I}^+$ , which reduces the asymptotic symmetry group from  $\text{Diff}(\mathcal{I})$  to the de Sitter group, is insufficient to characterize gravitational fluxes and is physically unreasonable.

To obtain guidance for the full non-linear theory with  $\Lambda > 0$ , linearized gravitational waves in de Sitter space-time are analyzed in detail. i) We show explicitly that conformal flatness of the boundary removes half the degrees of freedom of the gravitational field by hand and is not justified by physical considerations; ii) We obtain gauge invariant expressions of energy-momentum and angular momentum fluxes carried by gravitational waves in terms of fields defined at  $\mathcal{I}^+$ ; iii) We demonstrate that the flux formulas reduce to the familiar ones in Minkowski space-time in spite of the fact that the limit  $\Lambda \rightarrow 0$  is discontinuous (since, in particular,  $\mathcal{I}^+$  changes its space-like character to null in the limit); iv) We obtain a generalization of Einstein’s 1918 quadrupole formula for power emission by a linearized source to include a positive  $\Lambda$ ; and, finally v) We show that, although energy of linearized gravitational waves can be arbitrarily negative in general, gravitational waves emitted by physically reasonable sources carry positive energy.

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# Chapter 1 | Introduction

## 1.1 Preamble

On 14 September 2015 gravitational waves were directly detected for the first time [16], 100 years after their existence was originally predicted by Einstein [1]. Einstein's general theory of relativity postulates that space-time is a dynamical entity whose curvature is dictated by matter that resides in it, and concurrently, the motion of matter is governed by space-time. Gravitational waves are ripples in space-time that are caused by motion of matter and propagate outward from their source at the speed of light. Although gravitational waves in general relativity were already in 1916 [1] discussed and subsequently refined and generalized [2–5], their reality was debated for a long time [6]. This was mainly due to the fact that the first studies treated gravitational waves as linear perturbations on a background Minkowski space-time metric. Since general relativity does not provide a canonical decomposition of a space-time metric into a non-dynamical background and a dynamical perturbation, it was unclear whether gravitational waves were true physical phenomena or mere coordinate artifacts that could be gotten rid of with a coordinate transformation. The first detailed study of fully non-linear gravitational radiation by examining the asymptotic structure of the gravitational field of isolated systems was carried out by Bondi, van der Burgh and Metzner and Sachs and elaborated by Penrose, Newman, Geroch and many others [7–10]. Their detailed analysis, which established the framework of asymptotic flatness, provided a coordinate-independent characterization of gravitational radiation and finally established its reality. This laid the foundation for advances in numer-

ous and diverse areas of study including geometric analysis, numerical relativity, quantum field theory on curved space-times and relativistic astrophysics. In particular, it led to the proof of positivity of energy, calculation of energy-momentum radiated in gravitational collapse and obtaining gravitational waveforms via numerical simulations. These developments were instrumental in the direct detection of gravitational waves emitted by GW150914 [16].

Despite its successes, the analysis of Bondi et al has two limitations. We begin with the smaller issue. In asymptotic flatness physical quantities such as energy and momentum are defined using asymptotic symmetries<sup>1</sup> of space-time. In particular, energy is the ‘charge’ associated with an asymptotic time translation symmetry and linear momentum with spatial translation. However, as we will see in section 1.2 below, the asymptotic symmetry group is much larger. In fact, it is the infinite-dimensional group called the Bondi-Metzner-Sachs (BMS) group. However, a well-defined tensorial expression for the charge of an *arbitrary* symmetry in this group was unavailable. In Chapter 3 we provide such an expression.

The second and more concerning issue is that the foundational framework of asymptotic flatness assumes a vanishing cosmological constant  $\Lambda$ . Until 1998, it was widely believed that the value of  $\Lambda$  was zero. In fact, Einstein himself discarded<sup>2</sup> the cosmological constant after first introducing it to model a static universe when he found out that our universe is, in fact, expanding [12]. However, cosmological observations over the last two decades have established that the universe is undergoing an *accelerated* expansion which is best explained by a *positive*  $\Lambda$  [13,14]. So it is prudent to ask whether the description of isolated systems such as stars and black holes, and the characterization of gravitational radiation obtained for  $\Lambda = 0$  is easily generalised when  $\Lambda > 0$ . It would be interesting to know if there are significant observable differences, particularly in light of the newly opened window of gravitational wave astronomy. The answer is highly non-trivial. In this dissertation, we will show that the standard framework of isolated systems does *not* extend from the  $\Lambda = 0$  case to the  $\Lambda > 0$  case in a physically useful manner. In particular, *in the non-linear regime, we do not have a coordinate-invariant notion of gravitational waves for  $\Lambda > 0$* . So, we are unable to calculate energy-momentum radiated in gravitational collapse. Furthermore, there are no positive energy theo-

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<sup>1</sup>As explained in section 1.2, asymptotic symmetries are symmetries of the *boundary* of space-time, and not necessarily of all of space-time.

<sup>2</sup>According to George Gamow [11], Einstein called it the ‘biggest blunder’ of his life.

rems. These drastic consequences arise from the fact that for any value of  $\Lambda > 0$ , *no matter how small*, the global causal structure of space-time is starkly different from that when  $\Lambda = 0$ . As we will see below in section 1.3, this poses profound challenges in the study of gravitational radiation in space-times with positive  $\Lambda$ .

Are the implications for the theory of gravitational radiation and observations, then, hopeless? Certainly not! We recall that for  $\Lambda = 0$ , the first discussions of gravitational waves, done by Einstein [15] in the linearized context, supplied many physical insights which later informed the analysis in the full theory. An important example is the fact that there is no monopole or dipole gravitational radiation due to conservation of mass and linear momentum. In similar fashion, it turns out that the study of linearized gravitational fields with  $\Lambda > 0$ , which is a very good model to study the asymptotic behaviour of isolated systems, provides useful tools which can be used to alleviate the problems encountered in the full theory. In particular, in the linearized theory we obtain a gauge-invariant description of gravitational waves and derive useful conservation laws of energy-momentum and angular momentum. We use these to provide an extension of Einstein's quadrupole formula for power emission by a source to include a positive  $\Lambda$  and obtain positivity of energy. Lastly, we address the subtle and very relevant issue of the limit of our results when  $\Lambda \rightarrow 0$ , thereby providing quantitative deviations due to a positive  $\Lambda$  with potentially observable consequences.

To elucidate these ideas in greater detail, we naturally divide the dissertation into two main parts as follows.

## 1.2 Asymptotic flatness and conserved charges at $\mathcal{I}$

In Chapters 2 and 3, we discuss gravitational radiation theory of space-times with vanishing cosmological constant  $\Lambda$ . This will highlight the main issues and successes in describing gravitational radiation in general relativity with  $\Lambda = 0$ . Furthermore, it will provide guidance for the discussion of the case with  $\Lambda > 0$ .

The main steps in the analysis of gravitational radiation with  $\Lambda = 0$  can be summarized as follows:

- (1) A notion of isolated systems is captured with the definition of *asymptotically flat space-times*. The key idea here is that as one moves further away from an

isolated object, such as a star or binary system, the space-time metric approaches the Minkowski metric, at a rate specified in the definition. Minkowski space-time is chosen because it represents an empty universe with no matter and  $\Lambda = 0$ . The precise definition uses Penrose’s method of conformally completing physical space-time which attaches points at infinity as a boundary to the conformally completed spacetime. The advantages of using this technique are that it is manifestly coordinate independent and instead of taking limits to infinity, one can use local differential geometry ‘at infinity’. The latter simplifies calculations enormously.

(2) Next, *asymptotic symmetries* are obtained as the group of diffeomorphisms of physical space-time that preserve the boundary conditions in the definition in (1) as follows. One of the consequences of the definition is that the boundary of an asymptotically flat spacetime is a null 3-surface  $\mathcal{I}$  with topology  $\mathbb{S}^2 \times \mathbb{R}$ .  $\mathcal{I}$  is endowed with a conformal class of pairs  $(q_{ab}, n^a)$  of an intrinsic degenerate metric  $q_{ab}$  of signature  $(0 + +)$  and null normal  $n^a$  which is also tangential to the surface  $\mathcal{I}$ . Furthermore, since we can always set  $\mathcal{L}_n q_{ab} = 0$  on  $\mathcal{I}$ ,  $\mathcal{I}$  is said to be ‘ruled’ by its null normals. This ruling has the important consequence that the asymptotic symmetry group is the *Bondi-Metzner-Sachs (BMS) group*  $\mathcal{B}$  and *not* the group of diffeomorphisms  $\text{Diff}(\mathcal{I})$ . The BMS group  $\mathcal{B}$  is smaller than  $\text{Diff}(\mathcal{I})$  and is the semi-direct product,  $\mathcal{B} = \mathcal{S} \ltimes \mathcal{L}$ , of the group  $\mathcal{S}$  of supertranslations with the Lorentz group  $\mathcal{L}$ . The infinite-dimensional group of supertranslations  $\mathcal{S}$  can be thought of as a group of ‘angle-dependent’ translations. A particularly useful feature of the BMS group  $\mathcal{B}$  is that it admits a unique 4-dimensional normal sub-group of translations  $\mathcal{T}$  [8], which can be used to obtain unambiguous notions of energy-momentum.

(3) The presence of gravitational radiation in a space-time is characterized on  $\mathcal{I}$  by the gauge-invariant ‘*Bondi news tensor*’  $N_{ab}$ . The news tensor is constructed out of the curvature tensor of the derivative operator  $D$  on  $\mathcal{I}$  which is compatible with the intrinsic metric  $q_{ab}$ . In particular, since  $\mathcal{I}$  is a 3-dimensional surface, the Weyl curvature vanishes, and the Riemann tensor  $\mathcal{R}_{abc}{}^d$  of  $D$  is completely determined by the Schouten tensor  $S_a{}^b$  and intrinsic metric  $q_{ab}$  as follows  $\mathcal{R}_{abc}{}^d = q_{c[a} S_{b]}{}^d - S_{c[a} \delta_{b]}^d$ . However, recall that there is considerable freedom in the choice of conformal factor in step (1), so only those quantities are physically meaningful which are invariant

under allowed conformal transformations. The Bondi news tensor  $N_{ab}$  captures precisely the conformally invariant part of the Schouten tensor  $S_{ab}$ , and thus, encodes information about gravitational radiation.<sup>3</sup> In addition, when Bondi news vanishes  $N_{ab} = 0$ , the BMS group  $\mathcal{B}$  reduces to the Poincaré group, as discussed in Chapter 2, which is the symmetry group of Minkowski space-time.

(4) Finally, formulas to quantify energy-momentum and angular momentum carried by gravitational waves are obtained using asymptotic symmetries and Hamiltonian methods as shown by Ashtekar and Streubel in [17]. To gain more insight into their method, recall from (1) that the topology of  $\mathcal{I}$  is  $\mathbb{S}^2 \times \mathbb{R}$ . The  $\mathbb{R}$  direction is taken to be the retarded time axis, and each 2-sphere cross-section  $C$  of  $\mathcal{I}$  represents an instant of retarded time. As Figure 3.1 illustrates,  $\Delta\mathcal{I}$  is a patch of  $\mathcal{I}$  between two ‘times’ represented by cross-sections  $C_1$  and  $C_2$ . Then, Ashtekar and Streubel show that to any BMS symmetry  $\xi^a$  and a patch  $\Delta\mathcal{I}$  of  $\mathcal{I}$  one can associate a quantity  $F_\xi[\Delta\mathcal{I}]$  which represents a flux carried away by gravitational waves. The physical meaning of the flux is derived from the symmetry it is associated with. For example, if  $\xi^a$  is an asymptotic time translation,  $F_\xi$  represents the energy flux of gravitational waves. Given this framework, one can now ask, is it possible to characterize more information than the flux of a quantity carried by gravitational waves across  $\Delta\mathcal{I}$ : Is it possible to know the instantaneous value of the physical observable at any one cross-section of  $\mathcal{I}$  in terms of fields *local* to the cross-section?

The answer was shown to be yes. For supertranslations Ashtekar and Streubel showed that their flux could be integrated using Stoke’s theorem to provide a 2-dimensional charge integral which exactly matched the expression for supermomentum obtained by Geroch [27]. Furthermore, for the special case when  $\xi^a$  is a time translation this quantity would just be the Bondi energy of space-time at a retarded instant of time. For a general BMS vector field Dray and Streubel [26] proposed a charge which is compatible with the Ashtekar-Streubel flux. However, their result is stated in terms of Newman-Penrose scalars, thereby obscuring the underlying tensorial and geometric structure. In addition, their charge formula depends on a decomposition of the BMS vector field into its supertranslation and

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<sup>3</sup>In Bondi frames the news tensor  $N_{ab}$  is the trace-free part of the Schouten tensor  $S_{ab}$ . In a Bondi frame,  $n^a$  is divergence-free i.e.,  $\nabla_a n^a \hat{=} 0$  where  $\hat{=}$  is equality on  $\mathcal{I}$  and the pull-back of  $q_{ab}$  to a 2-sphere cross-section of  $\mathcal{I}$  is a round metric.

Lorentz parts. We overcome these limitations by providing a new tensorial expression for the 2-sphere charge integral of an arbitrary BMS field  $\xi^a$  on  $\mathcal{I}$ . The attractive features of our charge include manifest linearity in the symmetry  $\xi$  and conformal invariance.

This part of the dissertation is organized as follows. In Chapter 2 we provide details of the contents of (1) - (3) above. Chapter 3 addresses (4) as well as discusses the new charge for BMS vector fields.

### 1.3 Asymptotics with a positive cosmological constant

In the second part of the dissertation we investigate the effects of the observed positive cosmological constant on gravitational radiation theory. We begin our analysis by generalizing steps (1) - (4) outlined above to include a positive  $\Lambda$ . We find that the generalization throws up many new conceptual issues and, thus, does not satisfactorily characterize gravitational radiation in space-times with  $\Lambda > 0$ , no matter how small its value is. The main issues are summarized below.

In the absence of matter, the de Sitter metric solves Einstein's equation with a positive  $\Lambda$ . So, to describe isolated systems when  $\Lambda > 0$  we define *asymptotically de Sitter space-times* using Penrose's methods. The specific boundary conditions of the definition accommodate familiar examples including the de Sitter, Friedmann-Lemaître-Robertson-Walker space-time, Schwarzschild-de Sitter and Vaidya-de Sitter space-times. The immediate consequence of the definition, as pointed out by Penrose [10], is that the space-time boundary  $\mathcal{I}$  is now *space-like*. This has several far-reaching effects. Firstly, the normal  $n^a$  of  $\mathcal{I}$  is no longer tangential to it as in the  $\Lambda = 0$  case, so  $\mathcal{I}$  is not ruled by its normals. Consequently, the diffeomorphisms of  $\mathcal{I}$  do not act on  $n^a$  and the asymptotic symmetry group of  $\mathcal{I}$  is not reduced as was the case for  $\Lambda = 0$ . It is the full diffeomorphism group  $\text{Diff}(\mathcal{I})$ . This group does not admit a unique translation sub-group, and hence, energy and momentum cannot be defined unambiguously.

A possible remedy to this problem is to strengthen boundary conditions by requiring that the intrinsic metric  $q_{ab}$  on  $\mathcal{I}$  be conformally flat. This is motivated by the fact that the de Sitter metric satisfies this condition. In addition, this condition

is routinely used in the  $\Lambda < 0$  case to allow for well-defined time evolution [44, 45, 47]. Conformal flatness reduces the asymptotic symmetric group from  $\text{Diff}(\mathcal{I})$  to the 10-dimensional de Sitter group. This opens up the possibility of defining energy-momentum and angular momentum using elements of the the de Sitter group. Indeed, one is able to associate a de Sitter charge  $Q_\xi[C] = \oint_C \mathcal{E}_{ab} \xi^a dS^b$  with any 2-sphere cross-section  $C$  of  $\mathcal{I}$  and any de Sitter symmetry vector field  $\xi$ , where  $\mathcal{E}_{ab}$  is the electric part of the leading-order Weyl tensor. In particular, in Kerr-de Sitter space-time, the only nonvanishing de Sitter charges are the (correctly normalized) mass and angular momentum.

However, further investigation shows that the condition is too strong in the following sense. Requiring conformal flatness is the same as setting the magnetic part  $\mathcal{B}_{ab}$  of the leading order Weyl tensor on  $\mathcal{I}$  to zero. Since  $\mathcal{I}$  is space-like, it is clear that this is a severe mathematical restriction [56]. A comparison with the Maxwell field shows that this condition is the same as setting half the degrees of freedom of the gravitational field to zero! Furthermore it turns out that, irrespective of the strengthening of the boundary conditions, there is no analog of the Bondi news tensor for  $\Lambda = 0$ . In absence of matter fields at  $\mathcal{I}$ , the charges  $Q_\xi[C]$  are absolutely conserved. That is, gravitational waves do not carry any de Sitter momenta away from a source! Thus, we seem to hit an impasse: The boundary conditions are either too weak to even allow unambiguous definitions of physical observables, or, they are too restrictive and do not allow any fluxes to be radiated away by gravitational waves.

The second main issue is that for a space-like  $\mathcal{I}$  *all* asymptotic symmetry vector fields are space-like in a neighborhood of  $\mathcal{I}$ . These would include any ‘time translation’ we may use to define energy. Consequently, for any value of  $\Lambda > 0$ , energy is no longer strictly positive as in the  $\Lambda = 0$  case. Thus, the lower bound on energy of gravitational waves is infinitely discontinuous at  $\Lambda = 0$ . Furthermore, if gravitational waves can carry away negative energy from a system, the source could gain arbitrarily large amount of energy, thereby precipitating an instability. Therefore, we see that a positive  $\Lambda$ , *no matter how small its value*, has profound implications for gravitational radiation theory. How is it, then, that, despite infinite discontinuities, observations such as that of the Hulse-Taylor binary pulsar and GW150914 are so well approximated by the  $\Lambda = 0$  theory while other observations clearly tell us that  $\Lambda$  is positive?

To probe this question further, we seek guidance from the study of linearized gravitational field on de Sitter space-time. This naturally supplies the boundary  $\mathcal{I}$  with the 10-dimensional de Sitter group of symmetries. The symmetries include one dilation, which we call de Sitter time translation, three spatial translations, three rotations and three ‘inverted’ translations. Given these symmetries, Hamiltonian methods on the phase space of linearized gravitational fields provide *gauge invariant formulas* for the energy, momentum and angular momentum carried by gravitational waves in terms of fields that are well-defined on  $\mathcal{I}$ . Furthermore, since the source-free equations are easily solved in the linearized case, we explicitly see that conformal flatness of the boundary removes half the solutions by hand and reduces all the gravitational fluxes to zero.

The situations of physical interest, such as a neutron star or black hole binary emitting gravitational waves, involve sources. For these situations we argue that it is sufficient to restrict ourselves to one ‘half’ of de Sitter space-time. To be precise, we consider a linearized source which is spatially compact for all times so that its worldline intersects  $\mathcal{I}^-$  at a single point  $i^-$  and, similarly,  $\mathcal{I}^+$  at  $i^+$ . The future null cone of the past time-like infinity  $i^-$  of the source then covers the future expanding Poincaré patch of de Sitter space-time. Thus, any radiation emitted by this source will be contained in this patch. It is then easy to show that this patch is *not* left invariant by inverted translations. Thus the symmetry group of  $\mathcal{I}$  is a 7-dimensional sub-group of the de Sitter group.

We then use the de Sitter time translation and the definition of energy flux to obtain a generalization of Einstein’s quadrupole formula for power emission in terms of fields that are well-defined on  $\mathcal{I}$ . We show that the energy radiated by gravitational waves away from the sources is positive. In addition, we develop a scheme to obtain the limit of our results as  $\Lambda \rightarrow 0$ . This is crucial to quantify the errors made by assuming  $\Lambda = 0$  in calculations of gravitational fluxes. Finally, this will explain why results from the  $\Lambda = 0$  case are such a good approximation of those with  $\Lambda > 0$  for current observation capabilities and for the small observed value of  $\Lambda^4$ .

The organization of topics is as follows. In Chapter 4, we examine a natural extension of the framework of asymptotic flatness to the non-zero  $\Lambda$  case and point

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<sup>4</sup>In units of inverse length squared  $\Lambda \approx 10^{-52} m^{-2} \approx (3.2 \text{ giga parsecs})^{-2}$ . For comparison, the Milky Way is about 30 kilo parsecs across, the luminosity distance to the recently discovered black hole binary merger event GW150914 is estimated at  $410_{-180}^{+160}$  mega parsecs.

out the various difficulties that arise in doing so. In Chapter 5 we collect results about source-free linearized perturbations about the de Sitter metric. In Chapter 6, we complete the analysis of the linearized perturbations by considering a compact source for such perturbations. We derive a modified quadrupole formula. Finally we conclude in Chapter 7 with a summary and outlook.

We use the following conventions. Throughout we assume that the underlying space-time is 4-dimensional and set speed of light  $c=1$ . The space-time metric has signature  $(- + + +)$ . The curvature tensors are defined via:  $2\nabla_{[a}\nabla_{b]}k_c = R_{abc}{}^d k_d$ ,  $R_{ac} = R_{abc}{}^b$  and  $R = R_{ab}g^{ab}$ . Throughout we use Penrose's abstract index notation [84, 95]:  $a, b, \dots$  will be the abstract indices labeling tensors while indices  $\bar{a}, \bar{b}, \dots$  will be numerical indices. In particular, components of a tensor field  $T_{ab}$  (in a specified chart) are denoted by  $T_{\bar{a}\bar{b}}$ .

Material in Chapters 4, 5 and 6 have appeared in publications co-authored with Abhay Ashtekar and Béatrice Bonga noted as publications 1, 3, 4 and 5 in the vita attached at the end of the dissertation. The content of Chapters 2 and 3 are currently being written for publication with Abhay Ashtekar, marked as publication 7 in the vita. In addition to the topics discussed in the dissertation, during my graduate studies I conducted research on topics in inflation. In publication 2 on the vita, with collaborators we studied some effects of the presence of modes of perturbation longer than the observable size of the universe on the observed power spectrum and on the 'shapes' of bi-spectrum. In a forthcoming publication, number 8 on the vita, with collaborators we explore the role played by inflationary dynamics in the emergence of classical behaviour from an underlying quantum description of perturbations of the space-time metric.

# Chapter 2 | Preliminaries

In this chapter, we recall several definitions and results pertaining to asymptotically flat space-times. This will serve to anchor both our discussion of the new expression for BMS charges in Chapter 3, and the subsequent discussion of asymptotic structure of space-time with a positive cosmological constant. We will proceed as follows. First, we describe a precise notion of isolated systems by defining asymptotic flatness. Next, we recall the universal structure of the boundary  $\mathcal{I}$  of asymptotically flat space-times and the definition and group structure of asymptotic symmetries. This is followed by a summary of the characterization of gravitational radiation by a coordinate-invariant tensor which is built from space-time curvature called the Bondi news tensor  $N_{ab}$ .

## 2.1 Asymptotic flatness at null infinity

The notion of asymptotic flatness captures the idea of an isolated system in general relativity with vanishing cosmological constant. It makes precise the expectation that in a universe with a single isolated system, such as a star or a galaxy, the further one moves away from the system, the more ‘flat’ space-time becomes. This expectation is borne out of the fact that the flat Minkowski metric, which solves vacuum Einstein’s equation with  $\Lambda = 0$  represents an empty universe. Even though cosmological observations show us that the universe is homogeneous and isotropic on the largest scales, the concept of an isolated system is still very useful to discuss our observation of the afore-mentioned subsystems of our universe. This is because the sources are at a great distance from us and the average matter density of interstellar medium in the universe is very low. These considerations motivated

the development of the framework of asymptotic flatness [7, 8, 10, 28].

Asymptotics of space-times were examined in two distinct regimes by moving away from an isolated system in two sets of directions - space-like, by Arnowitt, Deser and Misner (ADM) [28–30], and null, by Bondi, Sachs, Penrose and others [7, 8, 31]. Since we are interested in the study of gravitational radiation, and because gravitational waves propagate along null cones in general relativity, we will restrict our discussion to asymptotic flatness at null infinity. In addition, in our discussion we will employ the conformal methods introduced by Penrose [10] to study asymptotic structure. The key idea of Penrose’s conformal method is to attach points at infinity as a boundary to space-time and study asymptotics using local differential geometry there. This has the advantage of being manifestly coordinate independent and, in addition, enormously simplifies calculations. In the language of conformal methods, then, asymptotically flat space-times are defined as follows.

**Definition 1** *A space-time  $(\hat{M}, \hat{g}_{ab})$  is said to be asymptotically flat at null infinity if there exists a manifold  $M$  with boundary  $\mathcal{I}$  and metric  $g_{ab}$ , with an embedding of  $\hat{M}$  into  $M$  such that:*

1. *there exists a smooth function  $\Omega$  on  $M$  such that  $g_{ab} = \Omega^2 \hat{g}_{ab}$  on  $\hat{M}$ ;  
 $\Omega = 0$  on  $\mathcal{I}$  and  $\nabla_a \Omega$  is nowhere vanishing on  $\mathcal{I}$ ;*
2.  *$\hat{g}_{ab}$  satisfies Einstein’s equation with zero cosmological constant,  
i.e.,  $\hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{g}_{ab} = 8\pi G \hat{T}_{ab}$ ; where  $\Omega^{-2} \hat{T}_{ab}$  has a smooth limit to  $\mathcal{I}$ ; and*
3.  *$\mathcal{I}$  is topologically  $\mathbb{S}^2 \times \mathbb{R}$  and the restriction to  $\mathcal{I}$  of  $n^a := g^{ab} \nabla_a \Omega$  is complete.*

As Penrose explains [10], physical space-time is conformally completed, as in the first condition, to attach points at infinity as a boundary to space-time and study asymptotics using local differential geometry there in a manifestly coordinate independent manner. We note that the boundary  $\mathcal{I}$  in the Definition 1 consists of future infinity  $\mathcal{I}^+$  and past infinity  $\mathcal{I}^-$ . Unless specified,  $\mathcal{I}$  will refer to both. Next, the requirement of non-vanishing of  $\nabla_a \Omega$  on  $\mathcal{I}$  ensures that  $\Omega$  can be used as a coordinate near  $\mathcal{I}$  to study asymptotic behaviour of fields. The second condition in the definition defines what we mean by an ‘isolated system’ in the physical space-time  $(\hat{M}, \hat{g}_{ab})$ . The specific fall-off of  $\hat{T}_{ab}$  used here is motivated by the analysis

of test matter fields in Minkowski space-time and ensures that a large class of examples are included in the definition. The condition on topology is inspired by the topology of  $\mathcal{I}$  in the standard conformal completion of Minkowski space-time. Completeness of  $n^a$  is required for reasonable definition of space-times with black hole regions. A space-time  $(M, g_{ab})$  is said to admit a black-hole region  $B$  if the past of  $\mathcal{I}^+$  does not cover  $M$ . If completeness of  $\mathcal{I}^+$  is not required, one can conjure up an unpleasant situation where Minkowski space-time contains a black hole region! Finally, note that the choice of conformal factor in the definition is not unique. Thus, only those quantities are physically meaningful which are invariant under a change of conformal factor.

The above definition admits many examples. These include stationary space-times such as Minkowski, Schwarzschild and Kerr space-times, and non-stationary ones such as Vaidya space-time. In addition, it also admits space-times with gravitational radiation [23].

We end this section by noting some consequences of the above definition.

1.  $\mathcal{I}$  is a null surface. This follows from rewriting Einstein's equation in terms of the conformally rescaled metric  $g_{ab}$ , and using the boundary condition of Definition 1 which yields  $n^a n_a \hat{=} 0$  i.e., the normal to  $\mathcal{I}$  is null. Here and throughout the disseration  $\hat{=}$  stands for equality at  $\mathcal{I}$ .
2.  $n^a$  can always be made 'divergence-free'. The choice of conformal factor in the above definition is not canonical. Under a conformal rescaling  $\Omega \rightarrow \Omega' = \omega\Omega$ , we have  $n'^a \hat{=} \omega^{-1}n^a$  and  $q'_{ab} \hat{=} \omega^2 q_{ab}$ . Using this freedom, again one find solutions to the equation  $\mathcal{L}_n \omega \hat{=} -\frac{1}{4}\omega \nabla_a n^a$  to obtain  $\nabla'_a n'^a \hat{=} 0$ . The conformal freedom is then reduced to  $\Omega' \hat{=} \omega\Omega$  where  $\mathcal{L}_n \omega \hat{=} 0$ .
3. The asymptotic Weyl tensor  $C_{abcd}$  of  $g_{ab}$  vanishes identically on  $\mathcal{I}$ .

One expresses the Schouten tensor  $\hat{S}_{ab} := \hat{R}_{ab} - (\hat{R}/6)\hat{g}_{ab}$  of  $\hat{g}_{ab}$  in terms of that of  $g_{ab}$ , and takes its 'curl' to obtain

$$\nabla_{[a}(\Omega \hat{S}_{b]c}) = \Omega \nabla_{[a} S_{b]c} + C_{abcd} n^d + g_{c[a} \hat{S}_{b]d} n^d, \quad (2.1)$$

where we have used the relation between the Riemann tensor and the Weyl tensor  $R_{abcd} = C_{abcd} + g_{a[c} S_{d]b} - g_{b[c} S_{d]a}$ .

Using Einstein's equation in physical space-time, one can rewrite (2.1) as

$$\Omega \nabla_{[a} S_{b]c} + C_{abcd} n^d = \nabla_{[a} (\Omega \tilde{T}_{b]c}) - g_{c[a} \tilde{T}_{b]d} n^d \quad (2.2)$$

where  $\tilde{T}_{ab} := 8\pi G \left( \hat{T}_{ab} - \frac{1}{3} \hat{T} \hat{g}_{ab} \right)$ . From the boundary condition imposed in Definition 1, we know that  $\Omega^{-2} \hat{T}_{ab}$  has a smooth limit to  $\mathcal{I}$ . So we conclude

$$C_{abcd} n^d \hat{=} 0. \quad (2.3)$$

For the final step in the argument of vanishing of  $C_{abcd}$  on  $\mathcal{I}$  we refer the reader to Theorem 11 of [27].

## 2.2 Universal structure on $\mathcal{I}$

The great utility of the Definition 1 is that it yields a universal structure to the boundaries of the entire class of space-times which are asymptotically flat at null infinity. This structure can be summarized as follows.

Since  $\mathcal{I}$  is a null surface, the null normal  $n^a$  is now also tangential to  $\mathcal{I}$  and the intrinsic metric  $q_{ab}$  on  $\mathcal{I}$  is degenerate, with signature  $(0 + +)$ . Due to the freedom to perform conformal rescalings,  $\mathcal{I}$  is endowed with an equivalence class of conformally related pairs  $(q_{ab}, n^a)$ . However, without loss of generality, and for calculational ease, we can restrict to 'divergence-free' conformal frames. Einstein's equations imply that in these conformal frames  $\mathcal{L}_n q_{ab} \hat{=} 0$ . Hence, the integral curves of  $n^a$  are referred to as *generators of  $\mathcal{I}$*  and it is said that  *$\mathcal{I}$  is ruled by its null normal  $n^a$* . One can now refer to the space  $S$  of generators of  $\mathcal{I}$ . The last condition of Definition 1 ensures that  $S$  is topologically  $\mathbb{S}^2$  i.e., there are 2-sphere worth of angular directions in which to move away an isolated system.

The universal asymptotic structure is therefore given by pairs  $(q_{ab}, n^a)$  of fields on a 3-manifold  $\mathcal{I}$  with topology  $\mathbb{S}^2 \times \mathbb{R}$  such that: i)  $q_{ab}$  is a degenerate metric of signature  $(0 + +)$  with  $q_{ab} n^b \hat{=} 0$  and  $\mathcal{L}_n q_{ab} \hat{=} 0$ ; ii) any two pairs  $(q_{ab}, n^a)$  and  $(q'_{ab}, n'^a)$  are related by  $q'_{ab} \hat{=} \omega^2 q_{ab}$  and  $n'^a \hat{=} \omega^{-1} n^a$  for some  $\omega$  satisfying  $\mathcal{L}_n \omega \hat{=} 0$ ; and, iii)  $n^a$  is complete. Because 2-spheres carry a unique conformal structure, the metrics  $q_{ab}$  in this class are all conformal to a unit 2-sphere metric.

## 2.3 Asymptotic symmetries

The asymptotic symmetry group  $\mathfrak{G}$  of asymptotically flat space-times is naturally defined as the group of diffeomorphisms on the physical space-time which preserve the universal structure of  $\mathcal{I}$  [8], or equivalently, preserve the boundary conditions of Definition 1. The BMS group  $\mathcal{B}$  is the group of diffeomorphisms preserving this universal structure. We will see below that  $\mathcal{B}$  is substantially smaller than  $\text{Diff}(\mathcal{I})$  and furthermore has rich, physically interesting structure.

At the infinitesimal level, elements of the Lie algebra  $\mathfrak{g}$  of  $\mathfrak{G}$  can be naturally represented by vector fields  $\xi^a$  which preserve the *ruling of  $\mathcal{I}$  by  $n^a$* . In particular,  $\xi^a$  maps one pair  $(q_{ab}, n^a)$  to another  $(q'_{ab}, n'^a)$  within the equivalence class of ‘divergence-free’ conformal frames and satisfies

$$\mathcal{L}_\xi q_{ab} = 2k q_{ab} \quad \text{and} \quad \mathcal{L}_\xi n^a = -k n^a \quad (2.4)$$

where  $k$  is any function on a 2-sphere cross-section of  $\mathcal{I}$ .

The BMS Lie algebra  $\mathfrak{b}$  has a semi-product structure very similar to that of the Poincaré algebra. First, we note that vector fields of the form  $\xi^a = f n^a$  (with  $\mathcal{L}_n f \doteq 0$ ) form a Lie ideal of the BMS Lie algebra  $\mathfrak{b}$ . This is the infinite-dimensional sub-Lie algebra  $\mathfrak{s}$  of *BMS supertranslations*. Next, note that every element of the quotient  $\mathfrak{b}/\mathfrak{s}$  can be characterized by the projection  $\bar{\xi}^a$  of a BMS vector field  $\xi^a$  on the 2-sphere  $S$  of generators of  $\mathcal{I}$  because of the condition  $\mathcal{L}_\xi n^a \doteq -k n^a$ . It is easily seen from the first equation of (2.4) that  $\bar{\xi}^a$  is a conformal Killing field on the space  $S$  of generators of  $\mathcal{I}$ . Recall that *the 2-sphere has a unique conformal structure*. Therefore, the quotient  $\mathfrak{b}/\mathfrak{s}$  is just the Lie algebra of conformal isometries of a round 2-sphere, which turns out to be isomorphic to the Lorentz Lie algebra in 4 dimensions. Thus,  $\mathcal{B}$  is the semi-direct product,  $\mathcal{B} = \mathcal{S} \ltimes \mathcal{L}$ , of the group  $\mathcal{S}$  of supertranslations with the Lorentz group  $\mathcal{L}$ . For the Poincaré group, the group of supertranslations is replaced by the 4-dimensional group of translations. Lastly, we note the non-trivial result that  $\mathcal{B}$  admits a unique 4-dimensional normal subgroup of translations  $\mathcal{T}$  [8]. We will see in Chapter 3 that this result will play an important role in extracting physically useful information from the asymptotic gravitational fields.

## 2.4 Bondi news tensor and gravitational radiation

In this section, we recall how information about gravitational radiation is encoded in the curvature of  $\mathcal{I}$ . In order to do this, we first define a derivative operator on  $\mathcal{I}$  as follows: Fix a conformal completion  $(M, g_{ab})$  of an asymptotically flat space-time  $(\hat{M}, \hat{g}_{ab})$  endowed with a pair of fields  $(q_{ab}, n^a)$  on  $\mathcal{I}$ . The next order structure is the pull-back  $D$  of the space-time connection  $\nabla$  compatible with  $g_{ab}$ . The pull-back is well defined because  $\nabla_a n_b \hat{=} 0$  in the divergence-free conformal frames, and satisfies  $D_a q_{bc} \hat{=} 0$  and  $D_a n^b \hat{=} 0$ . However,  $D$  is not uniquely determined by these properties because it is degenerate. What physical information does it encode? Recall first that there is still considerable restricted conformal freedom. Therefore one is naturally led to consider equivalence classes  $\{D\}$  of conformally related derivative operators  $D$ . The difference between the  $\{D\}$  arising from any two space-times is characterized by a trace-free symmetric tensor  $\Sigma_{ab}$  defined intrinsically on  $\mathcal{I}$  which is transverse to  $n^a$  (in the sense that  $\Sigma_{ab} n^b \hat{=} 0$ ). The two independent components of  $\Sigma_{ab}$ , encoded in a symmetric, transverse, trace-free tensor  $N_{ab}$ , called the *Bondi news tensor*, correspond to the two radiative modes of the gravitational field in full, non-linear general relativity [39].

Now we obtain an explicit expression for the News tensor. Since  $\mathcal{I}$  is 3-dimensional, the Riemann curvature  $R_{abc}{}^d$  of  $D$  is completely captured in a second rank Schouten tensor  $S_a{}^b$ :

$$R_{abc}{}^d \hat{=} \frac{1}{2}(q_{c[a} S_{b]}{}^d - S_{c[a} \delta_{b]}{}^d) \quad (2.5)$$

where  $S_{ab} = S_a{}^c q_{bc}$ .

However,  $S_a{}^b$  cannot be used to characterize the presence of gravitational radiation in physical space-time because it is not invariant under conformal transformations. In a new conformal frame  $(q'_{ab}, n'^a) = (\omega^2 q_{ab}, \omega^{-1} n^a)$  the tensors  $S_a{}^b$  and  $S_{ab}$  take the following form:

$$S'_a{}^b = \omega^{-2} S_a{}^b - 2\omega^{-3} D_a(q^{bc} D_c \omega) + 4\omega^{-4} (D_a \omega) q^{bc} D_c \omega - \omega^{-4} \delta_a{}^b (q^{cd} D_c \omega D_d \omega) \quad (2.6)$$

$$S'_{ab} = S_{ab} - 2\omega^{-1} D_a D_b \omega + 4\omega^{-2} (D_a \omega)(D_b \omega) - \omega^{-2} q_{ab} (q^{cd} D_c \omega D_d \omega). \quad (2.7)$$

However, using the fact that the space of integral curves of  $n^a$  is topologically  $\mathbb{S}^2$ , it

is possible to extract the conformally invariant information in  $S_a{}^b$ . To obtain this conformally invariant quantity from curvature, we need to use one more structure that is available on  $\mathcal{I}$ . Geroch [27] showed that for any asymptotically flat space-time, there exists a unique symmetric field  $\rho_{ab}$  on  $\mathcal{I}$  in any conformal frame such that

$$\rho_{ab}n^b = 0, \quad D_{[a}\rho_{b]c} = 0, \quad \rho_{ab}q^{ab} =: \bar{R} \quad (2.8)$$

Additionally,  $\rho_{ab}$  has the same conformal transformation property as  $S_{ab}$ .

$$\rho'_{ab} = \rho_{ab} - 2\omega^{-1}D_a D_b \omega + 4\omega^{-2}(D_a \omega)(D_b \omega) - \omega^{-2}q_{ab}(q^{cd}D_c \omega D_d \omega). \quad (2.9)$$

This enables us to define the Bondi news tensor as [27]:

$$N_{ab} = S_{ab} - \rho_{ab} \quad (2.10)$$

which is easily seen to be conformally invariant. A common strategy is to use a ‘Bondi conformal frame’ in which ( $n^a$  is divergence-free and in addition)  $q_{ab}$  is the unit 2-sphere metric. In a Bondi frame,  $N_{ab}$  is simply the trace-free part of  $S_{ab}$ .

The second tensor field which encodes gravitational radiation is the ‘magnetic part’  $\mathcal{B}^{ac}$  of the asymptotic Weyl curvature:

$$\mathcal{B}^{ac} = {}^*K^{abcd}n_b n_d \quad \text{where} \quad K^{abcd} = \lim_{\rightarrow \mathcal{I}} \Omega^{-1} C^{abcd}. \quad (2.11)$$

It is related to the News tensor by the relation:

$$\mathcal{B}^{ab} = 2\epsilon^{amn} D_m N_n{}^b \quad (2.12)$$

In Appendix A it is shown that if  $\mathcal{B}_{ab}$  vanishes on  $\mathcal{I}$ , then the Bondi news  $N_{ab}$  vanishes too. Therefore, if  $\mathcal{B}^{ab} \doteq 0$  on  $\mathcal{I}$ , connections  $\{D\}$  have trivial curvature. In space-times with  $\mathcal{B}^{ab} = 0$  at  $\mathcal{I}$ , it is then natural to include the ‘trivial’ equivalence class of connections  $\{\mathring{D}\}$  in the list of universal structures at  $\mathcal{I}$ . For this family of space-times, the asymptotic symmetry group is the sub-group of the BMS group  $\mathcal{B}$  that also leaves this  $\{\mathring{D}\}$  invariant. It turns out that this reduces the infinite dimensional BMS group  $\mathcal{B}$  to a 10-dimensional Poincaré sub-group thereof.

More importantly, we will see in Chapter 4 that, since  $\mathcal{B}^{ab} = 0$  implies that the Bondi news tensor  $N_{ab}$  must vanish, these space-times *admit no gravitational*

waves.

### 2.4.1 Properties of BMS fields

Finally, we end this chapter by elucidating some useful properties of BMS symmetry vector fields which follow from their definition. Given any conformal frame and a foliation by two-sphere cross-sections, we introduce a chart  $(u, \theta, \phi)$  such that each cross-section is a constant  $u$  surface and  $(\theta, \phi)$  coordinatize the surface of the cross-section. Also,  $\ell_a = -D_a u$  and normalization is fixed to be  $n^a \ell_a = -1$ . On this foliation, an arbitrary BMS field  $\xi^a$  can be decomposed into vertical and horizontal parts as follows:

$$\xi^a = v^a + h^a := \alpha(\theta, \phi) n^a + u k(\theta, \phi) n^a + h^a \quad (2.13)$$

such that

$$h_a n^b = 0 \quad \mathcal{L}_n \alpha = 0 \quad \mathcal{L}_\xi n^a = -k n^a \quad \mathcal{L}_\xi q_{ab} = 2k q_{ab} \quad (2.14)$$

where  $v^a := \alpha(\theta, \phi) n^a + u k(\theta, \phi) n^a$  is the vertical part and  $h^a$  the horizontal part. One can now characterize the BMS fields into the following different classes:

$$\begin{aligned} \text{Supertranslations} & \quad k = 0; h^a = 0 \\ \text{Rotations} & \quad k = 0; \alpha = 0; \mathcal{L}_h q_{ab} = 0; \mathcal{L}_h n^a = 0 \\ \text{Boosts} & \quad k \neq 0; \alpha = 0; \mathcal{L}_h q_{ab} = 2k q_{ab}; \mathcal{L}_h n^a = 0 \end{aligned} \quad (2.15)$$

An important consequence of the definition of the BMS fields is that the symmetric trace-free (STF) part of the tensor  $(D_a D_b k + \frac{1}{2} \mathcal{L}_h \rho_{ab})$  vanishes. It can be proven in the following manner. Choose a null basis  $(n^a, m^a, \bar{m}^a)$  on  $\mathcal{I}$  such that  $\bar{m}^a$  is the complex conjugate of  $m^a$ ,  $m \cdot \bar{m} = 1$  and  $n \cdot m = 0$ . Then, the trace-free, symmetric part of a tensor  $T_{ab}$  which obeys  $T_{ab} n^a = 0$  is captured in the component  $T_{ab} m^a m^b$  and its complex conjugate. Now, suppose in some Bondi frame  $\mathring{q}_{ab}$ , it is true that the horizontal part is an exact symmetry i.e.,  $\mathcal{L}_h \mathring{q}_{ab} = 0$ . Also, in a Bondi frame,  $\rho_{ab} = \frac{\bar{R}}{2} \mathring{q}_{ab}$  where  $\bar{R}$  is the scalar curvature of the 2-sphere. In particular, in this frame,  $\mathcal{L}_h \mathring{\rho}_{ab} = 0$ . Then in any other frame with  $q_{ab} = \omega^2 \mathring{q}_{ab}$

using the conformal transformation property of  $\rho_{ab}$  from Eq. 2.9 we obtain,

$$\begin{aligned} m^a m^b \mathcal{L}_h \rho_{ab} &= m^a m^b \left[ -2\mathring{D}_a \mathring{D}_b k + 4\mathring{D}_a k \mathring{D}_b \ln \omega \right] \\ m^a m^b D_a D_b k &= m^a m^b \left[ \mathring{D}_a \mathring{D}_b k - 2\mathring{D}_{(a} k \mathring{D}_{b)} \ln \omega + \omega^{-1} \mathring{q}_{ab} \omega^m D_m k \right]. \end{aligned} \quad (2.16)$$

Using the above two the following is manifestly true,

$$m^a m^b \left[ D_a D_b k + \frac{1}{2} \mathcal{L}_h \rho_{ab} \right] = 0. \quad (2.17)$$

Thus if  $h^a$  is chosen so that in some constant curvature conformal frame,  $\mathcal{L}_h \mathring{q}_{ab} = 0$ , then it is true that in *any* conformal frame, the symmetric trace-free (STF) part of  $(D_a D_b k + \frac{1}{2} \mathcal{L}_h \rho_{ab})$  vanishes.

In the next chapter, we use the structures described in this chapter to describe physics of isolated systems emitting gravitational radiation.

# Chapter 3 |

## Conserved quantities in asymptotically flat space-times

Gravitational radiation emitted by isolated bodies such as stars and other compact objects is a rich source of information about the dynamical processes that govern their evolution. This information can only be extracted if there are available, (i) well-defined notions of basic physical observables of a system such as energy-momentum and angular momentum and, (ii) methods to quantify how much of these physical quantities are carried away by gravitational radiation.

In general relativity, gravitational fields lack a gauge-invariant, local stress-energy tensor. So, quantities of physical interest such as energy-momentum and angular momentum are constructed using the Hamiltonian framework and asymptotic symmetries.<sup>1</sup> In pioneering work, Arnowitt, Deser and Misner (ADM) [28] used asymptotic symmetries to construct definitions of energy-momentum and angular momentum of asymptotically flat space-times at spatial infinity. The key idea in the Hamiltonian method is the following: Any given symmetry vector field  $\xi$  in space-time generates a canonical transformation on the gravitational phase space which leaves the naturally available symplectic form invariant. Each such canonical transformation is generated by a so-called ‘Hamiltonian’ function  $\mathcal{H}_\xi$  on the phase space. If the vector field is a (asymptotic) symmetry of space-time, this function is (asymptotically) ‘conserved’ in space-time. In the work of [28], space-time is foliated by a family of space-like hypersurfaces  $\Sigma_t$  labeled by their time coordinate  $t$ . The Hamiltonian associated to a symmetry vector field

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<sup>1</sup>This method also gets strong support from the analysis of test matter fields, for which, results using the Hamiltonian method match with those obtained using the stress-energy tensor.

is a 3-dimensional integral over the entirety of one such space-like hypersurface. Conservation means that the value of the Hamiltonian is independent of the choice of the hypersurface  $\Sigma_t$  on which it is evaluated. The interpretation of a Hamiltonian as a physical quantity is inspired by Noether's theorem, and hence, is derived from the symmetry it is associated with: for instance, the Hamiltonian corresponding to time translation symmetry represents the total energy of space-time. Likewise, the Hamiltonian of spatial translations is total momentum and that of a rotational symmetry about an axis is the total angular momentum of space-time about that axis.

We now point out two features of the ADM formalism. First, the ADM quantities, although defined as 3-surface integrals, can be simplified using Gauss' law to be written as 2-surface integrals over spheres 'at infinity'. Second, the ADM quantities thus defined at spatial infinity do not change with time, and hence, do not register information about the dynamics of space-time. Thus, in particular, the ADM formalism is insufficient to study gravitational radiation in space-time.

The situation at null infinity  $\mathcal{I}$  of space-time is different. Gravitational waves carry away energy and momentum from isolated systems out to  $\mathcal{I}$ , so these quantities are no longer absolutely conserved. Hence, to study gravitational radiation one seeks conservation laws in the following sense. Recall from Chapter 2 that  $\mathcal{I}$  of asymptotically flat space-times is a null 3-surface with topology  $\mathbb{S}^2 \times \mathbb{R}$ . Here the  $\mathbb{R}$  direction on  $\mathcal{I}$  is considered as the 'retarded time' direction and the two 2-sphere cross-sections represent two different instants of retarded time. Consider now a patch  $\Delta\mathcal{I}$  of  $\mathcal{I}$  as illustrated in Figure 3.1 which is bound by two 2-sphere cross-sections  $C_1$  and  $C_2$ . The conservation law we seek is of the form

$$F_\xi[\Delta\mathcal{I}] = Q_\xi[C_2] - Q_\xi[C_1]. \quad (3.1)$$

On the left hand side  $F_\xi[\Delta\mathcal{I}]$  denotes the flux across  $\Delta\mathcal{I}$  of a physical observable associated with symmetry  $\xi$ . On the right hand side  $Q_\xi[C_1]$  represents the value of the physical observable at the retarded instant of time represented by  $C_1$ . The quantity  $Q_\xi[C]$  is called the charge associated with symmetry  $\xi^a$  on the cross-section  $C$ . Thus, if the equality holds,  $F_\xi[\Delta\mathcal{I}]$  is the flux of  $Q_\xi[C]$  between the two 'times' represented by the cross-sections  $C_1$  and  $C_2$ . At  $\mathcal{I}$ , Ashtekar and Streubel [17] showed that the Hamiltonian method can be effectively used to obtain

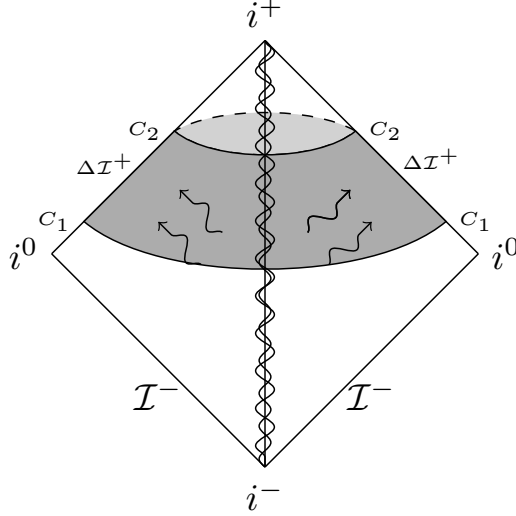
a formula for the flux  $F_\xi$  as a 3-dimensional integral over fields local to  $\mathcal{I}$ . The natural question, then, is whether one can integrate the flux  $F_\xi$  using Stoke's theorem to obtain expressions for the corresponding charge  $Q_\xi$ .

Yes. For the case when  $\xi$  is a supertranslation Ashtekar and Streubel showed that their flux could be integrated to provide a 2-sphere charge integral. The resulting expression exactly matched Geroch's supermomentum [27]. Specifically, for a time translation this quantity would just be the Bondi energy of space-time at a retarded instant of time. The case of an arbitrary  $\xi$  was treated by Dray and Streubel in [26]. However, their charge formula is stated in terms of Newman-Penrose scalars, thereby obscuring the underlying tensorial and geometric structure. In addition, their charge formula is strongly tied to the foliation of  $\mathcal{I}$  by a family of  $u = \text{constant}$  cross-sections. In particular, this makes it rather awkward to discuss conformal invariance of their formula except on the  $u = 0$  cross-section. Furthermore, their formula depends on a decomposition of the BMS vector field into its supertranslation and Lorentz parts. We overcome these issues by providing a new tensorial expression for the 2-sphere charge integral of an arbitrary BMS field  $\xi^a$ . The attractive features of our charge include manifest linearity in the symmetry  $\xi^a$  and conformal invariance. More importantly, it clarifies the role played by various asymptotic fields on  $\mathcal{I}$  in the charge formula. In addition, we present explicit calculations that relate our charge to the Ashtekar-Streubel flux for an arbitrary BMS vector field.

The chapter is organized as follows. In section 3.1 we summarize the main results of Ashtekar and Streubel [17] recalling their formula for the Hamiltonian flux associated with an arbitrary BMS symmetry vector field on  $\mathcal{I}$ . In section 3.2 we provide a new tensorial expression for the charge integral associated with an arbitrary BMS vector field and discuss its important features. We show how our new charge yields the Ashtekar-Streubel flux for any BMS vector field in section 3.3. Finally we conclude with a discussion of our results in section 3.4.

### 3.1 Hamiltonian framework

In this section we recall the formula for flux  $F_\xi[\Delta\mathcal{I}]$  associated to an arbitrary BMS vector field  $\xi^a$  provided by Ashtekar and Streubel [17]. The gravitational phase space  $\Gamma$  is naturally endowed with a non-degenerate closed anti-symmetric



**Figure 3.1.** bounded by cross-sections  $C_1$  and  $C_2$ .

symplectic form,  $\Omega_{ab}$ . A symmetry of the phase space refers to a vector field  $\xi^a$  on the manifold which preserves the symplectic form, i.e.,  $\mathcal{L}_\xi \Omega_{ab} = 0$ . As seen in Chapter 2, for asymptotically flat space-times the asymptotic symmetry group is the BMS group. Any symmetry vector field  $\xi^a$  generates a canonical transformation on the manifold which is generated by a Hamiltonian. Using the Hamiltonian, Ashtekar and Streubel provide the following formula for flux associated with an arbitrary BMS symmetry  $\xi^a$  and a patch  $\Delta\mathcal{I}$  of  $\mathcal{I}$ :

$$F_\xi[\Delta\mathcal{I}] = -\frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V N_{cd} q^{ac} q^{bd} [(\mathcal{L}_\xi D_a - D_a \mathcal{L}_\xi) \ell_b + 2 \ell_{(a} D_{b)} k] \quad (3.2)$$

where  $\kappa = 8\pi G$ . Here  $\ell_b$  is a covector field on  $\mathcal{I}$  satisfying  $\ell \cdot n = -1$ . The other symbols are as defined in Chapter 2:  $N_{ab}$  is the Bondi news tensor,  $D$  is an intrinsic derivative operator such that  $D_a q_{bc} = 0$  and  $k$  is a function on the 2-sphere which satisfies  $\mathcal{L}_\xi q_{ab} = k q_{ab}$ ,  $\mathcal{L}_\xi n^a = -k n^a$ . As explained in the beginning of the chapter,  $F_\xi[\Delta\mathcal{I}]$  is interpreted as the flux associated with the BMS vector field  $\xi^a$  radiated away through  $\Delta\mathcal{I}$  by gravitational waves. We now wish to associate an instantaneous charge to this flux for an arbitrary BMS field  $\xi^a$ . In the following section we provide a new tensorial expression for such a charge.

## 3.2 Charge of general BMS vector field

For an arbitrary BMS vector field  $\xi$  and an arbitrary cross-section  $C$  of  $\mathcal{I}$  the charge associated with  $\xi$  is given by the following 2-sphere integral:

$$Q_\xi[C] = -\frac{1}{\kappa} \oint_C d^2S \left[ -W_{ab} n^a \xi^b + (R_{abc}{}^d - \mathring{R}_{abc}{}^d) \bar{q}^{am} \bar{q}^{cn} (D_m \ell_n) \xi^b \ell_d - \frac{1}{6} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \quad (3.3)$$

where  $\kappa = 8\pi G$  and  $\bar{q}^{ab}$  is the inverse of the pull-back  $\bar{q}_{ab}$  of the metric  $q_{ab}$  to the 2-sphere cross-section  $C$ .  $\ell_a$  is a null covector which is orthogonal to the surface  $C$  i.e.,  $\ell \cdot v = 0$  for every  $v^a$  tangent to  $C$ , and it is normalized as  $\ell \cdot n = -1$ . In addition, the freedom in choice of conformal factor discussed in section 2.1 is used to make  $\ell_a$  divergence-free i.e.,  $\bar{q}^{ab} D_a \ell_b = 0$ .<sup>2</sup> In particular, this condition restricts the behaviour of the conformal factor *off*  $\mathcal{I}$ . Any solution  $\omega$  to the equation  $\bar{q}^{ab} D_a \ell_b + 2g^{cd} \ell_c D_d \omega = 0$  can be used to go to a frame  $g'_{ab} = \omega^2 g_{ab}$  in which  $\bar{q}'^{ab} D'_a \ell'_b = 0$ . This does not exhaust conformal freedom, and the remaining restricted conformal freedom is  $\mathcal{L}_\ell \omega = 0$  and  $\mathcal{L}_n \omega = 0$  (from  $\nabla_a n^a = 0$ ).

Now we move to the curvature terms that appear in the formula (3.3). The tensor  $W_{ab}$  is defined as  $W_{ab} := K_{acbd} \ell^c \ell^d$  in terms of the asymptotic Weyl curvature  $K_{abc}{}^d := \Omega^{-1} C_{abc}{}^d$  which has a good limit to  $\mathcal{I}$ . The next tensor to appear is the Riemann tensor  $R_{abc}{}^d$  of the intrinsic derivative operator  $D$ . As discussed in Chapter 2, this tensor has a complicated conformal behaviour. The conformally invariant part is extracted by adding precisely the last two terms in the formula (3.3). The tensor  $\mathring{R}_{abc}{}^d$  is defined as

$$\mathring{R}_{abc}{}^d \hat{=} \frac{1}{2} (q_{c[a} \rho_{b]}{}^d - \rho_{c[a} \delta_{b]}{}^d) \quad (3.4)$$

where  $\rho_{ab}$  is Geroch's symmetric tensor of section 2.4 with the properties listed in Eq. (2.8).

Next, we note some properties of the charge formula in (3.3). Firstly, the charge is manifestly linear in the BMS vector field  $\xi$ . Secondly, as mentioned above, the charge is conformally invariant.<sup>3</sup> Lastly, for familiar examples such as Kerr and

<sup>2</sup>In terms of Newman-Penrose scalars the condition becomes  $\rho = 0 = \bar{\rho}$  (see Appendix A).

<sup>3</sup>See Appendix A for a detailed derivation of this property.

Vaidya space-times, it yields the expected results. Specifically, for Kerr space-time the only non-zero charges are the mass and the (correctly normalized) angular momentum of the black hole. These are independent of the cross-section  $C$  of  $\mathcal{I}$  on which they are evaluated. For radiating Vaidya space-time the only non-zero charge is its mass, but now it is a function of the retarded time coordinate  $u$ .

### 3.3 From charges to fluxes

In this section we sketch the important steps to show that the flux of the charge (3.3) carried by gravitational waves across  $\mathcal{I}$  exactly matches that provided by Ashtekar and Streubel in [17]. For this, it is first convenient to rewrite the charge (3.3) as follows:

$$Q_\xi[C] = \frac{1}{\kappa} \oint_C dS \left[ W_{ab} n^a \xi^b + \frac{1}{2} (\ell_d \xi^d) N_{ac} (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn} - \frac{1}{2} q_{cb} \xi^b S_a{}^d \ell_d (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn} + \frac{1}{6} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \quad (3.5)$$

To clarify the main steps in the procedure, it is convenient to consider, first, the case when the BMS vector field  $\xi$  is a supertranslation. The computations in the case of Lorentz fields, although more involved, are conceptually similar.

#### 3.3.1 Supertranslations

Consider a BMS supertranslation  $\xi^a = \alpha n^a$  such that  $\mathcal{L}_n \alpha = 0$ . From (3.5), we can read off the charge associated with it on a cross-section  $C$  of  $\mathcal{I}$  as

$$Q_\xi[C] = \frac{1}{\kappa} \oint [\alpha W_{ab} n^a n^b - \frac{1}{2} \alpha N_{ac} (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn}] \quad (3.6)$$

The flux across  $\mathcal{I}$  between two cross-sections  $C_1$  and  $C_2$  is given by  $\int_{C_1}^{C_2} du \mathcal{L}_n Q_\xi$  where  $u$  is the affine parameter of  $n^a$ .

$$F_\xi^{(Q)}[C_1, C_2] = \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \mathcal{L}_n [\alpha W_{ab} n^a n^b - \frac{1}{2} \alpha N_{ac} (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn}]. \quad (3.7)$$

We need to find the action of the Lie derivative on the Weyl tensor,  $K_{abcd}$ , the news  $N_{ac}$  and the shear term  $D_m \ell_n$ . They are given by:

$$\begin{aligned}
\mathcal{L}_n(W_{ab} n^a n^b) &\hat{=} \bar{q}^{fd} \nabla_f K_{abcd} n^a \ell^b n^c \\
(\mathcal{L}_\xi D_m - D_m \mathcal{L}_\xi) \ell_n &= \xi^c (q_{n[c} S_m]{}^d + S_{n[c} \delta_m]{}^d) \ell_d - \ell_d D_m D_n \xi^d \\
\mathcal{L}_n D_m \ell_n &\hat{=} -\frac{1}{2} [q_{mn} S_c{}^d n^c \ell_d + (n \cdot \ell) S_{mn}] \\
\mathcal{L}_n N_{ac} &\hat{=} 2K_{abcd} n^b n^d.
\end{aligned} \tag{3.8}$$

The first equation follows from the contraction with  $n^a \ell^b n^c$  and a rewriting of the contracted Bianchi identity

$$\nabla^d K_{abcd} \hat{=} 0. \tag{3.9}$$

The third equation follows from the second because  $\mathcal{L}_{cn} \ell \hat{=} 0$ .

The last equation is obtained from the contracted Bianchi identity as follows:

$$\begin{aligned}
\nabla_d C_{abc}{}^d + \nabla_{[a} S_{b]c} &= 0 \\
\implies \Omega \nabla_d K_{abc}{}^d + K_{abc}{}^d \nabla_d \Omega + \nabla_{[a} S_{b]c} &= 0.
\end{aligned} \tag{3.10}$$

Contract with  $n^a = g^{ab} \nabla_b \Omega$  and evaluate on  $\mathcal{I}$  where  $\Omega \hat{=} 0$  to obtain

$$\mathcal{L}_n S_{ac} \hat{=} 2 K_{abcd} n^b n^d + n^b \nabla_a S_{bc}. \tag{3.11}$$

Pull-back indices to  $\mathcal{I}$  to obtain the last equation of (3.8).

Using the formulas in (3.8), we see that the Lie derivative of the news cancels out with part of the Lie derivative of the Weyl tensor. The remaining portion can then be integrated by parts and the total flux can be written as

$$F_\xi^{(Q)}[C_1, C_2] = \frac{1}{\kappa} \int_{\mathcal{I}} d^3V \left[ \frac{1}{4} \alpha N_{ac} \bar{q}^{am} \bar{q}^{cn} S_{mn} - K_{abcd} n^a \ell^b n^c \bar{q}^{fd} \nabla_f \alpha + \bar{D}_f t_1^f \right] \tag{3.12}$$

where the 2-surface boundary term is

$$t_1^f = \alpha K_{abcd} n^a \ell^b n^c \bar{q}^{df}. \tag{3.13}$$

Next, project the indices of (3.10) to the 2-surface to obtain

$$\bar{q}_a{}^m \bar{q}_b{}^n \bar{q}_c{}^s [K_{mnsd} n^d + \nabla_{[m} N_{n]s}] \hat{=} 0. \quad (3.14)$$

Using (3.14), simplify the second term in the flux as follows:

$$-K_{abcd} n^a \ell^b n^c \bar{q}^{fd} \nabla_f \alpha = -K_{abcd} n^a \bar{q}^{bc} \bar{q}^{fd} \nabla_f \alpha = \nabla_{[d} N_{c]b} \bar{q}^{bc} \bar{q}^{fd} \nabla_f \alpha. \quad (3.15)$$

With one more integration by parts, rewrite the flux as

$$F_\xi^{(Q)}[C_1, C_2] = \frac{1}{\kappa} \int_{\mathcal{I}} d^3V \left[ \frac{1}{4} \alpha N_{ac} \bar{q}^{am} \bar{q}^{cn} S_{mn} + \frac{1}{2} N_{ac} \bar{q}^{am} \bar{q}^{cn} \bar{D}_m \bar{D}_n \alpha + \bar{D}_f t_1^f + \bar{D}_f t_2^f \right] \quad (3.16)$$

where

$$t_1^f = \alpha K_{abcd} n^a \ell^b n^c \bar{q}^{df} \quad t_2^f = -\frac{1}{2} N_{ac} \bar{q}^{am} \bar{q}^{cf} \bar{D}_m \alpha. \quad (3.17)$$

It is now straight-forward to see that the Ashtekar-Streubel flux, which is given by

$$\begin{aligned} F_\xi[\Delta\mathcal{I}] &= \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ N_{ac} (\mathcal{L}_\xi D_m - D_m \mathcal{L}_\xi) \ell_n \bar{q}^{am} \bar{q}^{cn} \right] \\ &= \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \frac{1}{4} \alpha N_{ac} \bar{q}^{am} \bar{q}^{cn} S_{mn} + \frac{1}{2} N_{ac} \bar{q}^{am} \bar{q}^{cn} D_m D_n \alpha \right] \end{aligned} \quad (3.18)$$

matches the flux of the new charge (3.16) up to boundary terms. Since the topology of the cross-section is  $\mathbb{S}^2$ , we can evaluate all the boundary terms using Stoke's theorem and find that they are zero.

### 3.3.2 Lorentz transformations

Consider a Lorentz transformation  $\xi^a = v^a + h^a$  where  $v^a = u k(\theta, \phi) n^a$ . From (3.5), we can read off the charge associated with it on a cross-section  $C$  as

$$Q_\xi[C] = \frac{1}{\kappa} \oint dS \left[ K_{abcd} \ell^a n^b \ell^c \xi^d + \frac{1}{2} u k(\theta, \phi) N^{ab} (D_a \ell_b) + \frac{1}{2} S_a{}^d \ell_d h^n (D_m \ell_n) \bar{q}^{am} + \frac{1}{6} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \quad (3.19)$$

$$= Q_v[C] + Q_h[C]. \quad (3.20)$$

The Ashtekar-Streubel Hamiltonian flux can be simplified using (3.8) as

$$\begin{aligned} F_\xi[\Delta\mathcal{I}] &= \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ N_{ac} (\mathcal{L}_\xi D_m - D_m \mathcal{L}_\xi) \ell_n \bar{q}^{am} \bar{q}^{cn} \right] \\ &= F_v[\Delta\mathcal{I}] + F_h[\Delta\mathcal{I}] \end{aligned} \quad (3.21)$$

where

$$F_v[\Delta\mathcal{I}] = \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \bar{N}^{ab} \left( \frac{1}{2} u k S_{ab} + u D_a D_b k + k \sigma_{ab} \right) \right] \quad (3.22)$$

$$F_h[\Delta\mathcal{I}] = \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \bar{N}^{ab} \left( \frac{1}{2} h_a S_b^d \ell_d - \ell_d D_a D_b h^d \right) \right] \quad (3.23)$$

where  $\sigma_{ab} = D_a \ell_b$  and  $\bar{N}^{ab} = \bar{q}^{ac} \bar{q}^{bd} N_{cd}$ .

Our aim is to show that

$$\int_{C_1}^{C_2} du \mathcal{L}_n Q_\xi[C] = F_\xi[\Delta\mathcal{I}]. \quad (3.24)$$

First, we note that

$$\begin{aligned} \mathcal{L}_n Q_v[C] &= Q_{kn}[C] + \frac{u}{2\kappa} \oint d^2S \left( \frac{k}{2} S_{ab} + D_a D_b k \right) \bar{N}^{ab} \\ &\quad - \frac{k}{3\kappa} \oint d^2S (\mathcal{L}_n D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \end{aligned} \quad (3.25)$$

This is easily seen by applying the Lie derivative and simplifying as was done to obtain (3.16). Further, using (3.8), we obtain:

$$\begin{aligned} \int_{\Delta\mathcal{I}} \mathcal{L}_n Q_v[C] &= F_v[C_1, C_2] + \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \frac{k}{6} S_{ab} \sigma^{ab} + k K_{abcd} \ell^a n^b \ell^c n^d \right] \\ &=: F_v[\Delta\mathcal{I}] + \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V (U_1 + U_2) \end{aligned} \quad (3.26)$$

Next we calculate  $\mathcal{L}_n Q_h[C]$  as follows

$$\begin{aligned} \int_{\Delta\mathcal{I}} \mathcal{L}_n Q_h[C] &= \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \mathcal{L}_n \left[ K_{abcd} \ell^a n^b \ell^c h^d + \frac{1}{2} S_a^d \ell_d (D_m \ell_n) h^n \bar{q}^{am} \right. \\ &\quad \left. + \frac{1}{6} (D_p h^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \end{aligned} \quad (3.27)$$

As shown in Appendix A, the integrand of  $\int_{\Delta\mathcal{I}} \mathcal{L}_n Q_h[C]$  in Eq. (3.27) is given by

$$\begin{aligned}
& \frac{1}{4} h^n N_{nm} \bar{q}^{ma} S_a{}^d \ell_d - \frac{1}{2} N_{ab} \bar{q}^{df} \bar{q}^{bc} \ell_m \bar{D}_c \bar{D}_f h^m \\
& + K_{abcd} \ell^b n^d \bar{D}^c h^a - \frac{1}{2} N_{ab} \bar{q}^{df} \bar{q}^{bc} \bar{D}_c \ell_m \bar{D}_f h^m \\
& + \frac{1}{4} h^n (\rho_{nm} + \beta q_{mn}) \bar{q}^{ma} S_a{}^d \ell_d \\
& - \frac{1}{2} h^n (D_m \ell_n) \bar{q}^{am} \nabla_a \beta + \frac{1}{2} k S_{ab} \bar{\sigma}^{ab} \\
& + \bar{D}_f t_1^f + \bar{D}_f t_2^f
\end{aligned} \tag{3.28}$$

The first line is equal to the integrand of  $F_h$  in (3.23).

Using two-dimensional identities, it is shown in Appendix A that the first term on the second line can be simplified to the following expression:

$$\begin{aligned}
K_{abcd} \ell^b n^d \bar{D}^c h^a &= -k K_{abcd} n^a \ell^b n^c \ell^d \\
& + \frac{1}{2} \bar{D}_a (\Theta S_c{}^b \ell_b) \bar{\epsilon}^{ac} + \frac{1}{2} (\bar{R} h_d + 2\bar{D}_d k) \bar{D}_c \bar{\sigma}^{cd} - \frac{1}{2} S_c{}^b \sigma_{ab} \bar{D}^{[a} h^{c]}
\end{aligned} \tag{3.29}$$

Combining the above results in Eqs. (3.26) - (3.29), we obtain

$$\begin{aligned}
F_\xi[\Delta\mathcal{I}] - \int_{C_1}^{C_2} d^3V \mathcal{L}_n Q_\xi[C] &= \int_{\Delta\mathcal{I}} \left[ -\frac{1}{2} h^m \bar{D}_a \rho_{mb} + \rho^m{}_b \bar{D}_{[m} h_{a]} - \bar{D}_b \bar{D}_a k \right] \bar{\sigma}^{ab} \\
& + \text{Boundary Terms}
\end{aligned} \tag{3.30}$$

Using the property  $D_{[a} \rho_{b]c} = 0$ , the term in the square brackets can be rewritten as

$$-\bar{D}_a \bar{D}_b k - \frac{1}{2} \mathcal{L}_h \rho_{ab} + k \rho_{ab}. \tag{3.31}$$

Now, it has been shown as a property of BMS fields in Chapter 2 Eq. (2.17) that the trace-free symmetric part of  $(\bar{D}_a \bar{D}_b k + \frac{1}{2} \mathcal{L}_h \rho_{ab})$  vanishes. Then the only term that survives in a contraction with the trace-free  $\bar{\sigma}^{ab}$  is  $k \rho_{ab}$ . But in a Bondi frame  $\rho_{ab} \propto q_{ab}$ , hence, this term also does not contribute to the contraction with  $\bar{\sigma}^{ab}$ . Thus, the flux of our charge for a Lorentz field matches the expression provided by Ashtekar and Streubel up to ‘boundary terms’. Once again, we can evaluate all the boundary terms using Stoke’s theorem and find that they are zero.

### 3.4 Discussion

To summarize, we provide a new tensorial expression for a charge integral  $Q_\xi[C]$  associated to an arbitrary BMS field  $\xi^a$  on a cross-section  $C$  of  $\mathcal{I}$ . The charge is a 2-sphere integral over fields that are local to the cross-section  $C$ , and does not require extraneous structure. In particular, it is constructed using components of the Weyl tensor  $W_{ab} = K_{abcd} \ell^c \ell^d$  and the conformally invariant part of the Riemann tensor  $R_{abc}{}^d$  of the intrinsic derivative operator  $D$ . The conformal invariance of the charge is easily shown. Furthermore, we have shown that the charge  $Q_\xi[C]$  yields the Ashtekar-Streubel flux  $F_\xi[\Delta\mathcal{I}]$  thus completing the conservation law

$$F_\xi[\Delta\mathcal{I}] = Q_\xi[C_2] - Q_\xi[C_1]. \quad (3.32)$$

Hence, we can interpret the flux  $F_\xi[\Delta\mathcal{I}]$  through a region  $\Delta\mathcal{I}$  of  $\mathcal{I}$  as providing the difference between the ‘instantaneous’ values of the charge  $Q_\xi[C]$  at the two ‘times’ represented by the cross-sections  $C_2$  and  $C_1$  which bound the region  $\Delta\mathcal{I}$ .

We now compare our result to earlier approaches. As already mentioned, for supertranslations our charge reduces to Geroch’s supermomentum. Next, we compare with the charge of Dray and Streubel [26] for an arbitrary BMS field. To construct their charge Dray and Streubel foliate  $\mathcal{I}$  by  $u = \text{constant}$  surfaces where the coordinate  $u$  is the affine parameter of the null normal  $n^a$  i.e.,  $n^a \nabla_a u = 1$ . The cross-section  $C$  is a surface with  $u = u_0$  in this family. Next, they choose a null tetrad adapted to this cross-section and parallel propagate on  $\mathcal{I}$  along  $n^a$ . Then, they split the BMS field  $\xi$  into a part that is tangential to the cross-section and one that is orthogonal i.e., proportional to  $n^a$ . Their resultant charge formula is thus dependent on this split. In particular, it is awkward to show conformal invariance of their charge except on the  $u = 0$  cross-section. In contrast, the charge of Eq. (3.5) is defined without recourse to structure extraneous to the cross-section  $C$ . Specifically, it does not require a foliation of  $\mathcal{I}$  nor do we split the BMS field. Due to these factors, it is easier to prove conformal invariance of the charge. However, if a specific null tetrad basis is chosen on the cross-section  $C$ , then the new charge Eq. (3.5) reduces to that of Dray and Streubel [26] after integrations by parts.

Next, how does our charge relate to that proposed by Wald and Zoupas [19]? Wald and Zoupas proposed a scheme to compute ‘conserved quantities’ for general

diffeomorphism covariant theories of gravity derivable from a Lagrangian, generalizing earlier works on ‘linkages’ [20]. Specifically, their proposal for obtaining a Hamiltonian flux yields is compatible with the flux formula provided by Ashtekar and Streubel. Furthermore, they argue that their scheme would yield a charge formula that agrees with that of Dray and Streubel. Thus our own charge is in agreement with their proposed scheme.

Finally, we note that the results provided in this chapter will be useful for research in allied areas. First, in addition to energy, our charge provides expressions for angular momentum carried by gravitational waves. Knowledge of the instantaneous values of these charges will be valuable in the study of astrophysical systems, particularly in view of the recent direct detection of gravitational waves [16]. Second, in quantum theory, the BMS group and various extensions thereof are directly related to the infrared issues associated with the full, non-linear gravitational field [21, 22, 39, 41]. These could play an important role in the analysis of the black hole evaporation process. Thus, we expect that the charge formula provided in this chapter will aid in further research.

# Chapter 4 |

## Asymptotics with $\Lambda > 0$

For  $\Lambda = 0$ , space-times are endowed with rich asymptotic structure that facilitate the study of gravitational radiation. In this chapter we will show that such is not the case when  $\Lambda > 0$ . While Penrose's construction of null infinity naturally generalizes, the boundary of space-time  $\mathcal{I}$  is now *space-like*. Consequently, we will see in detail that the asymptotic symmetry group—the direct analog of the BMS group  $\mathcal{B}$ —is now the full diffeomorphism group  $\text{Diff}(\mathcal{I})$  of  $\mathcal{I}$ . The  $\text{Diff}(\mathcal{I})$  group is unable to provide unambiguous notions of energy and angular momentum. Furthermore, *there is no natural analog of the Bondi news to characterize gravitational radiation in the full non-linear context.*

This situation can be improved by imposing an extra requirement: demand that the intrinsic geometry of  $\mathcal{I}$  be conformally flat. This stronger boundary condition has support in two observations which make it a natural condition to impose. First, the de Sitter space-time metric (which is the analog of Minkowski space-time when  $\Lambda > 0$ ) satisfies this condition. Second, for  $\Lambda < 0$ , since  $\mathcal{I}$  is *time-like*, this condition is required to allow for well-defined time-evolution. The stronger boundary condition has the attractive feature that it reduces the infinite dimensional asymptotic group  $\text{Diff}(\mathcal{I})$  to the 10-dimensional de Sitter group  $G_{\text{dS}}$ ! This provides an avenue to define Bondi-like charges and fluxes for energy and angular momentum unambiguously on  $\mathcal{I}$ .

However, for  $\Lambda > 0$ , the stronger boundary condition is greatly restrictive. We will see that this condition is equivalent to *removing, by fiat, half the gravitational degrees of freedom!* Furthermore, it will be shown that the remaining degrees of freedom are insufficient to provide non-zero fluxes of charges. Thus, with conformal flatness of the boundary imposed, *gravitational waves do not carry away energy or*

*momentum across  $\mathcal{I}$ ! And without this condition, we do not have unambiguous notions of energy, momentum and angular momentum.*

In this chapter, we describe how the difficulties discussed above arise. The chapter is organized as follows. In section 4.1 we summarize the notion of asymptotically de Sitter space-times. The basic definition due to Penrose is extended to incorporate isolated systems including black holes and the cosmological space-times as they are commonly treated in the literature. We also discuss the asymptotic fields and equations they satisfy at  $\mathcal{I}$ . All this structure will be useful in the sections and chapters that follow. Section 4.2 discusses examples. The Vaidya-de Sitter solution is particularly interesting because it brings out certain features associated with non-trivial dynamics. Section 4.3 discusses symmetries and the associated definitions of conserved charges. In particular, the role played by the additional requirement of conformal flatness of the intrinsic metric on  $\mathcal{I}$  is spelled out. At first this framework seems satisfactory. Section 4.4 explains in detail why this is not the case. Section 4.5 summarizes the results and discusses other conceptual issues that arise in the passage from  $\Lambda = 0$  to  $\Lambda > 0$  cases. These can be addressed in the linearized theory as will be demonstrated in Chapters 5 and 6.

## 4.1 Asymptotically de Sitter space-times

This section is divided in two parts. In the first, we present definitions of asymptotically de Sitter space-times [44, 47]. While the basic underlying idea is completely parallel to that used in the  $\Lambda = 0$  case, it is now natural to allow for three different topologies of  $\mathcal{I}$  which arise in the most common applications. In the second part, we summarize the basic consequences of the conditions in the definition. These results will be used in sections 4.3-4.4.

### 4.1.1 Definitions

**Definition 2** *A space-time  $(\hat{M}, \hat{g}_{ab})$  will be said to be weakly asymptotically de Sitter if there exists a manifold  $M$  with boundary  $\mathcal{I}$  equipped with a metric  $g_{ab}$  and a diffeomorphism from  $\hat{M}$  onto  $(M \setminus \mathcal{I})$  of  $M$  (using which we identify  $\hat{M}$  and  $(M \setminus \mathcal{I})$ , the interior of  $M$ ) such that:*

1. *there exists a smooth function  $\Omega$  on  $M$  such that  $g_{ab} = \Omega^2 \hat{g}_{ab}$  on  $\hat{M}$ ;  $\Omega = 0$*

on  $\mathcal{I}$ ;

and  $n_a := \nabla_a \Omega$  is nowhere vanishing on  $\mathcal{I}$ ; and

2.  $\hat{g}_{ab}$  satisfies Einstein's equations with a positive cosmological constant, i.e.,  $\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} + \Lambda\hat{g}_{ab} = 8\pi G \hat{T}_{ab}$  with  $\Lambda > 0$ ; where  $\Omega^{-1}\hat{T}_{ab}$  has a smooth limit to  $\mathcal{I}$ .

The two conditions in this definition are direct generalizations of those used in the definition of asymptotically flat space-times of Definition 1. The first ensures that  $(M, g_{ab})$  is a conformal completion of the physical space-time  $(\hat{M}, \hat{g}_{ab})$  in which only the interior of  $M$  is diffeomorphic to the physical space-time and that the boundary  $\mathcal{I}$  is at infinity with respect to the physical metric  $\hat{g}_{ab}$ ;  $\hat{g}_{ab} = \Omega^{-2}g_{ab}$  is not well-defined on  $\mathcal{I}$  where  $\Omega = 0$ . The condition  $\nabla_a \Omega \neq 0$  ensures that  $\Omega$  can be used as a coordinate on  $M$ ; in particular, near the boundary, we can perform Taylor expansions in  $\Omega$  to capture the degree of fall-off of physical fields. In terms of the physical space-time  $(\hat{M}, \hat{g}_{ab})$ , it ensures that  $\Omega$  has the same asymptotic behavior as in de Sitter space. The second condition about fall-off rate of matter fields defines what we mean by an ‘isolated system’ in the physical space-time  $(\hat{M}, \hat{g}_{ab})$ . The specific fall-off of  $\hat{T}_{ab}$  used here is motivated by the analysis of test fields in de Sitter space-times such as the conformally coupled scalar field and the Maxwell field, and phenomenological matter fields more relevant to cosmology such as dust and fluids. (In fact null fluids and Maxwell fields fall-off faster;  $\Omega^{-2}\hat{T}_{ab}$  has a smooth limit to  $\mathcal{I}$ .)

One can further strengthen Definition 2 by requiring that  $\mathcal{I}$  be complete to yield the following definition:

**Definition 3** *A weakly asymptotically de Sitter space-time  $(\hat{M}, \hat{g}_{ab})$  will be said to be asymptotically de Sitter if  $\mathcal{I}$  is geodesically complete w.r.t.  $g_{ab}$ .*

Completeness is essential for reasonable definition of space-times with black hole regions. As in asymptotic flatness, a space-time  $(M, g_{ab})$  is said to admit a black-hole region  $B$  if the past of  $\mathcal{I}^+$  does not cover  $M$ . If completeness of  $\mathcal{I}^+$  is not required, one can conjure up an unpleasant situation where the de Sitter space-time contains a black hole region.

The topologies of  $\mathcal{I}$  that are most relevant for physical applications fall into three classes which prompts the following categorization of asymptotically de Sitter space-times.

- $(\hat{M}, \hat{g}_{ab})$  will be said to be *Globally asymptotically de Sitter* if it admits a conformal completion satisfying conditions of Definition 2 in which  $\mathcal{I}$  has the topology of a 3-sphere  $\mathbb{S}^3$ . de Sitter space-time with its standard completion belongs to this class.
- $(\hat{M}, \hat{g}_{ab})$  will be said to be *Asymptotically de Sitter in a Poincaré patch* if it admits a conformal completion satisfying conditions of Definition 2 in which its  $\mathcal{I}$  has topology  $\mathbb{R}^3 \simeq \mathbb{S}^3 \setminus \{p\}$ . The standard completions of the Friedmann-Lemaître cosmologies, for example, belong to this class where the point  $p$  represents spatial infinity,  $i^\circ$ . Therefore, this topology is of interest particularly in cosmological applications.
- $(\hat{M}, \hat{g}_{ab})$  will be said to be *Asymptotically Schwarzschild-de Sitter* if it admits a conformal completion satisfying conditions of Definition 2 in which its  $\mathcal{I}$  has topology  $\mathbb{S}^2 \times \mathbb{R} \simeq \mathbb{S}^3 \setminus \{p_1, p_2\}$  on  $\mathcal{I}$ . The standard completion of Schwarzschild-de Sitter space-time falls in this class, where the point  $p_2$  again represents spatial infinity  $i^\circ$  and the point  $p_1$  represents future or past time-like infinity  $i^\pm$  on  $\mathcal{I}^\pm$ . This topology is of interest in the discussion of compact isolated systems such as stars and black holes.

Note that a physical space-time can belong to more than one class, depending on the choice of the conformal factor. For example, given the standard conformal completion of de Sitter space-time in which  $\mathcal{I}$  has  $\mathbb{S}^3$  topology, one can choose another conformal factor  $\omega' = \alpha\Omega$  to obtain a completion in which it has  $\mathbb{R}^3$  topology: the required  $\alpha$  will simply diverge at a point  $p$  on  $\mathbb{S}^3$  at an appropriate rate, ‘opening up’  $\mathbb{S}^3$ , and the point  $p$  would then have the interpretation of being the point  $i^\circ$  at spatial infinity. The new completion would make  $(M, g_{ab})$  ‘asymptotically de Sitter in a Poincaré patch’. Similarly, given the standard  $\mathbb{S}^3$  completion, we can choose a conformal factor  $\Omega' = \beta\Omega$  which diverges at appropriate rates at two *points* which will represent  $i^\circ$  and  $i^\pm$  on  $\mathcal{I}^\pm$  of the resulting completion.

However, given the standard completion of the Schwarzschild-de Sitter space-time in which  $\mathcal{I}$  is topologically  $\mathbb{S}^2 \times \mathbb{R}$ , it is not possible to choose a conformal rescaling  $\beta$  to obtain a smooth rescaled metric  $g_{ab}$  and a  $\mathbb{S}^3$  topology for  $\mathcal{I}$ . The detailed discussion contained in the subsequent sections will make these features

transparent. Here we only note that in the asymptotically flat context, the topology of  $\mathcal{I}$  is always  $\mathbb{S}^2 \times \mathbb{R}$ , irrespective of whether the physical space-time of interest is just the Minkowski space-time or represents a star or a black hole. This difference arises because, as we will discuss in sections 4.2 and 4.3, whereas  $\mathcal{I}$  is naturally ruled (by null geodesics) if  $\Lambda = 0$ , there is no such ruling in the case when  $\Lambda > 0$ . One can also consider a topology  $\mathbb{S}^3 \setminus \{p_1, p_2, \dots, p_n\}$ . In this case, one of the punctures will represent the point  $i^\circ$  at spatial infinity and the remaining  $n - 1$  punctures would represent compact objects. However, since these are distinct points on  $\mathcal{I}^\pm$ , the physical distance between any two of them grows unboundedly in the distant future/past. Therefore these cases will not be relevant to the study of individual isolated systems normally considered in mathematical and numerical general relativity.

Finally, a stronger boundary condition can be imposed analogous to what is done in the  $\Lambda < 0$  case, leading to the final definition. When  $\Lambda < 0$ ,  $\mathcal{I}$  is time-like. To obtain well-defined evolution, conformal flatness of the boundary can be regarded as a natural ‘reflective boundary condition’ [44, 45, 47].

**Definition 4** *A space-time  $(\hat{M}, \hat{g}_{ab})$  will be said to be strongly asymptotically de Sitter if in a conformal completion satisfying conditions of Definition 3, the intrinsic metric  $q_{ab}$  on  $\mathcal{I}$  is conformally flat.*

As we will see below, the condition of conformal flatness of boundary is equivalent to setting the leading order magnetic part of the Weyl tensor to zero, and has profound physical consequences for gravitational waves.

### 4.1.2 Asymptotic fields and their equations

In this sub-section, we will collect the immediate implications of Definition 2. In this discussion, the requirement of completeness in Definition 3 and the choice of topology will not play any role as the considerations of this sub-section are local to  $\mathcal{I}$ .

1.  $\mathcal{I}$  is a space-like surface.

Expressing Einstein’s equation satisfied by  $\hat{g}_{ab}$  in terms of the conformally

rescaled metric  $g_{ab}$ :

$$R_{ab} - \frac{1}{2}g_{ab}R + 2\Omega^{-1}(\nabla_a n_b - g_{ab}\nabla^c n_c) + 3\Omega^{-2}g_{ab}n^c n_c + \Omega^{-2}\Lambda g_{ab} = 8\pi G\hat{T}_{ab}, \quad (4.1)$$

where, as before  $n_a := \nabla_a \Omega$ . Multiplying (4.1) by  $\Omega^2$  and evaluating the resulting expression on  $\mathcal{I}$  using our boundary conditions in Definition 2, we obtain

$$n^a n_a \hat{=} -\frac{\Lambda}{3} =: -\frac{1}{\ell^2} \quad (4.2)$$

where  $\hat{=}$  stands for equality at  $\mathcal{I}$  and  $\ell$  denotes the cosmological radius. Thus,  $n^a$  is time-like on  $\mathcal{I}$  and consequently  $\mathcal{I}$  is space-like. Hence condition iii) of geodesic completeness in Definition 3 is equivalent to completeness with respect to the Riemannian metric  $q_{ab}$  on  $\mathcal{I}$  induced by  $g_{ab}$ . The space-like character of  $\mathcal{I}$  is the first big departure from asymptotically flat space-times. This immediately gives rise to some conceptual complications. For example the ‘obvious’ strategy to impose the no incoming boundary condition fails already in the case of Maxwell fields and one cannot repeat the proofs of ‘peeling theorems’ that were used heavily in the early stages of the analysis of gravitational radiation in the Bondi et al program (see, e.g., [49, 50]).

2.  *$n^a$  can always be made ‘divergence-free’.*

The choice of conformal factor in the above definition is not canonical. Under a conformal rescaling  $\Omega \rightarrow \Omega' = \omega\Omega$ , we have  $n'^a \hat{=} \omega^{-1}n^a$  and  $q'_{ab} \hat{=} \omega^2 q_{ab}$ . Using this freedom, again one finds solutions to the equation  $\mathcal{L}_n \omega \hat{=} -\frac{1}{4}\omega \nabla_a n^a$  to obtain  $\nabla'_a n'^a \hat{=} 0$ . The residual conformal freedom is then  $\Omega' \hat{=} \omega\Omega$  where  $\mathcal{L}_n \omega \hat{=} 0$ . Throughout our analysis we will make this restriction because this choice simplifies calculations considerably [47]. In particular, Eq. (4.1) now implies that  $\nabla_a n_b \hat{=} 0$ , whence, in particular, the extrinsic curvature,  $k_{ab}$  of  $\mathcal{I}$  vanishes.

3. *The asymptotic Weyl tensor  $C_{abcd}$  of  $g_{ab}$  vanishes identically on  $\mathcal{I}$ .*

The argument follows the same steps outlined in section 2.1 up to Eq. (2.3). Now, since  $\mathcal{I}$  is space-like, the Weyl tensor  $C_{abcd}$  is completely determined by its electric and magnetic parts,  $E_{ac} := l^2 C_{abcd} n^b n^d$  and  $B_{ac} := l^2 \star C_{abcd} n^b n^d$

and both these fields vanish on  $\mathcal{I}$  because of (2.3). Therefore, we conclude that the *full* Weyl tensor must vanish on  $\mathcal{I}$ :

$$C_{abcd} \hat{=} 0. \quad (4.3)$$

Note, however, that since this equality holds only on the 3-dimensional surface  $\mathcal{I}$ , it does *not* imply that the metric  $g_{ab}$  —or even the metric  $q_{ab}$  it induces on  $\mathcal{I}$ — is conformally flat.

4. *Only one component,  $\Omega^{-1}\hat{T}_{ab}n^an^b$  is non-zero in the limit.*

This is a consequence of Bianchi identities. Consider the contracted Bianchi identity

$$\nabla^d C_{abcd} + \nabla_{[a} S_{b]c} = 0. \quad (4.4)$$

Using the definition  $K_{abcd} = \Omega^{-1} C_{abcd}$  of the asymptotic Weyl curvature, it therefore follows that

$$\lim_{\rightarrow\mathcal{I}} \left[ \nabla_{[a} S_{b]c} + \Omega^{-1} C_{abcd} n^d \right] \hat{=} 0. \quad (4.5)$$

(2.2) now immediately implies that

$$\lim_{\rightarrow\mathcal{I}} \left[ \Omega^{-1} \nabla_{[a} \left( \Omega \tilde{T}_{b]c} \right) - \Omega^{-1} g_{c[a} \tilde{T}_{b]d} n^d \right] \hat{=} 0. \quad (4.6)$$

Our boundary conditions on the physical stress energy tensor imply that  $\Omega^{-1}\hat{T}_{ab}$  has a smooth limit to  $\mathcal{I}$ . Therefore, by writing (4.6) in terms of quantities that have a limit to  $\mathcal{I}$  we find that

$$\lim_{\rightarrow\mathcal{I}} \left[ 2n_{[a} \Omega^{-1} \hat{T}_{b]c} - g_{c[a} \Omega^{-1} \hat{T}_{b]d} n^d - \Omega^{-1} \hat{T}_{pq} g^{pq} n_{[a} g_{b]c} \right] \hat{=} 0. \quad (4.7)$$

The projections of this equation along  $n^an^c q_m^b$  and  $n^a q_m^b q_n^c$  will have direct implications to our later discussion:

$$\lim_{\rightarrow\mathcal{I}} \Omega^{-1} \hat{T}_{ab} n^a q_m^b \hat{=} 0, \quad (4.8)$$

$$\lim_{\rightarrow\mathcal{I}} \Omega^{-1} \hat{T}_{ab} q_m^a q_n^b \hat{=} 0. \quad (4.9)$$

Thus, while our basic definition only asked that  $\Omega^{-1}\hat{T}_{ab}$  should have a smooth

limit to  $\mathcal{I}$ , Einstein's equations and Bianchi identities imply that only one component,  $\Omega^{-1}\hat{T}_{ab}n^an^b$ , of this limit can be non-zero.

5. *Conformal flatness of  $\mathcal{I}$  implies magnetic part of leading order Weyl tensor vanishes on  $\mathcal{I}$ .*

The definition of *strongly asymptotically de Sitter space-times* requires that the intrinsic metric  $q_{ab}$  on  $\mathcal{I}$  be conformally flat. Note that because the extrinsic curvature  $k_{ab}$  of  $\mathcal{I}$  vanishes in our choice of conformal frame, we can easily express the Riemann tensor  $\mathcal{R}_{abcd}$  of the 3-metric  $q_{ab}$  in terms of the Riemann tensor  $R_{abcd}$  of  $g_{ab}$  as follows

$$\mathcal{R}_{abcd} \hat{=} q_a{}^k q_b{}^l q_c{}^m q_d{}^n R_{klmn}. \quad (4.10)$$

Therefore, using the relation between Riemann and Weyl tensors and the equation  $C_{abcd} \hat{=} 0$ , the Ricci-tensor  $\mathcal{R}_{ab}$  of  $q_{ab}$  can be expressed in terms of the Schouten tensor  $S_{ab}$  as

$$\mathcal{R}_{ab} - \frac{1}{4}\mathcal{R}q_{ab} \hat{=} \frac{1}{2}q_a{}^m q_b{}^n S_{mn}. \quad (4.11)$$

Recall that the metric  $q_{ab}$  is conformally flat if and only if its (Cotton or) Bach tensor  $B_{abc} := D_{[a}(\mathcal{R}_{b]c} - \frac{1}{4}q_{b]c}\mathcal{R})$  vanishes. We can now relate  $B_{abc}$  to the asymptotic Weyl curvature as follows:

$$B_{abc} \hat{=} \frac{1}{2}q_a{}^m q_b{}^n q_c{}^l D_{[m}S_{n]l} \hat{=} -\frac{1}{2}q_a{}^m q_b{}^n q_c{}^l K_{mnlp}n^p, \quad (4.12)$$

where, in the last step we have used (2.1) and as before, denoted the leading order Weyl curvature at  $\mathcal{I}$  by

$$K_{abcd} := \Omega^{-1}C_{abcd}. \quad (4.13)$$

It is easy to check that the right side of (4.12) vanishes if and only if the leading order magnetic part

$$\mathcal{B}^{ac} := {}^*K^{abcd}\hat{n}_b\hat{n}_d = \frac{3}{\Lambda}{}^*K^{abcd}n_b n_d \quad (4.14)$$

vanishes, where  $\hat{n}^a$  is the unit future pointing normal to  $\mathcal{I}$ . Thus, the addi-

tional restriction that  $q_{ab}$  be conformally flat is equivalent to demanding that the asymptotic Weyl curvature  $K_{abcd}$  at  $\mathcal{I}$  have no magnetic part. Now, in electrodynamics, a restriction to Maxwell fields  $F_{ab}$  whose magnetic parts  $B_a$  vanish on a given space-like surface would be severe, as it would cut the space of allowable Maxwell fields by half. We will see in section 4.4, and in Chapter 5, that the situation is similar in the gravitational case; the requirement of conformal flatness of  $q_{ab}$  severely restricts permissible space-times and this restriction has no physical justification.

## 4.2 Examples

We now discuss several examples of strongly asymptotically de Sitter space-times, both stationary and dynamical, in which the three topologies of  $\mathcal{I}$  discussed in section 4.1 are realised.

### 4.2.1 de Sitter space-time

The simplest example of Definition 2 is provided by de Sitter space-time itself. We will discuss it briefly, mainly to compare and contrast with other examples.

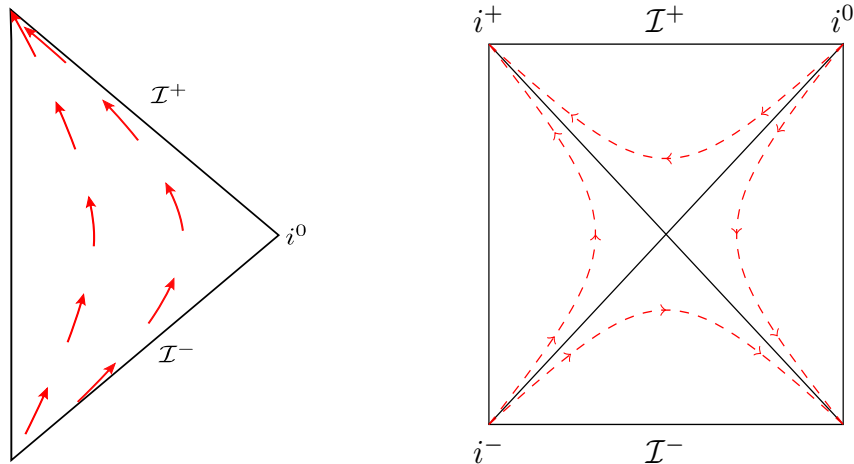
The standard ‘global coordinates’  $\tau, \chi, \theta, \phi$  on de Sitter space-time are introduced using the fact that de Sitter space-time can be realized as the unit time-like hyperboloid in 5-dimensional Minkowski space-time, and one can express its metric as:

$$d\hat{s}^2 = -d\tau^2 + \ell^2 \cosh^2\left(\frac{\tau}{\ell}\right) \left(d\chi^2 + \sin^2\chi d\omega_2^2\right), \quad (4.15)$$

where  $d\omega_2^2$  denotes the unit 2-sphere metric and  $\ell^2 = 3/\Lambda$ . In these coordinates, de Sitter space-time is foliated by spatial sections  $\tau = \text{const}$  that are round 3-spheres. Surfaces of future and past infinity  $\mathcal{I}^\pm$  correspond to  $\tau = \pm\infty$ , where the radius of the 3-sphere,  $\cosh(\tau/\ell)$  diverges. This suggests a natural choice of conformal factor for conformal completion,  $\Omega = [\cosh(\tau/l)]^{-1}$ . The line element using the rescaled metric then becomes

$$ds^2 = \Omega^2 d\hat{s}^2 = -\frac{\ell^2}{1 - \Omega^2} d\Omega^2 + \ell^2 (d\chi^2 + \sin^2\chi d\omega_2^2). \quad (4.16)$$

The conformally rescaled metric  $g_{ab}$  is manifestly well-defined at the boundary



**Figure 4.1.** Isometries near  $\mathcal{I}$  of Minkowski and de Sitter space-times. *Left Panel:* Minkowski space-time. The time translation Killing fields are time-like in a neighborhood of  $\mathcal{I}$  and null on  $\mathcal{I}$ . *Right Panel:* de Sitter space-time. Since  $\mathcal{I}$  is now space-like, all Killing fields of de Sitter are space-like near and on  $\mathcal{I}$ . The arrows represent a ‘time translation’ which changes its time-like versus space-like character across cosmological horizons.

$\Omega = 0$  which now has two disconnected components  $\mathcal{I}^\pm$ . Each of these components has topology  $\mathbb{S}^3$ , is endowed with a metric  $q_{ab}$  of a round sphere of radius  $\ell$  and is geodesically complete. It is easily checked that  $\nabla_a \Omega$  is non-zero at these boundaries because  $g^{ab} \nabla_a \Omega \nabla_b \Omega \hat{=} -\ell^{-2}$ , and as expected, the boundaries are space-like. Furthermore, since  $\hat{g}_{ab}$  is a metric of constant curvature, i.e.,  $\hat{R}_{ab} = (3/\ell^2)\hat{g}_{ab}$ , the stress-energy tensor  $\hat{T}_{ab}$  as well as the Weyl tensor  $\hat{C}_{abc}{}^d$  vanish identically everywhere in the physical space-time. Therefore, de Sitter space-time satisfies Definition 4 and, this completion makes it *strongly* (and globally) asymptotically de Sitter.

We conclude with a remark about the symmetries of this metric. Every global Killing field of the physical space-time admits an extension to the boundary and is tangential to  $\mathcal{I}$ .<sup>1</sup> Thus, in striking contrast to the case in Minkowski space-time, all ten de Sitter symmetries are space-like on  $\mathcal{I}^\pm$  (see Fig. 4.1). In particular, Minkowski space-time admits a time-like Killing field which is null and future pointing on  $\mathcal{I}^+$ , which mandates that energy flux of a matter field at  $\mathcal{I}^+$  is nec-

<sup>1</sup>Consider a Killing vector field  $\xi$  of the physical space-time metric  $\hat{g}_{ab}$  such that  $\mathcal{L}_\xi \hat{g}_{ab} = 0$ . Rewriting this in terms of the rescaled metric  $g_{ab} = \Omega^2 \hat{g}_{ab}$  one obtains  $\Omega^{-1} \hat{g}_{ab} \mathcal{L}_\xi \Omega = 0$ . Thus  $\mathcal{L}_\xi \Omega = 0$  everywhere in space-time, and in particular on  $\mathcal{I}$ .



and  $M$  is the Schwarzschild mass. Set  $\Omega = \ell/r$  (so it is dimensionless) and consider the conformally rescaled metric:

$$ds^2 := \Omega^2 d\hat{s}^2 = - \left( \Omega^2 - \frac{2M}{\ell} \Omega^3 - 1 \right) dt^2 + \frac{\ell^2 d\Omega^2}{\Omega^2 - \frac{2M}{\ell} \Omega^3 - 1} + \ell^2 d\omega_2^2. \quad (4.19)$$

Since the rescaled space-time metric  $g_{ab}$  is well-defined at  $\Omega = 0$  we can extend the physical space-time manifold  $\hat{M}$  to a manifold  $M$  by attaching the  $\Omega = 0$  surface which represents  $\mathcal{I}^+$ . Since  $g^{ab} \nabla_a \Omega \nabla_b \Omega \hat{=} -\ell^{-2}$ ,  $\nabla_a \Omega$  is non-zero at  $\mathcal{I}^+$  and  $\mathcal{I}^+$  is space-like. Furthermore,  $\mathcal{I}$  is topologically  $\mathbb{S}^2 \times \mathbb{R}$  because it is coordinatized by  $(r, \theta, \phi)$ . The intrinsic metric  $q_{ab}$  on  $\mathcal{I}^+$  is given by:

$$q_{ab} dx^a dx^b = dt^2 + \ell^2 d\omega_2^2. \quad (4.20)$$

Since  $t \in (-\infty, \infty)$ , it is clear that  $(\mathcal{I}^+, q_{ab})$  is geodesically complete. The end  $t = -\infty$  represents  $i^o$  while the end  $t = \infty$  represents  $i^+$ . We see explicitly that all the four Killing fields of  $\hat{g}_{ab}$  are tangential to  $\mathcal{I}^+$ , as they must be on general grounds. In particular, the generator of the ‘time-translation’ symmetry is *space-like* on  $\mathcal{I}^+$ , and indeed in the entire neighborhood of  $\mathcal{I}^+$  that the ‘static’ coordinates cover. Once again, this is in striking contrast to the  $\Lambda = 0$  case where  $\partial/\partial t$  Killing vector field is null on  $\mathcal{I}$  and time-like in its neighborhood covered by the static chart. However, in the  $\Lambda > 0$  case we will show in section 4.4.3 that the conserved quantity associated with  $\partial/\partial t$  is again the Schwarzschild mass  $M$ .

Finally, since  $\hat{g}_{ab}$  is a solution to source-free Einstein’s equations, the condition on the stress energy tensor in Definition 2 is trivially satisfied. Thus, we have obtained a conformal completion of  $(\hat{M}, \hat{g}_{ab})$  in which it is asymptotically Schwarzschild-de Sitter. Furthermore,  $q_{ab}$  can be recast as a conformally flat metric explicitly:

$$q_{ab} dx^a dx^b = \frac{\ell^2}{\tau^2} \left( d\tau^2 + \tau^2 d\omega_2^2 \right) \quad (4.21)$$

where  $\tau = e^{t/\ell}$ . Hence the space-time is also *strongly* asymptotically de Sitter.

*Remark:* Note that if we set  $M = 0$ , the physical metric  $\hat{g}_{ab}$  reduces to the de Sitter metric. Therefore, if we again use the conformal factor  $\Omega = \ell/r$ , the topology of  $\mathcal{I}^+$  would be  $\mathbb{S}^2 \times \mathbb{R}$  and the completion would be conceptually different from that considered in section 4.2.1. This is because the static coordinates underlying this

construction —and hence the conformal completion we obtain by setting  $\Omega = \ell/r$ — are tied to a specific ‘time translation’ Killing field  $\partial/\partial t$ . In de Sitter space-time, of course, there is no preferred rest frame; we have a 10-parameter group of isometries. In particular the ends  $i^+$  and  $i^o$  of  $\mathcal{I}^+$  of this conformal completion are not left invariant by the full isometry group, whence this completion is unnatural from the perspective of the full structure of the de Sitter space-time.

### 4.2.3 The Kerr-de Sitter solution

The discussion of the rotating black hole solution closely parallels that of the non-rotating case of the previous example and is conceptually the same. However, the expressions involved are more complicated. In Boyer-Lindquist type coordinates the physical metric is given by [51, 52]:

$$\begin{aligned} d\hat{s}^2 &= (r^2 + a^2 \cos^2 \theta) \left[ \frac{dr^2}{\Delta_r} + \frac{d\theta^2}{1 + \frac{a^2}{\ell^2} \cos^2 \theta} \right] + \sin^2 \theta \frac{1 + \frac{a^2}{\ell^2} \cos^2 \theta}{r^2 + a^2 \cos^2 \theta} \left[ \frac{adt - (r^2 + a^2)d\phi}{1 + \frac{a^2}{\ell^2}} \right]^2 \\ &\quad - \frac{\Delta_r}{r^2 + a^2 \cos^2 \theta} \left[ \frac{dt - a \sin^2 \theta d\phi}{1 + \frac{a^2}{\ell^2}} \right]^2 \end{aligned} \quad (4.22)$$

where  $\Delta_r = -\frac{r^4}{\ell^2} + (1 - \frac{a^2}{\ell^2})r^2 - 2Mr + a^2$ . In the limit  $a \rightarrow 0$ , one recovers the Schwarzschild-de Sitter metric as expected. Once again we focus on a future asymptotic region and choose the conformal factor  $\Omega = \ell/r$ . In the  $(t, \Omega, \theta, \phi)$  coordinates the conformally rescaled metric is then given by:

$$\begin{aligned} ds^2 &= \Omega^2 d\hat{s}^2 \\ &= \ell^2 \left( 1 + \Omega^2 \frac{a^2}{\ell^2} \cos^2 \theta \right) \left[ \frac{d\Omega^2}{-1 + \left( 1 - \frac{a^2}{\ell^2} \right) \Omega^2 - \frac{2M}{\ell} \Omega^3 + \frac{a^2}{\ell^2} \Omega^4} + \frac{d\theta^2}{1 + \frac{a^2}{\ell^2} \cos^2 \theta} \right] \\ &\quad + \sin^2 \theta \frac{1 + \frac{a^2}{\ell^2} \cos^2 \theta}{1 + \Omega^2 \frac{a^2}{\ell^2} \cos^2 \theta} \left[ \frac{\Omega^2 \frac{a}{\ell} dt - \ell \left( 1 + \Omega^2 \frac{a^2}{\ell^2} \right) d\phi}{1 + \frac{a^2}{\ell^2}} \right]^2 \\ &\quad - \frac{-1 + \left( 1 - \frac{a^2}{\ell^2} \right) \Omega^2 - \frac{2M}{\ell} \Omega^3 + \frac{a^2}{\ell^2} \Omega^4}{1 + \Omega^2 \frac{a^2}{\ell^2} \cos^2 \theta} \left[ \frac{dt - a \sin^2 \theta d\phi}{1 + \frac{a^2}{\ell^2}} \right]^2. \end{aligned} \quad (4.23)$$

The rescaled metric,  $g_{ab}$ , is manifestly well defined when  $\Omega = 0$  i.e., on  $\mathcal{I}^+$ ,  $g^{ab}\nabla_a\Omega\nabla_b\Omega \hat{=} -\ell^{-2}$ , so  $\nabla_a\Omega$  is nowhere vanishing on  $\mathcal{I}^+$  and  $\mathcal{I}^+$  is space-like. The intrinsic 3-metric  $q_{ab}$  at  $\mathcal{I}^+$  is now given by:

$$q_{ab} dx^a dx^b \hat{=} \frac{1}{\left(1 + \frac{a^2}{\ell^2}\right)^2} dt^2 - \frac{2a \sin^2 \theta}{\left(1 + \frac{a^2}{\ell^2}\right)^2} dt d\phi + \frac{\ell^2}{1 + \frac{a^2}{\ell^2} \cos^2 \theta} d\theta^2 + \frac{\ell^2 \sin^2 \theta}{1 + \frac{a^2}{\ell^2}} d\phi^2, \quad (4.24)$$

and is conformally flat because its Bach tensor vanishes. In this completion, Kerr-de Sitter is asymptotically Schwarzschild-de Sitter and *strongly* asymptotically de Sitter. We will see in section 4.4.3 that the conserved quantity associated with the Killing field  $\partial/\partial t$  is  $(1 + (a^2/\ell^2))^{-2}M$  and that associated with  $\partial/\partial\phi$  is  $-(1 + (a^2/\ell^2))^{-2}Ma$ .

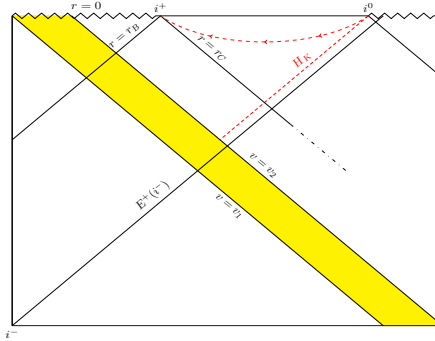
#### 4.2.4 The Vaidya-de Sitter solution

While the examples considered so far are important as they represent physically interesting equilibrium configurations of isolated systems, they do not encode dynamics. The simplest dynamical example is the collapse of a spherically symmetric null fluid to form a black hole, or its time reverse, the evaporation of a white hole through emission of a spherically symmetric null fluid, described by the Vaidya-de Sitter solutions [53]. While these processes are over-idealized from an astrophysical perspective, the example is conceptually interesting because it offers the first glimpses of the effects of non-trivial dynamics on the structure of cosmological horizons and asymptotic symmetries without recourse to numerical simulations. These lessons will be important for the later part of this program dealing with general isolated systems in full, non-linear general relativity [46].

Here we will focus on a Vaidya-de Sitter solution that describes black hole formation (see Fig. 4.3). In terms of the advanced null coordinate  $v$  and the spherical coordinates  $r, \theta, \phi$ , the metric can be expressed as

$$d\hat{s}^2 = -\left(1 - \frac{2M(v)}{r} - \frac{r^2}{\ell^2}\right)dv^2 + 2dvdr + r^2 d\omega_2^2, \quad (4.25)$$

where the mass function  $M(v)$  has the following properties:  $M(v) = 0$  for  $v < v_1$ , it increases monotonically from zero to a value  $M$  during the interval  $v_1 \leq v \leq v_2$ ,



**Figure 4.3.** Conformal diagram of the Vaidya-de Sitter space-time describing the gravitational collapse of an in-falling null fluid from  $\mathcal{I}^-$ . The in-falling null fluid is indicated by the shaded (yellow) region  $v_1 \leq v \leq v_2$ . In contrast to the asymptotically flat case, now the dynamical nature of the space-time geometry modifies the structure even at  $\mathcal{I}^+$ . While in the Schwarzschild-de Sitter space-time the Killing and the cosmological horizon  $E^+(i^-)$  coincide near  $\mathcal{I}^+$  they are now distinct; one intersects  $\mathcal{I}^+$  and the other meets the singularity. Furthermore, the natural identification that allowed us to consider a single black hole (and the associated white hole) in the Schwarzschild-de Sitter case is no longer available.

and  $M(v) = M$  for  $v > v_2$ .  $\hat{g}_{ab}$  is a solution to Einstein's equation in presence of a stress-energy tensor

$$\hat{T}_{ab} = \frac{\dot{M}}{4\pi r^2} \nabla_a v \nabla_b v, \quad (4.26)$$

where  $\dot{M} = dM/dv$ . The space-time naturally splits into three regions: de Sitter before collapse ( $v < v_1$ ), dynamical region during collapse ( $v_1 \leq v \leq v_2$ ), and Schwarzschild-de Sitter after collapse ( $v_2 < v$ ). The physical metric  $\hat{g}_{ab}$  is spherically symmetric in all three regions. What is the structure of  $\mathcal{I}$ ? Since in a neighborhood of  $\mathcal{I}^+$  the physical space-time is isometric to the Schwarzschild-de Sitter space-time of mass  $M$ , the structure of  $\mathcal{I}^+$  is the same as that in section 4.2.2.

On the other hand, because of the incoming radiation, the structure of  $\mathcal{I}^-$  is different from that in the Schwarzschild-de Sitter space-time. Let us discuss it in some detail. One can again carry out a conformal completion using  $\Omega = l/r$  to attach  $\mathcal{I}^-$  as the (past) boundary to the physical space-time. The form (4.26) of the stress-energy tensor implies that not only does  $\Omega^{-1}\hat{T}_{ab}$  admit a limit to  $\mathcal{I}^-$ , but that the limit is in fact zero, in spite of the incoming radiation. Furthermore, although the 'time translation' Killing field does not extend to the dynamical

region, the affine parameter  $t$  of the space-like geodesics orthogonal to the three rotational Killing vectors again runs from  $t = -\infty$  to  $t = \infty$ . Therefore  $\mathcal{I}^-$  is also geodesically complete. Next, the asymptotic Weyl curvature  $K_{abcd}$  vanishes on the portion of  $\mathcal{I}^-$  with  $v < v_1$ , and an explicit calculation shows that it has only a non-zero electric part  $\mathcal{E}^{ab}$  for  $v > v_1$ . Thus, in spite of the incoming radiation from  $\mathcal{I}^-$ , the conformal completion with  $\Omega = l/r$  endows the Vaidya space-time with the structure of a strongly asymptotically de Sitter space-time with  $\mathcal{I}^-$  of the asymptotically Schwarzschild-de Sitter type.

The dynamical nature of  $\hat{g}_{ab}$  has an interesting consequence for the causal structure of space-time, which will be important to our framework describing general isolated systems in presence of a positive  $\Lambda$  [46]. Let us first examine space-time geometry in the non-dynamical examples discussed in the last two sub-sections. Of particular interest are the event horizons associated with  $i^\mp$ . Their structure is the same in de Sitter and Schwarzschild-de Sitter space-times: the future horizon  $E^+(i^-)$  of  $i^-$  and the past horizon  $E^-(i^+)$  of  $i^+$  are both null 3-surfaces and Killing horizons for the ‘time translation Killing field’,  $\partial/\partial t$ . (In the de Sitter space-time, this is the ‘time translation’ adapted to the chosen points  $i^\mp$  on  $\mathcal{I}^\mp$ .) These Killing horizons intersect at a bifurcate horizon 2-surface where the ‘time-translation’ Killing vector field vanishes. In our dynamical example, the past event horizon  $E^-(i^+)$  of  $i^+$  is again a Killing horizon because it lies entirely in the  $v > v_2$  region. But since  $E^+(i^-)$  does intersect the dynamical region, it is no longer a Killing horizon to the future of the  $v = v_1$  surface. In the region  $v > v_2$  we do have a Killing horizon  $H_K$  for the ‘time-translation’ of the Schwarzschild-de Sitter metric. Like  $E^+(i^-)$ , it is a null surface. *However,  $H_K$  lies strictly to the future of  $E^+(i^-)$*  (see Fig. 4.3). The Killing field is now *transversal* to the portion of  $E^+(i^-)$  that lies to the future of  $v = v_2$ . This split between  $E^+(i^-)$  and the Killing horizon  $H_K$  has interesting implications for the conceptual framework describing non-stationary isolated systems with positive  $\Lambda$  both in the classical [46] and quantum regimes.

### 4.2.5 Friedmann-Lemaître cosmology

The notion of asymptotically de Sitter space-times is useful not only to the study of isolated systems but also in the description of the late time behavior of the

universe in Friedmann-Lemaître cosmology with positive  $\Lambda$ . In this case, Einstein's equations inform us that if matter obeys the strong energy condition, then the result of the expansion is that the cosmological constant dominates at late times, irrespective of how small its value is. Current observations imply that today the source of the Hubble parameter has two predominant components, modeled by dust ( $\sim 30\%$ ) and  $\Lambda$  ( $\sim 70\%$ ). Since the expansion of the universe dilutes the energy density of dust as  $a^{-3}$  (where  $a$  is the co-moving scale factor) and the energy density of  $\Lambda$  remains constant, given sufficient time, the universe will be naturally driven to become an asymptotically de Sitter space-time. We will now make these qualitative considerations more precise.

Let us first use Einstein's equations together with observational inputs to construct a space-time metric to describe the late stages of the large scale dynamics of our universe. For spatially flat,  $k = 0$  universe, the physical metric has the form

$$d\hat{s}^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2) \quad (4.27)$$

and Einstein's equations allow us to relate the matter content of the universe with the time dependence of the scale factor  $a$ :

$$H_0(t - t_\star) = \int_{a_\star}^{a(t)} \frac{d\tilde{a}}{\sqrt{\frac{\Omega_{r,0}a_0^4}{\tilde{a}^2} + \frac{\Omega_{d,0}a_0^3}{\tilde{a}} + \Omega_{\Lambda,0}\tilde{a}^2 + (1 - \Omega_c)}} \quad (4.28)$$

Here  $H = \dot{a}/a$  is the Hubble parameter; the subscript 0 refers to today's values and the subscript  $\star$  to values at any chosen 'initial' time; and  $\Omega_i = (8\pi G/3H^2) \rho_i$  are the fractional density parameters, with the subscript  $r$  referring to radiation,  $d$  to dust, and  $c$  to the critical density  $\rho_c := (3H^2/8\pi G)$ . Using the following approximate values of the density parameters:  $\Omega_{r,0} \sim 0$ ,  $\Omega_{d,0} \sim 0.3$ ,  $\Omega_{\Lambda,0} \sim 0.7$  and  $\Omega_c \sim 1$ ,<sup>2</sup> and setting  $t_\star \gg t_0$ , we obtain

$$H_0(t - t_\star) = \frac{2}{3\sqrt{0.7}} \ln \left[ \frac{\sqrt{0.7a^3} + \sqrt{0.3a_0^3 + 0.7a^3}}{\sqrt{0.7a_\star^3} + \sqrt{0.3a_0^3 + 0.7a_\star^3}} \right]. \quad (4.29)$$

Since the expansion is dominated by  $\Lambda$  at late times, the leading-order behavior of the scale factor is expected to be exponential. This is borne out by inverting

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<sup>2</sup>More accurate values for these parameters can be obtained from the results of Planck [13], but our conclusions do not change conceptually.

(4.29) to obtain the scale factor as a function of time as follows:

$$a(t) = \frac{e^{-\beta\Delta t}}{(2.8)^{1/3}} [0.3a_0^3(1 - e^{3\beta\Delta t})^2 + 1.4a_\star^3(1 + e^{6\beta\Delta t}) - 2\sqrt{(0.7a_\star^3)(0.3a_0^3 + 0.7a_\star^3)(1 - e^{6\beta\Delta t})}]^{1/3} \quad (4.30)$$

where the parameter  $\beta := \sqrt{0.7}H_0$  conveniently captures the residual effect of the presence of dust, and  $\Delta t := (t - t_\star)$ . At late times, the scale factor simplifies to:

$$a(t) \rightarrow \Gamma a_\star e^{\beta\Delta t} \quad (4.31)$$

where  $\Gamma = \left[ \frac{1}{2} + \frac{0.3a_0^3}{4 \cdot 0.7a_\star^3} + \frac{1}{2} \sqrt{1 + \frac{0.3a_0^3}{0.7a_\star^3}} \right]^{1/3}$ . Note that in the absence of dust we would have de Sitter space-time where  $\Gamma = 1$  and  $\beta = H_0$ .

For the conformal completion we wish to carry out, it is convenient to re-express the scale factor as a function of the conformal time  $\eta$ , related to the co-moving time  $t$  via  $d\eta = dt/a$ :

$$a(\eta) = -\frac{1}{\beta\eta}. \quad (4.32)$$

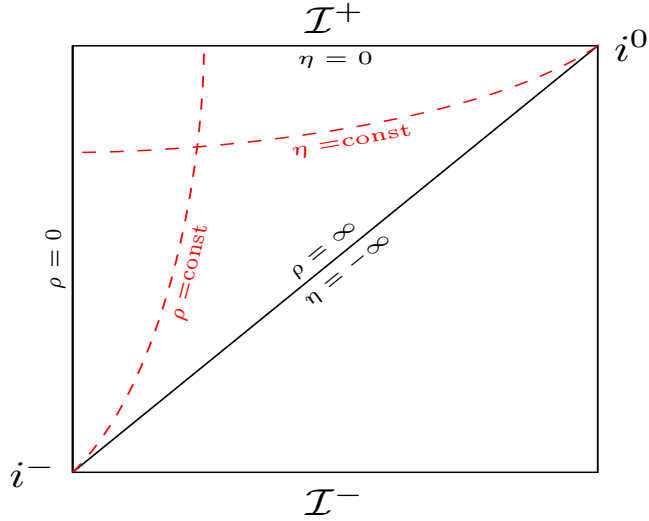
Thus, at late times, we can write the physical metric as

$$d\hat{s}^2 = \frac{1}{\beta^2\eta^2} \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right). \quad (4.33)$$

We can now carry out the conformal completion to verify if conditions in Definition 3 are met. The form (4.33) suggests that we set  $\Omega = -\beta\eta$  so that the conformally rescaled metric is given by:

$$ds^2 = -\beta^{-2}d\Omega^2 + dx^2 + dy^2 + dz^2. \quad (4.34)$$

On  $\mathcal{I}$ , where  $\Omega = 0$ ,  $\nabla_a\Omega$  is non-zero because  $g^{ab}\nabla_a\Omega\nabla_b\Omega \hat{=} -\beta^2$ . The physical metric  $\hat{g}_{ab}$  satisfies Einstein's equations with  $\hat{T}_{ab} = \hat{\rho}\hat{u}_a\hat{u}_b$  with  $\hat{u}^a$  the unit 4-velocity of a co-moving observer and  $\hat{\rho} = \hat{\rho}_d = 0.3\hat{\rho}_{c,0}(a_0^3/a(t)^3)$ . Therefore, in terms of fields which have well-defined limits to  $\mathcal{I}^+$ , we have  $\Omega^{-1}\hat{T}_{ab} = \text{const } u_a u_b$  (with  $g^{ab}u_a u_b = -1$ ), which admits a smooth limit to  $\mathcal{I}^+$ . Finally, by inspection, the



**Figure 4.4.** Conformal diagram of the Friedmann-Lemaître space-time with positive  $\Lambda$ . This space-time corresponds only to the Poincaré patch of de Sitter space-time because of the big-bang singularity along the event horizon  $E^+(i^-)$  (i.e.,  $\eta = -\infty$ , where  $\eta$  is the conformal time).

induced metric on  $\mathcal{I}^+$  is:

$$q_{ab}dx^a dx^b = dx^2 + dy^2 + dz^2. \quad (4.35)$$

Hence  $\mathcal{I}^+$  is geodesically complete and (conformally) flat. The fact that it is coordinatized by  $x, y, z$  shows that its topology is  $\mathbb{R}^3$ . Thus our conformal completion makes  $(\hat{M}, \hat{g}_{ab})$  *strongly asymptotically de Sitter in a Poincaré patch*. This is just what one would expect because, since the future event horizon  $E^+(i^-)$  of  $i^-$  in de Sitter space-time corresponds to the big bang singularity,  $(\hat{M}, \hat{g}_{ab})$  is conformally isometric to the expanding Poincaré patch, the ‘upper triangle’, of de Sitter space-time. Finally, if we set  $\hat{\rho} = 0$ , locally the solution reduces to the de Sitter space-time. Therefore we could have used  $\Omega = -\beta\eta$  also in that case. The resulting conformal completion would be inequivalent to the natural conformal completion discussed in section 4.2.1 where  $\mathcal{I}^+$  is topologically  $\mathbb{S}^3$  and includes the point  $i^o$  at spatial infinity.

Note that the ‘genuinely dynamical’ nature of this space-time distinguishes it from de Sitter space-time in three respects. First, as noted above, even though the conformally rescaled metric  $g_{ab}$  at  $\mathcal{I}^+$  is the same as that in de Sitter space-time,

the conformal factor  $\Omega = -\beta \eta = -\sqrt{0.7} H_0 \eta$  retains a memory of the matter content. Second difference is more important: whereas the area of the event horizon  $E^-(p)$  of any point  $p$  on  $\mathcal{I}^+$  is constant in de Sitter space-time, it increases monotonically as one approaches  $\mathcal{I}^+$  in the Friedmann-Lemaître space-time because of the matter content. Finally, this space-time has a big-bang singularity at  $t = 0$  or  $\eta = -\infty$ . Therefore it corresponds only to the ‘upper triangle’ or future Poincaré patch of de Sitter space-time (see Fig. 4.4).

We conclude with two remarks:

- 1) Recall from section 4.1.2 that although in Definition 2 the explicit requirement is only that  $\Omega^{-1}\hat{T}_{ab}$  should have a smooth limit to  $\mathcal{I}$ , field equations and Bianchi identities ensure that the space-space and space-time components of this limit vanish and only the time-time component can be non-zero. If the matter consists of Yang-Mills fields or a null fluid of the Vaidya-de Sitter solution, or radiation filled Friedmann-Lemaître cosmology, even this component vanishes. In the definition of asymptotically flat space-times one routinely requires that  $\Omega^{-2}\hat{T}_{ab}$  have a smooth limit to  $\mathcal{I}$ . We did not impose this stronger fall-off condition because in the dust filled Friedmann-Lemaître solution we just discussed, the limit of the time-time component of  $\Omega^{-1}\hat{T}_{ab}$  is smooth but non-zero.
- 2) We focused on the Friedmann-Lemaître model because it is used very widely in the contemporary cosmological literature. However, we expect that the early investigations by Wald [54] of Bianchi models will provide additional examples once appropriate restrictions are made on matter fields. We also expect that the much more general results obtained recently by Ringström [55] on absence of cosmological hair will provide a large class of examples satisfying our Definition 3 of asymptotically de Sitter space-times. However, a detailed analysis is necessary to relate the results obtained in these references in physical space-times to extract the precise asymptotic behavior of various fields after appropriate conformal completions.

### 4.3 Asymptotic symmetries

Given a conformally completed spacetime,  $(M, g_{ab})$ , asymptotic symmetries are those diffeomorphisms of the physical space-time  $(\hat{M}, \hat{g}_{ab})$  that preserve the bound-

ary conditions. The discussion for asymptotically de Sitter space-times will follow a structure similar to that for asymptotically flat space-times of Chapter 2. This will enable us to highlight the important differences and challenges that arise in the  $\Lambda > 0$  case.

### 4.3.1 Asymptotically de Sitter space-times

Analogous to the asymptotically flat case, the universal structure of  $\mathcal{I}$  consists of a pair consisting of an intrinsic 3-metric and a normal to  $\mathcal{I}$ ,  $(q_{ab}, n^a)$ . From Einstein's equation it follows that  $n^a$  is time-like and  $q_{ab}$  carries positive signature  $(+++)$  because  $\Lambda > 0$ . Thus, in contrast to the  $\Lambda = 0$  case,  $n^a$  is no longer tangential to  $\mathcal{I}$  for  $\Lambda > 0$ , and hence does not restrict the diffeomorphisms on  $\mathcal{I}$  in any way. The freedom available in the universe structure is that of conformal rescaling,  $q_{ab} \rightarrow q'_{ab} = \omega^2 q_{ab}$  and  $n^a \rightarrow n'^a = \omega^{-1} n^a$ . This appears to be the same freedom as in the asymptotically flat case, but there is an important difference. For  $\Lambda = 0$ , the intrinsic metric signature is  $(0++)$  on  $\mathbb{S}^2 \times \mathbb{R}$  topology; every metric on a 2-sphere is conformally equivalent to a constant curvature metric and the conformal group of a 2-sphere is isometric to the Lorentz group. But, for  $\Lambda > 0$ , there is no longer such a conformal structure available for 3-dimensional Riemannian metrics. For instance, if we fix the topology<sup>3</sup> of  $\mathcal{I}$  to be  $\mathbb{S}^3$ , the definitions of section 4.1 do not guarantee that the intrinsic 3-metric is conformally flat. Thus, a priori, the universal structure at  $\mathcal{I}$  would consist of all 3-metrics  $q_{ab}$  on  $\mathbb{S}^3$  of signature  $(+++)$  and the asymptotic symmetry group would be the group of all diffeomorphisms,  $\text{Diff}(\mathcal{I})$ . This group does not admit any preferred 'translation' and 'rotation' sub-groups that can be used to define unambiguous physical notions of energy-momentum and angular momentum. Thus, the rich structure made available by the BMS group  $\mathcal{B}$  in asymptotically flat space-times which allows the definition of charges and fluxes of physical quantities *disappears* once there is a positive  $\Lambda$ , *however small*.

### 4.3.2 Strongly asymptotically de Sitter space-times

We now consider space-times with conformally flat boundaries, as in Definition 4, or equivalently, the magnetic part  $\mathcal{B}^{ab}$  of whose leading order Weyl curvature is

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<sup>3</sup>The argument works for any of the three topologies of interest.

zero on  $\mathcal{I}$ . We will discuss the three topologies introduced in section 4.1.1 in turn.

#### 4.3.2.1 $\mathbb{S}^3$ topology

Now the available freedom of conformal transformation is restricted to the class of conformally flat intrinsic 3-metrics on  $\mathcal{I}$  with  $\mathbb{S}^3$  topology. The asymptotic symmetry group  $\mathfrak{G}$  is, thus, the de Sitter group  $G_{\text{dS}} \equiv \text{SO}(1, 4)$ . Conformal flatness of  $q_{ab}$  is a strong requirement that reduces the infinite dimensional  $\text{Diff}(\mathcal{I})$  to a 10-dimensional group. One can give a convenient description of  $\mathfrak{G}$  using the unit round metric  $\mathring{q}_{ab}$  in this conformal class:

$$\mathring{q}_{ab} dx^a dx^b = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.36)$$

where  $\chi \in [0, \pi]$ ,  $\theta \in [0, \pi]$  and  $\phi \in [0, 2\pi)$ . The Lie algebra  $\mathfrak{g}$  of  $\mathfrak{G}$  is spanned by conformal Killing fields on  $(\mathcal{I}, \mathring{q}_{ab})$ . The 6 Killing vectors of  $\mathring{q}_{ab}$  provide us with the Lie algebra  $\mathfrak{so}(4)$  of  $\text{SO}(4)$  which naturally splits into two  $\mathfrak{so}(3)$  sub-algebras:  $\mathfrak{so}(4) = \mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ . One can choose a basis in the 10-dimensional Lie algebra  $\mathfrak{so}(1, 4)$  such that the remaining 4-dimensional space is spanned by ‘pure’ conformal Killing fields  $C^a$  on  $(\mathcal{I}, \mathring{q}_{ab})$ , i.e., vector fields on  $\mathcal{I}$  which satisfy not only  $\mathcal{L}_C \mathring{q}_{ab} \hat{=} 2\alpha \mathring{q}_{ab}$  for some smooth function  $\alpha$  but also  $\mathring{D}_{[a} C_{b]} \hat{=} 0$ . (Recall that the Killing fields  $K^a$  satisfy  $\mathring{D}_{(a} K_{b)} = 0$ ; hence the terminology ‘pure’ conformal for the vector fields  $C^a$ .)

Let us embed  $(\mathcal{I}, \mathring{q}_{ab})$  as the unit 3-sphere in an abstractly defined 4-dimensional Euclidean space  $(\mathbb{R}^4, \mathring{e}_{IJ})$ . Then, it turns out that there is a natural 1-1 correspondence between the 10 Killing fields of  $(\mathbb{R}^4, \mathring{e}_{IJ})$  and elements of  $\mathfrak{g}$ . Each of the 6 Killing fields of  $\mathring{q}_{ab}$  is of course just the restriction of a rotational Killing field of  $(\mathbb{R}^4, \mathring{e}_{IJ})$  to  $\mathcal{I}$ , and is therefore labelled by a 2-form  $\mathring{k}_{IJ}$  on  $\mathbb{R}^4$ :

$$K^a = \mathring{q}^{aI} \mathring{k}_{IJ} x^J, \quad (4.37)$$

where  $\mathring{q}^{aI}$  is the projection operator on the unit 3-sphere and  $x^J$  are the position vectors of points in  $\mathbb{R}^4$ . The pure conformal Killing fields on  $\mathcal{I}$  turn out to be just the projections of the 4 translational Killing fields  $\mathring{c}_I$  on  $(\mathbb{R}^4, \mathring{e}_{IJ})$  to  $\mathcal{I}$ :

$$C^a = \mathring{q}^a_I \mathring{c}^I. \quad (4.38)$$

Thus, each form  $\mathring{k}_{IJ}$  on  $\mathbb{R}^4$  defines a Killing field  $K^a$  and each vector  $\mathring{c}^I$  defines a ‘pure’ conformal Killing field  $C^a$  on  $(\mathcal{I}, \mathring{q}_{ab})$ . The commutation relations are then given by:

$$\begin{aligned} [K, K'] &= K'', \quad \text{where} \quad \mathring{k}''^J = \mathring{k}_I{}^L \mathring{k}'^J - \mathring{k}'^L \mathring{k}_L{}^J \\ [K, C] &= C', \quad \text{where} \quad \mathring{c}'_I = -\mathring{k}_{IJ} \mathring{c}^J, \text{ and} \\ [C, C'] &= K, \quad \text{where} \quad \mathring{k}_{IJ} = 2 \mathring{c}'_{[I} \mathring{c}_{J]}. \end{aligned} \tag{4.39}$$

This is a convenient basis for calculations. For example, the 4 ‘pure’ conformal Killing fields  $C^a$  are generally taken to represent ‘translations’ in  $\mathfrak{g}$ , and are used to define the de Sitter energy-momentum, analogs of the more familiar energy-momentum 4-vectors in asymptotically flat space-times. However, note that while the first two brackets in (4.39) mimic the familiar commutation relations between rotations and between rotations and translations, the last bracket does not: while the 4 translations  $\mathring{c}^I$  commute on  $(\mathbb{R}^4, \mathring{e}_{IJ})$ , the 4 ‘pure’ conformal Killing fields  $C^a$  in  $\mathfrak{g}$  do not.

### 4.3.2.2 $\mathbb{R}^3$ topology

As we just saw, when the topology of  $\mathcal{I}$  is  $\mathbb{S}^3$ , the full de Sitter group  $G_{\text{dS}}$  constitutes the asymptotic symmetry group  $\mathfrak{G}$ . For other topologies, the local structure is the same. In particular, when  $\mathcal{B}^{ab} \hat{=} 0$ , the intrinsic metric on  $\mathcal{I}$  is again conformally flat and, given any conformal completion, we are led to consider the 10 conformal Killing fields of  $\mathring{q}_{ab}$ . Recall however that in Definition 3 of asymptotically de Sitter space-times we also required that  $(\mathcal{I}, q_{ab})$  be complete. Therefore the question is if the 3-manifold  $\mathcal{I}$  we began with continues to be complete also with respect to the image of  $q_{ab}$  under diffeomorphisms generated by all 10 conformal Killing fields. When  $\mathcal{I}$  is topologically  $\mathbb{S}^3$  as in section 4.3.2.1, this is assured by compactness of  $\mathcal{I}$ . But in non-compact topologies this issue has to be analyzed case by case. The asymptotic symmetry group can be smaller if  $\mathcal{I}$  fails to remain complete with respect to the image of  $q_{ab}$  under some conformal isometries. We will find that this does happen. When the topology is  $\mathbb{R}^3$ , the group is reduced to a 7-dimensional sub-group and when it is  $\mathbb{S}^2 \times \mathbb{R}$  to a 4-dimensional sub-group.

Before entering the calculations of completeness, it is instructive to return to the example of de Sitter space-time with a conformal completion with  $\mathbb{R}^3$  topology

on  $\mathcal{I}$  (as in section 4.2.1). We restrict to the future Poincaré patch of de Sitter space-time, i.e., on the causal future of a chosen point  $i^-$  on  $\mathcal{I}^-$ , which represents the past time-like infinity of a family of observers in de Sitter space-time (see Fig. 4.4). This patch is covered by coordinates  $(\eta, x, y, z)$  where  $(x, y, z) \in (-\infty, \infty)$  and the conformal time  $\eta \in (-\infty, 0)$ . The past boundary of this region is the event horizon  $E^+(i^-)$  of  $i^-$  which is *not* part of the Poincaré patch because  $\eta = -\infty$  there. Now, since the metric in the Poincaré patch is just the de Sitter metric, *locally* it admits 10 Killing fields. However, since the Poincaré patch is only a part of the de Sitter space-time, only those symmetries are permissible that map the Poincaré patch to itself, i.e., those that are tangential to  $E^+(i^-)$ .

Geometrically, it is simplest to characterize this restriction by embedding de Sitter space-time as a hyperboloid  $\mathcal{H}$  in a 5-dimensional Minkowski space  $\mathcal{M}_5$ . The 10 Killing fields of de Sitter space-time are just the Lorentz generators in  $\mathcal{M}_5$  and the event horizon  $E^+(i^-)$  is realized as the intersection of  $\mathcal{H}$  with a 4-dimensional null hyperplane  $\mathcal{N}$  passing through the origin of Minkowski space  $\mathcal{M}_5$ . Therefore, the isometry group of the Poincaré patch is generated by those Lorentz Killing fields in  $\mathcal{M}_5$  that are tangential to  $\mathcal{N}$ . This is a 7-dimensional sub-group of  $G_{\text{dS}}$ . In the Poincaré patch the generators are given by the three space-translations and three 3-rotations on the  $\eta = \text{const}$  surfaces and the ‘dilation’ (We will refer to  $T^a$  as the de Sitter *time translation* for reasons explained in Chapter 5.)

$$T = -\frac{1}{\ell} \left[ \eta \frac{\partial}{\partial \eta} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right]. \quad (4.40)$$

Therefore, one would anticipate that when  $\mathcal{B}^{ab} \triangleq 0$ , the asymptotic symmetry group  $\mathfrak{G}$  of space-times that are *asymptotically de Sitter in a Poincaré patch* would not be the full 10-dimensional  $G_{\text{dS}}$  but rather this 7-dimensional sub-group thereof. We will now use completeness of  $\mathcal{I}$  to arrive at this result using only the intrinsic structure at  $\mathcal{I}$ .

Let us then consider *any* strongly asymptotically de Sitter space-time in which  $\mathcal{I}$  has  $\mathbb{R}^3$  topology. Since  $\mathcal{I}$  is equipped with a class of conformally flat metrics  $q_{ab}$ , it is now convenient to work with a flat metric  $\hat{q}_{ab}$  in this class and the associated set of Cartesian coordinates  $(x_i = x, y, z)$  with  $(i = 1, 2, 3)$ . We can then introduce a convenient basis in the 10-dimensional Lie algebra of  $G_{\text{dS}}$ : 3 translations  $T_i^a$  with  $(i = 1, 2, 3)$ , 3 rotations  $R_i^a$ , 1 dilation  $T^a$  and 3 ‘inverted translations’  $\tilde{T}_i^a$  (which

are also known as ‘special conformal transformations’ in cosmology literature). The dilation is just the extension to  $\mathcal{I}$  of the vector field (4.40) and satisfies  $\mathcal{L}_T \mathring{q}_{ab} = 2 \mathring{q}_{ab}$ . The inverted translations  $\tilde{T}^a$  are:

$$\tilde{T}_i^a := 2(T_{ib}x^b) x^a - (x_b x^b) T_i^a \quad \text{satisfying} \quad \mathcal{L}_{\tilde{T}_i} \mathring{q}_{ab} = 4(T_{ic}x^c) \mathring{q}_{ab}, \quad (4.41)$$

$x^a \partial_a := \Sigma_i x_i \partial_{x_i}$  is the spatial dilation field of  $\mathring{q}_{ab}$ . The question is if these 10 conformal Killing fields of  $\mathring{q}_{ab}$  preserve completeness of  $\mathcal{I}$ . That is, to begin with, we know that  $\mathcal{I}$  is complete with respect to the given conformal class of metrics  $q_{ab}$  (to which  $\mathring{q}_{ab}$  belongs). Under the 1-parameter family  $d_V(\lambda)$  of diffeomorphisms generated by any of these 10 conformal Killing fields  $V^a$  we have:  $q_{ab} \rightarrow q_{ab}(\lambda) = d_V^*(\lambda) q_{ab} = \omega_\lambda^2 q_{ab}$  for an appropriate  $\omega_\lambda$ , and the issue is whether the given manifold  $\mathcal{I}$  continues to be complete with respect to these metrics  $q_{ab}(\lambda)$ .

To analyze this issue, it suffices to focus just on  $\mathring{q}_{ab}$ . Since  $\mathring{q}_{ab}$  is left invariant under the action of the 6 Killing fields and rescaled just by a constant,  $e^{2\lambda}$ , under the action of the dilation, it follows that the diffeomorphisms generated by these 7 symmetry vector fields do preserve completeness of  $\mathcal{I}$ . Therefore, we need to examine only the 3 inverted translations  $\tilde{T}^a$  in some detail.

Let us begin by recalling the geometric meaning of the inverted translations. Consider the ‘inverted’ coordinates  $\tilde{x}_i = x_i/r^2$  which send the origin to the point  $i^\circ$  at spatial infinity and  $i^\circ$  to the origin. (Here  $r^2 = x^2 + y^2 + z^2$ .) Using the  $\tilde{x}_i$  coordinates we can carry out a one point compactification of  $\mathcal{I} \equiv \mathbb{R}^3$  to obtain  $\mathbb{S}^3$ . Coordinates  $x_i$  cover all of  $\mathbb{S}^3$  except the ‘north pole’  $i^\circ$  and the coordinates  $\tilde{x}_i$  cover all of  $\mathbb{S}^3$  except the south pole (the origin in  $x^a$  coordinates). The  $\tilde{T}^a$  are simply the translations of  $\mathring{q}_{ab}$ ; their components are constant in the  $\tilde{x}^a$  coordinates (which explains the term ‘inverted translations’). Therefore it is clear that starting from any point (other than the origin) of the original  $\mathbb{R}^3$  we can reach  $i^\circ$  by moving along the integral curves of  $\tilde{T}^a$  through a finite affine parameter. This suggests that these diffeomorphisms may not preserve completeness of  $\mathcal{I}$ .

This is indeed the case. To be specific, let us consider the inverted translation  $\tilde{Z}^a$  along z-direction (i.e., set  $\tilde{t}^a \partial_a = \partial/\partial z$ ). Then, it is easy to verify that the image  $\mathring{q}_{ab}(\lambda)$  of  $\mathring{q}_{ab}$  under the 1-parameter family of diffeomorphisms  $d_{\tilde{Z}}(\lambda)$  is given by

$$\mathring{q}_{ab}(\lambda) = \omega^2(\lambda) \mathring{q}_{ab} = \frac{1}{(1 + 2\lambda z + \lambda^2 r^2)^2} \mathring{q}_{ab}. \quad (4.42)$$

Because  $\omega(\lambda)$  goes to zero sufficiently fast as one approaches  $i^o$ , the proper length of, say, the curve  $y = 0, z = 0$  with respect to  $\mathring{q}_{ab}(\lambda)$  is *finite*, equal to  $\pi/\lambda$ . Thus, our given 3-manifold  $\mathcal{I}$  is incomplete with respect to  $\mathring{q}_{ab}(\lambda)$  if  $\lambda \neq 0$ , whence the inverted translation  $\tilde{Z}^a$  is not a permissible symmetry for our completion that endows  $\mathcal{I}$  with  $\mathbb{R}^3$  topology. Clearly the same result holds for any inverted translation.

To summarize, as we expected from the conformal completion of the Poincaré patch of de Sitter space-time, for conformal completions that make physical space-times asymptotically de Sitter in a Poincaré patch, inverted translations are not part of the asymptotic symmetry group  $\mathfrak{G}$ . This group is now a 7-dimensional subgroup of  $G_{\text{dS}}$ . If we introduce a basis  $T_i, R_i, D$  with  $i = 1, 2, 3$  in the Lie algebra  $\mathfrak{g}$ , then the commutation relations are given by:

$$[T, T_i] = \frac{1}{\ell} T_i, \quad [T, R_i] = 0, \quad [T_i, R_j] = \epsilon_{ij}{}^k T_k, \quad \text{and} \quad [R_i, R_j] = \epsilon_{ij}{}^k R_k. \quad (4.43)$$

*Remark:*

We have presented the argument for the reduction of  $\mathfrak{G}$  from 10 to 7 dimensions in some detail because the issue is somewhat subtle. The metric  $\mathring{q}_{ab}(\lambda)$  is flat because it is just the image of a flat metric  $\mathring{q}_{ab}$  under a diffeomorphism. Therefore, each of its Cartesian coordinates can be extended to assume the full range from  $-\infty$  to  $\infty$ . By construction, this extended manifold would be complete w.r.t.  $\mathring{q}_{ab}(\lambda)$ . But this extension is not relevant for us. We are given a conformal completion with  $(\mathcal{I}, \mathring{q}_{ab})$  as the boundary. We ask if an inverted translation of  $\mathring{q}_{ab}$  is an asymptotic symmetry, which requires in particular that the manifold  $\mathcal{I}$  we begin with remain complete with respect to the metrics  $\mathring{q}_{ab}(\lambda)$ . What we showed is that it does not. The discussion of isometries of the Poincaré patch and of the geometrical meaning of inverted translation is, strictly speaking, not needed for the final result. But it brings out the reason as to why certain symmetries cease to be permissible as we move from  $\mathbb{S}^3$  topology to  $\mathbb{R}^3$  topology.

#### 4.3.2.3 $\mathbb{S}^2 \times \mathbb{R}$ topology

$\mathbb{S}^2 \times \mathbb{R}$  topology is obtained by removing a point —say, the origin— from  $\mathbb{R}^3$ . Since translations  $T_i^a$  do not leave the origin invariant, they are inadmissible as asymptotic symmetries in this case. Below, we explicitly show that the asymptotic symmetry group is generated by the three rotations  $R_i^a$  and the de Sitter time

translation or dilation  $T^a$ .

Consider the manifestly conformally flat 3-metric  $q_{ab}$  on  $\mathcal{I}$  of the Schwarzschild-de Sitter space-time discussed in section 4.2.2 for which  $\mathcal{I}$  is complete:

$$q_{ab}dx^a dx^b = \frac{\ell^2}{r^2} \left( dr^2 + r^2 d\omega_2^2 \right) = \frac{\ell^2}{r^2} \mathring{q}_{ab} dx^a dx^b . \quad (4.44)$$

The origin of the flat metric  $\mathring{q}_{ab}$  where  $r = 0$  is not in  $\mathcal{I}$  because  $q_{ab}$  is ill-defined there. Consider the metric under the action of a 1-parameter family of diffeomorphisms generated by, say, the  $z$ -directional translation  $Z^a$ :

$$q_{ab}(\lambda) = \frac{\ell^2}{x^2 + y^2 + (z + \lambda)^2} \mathring{q}_{ab} . \quad (4.45)$$

The distance between the point, say,  $(x = 0, y = 0, z = z_o)$  and the origin  $(x = 0, y = 0, z = 0)$  with respect to  $q_{ab}(\lambda)$  is *finite*, given by  $\ell \int_0^{z_o} dz/(z + \lambda) = \ell \ln((z_o + \lambda)/\lambda)$ . Thus, the origin is infinitely far away only when  $\lambda = 0$ , and in all other cases the manifold  $\mathcal{I}$  we began with is no longer complete. Therefore the translations  $T_i^a$  are not permissible symmetries. The three rotations  $R_i$  and the de Sitter time translation  $T$ , with commutation relations given by (4.43), are exact symmetries of the above metric and hence generate the asymptotic symmetry group. Now, there is a preferred time translation, represented by  $T$ .<sup>4</sup>

In summary, when the leading order term of the magnetic part of the Weyl tensor  $\mathcal{B}^{ab}$  is set to zero, the intrinsic metric on  $\mathcal{I}$  is conformally flat. This condition is strong enough to reduce the group of asymptotic symmetries from the infinite dimensional  $\text{Diff}(\mathcal{I})$  to a finite dimensional group. Furthermore, a new feature when  $\Lambda > 0$  is that the requirement of completeness of  $\mathcal{I}$  ties the topology of  $\mathcal{I}$  to the dimensionality of the asymptotic symmetry group,  $\mathfrak{G}$ . Thus,  $\mathfrak{G}$  is the 10-dimensional de Sitter group for  $\mathbb{S}^3$  and a 7- and 4-dimensional subgroup thereof for  $\mathbb{R}^3$  and  $\mathbb{S}^2 \times \mathbb{R}$  topologies respectively.

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<sup>4</sup>Strictly, we should also check that the inverted translations of  $\mathring{q}_{ab}$  are also not symmetries but the argument is essentially the same as that in section 4.3.2.2.

## 4.4 The $\mathcal{B}^{ab} \hat{=} 0$ condition

In this section we will probe the geometrical and physical meaning of the stronger boundary condition  $\mathcal{B}^{ab} \hat{=} 0$  at  $\mathcal{I}$  and show that it is a severe restriction which cannot be justified, or even motivated, on physical grounds. Since this is an important issue, we will proceed in three steps.

We will begin in section 4.4.1 with the  $\Lambda = 0$  case and analyze the implication of the  $\mathcal{B}^{ab} \hat{=} 0$  condition at  $\mathcal{I}$  in the well-understood asymptotically flat context. The additional restriction now reduces the BMS group to the 10-dimensional Poincaré group, just as it reduced  $\text{Diff}(\mathcal{I})$  to the 10-dimensional de Sitter group in section 4.3.2.1. However, we will find that the condition implies that there is *no* gravitational radiation at  $\mathcal{I}$ ! Given the close parallel between the reductions of symmetry groups in the two cases, the last result can be taken to be an indication that the condition  $\mathcal{B}^{ab} \hat{=} 0$  is inadmissible in the  $\Lambda > 0$  case as well. But recall that since  $\mathcal{I}$  is null in the  $\Lambda = 0$  case, the electric and magnetic parts of the Weyl tensor are not independent. Therefore the question naturally arises: Is this implication of absence of gravitational radiation in the  $\Lambda = 0$  case perhaps tied with the fact that  $\mathcal{I}$  is null in this case?

To probe this issue, in section 4.4.2 we examine test Yang-Mills fields in de Sitter space-time *where  $\mathcal{I}$  is space-like* and analyze the implications of the analogous additional condition  $B_i^a \hat{=} 0$  in the Yang-Mills sector. This analysis enables one to separate effects that can be attributed primarily to the null nature of  $\mathcal{I}$  in the  $\Lambda = 0$  case from those that originate from the non-Abelian character of the interaction — i.e., the fact that in the Yang-Mills theory (and general relativity), fields act as their own source. We will show that the condition  $B_i^a \hat{=} 0$  does reduce the local gauge group at  $\mathcal{I}$  to the global gauge group, just as  $\mathcal{B}^{ab} \hat{=} 0$  reduces  $\text{Diff}(\mathcal{I})$  to  $G_{\text{dS}}$  in the gravitational case. This enables one to define Yang-Mills charges unambiguously. However, they are absolutely conserved; even though Yang-Mills fields are sources of their own charges, there is no leakage through  $\mathcal{I}$ . Furthermore, while a restricted class of Yang-Mills waves is permissible at  $\mathcal{I}$ , they do not carry de Sitter energy-momentum or angular momentum even locally. Thus the requirement  $\mathcal{B}_i^a \hat{=} 0$  is again a strong restriction that cannot be justified. Finally, in section 4.4.3, we consider the full, non-linear gravitational field in the  $\Lambda > 0$  case. The discussion of the first two sub-sections provides a deeper understanding of the structures at play

in this case. We will find that the technical implication of the condition  $\mathcal{B}^{ab} \hat{=} 0$  is closer to that of  $B_i^a \hat{=} 0$  in the Yang-Mills case: We will find that while the  $\mathcal{B}_{ab} = 0$  condition for the full non-linear gravitational case does not completely rule out gravitational waves at  $\mathcal{I}$ , it is a severe restriction because it implies that the local fluxes of energy-momentum and angular momentum carried by gravitational waves across  $\mathcal{I}$  must all vanish.

#### 4.4.1 Asymptotically flat space-times

Recall from Chapter 2 that gravitational radiation in space-times with  $\Lambda = 0$  is characterized by two tensor fields: the Bondi news tensor  $N_{ab}$  and the ‘magnetic part’  $\mathcal{B}^{ac}$  of the asymptotic Weyl curvature  $K^{abcd}$ . The two tensor fields are related by

$$\mathcal{B}^{ab} = 2 \epsilon^{amn} D_m N_n{}^b \quad (4.46)$$

Now consider space-times for which  $\mathcal{B}^{ab} \hat{=} 0$ . Then we have  $D_{[m} N_{n]p} \hat{=} 0$ . Transvecting this equation with  $n^a$  and using  $n^a N_{ab} \hat{=} 0$  and  $D_a n^b \hat{=} 0$ , one obtains  $\mathcal{L}_n N_{ab} \hat{=} 0$ . Therefore  $N_{ab}$  admits an unambiguous projection to the 2-sphere  $S$  of generators of  $\mathcal{I}$ , which we denote by  $\bar{N}_{ab}$ . Without loss of generality one can work in a Bondi frame, in which  $n^a$  is divergence-free and the 2-sphere metric is the round unit 2-sphere metric. Then, on a unit 2-sphere  $S$ , we have a tensor field  $\bar{N}_{ab}$  satisfying

$$\bar{N}_{[ab]} = 0, \quad \bar{N}_{ab} \bar{q}^{ab} = 0, \quad \text{and} \quad \bar{D}_{[a} \bar{N}_{b]c} = 0, \quad (4.47)$$

where  $\bar{q}_{ab}$  is the unit 2-sphere metric on  $S$  induced by  $q_{ab}$  on  $\mathcal{I}$ , and  $\bar{D}$  is its torsion-free derivative operator. Appendix A shows that the only solution to these equations is  $\bar{N}_{ab} = 0$  on  $S$  which immediately implies  $N_{ab} \hat{=} 0$  on  $\mathcal{I}$ .<sup>5</sup> It turns out that this reduces the infinite dimensional BMS group  $\mathcal{B}$  to a 10-dimensional Poincaré sub-group thereof. This consequence of the  $\mathcal{B}^{ab} = 0$  condition in the  $\Lambda = 0$  case is completely analogous to that in the  $\Lambda > 0$  case. However, since  $\mathcal{B}^{ab} = 0$  implies that the Bondi news tensor  $N_{ab}$  must also vanish, these space-times *admit no gravitational waves*. Therefore, outside the limited context of stationary

<sup>5</sup>Since  $\mathcal{L}_n N_{ab} \hat{=} 0$ , if  $N_{ab}$  were not to vanish and  $\mathcal{I}$  were to be complete, then the total flux of Bondi energy over  $\mathcal{I}$  would be infinite. Therefore invoking the obvious physical considerations one can conclude that  $N_{ab}$  must vanish on  $\mathcal{I}$ . Appendix A establishes this result without assuming completeness of  $\mathcal{I}$ . Also an intermediate result in the proof holds in higher dimensions and has been used in different contexts, including the analysis of the structure at spatial infinity [29,30,59].

space-times, the requirement is extremely strong.

#### 4.4.2 Yang-Mills fields in de Sitter space-time

Let us now consider source-free Yang-Mills fields  $\hat{F}_{ab}^i$  in the de Sitter space-time  $(\hat{M}, \hat{g}_{ab})$ , where the index  $i$  refers to the Lie algebra of the internal group which we will take to be an  $n$ -dimensional compact group  $G$ . Since the Yang-Mills equation is conformally invariant, if we denote the Yang-Mills connection by  $\hat{A}_a^i$ , then  $A_a^i = \hat{A}_a^i$  satisfies the Yang-Mills equation on the conformally rescaled space-time  $(M, g_{ab})$ . We assume that  $A_a^i$  admits a smooth extension to  $\mathcal{I}$  (which is compatible with the requirement on the stress-energy tensor in Definition 2).

At  $\mathcal{I}$ , we can define two types of physical quantities for Yang-Mills fields. The first of these are associated with isometries of de Sitter space-time: given a Killing field  $\xi^a$ ,

$$\begin{aligned} \mathcal{F}_\xi[\Delta\mathcal{I}] &:= \int_{\Delta\mathcal{I}} T_{ab} \xi^a \hat{n}^b \, d^3V \\ &= -\frac{1}{4\pi} \int_{\Delta\mathcal{I}} \epsilon_{abc} E_i^a B^{bi} \xi^c \, d^3V \end{aligned} \quad (4.48)$$

defines the flux of the ‘de Sitter momentum’ across any patch  $\Delta\mathcal{I}$  of  $\mathcal{I}$ . Here  $\hat{n}^a$  is the *unit* normal to  $\mathcal{I}$ ,  $T_{ab}$  denotes the stress-energy tensor of the Yang-Mills fields,  $E_i^a = F_i^{ab} \hat{n}_b$ ,  $B_i^a = {}^*F_i^{ab} \hat{n}_b$ , and in the second step we have used the fact that every Killing field  $\xi^a$  is tangential to  $\mathcal{I}$ . The second type of physical quantity arises from the fact that, because of its non-Abelian nature, the Yang-Mills field carries its own charge. Therefore, given any 2-sphere  $S$  and a Lie-algebra-valued field  $\zeta^i$  on it, we can define electric and magnetic type charges:

$$Q[S, \zeta] := \frac{1}{4\pi} \oint_S E_i^a \zeta^i \, dS_a, \quad \text{and} \quad Q^*[S, \zeta] := \frac{1}{4\pi} \oint_S B_i^a \zeta^i \, dS_a. \quad (4.49)$$

However, because of the local gauge freedom in choosing the generator  $\zeta^i$ , in each class we have an infinite family of charges. If we vary  $S$ , the values of these charges change because of two reasons. First, there is a leakage of the Yang-Mills fields between any two 2-spheres and, second, the generators  $\zeta^i$  can change in an arbitrary fashion from one 2-sphere to another.

To compare charges in a useful manner one needs a rigid prescription to choose

$\zeta^i$  on any 2-sphere  $S$  to enable one to say that one is comparing the values of the ‘same’ charge on two different 2-spheres. This can be achieved by restricting oneself only to those Yang-Mills fields for which  $B_i^a$  vanishes at  $\mathcal{I}$ . Then the pull-back  $F_{ab}^i$  to  $\mathcal{I}$  of the Yang-Mills field vanishes and we can introduce *covariantly constant* Lie-algebra valued fields  $\zeta^i$ . Thus, the gauge group is reduced from the infinite dimensional group  $\text{Loc}[G]$  of local gauge transformations to the  $n$ -dimensional group  $G$  of global gauge transformations, just as the infinite dimensional  $\text{Diff}(\mathcal{I})$  is reduced to the 10-dimensional  $G_{\text{dS}}$  by imposing  $\mathcal{B}^{ab} \doteq 0$ . Furthermore, using these covariantly constant  $\zeta^i$ , we can define  $n$  charges  $Q_\zeta[S]$  on any 2-sphere, associated with the global gauge group. Therefore, it is now meaningful to compare charges on two different 2-spheres  $S_1$  and  $S_2$  and attribute the difference solely to the leakage of the Yang-Mills fields between  $S_1$  and  $S_2$ . In view of this nice structure, it is tempting to regard the additional condition  $B_i^a \doteq 0$  as a natural restriction.

However, from (4.48) we see that if  $B_i^a \doteq 0$ , then *all 10 de Sitter fluxes vanish identically*. Furthermore, this happens across *any* local region  $\Delta\mathcal{I}$  of  $\mathcal{I}$ ; i.e., not because of a subtle cancellation between different regions. Secondly, while now we have well-defined global charges  $Q_\zeta[S]$ , because we can now choose a gauge such that  $A_a^i = 0$  in the region enclosed by any given two 2-spheres  $S_1$  and  $S_2$ , we have

$$\begin{aligned} Q_\zeta[S_1] - Q_\zeta[S_2] &= \int_{\Delta\mathcal{I}} (D_a E_i^a) \zeta^i d^3V \\ &= \int_{\Delta\mathcal{I}} \text{Div}_A(E_i) \zeta^i d^3V = 0 \end{aligned} \quad (4.50)$$

where  $\Delta\mathcal{I}$  is the region of  $\mathcal{I}$  bounded by  $S_1$  and  $S_2$ ,  $\text{Div}_A$  is the gauge covariant divergence, and where in the last step we have used the Gauss law. Thus, the charges  $Q_\zeta$  are independent of the choice of the 2-sphere  $S$ . Finally, since  $\mathcal{I}$  is topologically  $\mathbb{S}^2$ , one can just contract any 2-sphere  $S$  indefinitely, whence  $Q_\zeta[S] = 0$  for charges and all  $S$ .

The vanishing of the fluxes and charges already shows that the restriction  $B_i^a \doteq 0$  is quite severe from a physical viewpoint. Mathematically, we can just refer to the Cauchy problem at  $\mathcal{I}$  to conclude that the restriction *removes half the degrees of freedom*. We can still have transverse electric fields  $E_i^a$  at  $\mathcal{I}$  but they will have zero local fluxes of de Sitter energy, momentum and angular momentum and all the electric charges will also vanish.

We conclude this section with two remarks.

1) Had we worked in the Poincaré patch of de Sitter space-time,  $\mathcal{I}$  would be topologically  $\mathbb{R}^3$ . We could also have considered space-times which are asymptotically Schwarzschild-de Sitter where  $\mathcal{I}$  is topologically  $\mathbb{S}^2 \times \mathbb{R}$ . In both cases, the same analysis shows that all the local fluxes (4.48) vanish and the n electric charges (4.49) are independent of the surface  $S$ . However, in the  $\mathbb{S}^2 \times \mathbb{R}$  case, these charges need not be zero. This is completely analogous to what happens in stationary space-times in the asymptotically flat context where the total 4-momentum and angular momentum of the space-time can be non-zero but fluxes of these ‘charges’ across any patch of  $\mathcal{I}$  vanish identically.

2) In the Yang-Mills case, charges (4.49) refer to the *internal* group while the fluxes (4.48) refer to the asymptotic *space-time* symmetries. In the gravitational case the charges are again 2-sphere integrals and fluxes are 3-surface integrals. But now they both refer to the asymptotic space-time symmetries and are therefore intertwined: fluxes account for the differences between the gravitational charges associated with different 2-spheres.

### 4.4.3 Non-linear gravitational fields with $\Lambda > 0$

Let us now consider strongly asymptotically de Sitter space-times. For simplicity of presentation, we will first consider the case when  $\mathcal{I}$  is topologically  $\mathbb{S}^3$  and then discuss the other, more interesting topologies.

#### 4.4.3.1 Bianchi identities and field equations

Since  $\mathcal{B}^{ab} \hat{=} 0$  and  $\mathcal{I}$  has  $\mathbb{S}^3$  topology, it is equipped with a 10-dimensional group of conformal isometries, isomorphic to  $G_{\text{dS}}$ . The natural question is whether, given a 2-sphere  $S$  on  $\mathcal{I}$ , we can associate with each generator  $\xi^a$  of these isometries a gravitational charge  $Q_\xi[S]$  and analyze the fluxes  $\mathcal{F}_\xi[\Delta\mathcal{I}]$  across regions  $\Delta\mathcal{I}$  bounded by two 2-spheres. These would be the analogs of the Bondi charges [40] and fluxes [17] in asymptotically flat space-times discussed in Chapters 2 and 3, and the charges and fluxes of Yang-Mills fields discussed in section 4.4.2. To probe this issue we need to find suitable consequences of the field equations and Bianchi identities that would motivate the definitions of charges and lead to the balance laws in terms of appropriate fluxes. This discussion will be parallel to that in

the asymptotically anti-de Sitter case [44] once the difference in the sign of  $\Lambda$  has been taken into account. However, there is an error in the intermediate steps of that discussion which led to the omission of terms involving the trace of the stress energy tensor of matter in the final result. We will take this opportunity to correct that error.

Let us begin by considering the contracted Bianchi identity to the next leading order from that considered in (4.4). Set

$$\underline{T}_a{}^b := \Omega^{-3} \hat{T}_a{}^b \quad (4.51)$$

which has a smooth limit to  $\mathcal{I}$  and treat it as a tensor field in the conformally completed space-time, whose indices are raised and lowered with the rescaled metric  $g^{ac}$  and  $g_{cb}$ . Then on  $\hat{M}$  we have the identity:

$$\begin{aligned} \nabla^m K_{abcm} = \\ \frac{8\pi G}{\Omega} \left[ -2n_{[a} \underline{T}_{b]}{}^m g_{mc} + \underline{T} n_{[a} g_{b]c} + g_{c[a} \underline{T}_{b]}{}^m n_m + \frac{1}{3} \Omega (\nabla_{[a} \underline{T}_{b]} g_{c]} - \Omega \nabla_{[a} \underline{T}_{b]}{}^m g_{mc} \right]. \end{aligned} \quad (4.52)$$

Transvecting this equation with  $n^a n^c$ , projecting the free index by  $q^{bp}$  and taking the limit to  $\mathcal{I}$  we obtain:

$$D_m \mathcal{E}^{mp} \hat{=} \lim_{\rightarrow \mathcal{I}} 4\pi G \left[ (1/\ell) J^p + \frac{1}{3} D^p \underline{T} + n^a \nabla_a \left( \underline{T}_{bc} q^{bp} \left( \frac{n^c}{n \cdot n} \right) \right) - D^p \left( \underline{T}_{ac} \left( \frac{n^a n^c}{n \cdot n} \right) \right) \right]. \quad (4.53)$$

Here

$$\mathcal{E}^{mp} \hat{=} \lim_{\rightarrow \mathcal{I}} K^{ampb} \hat{n}_a \hat{n}_b, \quad \text{and} \quad J^p \hat{=} \lim_{\rightarrow \mathcal{I}} \Omega^{-4} \hat{T}_a{}^b q^{ap} \hat{n}_b \quad (4.54)$$

and  $\hat{n}^a \hat{=} \ell n^a$  is the unit (future pointing) normal to  $\mathcal{I}$ .  $\mathcal{E}^{mp}$  is the leading order, electric part of Weyl curvature, and  $J^p$  is the leading order matter current at  $\mathcal{I}$ . (Recall that the limit to  $\mathcal{I}$  of  $\Omega^{-3} \hat{T}_a{}^b q^{ap} n_b$  vanishes.) Eq. (4.53) simplifies considerably on using again the fall-off conditions (4.8) which tell us that only the component  $\underline{T}_{ab} n^a n^b$  of  $\underline{T}_{ab}$  can be non-zero at  $\mathcal{I}$ . We obtain:

$$D_m \mathcal{E}^{mp} \hat{=} 8\pi G \left( \frac{1}{\ell} J^p - \frac{1}{3} D^p \underline{T} \right). \quad (4.55)$$

This will serve as the key equation for defining charges and fluxes at  $\mathcal{I}$ .

In conclusion we note that Eqs. (4.52), (4.53) and (4.55) are conformally covariant, as they must be: under  $g_{ab} \rightarrow g'_{ab} = \omega^2 g_{ab}$  (with  $n^a \nabla_a \omega \hat{=} 0$ ), they just get multiplied by a power of  $\omega$ . However, explicit checks of this behavior is rather subtle for the last two of these equations because various components of  $\underline{T}_a{}^b$  have different asymptotic behavior.

#### 4.4.3.2 Gravitational Charges

We can now obtain appropriate balance laws starting from Eq. (4.55). Recall that if  $\mathcal{B}^{ab} \hat{=} 0$ , infinitesimal asymptotic symmetries are represented by conformal Killing fields on  $\mathcal{I}$ . Let us transvect (4.55) with a conformal Killing field  $\xi^a$  and integrate over a portion of  $\mathcal{I}$  bounded by two 2-spheres (or, more generally, any two 2 compact surfaces). By integrating by parts and using the fact that  $\mathcal{E}^{ab}$  is symmetric and trace-free, we obtain:

$$\frac{l}{8\pi G} \left( \oint_{S_2} - \oint_{S_1} \right) (\mathcal{E}_{ab} + \frac{8\pi G}{3} \underline{T} q_{ab}) \xi^a dS^b = \int_{\Delta\mathcal{I}} (J_a \xi^a + \alpha \underline{T} \ell) d^3V \quad (4.56)$$

where the function  $\alpha$  is given by

$$\mathcal{L}_\xi q_{ab} = 2\alpha q_{ab}. \quad (4.57)$$

At first glance, the first term on the right side of (4.56) would appear to be the matter flux associated with  $\xi^a$  through the region  $\Delta\mathcal{I}$ . However, it is not conformally covariant: under  $g_{ab} \rightarrow g'_{ab} = \omega^2 g_{ab}$  (with  $n^a \nabla_a \omega \hat{=} 0$ ), on  $\mathcal{I}$  we have:

$$J^p \xi'_p \hat{=} \omega^{-3} \left( J^p \xi_p - (\omega^{-1} \mathcal{L}_\xi \omega) \ell \underline{T} \right). \quad (4.58)$$

The second term transforms as

$$\alpha' \underline{T}' \ell \hat{=} \omega^{-3} \left( \alpha + \omega^{-1} \mathcal{L}_\xi \omega \right) (\underline{T} \ell). \quad (4.59)$$

Therefore the sum of the two terms on the right of (4.56) is conformally covariant and, since  $d^3V' = \omega^3 d^3V$ , the integral on the right is conformally *invariant*. Thus, to define a physically viable matter current we need the second term on the right side; it is the sum that represents  $\mathcal{F}_\xi^{\text{matt}}[\Delta\mathcal{I}]$ , the flux of the  $\xi^a$  component of the

de Sitter momentum across  $\Delta\mathcal{I}$ , carried by matter.

In view of (4.56) we are now led to define the 2-sphere integrals on the left hand side as charges  $Q_\xi[S]$  associated with the generator  $\xi^a$  of the asymptotic symmetry. Therefore we set:

$$Q_\xi[S] := -\frac{\ell}{8\pi G} \oint_S \left( \mathcal{E}^{ab} + \frac{8\pi G}{3} \underline{T} q^{ab} \right) \xi_a \mathring{r}_b d^2V \quad (4.60)$$

where  $\mathring{r}^a$  is the unit normal to  $S$  and  $d^2V$  the volume element on  $S$ . This is the gravitational charge associated with the asymptotic symmetry generator  $\xi^a$  and the 2-surface  $S$ . Again, for these charges to have physical content, they should refer only to the physical space-time under consideration and not to the choice of the conformal completion made in their definition. Under a rescaling we have:

$$\mathcal{E}'_{ab} \hat{=} \omega^{-1} \mathcal{E}_{ab}, \quad \underline{T}' q'_{ab} = \omega^{-1} \underline{T} q_{ab}, \quad \xi'^a = \xi^a, \quad \mathring{r}'^b \hat{=} \omega^{-1} \mathring{r}^b \quad \text{and} \quad d^2V' \hat{=} \omega^2 d^2V. \quad (4.61)$$

Therefore the charges (4.60) are indeed conformally invariant. (Similarly, the flux integral on the right side of (4.56) is also conformally invariant, as it must be for consistency.) Thus, the charge integrals are indeed insensitive to the choice of conformal completion.

The charge integrals (4.60) and the balance laws (4.56) have two novel features from the viewpoint of asymptotically flat space-times. We will conclude this subsection with a discussion of these differences.

The first feature is the appearance of a matter term in the integrand of the gravitational charge integral. However, recall that in the asymptotically flat space-times, one asks for a stronger fall-off for the stress energy tensor, namely that  $\Omega^{-2} \hat{T}_{ab}$  should have a limit to  $\mathcal{I}$  (rather than  $\Omega^{-1} \hat{T}_{ab}$  as in Definition 2). Had we imposed this stronger condition, then the field  $\underline{T}$  would vanish on  $\mathcal{I}$  and the extra term would disappear. As discussed in section 4.1.1, the stronger condition is in fact natural for Yang-Mills fields and null fluids and the gravitational charge is then expressed entirely in terms of geometry. However, as we also pointed out in section 4.2, the weaker condition is necessary to accommodate cosmological solutions. In the Friedman-Lemaître-Robertson-Walker cosmologies, it turns out that the extra term integrates out to zero for all conformal Killing fields in the charge integrals as well as balance laws if we restrict ourselves to round 2-spheres. However, in the

general cosmological contexts, such as those considered in [55], they would play an interesting role.

The second difference from the asymptotically flat case is more significant. In absence of matter sources, charges (4.60) are *absolutely conserved*, i.e, do not depend on the choice of the 2-sphere  $S$  used in their evaluation. Therefore, if  $\mathcal{I}$  is topologically  $\mathbb{S}^3$  or  $\mathbb{R}^3$ , we can just continuously shrink the 2-sphere to a point to conclude  $Q_\xi[S] = 0$  for any asymptotic symmetry generator  $\xi^a$  and any 2-sphere  $S$ . In particular, as one would expect physically, all gravitational charges vanish identically in de Sitter space-time. Interesting cases correspond to isolated systems where the topology of  $\mathcal{I}$  is  $\mathbb{S}^2 \times \mathbb{R}$  and the asymptotic symmetry group is 4-dimensional, with one time translation (or dilation) and three rotations.

#### 4.4.3.3 Examples

A first viability test of the definition (4.60) is provided by computing these charges in simple examples. In the Schwarzschild-de Sitter space-time, using the conformal completion discussed in section 4.2.2, we obtain:

$$\mathcal{E}_{ab} \hat{=} - \frac{3GM}{\ell^3} \left( D_a t D_b t - \frac{1}{3} q_{ab} \right). \quad (4.62)$$

Since the matter current  $J^a$  and  $\underline{T}$  vanish identically, the charge integrals are independent of the choice of the 2-sphere  $S$ . Therefore we can evaluate them on round 2-spheres. A simple calculation gives:

$$Q_t[S] = M \quad \text{and} \quad Q_R[S] = 0, \quad (4.63)$$

where  $t^a$  is the time translation,  $t^a \partial_a = \partial/\partial t$  and  $R^a$  any rotational Killing vector.

Next, consider the conformal completion of the Kerr space-time discussed in section 4.2.2. Now, the expressions are much more involved:

$$\mathcal{E}_{ab} \hat{=} - \frac{GM}{\ell(1 + \frac{a^2}{\ell^2})^2} \left( \frac{2}{\ell^2} \nabla_a t \nabla_b t - \frac{\left(1 + \frac{a^2}{\ell^2}\right)^2}{1 + \frac{a^2}{\ell^2} \cos^2 \theta} \nabla_a \theta \nabla_b \theta \right) \quad (4.64)$$

$$- \sin^2 \theta \left[ 1 - \frac{a^2}{\ell^2} \left( \frac{1}{2} - \frac{3}{2} \cos 2\theta \right) \right] \nabla_a \phi \nabla_b \phi - \frac{4a}{\ell^2} \sin^2 \theta \nabla_{(a} t \nabla_{b)} \phi \Big). \quad (4.65)$$

The unit normal  $\hat{r}^a$  to the 2-spheres,  $t = \text{const}$ , and the intrinsic volume element

$d^2V$  on them are given by:

$$\dot{r}^a \partial_a \hat{=} \sqrt{\frac{1 + \frac{a^2}{\ell^2}}{1 + \frac{a^2}{\ell^2} \cos^2 \theta}} \left( \left(1 + \frac{a^2}{\ell^2}\right) \left(\frac{\partial}{\partial t}\right) + \frac{a}{\ell^2} \left(\frac{\partial}{\partial \phi}\right) \right), \quad (4.66)$$

$$d^2V \hat{=} \frac{\ell^2 \sin \theta}{\sqrt{1 + \frac{a^2}{\ell^2} \cos^2 \theta} \sqrt{1 + \frac{a^2}{\ell^2}}} d\theta d\phi. \quad (4.67)$$

A direct evaluation of the charge integrals  $Q_\xi[S]$  of (4.60) shows that, as one would expect, only those corresponding to the time translation Killing field,  $\xi^a \partial_a = t^a \partial_a \equiv \partial/\partial t$  and the rotational Killing field  $\xi^a \partial_a = \phi^a \partial_a \equiv \partial/\partial \phi$  are non-zero. They are given by:

$$Q_t[S] = \frac{M}{\left(1 + \frac{a^2}{\ell^2}\right)^2} \quad \text{and} \quad Q_\phi[S] = -\frac{Ma}{\left(1 + \frac{a^2}{\ell^2}\right)^2}. \quad (4.68)$$

These values are the  $\Lambda > 0$  counterparts of the Kerr-anti-de Sitter charges obtained in [60, 61]. Note that in the limit  $\Lambda \rightarrow 0$  we have  $\ell \rightarrow \infty$  and the values of charges reduce to the standard Kerr charges.<sup>6</sup> However, in the presence of a cosmological constant, the metric has three parameters  $M, a$  and  $\ell$ , and the parameter  $M$  is *not* the charge generating time translation  $\partial/\partial t$ . It generates a rescaled time translation  $\partial/\partial \bar{t}$  where  $\bar{t} = (1 + (a/\ell)^2)^2 t$ , where the rescaling varies from one space-time to another. In asymptotically flat space-times, by ‘the’ time translation, one means the one which is normalized to have unit norm at infinity with respect to the physical metric. In the presence of a cosmological constant, the norm of the ‘time translation’ with respect to the physical metric diverges and so a canonical normalization for all space-times is not available. However, the first law of black hole thermodynamics restricts the freedom to change the normalization as one moves from one space-time to another [62]. Deruelle has shown that the (anti-de Sitter analogs of the) charges obtained here do satisfy the first law [60].

In this sub-section we focused only on the time translation and three rotations because, as we discussed in section 4.3, the completeness requirement in Definition 3 reduces  $G_{\text{dS}}$  to this 4-dimensional group. However, because  $q_{ab}$  is conformally flat, it admits 10 conformal Killing vectors  $\xi^a$ ; it is just that the finite diffeomorphisms they generate fail to preserve the completeness condition. Since definitions of

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<sup>6</sup>The negative sign in front of the angular momentum charge is present also in the calculation at  $\mathcal{I}$  in asymptotically flat space-times if one uses the convention  $\ell \cdot n = -1$  at  $\mathcal{I}$ . The value of the mass is insensitive to this choice.

charges and fluxes do not refer to completeness, we can still use any of the 10  $\xi^a$  to define conserved charges  $Q_\xi[S]$  and fluxes  $\mathcal{F}_\xi^{\text{matt}}[\Delta\mathcal{I}]$ . In the Kerr-de Sitter case, the additional 6 charges vanish on any  $S$ .

#### 4.4.3.4 Fluxes and balance laws

Let us begin with the dynamical collapse described by the Vaidya-de Sitter solution. The situation at  $\mathcal{I}^+$  is the same as in the Schwarzschild-de Sitter solution. However, since there is matter flux at  $\mathcal{I}^-$ , the charge integrals are not conserved in the range  $v_1 < v < v_2$ . (But because the source is a null fluid  $\underline{T}$  vanishes on  $\mathcal{I}^-$  even in the dynamical region.) The leading term in the electric part of the Weyl tensor,  $\mathcal{E}^{ab}$  has the same form as in the Schwarzschild-de Sitter space-time except that  $M$  is not a constant but a function of  $v$ . Therefore, for any 2-sphere lying in the region  $v < v_1$ , we have  $\mathcal{E}^{ab} = 0$  and all charges vanish. In the dynamical region, the non-trivial balance law (4.56) becomes relevant and the values of the ‘energy’ charge integral increase in time in response to the matter flux  $J^a$  flowing into the space-time across  $\mathcal{I}^-$ . For  $v > v_2$  the charge integral remains constant, and equals  $M$ . The charge integrals and the balance laws faithfully capture the energetics of the Vaidya solution because the underlying spherical symmetry implies that there are no gravitational waves, whence one knows that the energy flux is entirely due to matter. The overall situation parallels that in the Vaidya solutions in the  $\Lambda = 0$  case.

However, more generally, in the  $\Lambda = 0$  case the Bondi energy-momentum and angular momentum change also because of the leakage of gravitational waves across  $\mathcal{I}$ . For example, for the charge corresponding to the time translation  $\xi^a = \alpha n^a$  at  $\mathcal{I}$ , the energy balance law reads

$$Q_\xi[S_1] - Q_\xi[S_2] = \int_{\Delta\mathcal{I}} \alpha \left[ \frac{1}{32\pi} |N_{ab}|^2 + \underline{T}_{ab} n^a n^b \right] d^3V, \quad (4.69)$$

where the first term on the right hand side describes the flux of energy carried by gravitational waves. There is no analog of this term in (4.56). Thus, if  $\mathcal{B}^{ab} \hat{=} 0$ , although the de Sitter charges are well-defined, there is no flux of de Sitter momentum due to gravitational waves even locally on  $\mathcal{I}$ ! In light of our discussion of the analogous condition in the  $\Lambda = 0$  case in section 4.4.1, this could have been anticipated. For, when  $\mathcal{B}^{ab} \hat{=} 0$  the Bondi news tensor  $N_{ab}$  vanishes identically in

the  $\Lambda = 0$  case and the gravitational contribution to the fluxes in the balance laws vanishes identically. That is, the balance law (4.56) is the direct analog of the one in the  $\Lambda = 0$  case with additional restriction  $\mathcal{B}^{ab} \hat{=} 0$ . The parallel runs quite deep. For example, in the  $\Lambda = 0$  case the expression for the energy-momentum and angular momentum charge integrals is the same as the first term in (4.60). The second term is absent simply because of the stronger fall-off of stress-energy tensor in the  $\Lambda = 0$  case.<sup>7</sup> However, there is also a key difference. In the  $\Lambda = 0$  case,  $\mathcal{B}^{ab} \hat{=} 0$  implies that there is *no* gravitational radiation at  $\mathcal{I}$  *at all*. For  $\Lambda > 0$ , this is not the case. Nothing prevents the electric part  $\mathcal{E}^{ab}$  of the asymptotic Weyl tensor on  $\mathcal{I}$  from having a ‘transverse-traceless’ piece in the decomposition of symmetric tensors into longitudinal, trace and transverse-traceless parts (that is often used in the initial value formulation of Einstein’s equations). In Chapter 5 we will see this feature in detail in the linearized approximation. In the  $\Lambda = 0$  case, by contrast, if  $\mathcal{B}^{ab} \hat{=} 0$ , the electric part  $\mathcal{E}^{ab}$  is (again traceless but) longitudinal.

Thus, the situation in the  $\Lambda > 0$  case is subtle. The condition  $\mathcal{B}^{ab} \hat{=} 0$  removes ‘half the radiative degrees of freedom’ in the gravitational field and, in addition, the gravitational waves it does allow can not carry any of the de Sitter momenta across  $\mathcal{I}$ .

We conclude this section with a brief comparison with some older works.

1) As we noted earlier, our discussion in this sub-section parallels that in the  $\Lambda < 0$  case of [44]. But in that case, the condition  $\mathcal{B}^{ab} \hat{=} 0$  can be regarded as a reflective boundary condition [45], an additional input that is needed to make the evolution well-defined because  $\mathcal{I}$  is time-like. The reflective nature of boundary conditions also explains why gravitational waves do not carry away energy-momentum or angular momentum across  $\mathcal{I}$ .

2) In the  $\Lambda > 0$  case, Abbott and Deser [63] have introduced a notion of gravitational charges in the metric framework. However, it is likely that their boundary conditions have to be refined to remove the analogs of the supertranslation ambiguity (briefly discussed in [44]). More recently, Kelly and Marolf [64] have provided a Hamiltonian framework based on Cauchy data on space-like surfaces, analogous to the  $\eta = \text{const}$  surfaces considered in section 4.2.5, without reference to  $\mathcal{I}$ . It is likely that their charges coincide with (4.60) when our 2-spheres  $S$  are chosen to

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<sup>7</sup>Also, the overall multiplicative factor  $\ell$  is absent simply because one normally chooses the conformal factor  $\Omega \sim 1/r$  with dimensions of inverse length in the  $\Lambda = 0$  case while in the  $\Lambda \neq 0$  case one chooses  $\Omega \sim \ell/r$  which is dimensionless.

lie in a neighborhood of  $i^o$  within  $\mathcal{I}$  in which there is no matter flux. However, to firmly establish this result, one would have to understand the relation between the approach to  $i^o$  along Cauchy surfaces and along  $\mathcal{I}$  more precisely.

3) A natural strategy to relate frameworks based on  $\mathcal{I}$  to those based on (partial) Cauchy surfaces that go to spatial infinity,  $i^o$ , would be to introduce a 4-dimensional treatment of spatial infinity along the lines used in [29, 30, 59, 65] for the  $\Lambda = 0$  case. In the Hamiltonian framework one works in the physical space-time and de Sitter charges arise as surface integrals on the 2-sphere boundaries of Cauchy surfaces at spatial infinity. Therefore, the treatment given in [65] seems to be best suited for comparison because it attaches to the physical space-time a 3-dimensional boundary rather than a single point  $i^o$ . We examined this possibility in detail. However, it turns out that the structure for  $\Lambda > 0$  is so different that the basic ideas used in the  $\Lambda = 0$  construction do not generalize. In particular we could not endow the 3-dimensional ‘hyperboloid’ at spatial infinity with a universal geometry as in the  $\Lambda = 0$  case. Therefore a space-time covariant treatment of spatial infinity remains an open problem.

4) In a Master’s thesis, Jäger [66] has introduced charge integrals at  $\mathcal{I}$ , following the procedure used in asymptotically anti-de Sitter space-times [44], just as we did in this section. However, as in [63, 64], a restriction was made to the source-free case. Therefore the charge integral did not have the second term in the expression (4.60), nor the right side of our balance law (4.56). Since the charges were absolutely conserved, the framework did not allow for situations analogous to the Vaidya collapse. Also the condition  $\mathcal{B}^{ab} \hat{=} 0$  and the associated symmetry reduction carried out in section 4.3 was not considered, nor the relation to the  $\Lambda = 0$  case.

5) In the literature inspired by the AdS/CFT correspondence, to define gravitational charges one generally introduces infinite counter terms to handle the blow up of the metric components at infinity in the commonly used charts [67]. By contrast, in the framework used here, everything is manifestly finite and no subtractions are necessary because the formulation is in terms of the Weyl curvature which remains smooth (and in fact vanishes!) at  $\mathcal{I}$ .

## 4.5 Discussion

We began in section 4.1 by summarizing and slightly extending the standard constructions that have been used in the literature to discuss asymptotically de Sitter space-times. In section 4.2 we presented several examples to illustrate the finer differences one finds in different physical situations, particularly in the topology of  $\mathcal{I}$ . These examples also brought out the differences from the more familiar asymptotically flat space-times.

In section 4.3 we discussed asymptotic symmetries in detail. We found that in general asymptotically de Sitter space-times, the asymptotic symmetry group  $\mathfrak{G}$  is just  $\text{Diff}(\mathcal{I})$  whence we cannot repeat the procedures used in the  $\Lambda = 0$  case to extract physics from the asymptotic behavior of the gravitational field at  $\mathcal{I}$ . This surprising outcome has not been appreciated in the literature in part because a stronger boundary condition is often introduced that requires the intrinsic 3-metric at  $\mathcal{I}$  to be conformally flat. Then, the symmetry group  $\mathfrak{G}$  reduces to the 10-dimensional de Sitter group  $G_{\text{dS}}$  if the topology of  $\mathcal{I}$  is  $\mathbb{S}^3$ . However, in physically interesting space-times the topology is generally different. In the cosmological context it is often  $\mathbb{R}^3$  and for isolated gravitating systems it is  $\mathbb{S}^2 \times \mathbb{R}$ . We showed that completeness requirement on  $\mathcal{I}$  reduces the symmetry group further to a 7-dimensional group in the  $\mathbb{R}^3$  case and to a 4-dimensional group of a ‘time’ translation and 3 rotations in the  $\mathbb{S}^2 \times \mathbb{R}$  case.

In section 4.4 we showed that given these asymptotic symmetries we can introduce conserved charges. In the Kerr-de Sitter space-time, and in the dynamical Vaidya-de Sitter solution depicting the simplest black hole formation via gravitational collapse, the conserved charges can be evaluated explicitly and provide the physically expected mass and angular momentum. However, we also showed that even in fully dynamical space-times where one expects gravitational waves near  $\mathcal{I}$ , the charges are absolutely conserved if there is no matter flux across  $\mathcal{I}$ . Thus, for  $\Lambda > 0$  *gravitational waves do not carry energy or angular momentum across  $\mathcal{I}$ !* This severe limitation comes from the stronger conformal flatness condition. We showed that it is equivalent to asking that the magnetic part  $\mathcal{B}^{ab}$  of the leading order Weyl curvature at  $\mathcal{I}$  must vanish. Thus, while the stronger condition seems attractive from symmetry considerations, it removes *by fiat* half the degrees of freedom. These results are surprising because they imply that we do not yet have

a strategy to extract physics of gravitational waves, and more generally, properties of isolated gravitating systems, in full general relativity with positive  $\Lambda$ . To better understand this limitation, we discussed in some detail the implications of the condition  $\mathcal{B}^{ab} = 0$  in the  $\Lambda = 0$  case, and of the analogous condition  $B_i^a \doteq 0$  on Yang-Mills fields on de Sitter space-time. We found that in both cases, the condition imposes unreasonably severe restrictions on permissible fields and constrains all the local fluxes of energy, momentum and angular momentum across  $\mathcal{I}$  to vanish identically.

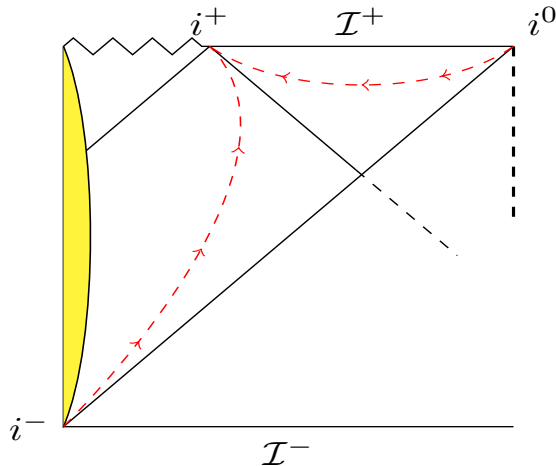
This leads to a quandary: if we drop the requirement of conformal flatness, the structure at  $\mathcal{I}$  is too weak to extract physics and if we keep it, we rule out the examples that are of primary interest to gravitational wave science and quantum gravity and of significant interest to geometric analysis.

There are further issues of prime physical interest in these three areas that are difficult to investigate using the currently available constructions. We will present one example in each area. At the interface of general relativity and geometric analysis, positive energy theorems are not only major landmarks but also serve as invaluable tools if  $\Lambda$  vanishes. However, in the  $\Lambda > 0$  case, we cannot even speak of de Sitter momentum unless we impose the stronger  $\mathcal{B}^{ab} \doteq 0$  condition, which eliminates the possibility of accounting for energy, momentum and angular momentum loss due to gravitational waves. Furthermore, now all symmetry vector fields are space-like near  $\mathcal{I}$  because  $\mathcal{I}$  itself is space-like. Therefore, one cannot hope to prove positive energy theorems either at  $\mathcal{I}$  or at spatial infinity  $i^o$ .<sup>8</sup> Indeed for test fields in de Sitter space-time, energy can be arbitrarily negative even when all the local energy conditions are satisfied simply because the ‘time translation’ vector fields are all space-like near and on  $\mathcal{I}$ . On the other hand, there is a time-translation Killing field which is time-like in the static patch of de Sitter space-time, whence the flux of energy of test fields across the future and the past cosmological horizons is guaranteed to be positive (see Fig. 4.1). Can one perhaps extend this idea to full non-linear general relativity and obtain positive energy theorems?

Next, let us consider gravitational collapse leading to the formation of a black hole. In the case of a Schwarzschild-de Sitter black hole, the well-known Kruskal conformal diagram of the  $\Lambda = 0$  case is replaced by Fig. 4.2. We now have

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<sup>8</sup>A related issue is the black hole uniqueness theorem in 4 dimensions which has also remained wide open for rotating black holes when  $\Lambda = 0$  even though (unlike in the  $\Lambda < 0$  case) the horizon is guaranteed to have a 2-sphere topology.



**Figure 4.5.** Conformal diagram depicting a spherical collapse. Shaded (yellow) region corresponds to the collapsing spherical star. The dashed (red) lines with arrows represent integral curves of the ‘static’ Killing field. Note that the space-time is incomplete to the right unless we add another collapsing star.

cosmological horizons and, furthermore, we have to carry out an identification if we want to avoid an infinite number of black hole and white hole regions. Because of this identification, Cauchy surfaces now have topology  $\mathbb{S}^2 \times \mathbb{S}^1$  (rather than  $\mathbb{S}^2 \times \mathbb{R}$ ). Now consider gravitational collapse leading to the formation of a Schwarzschild-de Sitter black hole depicted in Fig. 4.5. Again, the space-time diagram continues indefinitely to the right but now because the collapsing region is dynamical, a natural identification is not possible unless one adds another collapsing star to the right. While one can do this in the spherical case [68], it seems difficult to envisage the analogous extension for a generic collapse. In any case, physically, one generally needs to consider the collapse of a single star, e.g., to study the Hawking effect. Then we are led to Fig. 4.5 which shows that it will not suffice to specify the incoming state at  $\mathcal{I}^-$  alone, since additional information can flow in from the time-like dashed line on the right of the diagram. And it is difficult to know what the appropriate additional data would be to capture the idea that the total incoming state be vacuum. Once the back reaction is included, further ambiguities arise at  $\mathcal{I}^+$ .

Finally, consider the problem of black hole coalescence. Detailed investigations we are aware of have been carried out in the  $\Lambda = 0$  case. In that case, the  $\mathcal{I}$  framework is rich and, in particular, it enables one to calculate the 3-momentum

that is radiated away across  $\mathcal{I}^+$  (in the center of mass frame defined by the ADM 4-momentum). To compensate for this loss, the final black hole recoils, giving rise to the celebrated ‘black hole kicks’ of astrophysical interest that have been studied in detail in numerical relativity [42]. If  $\Lambda > 0$ , gravitational waves do not carry away any energy or momentum or angular momentum across  $\mathcal{I}^+$ . Does this then mean that in our real universe with a positive  $\Lambda$  there are no kicks? More generally, what are the implications to our actual universe with  $\Lambda > 0$  of all the beautiful simulations in the  $\Lambda = 0$  case that have provided us with detailed estimates of energy and angular momentum loss across  $\mathcal{I}^+$  in binary coalescences? These are important issues that must be addressed now, since we are now in the golden era of gravitational wave science through the global networks of advanced detectors.

Now is an opportune time to address these issues and develop systematic techniques to estimate errors one makes by setting  $\Lambda = 0$  from the start. In view of the smallness of the observed value of  $\Lambda$ , we expect these effects to be small, but precise calculations must confirm this. As we shall see in Chapters 5 and 6, in the linearized approximation, these calculations can be performed with the current theoretical tools.

# Chapter 5 |

## Linearised theory

As seen in the previous chapter, the extension of Penrose's conformal infinity methods, which provided a physically useful notion of isolated systems and gravitational waves emitted by them when  $\Lambda = 0$ , does not yield a gauge invariant characterization of gravitational waves for general relativity when  $\Lambda > 0$ ! The boundary conditions are either too weak to pick out uniquely physically meaningful asymptotic symmetries or they are too strong and set half the gravitational degrees of freedom to zero by the condition  $B_{ab} = 0$ , thereby not capturing any interesting information about gravitational radiation. In this chapter we will analyse source-free, linearized gravitational waves in de Sitter space-time to obtain clues about how to alleviate the afore-mentioned problems in the full theory.

This is done in three steps.

We begin by considering linearized Einstein's equations on a background de Sitter spacetime and demonstrate the unreasonable restrictiveness of the conformal flatness boundary condition of  $\mathcal{I}^+$  in sections 5.1 and 5.2. We restrict our calculations to a future Poincaré patch of de Sitter spacetime because only this patch is relevant in astrophysical scenarios due to the presence of cosmological horizons in de Sitter spacetime. This consideration is elaborated in the next chapter on linearisation of Einstein's equations with sources. The linearized equations can be solved explicitly in the Fourier space and the two independent solutions are examined at  $\mathcal{I}^+$ . We will show that the condition of conformal flatness of  $\mathcal{I}^+$ , or equivalently,  $\mathcal{B}_{ab} = 0$  at  $\mathcal{I}^+$  forces us to set the coefficient of one independent solution to zero, thus removing half the degrees of freedom of gravitational waves.

Next, in section 5.3 we lay out the method to define the total 4-momentum and angular momentum carried by linearized gravitational waves. These require

symmetries and a formula local in the linearized gravitational fields. The symmetries used in these definitions are readily obtained by considering the 7-dimensional subgroup of the 10-dimensional de Sitter group that leaves the Poincaré patch invariant, consisting of 4 (de Sitter) translations and 3 rotations. However, linearized gravitational fields lack a gauge-invariant, local stress-energy tensor because general relativity offers no canonical way to split the gravitational field into a non-dynamical background and a dynamical perturbation. So we take cue from test matter such as scalar, Maxwell or Yang-Mills fields: For these fields conserved quantities can be constructed either using their stress-energy tensors, or equivalently, using the covariant Hamiltonian framework. In the absence of the former method for gravitational fields, we use the latter.

The phase space  $\Gamma_{\text{Cov}}$  of linearized gravity consists of solutions to linearized Einstein's equation.  $\Gamma_{\text{Cov}}$  is naturally endowed with a symplectic structure  $\omega$ , which is preserved under diffeomorphisms generated by spacetime symmetries. Energy-momentum and angular momentum carried by gravitational waves are given by the Hamiltonians corresponding to the respective space-time symmetries, in similar fashion to the procedure in Chapter 3. These formulas are quadratic in the perturbations and will be expressed in terms of fields that are well-defined on  $\mathcal{I}^+$ . These expressions will be needed in the derivation of the energy loss due to a time-changing quadrupole moment in the  $\Lambda > 0$  case, derived in Chapter 6. We will argue that although positivity of energy of gravitational waves in de Sitter space-time is not guaranteed, for the class of solutions that are of direct physical interest in the investigation of isolated systems, they carry positive energy. A detailed analysis of the quadrupole formula for  $\Lambda > 0$  in Chapter 6 confirms this intuition.

Finally, we discuss the  $\Lambda \rightarrow 0$  limit. This is the first step towards quantifying the errors one makes by assuming  $\Lambda = 0$ . Physically one expects that in this limit energy-momentum and angular momentum expressions should reduce to the well-known ones for linear gravitational waves in Minkowski space. However, the discontinuous change in global space-time geometry between zero and non-zero  $\Lambda$  makes this limit non-trivial. In particular,  $\mathcal{I}$  is space-like for  $\Lambda > 0$  and null for  $\Lambda = 0$ . Hence, while energy can be negative in the former case, it is strictly positive in the latter. Our analysis provides a systematic method to obtain the limit and reproduces the Minkowski results in the limit.

## 5.1 Preliminaries: The Poincaré patch

In the  $\Lambda = 0$  case, the Minkowski metric solves Einstein's equations with no matter and is maximally symmetric with 10 symmetries. Hence, one studies isolated systems in the weak field limit by considering linearized gravitational fields in Minkowski space-time. The analogous maximally symmetric space-time metric which solves matter-free Einstein's equations when  $\Lambda > 0$  is de Sitter space-time metric. So, for the  $\Lambda > 0$  case, it is natural to study linearized gravity on de Sitter space-time. However, an important new feature of de Sitter space-time is the presence of cosmological horizons. Consider the illustration of a spatially bounded isolated system in Figure 5.1, the past and future time-like infinity of whose world-tube is denoted by  $i^\pm$ . While in the  $\Lambda = 0$  case the future of  $i^-$  is the entire Minkowski space-time, for  $\Lambda > 0$ , it is only the future Poincaré patch of de Sitter space-time. No observer whose world-line is confined to the past Poincaré patch can see the isolated system or detect the radiation it emits. Therefore, to study this system, it suffices to restrict oneself just to the future Poincaré patch rather than the full de Sitter space-time. Next, as discussed in Chapter 4, we will use coordinates  $(\eta, x, y, z)$  (with  $\eta \in (-\infty, 0)$  and  $x, y, z \in (-\infty, \infty)$ ) to express the de Sitter metric in the conformally flat form as:

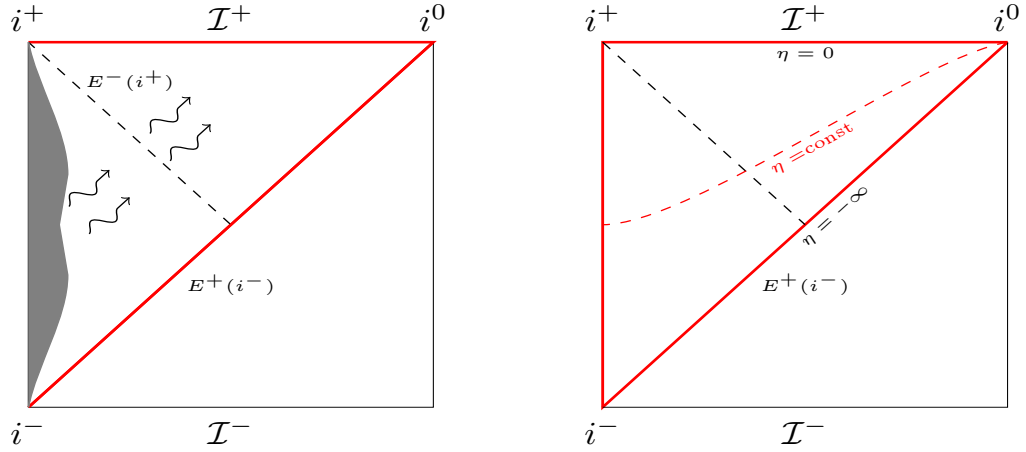
$$\bar{g}_{ab} dx^a dx^b = (a(\eta))^2 \left( -d\eta^2 + d\vec{x}^2 \right) =: (a(\eta))^2 \dot{g}_{ab} dx^a dx^b, \quad (5.1)$$

where the scale factor  $a(\eta) = -1/(H\eta)$  and  $H := \sqrt{\Lambda/3} =: 1/\ell$  is the Hubble parameter, the inverse of the cosmological radius  $\ell$ .<sup>1</sup> It is reasonable to expect that the de Sitter metric yields a Minkowski metric in the limit  $\Lambda \rightarrow 0$ . However, it is clear that the metric (5.1) is ill-defined in that limit as the scale factor  $a(\eta)$  blows up. The right limit to Minkowski space-time is obtained by using proper time  $t$ , which is related to the conformal time  $\eta$  via  $H\eta = -e^{-Ht}$ . In terms of  $t$ , the de Sitter metric becomes

$$\bar{g}_{ab} dx^a dx^b = -dt^2 + e^{2Ht} d\vec{x}^2 \quad (5.2)$$

---

<sup>1</sup>These coordinates –as well as the coordinates  $(t, \vec{x})$  discussed below– have the disadvantage that they do not cover the past boundary of our Poincaré patch, i.e., the event horizon  $E^+(i^-)$  of  $i^-$ . But this limitation will not affect our considerations.



**Figure 5.1.** *Left Panel:* The Penrose diagram of a spherical isolated star in general relativity with  $\Lambda > 0$ . The solid diagonal line denotes  $E^+(i^-)$ , the future event horizon of  $i^-$ . The star and the radiation it emits are invisible to all observers whose world-lines are confined to the lower portion of the de Sitter space-time below  $E^+(i^-)$ . Therefore in the discussion of this isolated system, it is natural to restrict oneself to the upper half. The dashed diagonal line is  $E^-(i^+)$ , the past event horizon of  $i^+$ . *Right Panel:* The Poincaré patch of de Sitter space-time of interest is the upper triangle, to the future of the event horizon  $E^+(i^-)$  where  $\eta = -\infty$ . The  $\eta = \text{const}$  lines denote the cosmological slices, i.e., flat Cauchy surfaces.

and it is manifest that as  $\Lambda \rightarrow 0$ , the Minkowski metric in the  $(t, \vec{x})$  chart is obtained. Therefore, to compare geometric structures in de Sitter space-time to those in Minkowski space-time, it is important to use the differential structure induced on the Poincaré patch by  $(t, \vec{x})$ , and *not* by  $(\eta, \vec{x})$ !

Next, we recall the symmetries of the Poincaré patch from Chapter 4, where it was shown that the Killing fields constitute a 7-dimensional sub-group of the 10-dimensional de Sitter group. These include 3 spatial translations  $T_i^a$  and 3 spatial rotations  $R_i^a$ , tangential to each  $\eta = \text{const}$  slice, and a 7th Killing field

$$T = -H \left[ \eta \frac{\partial}{\partial \eta} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right]. \quad (5.3)$$

We will refer to  $T^a$  as the de Sitter *time translation* because: i) it is the limit of the time translation Killing field in the Schwarzschild-de Sitter space-time as the mass goes to zero, and, ii) in the  $(t, \vec{x})$  coordinates, it reduces to a time-translation in Minkowski space-time as  $\Lambda \rightarrow 0$ .<sup>2</sup> The commutation relations between these

<sup>2</sup>Limit  $\Lambda \rightarrow 0$  of  $T^a$  illustrates the importance of using the correct differential structure to

seven Killing fields are given by:

$$[T, T_i] = H T_i, \quad [T, R_i] = 0, \quad [T_i, R_j] = \epsilon_{ij}{}^k T_k, \quad [R_i, R_j] = \epsilon_{ij}{}^k R_k. \quad (5.4)$$

(Note that the time translation does *not* commute with space-translations.) We will denote this 7-dimensional Lie-algebra of symmetries of the Poincaré patch by  $\mathfrak{g}_{\text{Poin}}$  and the Lie group it generates by  $\mathfrak{G}_{\text{Poin}}$ .<sup>3</sup> Finally, in the standard conformal completion of the Poincaré patch,  $\mathcal{I}^+$  has  $\mathbb{R}^3$  topology and this 7-dimensional group preserves the completeness of the allowed class of metrics on  $\mathcal{I}^+$ .

## 5.2 Linearized gravitational fields

In this section, we discuss the equation of motion of linear perturbations of the gravitational field and its solutions, and demonstrate why the  $\mathcal{B}_{ab} = 0$  boundary condition of Definition 4 of Chapter 4 is not well-motivated from a physical perspective.

### 5.2.1 Linearized field equations and solutions

We will use the  $(\eta, \vec{x})$  chart and the form (5.1) of the de Sitter metric  $\bar{g}_{ab}$  in the Poincaré patch. The perturbed metric will be denoted by  $g_{ab}$ ,

$$g_{ab} = \bar{g}_{ab} + \epsilon \gamma_{ab} \quad (5.5)$$

where  $\epsilon$  is the smallness parameter and  $\gamma_{ab}$  denotes the first order perturbation. Then, in the Lorentz and radiation gauge, i.e., when the gauge freedom is exhausted by requiring that  $\gamma_{ab}$  satisfy

$$\bar{\nabla}^a \gamma_{ab} = 0; \quad \gamma_{ab} \eta^a = 0; \quad \text{and} \quad \gamma_{ab} \bar{g}^{ab} = 0, \quad (5.6)$$

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take this limit. Had we used the differential structure provided by  $(\eta, \vec{x})$  we would have concluded from (5.3) that  $T^a$  vanishes in the limit. But this procedure would have been incorrect because the metric  $\bar{g}_{ab}$  diverges in this limit (although it reduces to the well-defined Minkowski metric if the limit is taken using the differential structure induced by  $(t, \vec{x})$ ). Note, incidentally, that  $T^a$  is sometimes referred to as ‘dilation’ because it is the conformal-Killing vector field representing a dilation with respect to the *flat* metric  $\dot{g}_{ab}$ .

<sup>3</sup>This is the group that leaves the point  $i^-$  on  $\mathcal{I}^-$  of de Sitter space-time invariant. As  $\Lambda \rightarrow 0$ ,  $\mathfrak{G}_{\text{Poin}}$  reduces to a well defined seven dimensional subgroup of the Poincaré group; the limit carries the memory of the preferred  $t = \text{const}$  slicing.

the linearized Einstein's equation simplifies to<sup>4</sup>

$$\square \gamma_{ab} - 2H^2 \gamma_{ab} = 0. \quad (5.7)$$

(Here  $\eta^a$  is a vector field normal to the cosmological slices with  $\eta^a \partial_a = \partial/\partial\eta$ .) To solve the above equation, it is convenient to rewrite (5.5) in terms of a mathematical field  $h_{ab}$  as

$$g_{ab} \equiv a^2(\eta) (\mathring{g}_{ab} + \epsilon h_{ab}) = \frac{1}{(H\eta)^2} (\mathring{g}_{ab} + \epsilon h_{ab}). \quad (5.8)$$

The gauge conditions (5.6) can now be written using the background *flat* geometry of  $\mathring{g}_{ab}$ :

$$\mathring{\nabla}^a h_{ab} = 0; \quad h_{ab} \eta^a = 0; \quad \text{and} \quad h_{ab} \mathring{g}^{ab} = 0, \quad (5.9)$$

and the linearized Einstein's equation becomes

$$\mathring{\square} h_{ab} - 2 \frac{a'}{a} h'_{ab} = \mathring{\square} h_{ab} + \frac{2}{\eta} h'_{ab} = 0, \quad (5.10)$$

where  $h'_{ab} \equiv \eta^c \mathring{\nabla}_c h_{ab}$ . Note that, in the  $(t, \vec{x})$  chart and in the limit  $\Lambda \rightarrow 0$ , the gauge conditions and linearized Einstein's equation satisfied by  $h_{ab}$  are the same as those satisfied by the linearized gravitational fields in Minkowski space-time. In particular, the extra term  $(2/\eta) h'_{ab}$  in the linearized Einstein's equation vanishes in the limit.

As in the case of linearized fields in Minkowski space-time, it is simplest to find explicit solutions using a Fourier transform:

$$h_{ab}(\vec{x}, \eta) \equiv \int \frac{d^3k}{(2\pi)^3} \sum_{(s)=1}^2 h_{\vec{k}}^{(s)}(\eta) \mathbf{e}_{ab}^{(s)}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \quad (5.11)$$

where  $(s)$  labels the two helicity states and  $\mathbf{e}_{ab}^{(s)}(\vec{k})$  are the polarization tensors, satisfying

$$\begin{aligned} \mathbf{e}_{[ab]}^{(s)}(\vec{k}) &= 0; \quad \mathbf{e}_{ab}^{(s)}(\vec{k}) k^b = 0; \quad \mathbf{e}_{ab}^{(s)}(\vec{k}) \mathring{q}^{ab} = 0; \\ (\mathbf{e}_{ab}^{(s)}(\vec{k}))^* &= \mathbf{e}_{ab}^{(s)}(-\vec{k}); \quad \mathbf{e}_{ab}^{(s)}(\vec{k}) \mathbf{e}_{cd}^{(s')}(-\vec{k}) \mathring{q}^{ac} \mathring{q}^{bd} = \delta_{(s),(s')}. \end{aligned} \quad (5.12)$$

Here, and in what follows,  $\mathring{q}_{ab}$  is the fixed spatial Euclidean metric on the cos-

<sup>4</sup>Details of calculations are provided in Appendix B.

mological slices, tailored to the co-moving coordinates  $\vec{x}$ , and  $\star$  denotes complex conjugation. The two functions  $h_{\vec{k}}^{(s)}(\eta)$  capture the gauge invariant information – the transverse traceless modes – of the linearized gravitational field. Since  $h_{ab}(\vec{x}, \eta)$  are real fields, it follows that

$$(h_{\vec{k}}^{(s)}(\eta))^\star = h_{-\vec{k}}^{(s)}(\eta). \quad (5.13)$$

The field equation (5.10) implies that the  $h_{\vec{k}}^{(s)}$  satisfy the ordinary differential equation (ODE):

$$(h_{\vec{k}}^{(s)})'' - \frac{2}{\eta}(h_{\vec{k}}^{(s)})' + k^2 h_{\vec{k}}^{(s)} = 0, \quad (5.14)$$

where the prime denotes differentiation with respect to  $\eta$ , and  $k^2 = \vec{k} \cdot \vec{k}$ . The second order ODE (5.14) can be readily solved to obtain the general solution

$$h_{\vec{k}}^{(s)}(\eta) = (-2H) \left[ E_{\vec{k}}^{(s)} (\eta \cos(k\eta) - (1/k) \sin(k\eta)) - B_{\vec{k}}^{(s)} (\eta \sin(k\eta) + (1/k) \cos(k\eta)) \right] \quad (5.15)$$

where  $E_{\vec{k}}^{(s)}$  and  $B_{\vec{k}}^{(s)}$  are arbitrary coefficients (in the Schwartz space), determined by the initial data of the solution. (These coefficients can also depend on  $\Lambda$ . We did not make this dependence explicit because in the main text we work with a fixed value of  $\Lambda$ .) Substituting (5.15) in (5.11) we obtain the general solutions  $h_{ab}$  representing first order perturbations.

Next, we discuss curvature.

### 5.2.2 $\mathcal{B}_{ab} = 0$ condition

Since the Weyl tensor of de Sitter space-time vanishes, the first order perturbations  ${}^{(1)}E_{ab}$  and  ${}^{(1)}B_{ab}$  of the electric and magnetic parts of the Weyl curvature are gauge invariant<sup>5</sup> and can be expressed directly in terms of the solutions  $h_{\vec{k}}(\eta)$  in (5.15). To find these expressions, we first note that, in exact general relativity, the electric and magnetic parts are related to the first and second fundamental forms  $q_{ab}$  and

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<sup>5</sup>A perturbed quantity  $\delta Q$  (indices suppressed) transforms under a gauge transformation  $\gamma_{ab} \rightarrow \gamma_{ab} + \nabla_a \xi_b$  as  $\delta Q \rightarrow \delta Q + \mathcal{L}_\xi \bar{Q}$  where  $\bar{Q}$  is the background quantity. If  $\bar{Q}$  is everywhere vanishing, then  $\delta Q$  is gauge invariant.

$K_{ab}$  on any space-like surface via

$$\begin{aligned} E_{ab} &= \mathcal{R}_{ab} - K_a{}^m K_{mb} + K K_{ab} - (1/2)(q_a{}^m q_b{}^n + q_{ab} q^{mn})({}^4R_{mn} - (1/6){}^4R g_{mn}) \\ B_{ab} &= \epsilon_{(a}{}^{mn} D_{|m|} K_{|n|b)} \end{aligned} \quad (5.16)$$

where  $D$ ,  $\epsilon_{abc}$  and  $\mathcal{R}_{ab}$  are the derivative operator, alternating tensor and the Ricci curvature of the 3-metric  $q_{ab}$ , and  ${}^4R_{ab}$  is the Ricci curvature of the space-time metric  $g_{ab}$ .

It is straightforward to linearize these equations using the cosmological foliation on the de Sitter background.<sup>6</sup> Calculations are simplified by noting that: (i)  $E_{ab}$  and  $B_{ab}$  are conformally invariant, and, (ii) a convenient conformal completion of de Sitter is provided by choosing the conformal factor  $\Omega = -H\eta$ , so that *the conformal metric  $\Omega^2 \bar{g}_{ab}$  that is well behaved at  $\mathcal{I}^+$  is just the Minkowski metric  $\dot{g}_{ab}$  in the  $(\eta, \vec{x})$  chart.* Therefore, in effect, linearization can be carried out using this flat background metric. The perturbed electric and magnetic parts of the Weyl tensor can be expressed using  $h_{ab}$  and geometric structures associated with the flat 3-metric  $\dot{q}_{ab}$  on each cosmological slice:

$${}^{(1)}E_{ab} = -\frac{1}{2} \left( \dot{D}^2 h_{ab} + \frac{1}{\eta} h'_{ab} \right), \quad \text{and} \quad {}^{(1)}B_{ab} = \frac{1}{2} \dot{\epsilon}_{(a}{}^{mn} \dot{D}_{|m|} h'_{|n|b)}. \quad (5.17)$$

Recall that the boundary conditions at  $\mathcal{I}^+$  imply that the Weyl curvature of an asymptotically de Sitter metric must vanish at  $\mathcal{I}^+$  [10]. Therefore, the first order perturbations  ${}^{(1)}E_{ab}$  and  ${}^{(1)}B_{ab}$  of Weyl curvature also vanish at  $\mathcal{I}^+$  and

$$\mathcal{E}_{ab} := \Omega^{-1} ({}^{(1)}E_{ab}), \quad \text{and} \quad \mathcal{B}_{ab} := \Omega^{-1} ({}^{(1)}B_{ab}) \quad (5.18)$$

admit smooth limits there. We will refer to  $\mathcal{E}_{ab}$  and the  $\mathcal{B}_{ab}$  as the *perturbed electric and magnetic parts of the Weyl curvature* as a short hand since it is these quantities that will feature in most of our discussion. Using explicit solutions (5.15) it is easy to verify that they do indeed admit smooth limits to  $\mathcal{I}^+$ :

$$\mathcal{E}_{ab}(\vec{x}, \eta) |_{\eta=0} = \int \frac{d^3 k}{(2\pi)^3} \sum_{(s)=1}^2 k^2 E_{\vec{k}}^{(s)} \mathbf{e}_{ab}^{(s)}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

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<sup>6</sup>Details of calculations are provided in Appendix B.

$$\mathcal{B}_{ab}(\vec{x}, \eta) |_{\eta=0} = \int \frac{d^3k}{(2\pi)^3} \sum_{(s)=1}^2 k^2 B_{\vec{k}}^{(s)} {}^* \mathbf{e}_{ab}^{(s)}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \quad (5.19)$$

where  ${}^* \mathbf{e}_{ab}^{(s)} = \epsilon_a{}^{mn} (k_n/k) \mathbf{e}_{mb}^{(s)}$  is the ‘dual’ of the polarization tensor. These formulas bring out the meaning of the coefficients  $E_{\vec{k}}^{(s)}$  and  $B_{\vec{k}}^{(s)}$  that feature in the expression (5.15) of a general solution to the linearized equations.  $E_{\vec{k}}^{(s)}$  directly determines the electric part of the perturbed Weyl tensor at  $\mathcal{I}^+$  and  $B_{\vec{k}}^{(s)}$  the magnetic part at  $\mathcal{I}^+$ .

It is therefore clear that the perturbed Weyl tensor has no magnetic part  $\mathcal{B}_{ab}(\vec{x}, \eta)$  at  $\mathcal{I}^+$  if and only if the solution  $h_{ab}$  has the form

$$h_{\vec{k}}^{(s)}(\eta) = (-2H) \left[ E_{\vec{k}}^{(s)} (\eta \cos k\eta - (1/k) \sin k\eta) \right] \quad (5.20)$$

everywhere, obtained by setting  $B_{\vec{k}}^{(s)} = 0$  in (5.15).<sup>7</sup> Thus, the condition that the magnetic part vanish at  $\mathcal{I}^+$  –or, that conformal flatness of the 3-metric at  $\mathcal{I}^+$  be preserved to first order– *removes, by fiat, half the degrees of freedom from consideration.*

We conclude this section with an important fact about the perturbed electric part of the Weyl tensor  $\mathcal{E}_{ab}$ . The linearized Weyl tensor satisfies conformally invariant equations. So its propagation on background de Sitter space-time is the same as that on background Minkowski – sharp i.e., it has support only on the light cone. On the other hand, the field  $h_{ab}$ , which can be thought of as propagating on Minkowski space-time, has support within the light cone as well. This can be seen from the presence of the ‘friction’ term  $(2/\eta) h'_{ab}$  in (5.10). However, interestingly, the time derivative  $h'_{ab}$  satisfies the conformally invariant equation,  $(\bar{\square} - ({}^4\bar{R}/6))h'_{ab} = 0$ . Equivalently, since  $\bar{g}_{ab} = (1/H^2\eta^2) \dot{g}_{ab}$ , it follows that  $\bar{\square} [(1/\eta) h'_{ab}] = 0$ . Therefore it follows that the propagation of  $(1/\eta) h'_{ab}$  on  $(M_{\text{P}}^+, \dot{g}_{ab})$ , and hence of  $h'_{ab}$  on  $(M_{\text{P}}^+, \bar{g}_{ab})$ , is in fact sharp, without any tail terms. This fact has an interesting implication in the discussion of the quadrupole formula in Chapter 6.

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<sup>7</sup>The explicit solution (5.15) shows that, as one approaches  $\mathcal{I}^+$  (i.e. as  $\eta \rightarrow 0$ ), the term associated with  $E_{\vec{k}}^{(s)}$  vanishes while the term associated with  $B_{\vec{k}}^{(s)}$  survives. In the cosmology literature, the first is referred to as the ‘decaying mode’ and the second as the ‘growing mode’. Thus, the requirement that the magnetic part of the perturbed Weyl curvature vanish at  $\mathcal{I}^+$  removes by fiat the growing mode and leaves only the decaying mode. These perturbations  $h_{ab}$  vanish at  $\mathcal{I}^+$ .

## 5.3 The Hamiltonian framework

This section is divided into two parts. In the first, we review the Hamiltonian framework for Maxwell fields on a de Sitter background. In the second, construct the covariant phase space of source-free, linearized gravitational fields on the de Sitter background.

### 5.3.1 Maxwell fields in de Sitter space-time

As is well-known, each Killing symmetry  $K^a$  leads to a conserved quantity. For matter fields –such as the Maxwell field  $F_{ab}$ – the standard procedure is to use the stress-energy tensor  $T_{ab} = \frac{1}{4\pi}(F_{am}F_{bn}\bar{g}^{mn} - (1/4)\bar{g}_{ab}F_{cd}F_{mn}\bar{g}^{cm}\bar{g}^{dn})$ . The conserved quantity associated with a Killing field  $K^a$  is given by

$$\mathcal{F}_K = \int_{\Sigma} T_{ab} K^a n^b d^3V_{\Sigma} \quad (5.21)$$

where the integral is taken over any Cauchy surface  $\Sigma$  with unit normal  $n^a$ .  $\mathcal{F}_K$  may be regarded as the ‘flux’ of the conserved quantity across  $\Sigma$ .

However, for the linearized gravitational field, we do not have a gauge invariant, locally defined stress-energy tensor. We will now show that, in the Maxwell theory, the expression (5.21) of  $\mathcal{F}_K$  can also be obtained using a covariant phase space framework without having to refer to the stress-energy tensor. In section 5.3 we will use this alternate method to calculate conserved quantities for the linearized gravitational field.

Consider a globally hyperbolic space-time,  $(M_{\mathbb{P}}^+, g_{ab})$  with a Killing field  $K^a$ . Denote by  $\Gamma_{\text{Cov}}^{\text{Max}}$  the space of all suitably regular, source-free solutions  $F_{ab}$  to Maxwell equations  $\bar{\nabla}_{[a}F_{bc]} = 0$  and  $\bar{g}^{ac}\bar{\nabla}_c F_{ab} = 0$ . Starting from the Maxwell Lagrangian, one can show that  $\Gamma_{\text{Cov}}^{\text{Max}}$  is naturally endowed with a symplectic structure (i.e., a closed, non-degenerate 2-form)  $\omega_{\text{Max}}$ :

$$\omega_{\text{Max}}(F, \underline{F}) = \frac{1}{4\pi} \int_{\Sigma} \bar{g}^{ac} [F_{ab}\underline{A}_c - \underline{F}_{ab}A_c] n^b d^3V_{\Sigma}. \quad (5.22)$$

Here  $F$  and  $\underline{F}$  are any two solutions to Maxwell equations,  $A_a$  is any vector potential for  $F_{ab}$  (i.e.,  $F_{ab} = 2\nabla_{[a}A_{b]}$ ) and  $\Sigma$  is again any Cauchy surface. Using Maxwell equations (and the suitable fall-off implicit in the regularity condition)

it is easy to verify that the right side is independent of the choice of the Cauchy surface  $\Sigma$  and is gauge invariant. The pair  $(\Gamma_{\text{Cov}}^{\text{Max}}, \omega_{\text{Max}})$  is the Maxwell covariant phase space. Each Killing field  $K^a$  on  $M_{\mathbb{P}}^+$  naturally defines a vector field  $\mathcal{K}$  on  $\Gamma_{\text{Cov}}^{\text{Max}}$  via:  $\mathcal{K}|_F \equiv \delta_K F := \mathcal{L}_K F_{ab}$ . Not surprisingly, the flow generated by  $\mathcal{K}$  on  $\Gamma_{\text{Cov}}^{\text{Max}}$  preserves the symplectic structure  $\omega_{\text{Max}}$ , i.e., defines a 1-parameter family of canonical transformations on  $(\Gamma_{\text{Cov}}^{\text{Max}}, \omega_{\text{Max}})$ . The Hamiltonian generating this flow is a function  $\mathcal{H}_K$  on  $\Gamma_{\text{Cov}}^{\text{Max}}$  given by:

$$\mathcal{H}_K := -\frac{1}{2}\omega_{\text{Max}}(F, \mathcal{L}_K F). \quad (5.23)$$

For any Killing field  $K^a$  one can verify that  $\mathcal{H}_K$  defined in (5.23) equals  $\mathcal{F}_K$  defined in (5.21). (For details on the covariant phase space of fields, including general relativity, see, e.g., [73].)

Let us illustrate this result for the Killing fields in the Poincaré patch. Let us first set  $K^a = S^a$ , where  $S^a$  stands for any one of the 6 Killing fields  $T_{(i)}^a$  and  $R_{(i)}^a$ , tangential to the space-like slices  $\Sigma$  given by  $\eta = \text{const}$ . Then, we have

$$\mathcal{F}_S = \frac{1}{4\pi} \int_{\Sigma} (F_{am} F_{bn} \bar{g}^{mn} S^a n^b) d^3V_{\Sigma} = \frac{1}{4\pi} \int_{\Sigma} (\epsilon_{abc} E^b B^c S^a) d^3V_{\Sigma} \quad (5.24)$$

where  $E_a := F_{ab} n^b$  and  $B_a := {}^*F_{ab} n^b$  are the electric and magnetic parts of the Maxwell field, and  $\epsilon_{abc}$  the alternating tensor on the slice  $\Sigma$ . Thus, as one would expect,  $\mathcal{F}_S$  is the flux of the  $S$ -component of the Poynting vector  $\epsilon_{abc} E^b B^c$  across  $\Sigma$ . Next, let us consider the Hamiltonian (5.23) generated by  $S$ :

$$\begin{aligned} \mathcal{H}_S &= -\frac{1}{8\pi} \int_{\Sigma} \bar{g}^{ac} [(\mathcal{L}_S F_{ab}) A_c - F_{ab} (\mathcal{L}_S A_c)] n^b d^3V_{\Sigma} \\ &= -\frac{1}{4\pi} \int_{\Sigma} \bar{g}^{bc} F_{ac} (\mathcal{L}_S A_b) n^a d^3V_{\Sigma} = \frac{1}{4\pi} \int_{\Sigma} (\epsilon_{abc} E^b B^c S^a) d^3V_{\Sigma} \end{aligned} \quad (5.25)$$

where in the first step we have integrated by parts and in the second step used Cartan identity and the Maxwell equation  $\bar{D}_a E^a = 0$ . Thus, using the covariant phase space we can recover the conserved quantity  $\mathcal{F}_S$  as the Hamiltonian  $\mathcal{H}_S$  defined by the Killing symmetry  $S^a$ . Because of conformal invariance of Maxwell equations, we can easily take the limit as  $\Sigma$  approaches  $\mathcal{I}^+$  and express the conserved flux as an integral over  $\mathcal{I}^+$ . The expression (5.24) brings out the fact that if the magnetic field vanishes at  $\mathcal{I}^+$ , then that electromagnetic wave carries no angular momentum

or linear momentum.

For the time translation  $T^a$ , the argument establishing the equality of  $\mathcal{F}_K$  and  $\mathcal{H}_K$  is the same but the calculation is a little more involved because  $T^a$  has components both along *and* orthogonal to the cosmological slices (see Eq.(5.3)). We find:

$$\mathcal{F}_T = \mathcal{H}_T = \frac{1}{8\pi} \int_{\Sigma} \left[ (E_a E_b + B_a B_b) \bar{g}^{ab} + 2\epsilon_{abc} E^b B^c T^a \right] d^3V_{\Sigma}. \quad (5.26)$$

In the limit as  $\Sigma$  approaches  $\mathcal{I}^+$ ,  $T^a$  becomes tangential to  $\mathcal{I}^+$  (since  $\eta = 0$  at  $\mathcal{I}^+$ ) and  $\bar{g}^{ab}$  vanishes. Therefore the expression of the conserved energy reduces to an integral of the component of the Poynting vector along  $T^a$ :

$$\mathcal{F}_T = \mathcal{H}_T = \frac{1}{4\pi} \int_{\mathcal{I}^+} (\epsilon_{abc} E^b B^c T^a) d^3V_{\mathcal{I}^+} \quad (5.27)$$

where the electric and magnetic fields and the alternating tensor are calculated using any conformally rescaled metric that is regular at  $\mathcal{I}^+$  (e.g.,  $\mathring{g}_{ab}$ ). This expression brings out two interesting facts. First, in de Sitter space-time while the energy carried by electromagnetic waves is conserved as in Minkowski space-time, now it can be *negative* and is *unbounded below*. Second, if we restrict ourselves to Maxwell fields whose magnetic field vanishes at  $\mathcal{I}^+$ , then those electromagnetic fields carry no energy either. Note that the second result is specific to  $\mathcal{I}^+$ : If the magnetic field vanishes on a cosmological slice  $\eta = \text{const} \neq 0$ , the energy of that Maxwell field does *not* vanish unless the Maxwell field itself vanishes identically. The 3-momentum and the angular momentum, on the other hand do vanish.

To summarize, for Maxwell fields, the conserved quantities associated with Killing fields in the Poincaré patch can be recovered as Hamiltonians on the covariant phase space, without any reference to the stress-energy tensor. Also, because all Killing fields  $K^a$  on de Sitter space-time are tangential to  $\mathcal{I}^+$  –and hence space-like– one can express every conserved quantity  $F_K$  as an integral across  $\mathcal{I}^+$  of the component of the Poynting vector along  $K^a$ . This expression brings out the fact that if we were to require that the magnetic field vanish at  $\mathcal{I}^+$ , we would be left with electromagnetic waves *that carry no 3-momentum or angular momentum, nor energy defined by de Sitter isometries!*

### 5.3.2 The covariant phase space of linearized gravitational fields

For linearized gravitational fields, the covariant phase space  $\Gamma_{\text{Cov}}$  can be taken to be the space of solutions  $\gamma_{ab}$  to the equations (5.6) and (5.7). For simplicity, we will assume that the solutions of interest have initial data in the Schwartz space of rapidly decreasing, smooth fields, although these conditions can be weakened considerably. The standard procedure (see, e.g. [73]) endows  $\Gamma_{\text{Cov}}$  with a symplectic structure  $\omega$ . Restricted to the cosmological slices  $\Sigma$  (given by  $\eta = \text{const}$ ), it becomes:

$$\omega(\gamma, \underline{\gamma}) \equiv \omega(h, \underline{h}) := \frac{a^2(\eta)}{4\kappa} \int_{\Sigma} d^3x \left( h_{ab} \underline{h}'_{cd} - h'_{ab} \underline{h}_{cd} \right) \dot{q}^{ac} \dot{q}^{bd}, \quad (5.28)$$

where  $h_{ab}$  is related to the physical metric perturbation  $\gamma_{ab}$  via  $\gamma_{ab} = a^2 h_{ab}$  (see Eq. (5.8)) and  $\kappa = 8\pi G$ . It is easy to verify that (5.9) and (5.10) imply that the integral is independent of the  $\eta = \text{const}$  slice on which it is evaluated. This form of the symplectic structure is useful in calculations within the Poincaré patch. Furthermore, as we will see in section 5.4, it is well-adapted for taking the limit  $\Lambda \rightarrow 0$ .

In the cosmology literature, one often works with the functions  $h_k^{(s)}(\eta)$  defined in (5.11) and their Fourier transforms

$$\phi^{(s)}(\vec{x}, \eta) := \frac{1}{\sqrt{4\kappa}} \int \frac{d^3k}{(2\pi)^3} h_k^{(s)}(\eta) e^{i\vec{k}\cdot\vec{x}} \quad (5.29)$$

in place of the tensor fields  $\gamma_{ab}$  or  $h_{ab}$ . (The factor of  $\sqrt{4\kappa}$  is introduced to endow  $\phi^{(s)}$  with the standard dimensions of a scalar field, so that the scalar and tensor perturbations can be treated in a completely parallel manner. See, e.g., section 3.D of [74].) These are referred to as the two *tensor modes*. It is straightforward to verify that these fields satisfy the wave equation in de Sitter space-time

$$\square \phi^{(s)}(\vec{x}, \eta) = 0. \quad (5.30)$$

Thus, each tensor mode  $\phi^{(s)}$  of the linearized gravitational field satisfies just the massless Klein-Gordon equation. It is clear that, given fixed polarization tensors  $\mathbf{e}_{ab}^{(s)}(\vec{k})$ , there is a natural isomorphism between the functions  $\phi^{(s)}(\vec{x}, \eta)$  and solutions  $h_{ab}(\vec{x}, \eta)$  to the linearized Einstein equation (5.10) and gauge conditions

(5.9). It is easy to check that the symplectic structure (5.28) on  $\Gamma_{\text{Cov}}$  translates to the standard symplectic structure on the covariant phase space  $\Gamma_{\text{Cov}}^{\text{KG}}$  consisting of pairs of solutions  $\phi \equiv \{\phi^{(s)}(\vec{x}, \eta)\}$  to the Klein-Gordon equation:

$$\omega_{\text{KG}}(\phi, \underline{\phi}) = a^2(\eta) \int_{\Sigma} d^3x \sum_{(s)=1}^2 \left( \phi^{(s)} (\underline{\phi}^{(s)})' - (\phi^{(s)})' \underline{\phi}^{(s)} \right). \quad (5.31)$$

This form of the symplectic structure is useful to compute expressions of fluxes of energy-momentum and angular momentum that are adapted to the ‘tensor modes’ used in the cosmological perturbation theory.

However, expressions (5.28) and (5.31) of the symplectic structure have one drawback: because of the multiplicative factor  $a^2(\eta) = (1/H^2\eta^2)$ , they are not well-suited to take the limit to  $\mathcal{I}^+$  (where  $\eta = 0$ ). While, the limit itself is well defined because the symplectic structure is independent of  $\eta$ , to express physical results –e.g. the formula of energy– in terms of fields that are well defined at  $\mathcal{I}^+$ , one has to be extremely careful in keeping track of terms in the integrand which tend to zero at the appropriate rate to compensate for the apparent blow up as  $1/\eta^2$  due to the pre-factor in front of the integral. Also, these expressions are not gauge invariant as they use specific gauge conditions (5.9). To overcome these limitations, it is convenient to recast the expression (5.28) using the relation between the perturbed electric part of the Weyl tensor  $\mathcal{E}_{ab}$  and the metric perturbation,

$$2\mathcal{E}_{ab}(\vec{x}, \eta) = \frac{1}{H\eta^2} \left( h'_{ab} + \eta \dot{D}^2 h_{ab} \right), \quad (5.32)$$

that holds on any cosmological slice. Substituting for  $h'_{ab}$  in terms of  $\mathcal{E}_{ab}$  and simplifying by performing integrations by parts, we obtain:

$$\omega(h, \underline{h}) = \frac{1}{2H\kappa} \int_{\Sigma} d^3x \left( h_{ab} \underline{\mathcal{E}}_{cd} - \underline{h}_{ab} \mathcal{E}_{cd} \right) \dot{q}^{ac} \dot{q}^{bd}. \quad (5.33)$$

We will use both expressions, (5.28) and (5.33), of the symplectic structure on  $\Gamma_{\text{Cov}}$  in our discussion of the conserved fluxes associated with the 7 Killing vectors. (The equivalent form (5.31) in terms of the Klein-Gordon fields  $\phi^{(s)}$  turns out not to be as useful in providing hints for the full, nonlinear theory.)

We will conclude this discussion by pointing out several consequences that follow immediately from the form (5.33) of the symplectic structure. First, it is

transparent that  $(1/2H\kappa)\mathcal{E}^{ab}$  can be regarded as the momentum that is canonically conjugate to the metric perturbation  $h_{ab}$ . Second, as we saw in section 5.2, the perturbations  $h_{ab}$  as well as the perturbed electric part of the Weyl tensor  $\mathcal{E}_{ab}$  admit well defined limits to  $\mathcal{I}^+$ . Therefore, one can take the limit  $\Sigma \rightarrow \mathcal{I}^+$  simply by evaluating the integral (5.33) on  $\mathcal{I}^+$ . This feature will facilitate our task of expressing energy, momentum and angular momentum in terms of asymptotic fields at  $\mathcal{I}^+$ . In turn, these expressions will be directly useful in Chapter 6 to obtain a formula for the energy emitted by a time changing quadrupole, and establishing its positivity. The third and more important feature is gauge invariance. Note first that  $\mathcal{E}_{ab}$  by itself is gauge invariant, it is tangential to the cosmological slices, and it is divergence- and trace-free. This fact enables us to drop the gauge fixing conditions (5.9) and consider general perturbations. For, if either  $\gamma_{ab}$  (or,  $\underline{\gamma}_{ab}$ ) is a pure gauge field –i.e. of the form  $\bar{\nabla}_{(a}\xi_{b)}$  for a space-time vector field  $\xi^a$ – properties of  $\mathcal{E}_{ab}$  ensure that the expression (5.33) of  $\omega(h, \underline{h})$  vanishes identically. Thus, the passage from  $h'_{ab}$  to  $\mathcal{E}_{ab}$  using (5.32) has provided us with a manifestly gauge invariant expression (5.33) of the symplectic structure. Finally, using the explicit solutions (5.15), we can re-express the symplectic structure in terms of the coefficients  $E_{\vec{k}}$  and  $B_{\vec{k}}$ :

$$\omega(h, \underline{h}) = \frac{1}{\kappa} \int \frac{d^3k}{(2\pi)^3} \sum_{(s)=1}^2 k \left( (B_{\vec{k}}^{(s)})^* \underline{E}_{\vec{k}}^{(s)} - (\underline{B}_{\vec{k}}^{(s)})^* E_{\vec{k}}^{(s)} \right), \quad (5.34)$$

where, as before,  $\star$  denotes complex-conjugation. Consequently, the pull-back of the symplectic structure to the subspace of  $\Gamma_{\text{Cov}}$  on which  $\mathcal{B}_{ab}$  vanishes on  $\mathcal{I}^+$  –or, alternatively, on which  $\mathcal{E}_{ab}$  vanishes on  $\mathcal{I}^+$ – is identically zero. These subspaces are among the maximal Lagrangian subspaces of  $\Gamma_{\text{Cov}}$ . In this respect the situation is again completely parallel to that in the Maxwell theory.

## 5.4 Energy, 3-momentum and angular momentum carried by gravitational waves

We can now calculate the Hamiltonians on  $\Gamma_{\text{Cov}}$  corresponding to the seven Killing fields on the Poincaré patch. Recall from (5.8) that the physical metric perturbation is  $\gamma_{ab} = a^2 h_{ab}$  and it satisfies the gauge conditions (5.6) and linearized

Einstein's equation (5.7) that refer only to the background de Sitter metric  $\bar{g}_{ab}$ . Therefore, if  $\gamma_{ab} \in \Gamma_{\text{Cov}}$ , then so is  $\gamma_{ab}^{(K)} := \mathcal{L}_K \gamma_{ab}$ , for any Killing field  $K^a$  of  $\bar{g}_{ab}$ . From the definition (5.8) of  $h_{ab}$ , it follows that

$$\gamma_{ab}^{(K)} = a^2 (\mathcal{L}_K h_{ab} + 2(a^{-1} \mathcal{L}_K a) h_{ab}) =: a^2 h_{ab}^{(K)} \quad (5.35)$$

with  $a = -1/(H\eta)$ . As in the Maxwell case, the isometries generated by each of the seven Killing fields  $K^a$  in  $\mathfrak{g}_{\text{Poin}}$  provide a 1-parameter family of canonical transformations on  $\Gamma_{\text{Cov}}$ . From general results on the covariant phase space [73] it follows that the corresponding Hamiltonian is again given by

$$\mathcal{H}_K := -\frac{1}{2} \omega(\gamma, \gamma^{(K)}) = -\frac{1}{2} \omega(h, h^{(K)}). \quad (5.36)$$

Recall from section 5.2 that *if  $\mathcal{B}_{ab} = 0$  at  $\mathcal{I}^+$ , then  $h_{ab}$  also vanishes there*. In this case, then, we have  $\mathcal{H}_S = 0$ . Thus, although there do exist linearized gravitational waves that retain conformal flatness of the induced geometry at  $\mathcal{I}^+$  to first order, *they carry no energy, 3-momentum or angular momentum*.

We will now compute the Hamiltonians (5.36) for the seven Killing fields in  $\mathfrak{g}_{\text{Poin}}$ .

### 5.4.1 3-momentum and angular momentum

As in the case of Maxwell fields discussed in section 5.3.1, the calculations are identical for the 3 spatial translations  $T_i^a$  and the 3 rotations  $R_i^a$ . Let us therefore again denote by  $S^a$  any of these six Killing fields and calculate the 3-momentum or angular momentum  $\mathcal{H}_S$ , and then discuss energy  $\mathcal{H}_T$  separately. For these six Killing fields, we have  $h_{ab}^{(S)} = \mathcal{L}_S h_{ab}$  since these fields are all tangential to the  $\eta = \text{const}$  surfaces. Furthermore, from (5.32) it follows that the corresponding perturbed electric part of the Weyl tensor,  $\mathcal{E}_{ab}^{(S)}$ , is given by:

$$\mathcal{E}_{ab}^{(S)} = \frac{1}{2H\eta^2} (\mathcal{L}_\eta (\mathcal{L}_S h_{ab}) + \eta \mathring{D}^2 (\mathcal{L}_S h_{ab})) = \mathcal{L}_S \mathcal{E}_{ab} \quad (5.37)$$

Therefore (5.36) becomes:

$$\mathcal{H}_S = -\frac{1}{2} \omega(h, h^{(S)}) = -\frac{1}{4H\kappa} \int_{\Sigma} d^3x (h_{ab} \mathcal{E}_{cd}^{(S)} - h_{ab}^{(S)} \mathcal{E}_{cd}) \mathring{q}^{ac} \mathring{q}^{cd}$$

$$= \frac{1}{2H\kappa} \int_{\Sigma} d^3x \left( \mathcal{E}_{ab} \mathcal{L}_S h_{cd} \right) \dot{q}^{ac} \dot{q}^{cd}, \quad (5.38)$$

where, in the second step we have integrated by parts. Thus, the expressions of 3-momentum and angular momentum mirror those in the Maxwell theory. Since the integrand in (5.38) refers only to the fields  $h_{ab}$ ,  $\mathcal{E}_{ab}$  and the metric  $\dot{q}_{ab}$ , all of which have smooth limits to  $\mathcal{I}^+$ , to take the limit  $\Sigma \rightarrow \mathcal{I}^+$  we just have to evaluate (5.38) on  $\mathcal{I}^+$ .

Finally, let us consider the limit  $\Lambda \rightarrow 0$  of  $\mathcal{H}_S$ . Since the Hubble parameter  $H$  tends to zero in this limit, from the form of (5.38), the limit seems divergent at first sight. However, this conclusion is incorrect because fields in the integrand also depend on  $H$ . Let us therefore analyze the limit more carefully. As explained in section 5.2, to take this limit, we should use the differential structure induced by the chart  $(t, \vec{x})$  on the Poincaré patch (and not by the chart  $(\eta, \vec{x})$ ). Then, for the background geometry, we find that as  $\Lambda \rightarrow 0$ , we have

$$\begin{aligned} \bar{g}_{ab} &\rightarrow \eta_{ab} = -\partial_a t \partial_b t + \partial_a x \partial_b x + \partial_a y \partial_b y + \partial_a z \partial_b z; \\ H\eta &\equiv -e^{-Ht} \rightarrow -1; \quad \frac{\partial}{\partial \eta} \rightarrow \frac{\partial}{\partial t} \equiv t^a \partial_a; \quad T^a \rightarrow t^a. \end{aligned} \quad (5.39)$$

Note that the Minkowski metric  $\eta_{ab}$  in (5.39) is distinct from the Minkowski metric  $\dot{g}_{ab}$  in (5.8). In the Poincaré patch, each of the Cartesian coordinates,  $(t, \vec{x})$  of  $\eta_{ab}$  takes the full range of values,  $(-\infty, \infty)$  (whereas  $\eta$ , of  $\dot{g}_{ab}$  only runs from  $(-\infty, 0)$ ). A second important point for the limit is that the  $(t, \vec{x})$  chart does not cover  $\mathcal{I}^+$  (where  $t = \infty$ ). Therefore, to take the  $\Lambda \rightarrow 0$  limit, we are led to evaluate the symplectic structure and Hamiltonians  $\mathcal{H}_S$  on a cosmological slice corresponding to a finite constant value of  $t$ .

Consider any 1-parameter family of smooth fields  $h_{ab}(\Lambda)$  which solve the gauge condition (5.9) and the linearized Einstein equation (5.10) for each  $\Lambda$  and admit a smooth limit  $\dot{h}_{ab}$  as  $\Lambda \rightarrow 0$  everywhere on the Poincaré patch. Then, using (5.39), it is straightforward to show that  $\dot{h}_{ab}$  satisfies the Lorentz and radiation gauge, as well as the linearized Einstein's equation with respect to the Minkowski metric  $\eta_{ab}$ :

$$\partial^a \dot{h}_{ab} = 0; \quad \dot{h}_{ab} \eta^{ab} = 0; \quad \dot{h}_{abt} = 0; \quad \text{and} \quad \partial^c \partial_c \dot{h}_{ab} = 0. \quad (5.40)$$

Denote the space of solutions  $\dot{h}_{ab}$  to these equations by  $\mathring{\Gamma}_{\text{cov}}$ . It is straightforward

to verify that in the limit  $\Lambda \rightarrow 0$ , the symplectic structure  $\omega$  on  $\Gamma_{\text{Cov}}$  goes over to the standard symplectic structure  $\dot{\omega}$  on  $\dot{\Gamma}_{\text{cov}}$ :

$$\dot{\omega}(\dot{h}, \dot{h}) = \frac{1}{4\kappa} \int_{\Sigma} d^3x \left( \dot{h}_{ab} \partial_t(\dot{h}_{cd}) - \partial_t(\dot{h}_{ab}) \dot{h}_{cd} \right) \dot{q}^{ac} \dot{q}^{bd}, \quad (5.41)$$

where the integral is taken on a  $t = \text{const}$  slice. Finally, the limit  $\dot{\mathcal{H}}_S$  of the Hamiltonian  $\mathcal{H}_S = (-1/2) \omega(h, h^{(S)})$  is given by:

$$\begin{aligned} \dot{\mathcal{H}}_S &= -\frac{1}{2} \dot{\omega}(\dot{h}, \mathcal{L}_S \dot{h}) \\ &= \frac{1}{4\kappa} \int_{\Sigma} d^3x \left( \partial_t(\dot{h}_{ab}) \mathcal{L}_S \dot{h}_{cd} \right) \dot{q}^{ac} \dot{q}^{bd}. \end{aligned} \quad (5.42)$$

This is precisely the expression of the linear and angular momentum of linearized gravitational waves in Minkowski space-time. Thus, although the procedure of taking the limit  $\Lambda \rightarrow 0$  is rather subtle, *the de Sitter 3-momentum and angular momentum (5.38) do reduce to the standard conserved fluxes in Minkowski space-time*. While taking the limit, we assumed the existence of a family  $h_{ab}(\Lambda)$ , satisfying the gauge conditions (5.9) and linearized Einstein equation (5.10) for each  $\Lambda$ , that admits a smooth limit  $\dot{h}_{ab}$  on the Poincaré patch as  $\Lambda \rightarrow 0$ . In Appendix B we construct an explicit example.

## 5.4.2 Energy

Next, let us consider the energy  $\mathcal{H}_T$  defined by the time translation  $T^a$  of (5.3). In this case, the calculation is not as straightforward because: (i) the vector field  $T^a$  is *not* tangential to the cosmological slices  $\eta = \text{const}$  except at  $\mathcal{I}^+$ ; (ii)  $h_{ab}^{(T)}$  has an extra term relative to  $h_{ab}^{(S)}$ ,

$$h_{ab}^{(T)} = \mathcal{L}_T h_{ab} + 2H h_{ab}; \quad (5.43)$$

and, (iii) a detailed calculation shows that  $\mathcal{E}_{ab}^{(T)}$  also has an extra term:

$$\mathcal{E}_{ab}^{(T)} = \mathcal{L}_T \mathcal{E}_{ab} - H \mathcal{E}_{ab}. \quad (5.44)$$

Once these differences are taken into account, the conserved energy-flux  $\mathcal{H}_T$  across  $\Sigma$  can be calculated using (5.36). We have:

$$\mathcal{H}_T = -\frac{1}{4H\kappa} \int_{\Sigma} d^3x \left( h_{ab} \mathcal{L}_T \mathcal{E}_{cd} - \mathcal{E}_{cd} \mathcal{L}_T h_{ab} - 3H h_{ab} \mathcal{E}_{cd} \right) \mathring{q}^{ac} \mathring{q}^{bd} \quad (5.45)$$

where, as before,  $\Sigma$  is any cosmological slice. However, since  $T$  is not tangential to  $\Sigma$ , on a general cosmological slice we cannot integrate by parts as we did for  $\mathcal{H}_S$ . Again the limit to  $\mathcal{I}^+$  is straightforward since all fields in the integrand have smooth limits to  $\mathcal{I}^+$ . Furthermore, in the limit  $T^a$  becomes tangential to  $\mathcal{I}^+$  enabling us to simplify (5.45) further:

$$\mathcal{H}_T = \frac{1}{2H\kappa} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathcal{L}_T h_{ab} + 2H h_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd} \quad (5.46)$$

$$= \frac{H}{\kappa} \int \frac{d^3k}{(2\pi)^3} k \sum_{(s)=1}^2 \left( E_{\vec{k}}^{(s)} \mathcal{L}_k (B_{\vec{k}}^{(s)})^* + 2 E_{\vec{k}}^{(s)} (B_{\vec{k}}^{(s)})^* \right), \quad (5.47)$$

where in the second step we have used the explicit solutions (5.15) for  $h_{ab}$  in terms of Fourier modes. Since  $\mathcal{B}_{ab} = 0$  if and only if  $B_{\vec{k}}^{(s)} = 0$ , the last expression makes it explicit that if a gravitational wave does not change conformal flatness of the intrinsic geometry at  $\mathcal{I}^+$  to first order, it does not carry energy. Finally, the expression (5.36) of  $\mathcal{H}_K$  is linear in  $K^a$  for all Killing fields. Therefore,  $\mathcal{H}_{\lambda T} = \lambda \mathcal{H}_T$  for all real numbers  $\lambda$ . For linearized gravitational waves on Minkowski space-time, energy is positive definite and vanishes if and only if the perturbation is pure gauge. On the de Sitter space-time, the conserved energy  $\mathcal{H}_T$  can have either sign and we have an infinite dimensional subspace of the physical, transverse-traceless modes for which the energy vanishes. From (5.46) it is clear that energy also vanishes if  $\mathcal{E}_{ab}$  vanishes on  $\mathcal{I}^+$ . (The other possibility,  $\mathcal{L}_T h_{ab} = -2H h_{ab}$  on  $\mathcal{I}^+$  is not realized because such perturbations would not be in the Schwartz space on  $\mathcal{I}^+$  of the Poincaré patch, which is topologically  $\mathbb{R}^3$ .)

Finally, let us consider the limit  $\Lambda \rightarrow 0$  of the conserved energy  $\mathcal{H}_T$ . For reasons given in section 5.1, we have to use the differential structure induced by the coordinates  $(t, \vec{x})$  and work with a cosmological slice in the Poincaré patch with  $\eta \neq 0$ . Let us again suppose that we have a 1-parameter family of perturbations  $h_{ab}(\Lambda)$  that satisfy the gauge conditions (5.9) and the linearized Einstein equation (5.10), and admit a smooth limit  $\mathring{h}_{ab}$  as  $\Lambda \rightarrow 0$ . As discussed above,  $\mathring{h}_{ab}$  is a metric

perturbation on the Minkowski metric  $\eta_{ab}$ , satisfying (5.40). The limit  $\mathring{\mathcal{H}}_t$  of the Hamiltonian  $\mathcal{H}_T = (-1/2) \omega(h, h^{(T)})$  is given by:

$$\begin{aligned} \mathring{\mathcal{H}}_T &= -\frac{1}{8\kappa} \int d^3x \left( \mathring{h}_{ab} \partial_t^2 \mathring{h}_{cd} - (\partial_t \mathring{h}_{ab})(\partial_t \mathring{h}_{cd}) \right) \mathring{q}^{ac} \mathring{q}^{bd} \\ &= \frac{1}{8\kappa} \int d^3x \left( (\partial_t \mathring{h}_{ab})(\partial_t \mathring{h}_{cd}) + (\mathring{D}_m \mathring{h}_{ab})(\mathring{D}_n \mathring{h}_{cd}) \mathring{q}^{mn} \right) \mathring{q}^{ac} \mathring{q}^{bd}. \end{aligned} \quad (5.48)$$

where in the second step we have used (5.40) and integrated by parts. This is precisely the conserved energy flux of the linearized gravitational field  $\mathring{h}_{ab}$  in Minkowski space-time. Thus, our energy expression (5.45) for linearized gravitational fields in de Sitter space-time does have the expected limit as  $\Lambda \rightarrow 0$ . Note that the limit is quite subtle and discontinuous: *While  $\mathcal{H}_T$  can be negative and arbitrarily large, no matter how small the positive  $\Lambda$  is, in the limit  $\Lambda \rightarrow 0$  we obtain  $\mathring{\mathcal{H}}_T$  which is positive definite!* Geometrically, this occurs because while the Killing field  $T^a$  of de Sitter metric  $\bar{g}_{ab}$  is space-like in the ‘upper half of the Poincaré patch’ for every  $\Lambda > 0$ , its limit, the Killing field  $t^a$  of  $\eta_{ab}$ , is time-like everywhere.

*Remarks:*

(i) In the cosmological literature, the discussion of ‘energy’ often refers to the Hamiltonians  $\mathcal{H}_\eta$  or  $\mathcal{H}_t$  that generate evolution along the conformal time  $\eta$  or proper time  $t$ . Since  $\eta^a$  and  $t^a$  are not Killing fields, these Hamiltonians are *not* conserved. Thus, they are unrelated to the conserved energy  $\mathcal{H}_T$  discussed above and are *not* the analogs of the standard notion of energy in Minkowski space-time, used in the gravitational radiation theory.

(ii) As discussed in section 5.2.1, in cosmology one often encodes the metric perturbations  $\gamma_{ab}(\vec{x}, \eta)$  in the two ‘tensor modes’  $\phi^{(s)}(\vec{x}, \eta)$  that satisfy the Klein-Gordon equation with respect to the de Sitter metric  $\bar{g}_{ab}$ . On the Klein-Gordon phase space  $\Gamma_{\text{Cov}}^{\text{KG}}$ , the isometry generated by any Killing field  $K^a$  again defines a 1-parameter family of transformations that preserve the symplectic structure  $\omega_{\text{KG}}$ . As one would expect, the corresponding Hamiltonians agree with the  $\mathcal{H}_K$  obtained above for all seven Killing fields. That is, our energy-momentum and angular momentum expressions  $\mathcal{H}_T$  and  $\mathcal{H}_S$  hold both for the metric perturbations  $\gamma_{ab}$  satisfying (5.6) and (5.7) and the ‘tensor modes’  $\phi^{(s)}$  satisfying the wave equation (5.30) in the Poincaré patch.

(iii) Finally, we note that the explicit solutions (5.15) are widely used in the cosmo-

logical literature on linearized gravitational waves. However, the primary interest there is on the effect of these gravitational waves on the polarization of the CMB electromagnetic waves. To our knowledge, this literature does not contain the analysis of the asymptotic behavior of these perturbations at  $\mathcal{I}^+$ , or the implications of the assumption that the perturbations preserve conformal flatness of  $\mathcal{I}^+$  to linear order, nor a discussion on the isometry group  $\mathfrak{G}_{\text{Poin}}$  that preserves the Poincaré patch, or the associated conserved fluxes  $\mathcal{H}_K$  given above.

## 5.5 Discussion

In the  $\Lambda = 0$  case, there is a well-developed theory of isolated systems and gravitational radiation in full, nonlinear general relativity that has played a dominant role in a number of areas of gravitational science. In Chapter 4, we showed that there are significant conceptual obstacles in extending this theory to allow a positive cosmological constant, however small, because the limit  $\Lambda \rightarrow 0$  is discontinuous. In particular, whereas  $\mathcal{I}$  is space-like, no matter how small  $\Lambda$  is, it is null when  $\Lambda$  vanishes. If  $\Lambda$  were zero and the accelerated expansion of the universe is caused by some matter field rather than a cosmological constant, that field would not have the asymptotic fall-off we are familiar with in the  $\Lambda = 0$  case, and space-time curvature far away from the sources would be similar to that in asymptotically de Sitter space-times. Therefore, difficulties discussed in Chapter 4 would persist also in the  $\Lambda = 0$  case if the observed accelerated expansion continues to infinite future. To overcome these obstacles, one needs a new framework. In this chapter we completed the first step to this goal by discussing linear gravitational waves in de Sitter space-time.

Motivated by considerations of isolated systems discussed in section 5.1, we focused on the upper Poincaré patch of de Sitter space-time. Isometries generated by 7 of the 10 de Sitter Killing fields leave this patch invariant. This group  $\mathfrak{G}_{\text{Poin}}$  is generated by 3 space-translations and 3 rotations that are tangential to the cosmological slices and a time translation that is transversal to them. Therefore, one expects well defined notions of linear and angular momentum, and energy, associated with any physical field on the Poincaré patch. We showed in section 5.3.1 that, in the case of Maxwell fields, these ‘conserved fluxes’ arise as the Hamiltonians generating canonical transformations induced by the action of Killing fields

on the covariant phase space  $\Gamma_{\text{Cov}}^{\text{Max}}$ . Furthermore, in the  $\Lambda = 0$  case, the Hamiltonian framework has been used very effectively also for gravitational waves in *full, nonlinear* general relativity: It leads to flux integrals corresponding to the Bondi-Metzner-Sachs (BMS) asymptotic symmetries [17]. Therefore, it is natural to use this strategy also in the  $\Lambda > 0$  case.

Since the covariant phase space consists of solutions to the field equations, in section 5.2.1 we discussed the asymptotic properties of solutions to linearized Einstein's equation in de Sitter space-time. In section 5.2 we constructed the covariant phase space  $(\Gamma_{\text{Cov}}, \omega)$  of these linear gravitational waves. Each of the 7 Killing fields  $K^a$  naturally defines flow on  $\Gamma_{\text{Cov}}$  that preserves the symplectic structure  $\omega$  thereon, and thus defines a Hamiltonian  $\mathcal{H}_K$ . These Hamiltonians provided us with the expressions (5.38) and (5.45) of fluxes of energy-momentum and angular momentum carried by gravitational waves. Furthermore, we could express these conserved fluxes in terms of fields defined on  $\mathcal{I}^+$ .

These results have a number of interesting features. First, to make the full nonlinear theory manageable in the  $\Lambda > 0$  case, at first it seems natural to strengthen the boundary conditions by requiring that the intrinsic geometry of  $\mathcal{I}^+$  be conformally flat, as in de Sitter space-time. However, almost 30 years ago Friedrich showed that the freely specifiable data at  $\mathcal{I}^-$  consists, up to arbitrary conformal rescalings, of a freely specifiable Riemannian metric and a trace-free, symmetric tensor field of valence two, which satisfies a divergence equation [56]. Therefore, by applying those results to  $\mathcal{I}^+$  (in place of  $\mathcal{I}^-$ ), it follows that demanding conformal flatness of the metric at  $\mathcal{I}^+$  removes by hand part of this free data. In the linearized approximation, we could sharpen the implication of this condition. First, because the perturbed electric and magnetic parts of the Weyl curvature are gauge invariant, we can discuss the physical or true degrees of freedom, not just the freely specifiable data. Second, we could parametrize the gauge invariant content of a general linearized solution in terms of 4 functions  $E^{(s)}$  and  $B^{(s)}$  on  $\mathcal{I}^+$  that capture these true (phase space) degrees of freedom. Finally, we showed that the additional condition at  $\mathcal{I}^+$  sets  $B^{(s)} = 0$ . Therefore, in the linear approximation one sees explicitly that this condition cuts the true degrees of freedom in gravitational waves exactly by half. Furthermore, the gravitational waves that do satisfy this condition carry no energy-momentum or angular momentum! Thus, although this strategy of gaining control over the nonlinear theory seems plausible

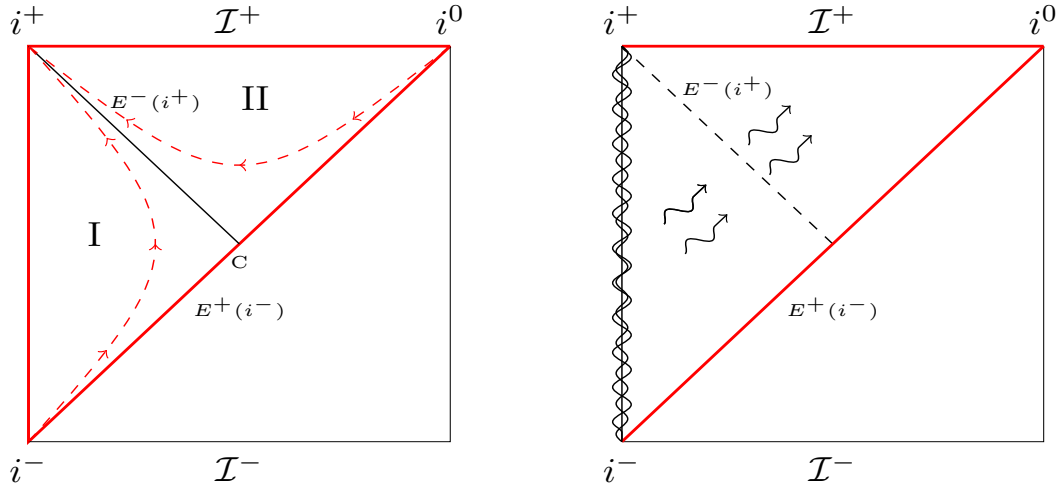
at first, it is simply not viable. By isolating the true degrees of freedom at  $\mathcal{I}^+$ , it should be possible to show that this sharper results holds also in full general relativity with positive  $\Lambda$ .

Second, we found that the conserved energy has a peculiar feature: For matter fields as well as linearized gravity, energy  $\mathcal{H}_T$  defined by the time translation  $T^a$  can have either sign and, furthermore, is *unbounded* below. Thus, there exist both electromagnetic and gravitational waves on de Sitter space-time which carry *arbitrarily negative* energy, no matter *how small* the positive  $\Lambda$  is! This is in striking contrast with the  $\Lambda = 0$  situation, where the corresponding waves carry strictly positive energy  $\mathring{\mathcal{H}}_t$  in Minkowski space-times. What happens to the infinitely many solutions with large negative energy in the limit  $\Lambda \rightarrow 0$ ?

To analyze this issue, let us first recall that, to take this limit, one has to use the differential structure induced by the coordinates  $(\vec{x}, t)$ . In this chart, the cosmological horizons which bound region I of Figure 5.2 lie at  $r^2 = (3/\Lambda)e^{-2Ht}$  (where  $r^2 = \vec{x} \cdot \vec{x}$ ). Therefore, in the limit  $\Lambda \rightarrow 0$ , region I in which  $T^a$  is time-like fills out the whole Minkowski space. This is the geometric reason why even though  $\mathcal{H}_T$  is unbounded below no matter how small the positive  $\Lambda$  is, the limiting  $\mathring{\mathcal{H}}_t$  is strictly positive. In the phase space language, as  $\Lambda$  changes, the covariant phase space  $\Gamma_{\text{Cov}}^{(\Lambda)}$ , on which the Hamiltonian  $\mathcal{H}_T$  are defined, *itself changes*. In the limit, the set of solutions  $h_{ab}$  on which  $\mathcal{H}_T$  is negative simply disappears!

To summarize, as we showed explicitly in section 5.1, there are families of metric perturbations  $\gamma_{ab}(\Lambda)$  that satisfy the gauge conditions (5.6) and field equations (5.7) for each  $\Lambda$ , and admit well-defined limits  $\mathring{h}_{ab}$  as  $\Lambda \rightarrow 0$  satisfying the standard gauge conditions and field equations (5.40) in Minkowski space-time. The limits  $\mathring{h}_{ab}$  span the entire phase space  $\mathring{\Gamma}_{\text{cov}}$  of metric perturbations in Minkowski space. Furthermore along any of these families, the energy  $\mathcal{H}_T |_{\gamma(\Lambda)}$  tends to the energy  $\mathring{\mathcal{H}}_T |_{\mathring{h}}$  of the limiting perturbation in Minkowski space. Nonetheless, the lower bound of the energy function on phase spaces  $\Gamma_{\text{Cov}}^{(\Lambda)}$  is discontinuous in the limit: It equals  $-\infty$  for every  $\Lambda$ , however small, but vanishes for  $\Lambda = 0$ .

Even though we do recover positivity of energy in the limit  $\Lambda \rightarrow 0$ , we are left with a conundrum because there is strong evidence that  $\Lambda$  is small but non-zero in our universe: Can realistic gravitational waves have arbitrarily large negative energy in de Sitter space-time or, in the nonlinear context, in asymptotically de Sitter space-times? To probe this issue let us first analyze in some detail the origin



**Figure 5.2.** *Left Panel:* Integral curves of the time translation Killing field  $T^a$  in the Poincaré patch.  $T^a$  is future directed and time like in region I and space-like in region II. It is future directed and null on portion of the event horizon  $E^-(i^+)$  to the future of the cross-over 2-sphere (bifurcate horizon)  $C$  and on the portion of the null event horizon  $E^+(i^-)$  to the past of  $C$ . It is *past directed* and null on the portion of  $E^+(i^-)$  to the future of  $C$ . *Right Panel:* A time changing quadrupole emitting gravitational waves. Radiation crossing the cosmological horizon  $E^-(i^+)$  and reaching  $\mathcal{I}^+$  originates in region I; there is no incoming radiation on  $E^+(i^-)$ .

of negative energy. Let us begin with Maxwell fields in de Sitter space-time. The stress-energy satisfies the dominant energy condition and the Killing field  $T^a$  is future pointing on the part of  $E^+(i^-)$  that lies in region I and past pointing on the part that lies in region II (see the left panel in Figure 5.2). Therefore, the energy flux across  $E^+(i^-)$  into region I is positive *but that into region II is negative*. It is because of this negative flux into region II that the total energy can be negative. Therefore, if the Maxwell field under consideration vanished on the part of the horizon  $E^+(i^-)$  that lies in region II, the energy of those electromagnetic waves would be necessarily positive. For gravitational waves, we do not have a stress-energy tensor. However, using the fact that the Killing field  $T^a$  is future directed and time-like in region I, it is easy to show that, if the initial data on any cosmological slice  $\Sigma$  were restricted to lie entirely in the intersection of  $\Sigma$  with region I, the energy (5.45) of that cosmological perturbation is necessarily positive.<sup>8</sup>

<sup>8</sup>This is seen by using the symplectic form in (5.28) to rewrite the energy as follows:

$$\mathcal{H}_T = -\frac{1}{8\kappa H\eta} \int d^3x \left( (\hat{r}^m - \eta^m) (\hat{r}^n - \eta^n) + s^{mn} + 2\left(1 + \frac{r}{\eta}\right) \hat{r}^m \eta^n \right) \overset{\circ}{\nabla}_m h_{ab} \overset{\circ}{\nabla}_n h_{cd} \overset{\circ}{q}^{ac} \overset{\circ}{q}^{bd} \quad (5.49)$$

In the limit  $\eta \rightarrow -\infty$ , the cosmological slice tends to  $E^+(i^-)$ . Therefore, it again follows that the conserved energy flux at  $\mathcal{I}^+$  can be negative only because there is a negative energy flux into the Poincaré patch across the part of  $E^+(i^-)$  that lies in region II. But in realistic situations gravitational waves from isolated systems would be generated entirely by a time changing quadrupole moment (depicted in the right panel of Figure 5.2), whence there would be no incoming flux across  $E^+(i^-)$  at all. The flux across  $\mathcal{I}^+$  would just equal that across the future horizon  $E^-(i^+)$  that separates regions I and II. Since the Killing field  $T^a$  is null and *future directed* on this horizon, this flux has to be positive. Indeed we will show this explicitly in Chapter 6. Thus, in terms of fields *at*  $\mathcal{I}^+$ , while general initial data *can* have arbitrarily large negative energies, *the initial data induced by gravitational waves produced by realistic sources is appropriately constrained for the energy flux across  $\mathcal{I}^+$  to be positive*. An interesting challenge in full nonlinear general relativity is to find the analogous constraints on fields at  $\mathcal{I}^+$  induced by gravitational waves produced by realistic sources, in absence of incoming radiation (at least from the portion of the event horizon  $E^+(i^-)$  that lies to the future of the cross-over surface  $C$ ). With these constraints at hand, one could hope to show that fluxes of energy carried by gravitational waves produced in physically realistic processes would be positive in full nonlinear general relativity with  $\Lambda > 0$ , as one physically expects.

Finally, note that our entire analysis –and in particular the limit to Minkowski space– was carried out by restricting ourselves to the future Poincaré patch of de Sitter space. As discussed in section 5.2, in the description of isolated systems, this restriction is motivated by direct physical considerations. However, one may still ask if the results can be extended to full de Sitter space-time. The explicit form of the solutions we presented is indeed restricted to the future Poincaré patch because of the heavy use of the cosmological slicing. But each of these solutions admits a well-defined extension to full de Sitter space-time simply because every solution in our covariant phase space  $\Gamma_{\text{Cov}}$  induces a well-defined initial data on the de Sitter Cauchy surfaces. For these extended solutions, our main results also hold on  $\mathcal{I}^-$ . The central formula (5.36) holds for all ten Killing fields  $K^a$  of de Sitter space-time in this extension.

These constructions and results provide further guidance for the development

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where  $\hat{q}_{ab}\hat{r}^a\hat{r}^b = 1$  and  $s^{mn} = \hat{q}^{mn} - \hat{r}^m\hat{r}^n$ . If the initial data is restricted to the intersection of  $\Sigma$  with region I we have  $|r/\eta| < 1$  and, consequently,  $\mathcal{H}_T$  is necessarily positive.

of the gravitational radiation theory in full nonlinear general relativity with  $\Lambda > 0$ . We will conclude this discussion with two examples.

Consider first the problem of defining the 2-sphere energy-momentum and angular momentum charge integrals, analogous to the Bondi 4-momentum in the  $\Lambda = 0$  case. For a given prescription for selecting asymptotic symmetries, considerations involving field equations and geometry of  $\mathcal{I}^+$  (discussed in Chapter 4) suggest a natural, candidate expression for these charges for  $\Lambda > 0$ . The difference between these integrals evaluated on any two 2-spheres on  $\mathcal{I}^+$  provides a candidate expression of fluxes in the full theory across the region of  $\mathcal{I}^+$  bounded by these 2-spheres. One can show that their linearization provides precisely the flux formulas (5.38) and (5.46) at  $\mathcal{I}^+$ , derived using completely independent Hamiltonian methods. This result provides a powerful hint for the charge integrals in the full nonlinear theory. The remaining open issue is the selection of appropriate asymptotic symmetries, *without assuming conformal flatness of the intrinsic geometry of  $\mathcal{I}^+$*  (which, as we showed, forces all fluxes to vanish).

A second issue in the full theory is the following. While observations strongly suggest that  $\Lambda$  is positive in our universe, almost all analytical calculations and numerical simulations in gravitational wave science set  $\Lambda$  to zero and work in the asymptotically Minkowskian context. (For notable exceptions, see [75–77].) Since the actual value of  $\Lambda$  is so small compared to the scales involved, say, in binary coalescences of astrophysical interest, it is natural to assume that setting  $\Lambda$  to zero is an excellent approximation. However, it is not completely clear that this is true for two reasons. First, as we pointed out, the limit  $\Lambda \rightarrow 0$  is discontinuous in important respects. Second, advanced LIGO will be eventually capable of detecting gravitational waves from sources that are  $\sim 1$  Gpc away, a distance that is approximately 20% of the cosmological radius. Therefore, apart from the intrinsic conceptual interest, it is important to be able to reliably calculate the ‘errors’ one makes by setting  $\Lambda$  to zero.<sup>9</sup> The details of the discussion of the  $\Lambda \rightarrow 0$  limit presented in this chapter will help significantly in streamlining these calculations.

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<sup>9</sup>For example, there may be subtle effects –analogous to the nonlinear memory in the  $\Lambda = 0$  case– that have remained under the radar so far.

# Chapter 6 |

## The modified quadrupole formula for $\Lambda > 0$

In 1916, Einstein described gravitational waves in general relativity by linearizing field equations off Minkowski background, in the presence of an external, time-changing source [1]. Two years later, he calculated the energy carried by these waves far away from the source. He found that the leading order contribution to the emitted power is proportional to the square of the third time derivative of the mass quadrupole moment [79]. Today we know that high precision measurements on the Hulse-Taylor binary pulsar have confirmed the existence of gravitational quadrupolar radiation to an accuracy of 3 parts in  $10^3$  [80, 81]. Furthermore, we also have confirmation of direct detection of gravitational waves emitted by two black holes in the final stages of their merger [16]. However, theoretical calculations so far have been largely limited to assuming  $\Lambda = 0$ . As explained in Chapter 1 and demonstrated in Chapters 4 and 5, a non-zero positive  $\Lambda$ , no matter how small, has significant departures from the standard  $\Lambda = 0$  case. This is because it effects an abrupt change to asymptotic structure.<sup>1</sup>In particular, the limit of observable quantities associated with gravitational waves can be *discontinuous*, whence smallness of  $\Lambda$  does not always translate to smallness of corrections to the  $\Lambda = 0$  results. The question then is whether one can reliably justify one's first instinct that Einstein's  $\Lambda = 0$  quadrupole formula can receive only negligible corrections, given the smallness of  $\Lambda$ .

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<sup>1</sup>Although for concreteness and simplicity we will refer to a cosmological constant, as in Chapters 4 and 5, our main results will not change if instead one has an unknown form of 'dark energy'.

To make this concern concrete, we consider a few illustrations of the qualitative differences in addition to the ones described in detail in Chapters 4 and 5. First, while wavelengths of linear fields are constant in Minkowski space-time, they increase as waves propagate on de Sitter space-time, and exceed the curvature radius in the asymptotic region of interest. Therefore, the commonly used geometric optics approximation fails in the asymptotic region. Also, one cannot carry over the standard techniques to specify ‘near and far wave zones’ from the  $\Lambda = 0$  case. Second, in Minkowski space-time one can approach  $\mathcal{I}^+$  –the arena on which properties of gravitational waves can be analyzed in a gauge invariant manner– using time-like  $r = r_0$  surfaces with larger and larger values of  $r_0$ . Therefore it is standard practice to use  $1/r$  expansions of fields in the analysis of gravitational waves (see, e.g., [88–90]). By contrast, in de Sitter space-time, such time-like surfaces approach a past cosmological horizon *across which there is no flux of energy or momentum for retarded solutions*. Furthermore, these surfaces intersect  $\mathcal{I}^+$  at a single point.  $\mathcal{I}^+$  of de Sitter space-time is approached by a family of space-like surfaces (on which *time* is constant) whence one cannot use the  $1/r$  expansions that dominate the literature on gravitational waves. Third, while  $\mathcal{I}^+$  is null in the asymptotically flat case, it is space-like if  $\Lambda$  is positive [10]. Consequently, unfamiliar features can arise as we move from  $\Lambda = 0$  to a tiny positive value both in full general relativity and in the linearized limit. In particular, for every  $\Lambda > 0$ , all asymptotic symmetry vector fields –including those corresponding to ‘time translations’– are also space-like in a neighborhood of  $\mathcal{I}^+$ . As a result, while the energy carried by electromagnetic and linearized gravitational waves is necessarily positive in the  $\Lambda = 0$  case, it can be *negative and of arbitrarily large magnitude* if  $\Lambda > 0$  as discussed in Chapter 5. Since this holds for every  $\Lambda > 0$ , however tiny, the lower bound on energy carried by these waves has an *infinite* discontinuity at  $\Lambda = 0$ . Now, if (electromagnetic or) gravitational waves produced in realistic physical processes *could* carry negative energy, we would be faced with a fundamental instability: the source could gain arbitrarily high energy simply by letting the emitted waves carry away negative energy. Thus a positive  $\Lambda$ , however small, opens up an unforeseen possibility, with potential to drastically change gravitational dynamics. Finally, yet another difference is that in the transverse (i.e., Lorentz) traceless gauge the linearized 4-metric field satisfies the *massive* Klein-Gordon equation (where the mass is proportional to  $\sqrt{\Lambda}$ ). While the mass is tiny, a priori it is possible that over

cosmological distances the difference from the propagation in the  $\Lambda = 0$  case could accumulate, creating an  $\mathcal{O}(1)$  difference in the linearized metric in the asymptotic region, far way from sources. Since Einstein’s quadrupole formula is based on the form of the metric perturbation in this ‘wave zone’, secular accumulation could then lead to  $\mathcal{O}(1)$  departures from that formula, even when  $\Lambda$  is tiny.

These considerations bring out the necessity of a systematic analysis to determine whether the Einstein’s quadrupole formula continues to be valid even though many of the key intermediate steps cannot be repeated for the de Sitter background. The goal of this chapter is to complete this task for linearized gravitational waves created by time changing (first order) sources on de Sitter background.

As in the  $\Lambda = 0$  case, the calculation involves two steps:

- (i) expressing metric perturbations far away from the source in terms of the quadrupole moments of the source, and,
- (ii) finding the energy radiated by this source in the form of gravitational waves.

However, the extension of the  $\Lambda = 0$  analysis introduces unforeseen issues in both steps. In step (i), since the background space-time is no longer flat, the meaning of ‘quadrupole moment’ is not immediately clear. The second subtlety concerns both steps. Specifically, due to the curvature of the de Sitter space-time, the gravitational waves back-scatter. This back-scattering introduces a tail term in the solutions to the linearized Einstein’s equation already in the first post-Newtonian order. That in and of itself is not problematic. However, if a tail term persisted in the formula for energy loss, one would need to know the history of the source throughout its evolution in order to determine the flux of energy emitted at any given retarded instant of time.<sup>2</sup> Third, as discussed above, the energy calculated in step (ii) could, in principle, be arbitrarily negative, in which case self-gravitating systems would be drastically unstable to emission of gravitational waves.

Thus, from a conceptual standpoint, the generalization of the quadrupole formula to include a positive  $\Lambda$  is both interesting and subtle. For example, the presence of the tail term opens a door to a new contribution to the ‘memory effects’ associated with gravitational waves [91–93]. In addition, as in the asymptotically flat case, it offers guidance in the development of the full, nonlinear framework.

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<sup>2</sup>In the  $\Lambda = 0$  case, back-scattering occurs only at higher post-Newtonian orders. These higher order corrections to the quadrupole formula are not needed to compare theory with observations for the Hulse-Taylor pulsar so far because the current observational accuracy is at the  $10^{-3}$  level rather than  $10^{-4}$ .

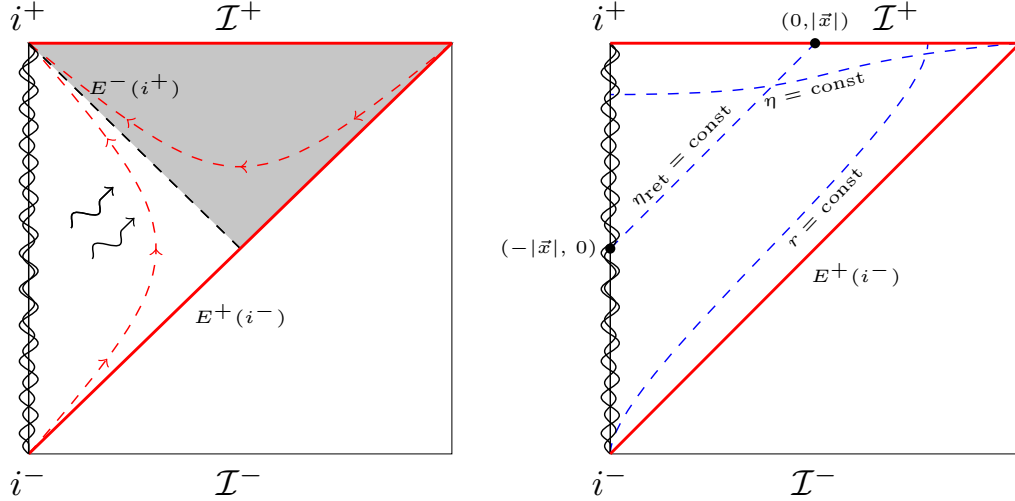
Finally, as we will see, this generalization also provides detailed control on the approximations involved in setting  $\Lambda$  to zero.

The chapter is organized as follows. In section 6.1, we introduce our notation and recall the linearized Einstein's equation on de Sitter background as well as their retarded solutions sourced by a (first order) stress-energy tensor. In section 6.2, we introduce the late time and post-Newtonian approximations and express the leading terms of solutions in terms of the quadrupole moments of sources. In section 6.3, we use these expressions to calculate the energy emitted by the source using Hamiltonian methods on the covariant phase-space of the linearized solutions introduced in Chapter 5, and then discuss in some detail the novel features that arise because of the presence of a positive  $\Lambda$ . We find that the energy carried away by the gravitational waves produced by a time changing source is necessarily positive. Detailed expressions bear out the expectation that, for sources of gravitational radiation currently under consideration by gravitational wave observatories, the primary modification to Einstein's formula can be incorporated by taking into account the expansion of the universe and the resulting gravitational red-shift. Section 6.4 contains a brief summary. Appendix C discusses the tail term in the retarded solution which makes the limit  $\Lambda \rightarrow 0$  limit quite subtle.

## 6.1 Linearized Einstein's equation with sources

The isolated system of interest is depicted in the left panel of Fig. 6.1 (and specified in greater detail in the beginning of section 6.2). It represents a matter source in de Sitter space-time whose spatial size is uniformly bounded in time. Such a source intersects  $\mathcal{I}^\pm$  at single points  $i^\pm$ . Examples are provided by isolated stars and coalescing binary systems. The causal future of such a source covers only the future Poincaré patch,  $M_P^+$ . No observer whose world-line is confined to the past Poincaré patch can see the isolated system or detect the radiation it emits. Therefore, to study this system, it suffices to restrict oneself just to  $M_P^+$ . The description of the Poincaré patch required for our study of the isolated system has been provided in Chapter 5 section 5.1.

To study the gravitational radiation emitted by an isolated system in the presence of positive  $\Lambda$ , we consider first order perturbations off de Sitter space-time.



**Figure 6.1.** *Left Panel:* A time-changing quadrupole emitting gravitational waves whose spatial size is uniformly bounded in time. The causal future of such a source covers only the future Poincaré patch  $M_P^+$  (the upper triangle of the figure). There is no incoming radiation across the past boundary  $E^+(i^-)$  of  $M_P^+$  because we use retarded solutions. The shaded region represents a convenient neighborhood of  $\mathcal{I}^+$  in which perturbations satisfy a homogenous equation and the approximation (6.30), discussed below, holds everywhere. The dashed (red) lines with arrows show the integral curves of the ‘time translation’ Killing field  $T^a$  (adapted to the rest frame of the source). *Right Panel:* The rate of change of the quadrupole moment at the point  $(-|\vec{x}|, \vec{0})$  on the source creates the retarded field at the point  $(0, \vec{x})$  on  $\mathcal{I}^+$ . The figure also shows the cosmological foliation  $\eta = \text{const}$  and the time-like surfaces  $r := |\vec{x}| = \text{const}$ . As  $r$  goes to infinity, the  $r = \text{const}$  surfaces approach  $E^+(i^-)$ . Therefore, in contrast with the situation in Minkowski space-time, for sufficiently large values of  $r$ , there is no flux of energy across the  $r = \text{const}$  surfaces.

The perturbed metric is denoted by  $g_{ab}$ ,

$$g_{ab} = \bar{g}_{ab} + \epsilon \gamma_{ab} =: a^2(\eta)(\dot{g}_{ab} + \epsilon h_{ab}), \quad (6.1)$$

where  $\epsilon$  is a smallness parameter. While  $\gamma_{ab}$  are the physical first order perturbations off de Sitter space-time, it is convenient –as will be clear shortly– to use the conformally related mathematical field  $h_{ab}$  while solving the linearized Einstein’s equation.

In terms of the trace-reversed metric perturbation  $\bar{\gamma}_{ab} := \gamma_{ab} - \frac{1}{2}\bar{g}_{ab}\gamma$ , the linearized Einstein’s equation in the presence of a (first order) linearized source

$T_{ab}$  can be written as

$$\square \bar{\gamma}_{ab} - 2\bar{\nabla}_{(a}\bar{\nabla}^c\bar{\gamma}_{b)c} + \bar{g}_{ab}\bar{\nabla}^c\bar{\nabla}^d\bar{\gamma}_{cd} - \frac{2}{3}\Lambda(\bar{\gamma}_{ab} - \bar{g}_{ab}\bar{\gamma}) = -16\pi G T_{ab} \quad (6.2)$$

where  $\bar{\nabla}$  and  $\square$  denote the derivative operator and the d'Alembertian defined by the de Sitter metric  $\bar{g}_{ab}$ .

The solutions to the linearized equation with sources on the future Poincaré patch  $(M_{\text{P}}^+, \bar{g}_{ab})$  are discussed in detail by de Vega et al. in [96] (see also [97] for a recent discussion). Here we will summarize their results, comment on the physical interpretation, and also discuss the limit  $\Lambda \rightarrow 0$ .

Denote by  $\eta^a$  the vector field normal to the cosmological slices  $\eta = \text{const}$  satisfying  $\eta^a \bar{\nabla}_a \eta = 1$  and let  $n^a := -H\eta \eta^a$  denote the future pointing unit normal to these slices. Then, it is convenient to solve (6.2) using the following gauge condition:

$$\bar{\nabla}^a \bar{\gamma}_{ab} = 2H n^a \bar{\gamma}_{ab}. \quad (6.3)$$

This is a generalization of the more familiar Lorentz gauge condition and, as with the Lorentz condition, it does not exhaust the gauge freedom. Nonetheless, in this gauge the linearized Einstein's equation (6.2) simplifies significantly when it is rewritten in terms of the field  $\bar{\chi}_{ab}$  which is related to the trace-reversed metric perturbations  $\bar{\gamma}_{ab}$  via  $\bar{\chi}_{ab} := a^{-2}\bar{\gamma}_{ab} = h_{ab} - \frac{1}{2}\hat{g}_{ab}\hat{g}^{cd}h_{cd}$ . Finally, it is easiest to obtain solutions to (6.2) by performing a decomposition of  $\bar{\chi}_{ab}$  and  $T_{ab}$ , adapted to the cosmological  $\eta = \text{const}$  slices:

$$\tilde{\chi} := (\eta^a \eta^b + \hat{q}^{ab}) \bar{\chi}_{ab}, \quad \chi_a := \eta^c \hat{q}_a^b \bar{\chi}_{bc}, \quad \chi_{ab} := \hat{q}_a^m \hat{q}_b^n \bar{\chi}_{mn}, \quad (6.4)$$

$$\tilde{\mathcal{T}} := (\eta^a \eta^b + \hat{q}^{ab}) T_{ab}, \quad \mathcal{T}_a := \eta^c \hat{q}_a^b T_{bc}, \quad \mathcal{T}_{ab} := \hat{q}_a^m \hat{q}_b^n T_{mn}, \quad (6.5)$$

where  $\hat{q}^{ab}$  is the (contravariant) spatial metric on a  $\eta = \text{const}$  slice induced by the flat metric  $\hat{g}_{ab}$ , i.e.,  $\hat{q}^{ab} = \hat{g}^{ab} + \eta^a \eta^b$ . (Note that unlike  $\bar{\chi}_{ab}$  in (6.4), the stress energy tensor  $T_{ab}$  in (6.5) has neither been rescaled by  $a^{-2}$  nor has it been trace-reversed.) In the  $(\eta, \vec{x})$  chart,  $-\frac{1}{4H\eta} \tilde{\chi}$  is the perturbed lapse function and  $(H\eta)^{-2} \hat{q}^{ab} \chi_b$  is the perturbed shift field.<sup>3</sup> Thus, as in the linearized theory off

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<sup>3</sup>Perturbed lapse is  $\delta N = \delta(\sqrt{-\eta^a \eta^b \bar{g}_{ab}}) = -\frac{H\eta}{2} \eta^a \eta^b \gamma_{ab} = -\frac{H\eta}{4} a^2(\eta) [\eta^a \eta^b \bar{\chi}_{ab} + \chi] = -\frac{1}{4H\eta} \tilde{\chi}$ . Perturbed shift is  $\hat{q}^{ab} \delta N_a$ ;  $\delta N_a = \eta^b \delta(g_{ab} + n_a n_b)$  and  $\delta n_a = -\frac{1}{2} \bar{n}_a (\bar{n}^c \bar{n}^d \gamma_{cd})$  where  $n_a = \nabla_a \eta / \sqrt{-g^{cd} \nabla_c \eta \nabla_d \eta}$ ,  $\bar{n}_a = -a(\eta) \nabla_a \eta$ ,  $\bar{n}^a = a^{-1} \eta^a$ .

Minkowski space-time, the physical degrees of freedom associated with radiation are encoded in the totally spatial projection  $\chi_{ab}$ .

It is convenient to regard the fields  $\tilde{\chi}$ ,  $\chi_a$  and  $\chi_{ab}$ , as living in the flat space-time  $(M_{\mathbb{P}}^+, \mathring{g}_{ab})$  because: (i) the gauge condition and field equations have a simple form in terms of the derivative operators defined by  $\mathring{g}_{ab}$ ; and (ii) these gauge conditions and field equations are well defined also at  $\mathcal{I}^+$  because, as we will see in section 6.3, the metric  $\mathring{g}_{ab}$  turns out to provide a viable conformal rescaling of  $\bar{g}_{ab}$  that is well-defined at  $\mathcal{I}^+$ . The gauge conditions (6.3) become

$$\mathring{D}^a \chi_{ab} = \partial_\eta \chi_b - \frac{2}{\eta} \chi_b, \quad \text{and} \quad \mathring{D}^a \chi_a = \partial_\eta (\tilde{\chi} - \chi) - \frac{1}{\eta} \tilde{\chi}, \quad (6.6)$$

where  $\mathring{D}$  is the derivative operator of the spatial metric  $\mathring{q}_{ab}$  and  $\chi = \mathring{q}^{ab} \chi_{ab}$ . In this gauge, the linearized Einstein's equation (6.2) is split into three as follows

$$\mathring{\square} \left( \frac{1}{\eta} \tilde{\chi} \right) = -\frac{16\pi G}{\eta} \tilde{\mathcal{T}}, \quad (6.7)$$

$$\mathring{\square} \left( \frac{1}{\eta} \chi_a \right) = -\frac{16\pi G}{\eta} \mathcal{T}_a, \quad (6.8)$$

$$\left( \mathring{\square} + \frac{2}{\eta} \partial_\eta \right) \chi_{ab} = -16\pi G \mathcal{T}_{ab}. \quad (6.9)$$

where  $\mathring{\square}$  is the d'Alembertian operator of the flat metric  $\mathring{g}_{ab}$ . Using the conservation of the first order stress-energy tensor,  $\bar{\nabla}^a T_{ab} = 0$ , it is easy to directly verify that the gauge conditions and the field equations are consistent, as they must be.

Since we wish to impose the 'no incoming radiation' boundary conditions, we will seek retarded solutions to these equations. The first two equations, (6.7) and (6.8), can be solved using the scalar retarded Green's function of  $\mathring{\square}$ :

$$G_R^{(M)}(x, x') = \frac{1}{4\pi |\vec{x} - \vec{x}'|} \delta(\eta - \eta' - |\vec{x} - \vec{x}'|) \quad (6.10)$$

to yield

$$\begin{aligned} \tilde{\chi}(\eta, \vec{x}) &= 4G \eta \int \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|} \frac{1}{\eta_{\text{Ret}}} \tilde{\mathcal{T}}(\eta_{\text{Ret}}, \vec{x}'), \quad \text{and} \\ \chi_{\bar{a}}(\eta, \vec{x}) &= 4G \eta \int \frac{d^3 \vec{x}'}{|\vec{x} - \vec{x}'|} \frac{1}{\eta_{\text{Ret}}} \mathcal{T}_{\bar{a}}(\eta_{\text{Ret}}, \vec{x}'), \end{aligned} \quad (6.11)$$

where  $\eta_{\text{Ret}}$  is the retarded time related to  $\eta$  and  $\vec{x}$  by  $\eta_{\text{Ret}} := \eta - |\vec{x} - \vec{x}'|$ . We could use the scalar Green's function of  $\square$  also in the second equation because  $\chi_{\bar{a}}$  refer to the Cartesian components of the vector perturbation. While we will use the solutions (6.11) in an intermediate step, the fluxes of energy, momentum and angular momentum turn out to depend only on  $\chi_{ab}$  because, as we noted above, the other components correspond to linearized lapse and shift fields.

One can use a *scalar* Green's function also for the Cartesian components of the spatial tensor field  $\chi_{ab}$ . However, since the operator on the left hand side of (6.9) has the extra term,  $(2/\eta)\partial_\eta$ , we cannot use the Green's function of the flat space wave operator  $\square$ . Instead, Ref. [96] provides the retarded Green's function satisfying

$$\left(\square + \frac{2}{\eta}\partial_\eta\right) G_R(x, x') = -(H^2\eta^2) \delta(x, x'). \quad (6.12)$$

In contrast to the flat space scalar Green's function, the solution to this equation has an additional term that extends its support to the region in which  $x, x'$  are time-like related:

$$G_R(x, x') = \frac{H^2 \eta \eta'}{4\pi |\vec{x} - \vec{x}'|} \delta(\eta - \eta' - |\vec{x} - \vec{x}'|) + \frac{H^2}{4\pi} \theta(\eta - \eta' - |\vec{x} - \vec{x}'|) \quad (6.13)$$

where  $\theta(x)$  is the step function which is 1 when  $x \geq 0$  and 0 otherwise. Therefore the solution  $\chi_{ab}$  is given by

$$\chi_{\bar{a}\bar{b}}(\eta, \vec{x}) = 16\pi G \int d^3\vec{x}' \int d\eta' G_R(x, x') \left(\frac{1}{H^2\eta'^2}\right) \mathcal{T}_{\bar{a}\bar{b}}(x'). \quad (6.14)$$

To simplify the solution, one uses the identity

$$\begin{aligned} & \left(\frac{1}{|\vec{x}-\vec{x}'|} \frac{\eta}{\eta'}\right) \delta(\eta - \eta' - |\vec{x} - \vec{x}'|) + \frac{1}{\eta'^2} \theta(\eta - \eta' - |\vec{x} - \vec{x}'|) \\ &= \frac{1}{|\vec{x}-\vec{x}'|} \delta(\eta - \eta' - |\vec{x} - \vec{x}'|) - \frac{\partial}{\partial \eta'} \left(\frac{1}{\eta'} \theta(\eta - \eta' - |\vec{x} - \vec{x}'|)\right), \end{aligned} \quad (6.15)$$

in (6.14), integrates by parts with respect to  $\eta'$ , and shows that the boundary terms do not contribute for any given  $(\eta, \vec{x})$ . Then everywhere on  $M_{\text{P}}^+$  the solution is given by

$$\chi_{\bar{a}\bar{b}}(\eta, \vec{x}) = 4G \int \frac{d^3\vec{x}'}{|\vec{x} - \vec{x}'|} \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{Ret}}, \vec{x}')$$

$$+ 4G \int d^3 \vec{x}' \int_{-\infty}^{\eta_{\text{Ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{\bar{a}\bar{b}}(\eta', \vec{x}') \quad (6.16)$$

$$\equiv \#_{\bar{a}\bar{b}}(\eta, \vec{x}) + \flat_{\bar{a}\bar{b}}(\eta, \vec{x}), \quad (6.17)$$

where  $\#_{\bar{a}\bar{b}}(\eta, \vec{x})$  denotes the *sharp* propagation term and  $\flat_{\bar{a}\bar{b}}(\eta, \vec{x})$ , the prolonged *tail* term. Note that this solution relates the Cartesian components of  $\chi_{ab}$  to those of  $\mathcal{T}_{ab}$ . Therefore, throughout the rest of the chapter, whenever we use this solution, we will restrict ourselves to components in the Cartesian chart.

The retarded solution (6.16) has an interesting feature. The first term  $\#_{ab}$  in this expression is identical to the solution for the trace-reversed perturbations which satisfy the linearized Einstein equation (with a first order source  $\mathcal{T}_{ab}$ , and in the Lorentz gauge) w.r.t. the Minkowski metric  $\mathring{g}_{ab}$  i.e.,  $\mathring{\square} \#_{ab} = -16\pi G \mathcal{T}_{ab}$ . The second term  $\flat_{ab}$ , which is absent in the Minkowski case, depends on the entire history of the behavior of the source up to time  $\eta_{\text{Ret}}$ . It results from the back-scattering of the perturbation by curvature in the de Sitter background. Thus, in contrast to the  $\Lambda = 0$  case, the propagation of the metric perturbation fails to be sharp already at the first post-Newtonian order. The retarded solutions (6.11) and (6.16) satisfy the equations of motion (6.7) - (6.9) by construction. However, to obtain a solution to the physical problem at hand, we need to make sure that they also satisfy the gauge conditions (6.6). One can verify that this is the case using conservation of the stress-energy tensor.

Finally, we discuss the limit  $\Lambda \rightarrow 0$ . From the solution (6.16) it is not obvious that the tail term will disappear in this limit. However, as stated above, to study this limit one needs to use the differential structure given not by the  $(\eta, \vec{x})$  chart, but by the  $(t, \vec{x})$  chart in which the de Sitter metric  $\bar{g}_{ab}$  of (5.2) admits a well defined limit to the Minkowski metric  $\mathring{\eta}_{ab}$  as  $\Lambda \rightarrow 0$ . Using the  $(t, \vec{x})$  chart, it is easy to show that the gauge condition (6.3) and the linearized Einstein's equation (6.2) reduce to the familiar Lorentz gauge condition and linearized Einstein's equation in Minkowski space-time, respectively,

$$\mathring{\nabla}^b \mathring{\gamma}_{ab} = 0, \quad \text{and} \quad \mathring{\square} \mathring{\gamma}_{ab} = -16\pi G T_{ab}, \quad (6.18)$$

where for notational coherence the metric perturbations off the Minkowski space-time metric  $\mathring{\eta}_{ab}$  are denoted by  $\mathring{\gamma}_{ab}$ . Note that, while in the de Sitter case different components of the perturbation satisfy different equations, (6.7)-(6.9), in the  $\Lambda \rightarrow$

0 limit these distinct equations collapse to a single flat space scalar wave equation for all Cartesian components of  $\dot{\gamma}_{ab}$ . Consequently, the Green's functions (6.10) and (6.13) used to solve for various components of the de Sitter perturbations, reduce to the scalar Green's function of the flat d'Alembertian operator  $\square$  of  $\dot{\eta}_{ab}$  (which, as we noted before, is distinct from the flat metric  $\dot{g}_{ab}$ ),

$$G_R^{(M)}(x, x') = \frac{1}{4\pi|\vec{x} - \vec{x}'|} \delta(t - t' - |\vec{x} - \vec{x}'|). \quad (6.19)$$

Therefore in the  $(t, \vec{x})$  chart the retarded solutions of (6.18) are given by

$$\dot{\gamma}_{\bar{a}\bar{b}}(t, \vec{x}) = 4G \int \frac{d^3\vec{x}'}{|\vec{x} - \vec{x}'|} T_{\bar{a}\bar{b}}(t - |\vec{x} - \vec{x}'|, \vec{x}'). \quad (6.20)$$

This also follows directly by first expressing the final solutions (6.11) and (6.16) in the  $(t, \vec{x})$  chart and then taking the  $\Lambda \rightarrow 0$  limit, as it must. Thus, our expectation that tail term should disappear in the limit  $\Lambda \rightarrow 0$  is explicitly borne out.

## 6.2 The retarded solution and quadrupole moments

In full general relativity with positive  $\Lambda$ , space-times describing isolated gravitating systems are asymptotically de Sitter. To compute the energy emitted in the form of gravitational waves, one would (numerically) solve Einstein's equations by imposing an appropriate 'no-incoming radiation' boundary condition, find the gravitational fields on  $\mathcal{I}^+$ , and extract the energy radiated by gravitational waves from these fields. This chapter, of course, restricts itself to a simplified version of this problem using the first post-de Sitter approximation. We have already incorporated the 'no incoming radiation' boundary condition through retarded Green's functions and our task is to extract physical information from the emitted gravitational waves by examining these solutions at  $\mathcal{I}^+$ . The calculation will be performed in two steps. In the first, carried out in this section, we use physically motivated approximations to simplify the retarded solution (6.16) in the asymptotic region near  $\mathcal{I}^+$  and relate the leading term to the time-variation of the source quadrupole moment. The second step will be carried out in section 6.3.

## 6.2.1 The late time and post-Newtonian approximations

To extract physical information from Eq. (6.16), we need to examine this solution in the asymptotic region near  $\mathcal{I}^+$ . In linearized gravity off Minkowski space-time, one can approach  $\mathring{\mathcal{I}}^+$  using a family of time-like tubes  $r = r_o$ , with ever increasing values of the constant  $r_o$ . Therefore, one focuses on the form of the retarded solutions at large distances from the source, keeping the leading order  $1/r$  contribution, and ignoring terms that fall-off as  $1/r^2$ . Since the conformal factor used to complete Minkowski space-time in order to attach the null boundary  $\mathcal{I}^+$  falls-off as  $1/r$ , this approximation is sufficient to recover the asymptotic perturbation on  $\mathcal{I}^+$  and extract energy, momentum and angular momentum carried by gravitational waves. In de Sitter space-time, by contrast, as mentioned earlier, the  $r = r_o$  time-like surfaces approach the cosmological horizon  $E^+(i^-)$ , rather than  $\mathcal{I}^+$  (see Fig. 6.1). And the flux of energy or momentum or angular momentum across  $E^+(i^-)$  vanishes identically for retarded solutions! Indeed, this is precisely the ‘no incoming radiation condition’. (Thus,  $E^+(i^-)$  is analogous to  $\mathring{\mathcal{I}}^-$  rather than  $\mathring{\mathcal{I}}^+$  in Minkowski space-time.) Therefore, contrary to the strong intuition derived from Minkowski space-time [88–90], the  $1/r$ -expansions are now ill-suited to study gravitational waves. (In particular, one cannot blindly take over well-understood notions such as the ‘wave zone’. All these differences occur also for test electromagnetic fields on de Sitter space-time.)

As explained in Chapter 4,  $\mathcal{I}^+$  of de Sitter space-time is space-like and corresponds to the surface  $\eta = 0$  (see also section 6.3.1). Therefore, it can be approached by a family of *space-like* surfaces. The first natural candidate is provided by the cosmological slices  $\eta = \text{const}$  used in section 5.1. Another possibility is to use the family of space-like 3-surfaces which lie in the shaded region of the left panel of Fig. 6.1 to which  $T^a$  and the three rotational Killing fields of  $(M_{\text{P}}^+, \bar{g}_{ab})$  are everywhere tangential. In this section we will use the cosmological slices and in the next section, the 3-surfaces in the shaded region. To summarize, to approach  $\mathcal{I}^+$  and extract the radiative part of the solution, we now need a *late time* approximation in place of the Minkowskian ‘far field’ approximation.

To introduce this approximation, we first need to sharpen our restrictions on the spatial support of the matter source. These conditions will capture the idea that the system under consideration is isolated, e.g., an oscillating star or a compact

binary, when  $\Lambda > 0$ . First, we will assume that the physical size  $D(\eta)$  of the system is uniformly bounded by  $D_o$  on all  $\eta = \text{const}$  slices. A particular consequence of this requirement is that the source punctures  $\mathcal{I}^+$  at a single point  $i^+$ , and  $\mathcal{I}^-$  at a single point  $i^-$ , as depicted in Fig. 6.1. Physically, this assumption will ensure that a binary, for example, remains compact in spite of the expansion of the universe. We further sharpen the ‘compactness’ restriction through a second requirement:  $D_o \ll \ell_\Lambda$ , where  $\ell_\Lambda (= 1/H)$  is the cosmological radius.<sup>4</sup> Finally, for simplicity, we assume that the system is stationary in the distant past and distant future, i.e.,  $\mathcal{L}_T T_{ab} = 0$  outside a finite  $\eta$ -interval. Such a system is dynamically active only for a finite time interval  $(\eta_1, \eta_2)$ . This simplifying assumption can be weakened substantially to allow  $\mathcal{L}_T T_{ab}$  to fall-off at a suitable rate in the approach to  $i^\pm$ . We use the stronger assumption just to ensure finiteness of various integrals without having to consider the fall-off conditions in detail at each intermediate step. Furthermore, given that we are primarily interested in calculating radiated power at a retarded instant of time, the assumption is not really restrictive.

With these restrictions, we can now obtain an approximate form of the solution (6.16) which is valid near  $\mathcal{I}^+$ . Consider, then, a cosmological slice,  $\eta = \text{const}$ , and choose the Cartesian coordinates  $\vec{x}$  such that the center of mass of the source lies at the origin. The right side of (6.17) expresses  $\chi_{ab}$  as a sum of a sharp term and tail term. We first simplify the sharp term. As in the standard linearized theory off Minkowski space-time [88], we first write it as

$$\sharp_{\bar{a}\bar{b}}(\eta, \vec{x}) = 4G \int d^3x' \int d^3y' \frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{Ret}}, \vec{y}')}{|\vec{x} - \vec{x}'|} \delta(\vec{x}', \vec{y}'), \quad (6.21)$$

and Taylor-expand the  $|\vec{x} - \vec{x}'|$  dependence of the integrand around  $\vec{x}' = 0$  (recall that the integral over  $\vec{x}'$  is over a compact region around the origin, the support of  $\mathcal{T}_{\bar{a}\bar{b}}$ ). In the Taylor expansion, each derivative  $\partial/\partial x'^{\bar{a}}$  can be replaced by  $-\partial/\partial x^{\bar{a}}$  because the  $\vec{x}'$ -dependence of the integrand of the last integral comes entirely from  $|\vec{x}' - \vec{x}|$ . Hence,

$$\sharp_{\bar{a}\bar{b}}(\eta, \vec{x}) = \frac{4G}{r} \left[ \int d^3x' \left( \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + \frac{x'^c \hat{r}_c}{r} \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + (x'^c \hat{r}_c) \partial_{\eta_{\text{ret}}} \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + \dots \right) \right]$$

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<sup>4</sup>Given that  $\ell_\Lambda$  is about 5 Gpc, the condition is easily met by sources of physical interest, such as an isolated oscillating star or a compact binary.

$$\begin{aligned}
&= \frac{4G}{r} \left[ \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + \left(\frac{x_1'^c \hat{r}_c}{r}\right) \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') \right. \\
&\quad \left. + (x_2'^c \hat{r}_c) \int d^3x' \partial_{\eta_{\text{ret}}} \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + \dots \right] \tag{6.22}
\end{aligned}$$

where we have carried out the integral over  $\vec{y}'$  and where the  $\dots$  denote higher order terms in the Taylor expansion. Note that we have replaced

$$\eta_{\text{Ret}} = (\eta - |\vec{x} - \vec{x}'|) \quad \text{by} \quad \eta_{\text{ret}} = \eta - r \tag{6.23}$$

because the coefficients of the Taylor expansion are evaluated at  $\vec{x}' = 0$ . In the second step we have used the mean value theorem and  $\vec{x}'_1$  and  $\vec{x}'_2$  are the points in the support of  $\mathcal{T}_{\bar{a}\bar{b}}$ , determined by this theorem. Next, using the fact that each of  $|x_1'^c \hat{r}_c / r|$  and  $|x_2'^c \hat{r}_c / r|$  is bounded by the coordinate radius of the source at  $\eta = \eta_{\text{ret}}$ ,

$$d(\eta_{\text{ret}}) := D(\eta_{\text{ret}}) / a(\eta_{\text{ret}}), \tag{6.24}$$

we conclude

$$\sharp_{\bar{a}\bar{b}}(\eta, \vec{x}) = \frac{4G}{r} \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') \left[ 1 + \frac{\mathcal{O}(d(\eta_{\text{ret}}))}{r} + \frac{\mathcal{O}(d(\eta_{\text{ret}}))}{\Delta\eta_{\text{ret}}} \right], \tag{6.25}$$

where  $\Delta\eta_{\text{ret}}$  is the dynamical time scale (measured in the  $\eta$  coordinate) in which the change in the source is of  $\mathcal{O}(1)$ . It will be clear from section 6.2.2 that this is the time scale in which the change in the quadrupole moments of the source is  $\mathcal{O}(1)$ . Note that  $d(\eta_{\text{ret}})/r$  is also the ratio of the proper size of the source  $D(\eta_{\text{ret}})$  to the proper distance  $[a(\eta_{\text{ret}}) r]$  at time  $\eta_{\text{ret}}$ .

Up to this point, the mathematical manipulations are essentially the same as those in the linear theory off Minkowski space-time [88]. The difference lies in the underlying assumptions and the physical meaning of the approximation scheme. A straightforward calculation relates the second and third terms in the square brackets in (6.25) to physical properties of the source. First, we have

$$\frac{d(\eta_{\text{ret}})}{r} = \frac{D(\eta_{\text{ret}})}{\ell_\Lambda} \frac{(-\eta_{\text{ret}})}{r} \leq \frac{D_o}{\ell_\Lambda} \left(1 - \frac{\eta}{r}\right). \tag{6.26}$$

Note that to study the asymptotic form of the solution on  $\mathcal{I}^+$ , unlike in the calculation off Minkowski space-time, we cannot use a large  $r$  approximation. Indeed, in

the calculation of the radiated energy in 6.3, we will need to integrate over a finite range of  $r$ .<sup>5</sup> While  $(1 - \eta/r)$  can be large, given *any*  $r_o \neq 0$ , we can choose a cosmological slice  $\eta = \text{const}$  sufficiently close to  $\mathcal{I}^+$  such that for all  $r > r_o$ ,  $(1 - \eta/r)$  is arbitrarily close to 1, whence  $d(\eta_{\text{ret}})/r$  is negligible. This is the late-time approximation. In particular, on  $\mathcal{I}^+$  (where  $\eta = 0$ ) we can ignore the second term in the square bracket in (6.25) for all  $r > 0$ . The third term can be re-expressed as

$$\frac{d(\eta_{\text{ret}})}{\Delta\eta_{\text{ret}}} = \frac{D(\eta_{\text{ret}})}{\Delta t_{\text{ret}}} \approx v \quad (6.27)$$

where  $D$  is the physical length scale of the source and  $\Delta t$  the interval in proper time in which the source changes by  $\mathcal{O}(1)$ , and where we have used the standard reasoning from Minkowski space-time to conclude that the ratio  $D(\eta_{\text{ret}})/\Delta t_{\text{ret}}$  can be identified with the velocity  $v$  of the source. We now use the slow motion approximation in which  $v \ll 1$  (in our  $c = 1$  units). Thus, within our assumptions the sharp term is given by

$$\sharp_{\bar{a}\bar{b}}(\eta, \vec{x}) = \frac{4G}{r} \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}'). \quad (6.28)$$

For the tail term  $\flat_{\bar{a}\bar{b}}(\eta, \vec{x})$  in (6.17), this procedure only replaces  $\eta_{\text{Ret}}$  by  $\eta_{\text{ret}}$ .

By adding the two contributions  $\sharp_{\bar{a}\bar{b}}$  and  $\flat_{\bar{a}\bar{b}}$ , we can express  $\chi_{ab}$  as follows:

$$\begin{aligned} \chi_{\bar{a}\bar{b}}(\eta, \vec{x}) &= \frac{4G}{r} \int d^3\vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') \left[ 1 + \mathcal{O}\left(\frac{D_o}{\ell_\Lambda} \left(1 - \frac{\eta}{r}\right)\right) + \mathcal{O}(v) \right] \\ &+ 4G \int_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \int d^3\vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta', \vec{x}'). \end{aligned} \quad (6.29)$$

(The error term arising from  $\eta_{\text{Ret}} \rightarrow \eta_{\text{ret}}$  in the tail term is included in the square bracket in the first term.) On any  $\eta = \eta_o$  slice, the second term in the square bracket can be neglected, in particular, for all  $r > -\eta_o$ , i.e., beyond the intersection of that slice with the cosmological horizon  $E^-(i^+)$ . On  $\mathcal{I}^+$ , it can be neglected for all  $r > 0$ .

We conclude by summarizing all the approximations that were made. First, in section 6.1, we presented the retarded solution to Einstein's equations in the first post-de Sitter approximation. We then assumed that the source is compact

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<sup>5</sup>On  $\mathcal{I}^+$  the energy flux will be non-zero in the interval  $-\eta_2 < r < -\eta_1$ , where  $(\eta_1, \eta_2)$  is the interval where the source is dynamical, i.e.,  $\mathcal{L}_T \mathcal{T}_{ab} \neq 0$ .

in the sense that the physical size  $D(\eta)$  of the support of the stress-energy tensor  $T_{ab}$  is uniformly bounded by  $D_o$ , with  $D_o \ll \ell_\Lambda$ . Finally, we used the first post-Newtonian approximation to set  $v \ll 1$  in our  $c = 1$  units. (If one were to restore  $c$ , then the overall factor  $4G$  would be replaced by  $4G/c^4$  in the first term and the  $\mathcal{O}(v)$  term would be of 1.5 post-Newtonian order.) Note that to obtain (6.30), we did not have to make any assumption relating the dynamical time scale  $\Delta t_{\text{ret}}$  of the system with the Hubble time  $t_H = 1/H$ . Astrophysical sources of greatest interest to the current gravitational wave observatories satisfy  $\Delta t_{\text{ret}} \ll t_H$ . We will simplify the final results using this approximation in section 6.3.2.

To avoid proliferation of symbols, from now on  $\chi_{ab}(\eta, \vec{x})$  will stand for the approximate solution obtained by ignoring the  $\mathcal{O}((D_o/\ell_\Lambda)(1 - \eta/r))$  and  $\mathcal{O}(v)$  terms relative to the  $\mathcal{O}(1)$  terms in (6.29). Thus, we will set

$$\chi_{\bar{a}\bar{b}}(\eta, \vec{x}) = \frac{4G}{r} \int d^3\vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') + 4G \int_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \int d^3\vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta', \vec{x}') \quad (6.30)$$

and again denote the sharp and the tail terms by  $\sharp_{ab}$  and  $\flat_{ab}$  respectively.

## 6.2.2 Expressing approximate solutions in terms of quadrupole moments

To make the relation between the energy carried by the gravitational perturbations and the behavior of the source transparent, we will now express the approximate solution in terms of multipole moments of the source. Both terms on the right side of (6.30) involve the integral  $\int d^3\vec{x}' \mathcal{T}_{\bar{a}\bar{b}}$  of spatial components of the stress energy tensor of the source. We can rewrite this integral in terms of time derivatives of other components, using the conservation of  $T_{ab}$ . Recall that this strategy is used in the  $\Lambda = 0$  case to express the integral entirely in terms of the second time derivative of the time-time component of  $T_{ab}$ , i.e., the energy density. Consequently, for perturbations off flat space, only the mass quadrupole moment is relevant in the far-field approximation. As we will now show, the situation is more complicated in the  $\Lambda > 0$  case because the conservation equation,  $\bar{\nabla}^a T_{ab} = 0$ , has additional terms due to the expansion of the scale factor of the de Sitter background.

In the  $(t, \vec{x})$  coordinates, projection of the conservation equation along  $t^a$

(where, as usual,  $t^a \partial_a := \partial/\partial t$ ) and  $\dot{q}_a^b$  yield, respectively,

$$\partial_t \rho - e^{-3Ht} \dot{D}^a \mathcal{T}_a + H(3\rho + p_1 + p_2 + p_3) = 0, \quad (6.31)$$

$$\partial_t \mathcal{T}_a - e^{-Ht} \dot{D}^b \mathcal{T}_{ab} + 2H \mathcal{T}_a = 0, \quad (6.32)$$

where the matter density and pressure are defined as usual via

$$\rho = T_{ab} n^a n^b \equiv H^2 \eta^2 T_{ab} \eta^a \eta^b, \quad \text{and} \quad p_{\bar{i}} = T^{ab} \partial_a x_{\bar{i}} \partial_b x_{\bar{i}}, \quad (6.33)$$

and where  $\dot{D}_a$  is the derivative operator compatible with the flat spatial metric  $\dot{q}_{ab}$ . (In the last equation, there is no sum over  $\bar{i}$ .) In this  $(t, \vec{x})$  chart it is manifest that when  $\Lambda \rightarrow 0$  (i.e.,  $H \rightarrow 0$ ), these equations reduce to the time and space projections of the conservation equation with respect to the Minkowski metric  $\dot{\eta}_{ab}$ . Extra terms proportional to  $H$  arise in de Sitter space-time due to the expansion of the scale factor. These, in particular, include all the pressure terms which appear more generally in *any* spatially homogeneous and isotropic space-time. Consequently, it will turn out that  $\int d^3 \vec{x} \mathcal{T}_{\bar{a}\bar{b}}$  is related not just to the second time derivative of the mass quadrupole moment of the source as in flat space-time, but also to the analogous pressure quadrupole moment. The exact dependence on the pressure terms will be derived below. But because they are multiplied by  $H$ , it is already clear that these terms will have fewer time derivatives than the corresponding terms involving density.

To recast  $\int d^3 \vec{x} \mathcal{T}_{\bar{a}\bar{b}}$  in the desired form, our first task is to introduce the notion of mass and pressure quadrupole moments on the de Sitter background. Being a physical attribute of the source, the quadrupole moment at any instant of time should only depend on the physical geometry and coordinate invariant properties of the source. Therefore, we define the mass quadrupole moment as follows:

$$Q_{\bar{a}\bar{b}}^{(\rho)}(\eta) := \int_{\Sigma} d^3 V \rho(\eta) \bar{x}_{\bar{a}} \bar{x}_{\bar{b}}, \quad (6.34)$$

where  $\Sigma$  denotes any  $\eta = \text{const}$  surface with proper volume element  $d^3 V$  and  $\bar{x}_{\bar{a}} := a(\eta) x_{\bar{a}}$  is the *physical* separation of the point  $\vec{x}$  from the origin. The pressure quadrupole moment is defined similarly:

$$Q_{\bar{a}\bar{b}}^{(p)}(\eta) := \int_{\Sigma} d^3 V (p_1(\eta) + p_2(\eta) + p_3(\eta)) \bar{x}_{\bar{a}} \bar{x}_{\bar{b}}. \quad (6.35)$$

We can now use the conservation of stress-energy equations (6.31) and (6.32) to relate the integral  $\int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}$  to these quadrupole moments and their time derivatives.

This derivation follows the same steps as in the calculation in Minkowski space-time. We begin by noting that  $\int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') = - \int d^3 \vec{x}' (\mathring{D}^{\bar{m}} \mathcal{T}_{\bar{m}(\bar{a})} x_{\bar{b}})$  because the boundary term that arises in the integration by parts vanishes since the stress-energy tensor has compact spatial support. Using the spatial projection (6.32) of the conservation equation, we can rewrite the integral as follows:

$$\begin{aligned} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') &= - \int d^3 \vec{x}' e^{Ht'} (\partial_{t'} + 2H) \mathcal{T}_{\bar{a}}(t', \vec{x}') x_{\bar{b}} \\ &= \frac{1}{2} \int d^3 \vec{x}' e^{Ht'} (\partial_{t'} + 2H) (\mathring{D}^{\bar{m}} \mathcal{T}_{\bar{m}}(t', \vec{x}')) x_{\bar{a}} x_{\bar{b}}. \end{aligned} \quad (6.36)$$

Next, we use (6.31), the projection of the conservation equation along  $t^a$ , to eliminate  $\mathcal{T}_a$  in favor of the energy density and pressure:

$$\begin{aligned} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') &= \\ \frac{1}{2} \int d^3 \vec{x}' e^{4Ht'} &\left[ \frac{\partial^2 \rho}{\partial t'^2} + H \frac{\partial}{\partial t'} (8\rho + p_1 + p_2 + p_3) + 5H^2 (3\rho + p_1 + p_2 + p_3) \right] x_{\bar{a}} x_{\bar{b}}. \end{aligned} \quad (6.37)$$

The last step in this derivation is to express the right side of (6.37) in terms of the quadrupole moments defined in (6.34) and (6.35). A simple calculation yields:

$$\mathring{e}_{\bar{a}}^{\bar{a}} \mathring{e}_{\bar{b}}^{\bar{b}} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') = \frac{1}{2a(t')} \left[ \partial_{t'}^2 Q_{ab}^{(\rho)} - 2H \partial_{t'} Q_{ab}^{(\rho)} + H \partial_{t'} Q_{ab}^{(p)} \right](t'), \quad (6.38)$$

where  $\mathring{e}_{\bar{a}}^{\bar{a}}$  are the basis co-vectors in the  $\vec{x}$ -chart. Finally, using the fact that Lie derivative of any tensor field  $Q_{ab}$  with respect to the time translation Killing vector field is given by  $\mathcal{L}_T Q_{ab} = T^c \mathring{\nabla}_c Q_{ab} - 2H Q_{ab}$ , it is straightforward to show that

$$\mathring{e}_{\bar{a}}^{\bar{a}} \mathring{e}_{\bar{b}}^{\bar{b}} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') = \frac{1}{2a(t')} \left[ \mathcal{L}_T \mathcal{L}_T Q_{ab}^{(\rho)} + 2H \mathcal{L}_T Q_{ab}^{(\rho)} + H \mathcal{L}_T Q_{ab}^{(p)} + 2H^2 Q_{ab}^{(p)} \right](t'), \quad (6.39)$$

Since one can readily take the limit  $\Lambda \rightarrow 0$  in the  $(t, \vec{x})$  chart, we see immediately

that in this limit one recovers the familiar expression

$$\dot{e}_a^{\bar{a}} \dot{e}_b^{\bar{b}} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') \rightarrow \frac{1}{2} [\mathcal{L}_t \mathcal{L}_t Q_{ab}^{(\rho)}] \quad (6.40)$$

from the discussion of the quadrupole formula in Minkowski space-time.

Let us return to Eq.(6.39). Note that it is an *exact* equality within the post-de Sitter approximation; in section 6.2.2 we have not used the assumption  $D_o \ll \ell_\Lambda$  on the size of the source, nor the post-Newtonian assumption  $v \ll 1$ . If we invoke, e.g., kinetic theory, then the pressure goes as  $p \sim \rho v^2$  and can then be ignored compared to the density  $\rho$ . Then (6.39) simplifies to

$$\dot{e}_a^{\bar{a}} \dot{e}_b^{\bar{b}} \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(t', \vec{x}') \approx \frac{1}{2a(t')} [\mathcal{L}_T \mathcal{L}_T Q_{ab}^{(\rho)} + 2H \mathcal{L}_T Q_{ab}^{(\rho)} + 2H^2 Q_{ab}^{(p)}](t'). \quad (6.41)$$

where we have retained the last term because so far we have not made any assumption on relative magnitudes of the dynamical time scale of the system and Hubble time  $1/H$ . Now, in the post-Minkowski analysis, one does not have to make the assumption  $p \ll \rho$  because the continuity equations (6.31) do not involve pressure terms in that case. Furthermore, in the  $\Lambda > 0$  case, it turns out that dropping the pressure term from the exact expression (6.39) obscures certain conceptually important features (see footnote 11). Therefore we will retain the full expression for now.

Finally we can express the solution (6.30) on  $(M_P^+, \bar{g}_{ab})$  in terms of the source quadrupole moments (after a simple transformation to the  $(\eta, \vec{x})$  chart). Denoting by an ‘overdot’ the Lie derivative with respect to  $T^a$ , we obtain:

$$\begin{aligned} \chi_{ab}(\eta, \vec{x}) &= \frac{2G}{r a(\eta_{\text{ret}})} [\ddot{Q}_{ab}^{(\rho)} + 2H \dot{Q}_{ab}^{(\rho)} + H \dot{Q}_{ab}^{(p)} + 2H^2 Q_{ab}^{(p)}](\eta_{\text{ret}}) \\ &+ 2G \int_{-\infty}^{\eta_{\text{ret}}} \frac{d\eta'}{\eta'} \partial_{\eta'} \frac{1}{a(\eta')} [\ddot{Q}_{ab}^{(\rho)} + 2H \dot{Q}_{ab}^{(\rho)} + H \dot{Q}_{ab}^{(p)} + 2H^2 Q_{ab}^{(p)}](\eta') \\ &=: \sharp_{ab}(\eta, \vec{x}) + \flat_{ab}(\eta, \vec{x}) \end{aligned} \quad (6.42)$$

This expression is a good approximation to the exact solution (6.16) everywhere on  $\mathcal{I}$  (except at  $r = 0$ ).

## 6.3 Time-varying quadrupole moment & energy flux

In this section, we will carry out the second main step spelled out at the start of this chapter: We will use the approximate solution (6.42) to generalize Einstein's quadrupole formula for the energy  $E_T$  carried away by gravitational waves across  $\mathcal{I}^+$ . Since linearized gravitational fields do not have a gauge invariant, local stress-energy tensor, we employ the covariant Hamiltonian framework used in Chapter 5 to compute this energy.

This section is divided into three parts. In the first, we will discuss the asymptotic behavior of the fields that enter the expression of energy  $E_T$ , in the second, we will derive the quadrupole formula, and in the third, we will discuss its properties.

### 6.3.1 $\mathcal{I}^+$ and the perturbed electric part $\mathcal{E}_{ab}$ of Weyl curvature

As in the  $\Lambda = 0$  case, it is simplest to obtain manifestly gauge invariant expressions of fluxes of energy-momentum and angular momentum carried away by gravitational waves using fields defined on  $\mathcal{I}^+$ . Therefore we need to carry out a future conformal completion of the background space-time  $(M_{\mathbb{P}}^+, \bar{g}_{ab})$ . It is natural to seek a completion that makes  $(M_{\mathbb{P}}^+, \bar{g}_{ab})$  asymptotically de Sitter in a Poincaré patch as defined in Chapter 4. Because the physical metric  $\bar{g}_{ab}$  has the form,

$$\bar{g}_{ab} = a^2 \mathring{g}_{ab} \equiv (H\eta)^{-2} \mathring{g}_{ab}, \quad (6.43)$$

it is easy to verify that such a conformal completion can be obtained by setting the conformal factor  $\Omega = -H\eta$ , so that the conformally rescaled 4-metric, which is smooth at  $\mathcal{I}^+$ , is simply the flat metric  $\mathring{g}_{ab}$ . We will use this completion because all our equations in the Cartesian chart of  $\mathring{g}_{ab}$  and the solution  $\chi_{ab}$  will then automatically hold on the conformally completed space-time, including on  $\mathcal{I}^+$ . The final results, of course, will be conformally invariant.

The formulas for fluxes of energy-momentum and angular momentum –spelled out in sections 6.3.2 and 6.4– involve the so-called *perturbed electric part of the Weyl tensor*,  $\mathcal{E}_{ab}$ , at  $\mathcal{I}^+$ , as shown in Chapter 5. Therefore, we will first express  $\mathcal{E}_{ab}$  in terms of the metric perturbations –for which we already have the explicit expression (6.42) in terms of the quadrupole moments– and then discuss its properties needed in the subsequent discussion.

Recall that the local conditions included in the definition of *weakly* asymptotically de Sitter space-times –and therefore satisfied by space-times that are asymptotically Sitter in a Poincaré patch– imply that the Weyl curvature of the conformally rescaled metric must vanish at  $\mathcal{I}$  and therefore  $\Omega^{-1}C_{abc}{}^d$  admits a smooth limit there [10]. Our conformally rescaled metric  $\mathring{g}_{ab}$  is flat, whence the limit of  $\Omega^{-1}\mathring{C}_{abc}{}^d$  also vanishes. Therefore, not only is the first order perturbation  ${}^{(1)}C_{abc}{}^d$  such that  $\Omega^{-1}({}^{(1)}C_{abc}{}^d)$  admits a limit to  $\mathcal{I}^+$ , but furthermore the limit is *gauge invariant*.<sup>6</sup> The field of interest is the limit to  $\mathcal{I}^+$  of its electric part,

$$\mathcal{E}_{ab} := \Omega^{-1}({}^{(1)}C_{amb}{}^n \eta^m \eta_n) = - (H\eta)^{-1}({}^{(1)}C_{amb}{}^n \eta^m \eta_n), \quad (6.44)$$

where, as before,  $\eta^a$  is the unit normal to the cosmological slices ( $\eta = \text{const}$ ) w.r.t. the conformal metric  $\mathring{g}_{ab}$  and the indices are raised and lowered also using  $\mathring{g}_{ab}$ . We need to express  $\mathcal{E}_{ab}$  in terms of the (trace-reversed, rescaled) metric perturbation  $\bar{\chi}_{ab}$  produced by the source. This can be accomplished using the expression of  ${}^{(1)}C_{abc}{}^d$  in terms of the metric perturbation  $\bar{\gamma}_{ab}$ , and the equation of motion (6.9). The final result is:

$$\mathcal{E}_{ab} = \frac{1}{2H\eta} (\mathring{q}_a{}^c \mathring{q}_b{}^d - \frac{1}{3} \mathring{q}_{ab} \mathring{q}^{cd}) \left[ \frac{1}{2} \mathring{D}_c \mathring{D}_d \tilde{\chi} - \mathring{D}_{(c} \mathring{D}^m \chi_{d)m} - \mathring{D}_{(c} \partial_\eta \chi_{d)} + (\partial_\eta^2 - \frac{1}{\eta} \partial_\eta) \chi_{cd} \right]. \quad (6.45)$$

Consider now the limit of each term to  $\mathcal{I}^+$ . Although we already know from general considerations that the left side of (6.45) admits a smooth limit to  $\mathcal{I}^+$ , some care is needed to evaluate the right hand side because there is a  $(1/\eta)$  pre-factor, and  $\eta = 0$  at  $\mathcal{I}^+$ . However, because the explicit retarded solutions (6.11) decay as  $\eta$ , one can show that the terms involving  $\tilde{\chi}$  and  $\chi_{\bar{a}}$  admit a smooth limit to  $\mathcal{I}$ . A more detailed calculation using (6.30) shows that the fourth term,  $(1/\eta)(\partial_\eta^2 - \frac{1}{\eta} \partial_\eta) \chi_{ab}$ , also has a smooth limit to  $\mathcal{I}^+$ :

$$\frac{1}{\eta} \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{\bar{a}\bar{b}} = \frac{4G}{r} \left[ \frac{1}{\eta_{\text{ret}}} \partial_\eta^2 \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') - \frac{1}{\eta_{\text{ret}}^2} \partial_\eta \int d^3 \vec{x}' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') \right]. \quad (6.46)$$

Thus, we have expressed  $\mathcal{E}_{ab}$  at  $\mathcal{I}^+$  in terms of the perturbed metric, as required. In particular, in spite of the presence of a  $(1/\eta)$ -pre-factor in (6.45), each of the

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<sup>6</sup>A perturbed quantity  $\delta Q$  (indices suppressed) transforms under a gauge transformation  $\gamma_{ab} \rightarrow \gamma_{ab} + \nabla_a \xi_b$  as  $\delta Q \rightarrow \delta Q + \mathcal{L}_\xi \bar{Q}$  where  $\bar{Q}$  is the background quantity. If  $\bar{Q}$  is everywhere vanishing, then  $\delta Q$  is gauge invariant.

four terms in that formula for  $\mathcal{E}_{ab}$  has well-defined limits to  $\mathcal{I}^+$ . Note, incidentally, that in this calculation not only does the tail term  $\flat_{ab}$  in  $\chi_{ab}$  contribute but the result would diverge at  $\eta = 0$  without it. However, the process of taking derivatives has made the integral over  $\eta'$  in  $\flat_{ab}$  disappear, showing that the propagation of the left side of (6.46) is sharp. These features and Eq. (6.46) in particular will play an important role in section 6.3.2.

We will now discuss the properties of  $\mathcal{E}_{ab}$  that will be needed in subsequent calculations. First, the field equations satisfied by the first order perturbation  ${}^{(1)}C_{abc}{}^d$  are conformally invariant. Since they are completely equivalent to the field equations satisfied by the first order Weyl tensor in the flat space-time  $(M_{\mathbb{P}}^+, \mathring{g}_{ab})$ , we know that the propagation of  ${}^{(1)}C_{abc}{}^d$  is sharp along the null cones of  $\mathring{g}_{ab}$  (which are the same as the null cones of the de Sitter metric  $\bar{g}_{ab}$ ). Therefore the expression of the field  $\mathcal{E}_{ab}$  at  $\mathcal{I}^+$  in terms of source quadrupole moments is also sharp. Indeed, one can verify this explicitly using the expression (6.45) and the exact solutions (6.11) and (6.16). Second, in any neighborhood of  $\mathcal{I}^+$  where there are no matter sources, the field  $\mathcal{E}_{ab}$  is divergence-free, as easily seen from (4.55)

$$\mathring{D}^a \mathcal{E}_{ab} = 0. \quad (6.47)$$

Thus,  $\mathcal{E}_{ab}$  is transverse, traceless on  $\mathcal{I}^+$ . This property will make the gauge invariance of our expression of energy flux transparent.

Finally, as one would expect from the fact that  $\mathcal{E}_{ab}$  is gauge invariant, only the transverse-traceless (TT) components of  $\chi_{ab}$  (in its decomposition into irreducible parts) contribute to  $\mathcal{E}_{ab}$ . To see this begin with a standard decomposition of the ten components of the (rescaled, trace-reversed) metric perturbation  $\bar{\chi}_{ab}$ :

$$\begin{aligned} \tilde{\chi} &:= (\eta^a \eta^b + \mathring{q}^{ab}) \bar{\chi}_{ab}, & \chi &:= \mathring{q}^{ab} \chi_{ab}, & \chi_a &:= \mathring{D}_a A + A_a^T, \\ \chi_{ab} &:= \frac{1}{3} \mathring{q}_{ab} \mathring{q}^{cd} \chi_{cd} + \left( \mathring{D}_a \mathring{D}_b - \frac{1}{3} \mathring{q}_{ab} \mathring{D}^2 \right) B + 2 \mathring{D}_{(a} B_{b)}^T + \chi_{ab}^{TT}, \end{aligned} \quad (6.48)$$

where  $A_a^T$  and  $B_a^T$  are transverse and  $\chi_{ab}^{TT}$  is transverse, trace-less,

$$\mathring{D}^a A_a^T = 0 \quad \mathring{D}^a B_a^T = 0 \quad \mathring{D}^a \chi_{ab}^{TT} = 0 \quad \mathring{q}^{ab} \chi_{ab}^{TT} = 0, \quad (6.49)$$

and  $\tilde{\chi}$ ,  $\chi$ ,  $B$ ,  $\mathring{D}_a A$  are the longitudinal modes. Using the gauge condition (6.3), one can show that, in the expression (6.45) of  $\mathcal{E}_{ab}$ , all contributions from the

longitudinal and trace parts of  $\bar{\chi}_{ab}$  cancel out and  $\mathcal{E}_{ab}$  depends only on  $\chi_{ab}^{TT}$ :

$$\mathcal{E}_{ab} = \frac{1}{2H\eta} \left[ \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right] \chi_{ab}^{TT}, \quad (6.50)$$

Since  $\mathcal{E}_{ab}$  and  $\chi_{ab}^{TT}$  are both gauge invariant, the final relation (6.50) holds in any gauge. The limit to  $\mathcal{I}^+$  of this equality will play an important role in the next two sub-sections.

Before we end this sub-section, we remark on the important issue of obtaining transverse-traceless parts of symmetric tensors. In the literature on gravitational perturbations off Minkowski space-time, there is often confusion regarding the decomposition of spatial, symmetric tensors such as  $\chi_{ab}$  into its irreducible parts. While studying vacuum solutions to linearized Einstein's equations, one generally uses the notion spelled out in Eq. (6.48) (see e.g. Box 5.7 in [88], or section 4.3 in [89], or section 35.4 of [90]). In particular, by  $\chi_{ab}^{TT}$  one means the trace-free and divergence-free part of (the spatial tensor)  $\chi_{ab}$  as in (6.48). This usage is standard in cosmology, e.g. in the presentation of results by BICEP and Planck collaborations. It is also used heavily in the (perturbative) quantum gravity literature; for example, the conclusion that the graviton has spin 2 is arrived at by calculating the Casimir operators of the Poincaré group on the 1-graviton Hilbert space constructed from the Minkowski space-time analog of  $\chi_{ab}^{TT}$ .

But then in the study of retarded fields produced by compact sources, one uses an entirely different decomposition: Here, the  $1/r$ -part of  $\chi_{ab}$  (i.e., the far field approximation) of the full retarded solution is projected into radial and the orthogonal spherical directions in *physical space*. Unfortunately, these projections are also referred to as the trace, longitudinal and transverse-traceless parts of  $\chi_{ab}$ . For concreteness, let us denote by  $P_a^c$  the projection operator in to the 2-sphere orthogonal to the radial direction in the physical space and set  $\chi_{ab}^{tt} = (P_a^c P_b^d - (1/2)P_{ab} P^{cd}) \chi_{cd}$ . In the literature, in place of *tt*, the symbol *TT* is used also for this projection (see, e.g., chapter 11 of [88], or section 4.5.1 in [89], or section 36.10 in [90]). This is confusing because the two notions of transverse traceless parts are distinct and inequivalent. The first notion is local in momentum space and the resulting  $\chi_{ab}^{TT}$  is exactly gauge invariant everywhere in space-time. The second notion, which we will continue to denote by  $\chi_{ab}^{tt}$ , is local in the physical

space and  $\chi_{ab}^{tt}$  is gauge invariant only in a weaker sense involving  $1/r$  fall-offs. Nonetheless, it is  $\chi_{ab}^{tt}$  that is well-tailored to the Bondi-Sachs formalism at null infinity of asymptotically flat space-times.

As we have seen in section 6.2.1, the  $1/r$ -expansion is not very useful at  $\mathcal{I}^+$  of de Sitter space-time. Therefore in the  $\Lambda > 0$  discussion we only use the first decomposition, spelled out explicitly in (6.48). We will refer to the second notion only in the discussion of the  $\Lambda \rightarrow 0$  limit.

### 6.3.2 Fluxes across $\mathcal{I}^+$

We now calculate the flux of energy associated with the time translation  $T^a$  across  $\mathcal{I}^+$ . Since  $T^a$  is a Killing field of the background space-time  $(M_{\mathbb{P}}^+, \bar{g}_{ab})$  we know that, for any choice of admissible conformal completion,  $T^a$  admits a smooth extension tangential to  $\mathcal{I}^+$ . For the choice  $\Omega = -H\eta$  of the conformal factor we made above,  $T^a$  also serves as the dilation w.r.t. the intrinsic 3-metric  $\mathring{q}_{ab}$  on  $\mathcal{I}^+$ :

$$T \hat{=} -H \left[ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right]. \quad (6.51)$$

From the detailed analysis of the covariant phase space  $\Gamma_{\text{Cov}}$  carried out in Chapter 5, the total energy flux  $E_T$  across  $\mathcal{I}^+$  is given by the Hamiltonian generating the time translation  $T^a$  on  $\Gamma_{\text{Cov}}$ . The result can be expressed most simply in terms of  $\mathcal{E}_{ab}$  and the Lie derivative of the metric perturbation w.r.t.  $T^a$  at  $\mathcal{I}^+$ :

$$E_T \hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathcal{L}_T \chi_{ab} + 2H \chi_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd}. \quad (6.52)$$

Note that because  $\mathcal{E}_{ab}$  is transverse-traceless ( $TT$ ), the integral automatically extracts the  $TT$  part of the term in the bracket and we have

$$\begin{aligned} E_T &\hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathcal{L}_T \chi_{ab} + 2H \chi_{ab} \right)^{TT} \mathring{q}^{ac} \mathring{q}^{bd} \\ &\hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( T^m \mathring{\nabla}_m \chi_{ab} \right)^{TT} \mathring{q}^{ac} \mathring{q}^{bd}, \end{aligned} \quad (6.53)$$

where in the second step we have used the fact that  $(\mathcal{L}_T \chi_{ab} + 2H \chi_{ab}) = T^m \mathring{\nabla}_m \chi_{ab}$ . We note on the side that, because the derivative  $T^m \mathring{\nabla}_m$  commutes with the oper-

ation of taking the  $TT$  part on  $\mathcal{I}^+$ , the integral can be rewritten as

$$E_T \hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( T^m \overset{\circ}{\nabla}_m \chi_{ab}^{TT} \right) \overset{\circ}{q}^{ac} \overset{\circ}{q}^{bd} \quad (6.54)$$

which is manifestly gauge invariant.

Next, we return to (6.53) and use (6.50) to express  $\mathcal{E}_{ab}$  in terms of the  $TT$ -part of  $\chi_{ab}$ . Using the fact that the operator  $(1/\eta)[\partial_\eta^2 - \frac{1}{\eta}\partial_\eta]$  commutes with the operation of taking the  $TT$  part, we have:

$$\begin{aligned} E_T &\hat{=} \lim_{\rightarrow \mathcal{I}} \frac{1}{32\pi GH^2} \int d^3x \left[ \frac{1}{\eta} \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{ab} \right]^{TT} \left[ T^m \overset{\circ}{\nabla}_m \chi_{cd} \right]^{TT} \overset{\circ}{q}^{ac} \overset{\circ}{q}^{bd} \\ &\hat{=} \lim_{\rightarrow \mathcal{I}} \frac{1}{32\pi GH^2} \int d^3x \left[ \frac{1}{\eta} \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{ab} \right] \left[ T^m \overset{\circ}{\nabla}_m \chi_{cd} \right]^{TT} \overset{\circ}{q}^{ac} \overset{\circ}{q}^{bd} \end{aligned} \quad (6.55)$$

where in the second step we removed the  $TT$  on the first square bracket because the second square bracket is already  $TT$  and therefore the integral automatically extracts only the  $TT$  part of the first square bracket. These expressions hold for any solution  $\chi_{ab}$  that is source-free in a neighborhood of  $\mathcal{I}^+$  (e.g. within the shaded region in the left panel of Fig. 6.1).

We now use the approximations  $D_o/\ell_\Lambda \ll 1$  and  $v \ll 1$  spelled out in section 6.2.1 and insert in (6.55) the convenient expression of  $\chi_{ab}$  given in (6.30). For the first square bracket we use (6.46) and  $\partial_\eta f(\eta - r) = -\partial_r f(\eta - r)$  and evaluate the expression at  $\mathcal{I}^+$  by setting  $\eta = 0$ . The result is:

$$\frac{1}{\eta} \left[ \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{\bar{a}\bar{b}} \right] (\vec{x}) \hat{=} -\frac{4G}{r} \partial_r \left( \frac{1}{r} \partial_r \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}') \right). \quad (6.56)$$

As we noted after (6.46), although the tail term  $b_{ab}$  in the expression (6.17) of  $\chi_{ab}$  does contribute to the result, the process of taking derivatives has made the integral over  $\eta$  in  $b_{ab}$  disappear and the result depends only on what the source does at time  $\eta = \eta_{\text{ret}}$ .

Next, consider the second square bracket in the integrand of (6.55). Since the term multiplying this bracket has a well-defined limit to  $\mathcal{I}^+$ , we can replace  $T^m$  by its limiting value  $-Hr\hat{r}^m$  at  $\mathcal{I}^+$ . Using (6.30) we again find that, although the tail term  $b_{ab}$  does contribute to the result, the integration over  $\eta$  disappears because

of the directional derivative along  $T^m$  and we obtain

$$\left[ T^m \overset{\circ}{\nabla}_m \chi_{\bar{c}\bar{d}} \right] (\vec{x}) = \frac{4GH}{r} \int d^3x' \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\text{ret}}, \vec{x}'). \quad (6.57)$$

Substituting (6.56) and (6.57) in (6.55), performing an integration by parts, and using Eq. (6.39) to express the integral over the stress-energy tensor in terms of quadrupole moments, we obtain

$$E_T \hat{=} \frac{G}{8\pi H} \int \frac{dr}{r} d^2S \left[ \left( \partial_r Hr \left( \ddot{Q}_{ab}^{(\rho)} + 2H \dot{Q}_{ab}^{(\rho)} + H \dot{Q}_{ab}^{(p)} + 2H^2 Q_{ab}^{(p)} \right) \right) \times \right. \\ \left. \left( \partial_r Hr \left( \ddot{Q}_{cd}^{(\rho)} + 2H \dot{Q}_{cd}^{(\rho)} + H \dot{Q}_{cd}^{(p)} + 2H^2 Q_{cd}^{(p)} \right)^{TT} \right) \right] \dot{q}^{ac} \dot{q}^{bd}, \quad (6.58)$$

where  $d^2S$  is the unit 2-sphere volume element of the flat metric  $\dot{q}_{ab}$  at  $\mathcal{I}^+$ , and, as before an ‘overdot’ denotes the Lie derivative w.r.t.  $T^a$ . Finally, using the fact that the operation  $r\partial_r$  commutes with the operation of extracting the  $TT$  part<sup>7</sup> and that the affine parameter  $T$  along the integral curves of  $T^a$  satisfies  $dT = dr/(rH)$  at  $\mathcal{I}^+$ ,<sup>8</sup> we obtain

$$E_T \hat{=} \frac{G}{8\pi} \int_{\mathcal{I}^+} dT d^2S \left[ \mathcal{R}_{ab}(\vec{x}) \mathcal{R}_{cd}^{TT}(\vec{x}) \dot{q}^{ac} \dot{q}^{bd} \right], \quad (6.59)$$

where the ‘radiation field’  $\mathcal{R}_{ab}(\vec{x})$  on  $\mathcal{I}^+$  is given by

$$\mathcal{R}_{ab}(\vec{x}) \hat{=} \left[ \ddot{Q}_{ab}^{(\rho)} + 3H \ddot{Q}_{ab}^{(\rho)} + 2H^2 \dot{Q}_{ab}^{(\rho)} + H \ddot{Q}_{ab}^{(p)} + 3H^2 \dot{Q}_{ab}^{(p)} + 2H^3 Q_{ab}^{(p)} \right] (\eta_{\text{ret}}), \quad (6.60)$$

where, as before,  $\eta_{\text{ret}} = \eta - r \hat{=} -r$ . Note that  $\mathcal{R}_{ab}$  is a *field* on  $\mathcal{I}^+$  because, given a point  $\vec{x}$  on  $\mathcal{I}^+$ , the quadrupole moments  $Q_{ab}^{(\rho)}$  and  $Q_{ab}^{(p)}$  are obtained by performing an integral over the source along the 3-surface  $\eta = \eta_{\text{ret}}$  and these 3-surfaces change as we change  $\vec{x}$  on  $\mathcal{I}^+$  (see Fig. 6.1). This occurs also in the standard quadrupole formula in flat space. There is, however one difference from the standard formula: (6.59) uses the  $TT$  decomposition rather than the  $tt$  decomposition. (Indeed, since

<sup>7</sup>This is most easily seen in the Fourier space where the projection operator  $\Gamma_{ab}{}^{cd}(\vec{k})$  to extract the  $TT$  part is  $\Gamma_{ab}{}^{cd}(\vec{k}) = P_a{}^c(\vec{k})P_b{}^d(\vec{k}) - \frac{1}{2}P_{ab}(\vec{k})P^{cd}(\vec{k})$  with  $P^{cd}(\vec{k}) = \dot{q}_k^{cd} - \hat{n}_k^c \hat{n}_k^d$ . Here  $\dot{q}_{\vec{k}ab}$  is the Euclidean 3-metric in the Fourier space,  $\hat{n}_{\vec{k}}^c$  is the unit normal in the radial  $\vec{k}$  direction.

The operator  $r\partial_r$  in physical space acts as  $\vec{k}^c D_c \equiv \Sigma_i k_i (\partial/\partial k_i)$  in Fourier space.

<sup>8</sup> $T^m \partial_m \hat{=} -Hr \partial_r$ , so on  $\mathcal{I}^+$ ,  $T^m \partial_m (\ln r/H) \hat{=} -1$ .

the  $tt$  decomposition used in the flat space analysis is tied to the  $1/r$ -expansion, it is not very useful in the de Sitter context.) One consequence is that the  $TT$  label appears only on the  $\mathcal{R}_{cd}$  term in (6.59); the term  $\mathcal{R}_{ab}$  is not automatically  $TT$  because the volume element in (6.59) is  $d^3x/(Hr)$  and not  $d^3x$  with respect to which TT is defined. Finally, while components of individual terms such as  $\ddot{Q}_{\bar{a}\bar{b}}^{(\rho)}(0, \vec{x})$  depend only on  $r \equiv |\vec{x}|$  at  $\mathcal{I}^+$  and not on angles, an angular dependence is introduced while taking the  $TT$  part. Therefore, the total integrand of (6.59) has a genuine angular dependence; otherwise one could have trivially performed the angular integral and replaced it just by a  $4\pi$  factor. Again, conceptually, this situation is the same as for the standard quadrupole formula in flat space-time the  $tt$  operation also introduces angular dependence.

Finally, as in the  $\Lambda = 0$  calculation, we extract power  $P_T$  radiated by the system at any ‘instant of time’  $T_0$  at  $\mathcal{I}^+$  (i.e., a 2-sphere cross-section of  $\mathcal{I}^+$ , orthogonal to the orbits of the ‘time-translation’  $T^a$ ):

$$P_T(T_0) \hat{=} \frac{G}{8\pi} \int_{T=T_0} d^2S \left[ \mathcal{R}^{ab}(\vec{x}) \mathcal{R}_{ab}^{TT}(\vec{x}) \right] \quad (6.61)$$

While the expression (6.52) of radiated energy is completely local in  $\chi_{ab}$  a degree of non-locality enters while casting it in terms of sources: (6.61) involves only the  $TT$ -part of one of the ‘radiation fields’. However, because the  $TT$ -part is taken only for one of the two ‘radiation fields’, one can show that if  $\mathcal{L}_T T_{ab} = 0$  at an instant  $\eta_o$ , then the power at  $\mathcal{I}^+$  vanishes at the cross-section  $T = T_0$  representing the intersection of  $\mathcal{I}^+$  with the null cone with vertex  $(\eta_o, \vec{x} = \vec{0})$ .

The expression (6.59) of radiated energy is the main result of this section. As in Einstein’s quadrupole formula, it has been derived using the first post-Newtonian approximation under the assumption that we have an externally specified, first order stress-energy tensor  $T_{ab}$  satisfying the conservation equation with respect to the background metric.

We close this section with a note about the covariant phase space in presence of sources. The covariant phase space  $\Gamma_{\text{Cov}}$  constructed and used in Chapter 5 to obtain flux formulas at  $\mathcal{I}^+$  consists of *homogeneous* solutions to linearized Einstein’s equations. In this chapter, we are considering retarded solutions with a first order source  $T_{ab}$ . However, in the shaded neighborhood of  $\mathcal{I}^+$  shown in the left panel of

Fig. 1, all (trace-reversed) metric perturbations  $\bar{\gamma}_{ab}$  satisfy the homogeneous equation and there is a family of Cauchy surfaces for this neighborhood that approach  $\mathcal{I}^+$ . Therefore, one can use the covariant phase space framework in this neighborhood to calculate fluxes of energy, momentum and angular momentum carried by the perturbations  $\bar{\gamma}_{ab}$  across  $\mathcal{I}^+$ . In this calculation, we used the leading-order terms in the expression (6.42) of  $\chi_{ab}$ , ignoring terms of order  $\mathcal{O}((D_o/\ell_\Lambda)(1-\eta/r))$  and  $\mathcal{O}(v)$  compared to terms of order  $\mathcal{O}(1)$ . However, as noted above, the simplified formula (6.30) for  $\chi_{ab}$  is valid in an entire neighborhood of  $\mathcal{I}^+$  (the shaded region in the left panel of Fig. 6.1). Finally note that, since the flux formula is gauge invariant, the calculation can be carried out in any gauge.

### 6.3.3 Properties of fluxes across $\mathcal{I}^+$

Our formula of the energy carried by gravitational waves across  $\mathcal{I}^+$  have several interesting features which we now discuss in some detail.

(1) First, the cosmological constant term does survive (through  $H = \sqrt{\Lambda/3}$ ) even at  $\mathcal{I}^+$ . Nonetheless, we explicitly see that, in this first post-Newtonian approximation, the radiated energy is still quadrupolar.

(2) As we discussed in section 6.3.1, because of its conformal properties, it is clear that  $\mathcal{E}_{ab}$  has sharp propagation. However, the fundamental formula (6.52) for the energy flux we started out with depends also on  $\chi_{ab}$  whose expression does contain an integral over all  $\eta'$  that extends all the way back to  $\eta' = -\infty$ . So, why is there no such integral in the final expressions of radiated energy? The reason is that what features in (6.52) is not  $\chi_{ab}$  itself but rather, *its derivative*,  $(\mathcal{L}_T + 2H)\chi_{ab} \hat{=} -Hr \partial_r \chi_{ab}$ . The integral over  $\eta'$  disappears while taking this derivative, as we saw in (6.57). This is why our quadrupole formula (6.59) does not contain an explicit tail term in spite of back-scattering due to the background de Sitter curvature. As in the asymptotically flat case, of course, tail terms will arise in higher post-Newtonian orders.

(3) In contrast to the Einstein formula, there is a contribution at the leading order from the time variation of the *pressure* quadrupole and, furthermore,

from the *pressure quadrupole itself*. It is well known from the Raychaudhuri equation in cosmology that pressure contributes to gravitational attraction in any Friedmann-Lemaître-Robertson-Walker universe. Eq. (6.59) shows that, if  $\Lambda > 0$ , it also *sources* gravitational waves already in the leading order post-Newtonian approximation. If  $p \ll \rho$  (in the  $c = 1$  units) as for Newtonian fluids, then the pressure terms  $H\ddot{Q}_{ab}^{(p)} + 3H^2\dot{Q}_{ab}^{(p)}$  can be neglected compared to the density terms  $3H\ddot{Q}_{ab}^{(\rho)} + 2H^2\dot{Q}_{ab}^{(\rho)}$  and the expression (6.60) of  $\mathcal{R}_{ab}$  simplifies to:

$$\mathcal{R}_{ab}(\vec{x}) = \left[ \ddot{Q}_{ab}^{(\rho)} + 3H\dot{Q}_{ab}^{(\rho)} + 2H^2\dot{Q}_{ab}^{(\rho)} + 2H^3Q_{ab}^{(p)} \right] (\eta_{\text{ret}}). \quad (6.62)$$

For compact binaries of immediate interest to the gravitational wave detectors, we also have  $(\Delta t_{\text{ret}})/t_H \ll 1$  where  $\Delta t_{\text{ret}}$  is the dynamical time scale in which the mass and pressure quadrupole change by factors of  $\mathcal{O}(1)$  and  $t_H$ , the Hubble scale.<sup>9</sup> Then the formula further simplifies and acquires a form similar to that of the  $\Lambda = 0$  Einstein formula:

$$\mathcal{R}_{ab}(\vec{x}) = \ddot{Q}_{ab}^{(\rho)}(\eta_{\text{ret}}) \quad (6.63)$$

When  $\Lambda$  is as tiny as the observations imply, the de Sitter quadrupole and its ‘overdots’ are extremely well approximated by those in Minkowski space-time and the  $\Lambda > 0$  first post-Newtonian approximation is extremely well-approximated by the standard one. The full expression (6.60) provides a precise control over the errors one makes while using the Einstein formula in presence of  $\Lambda$ .

(4) Positivity of energy flux is not transparent because the integrand of (6.59) is not manifestly positive, as it is in Einstein’s formula for flat space. However, one can establish positivity as follows. First, properties of the retarded Green’s function imply that the  $\chi_{ab}^{TT}(\eta, \vec{x})$  can be expressed using the  $TT$  part  $\mathcal{T}_{ab}^{TT'}$  of  $\mathcal{T}_{ab}(\eta, \vec{x}')$ , where the prime in  $TT'$  emphasizes that the transverse traceless part refers to the argument  $\vec{x}'$ :

$$\chi_{\bar{a}\bar{b}}^{TT}(\eta, \vec{x}) = 4G \int \frac{d^3\vec{x}'}{|\vec{x} - \vec{x}'|} \mathcal{T}_{\bar{a}\bar{b}}^{TT'}(\eta_{\text{Ret}}, \vec{x}') + 4G \int d^3\vec{x}' \int_{-\infty}^{\eta_{\text{Ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{\bar{a}\bar{b}}^{TT'}(\eta', \vec{x}'). \quad (6.64)$$

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<sup>9</sup>This need not be the case for the *very* long wavelength emission due to the coalescence of supermassive black holes at the center of galaxies.

(The  $TT$  in  $\chi_{\bar{a}\bar{b}}^{TT}(\eta, \vec{x})$  on the left side refers to  $\vec{x}$ .) Next, we rewrite the expression (6.55) in terms of  $\chi_{ab}^{TT}$

$$E_T \doteq \lim_{\rightarrow \mathcal{I}} \frac{1}{32\pi GH^2} \int d^3x \left[ \frac{1}{\eta} \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{ab}^{TT} \right] \left[ T^m \overset{\circ}{\nabla}_m \chi_{cd}^{TT} \right] \overset{\circ}{q}^{ac} \overset{\circ}{q}^{bd} \quad (6.65)$$

where we have used the fact that  $\partial_\eta$  and  $T^m \overset{\circ}{\nabla}_m$  commute with the operation of taking the  $TT$  part. Finally, we substitute (6.64) in (6.65) and simplify following the procedure of section 6.2.1<sup>10</sup> to obtain:

$$E_T \doteq \frac{G}{2\pi} \int_{\mathcal{I}^+} dT d^2S \left[ \partial_r \int d^3x' \mathcal{T}_{ab}^{TT'}(\eta_{\text{ret}}, \vec{x}') \right] \left[ \partial_r \int d^3x' \mathcal{T}_{cd}^{TT'}(\eta_{\text{ret}}, \vec{x}') \right] \overset{\circ}{q}^{ac} \overset{\circ}{q}^{cd}, \quad (6.66)$$

which is manifestly positive.

As we discussed in the beginning of the chapter, de Sitter space-time admits gravitational waves whose energy can be arbitrarily negative in the linearized approximation because the time translation Killing field  $T^a$  is space-like in a neighborhood of  $\mathcal{I}^+$ . Indeed, for systems under consideration, gravitational waves satisfy the homogeneous, linearized Einstein's equations in a neighborhood of  $\mathcal{I}^+$  and there is an *infinite dimensional subspace of these solutions for which the total energy is negative*. What, then, is the physical reason behind the positivity of our  $E_T$ ? Consider the shaded triangular region in the left panel of Fig. 6.1. It is bounded by  $\mathcal{I}^+$ , upper half of  $E^+(i^-)$  and  $E^-(i^+)$ . The time translation vector field  $T^a$  is tangential to all these three boundaries, being space-like on  $\mathcal{I}^+$ , null and *past directed* on the upper half of  $E^+(i^-)$ , and null and *future directed* on  $E^-(i^+)$ . As a result, for *any* solution, the energy flux across the upper half of  $E^+(i^-)$  is negative, that across  $E^-(i^+)$  is positive, and that across  $\mathcal{I}^+$  is the sum of the two, which can have either sign and arbitrary magnitude. Thus, the potentially negative energy contribution at  $\mathcal{I}^+$  can be traced directly to the incoming gravitational waves across the upper half of  $E^+(i^-)$ . Now, in the present calculation, physical considerations led us to the *retarded* metric perturbation created by the time varying quadrupoles. Therefore there is no flux of energy across the cosmological horizon  $E^+(i^-)$ ; the potential negative energy flux across  $\mathcal{I}^+$  is simply absent. The entire energy flux across  $\mathcal{I}^+$  equals the energy flux across  $E^-(i^+)$  which is always

<sup>10</sup>We assume that integrals involving  $\mathcal{T}_{ab}^{TT'}$  are all well-defined. This is a plausible assumption since  $\mathcal{T}_{ab}$  is smooth and of compact support whence its Fourier transform is in Schwartz space.

positive because  $T^a$  is future directed there. To summarize then, while in general the energy flux across  $\mathcal{I}^+$  can have either sign, the metric perturbation  $\bar{\chi}_{ab}$  at  $\mathcal{I}^+$  created by physically reasonable sources are so constrained that the energy carried by gravitational waves across  $\mathcal{I}^+$  is necessarily positive.

(5) The fifth feature concerns time dependence of the source. Equations satisfied by the full (trace-reversed, rescaled) metric perturbation  $\bar{\chi}_{ab}$  refer only to the background metric  $\bar{g}_{ab}$  and  $T^a$  is a Killing field of  $\bar{g}_{ab}$  which is time-like in the region in which the source  $T_{ab}$  resides. Therefore, it follows that if the source is static, i.e., if  $\mathcal{L}_T T_{ab} = 0$ , then the retarded solution  $\bar{\chi}_{ab}$  must satisfy  $\mathcal{L}_T \bar{\chi}_{ab} = 0$ . Physically, one would expect there to be no flux of energy across  $\mathcal{I}^+$ . But this is not manifest in Eq. (6.59) since it contains a term  $Q_{ab}^{(p)}$  that does *not* involve a time derivative. Let us therefore examine the fields that enter the definitions (6.34) and (6.35) of quadrupole moments. A simple calculation shows that, if  $\mathcal{L}_T T_{ab} = 0$ , the fields that enter the definitions of quadrupole moments satisfy

$$\mathcal{L}_T \rho = 0; \quad \mathcal{L}_T p = 0; \quad \mathcal{L}_T a(\eta)x_{\bar{b}} = 0; \quad \mathcal{L}_T \dot{e}_{\bar{a}} = -H \dot{e}_{\bar{a}}; \quad (6.67)$$

and the 3-dimensional volume element  $dV$  is preserved under the isometry generated by  $T^a$ . Note that the last two equations in (6.67) hold always. Therefore we have:

$$\mathcal{L}_T Q_{ab}^{(\rho)} = -2H Q_{ab}^{(\rho)} \quad \text{and} \quad \mathcal{L}_T Q_{ab}^{(p)} = -2H Q_{ab}^{(p)}. \quad (6.68)$$

Thus, in contrast to what happens in the Minkowski space-time calculation, because of the expansion of the de Sitter scale factor, now  $\mathcal{L}_T T_{ab} = 0$  does not imply that quadrupoles are left invariant by the flow generated by  $T^a$ . However, using (6.42), (6.59), (6.61) and (6.68), it immediately follows that

$$\text{if } \mathcal{L}_T T_{ab} = 0 \text{ everywhere, then } E_T \hat{=} 0, \quad \text{and} \quad P_T(T_0) \hat{=} 0 \quad (6.69)$$

for all  $T_0$ . (In fact, it follows from Eq. (6.61) that an ‘instantaneous’ result also holds: if  $\mathcal{L}_T T_{ab} |_{\eta=\eta_{\text{ret}}} = 0$ , then  $P_T(T_0) \hat{=} 0$  where  $\eta_{\text{ret}} = \eta - r_0 \equiv -r_0$  and  $T_0 = \ln(r_0 H)$ .) Thus, the presence of the term without a time derivative of the pressure quadrupole  $Q_{ab}^{(p)}$  is, in fact, essential to ensure that if  $\mathcal{L}_T T_{ab} = 0$  then  $E_T$

and  $P_T(T_0)$  vanish on  $\mathcal{I}^+$ .<sup>11</sup>

(6) Next, let us consider the limit  $\Lambda \rightarrow 0$ . As discussed in Chapter 5, the limit is subtle and has to be taken in the  $(t, \vec{x})$  (rather than the  $(\eta, \vec{x})$ ) chart. Since the  $(t, \vec{x})$  chart breaks down at  $\mathcal{I}^+$  (where  $\eta = 0$  but  $t = \infty$ ), we cannot directly take the limit of our final expression of the energy flux at  $\mathcal{I}^+$  of de Sitter space-time. Rather, we have to ‘pass through’ the physical space-time as in Chapter 5 and use results from the covariant phase space framework relating expressions involving the  $TT$  and  $tt$  decompositions in Minkowski space-time. As a result, the procedure is rather long and we will only summarize the main steps here.

Consider the 1-parameter family of de Sitter backgrounds  $\bar{g}_{ab}^{(\Lambda)}$ , parametrized by  $\Lambda$ , with a 1-parameter family  $T_{ab}^{(\Lambda)}$  of stress-energy tensors, each satisfying the conservation law with respect to the respective  $\bar{g}_{ab}^{(\Lambda)}$  and the condition  $\mathcal{L}_T T_{ab}^{(\Lambda)} = 0$  outside a compact time interval. Let  $\chi_{ab}^{(\Lambda)}(t, \vec{x})$  denote the retarded solutions (6.30) to the field equations and gauge conditions. For each  $\Lambda$ , one can express this solution in terms of the source quadrupoles as in (6.42). The question is whether as  $\Lambda \rightarrow 0$  this 1-parameter family of solutions has a well-defined limit  $\check{\chi}_{ab}(t, \vec{x})$ . If so, the analysis in section 5.2 of Chapter 5 shows that: i)  $\check{\chi}_{ab}(t, \vec{x})$  satisfies the dynamical equation and gauge conditions w.r.t. the Minkowski metric  $\check{\eta}_{ab}$ ; and, ii) the expression (6.59) of energy in the gravitational waves has a well-defined limit, which is furthermore precisely the energy in the solution  $\check{\chi}_{ab}(t, \vec{x})$ , calculated in Minkowski space-time.

We have already shown in section 6.1 that the exact retarded solutions do tend to the exact retarded solution in Minkowski space-time. We will now show that this is also the case for the approximate solutions (6.42). In the  $(t, \vec{x})$  chart, one can perform the integral in the tail term  $\flat_{ab}(t, \vec{x})$  in the solutions (6.42) to find that  $\flat_{ab}(t, \vec{x})$  has an explicit overall factor of  $H$  whence, as one would expect, the limit  $\Lambda \rightarrow 0$  of this term vanishes (see Appendix C). Next consider the sharp term  $\sharp_{ab}(t, \vec{x})$  in (6.42). In the  $\Lambda \rightarrow 0$  limit, we have  $T^a \rightarrow t^a$ , a time translation in Minkowski metric  $\check{\eta}_{ab}$ ;  $\mathcal{L}_T \rightarrow \mathcal{L}_t$ ;  $a(t) \rightarrow 1$  and  $Q_{ab}^{(\rho)} \rightarrow \check{Q}_{ab}^{(\rho)}$ , the mass quadrupole moment constructed from the limiting stress-energy tensor  $\check{T}_{ab}$  using the Minkowski

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<sup>11</sup>This consistency would have been obscured if we had ignored the pressure terms relative to the density terms in (6.39), and used the resulting approximation (6.41) to arrive at the expression of  $\chi_{ab}$ . That is why we kept all the pressure quadrupole terms even though they can be ignored relative to the analogous density quadrupole terms for Newtonian fluids.

metric  $\mathring{\eta}_{ab}$ . Therefore, the limiting solution is given by

$$\lim_{\Lambda \rightarrow 0} \chi_{ab}^{(\Lambda)}(t, \vec{x}) = \frac{2G}{r} \mathcal{L}_t \mathcal{L}_t \mathring{Q}_{ab}^{(\rho)}(t_{\text{ret}}) =: \mathring{\chi}_{ab}(t, \vec{x}) \quad (6.70)$$

for all  $r \gg d(t)$ , where  $d$  is the physical size of the source with respect to the Minkowski metric  $\mathring{\eta}_{ab}$ . Now, since by assumption the source is active for a finite time interval, on a  $t = \text{const}$  surface sufficiently in the future, the support of the initial data of  $\mathring{\chi}_{ab}(t, \vec{x})$ , which has sharp propagation in Minkowski space-time, is entirely in a region where the approximation holds. Consider only the future of this slice. In that space-time region we have a 1-parameter family of solutions  $\chi_{ab}^{(\Lambda)}(t, \vec{x})$  to the source-free equations whose total energy is given by (6.59) for each  $\Lambda > 0$ . The limit  $\mathring{\chi}_{ab}(t, \vec{x})$  is well-defined, as required. Therefore, in the  $\Lambda \rightarrow 0$  limit the energy expression (6.59) goes over to the energy in  $\mathring{\chi}_{ab}(t, \vec{x})$  with respect to  $t^a$  in Minkowski space (see Eq (4.24) of Chapter 5). And we know that this energy is given by the Einstein formula. Thus, in the limit  $\Lambda \rightarrow 0$  one recovers the standard quadrupole formula in Minkowski space-time.

To summarize, our energy expression (6.59) arises as the Hamiltonian on the covariant phase space of linearized solutions on de Sitter space-time, and using results from Chapter 5 we can conclude that it tends to the expression of the Hamiltonian in Minkowski space in the  $\Lambda \rightarrow 0$  limit, which in turn reduces to the Einstein flux formula at  $\mathring{\mathcal{I}}^+$ . The argument is indirect mainly because in linearized gravity off *Minkowski space-time* we do not know the relation between the  $TT$  and  $tt$  decompositions. What we know is only the equality between the two expressions of energy, the first evaluated on space-like planes in terms of the  $TT$  decomposition and the second, evaluated at  $\mathcal{I}^+$  in terms of  $tt$ . (For definitions of  $TT$  and  $tt$  fields see the end of section 6.3.2).

(7) So far we have focussed on the energy carried by gravitational waves. We now discuss the flux of 3-momentum across  $\mathcal{I}^+$ . The component of the 3-momentum along a space translation  $S^{a_{\bar{i}}}$  is given by Chapter 5

$$P_{\bar{i}} \hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathcal{L}_{S_{\bar{i}}} \chi_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd} \quad (6.71)$$

We can again use (6.50) to express  $\mathcal{E}_{cd}$  in terms of  $\chi_{cd}$ :  $\mathcal{E}_{ab} = [\frac{1}{\eta} (\partial_{\eta}^2 - \frac{1}{\eta} \partial_{\eta}) \chi_{ab}]^{TT}$ .

Now, it is clear from the expression (6.30) of  $\chi_{ab}$  that its dependence on  $\vec{x}$  comes entirely from  $\eta_{\text{ret}}$ . Therefore,  $\chi_{ab}$  is invariant under the parity operation  $\Pi : \vec{x} \rightarrow -\vec{x}$ , whence  $\frac{1}{\eta} \left( \partial_\eta^2 - \frac{1}{\eta} \partial_\eta \right) \chi_{ab}$  is also invariant. Since the operation of taking the  $TT$ -part refers only to the 3-metric  $\mathring{q}_{ab}$ , it also commutes with  $\Pi$ . Hence  $\mathcal{E}_{ab}$  is even under  $\Pi$ . The second term,  $S^m_{\bar{i}} \mathring{D}_m \chi_{ab}$  is manifestly odd under  $\Pi$  since  $S^a$  is odd but  $\chi_{ab}$  is even. Therefore the integral on the right side of (6.71) vanishes. Thus, as in the  $\Lambda = 0$  case, the gravitational waves sourced by a time changing quadrupole do not carry 3-momentum in the post-de Sitter, first post-Newtonian approximation so long as  $D_o \ll \ell_\Lambda$ .

(8) Finally, we consider angular momentum. The flux of angular-momentum in the  $\bar{i}$ -direction is given by Chapter 5:

$$J_{\bar{i}} \hat{=} \frac{1}{16\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathcal{L}_{R_{\bar{i}}} \chi_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd} \quad (6.72)$$

where  $R^m_{\bar{i}}$  is the rotational Killing field in the  $\bar{i}$ -th spatial direction. Now, since the  $\vec{x}$ -dependence in  $\chi_{ab}$  is derived entirely through  $\eta_{\text{ret}}$ , we have

$$\mathcal{L}_{R_{\bar{i}}} \chi_{ab} = \chi_{mb} \mathring{D}_a R^m_{\bar{i}} + \chi_{am} \mathring{D}_b R^m_{\bar{i}} = -2\chi_{m(b} \mathring{\epsilon}_a)^m \mathring{e}^n_{\bar{i}}. \quad (6.73)$$

Hence,

$$J_{\bar{i}} \hat{=} - \frac{1}{8\pi GH} \int_{\mathcal{I}^+} d^3x \mathcal{E}_{cd} \left( \mathring{\epsilon}_{am}^n \mathring{e}^m_{\bar{i}} \chi_{nb} \right) \mathring{q}^{ac} \mathring{q}^{bd} \quad (6.74)$$

Since  $\chi_{cd}$  now appears without a derivative in (6.74), there is a major difference between the calculations of energy and 3-momentum fluxes across  $\mathcal{I}^+$ : Now the integral over  $\eta'$  in the tail term  $b_{ab}$  in the expression (6.42) of  $\chi_{ab}$  persists. To evaluate the right side of (6.74), for  $\chi_{ab}$  we simplify the tail term  $b_{ab}$  in (6.30) by carrying out the integral over  $\eta'$  (see Appendix C), and for  $\mathcal{E}_{ab}$  we use Eqs (6.50) and (6.56) as in the calculation of the energy flux. These simplifications lead to:

$$J_{\bar{i}} \hat{=} \frac{G}{4\pi} \int_{\mathcal{I}^+} dT d^2S \left[ \mathcal{R}^{ab} \right] \left[ \mathring{\epsilon}_{am}^n \mathring{e}^m_{\bar{i}} \left( \ddot{Q}_{nb}^{(\rho)} + H \dot{Q}_{nb}^{(\rho)} + H \dot{Q}_{nb}^{(p)} + H^2 Q_{nb}^{(p)} \right) \right]^{TT}, \quad (6.75)$$

where, as before  $T$  is the affine parameter along the integral curves of the ‘time translation’ Killing field  $T^a$  and  $\mathcal{R}^{ab}$  is defined in (6.60). Note that if the stress-energy satisfies  $\mathcal{L}_T T_{ab} = 0$  at some time  $\eta = \eta_o$  then the ‘radiation field’  $\mathcal{R}_{ab}$

vanishes on the cross-section  $r = \eta_o$  on  $\mathcal{I}^+$ , whence the flux of (energy and) angular momentum vanish on that cross-section. Similarly if  $\mathcal{L}_{R_{\bar{i}}} T_{ab}$  vanishes at  $\eta = \eta_o$ , then the flux of angular momentum vanishes on the cross-section  $r = \eta_o$ . Finally, in the limit  $\Lambda \rightarrow 0$ , using the same argument as that used for energy, one can show that (6.74) reduces to the standard formula in Minkowski space-time. Again the argument is indirect because the expression of the Hamiltonian generating rotations on the covariant phase space in Minkowski space-time involves the  $TT$  part of the solution while the standard expression of angular momentum at null infinity involves the  $tt$ -part and the explicit relation between the two is not yet known.

## 6.4 Discussion

Einstein's quadrupole formula has played a seminal role in the study of gravitational waves emitted by astrophysical sources. His analysis was carried out only to the leading post-Newtonian order, assuming that the time-changing quadrupole is a first order, external source in Minkowski space-time. In spite of these restrictions, his quadrupole formula sufficed to bring to forefront the extreme difficulty of detecting these waves. However, thanks to the richness of our physical universe and ingenuity of observers, impressive advances have occurred over the last four decades. First, the careful monitoring of the Hulse-Taylor pulsar has provided clear evidence for the validity of the quadrupole formula to a  $10^{-3}$  level accuracy. Furthermore the recent detection of gravitational waves has heralded the era of gravitational wave astronomy. Therefore it is now all the more important that our theoretical understanding of gravitational waves be sufficiently deep to do full justice to the impressive status of the field on the observational front.

Since the observed value of the cosmological constant is so small, one's first reaction is just to ignore its presence. However, as seen in Chapter 4, any value of  $\Lambda > 0$ , however tiny, abruptly changes the conceptual setup that is used to analyze gravitational waves. As a result, the limit  $\Lambda \rightarrow 0$  is not necessarily continuous; indeed, some physical quantities –such as the lower bound of the energy carried by gravitational waves– can be *infinitely* discontinuous. Therefore, without a systematic analysis, one can not be confident that the quadrupole formula would continue to be valid in presence of a positive cosmological constant.

Indeed, our analysis revealed that the presence of a cosmological constant does modify Einstein's analysis in unforeseen ways. In particular:

(i) The propagation equation for metric perturbations in the transverse-traceless gauge is not the wave equation as in Minkowski space-time, but has an effective mass term (see (6.2)). Although this mass is tiny, there is potential for the differences from Minkowskian propagation to accumulate over cosmological distances to produce  $\mathcal{O}(1)$  departures in the value of the metric perturbation in the asymptotic region;

(ii) The retarded field does not propagate sharply along the null cone of the de Sitter metric. Although the de Sitter metric is conformally flat, since the equation satisfied by the metric perturbation is not conformally invariant, its expression acquires a tail term due to the back-scattering by de Sitter curvature.<sup>12</sup> As shown in the Appendix C, even in the asymptotic region, the cumulative effects make the tail term comparable in magnitude to the sharp term (which has the same form as in Minkowski space-time);

(iii) Since the radial  $r$  coordinate goes to infinity  $\mathcal{I}^+$  of Minkowski space-time, the analysis of waves makes heavy use of  $1/r$  expansions. These can no longer be used in de Sitter space-time because  $r$  ranges over the entire positive real axis on de Sitter  $\mathcal{I}^+$ . In particular, the  $tt$ -decomposition, that is local in space being tailored to the  $1/r$  expansions in Minkowski space-time, is no longer meaningful near de Sitter  $\mathcal{I}^+$ .

(iv) The retarded, first order metric perturbation depends not only on the mass quadrupole as in Einstein's calculation but also on the pressure quadrupole. Also, while only the third time derivative of the mass quadrupole features in Einstein's calculation, now we also have a contribution from lower time derivatives of the two quadrupoles, as well as the pressure quadrupole itself;

(v) The *physical* wavelengths  $\lambda_{\text{phys}}$  of perturbations grow exponentially as the wave propagates and vastly exceed the curvature radius  $\ell_\Lambda = H^{-1} \equiv \sqrt{3/\Lambda}$  in the asymptotic region near  $\mathcal{I}^+$ . Therefore, the geometric optics approximation often used to study the effect of background curvature on propagation of gravitational

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<sup>12</sup>Since the propagation of the linearized Weyl curvature is sharp, one might wonder if the tail term in the retarded solution (6.16) is a gauge artifact. It is not. Since the *gauge invariant part*  $\chi_{ab}^{\text{TT}}$  of the metric perturbation satisfies  $(\square + (2/\eta)\partial_\eta)\chi_{ab}^{\text{TT}} = 16\pi G\mathcal{T}_{ab}^{\text{TT}}$  –i.e., the same equation as  $\chi_{ab}$  but with  $T_{ab}$  on the right hand side replaced by its TT-part– it follows that  $\chi_{ab}^{\text{TT}}$  is given by replacing  $\mathcal{T}_{ab}$  with  $\mathcal{T}_{ab}^{\text{TT}}$  in (6.16). Therefore it also has a non-trivial tail term.

waves [98] fails even for waves produced by ‘tame sources’ such as a compact binary. Since waves ‘experience’ the curvature, their propagation is quite different from that in flat space. Also, since the expression (6.55) involves the metric perturbation evaluated in the zone where  $\lambda_{\text{phys}} > \ell_{\Lambda}$ , a priori the effect of  $\Lambda$  on radiated energy could be non-negligible;

(vi)  $\mathcal{I}^+$ , the arena used to analyze properties of gravitational waves unambiguously changes its character from being a null future boundary of space-time to a space-like one. As a result, all Killing fields of the background de Sitter space-time –including the ‘time translation’ used to define energy– are space-like in a neighborhood of  $\mathcal{I}^+$ . Consequently, while linearized gravitational waves carry positive energy in Minkowski space-time, de Sitter space-time admits gravitational waves carrying arbitrarily large negative energy.

These differences are sufficiently striking to cast a doubt on one’s initial intuition that the cosmological constant will have no role in the study of compact binaries. For example, they open up the possibility that Einstein’s quadrupole formula could receive significant corrections –e.g., of the order  $\mathcal{O}(H\lambda_{\text{phy}})$ – even though the observed value of  $H$  is so small. Interestingly, the final expression (6.61) of radiated power shows that this does not happen for astrophysical processes such as the Hulse-Taylor binary pulsar, or the compact binary mergers that are of greatest interest to the current ground based gravitational wave observatories. How does this come about? Why do the qualitative differences noted in the last paragraph not matter in the final result for these systems? The physical reasons can be summarized as follows:

(a) First, while the propagation of  $\chi_{ab}$  is indeed not sharp, what matters for radiated energy are certain *derivatives* of  $\chi_{ab}$  and these do have sharp propagation.

(b) Second, while the final expressions (6.59) and (6.75) of radiated energy and angular momentum are evaluated at  $\mathcal{I}^+$ , the integrand refers to the time derivatives of quadrupole moments evaluated at retarded instants of time. In our  $c = 1$  units, even though (6.59) and (6.75) involve fields at late times, the time scales in the ‘dots’ in these expressions are determined by  $\lambda_{\text{phy}}^{\text{source}}$ , the wave length evaluated at the source, and *not* by the exponentially larger physical wavelengths  $\lambda_{\text{phy}}^{\text{asym}}$  in the asymptotic region. Therefore for the sources on which gravitational wave observatories will focus in the foreseeable future,  $H\ddot{Q}_{ab}^{(\rho)}$ , for example, is suppressed relative to  $\ddot{Q}_{ab}^{(\rho)}$  by the factor  $H\lambda_{\text{phy}}^{\text{source}}$  (rather than enhanced by the factor  $H\lambda_{\text{phy}}^{\text{asym}}$ ) and

$\ddot{Q}_{ab}^{(\rho)}$  completely dominates over the remaining 5 terms (which have  $H, H^2$  or  $H^3$  as coefficient). In particular, the pressure quadrupole can be neglected for these sources. Had our expression of power referred to time scales associated with the asymptotic values of  $\lambda_{\text{phy}}$ , effects discussed in the previous paragraph would have completely altered the picture. Then, the terms with the highest powers of  $H$  –in particular the pressure quadrupole  $Q_{ab}^{(p)}$  term– would have dominated and the contribution due to  $\ddot{Q}_{ab}^{(\rho)}$  would have been completely negligible!

(c) Third, while a neighborhood of  $\mathcal{I}^+$  in the Poincaré patch  $(M_{\text{P}}^+, \bar{g}_{ab})$  does admit gravitational waves carrying arbitrarily large negative energies, our calculation showed that such waves can not result from time-changing quadrupoles. The reason is simplest to explain using the shaded region in the left panel of Fig. 6.1. Negative contribution to the energy at  $\mathcal{I}^+$  can come only from the waves that arrive from the upper half of  $E^+(i^-)$ . But the physics of the problem led us to consider *retarded* solutions with the given  $T_{ab}$  as source and for these solutions there is no energy flux at all across  $E^+(i^-)$ . This is why our energy flux (6.59) across  $\mathcal{I}^+$  is necessarily positive.

Because of these reasons, for binary coalescences that are of greatest interest to the current gravitational wave observatories, energy and power are determined essentially by the third time derivative of the mass quadrupole, as in Einstein’s formula. This quadrupole moment (6.34) is calculated using the physical de Sitter geometry and the time derivative ‘overdot’ refers to the Lie derivative with respect to the de Sitter time translation  $T^a$  specified in (5.3). However, in the limit  $\Lambda \rightarrow 0$ , it goes over the mass-quadrupole used in Einstein’s formula. Therefore, for compact binaries of interest to the current gravitational wave observatories, the difference is again negligible.

# Chapter 7 |

## Conclusions and Outlook

### 7.1 Summary

A highly successful theory of gravitational radiation exists in general relativity with a vanishing cosmological constant  $\Lambda$ . The gauge invariant description of gravitational waves it provides is a point of departure for research in many varied fields including geometric analysis, numerical relativity, data analysis and direct detection of gravitational waves and also computations of quantum evaporation of black holes. However, there is strong evidence from observational cosmology for a non-zero positive value for the cosmological constant [13]. So it is time to generalize gravitational radiation theory to incorporate this fact. Can the framework available for  $\Lambda = 0$  be extended to include a positive  $\Lambda$ ? In this dissertation we find that the issue of extension is surprisingly subtle and the commonly used strategy does *not* provide a physically interesting theory of gravitational waves for  $\Lambda > 0$ . In the simplified context of linearized gravitational fields we were able to construct a satisfactory framework and furthermore, provide the generalization of Einstein's quadrupole formula for power radiation. These results also provide guidance for the full non-linear theory. Our findings are summarized below.

Following the standard framework, we define asymptotically de Sitter space-times in Definition 3. The fall-off rate of the physical stress-energy tensor and the various choices of boundary topologies allow the description of many space-times of physical interest including black hole space-times (Kerr-de Sitter, Vaidya de Sitter) and homogeneous isotropic cosmological space-times (Friedmann-Lemaître-Robertson-Walker). For space-times which satisfy the definition  $\mathcal{I}$  is space-like [10]

and the asymptotic symmetry group is  $\text{Diff}(\mathcal{I})$ . Thus, in contrast to the BMS group  $\mathcal{B}$  of the  $\Lambda = 0$  case,  $\text{Diff}(\mathcal{I})$  does not admit any preferred canonical sub-group of translations or rotations. Consequently, one cannot introduce physically meaningful 2-sphere charges analogous to the Bondi 4-momentum at  $\mathcal{I}$  [7, 9], or calculate fluxes of energy, momentum and angular momentum carried away by gravitational waves [17]. A common strategy to reduce  $\text{Diff}(\mathcal{I})$  to the 10-dimensional de Sitter group is to strengthen the boundary conditions by requiring that the intrinsic 3-metric  $q_{ab}$  on  $\mathcal{I}$  be conformally flat. This additional restriction seems natural, because the condition is satisfied in the familiar examples, including the Kerr-de Sitter and Friedmann-Lemaître-Robertson-Walker space-times, and is also imposed when  $\Lambda < 0$  for well-defined time evolution. Furthermore, the 2-sphere charges at  $\mathcal{I}$  associated with the Kerr-de Sitter time-translation and rotation yield the expected mass and angular momentum.

However, we showed that the additional boundary condition is equivalent to demanding that *the magnetic part  $\mathcal{B}_{ab}$  of the leading order asymptotic Weyl curvature must vanish at  $\mathcal{I}$* . Now, in the case of Maxwell fields on asymptotically de Sitter space-times, the analogous requirement would be that the magnetic field  $\mathcal{B}_a$  should vanish at  $\mathcal{I}$ . This requirement would remove half the space of solutions *by fiat!* By analogy, in the gravitational case, the strengthening of the boundary conditions appears to be physically unjustifiable. Furthermore, irrespective of whether one strengthens the boundary conditions in this manner or not, one does not have the analog of the Bondi news tensor, nor expressions of fluxes of energy-momentum and angular momentum carried away by gravitational waves. Thus we seem to have too little structure on  $\mathcal{I}$  or too restrictive boundary conditions, thereby unable to obtain a physically useful description of gravitational waves in the full non-linear theory with  $\Lambda > 0$ .

Linearizing the gravitational field about a background de Sitter space-time addresses all of the issues above. For physically relevant situations, we consider a linearized source which is spatially compact for all times so that its worldline intersects  $\mathcal{I}^-$  at a single point  $i^-$  and, similarly,  $\mathcal{I}^+$  at  $i^+$ . We then restrict ourselves to the future Poincaré patch i.e., the future light cone of  $i^-$  because no observer restricted to the past Poincaré patch will receive any signal from the source. The linearized source produces a perturbation  $\gamma_{ab}$  on de Sitter space-time. The symmetries of the Poincaré patch include one dilation, which we call de Sitter time

translation, three spatial translations and three rotations. Using these symmetries and Hamiltonian methods we provide *gauge invariant formulas* for the energy, momentum and angular momentum carried by the linearized field. In terms of the rescaled metric perturbation field  $h_{ab} = a^{-2} \gamma_{ab}$  and the symmetry vector field  $\xi^a$  the formula is

$$H_\xi = \frac{1}{2H\kappa} \int_\Sigma d^3x \left( h_{ab} \mathcal{E}_{cd}^{(\xi)} - h_{ab}^{(\xi)} \mathcal{E}_{cd} \right) \dot{q}^{ac} \dot{q}^{bd} \quad (7.1)$$

where  $\mathcal{E}_{ab}$  is the electric part of the leading order perturbed Weyl tensor and the super-script  $(\xi)$  refers to the perturbation  $\mathcal{L}_\xi \gamma_{ab}$ .

This formula is then used to obtain a generalization of Einstein's quadrupole formula for power emission.

$$P_T(T_0) \doteq \frac{G}{8\pi} \int_{T=T_0} d^2S \left[ \mathcal{R}^{ab}(\vec{x}) \mathcal{R}_{ab}^{TT}(\vec{x}) \right] \quad (7.2)$$

where the 'radiation field'  $\mathcal{R}_{ab}(\vec{x})$  on  $\mathcal{I}^+$  is given by

$$\mathcal{R}_{ab}(\vec{x}) \doteq \left[ \ddot{Q}_{ab}^{(\rho)} + 3H\ddot{Q}_{ab}^{(\rho)} + 2H^2\dot{Q}_{ab}^{(\rho)} + H\ddot{Q}_{ab}^{(p)} + 3H^2\dot{Q}_{ab}^{(p)} + 2H^3Q_{ab}^{(p)} \right] (\eta_{\text{ret}}). \quad (7.3)$$

We have shown that under the physically reasonable assumption of 'no incoming radiation' condition on the cosmological horizon  $E^+(i^-)$  that bounds the future Poincaré patch (see Figure 5.2) the energy radiated across  $\mathcal{I}$  is positive. This is easily seen by recognizing that the contribution to energy flux on  $\mathcal{I}$  comes from two parts, the source and the cosmological horizon  $E^+(i^-)$ . Due to the de Sitter time translation being *null and future pointing* on the top half of the past horizon  $E^-(i^+)$  of future time-like infinity  $i^+$  (see Figure 5.2) the energy emitted by the source is purely positive. The only negative contribution can come from the top half of  $E^+(i^-)$  and it is physically reasonable to set this flux to zero. Hence, energy radiated away by gravitational waves at  $\mathcal{I}$  is positive.

Furthermore, we have analyzed the limit  $\Lambda \rightarrow 0$  carefully. This has enabled us to compare physical results when  $\Lambda = 0$  with those of a positive  $\Lambda$ . In particular, we precisely quantify the differences between the two cases and explain why the small observed value of  $\Lambda$  does not have a significant effect on current observations of isolated systems.

In addition to our analysis of asymptotic structure of space-times with a positive

$\Lambda$ , we provided a new result in the fully non-linear regime of the  $\Lambda = 0$  theory. We obtained a tensorial expression for a 2-sphere charge integral associated with an arbitrary symmetry of the asymptotic symmetry group, the Bondi-Metzner-Sachs (BMS) group, at  $\mathcal{I}$ . It has the attractive feature that its flux is the same as the flux obtained using Hamiltonian methods for an arbitrary BMS vector field by Ashtekar and Streubel in [17]. Furthermore, it is in agreement with several earlier results: In particular, it yields Bondi 4-momentum for translations; and, when the BMS field is decomposed into supertranslation and Lorentz parts, our charge agrees with the charge expression of Dray and Struebel [26].

## 7.2 Outlook

The question of existence of a consistent theory of gravitational radiation with  $\Lambda > 0$  is of primary conceptual importance. Our work is the first step towards building such a theory of gravitational radiation for full non-linear general relativity with  $\Lambda > 0$ . It opens up many avenues for future research, a sample of which are described below.

The observed value of  $\Lambda$  completely changes the theoretical paradigm of isolated systems, but its smallness has meant that current observations are very well modeled by the standard results from the  $\Lambda = 0$  theory. However, there are circumstances in which the differences between the  $\Lambda = 0$  and  $\Lambda > 0$  could be significant. First, consider the tail term in the expression (6.42) of  $\chi_{ab}$ . Since it arises because of back-scattering due to de Sitter curvature, it is proportional to  $H$ . However, it involves an integral over a cosmologically large time interval which could compensate the smallness of  $H$  and make the tail term comparable to the one that arises from sharp propagation. The tail term could then yield a significant new contribution to the memory effect [91–94]. A second example is provided by mergers of supermassive black holes at the centers of two different galaxies. Since the time scales associated with such galactic coalescences are cosmological, gravitational waves created in this process will have *extremely* long wavelength already at inception, making the departures from Einstein’s quadrupole formula significant. While these waves will not be detected directly in any foreseeable future, they provide a background which could have indirect influences [100].

Another avenue of interest is to extend our work on linearized gravity to

a Friedmann-Lemaître-Robertson-Walker background space-time which describes our universe more faithfully than the de Sitter metric does. However, since both space-times are asymptotically exponentially expanding, our results must qualitatively agree. It will also be of interest to perform analytical calculations involving the adiabatic phase of binary compact object mergers and Extreme Mass Ratio Inspirals (EMRI). One could potentially observe secular effects for such phenomena as the rate at which a binary system loses orbit eccentricity [106], or the distribution of spin orientations of a binary at the end of the adiabatic inspiral phase [107].

Our work also provides hints for full theory with  $\Lambda > 0$  which would be of interest to geometric analysis, because of issues such as the positivity of total energy. As described earlier, for a physical source of gravitational radiation, it is sufficient to consider one ‘half’ of space-time, the future of past time-like infinity  $i^-$  defined by the source. Then, the ‘no-incoming radiation’ boundary condition will have to be imposed on the past boundary,  $E^+(i^-)$ . Since this is an event horizon, a natural strategy would be to demand that it be a weakly isolated horizon [101–103]. It would be interesting to analyze if this condition would suffice to ensure that the flux of energy across  $\mathcal{I}^+$  is positive, as in the weak field limit discussed in Chapter 6. If so, one would have the desired generalization of the celebrated result due to Bondi and Sachs that gravitational waves carry away *positive* energy, in spite of the fact that the corresponding asymptotic ‘time translation’ on  $\mathcal{I}^+$  would now be space-like for  $\Lambda > 0$ . Furthermore, results of Chapter 4 and Chapter 5 suggest that there will be a 2-sphere ‘charge integral’ –representing the generalization of the notion of Bondi-Sachs energy to the  $\Lambda > 0$  case– and the difference between charges associated with two different 2-spheres will equal the energy flux across the region bounded by the two 2-spheres. A natural question is whether this charge is also positive.<sup>1</sup> Finally, in the linear approximation considered in this chapter, the past cosmological event horizon  $E^-(i^+)$  of the point at future time-like infinity could be taken to lie in the ‘far zone’. Furthermore, since there is no incoming radiation across  $E^+(i^-)$  from (the shaded portion of the left panel of) Fig. 5.1 it follows that the flux of energy across  $E^-(i^+)$  equals that across  $\mathcal{I}^+$  and is, in

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<sup>1</sup>These Bondi-type charge integrals will also refer to an asymptotic ‘time-translation’. They will be distinct from the ADM-type charge-integral associated with a *conformal* –rather than time-translation– symmetry discussed in [104], and the intriguing 2-sphere integral recently discovered [105], both of which are known to be positive.

particular, positive. In full, non-linear general relativity, then,  $E^-(i^+)$  may well serve as an ‘approximate’  $\mathcal{I}^+$  to analyze gravitational waves. Because this surface is null, it may be easier to compare results in the  $\Lambda > 0$  case with those in the  $\Lambda = 0$  case in full general relativity.

Thus, the study of asymptotic structure of the gravitational field with a positive cosmological constant  $\Lambda$  presents exciting challenges and opportunities for research in gravitational science.

# Appendix A |

## Some results for asymptotically flat space-times

### A.1 Newman-Penrose notation

The Newman-Penrose spin coefficients are defined using an orthonormal tetrad of null vector fields [24]. If  $\mathcal{I}$  is coordinatized by  $(u, \theta, \phi)$  then the basis is given by:

$$\begin{aligned} n^a &= g^{ab} \nabla_b \Omega; & \ell_a &= -\nabla_a u; \\ m^a \partial_a &= \frac{1}{\sqrt{2}} (\partial_\theta + \frac{i}{\sin \theta} \partial_\phi); & \bar{m}^a \partial_a &= \frac{1}{\sqrt{2}} (\partial_\theta - \frac{i}{\sin \theta} \partial_\phi). \end{aligned} \quad (\text{A.1})$$

The Newman-Penrose scalars relevant to our discussion in Chapter 3 are:

(i) Components of  $D_a \ell_b$

$$\sigma = m^a \bar{m}^b D_b \ell_a; \quad \tau = m^a n^b D_b \ell_a; \quad \rho = m^a \bar{m}^b D_b \ell_a; \quad (\text{A.2})$$

(ii) Derivatives of the null basis vectors

$$2\alpha = m^a \bar{m}^b D_b \bar{m}_a; \quad 2\gamma = -\bar{m}^a n^b D_b m_a; \quad (\text{A.3})$$

(iii) Components of the leading order Weyl tensor  $K_{abcd} := \Omega^{-1} C_{abcd}$

$$\begin{aligned} \psi_0 &= K_{abcd} \ell^a m^b \ell^c m^d; \\ \psi_1 &= K_{abcd} \ell^a n^b \ell^c m^d; \\ \psi_2 &= K_{abcd} \bar{m}^a n^b \ell^c m^d; \end{aligned}$$

$$\begin{aligned}
\psi_3 &= K_{abcd} \bar{m}^a n^b \ell^c n^d; \\
\psi_4 &= K_{abcd} \bar{m}^a n^b \bar{m}^c n^d;
\end{aligned} \tag{A.4}$$

(iv) Components of the Bondi news tensor

$$N_{ab} = 2(N m_a m_b + \bar{N} \bar{m}_a \bar{m}_b) \tag{A.5}$$

The spin weight,  $s$ , of an object  $c$  on a 2-sphere is defined as follows: If, under a rotation on the 2-sphere, which is denoted by  $m'^a = e^{i\chi} m^a$ ,  $c$  transforms as  $c' = e^{is\chi} c$ ,  $c$  has spin weight  $s$ .

The complex angular derivative operator  $\bar{\partial}$  acting on  $c$  is defined as:

$$\bar{\partial}c = (\sin \theta)^s (\partial_\theta + \frac{i}{\sin \theta} \partial_\phi) [(\sin \theta)^{-s} c]; \tag{A.6}$$

$$\bar{\partial}\bar{c} = (\sin \theta)^{-s} (\partial_\theta - \frac{i}{\sin \theta} \partial_\phi) [(\sin \theta)^s \bar{c}]; \tag{A.7}$$

$$\bar{\partial}A = (m^a D_a + 2s\bar{\alpha})A; \quad \bar{\partial}\bar{A} = (\bar{m}^a D_a - 2s\alpha)A; \tag{A.8}$$

where  $s$  is the spin weight of  $A$  defined as  $A' = e^{is\chi} A$  under a rotation  $m'^a = e^{i\chi} m^a$ .

## A.2 Conformal invariance of new charge

In this section we will show that the charge formula Eq. (3.5) given in Chapter 3,

$$\begin{aligned}
Q_\xi[C] = \frac{1}{\kappa} \oint_C dS \quad & \left[ W_{ab} n^a \xi^b + \frac{1}{2} (\ell_d \xi^d) N_{ac} (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn} \right. \\
& \left. - \frac{1}{2} q_{cb} \xi^b S_a{}^d \ell_d (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn} + \frac{1}{6} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right]
\end{aligned} \tag{A.9}$$

is invariant under the following conformal transformation

$$g'_{ab} = \omega^2 g_{ab} \tag{A.10}$$

such that  $\mathcal{L}_\ell \omega = 0$ . Recall that this restriction comes from the requirement that  $\ell_a$  be divergence-free.

We begin by listing out the conformal transformation properties of various fields

that appear in the expression.

$$q'_{ab} = \omega^2 q_{ab}; \quad \bar{q}'_{ab} = \omega^2 \bar{q}_{ab}. \quad (\text{A.11})$$

The null basis transforms to another mutually orthogonal null basis preserving normalization:

$$n'^a = \omega^{-1} n^a; \quad \ell'_a = \omega \ell_a; \quad m'^a = \omega^{-1} m^a; \quad m'_a = \omega m_a. \quad (\text{A.12})$$

The symmetry field  $\xi^a$  is unchanged.

The Christoffel symbols relating the transformed derivative operator  $\nabla'$  to the original metric-compatible derivative operator  $\nabla$  are given by

$$C^c{}_{ab} = \omega^{-1} \left[ 2\delta_{(a}^c \nabla_{b)} \omega - g_{ab} \omega^c \right] \quad (\text{A.13})$$

where  $\omega^c := g^{cd} \nabla_d \omega$ .

Using this one can work out the transformation of the tensor  $D_a \ell_b$  to be:

$$D'_a \ell'_b = \omega D_a \ell_b - \ell_a D_b \omega + q_{ab} (\omega^c \ell_c). \quad (\text{A.14})$$

The last term vanishes for transformations that preserve the divergence-free property of  $\ell_a$ . We will assume this for the rest of the discussion.

Next, we note that the divergence of the symmetry  $\xi^a$  transforms as:

$$D'_a \xi'^a = D_a \xi^a + 3\mathcal{L}_\xi \omega \quad (\text{A.15})$$

Next, we consider the curvature tensors. The Bondi news tensor  $N_{ab}$  is, of course, conformally invariant. But the Schouten tensor has a more complicated transformation law, as stated in Chapter 2 Eq. 2.6:

$$S'^a{}^b = \omega^{-2} S_a{}^b - 2\omega^{-3} D_a (q^{bc} D_c \omega) + 4\omega^{-4} (D_a \omega) q^{bc} D_c \omega - \omega^{-4} \delta_a{}^b (q^{cd} D_c \omega D_d \omega) \quad (\text{A.16})$$

The tensor  $W_{ab}$  transforms by a simple rescaling due to the conformal invariance of the Weyl tensor  $C_{abc}{}^d$  and the fact that the combination  $\ell_c \ell^d$  is conformally

invariant, i.e.,

$$W'_{ab} = K'_{acb}{}^d \ell'_c \ell'^d = \omega^{-1} K_{acb}{}^d \ell_c \ell^d = \omega^{-1} W_{ab}. \quad (\text{A.17})$$

Now we examine the conformal transformation property of the charge of an arbitrary BMS field  $\xi^a$  as given by Eq. (A.9). First, we note that under a conformal transformation, the volume element is rescaled as  $d^2S \rightarrow \omega^2 d^2S$ . Then it is easy to see that the first term of (A.9) is conformally invariant. Next, since the news tensor  $N_{ab}$  is conformally invariant and trace-free, the second term is also conformally invariant. This leaves us with the last two terms.

The third term in the integrand of Eq. (A.9), which we name  $t_3 := -\frac{1}{2\kappa} q_{cb} \xi^b S_a{}^d \ell_d (D_m \ell_n) \bar{q}^{am} \bar{q}^{cn}$  transforms as

$$t'_3 = \omega^{-2} t_3 - \frac{1}{\kappa} \omega^{-3} \bar{x}i^a (D_a \ell_b) \bar{q}^{bd} (D_c \ell_d) \omega^c \quad (\text{A.18})$$

where  $\bar{\xi}^a = \bar{q}^a{}_b \xi^b$  and we have used the condition that  $\omega^c \ell_c = 0$ .

Similarly, the fourth term  $t_4 := \frac{1}{6\kappa} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd}$  of Eq. (A.9) transforms as

$$t'_4 = \omega^{-2} t_4 + \frac{1}{2\kappa} \omega^{-3} (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \mathcal{L}_\xi \omega. \quad (\text{A.19})$$

Now, adding Eqs. (A.18) and (A.19),

$$\begin{aligned} t'_3 + t'_4 &= \omega^{-2} (t_3 + t_4) + \frac{1}{2\kappa} \omega^{-3} (D_a \ell_b) (D_c \ell_d) \bar{q}^{bd} [-2 \bar{x}i^a \omega^c + \mathcal{L}_\xi \omega \bar{q}^{ac}] \\ &= \omega^{-2} (t_3 + t_4) + \frac{1}{\kappa} \omega^{-3} \sigma \bar{\sigma} [-\bar{\xi}^a D_a \omega + \xi^a D_a \omega] \end{aligned} \quad (\text{A.20})$$

where in the second line we have simplified using Newman-Penrose scalars and once again, used that  $\bar{q}^{ab} D_a \ell_b = 0$ . Now, the term in the square brackets is clearly zero because  $\mathcal{L}_n \omega = 0 \implies \mathcal{L}_\xi \omega = \mathcal{L}_{\bar{\xi}} \omega$ . Thus, the conformal invariance of the charge formula Eq. (A.9) is proven.

### A.3 Lorentz transformations

In this section we provide details of the calculations of section 3.3.2 which show that the charge  $Q_\xi[C]$  associated with a Lorentz transformation  $\xi$  yields a flux that agrees with the Ashtekar-Streubel flux.

The charge associated with a Lorentz transformation  $\xi^a = v^a + h^a$  where  $v^a = u k(\theta, \phi) n^a$  on a cross-section  $C$  is

$$Q_\xi[C] = \frac{1}{\kappa} \oint dS \left[ K_{abcd} \ell^a n^b \ell^c \xi^d + \frac{1}{2} u k(\theta, \phi) N^{ab} (D_a \ell_b) \right. \\ \left. + \frac{1}{2} S_a{}^d \ell_d h^n (D_m \ell_n) \bar{q}^{am} + \frac{1}{6} (D_p \xi^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \quad (\text{A.21})$$

$$= Q_v[C] + Q_h[C]. \quad (\text{A.22})$$

The Ashtekar-Streubel Hamiltonian flux can be simplified using (3.8) as

$$F_\xi[\Delta\mathcal{I}] = \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ N_{ac} (\mathcal{L}_\xi D_m - D_m \mathcal{L}_\xi) \ell_n \bar{q}^{am} \bar{q}^{cn} \right] \\ = F_v[\Delta\mathcal{I}] + F_h[\Delta\mathcal{I}] \quad (\text{A.23})$$

where

$$F_v[\Delta\mathcal{I}] = \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \bar{N}^{ab} \left( \frac{1}{2} u k S_{ab} + u D_a D_b k + k \sigma_{ab} \right) \right] \quad (\text{A.24})$$

$$F_h[\Delta\mathcal{I}] = \frac{1}{2\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \bar{N}^{ab} \left( \frac{1}{2} h_a S_b{}^d \ell_d - \ell_d D_a D_b h^d \right) \right] \quad (\text{A.25})$$

where  $\sigma_{ab} = D_a \ell_b$  and  $\bar{N}^{ab} = \bar{q}^{ac} \bar{q}^{bd} N_{cd}$ .

Our aim is to show that

$$\int_{C_1}^{C_2} du \mathcal{L}_n Q_\xi[C] = F_\xi[\Delta\mathcal{I}]. \quad (\text{A.26})$$

First, we note that

$$\mathcal{L}_n Q_v[C] = Q_{kn}[C] + \frac{u}{2\kappa} \oint d^2S \left( \frac{k}{2} S_{ab} + D_a D_b k \right) \bar{N}^{ab} \\ - \frac{k}{3\kappa} \oint d^2S (\mathcal{L}_n D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \quad (\text{A.27})$$

This is easily seen by applying the Lie derivative and simplifying as was done to

obtain (3.16). Further, using (3.8), we obtain:

$$\begin{aligned} \int_{\Delta\mathcal{I}} \mathcal{L}_n Q_v[C] &= F_v[\Delta\mathcal{I}] + \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \left[ \frac{k}{6} S_{ab} \sigma^{ab} + k K_{abcd} \ell^a n^b \ell^c n^d \right] \\ &=: F_v[\Delta\mathcal{I}] + \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V (U_1 + U_2) \end{aligned} \quad (\text{A.28})$$

Next we calculate  $\mathcal{L}_n Q_h[C]$  as follows

$$\begin{aligned} \int_{\Delta\mathcal{I}} \mathcal{L}_n Q_h[C] &= \frac{1}{\kappa} \int_{\Delta\mathcal{I}} d^3V \mathcal{L}_n \left[ K_{abcd} \ell^a n^b \ell^c h^d + \frac{1}{2} S_a^d \ell_d (D_m \ell_n) h^n \bar{q}^{am} \right. \\ &\quad \left. + \frac{1}{6} (D_p h^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \right] \end{aligned} \quad (\text{A.29})$$

We split the integrand of (A.29) into four terms

$$\begin{aligned} T_1 &= \mathcal{L}_n K_{abcd} \ell^a n^b \ell^c h^d \\ T_2 &= \frac{1}{2} h^n S_a^d \ell_d \mathcal{L}_n (D_m \ell_n) \bar{q}^{am} \\ T_3 &= \frac{1}{2} h^c (D_c \ell_d) \bar{q}^{ad} \ell_b \mathcal{L}_n S_a^b \\ T_4 &= \frac{1}{6} \mathcal{L}_n [(D_p h^p) (D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd}] \end{aligned} \quad (\text{A.30})$$

We first note that since  $D_p h^p = 2k$ ,  $T_4 = 2U_1$  in (A.28) since

$$T_4 = \frac{2}{3} k \mathcal{L}_n [(D_a \ell_b) (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd}] = \frac{1}{3} k S_{ab} (D_c \ell_d) \bar{q}^{ac} \bar{q}^{bd} \quad (\text{A.31})$$

where in the last step we have used (3.8) and the fact that  $\bar{q}^{cd} D_c \ell_d = 0$ .

Next, we simplify  $T_2$ . Using (3.8)

$$\begin{aligned} T_2 &= \frac{1}{4} h^n N_{nm} \bar{q}^{ma} S_a^d \ell_d + \frac{1}{4} h^n (\rho_{nm} + \beta q_{mn}) \bar{q}^{ma} S_a^d \ell_d \\ &:= T_{2a} + T_{2b} \end{aligned} \quad (\text{A.32})$$

where we used the definition

$$S_p^q n^p := \beta n^q := -\frac{\bar{R}}{2} n^q. \quad (\text{A.33})$$

Note that  $T_{2a}$  is the same as the first term in the integrand of  $F_h[\Delta\mathcal{I}]$  in (A.25).

Next, we simplify  $T_3$ . Using (3.11) and (A.33)

$$\mathcal{L}_n S_a{}^b = 2 K_{man}{}^b n^m n^n + n^b \nabla_a \beta \quad (\text{A.34})$$

$$\begin{aligned} T_3 &= \frac{1}{2} h^c (D_c \ell_d) \bar{q}^{ad} \ell_b (2 K_{man}{}^b n^m n^n + n^b \nabla_a \beta) \\ &= -\frac{1}{2} h^c (D_c \ell_d) \bar{q}^{ad} \nabla_a \beta + h^c (D_c \ell_d) \bar{q}^{ad} K_{paq}{}^b n^p n^q \ell_b \\ &:= T_{3a} + T_{3b} \end{aligned} \quad (\text{A.35})$$

To simplify  $T_1$ , contract (3.9) with  $h^a \ell^b n^c$

$$\begin{aligned} T_1 &= \mathcal{L}_n K_{abcd} \ell^a n^b \ell^c h^d \\ &= -K_{abcd} \bar{q}^{df} h^a n^c \nabla_f \ell^b - K_{abcd} \bar{q}^{df} \ell^b n^c \nabla_f h^a + \bar{q}^{df} \nabla_f (K_{abcd} h^a \ell^b n^c) \\ &:= T_{1a} + T_{1b} + T_{1c} \end{aligned} \quad (\text{A.36})$$

First note that  $T_{1c} = \bar{D}_f t_1^f = \bar{q}^{df} \nabla_f (K_{abcm} h^a \ell^b n^c \bar{q}_d{}^m)$  is a boundary term.

Next, since  $\bar{q}^{df} \nabla_f \ell^b = \bar{q}^{df} \bar{q}^{bm} \nabla_f \ell_m$ , we can use (3.14) to write

$$T_{1a} = -\bar{D}_{[a} N_{b]d} h^a \bar{q}^{df} \bar{q}^{bm} \nabla_f \ell_m \quad (\text{A.37})$$

It can be shown that  $T_{1a}$  cancels  $T_{3b}$ . This is similar to the cancellation which happened for the supertranslation—the Lie derivative of news cancels out part of the Lie derivative of the Weyl tensor.

Next, in  $T_{1b}$ , expand  $\nabla_f h^a = g^a{}_m \nabla_f h^m$  and use (3.14) again to get

$$\begin{aligned} &T_{1b} \\ &= -K_{abcd} \bar{q}^{df} \ell^b n^c \nabla_f h^a \\ &= -K_{abcd} \ell^b n^c \bar{D}^d h^a + 2K_{abcd} \bar{q}^{df} n^a \ell^{(b} n^{c)} \ell_m \nabla_f h^m \\ &= K_{abcd} \ell^b n^d \bar{D}^c h^a + K_{dcba} n^a \bar{q}^{df} \bar{q}^{bc} \ell_m \nabla_f h^m \\ &= K_{abcd} \ell^b n^d \bar{D}^c h^a + \frac{1}{2} \bar{D}_c N_{db} \bar{q}^{df} \bar{q}^{bc} \ell_m \nabla_f h^m \\ &= K_{abcd} \ell^b n^d \bar{D}^c h^a - \frac{1}{2} N_{db} \bar{q}^{df} \bar{q}^{bc} \bar{D}_c \ell_m \bar{D}_f h^m - \frac{1}{2} N_{db} \bar{q}^{df} \bar{q}^{bc} \ell_m \bar{D}_c \bar{D}_f h^m + \bar{D}_f t_2^f \end{aligned} \quad (\text{A.38})$$

where  $t_2^f = \frac{1}{2}N_{db}\bar{q}^{dc}\bar{q}^{bf}\ell_m\bar{D}_c h^m$ . Also note that the last non-boundary term above is the same as the second term in the integrand of (A.25).

The integrand of  $\int_{\Delta\mathcal{I}}\mathcal{L}_n Q_h[C]$  is given by

$$\begin{aligned}
T_1 + T_2 + T_3 + T_4 &= \frac{1}{4}h^n N_{nm}\bar{q}^{ma}S_a{}^d\ell_d - \frac{1}{2}N_{db}\bar{q}^{df}\bar{q}^{bc}\ell_m\bar{D}_c\bar{D}_f h^m \\
&+ K_{abcd}\ell^b n^d\bar{D}^c h^a - \frac{1}{2}N_{db}\bar{q}^{df}\bar{q}^{bc}\bar{D}_c\ell_m\bar{D}_f h^m \\
&+ \frac{1}{4}h^n(\rho_{nm} + \beta q_{mn})\bar{q}^{ma}S_a{}^d\ell_d \\
&- \frac{1}{2}h^n(D_m\ell_n)\bar{q}^{am}\nabla_a\beta + \frac{1}{2}kS_{ab}\bar{\sigma}^{ab} \\
&+ \bar{D}_f t_1^f + \bar{D}_f t_2^f
\end{aligned} \tag{A.39}$$

The first line is equal to the integrand of  $F_h$  in (A.25).

In order to simplify the rest of the terms, we require the use of two-dimensional identities. Let

$$\bar{D}^c h^a = k\bar{q}^{ca} + \Theta\bar{\epsilon}^{ca} \tag{A.40}$$

Since  $K_{abcd}$  is trace-free, we substitute  $\bar{q}^{ca}$  with  $-2\ell^{(a}n^{c)}$

$$\begin{aligned}
&K_{abcd}\ell^b n^d\bar{D}^c h^a \\
&= -kK_{abcd}n^a\ell^b n^c\ell^d + \Theta K_{abcd}\ell^b n^d\bar{\epsilon}^{ca} \\
&= -kK_{abcd}n^a\ell^b n^c\ell^d + \Theta\nabla_{[a}S_{b]c}\ell^b\bar{\epsilon}^{ac} \\
&= -kK_{abcd}n^a\ell^b n^c\ell^d \\
&\quad + \frac{1}{2}\bar{D}_a(\Theta S_c{}^b\ell_b)\bar{\epsilon}^{ac} - \frac{1}{2}(\bar{D}_a\Theta)S_c{}^b\ell_b\bar{\epsilon}^{ac} - \frac{1}{2}S_c{}^b(\nabla_a\ell_b)\Theta\bar{\epsilon}^{ac}
\end{aligned} \tag{A.41}$$

Note that the first term above cancels out a term in (A.28).

For a conformal Killing field  $h^d$ ,

$$\bar{D}_a\bar{D}_b h_c = \bar{R}_{cba}{}^d h_d + (\bar{D}_a k)\bar{q}_{bc} + (\bar{D}_b k)\bar{q}_{ac} - (\bar{D}_c k)\bar{q}_{ab} \tag{A.42}$$

The above relation is obtained by alternately acting on  $\bar{D}_a\bar{D}_b h_c$  with the conformal Killing field relation  $\bar{D}_{(a}h_{b)} = k\bar{q}_{ab}$  and the definition of Riemann tensor  $\bar{D}_a\bar{D}_b h_c = \bar{D}_b\bar{D}_a h_c + \bar{R}_{abc}{}^d h_d$  repeatedly.

Next, using  $\bar{D}_{[b}h_{c]} = \Theta \bar{\epsilon}_{bc}$

$$\bar{D}_a \Theta \bar{\epsilon}_{bc} = \bar{R}_{cba}{}^d h_d + (\bar{D}_b k) \bar{q}_{ac} - (\bar{D}_c k) \bar{q}_{ab} \quad (\text{A.43})$$

$$= \frac{1}{2} \bar{R} \bar{\epsilon}_{cb} \bar{\epsilon}_a{}^d h_d + (\bar{D}_b k) \bar{q}_{ac} - (\bar{D}_c k) \bar{q}_{ab} \quad (\text{A.44})$$

$$\implies \bar{D}_a \Theta = -\frac{1}{2} \bar{\epsilon}_a{}^d (\bar{R} h_d + 2\bar{D}_d k) \quad (\text{A.45})$$

Using this, we can write the second term of (A.41) as

$$-\frac{1}{2} (\bar{D}_a \Theta) S_c{}^b \ell_b \bar{\epsilon}^{ac} = \frac{1}{4} \bar{q}^{dc} (\bar{R} h_d + 2\bar{D}_d k) S_c{}^b \ell_b = \frac{1}{2} (\bar{R} h_d + 2\bar{D}_d k) \bar{D}_c \bar{\sigma}^{cd} \quad (\text{A.46})$$

where  $\bar{\sigma}^{cd} = \bar{q}^{ac} \bar{q}^{bd} \bar{\sigma}_{ab}$  and we have used the following identity

$$4\bar{q}^{ab} \bar{q}_m{}^c D_{[a} D_{b]} \ell_c = 2\bar{D}^a \sigma_{ab} = \bar{q}_m{}^c S_c{}^d \ell_d. \quad (\text{A.47})$$

Thus

$$\begin{aligned} K_{abcd} \ell^b n^d \bar{D}^c h^a &= -k K_{abcd} n^a \ell^b n^c \ell^d \\ &\quad + \frac{1}{2} \bar{D}_a (\Theta S_c{}^b \ell_b) \bar{\epsilon}^{ac} - \frac{1}{2} (\bar{D}_a \Theta) S_c{}^b \ell_b \bar{\epsilon}^{ac} - \frac{1}{2} S_c{}^b (\nabla_a \ell_b) \Theta \bar{\epsilon}^{ac} \\ &= -k K_{abcd} n^a \ell^b n^c \ell^d \\ &\quad + \frac{1}{2} \bar{D}_a (\Theta S_c{}^b \ell_b) \bar{\epsilon}^{ac} + \frac{1}{2} (\bar{R} h_d + 2\bar{D}_d k) \bar{D}_c \bar{\sigma}^{cd} - \frac{1}{2} S_c{}^b \sigma_{ab} \bar{D}^{[a} h^{c]} \end{aligned} \quad (\text{A.48})$$

Combining the above results,

$$\begin{aligned} F_\xi[\Delta \mathcal{I}] - \int_{\Delta \mathcal{I}} d^3 V \mathcal{L}_n Q_\xi[C] &= \frac{1}{2} (\bar{R} h_d + 2\bar{D}_d k) \bar{D}_c \bar{\sigma}^{cd} \\ &\quad - \frac{1}{2} S_c{}^b \sigma_{ab} \bar{D}^{[a} h^{c]} - \frac{1}{2} \bar{N}^{bc} \bar{D}_c \ell_m \bar{D}_b h^m \\ &\quad + \frac{1}{4} h^n (\rho_{nm} - \frac{1}{2} R q_{mn}) \bar{q}^{ma} S_a{}^d \ell_d \\ &\quad + \frac{1}{4} h^n (D_m \ell_n) \bar{q}^{am} \bar{D}_a R + \frac{1}{2} k S_{ab} \bar{\sigma}^{ab} \\ &\quad + \bar{D}_f t_1^f + \bar{D}_f t_2^f + \bar{D}_f t_3^f \end{aligned} \quad (\text{A.49})$$

In the first term, the derivative is moved over from  $\bar{\sigma}^{cd}$ . In the second line, use the relations  $\bar{D}^{[c} h^{a]} = \bar{D}^c h^a - k q^{ca}$  and  $S_{bc} - N_{bc} = \rho_{bc}$  to simplify. In the third line,

use the identity in (A.47), and then move the derivative over from  $\bar{\sigma}_{ab}$ . We can then simplify the above expression to the following:

$$F_\xi[\Delta\mathcal{I}] - \int_{\Delta\mathcal{I}} d^3V \mathcal{L}_n Q_\xi[C] = \int_{\Delta\mathcal{I}} \left[ -\frac{1}{2} h^m \bar{D}_a \rho_{mb} + \rho^m{}_b \bar{D}_{[m} h_{a]} - \bar{D}_b \bar{D}_a k \right] \bar{\sigma}^{ab} + \text{Boundary Terms} \quad (\text{A.50})$$

Using the property  $D_{[a}\rho_{b]c} = 0$ , the term in the square brackets can be rewritten as

$$-\bar{D}_a \bar{D}_b k - \frac{1}{2} \mathcal{L}_h \rho_{ab} + k \rho_{ab}. \quad (\text{A.51})$$

Now, it has been shown as a property of BMS fields in (2.17) that the trace-free symmetric part of  $(\bar{D}_a \bar{D}_b k + \frac{1}{2} \mathcal{L}_h \rho_{ab})$  vanishes. Then the only term that survives in a contraction with the trace-free  $\bar{\sigma}^{ab}$  is  $k \rho_{ab}$ . But in a Bondi frame  $\rho_{ab} \propto q_{ab}$ , hence, this term also does not contribute to the contraction with  $\bar{\sigma}^{ab}$ . Thus, the flux of our charge for a Lorentz field matches the expression provided by Ashtekar and Streubel up to ‘boundary terms’. Once again, we can evaluate all the boundary terms using Stoke’s theorem and find that they are zero.

## A.4 Bondi news vanishes if $\mathcal{B}^{ab} \hat{=} 0$

In this section we prove a technical result that was used as an intermediate step in Chapter 4 section 4.4.1 to show that, in asymptotically flat space-times, if  $\mathcal{B}^{ab} \hat{=} 0$  then the Bondi news tensor must vanish, i.e.,  $N_{ab} \hat{=} 0$ .

In section 4.4.1 we saw that  $\mathcal{B}^{ab} \hat{=} 0$  implies that  $N_{ab}$  is a lift to  $\mathcal{I}$  of a tensor field  $\underline{N}_{ab}$  on the 2-sphere  $S$  of generators of  $\mathcal{I}$ , satisfying:

$$\underline{N}_{[ab]} = 0, \quad \underline{N}_{ab} \underline{q}^{ab} = 0, \quad \text{and} \quad \underline{D}_{[a} \underline{N}_{b]c} = 0 \quad (\text{A.52})$$

where  $\underline{q}_{ab}$  is the metric on  $S$  induced by the metric  $q_{ab}$  on  $\mathcal{I}$  and  $\underline{D}$  its derivative operator. While this result holds in any conformal frame, it is easiest to extract its consequences by working in a Bondi conformal frame where  $\underline{q}_{ab}$  is the unit 2-sphere metric. In the rest of this Appendix, we make this assumption.

Let us embed  $(S, \underline{q}_{ab})$  as the unit 2-sphere in  $\mathbb{R}^3$  and denote the Cartesian coordinates in  $\mathbb{R}^3$  by  $x^i$ . Then, the projections  $\underline{C}^a := \underline{q}^a{}_b \underline{e}^b$  of constant vector fields

$\dot{c}^b$  on  $\mathbb{R}^3$  provide us with ‘pure’ conformal Killing fields on  $(S, \underline{q}_{ab})$ :

$$\underline{C}_a = \underline{D}_a \dot{c}^b x_b = \underline{q}_a^c \partial_c (\dot{c}_b x^b), \quad \text{and} \quad \underline{D}_a \underline{C}_b = -(\dot{c}_c x^c) \underline{q}_{ab}. \quad (\text{A.53})$$

(Restrictions of functions  $\dot{c}_c x^c$  to  $S$  are linear combinations of the first three spherical harmonics  $Y_{1,m}$ .) It is easy to verify that  $\underline{D}_{[a} (\underline{N}_{b]c} \underline{C}^c) = 0$ . Therefore  $\underline{N}_{bc} \underline{C}^c$  is a gradient  $\underline{D}_b h$  on  $S$  and we can eliminate the freedom of adding a constant to its potential  $h$  by requiring  $\oint_S d^2V h = 0$ .

Thus,  $\underline{N}_{ab}$  provides us with a linear map from the 3-dimensional space of ‘pure conformal Killing fields’  $\underline{C}^a$  on  $(S, \underline{q}_{ab})$  to the space of functions  $h$  on  $S$  (satisfying  $\oint_S d^2V h = 0$ ). But there is also a natural isomorphism between the vector space of these  $\underline{C}^a$  and the vector space of their conformal Killing data [69] at any point  $p$  on  $S$ ,

$$\left( \underline{C}^a, \underline{D}_{[a} \underline{C}_{b]}, \underline{D}_a \underline{C}^a, \underline{D}^a \underline{D}_b \underline{C}^b \right)_p = (\underline{q}^{ab} \underline{D}_b \dot{c} \cdot x, 0, (\dot{c} \cdot x), \underline{q}^{ab} \underline{D}_b (\dot{c} \cdot x))_p. \quad (\text{A.54})$$

Since the conformal Killing data is determined by the values of  $\underline{C}^a$  and  $(\dot{c} \cdot x)$  in the tangent space of every point  $p$  on  $S$ , we have a number  $\underline{f}$  and a vector  $\underline{V}_a$  such that the function  $h$  determined by  $\underline{N}_{ab}$  is given at  $p$  by:

$$h = [\underline{q}^{ac} \underline{D}_c (\dot{c} \cdot x)] \underline{V}_a + [\dot{c} \cdot x] \underline{f}. \quad (\text{A.55})$$

Now, using the relation  $\underline{N}_{ab} \underline{C}^b = \underline{D}_a h$  and the fact that  $\underline{N}_{ab}$  is tangential to  $S$  and therefore satisfies  $\underline{N}_{ab} x^b = 0$ , one concludes  $\underline{V}_a = \underline{D}_a \underline{f}$  and hence

$$\underline{N}_{ab} = \underline{D}_a \underline{D}_b \underline{f} + \underline{f} \underline{q}_{ab}. \quad (\text{A.56})$$

This is the key consequence of the first and the third equations in (A.52). Finally we use the second equation. The fact that  $\underline{N}_{ab}$  is trace-free implies that  $\underline{f}$  must satisfy  $\underline{D}^2 \underline{f} + 2\underline{f} = 0$  whence it is necessarily of the form  $\underline{f} = \dot{c} \cdot x$  for some  $\dot{c}$ . Substituting this back in (A.56) implies  $\bar{N}_{ab} \hat{=} 0$ . Since  $N_{ab}$  is the pull-back to  $\mathcal{I}$  of  $\bar{N}_{ab}$ , we conclude  $N_{ab} \hat{=} 0$ .

Finally, we note that since  $S$  has the topology of  $\mathbb{S}^2$ , the conclusion  $\underline{N}_{ab} = 0$  follows already from  $\underline{N}_{[ab]} = 0$  and  $\underline{D}^a \underline{N}_{ab} = 0$  on  $S$  which are implied by (A.52)

[70]. The proof we presented here is of interest because of the intermediate result (A.56) which follows only from the first and the third equation in (A.52), and *more importantly* holds also on surfaces of constant curvature with higher dimensions and non-positive definite signature. (However, if the signature is not positive definite, the trace-free property of  $\underline{N}_{ab}$  does not then imply  $\underline{N}_{ab} = 0$ .) The higher dimensional result is used in the analysis of spatial infinity [29, 30, 59].

# Appendix B |

## Linearised gravitational fields on de Sitter space-time

**Notation:**

Background: Expanding Poincaré patch of deSitter spacetime with conformal time,  $\eta$ , and scale factor,  $a(\eta) = -(H\eta)^{-1}$

$$\bar{g}_{ab} = a^2(\eta)(-\nabla_a\eta\nabla_b\eta + \nabla_ax\nabla_bx) \quad (\text{B.1})$$

Perturbation:  $\dot{g}_{ab}$  is the conformally related Minkowski metric.

$$g_{ab}(\epsilon) = \bar{g}_{ab} + \epsilon\gamma_{ab} \quad (\text{B.2})$$

### B.1 Derivation of linearised equations

We linearise Einstein's equation with a positive cosmological constant:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad (\text{B.3})$$

Consider a one-parameter family of metrics,  $g_{ab}(\epsilon)$  such that  $g_{ab}(\epsilon)|_{\epsilon=0} = \bar{g}_{ab}$  is the deSitter metric. So this means  $T_{ab}(\epsilon)|_{\epsilon=0} = 0$ .

**Linearised metric and inverse:**

$$g_{ab}(\epsilon) = \bar{g}_{ab} + \epsilon\gamma_{ab} \quad g^{ab}(\epsilon) = \bar{g}^{ab} - \epsilon\gamma^{ab} \quad (\text{B.4})$$

**Linearised Christoffel symbols:**

These measure the deviation from the derivative operator compatible with the background metric i.e.,  $\Gamma^c{}_{ab}|_{\epsilon=0} = 0$ .

$$\begin{aligned} \frac{d}{d\epsilon} \nabla_a k_b(\epsilon)|_{\epsilon=0} &:= \frac{d}{d\epsilon} (\bar{\nabla}_a k_b(\epsilon) - \Gamma^c{}_{ab}(\epsilon) k_c)|_{\epsilon=0} := -C^c{}_{ab} k_c \\ \frac{d}{d\epsilon} \Gamma^c{}_{ab}(\epsilon)|_{\epsilon=0} &:= C^c{}_{ab} = \frac{1}{2} \bar{g}^{cd} (\bar{\nabla}_a \gamma_{bd} + \bar{\nabla}_b \gamma_{ad} - \bar{\nabla}_d \gamma_{ab}) \end{aligned} \quad (\text{B.5})$$

**Linearised Riemann tensor:**

$$\begin{aligned} \frac{d}{d\epsilon} R_{abc}{}^d(\epsilon) k_d|_{\epsilon=0} &= 2 \frac{d}{d\epsilon} \nabla_{[a} \nabla_{b]} k_c(\epsilon)|_{\epsilon=0} = 2 \frac{d}{d\epsilon} [(\bar{\nabla}_{[a} \nabla_{b]} k_c - \Gamma^d{}_{c[a} \nabla_{b]} k_d)(\epsilon)]|_{\epsilon=0} \\ &= -2 \bar{\nabla}_{[a} C^d{}_{b]c} k_d \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \frac{d}{d\epsilon} R_{abc}{}^d(\epsilon)|_{\epsilon=0} &= -2 \bar{\nabla}_{[a} C^d{}_{b]c} = -\bar{\nabla}_{[a} \bar{\nabla}_{b]} \gamma_c{}^d - \bar{\nabla}_{[a} \bar{\nabla}_{|c|} \gamma_{b]}{}^d + \bar{\nabla}_{[a} \bar{\nabla}^d \gamma_{b]c} \\ &= H^2 [\gamma_{[a}{}^d \bar{g}_{b]c} + \gamma_{c[a} \bar{g}_{b]}{}^d] - \bar{\nabla}_{[a} \bar{\nabla}_{|c|} \gamma_{b]}{}^d + \bar{\nabla}_{[a} \bar{\nabla}^d \gamma_{b]c} \end{aligned} \quad (\text{B.7})$$

**Linearised Ricci tensor:**

$$\frac{d}{d\epsilon} R_{ac}(\epsilon)|_{\epsilon=0} = \frac{1}{2} [\bar{\nabla}_d \bar{\nabla}_a \gamma_c{}^d - \bar{\nabla}_a \bar{\nabla}_c \gamma + \bar{\nabla}_d \bar{\nabla}_c \gamma_a{}^d - \bar{\square} \gamma_{ac}] \quad (\text{B.8})$$

**Linearised Ricci scalar:**

$$\begin{aligned} \frac{d}{d\epsilon} [g^{ac}(\epsilon) R_{ac}(\epsilon)]|_{\epsilon=0} &= \bar{g}^{ac} \frac{d}{d\epsilon} [(\epsilon) R_{ac}(\epsilon)]|_{\epsilon=0} - \gamma^{ac} \bar{R}_{ac} \\ &= -\bar{\square} \gamma + \bar{\nabla}^d \bar{\nabla}^c \gamma_{cd} - 3 H^2 \gamma \end{aligned} \quad (\text{B.9})$$

In the above equations  $\Lambda = 3 H^2$ , where  $H$  is the Hubble parameter and  $\gamma = \bar{g}^{ab} \gamma_{ab}$  is the trace of the metric perturbation  $\gamma_{ab}$ .

**Linearized Einstein's equation:**

Define the trace-reversed metric perturbation as  $\bar{h}_{ab} := \gamma_{ab} - \frac{1}{2} \bar{g}_{ab} \gamma$ ,  $\bar{h} = \bar{g}^{ab} \bar{h}_{ab}$ . Then the linearization of Einstein's equation is given by

$$\begin{aligned} &-\frac{1}{2} \bar{\square} \gamma_{ab} + \bar{\nabla}^d \bar{\nabla}_{(a} \gamma_{b)d} - \frac{1}{2} \bar{\nabla}_a \bar{\nabla}_b \gamma - \Lambda \gamma_{ab} \\ &-\frac{1}{2} \bar{g}_{ab} (-\bar{\square} \gamma + \bar{\nabla}^m \bar{\nabla}^n \gamma_{mn} - \Lambda \gamma) = 8\pi G \frac{d}{d\epsilon} T_{ac}(\epsilon)|_{\epsilon=0} \end{aligned} \quad (\text{B.10})$$

In terms of the trace-reversed perturbations  $\bar{h}_{ab}$ , the equation reads

$$-\frac{1}{2}\bar{\square}\bar{h}_{ac} + \bar{\nabla}^d\bar{\nabla}_{(a}\bar{h}_{c)d} - \frac{1}{2}\bar{g}_{ac}\bar{\nabla}^d\bar{\nabla}^m\bar{h}_{dm} - 3H^2\bar{h}_{ac} = 8\pi G\frac{d}{d\epsilon}T_{ac}(\epsilon)|_{\epsilon=0} \quad (\text{B.11})$$

We make the standard transverse, traceless gauge choice:  $\bar{\nabla}^b\bar{h}_{ab} = 0$ ,  $\bar{h} = 0$ .

$$\bar{\nabla}_d\bar{\nabla}_a\bar{h}_c{}^d = \bar{\nabla}_a\bar{\nabla}_d\bar{h}_c{}^d + \bar{R}_{dac}{}^m\bar{h}_m{}^d + \bar{R}_{da}{}^d{}_m\bar{h}_c{}^m$$

Since  $\bar{R}_{abcd} = \frac{2}{3}\Lambda\bar{g}_{c[a}\bar{g}_{b]d}$ , this gives

$$\bar{\nabla}_d\bar{\nabla}_a\bar{h}_c{}^d = \bar{\nabla}_a\bar{\nabla}_d\bar{h}_c{}^d + \frac{4}{3}\Lambda\bar{h}_{ac} - \frac{1}{3}\Lambda\bar{h}\bar{g}_{ac} \quad (\text{B.12})$$

$$\bar{\square}\bar{h}_{ac} - 2H^2\bar{h}_{ac} = -16\pi G\frac{d}{d\epsilon}T_{ac}(\epsilon)|_{\epsilon=0} \quad (\text{B.13})$$

Note that in the transverse-traceless gauge,  $\gamma_{ab} = \bar{h}_{ab}$ . So the linearized Einstein's equation can be written as

$$\bar{\square}\gamma_{ac} - 2H^2\gamma_{ac} = -16\pi G\frac{d}{d\epsilon}T_{ac}(\epsilon)|_{\epsilon=0} \quad (\text{B.14})$$

### B.1.1 Making contact with cosmology

Rewrite Eq. B.14 in terms of the field  $h_{ab} := a^{-2}\gamma_{ab}$  and fix gauge using  $\eta^a\gamma_{ab} = 0$  where  $\eta^a\partial_a = \partial/\partial\eta$ .

$$\begin{aligned} & \bar{\square}\gamma_{ac} - 2H^2\gamma_{ac} \\ &= 2\bar{g}^{pq}(\bar{\nabla}_p a^2)(\bar{\nabla}_q h_{ac}) + a^2\bar{g}^{pq}\bar{\nabla}_p\bar{\nabla}_q h_{ac} + h_{ac}\bar{g}^{pq}\bar{\nabla}_p\bar{\nabla}_q a^2 - 2H^2\gamma_{ac} \end{aligned} \quad (\text{B.15})$$

The first term of (B.15) can be simplified using  $C^m{}_{ca}\eta^c = H a \delta^m{}_a$  to yield:

$$2\bar{g}^{pq}(\bar{\nabla}_p a^2)(\bar{\nabla}_q h_{ac}) = -4\frac{a'}{a}h'_{ac} + 8H^2 a^2 h_{ac} \quad (\text{B.16})$$

Next, using  $a' = H a^2$ ,  $a'' = 2H^2 a^3$ , the third term of (B.15) also simplifies:

$$h_{ac}\bar{\square}a^2 = -10H^2 a^2 h_{ac} \quad (\text{B.17})$$

Using  $\bar{g}^{pq}C^m{}_{pq} = (2H/a)(\eta)^m$  and the gauge condition  $\bar{\nabla}^a h_{ab} = 0$ ,  $h := \dot{g}^{ab}h_{ab} = 0$  the second term of (B.15) now simplifies to

$$-h''_{ac} + \dot{D}^2 h_{ac} + 2\frac{a'}{a}h'_{ac} + 4H^2 a^2 h_{ac} \quad (\text{B.18})$$

Using equations (B.16), (B.17) and (B.18), the linearised Einstein equation (B.14) reduces to

$$h''_{ac} - \dot{D}^2 h_{ac} + 2\frac{a'}{a}h'_{ac} = 16\pi G \frac{d}{d\epsilon} T_{ac}(\epsilon)|_{\epsilon=0} \quad (\text{B.19})$$

### B.1.2 Gauge considerations

In order for the above equations to be taken seriously and their solutions studied, one must ensure that all the gauge choices for the metric perturbation made during the derivation are allowed in general relativity. We recall the transverse, traceless and perpendicular gauge conditions:

$$\bar{\nabla}^a \bar{h}_{ab} = 0; \quad \bar{h} = 0; \quad \eta^a \bar{h}_{ab} = 0 \quad (\text{B.20})$$

where  $\bar{h}_{ab} := \gamma_{ab} - \frac{1}{2}\bar{g}_{ab}\gamma$ .

In general relativity, the gauge freedom corresponding to the diffeomorphism group implies that in linearised gravity, two tensors represent the same physical perturbation if they are related by a diffeomorphism.

$$\gamma_{ab} \rightarrow \tilde{\gamma}_{ab} = \gamma_{ab} + 2\bar{\nabla}_{(a}\xi_{b)} \quad (\text{B.21})$$

Let us first impose transverseness on the transformed trace-reversed perturbation  $\tilde{\bar{h}}_{ab} := \tilde{\gamma}_{ab} - \frac{1}{2}\bar{g}_{ab}\tilde{\gamma}$ . Then,

$$\bar{\nabla}^a \tilde{\bar{h}}_{ab} = 0 \implies \bar{\square}\xi_b + \Lambda\xi_b = -\bar{\nabla}^a \bar{h}_{ab} \quad (\text{B.22})$$

Thus, given any perturbation  $\gamma_{ab}$ , transverse gauge can be imposed using a diffeomorphism generated by the vector field  $\xi^a$  that satisfies eqn (B.22).

Once in the transverse gauge, there is residual gauge freedom as long as the

diffeomorphism leaves the new perturbation transverse:

$$\tilde{\gamma}_{ab} \rightarrow \bar{\Gamma}_{ab} = \tilde{\gamma}_{ab} + 2\bar{\nabla}_{(a}\zeta_{b)}; \quad \text{where} \quad \bar{\square}\zeta_b + \Lambda\zeta_b = 0 \quad (\text{B.23})$$

In the conformally flat Cartesian coordinates we use, this condition can be written in terms of the components of  $\zeta_a$  with index  $i$  denoting spatial components and 0 refers to a contraction with  $\eta^a$ :

$$\begin{aligned} \bar{\square}\zeta_0 + \frac{2}{\eta}\dot{\nabla}^i\zeta_i + \frac{8}{\eta^2}\zeta_0 &= 0 \\ \bar{\square}\zeta_i + \frac{2}{\eta}\dot{\nabla}_i\zeta_0 + \frac{6}{\eta^2}\zeta_i &= 0 \end{aligned} \quad (\text{B.24})$$

This freedom can be used to make  $\bar{\Gamma}_{ab}$  traceless as follows:

$$\bar{\Gamma} = 0 \implies \frac{\partial\zeta_0}{\partial\eta} - \dot{\nabla}^i\zeta_i + 2Ha\zeta_0 = \frac{a^2}{2}\tilde{\gamma} \quad (\text{B.25})$$

$$\frac{\partial\bar{\Gamma}}{\partial\eta} = 0 \implies \dot{\nabla}^2\zeta_0 + 6H^2a^2\zeta_0 - \frac{\partial}{\partial\eta}(\dot{\nabla}^i\zeta_i) = a^2\frac{\partial\tilde{\gamma}}{\partial\eta} \quad (\text{B.26})$$

Equation (B.26) is obtained by taking the time ( $\eta$ ) derivative of the first, using (B.24) to replace  $\frac{\partial^2\zeta_0}{\partial\eta^2}$  and using (B.25) again.

We also impose the condition of perpendicularity  $\bar{\Gamma}_{ai}\eta^a =: \bar{\Gamma}_{0i} = 0$

$$\bar{\Gamma}_{0i} = 0 \implies \frac{\partial\zeta_i}{\partial\eta} + \frac{\partial\zeta_0}{\partial x^i} - 2Ha\zeta_i = -\tilde{\gamma}_{0i} \quad (\text{B.27})$$

$$\frac{\partial\bar{\Gamma}_{0i}}{\partial\eta} = 0 \implies \dot{\nabla}^2\zeta_i + \dot{\nabla}_i\frac{\partial\zeta_0}{\partial\eta} = -2Ha\tilde{\gamma}_{0i} - \frac{\partial\tilde{\gamma}_{0i}}{\partial\eta} \quad (\text{B.28})$$

Equation (B.28) is obtained by taking the time ( $\eta$ ) derivative of the first, using (B.24) to replace  $\frac{\partial^2\zeta_i}{\partial\eta^2}$  and using (B.27) again.

The goal is to find  $\zeta_a$  which solves Eqs. (B.25) - (B.28). Start by rewriting equation (B.26) using (B.27) to substitute for  $\frac{\partial\zeta_i}{\partial\eta}$ :

$$-\dot{\nabla}^2\zeta_0 - 3H^2a^2\zeta_0 + Ha\dot{\nabla}^i\zeta_i = \frac{1}{4}\left[2\dot{\nabla}^i\tilde{\gamma}_{0i} - a^2\frac{\partial\tilde{\gamma}}{\partial\eta}\right] \quad (\text{B.29})$$

Rewrite equation (B.28) using (B.25) to substitute for  $\frac{\partial \zeta_0}{\partial \eta}$  and take divergence:

$$2\overset{\circ}{\nabla}^2\overset{\circ}{\nabla}^i\zeta_i - 2Ha\overset{\circ}{\nabla}^2\zeta_0 = -\frac{1}{2}a^2\overset{\circ}{\nabla}^2\tilde{\gamma} - 2Ha\overset{\circ}{\nabla}^i\tilde{\gamma}_{0i} - \overset{\circ}{\nabla}^i\frac{\partial\tilde{\gamma}_{0i}}{\partial\eta} \quad (\text{B.30})$$

Substitute for  $\overset{\circ}{\nabla}^j\zeta_j$  from (B.29) to obtain:

$$\overset{\circ}{\nabla}^2(\overset{\circ}{\nabla}^2\zeta_0) + 2H^2a^2\overset{\circ}{\nabla}^2\zeta_0 = S \quad (\text{B.31})$$

where

$$S = -\frac{1}{8}\left(2\overset{\circ}{\nabla}^i\tilde{\gamma}_{0i} - a^2\frac{\partial\tilde{\gamma}}{\partial\eta}\right) - \frac{1}{4}Ha^3\overset{\circ}{\nabla}^2\tilde{\gamma} - H^2a^2\overset{\circ}{\nabla}^i\tilde{\gamma}_{0i} - \frac{1}{2}Ha\overset{\circ}{\nabla}^i\frac{\partial\tilde{\gamma}_{0i}}{\partial\eta}.$$

Now the equations can be solved in the following sequence.

1. Solve (B.31) for  $\overset{\circ}{\nabla}^2\zeta_0$ . Substitute solution in (B.30).
2. Solve (B.30) for  $\overset{\circ}{\nabla}^i\zeta_i$ . Substitute solution in (B.29) and (B.25).
3. Solve (B.29) for  $\zeta_0$ . Substitute solution in (B.25).
4. Solve (B.25) for  $\frac{\partial\zeta_0}{\partial\eta}$ . Substitute solution in (B.28).
5. Solve (B.28) for  $\zeta_i$ . Substitute solution in (B.27).
6. Solve (B.27) for  $\frac{\partial\zeta_i}{\partial\eta}$ .

Thus, the gauge conditions of Eq. B.20 can be imposed.

## B.2 Extrinsic curvature, Electric and Magnetic parts of Weyl tensor at $\mathcal{I}^+$

**Conformal completion:**

$$\hat{g}_{ab} = \omega^2 g_{ab}; \quad \omega = a^{-1}(\eta) \implies \hat{g}_{ab} = \hat{g}_{ab} + h_{ab} \quad (\text{B.32})$$

Consider a foliation of space-time by constant conformal time slices  $\eta = \text{constant}$ . Since the perturbation is purely spatial, the unit normal and three-metric on the

slices are given by:

$$\hat{n}^a = \eta^a; \quad \hat{q}_{ab} = \hat{g}_{ab} + \hat{n}_a \hat{n}_b \quad (\text{B.33})$$

### Extrinsic curvature

The extrinsic curvature of a slice and its linearization are:

$$\hat{K}_{ab} = \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{q}_{ab}; \quad \frac{d}{d\epsilon} \hat{K}_{ab}|_{\epsilon=0} = \frac{1}{2} \mathcal{L}_{\hat{n}} h_{ab} = \frac{1}{2} h'_{ab} \quad (\text{B.34})$$

Recall from Eq. (5.15) that the most general solution to the homogeneous linearized equation, in terms of Fourier modes for each polarization is given by

$$h_{\vec{k}}(\eta) = (-2H) \left[ E_{\vec{k}}(\eta \cos(k\eta) - (1/k) \sin(k\eta)) - B_{\vec{k}}(\eta \sin(k\eta) + (1/k) \cos(k\eta)) \right] \quad (\text{B.35})$$

$$h'_{\vec{k}} = (2H) \left[ B_{\vec{k}}(k\eta \cos(k\eta)) - E_{\vec{k}}(k\eta \sin(k\eta)) \right] \quad (\text{B.36})$$

On  $\mathcal{I}_{dS}^+$ ,  $\eta = 0$ , the linearized extrinsic curvature vanishes for both the channels. But  $\omega^{-1} \frac{d}{d\epsilon} \hat{K}_{ab}|_{\epsilon=0}$  is non-zero only for channel  $B_{\vec{k}}$ .

### Electric part of Weyl

From Geroch's work [27], Eq. (102):

$$\hat{E}_{ab} = {}^3\hat{R}_{ab} - \hat{K}_a{}^m \hat{K}_{bm} + \hat{K} \hat{K}_{ab} - \frac{1}{2} (\hat{q}_a{}^m \hat{q}_b{}^n \hat{L}_{mn} + \hat{q}_{ab} \hat{L}_{mn} \hat{q}^{mn}) \quad (\text{B.37})$$

where  $\hat{L}_{mn} := \hat{R}_{ab} - (1/6) \hat{R} \hat{g}_{ab}$  is the Schouten tensor.

We recall that the hatted quantities refer to metric perturbations off Minkowski space-time. So we proceed to linearise about the Minkowski metric. Since  $\hat{K}|_{\epsilon=0} = 0$ ,  $\hat{K}_{ab}|_{\epsilon=0} = 0$  and  $\hat{L}_{ab}|_{\epsilon=0} = 0$ :

$$\frac{d}{d\epsilon} \hat{E}_{ab}|_{\epsilon=0} = \frac{d}{d\epsilon} {}^3\hat{R}_{ab}|_{\epsilon=0} - \frac{1}{2} (\hat{q}_a{}^m \hat{q}_b{}^n \frac{d}{d\epsilon} \hat{L}_{mn} + \hat{q}_{ab} \frac{d}{d\epsilon} \hat{L}_{mn} \hat{q}^{mn})|_{\epsilon=0}. \quad (\text{B.38})$$

The linearized fields are as follows, where the second equality is in the transverse-traceless gauge:

$$\frac{d}{d\epsilon} \hat{R}_{ab}|_{\epsilon=0} = \overset{\circ}{\nabla}_{(a} \overset{\circ}{\nabla}{}^m h_{b)m} - \frac{1}{2} \overset{\circ}{\nabla}{}^m \overset{\circ}{\nabla}{}_m h_{ab} - \frac{1}{2} \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b h = -\frac{1}{2} \overset{\circ}{\nabla}{}^m \overset{\circ}{\nabla}{}_m h_{ab} \quad (\text{B.39})$$

$$\frac{d}{d\epsilon}\hat{R}|_{\epsilon=0} = \overset{\circ}{\nabla}{}^c\overset{\circ}{\nabla}{}^m h_{cm} - \overset{\circ}{\nabla}{}^m\overset{\circ}{\nabla}{}_m h = 0 \quad (\text{B.40})$$

$$\frac{d}{d\epsilon}\hat{L}_{mn}|_{\epsilon=0} = \frac{d}{d\epsilon}(\hat{R}_{mn} - \frac{1}{6}\hat{R}\hat{g}_{mn})|_{\epsilon=0} = -\frac{1}{2}\overset{\circ}{\nabla}{}^2 h_{mn} \quad (\text{B.41})$$

$$\frac{d}{d\epsilon}{}^3\hat{R}_{ab}|_{\epsilon=0} = \overset{\circ}{D}{}_{(a}\overset{\circ}{D}{}^m h_{b)m} - \frac{1}{2}\overset{\circ}{D}{}^m\overset{\circ}{D}{}_m h_{ab} - \frac{1}{2}\overset{\circ}{D}{}_a\overset{\circ}{D}{}_b h = -\frac{1}{2}\overset{\circ}{D}{}^m\overset{\circ}{D}{}_m h_{ab} \quad (\text{B.42})$$

Putting the above together:

$$\begin{aligned} \frac{d}{d\epsilon}\hat{E}_{ab}|_{\epsilon=0} &= -\frac{1}{2}\overset{\circ}{D}{}^2 h_{ab} + \frac{1}{4}\overset{\circ}{\nabla}{}^2 h_{ab} = \frac{1}{4}(-h''_{ab} - \overset{\circ}{D}{}^2 h_{ab}) \\ &= \frac{1}{4}(2\frac{a'}{a}h'_{ab} - 2\overset{\circ}{D}{}^2 h_{ab}) = -\frac{1}{2}(h''_{ab} - \frac{1}{\eta}h'_{ab}) \end{aligned} \quad (\text{B.43})$$

where the last equality was obtained using the equation of motion for  $h_{ab}$ .

Now define  $\mathcal{E}_{ab} := \omega^{-1}E_{ab} = -\frac{1}{H\eta}E_{ab}$  to obtain

$$\mathcal{E}_{ab} = \frac{1}{2H\eta}(\overset{\circ}{D}{}^2 h_{ab} + \frac{1}{\eta}h'_{ab}) \quad (\text{B.44})$$

In terms of the Fourier modes,

$$\frac{1}{2H\eta}(h''_k - \frac{1}{\eta}h'_k) = -k^2(B_{\vec{k}}\sin(k\eta) - E_{\vec{k}}\cos(k\eta)) \quad (\text{B.45})$$

Thus, at  $\mathcal{I}^+$  where  $\eta = 0$ ,  $\hat{\mathcal{E}}_{ab}$  is non-zero only for channel  $E_{\vec{k}}$ , the channel for which  $h_{ab}$  vanishes.

### Magnetic part of Weyl

From Geroch [27], (103),

$$\hat{B}_{ab} = \hat{\epsilon}_{mn(a}\hat{D}{}^m\hat{K}{}^n{}_{b)} \quad (\text{B.46})$$

Define  $\hat{\beta}_{ab} = \omega^{-1}\hat{B}_{ab}$ . Then from (B.34) and (B.36), it is seen that the linearization of  $\hat{\beta}_{ab}$ ,  $\frac{d}{d\epsilon}\hat{\beta}_{ab}|_{\epsilon=0}$  is non-zero only for channel  $B_{\vec{k}}$ , the same channel for which the perturbation  $h_{ab}$  and  $\omega^{-1}\frac{d}{d\epsilon}\hat{K}_{ab}|_{\epsilon=0}$  are non-zero at  $\mathcal{I}^+$ .

In summary, channel  $B_{\vec{k}}$  has non-vanishing  $h_{ab}$  and  $\beta_{ab}$  and vanishing  $\mathcal{E}_{ab}$ , channel  $E_{\vec{k}}$  has non-vanishing  $\mathcal{E}_{ab}$  and vanishing  $h_{ab}$  and  $\beta_{ab}$  at  $\mathcal{I}^+$ .

### B.3 The Minkowski limit

To study the Minkowski limit, make a coordinate transformation from conformal time  $\eta$  to proper time  $t$  and then take the limit  $H \rightarrow 0$ . In these coordinates the metric and Killing vector fields of de Sitter space-time have good limits to Killing vector fields of Minkowski space-time.

$$\begin{aligned} -H\eta &= e^{-Ht}; & ds^2 &= -dt^2 + e^{2Ht}d\vec{x}^2 \\ \text{Dilation : } D^a &= t^a - HX^a \text{ where } t^a = (\partial_t)^a \end{aligned} \quad (\text{B.47})$$

#### Equations of motion and solutions:

The equation of motion in  $(t, x)$  coordinates is:

$$e^{2Ht}\ddot{h}_{ab} - \dot{D}^2 h_{ab} + 3He^{2Ht}\dot{h}_{ab} = 0 \quad (\text{B.48})$$

where overdot refers to derivative with respect to proper time,  $t$ . Solutions for each Fourier mode in each polarization ( $s$ ) are:

$$\begin{aligned} h_{\vec{k}} = & 2B_{\vec{k}} \left[ \frac{H}{k} \cos\left(e^{-Ht} \frac{k}{H}\right) + e^{-Ht} \sin\left(e^{-Ht} \frac{k}{H}\right) \right] \\ & - 2E_{\vec{k}} \left[ -\frac{H}{k} \sin\left(e^{-Ht} \frac{k}{H}\right) + e^{-Ht} \cos\left(e^{-Ht} \frac{k}{H}\right) \right] \end{aligned} \quad (\text{B.49})$$

Limit  $H \rightarrow 0$  of (B.48) easily yields the equation of motion of tensor perturbations off of Minkowski spacetime (in the transverse, traceless, radiation gauge)

$$\square h_{ab} = 0 \quad (\text{B.50})$$

The most general solutions for each Fourier mode is

$$h_{\vec{k}} = 2C_{\vec{k}} \sin(kt) + 2D_{\vec{k}} \cos(kt) \quad (\text{B.51})$$

We now describe how (B.51) is recovered from (B.49) in the limit  $H \rightarrow 0$ .

We first note that rescaling  $k \rightarrow \tilde{k} = H^{-1}k$  does not make the limit of the equation of motion and solutions transparent because once the factors of  $H$  are absorbed it is unclear how to take the limit. Instead we Taylor expand our solution in (B.52) around  $H = 0$ .

$$\begin{aligned}
h_{\tilde{k}} &= 2B_{\tilde{k}} \left[ \frac{H}{k} \cos \left( \frac{k}{H}(1 - Ht + \dots) \right) + (1 - Ht + \dots) \sin \left( \frac{k}{H}(1 - Ht + \dots) \right) \right] \\
&\quad - 2E_{\tilde{k}} \left[ -\frac{H}{k} \sin \left( \frac{k}{H}(1 - Ht + \dots) \right) + (1 - Ht + \dots) \cos \left( \frac{k}{H}(1 - Ht + \dots) \right) \right] \\
&= 2B_{\tilde{k}} \sin \left( \frac{k}{H}(1 - Ht + \dots) \right) + 2E_{\tilde{k}} \cos \left( \frac{k}{H}(1 - Ht + \dots) \right) \\
&\quad + H 2B_{\tilde{k}} \left[ \frac{1}{k} \cos \left( \frac{k}{H}(1 - Ht + \dots) \right) - t \sin \left( \frac{k}{H}(1 - Ht + \dots) \right) \right] \\
&\quad + H 2E_{\tilde{k}} \left[ \frac{1}{k} \sin \left( \frac{k}{H}(1 - Ht + \dots) \right) + t \cos \left( \frac{k}{H}(1 - Ht + \dots) \right) \right] \\
&\quad + O(H^2)
\end{aligned} \tag{B.52}$$

At leading order in  $H$ :

$$\begin{aligned}
h_k &= \sin(kt) \left[ -2B_{\tilde{k}} \cos \left( \frac{k}{H} \right) + 2E_{\tilde{k}} \sin \left( \frac{k}{H} \right) \right] \\
&\quad + \cos(kt) \left[ 2B_{\tilde{k}} \sin \left( \frac{k}{H} \right) + 2E_{\tilde{k}} \cos \left( \frac{k}{H} \right) \right] \\
&= 2C_{\tilde{k}} \sin(kt) + 2D_{\tilde{k}} \cos(kt) + O(H)
\end{aligned} \tag{B.53}$$

where we have defined the coefficients as:

$$\begin{aligned}
C_{\tilde{k}} &= E_{\tilde{k}} \sin(k/H) - B_{\tilde{k}} \cos(k/H) \\
D_{\tilde{k}} &= E_{\tilde{k}} \cos(k/H) + B_{\tilde{k}} \sin(k/H)
\end{aligned} \tag{B.54}$$

**Remarks:**

1. When taking the Minkowski limit  $H \rightarrow 0$ , the basis functions,  $\cos \left( e^{-Ht} \frac{k}{H} \right)$

and  $\sin\left(e^{-Ht}\frac{k}{H}\right)$  are no longer well behaved because the argument is divergent due to  $k/H$ . So carefully expand around  $H = 0$  to obtain basis functions  $\sin(kt)$  and  $\cos(kt)$  whose coefficients at leading order in  $H$ ,  $C_{\vec{k}}$  and  $D_{\vec{k}}$  are defined as functions of  $k$ ,  $H$ ,  $E_{\vec{k}}$  and  $B_{\vec{k}}$ .

2. A solution in de Sitter is said to have a good limit to Minkowski if there exists a sequence in  $H$  along which the coefficients  $C_{\vec{k}}$  and  $D_{\vec{k}}$  have a good limit as  $H \rightarrow 0$  or are independent of  $H$ . For such solutions,  $B_{\vec{k}}$  and  $E_{\vec{k}}$  depend on  $H$ . We can immediately construct two interesting solutions:

$$\begin{aligned} B_{\vec{k}} = -\dot{C}_{\vec{k}} \cos(k/H), E_{\vec{k}} = \dot{D}_{\vec{k}} \sin(k/H) &\implies (C_{\vec{k}}, D_{\vec{k}}) = (\dot{C}_{\vec{k}}, 0) \\ B_{\vec{k}} = \dot{D}_{\vec{k}} \sin(k/H), E_{\vec{k}} = \dot{C}_{\vec{k}} \cos(k/H) &\implies (C_{\vec{k}}, D_{\vec{k}}) = (0, \dot{D}_{\vec{k}}) \end{aligned} \quad (\text{B.55})$$

Not all solutions in de Sitter space can have well-defined limits to Minkowski. In particular, a solution in de Sitter with either  $E_{\vec{k}} = 0$  or  $B_{\vec{k}} = 0$  has no well-defined limit to Minkowski, because  $C_{\vec{k}}$  and  $D_{\vec{k}}$  are necessarily  $H$ -dependent, unless they both trivially vanish. This is another indication that requiring conformal flatness is a very strong boundary condition. Recall that conformal flatness of the boundary implies  $\beta_{ab} \hat{=} 0$  on  $\mathcal{I}$ , i.e.,  $B_{\vec{k}} = 0 \forall \vec{k}$ . Thus, these solutions do not have a good limit to Minkowski.

3. Since the coefficients  $C_{\vec{k}}$  and  $D_{\vec{k}}$  are related by a rotation through angle  $(k/H)$  to the coefficients  $E_{\vec{k}}$  and  $B_{\vec{k}}$

$$|C_{\vec{k}}|^2 + |D_{\vec{k}}|^2 = |E_{\vec{k}}|^2 + |B_{\vec{k}}|^2 \quad (\text{B.56})$$

Thus, if a solution in de Sitter space has amplitudes which are square integrable in the momentum space, then its limit in Minkowski space, if it exists, also has square integrable amplitudes in momentum space. Thus the solution is square integrable.

# Appendix C

## The tail term

A qualitative difference between the  $\Lambda > 0$  and  $\Lambda = 0$  cases is the presence of the tail term in the retarded solution. In this section we will discuss some properties of this term. The first natural question is whether it disappears in the  $\Lambda \rightarrow 0$  limit, i.e., whether the limit is continuous. The second conceptually important question is whether  $b_{ab}$  is negligible compared to the sharp term  $\sharp_{ab}$  if  $\Lambda \neq 0$  but tiny. We will now show that the answer to the first question is in the affirmative but that to the second question is in the negative. This is another illustration of the subtlety of the limit  $\Lambda \rightarrow 0$ .

To answer these questions, it is most convenient to work in the  $(t, \vec{x})$  chart. Now the tail term assumes the form

$$b_{ab}(t, \vec{x}) = -2GH \int_{-\infty}^{t_{\text{ret}}} dt' \left[ \ddot{Q}_{ab}^{(\rho)} + 3H\ddot{Q}_{ab}^{(p)} + 2H^2\dot{Q}_{ab}^{(\rho)} + H\ddot{Q}_{ab}^{(p)} + 3H^2\dot{Q}_{ab}^{(p)} + 2H^3Q_{ab}^{(p)} \right](t'). \quad (\text{C.1})$$

In the  $\Lambda \rightarrow 0$  limit, the ‘overdot’ tends to the well-defined Lie derivative with respect to a time translation Killing vector field in Minkowski space-time. Therefore the overall multiplicative factor  $H$  in (C.1) makes it transparent that  $b_{ab}$  does vanish in the  $\Lambda \rightarrow 0$  limit.

To answer the second question, let us use the fact that  $\dot{Q}_{ab} = \partial_t Q_{ab} - 2H Q_{ab}$  to carry out the integral over  $t$  in (C.1). Then, we have:

$$b_{ab}(t, \vec{x}) = -2GH \left[ \ddot{Q}_{ab}^{(\rho)} + H\dot{Q}_{ab}^{(\rho)} + H\dot{Q}_{ab}^{(p)} + H^2Q_{ab}^{(p)} \right]_{-\infty}^{t_{\text{ret}}}. \quad (\text{C.2})$$

As shown in section 6.3.3, the assumption  $\mathcal{L}_T T_{ab} = 0$  in the distant past implies  $\dot{Q}_{ab}^{(\rho)} = -2H Q_{ab}^{(\rho)}$  there (and similarly for the pressure quadrupole). Therefore, we

have

$$\mathfrak{b}_{ab}(t, \vec{x}) = -2GH \left[ \ddot{Q}_{ab}^{(\rho)} + H\dot{Q}_{ab}^{(\rho)} + H\dot{Q}_{ab}^{(p)} + H^2 Q_{ab}^{(p)} \right] (t_{\text{ret}}) + 2GH^3 C_{ab}, \quad (\text{C.3})$$

where  $C_{ab}$  is just a constant term. It does not play any role in the calculation of energy flux because in the expression (5.46) only derivatives of  $\chi_{ab}$  appear. In the expression (6.72) of the flux of angular momentum,  $\chi_{ab}$  does appear without a derivative but the constant term does not contribute because it is integrated against  $\mathcal{E}^{ab}$  which is of compact support and divergence-free on  $\mathcal{I}^+$ . Finally, since it is constant, it will not feature in the analysis of the memory effect as well.

With this simplification of the tail term, we can return to (6.42) and, for  $r > -\eta$ , write  $\chi_{ab}$  as

$$\chi_{ab}(\eta, \vec{x}) = \frac{2G}{R(\eta_{\text{ret}})} \left[ \left(1 - \frac{r}{r - \eta}\right) \ddot{Q}_{ab}^{(\rho)} \right] + \mathcal{O}(H) \quad (\text{C.4})$$

where  $R(\eta_{\text{ret}}) = ra(\eta_{\text{ret}})$  is the physical distance between the source and the point  $\vec{x}$  at time  $\eta = \eta_{\text{ret}}$  (and terms  $\mathcal{O}(H)$  vanish in the limit  $\Lambda \rightarrow 0$ ). The factor 1 in the square bracket comes from the sharp term while the factor  $r/(r - \eta)$  comes from the tail term. At late times the two contributions are comparable and at  $\mathcal{I}^+$  they are in fact equal in magnitude but opposite in sign. This occurs no matter how small  $\Lambda$  is! The remainder –i.e., the  $\mathcal{O}H$  term– at  $\mathcal{I}^+$  has contributions from both the sharp and the tail terms:

$$\chi_{ab}(\vec{x}) \hat{=} 2GH^2 [\dot{Q}_{ab}^{(\rho)} + H\dot{Q}_{ab}^{(p)}] + 2H^3 C_{ab}. \quad (\text{C.5})$$

This analysis provides the precise sense in which the back-scattering effects encoded in the tail term –which can also be thought of as arising from the addition of a mass term to the propagation equation of  $\bar{\gamma}_{ab}$ – provide an  $\mathcal{O}(1)$  contribution to the metric perturbation  $\chi_{ab}$  near  $\mathcal{I}^+$ . This is a concrete realization of the non-trivial outcome of the secular accumulation of small effects we referred to at the start of Chapter 6. Finally, as mentioned after Eq. (6.46), the tail term is essential to make the field  $\mathcal{E}_{ab}$  finite at  $\mathcal{I}^+$ . As a result, it contributes on an equal footing as the sharp term to the expression of energy and angular momentum radiated across  $\mathcal{I}^+$ .

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