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Algebraic aspects of conformal field theory

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Abstract

This thesis presents an algebraic study of two areas of conformal field theory. The first part extends the theory of Galilean conformal algebras. These algebras are extended conformal symmetry algebras for two-dimensional quantum field theories, formed through a process known as Galilean contraction. Analogous to an Inonu-Wigner contraction, the Galilean contraction procedure takes two conformal symmetry algebras, equivalent up to central charge, as input and produces a new conformal symmetry algebra. We extend the theory of Galilean contractions in several directions. First, we develop the theory to allow input of any number of symmetry algebras. These generalised algebras have a truncated graded structure. We develop a theory for multi-graded Galilean algebras, whereby we can extend structures which are graded by sequences. Finally, we present a comprehensive analysis of the possible algebras which arise when the input algebras are no longer required to be equivalent. We refer to this as the asymmetric Galilean contraction. For each stage of generalisation, we present several pertinent examples, discuss the Sugawara construction of a Galilean Virasoro algebra given an affine Lie algebra, and apply our results to the W-algebra W_3.

The second part of this thesis presents an exploratory study of reducible but indecomposable modules of the $N = 2$ Superconformal algebras, known as staggered modules. These modules are characterised by a non-diagonalisable action of L_0, the Virasoro zero-mode. Using recent results on the coset construction of $N = 2$ minimal models, we are able to construct the first examples of staggered modules for the $N = 2$ algebras. We determine the structure of a family of such modules for all admissible values of the central charge. Furthermore, we investigate the action of symmetries such as spectral flow on these modules. We present the results along with a range of examples, and discuss possible paths towards a classification of such modules for the Neveu-Schwarz and Ramond $N = 2$ superconformal algebras.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

I have clearly stated the contribution of others to my thesis as a whole, including statistical assistance, survey design, data analysis, significant technical procedures, professional editorial advice, financial support and any other original research work used or reported in my thesis. The content of my thesis is the result of work I have carried out since the commencement of my higher degree by research candidature and does not include a substantial part of work that has been submitted to qualify for the award of any other degree or diploma in any university or other tertiary institution. I have clearly stated which parts of my thesis, if any, have been submitted to qualify for another award.

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Publications included in this thesis

Publications included in this thesis are the following

1. [1] J. Rasmussen, C. Raymond, *Galilean contractions of W-algebras*, Nucl. Phys. **B922** (2017), 435–479, arXiv:1701.04437 [hep-th].
2. [2] J. Rasmussen, C. Raymond, *Higher-order Galilean contractions*, Nucl. Phys. **B945** (2019), 114680, arXiv:1901.06069 [hep-th].
3. [3] E. Ragoucy, J. Rasmussen, C. Raymond, *Multi-graded Galilean conformal algebras*, Nucl. Phys. **B957**, 115092, arXiv:2002.08637 [hep-th].

Submitted manuscripts included in this thesis

No manuscripts submitted for publication.

Other publications during candidature

No other published articles are included in this thesis.

Contributions by others to the thesis

The work on Galilean algebras was conceptualised, and guided by Jørgen Rasmussen. The initial work on multi-graded, and asymmetric, Galilean algebras came from correspondence between Jørgen Rasmussen and Eric Ragoucy. Jørgen Rasmussen and Eric Ragoucy contributed to technical results on the product structure of multi-graded and asymmetric Galilean algebras.

The project on $N = 2$ superconformal staggered modules was conceived and designed in conversations with David Ridout and Jørgen Rasmussen, designed to extend previous work of David Ridout and collaborators. David Ridout contributed to early module calculations, and interpretation of the resulting modules that were found.

Jørgen Rasmussen, David Ridout, and Eric Ragoucy have provided feedback on the manuscript for clarity and accuracy.

Statement of parts of the thesis submitted to qualify for the award of another degree

The beginning of Chapter 2 contains background theory and instructional examples which were previously submitted in [4] for the degree of M.Phil at The University of Queensland. We do not report these results as new in this thesis, and include them only as background necessary to understand further work based upon them.

Research involving human or animal subjects

No animal or human subjects were involved in this research.

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Contents

Chapter 1

Introduction to conformal field theory

We begin by introducing the reader to fundamental concepts in conformal field theory which will be relevant for the thesis. We note however, that this introduction is not exhaustive. Moreover, the introduction presents material in a way that best relates to the results featured later on. We make this remark because of the many different presentations of conformal field theory. We will begin by introducing the Virasoro algebra, the algebra which generates infinitesimal conformal symmetry, along with a discussion of highest-weight modules. This is followed by a discussion of the fields of a conformal field theory, and the state-field correspondence. Following this we introduce the Virasoro unitary minimal models, and rational conformal field theory. We briefly discuss the important example of conformal field theories given by affine Lie algebras, and introduce the coset construction of conformal symmetry algebras. Finally, we conclude with a brief introduction to the extended symmetry algebras known as W-algebras. Throughout the introduction we draw primarily on the sources [5–9]. All of these texts cover almost all of the included information in the chapter. We remark here that our introduction in this section is focused primarily on important results, and not on the derivation of those results. The derivations of these results can be found in the referenced papers, as well as in the texts mentioned earlier.

1.1 Conformal symmetry in two dimensions

For physical quantum systems, the symmetries of the system give rise to conserved currents. The modes of those currents form a Lie algebra, encoding the infinitesimal symmetries of the system. The state spaces of the quantum system are then representations of the underlying Lie algebra.

The study of conformal field theory is the study of field theories (not necessarily quantum, but we will focus on quantum systems) for which there is conformal symmetry. A surprising property of such systems, discovered in [10], was that when restricted to two dimensions, these quantum field theories have an infinite number of conserved quantities. Correspondingly, the Lie algebra of symmetries is infinite dimensional. Moreover, having an infinite number of conserved quantities implies that the corresponding conformal field theories are likely to be exactly solvable. That is, the spectrum of states for such theories can be completely deduced using the action of the algebra.

The infinite-dimensional Lie algebra which generates the infinitesimal conformal symmetry is known as the Virasoro algebra. The Virasoro algebra, denoted $\mathfrak{Vir}$, is a Lie algebra over the complex field (we will always consider algebras over the field $\mathbb{C}$), spanned by elements $\{L_n, c \mid n \in \mathbb{Z}\}$, with defining commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{1.1}$$

The element c is central in the Lie algebra, and is referred to as the central charge.

1.2 The Virasoro algebra and its representation theory

Given that the Virasoro algebra encodes the symmetries of the physical systems we seek to understand, the study of representations of the Virasoro algebra is fundamental to conformal field theory. We begin by introducing the so-called Verma modules. The Verma modules are highest-weight representations of the algebra.

The standard construction of the Verma modules, labelled by $V_{h,c}$ for $h, c \in \mathbb{C}$, begins by observing that the algebra has a triangular decomposition

$$\mathfrak{Vir} = \mathfrak{Vir}_- \oplus \mathfrak{Vir}_0 \oplus \mathfrak{Vir}_+, \tag{1.2}$$

where $\mathfrak{Vir}_+ = \{L_n \mid n > 0\}$, $\mathfrak{Vir}_- = \{L_n \mid n < 0\}$ and $\mathfrak{Vir}_0 = \{L_0,\ c\}$, are the positive-, negative-, and zero-mode subalgebras respectively.

We begin with a one-dimensional representation $\mathbb{C}_{h,c}$ of the Borel subalgebra $\mathfrak{b} = \mathfrak{Vir}_+ \oplus \mathfrak{Vir}_0$, labelled by $h, c \in \mathbb{C}$, the conformal weight and central charge respectively.

The representation $\mathbb{C}_{h,c}$ is spanned by a vector $|v\rangle$ such that

$$L_0 |v\rangle = h |v\rangle \quad \text{and} \quad c |v\rangle = c |v\rangle, \quad L_n |v\rangle = 0,\ n > 0. \tag{1.3}$$

We mention that the nomenclature "central charge" does not distinguish between the element of the algebra, and the value that the element takes on the highest-weight vector, however the distinction should be clear from context.

One can then induce from this one dimensional representation to a representation of the full Virasoro algebra by free left action, that is

$$V_{h,c} = U(\mathfrak{Vir}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{h,c}, \tag{1.4}$$

where $U(\cdot)$ denotes the corresponding universal enveloping algebra. Thus, the Verma modules of the Virasoro algebra are highest-weight modules, with highest-weight vector $|v\rangle$. The Verma module has a basis given by

$$\{|v\rangle\} \cup \{L_{-n_j} \dots L_{-n_1} |v\rangle \mid 1 \leq n_1 \leq \dots \leq n_j \in \mathbb{Z}\}. \tag{1.5}$$

The Verma modules are graded vector spaces, where the grading is with respect to eigenspaces of the operator L_0. The Verma module $V_{h,c}$ decomposes into weight spaces

$$V_{h,c} = \bigoplus_{n=0}^{\infty} M_{h+n,c}, \tag{1.6}$$

where $M_{h+n,c}$ is the weight space with L_0 eigenvalue $h+n$, and c eigenvalue c. The action of L_0 on an element of the basis is given by

$$L_0 L_{-n_j} \dots L_{-n_1} |v\rangle = \left(h + \sum_{i=1}^{j} n_i\right) L_{-n_1} \dots L_{-n_j} |v\rangle . \tag{1.7}$$

Each space $M_{h+n,c}$ is given by the span of states such that their mode indices add to $-n$. This leads to the observation that the states in a given weight space are in correspondence with partitions of the integer n, where the mode indices of the generators are the negative of the parts of n. The value of n is commonly referred to as the "level", so that a vector $|w\rangle \in M_{h+n,c}$ is a vector at level n.

We can associate a generating function, known as the character, to a Verma module. The character of a Verma module $\chi_{h,c}(q)$ is a formal series in the variable q, defined by

$$\chi_{h,c}(q) = \mathrm{Tr}_{V_{h,c}} \left(q^{L_0 - \frac{c}{24}}\right) = \sum_{n=0}^{\infty} \dim(M_{h+n}) q^{n+h-c/24}. \tag{1.8}$$

The trace is performed over all states in the module $V_{h,c}$. The inclusion of the overall factor $q^{-c/24}$ is related to the action of the modular group on the character functions. We will not discuss this here, but there is a thorough description in [5]. As we have already observed, the dimensions of the weight spaces are given by partitions of n, and as such there is a natural link between the function $\chi_{h,c}(q)$ and the generating function for partitions

$$\frac{1}{\phi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n) q^n, \tag{1.9}$$

where $p(n)$ is the number of partitions of n. We can write the Virasoro Verma module character function as

$$\chi_{h,c}(q) = \frac{q^{h-c/24}}{\phi(q)} = q^{h-c/24} \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{1.10}$$

1.3 The state-field correspondence and the operator product expansion

So far, we have focused on the description of modules over the Virasoro algebra, ignoring the field content of conformal field theories. Here we will introduce the fields of our theories, but we will require some preliminaries.

It is customary to work with complex co-ordinates $z, \bar{z}$, which are Wick rotations of the usual spacetime co-ordinates in two dimensions (t,x), that is, $z = x + it$, $\bar{z} = x - it$. We remark that generally the full symmetry algebra of a given two dimensional quantum system with conformal symmetry is given by two copies of the Virasoro algebra. We will proceed only considering the case when these two copies decouple, giving separate sectors. In the so-called holomorphic sector, the Virasoro algebra acts as holomorphic transformations with respect to z, similarly in anti-holomorphic sector, the second copy of the algebra acts as anti-holomorphic transformations with respect to $\bar{z}$. Focusing on a single sector of the decoupled picture is known as chiral conformal field theory. The two algebras need not decouple, however it greatly simplifies the exposition, allowing us to focus only on the holomorphic sector. A useful text for understanding CFTs when these two sectors do not decouple is [11].

In these rotated co-ordinates we think of positions x as lying on concentric circles centred at the origin, where time t determines the radius of the circle. As such, the co-ordinate $z = 0$ corresponds to $t \to -\infty$, and time-ordering of events becomes radial ordering.

A quantised field $\phi(z)$ is an operator valued function in the co-ordinate z. We will consider it from the perspective of its series expansion

$$\phi(z) = \sum_{n \in \mathbb{Z}-h} \phi_n z^{-n-h}, \tag{1.11}$$

where the expansion modes are operators $\phi_n \in \mathrm{End}(V)$, and where V is a vector space often referred to as the vacuum module. The quantity h is referred to as the conformal dimension or conformal weight of a field, and is related to the conformal transformation properties of the field. The operators in the formal series are referred to as the modes of the field, and they form the modes of the underlying Lie algebra of symmetries.

For the Virasoro algebra, there is a generating field $T(z)$, often referred to as the stress-energy tensor due to its applications in physics, the mode expansion for which is given by

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \tag{1.12}$$

where the L_n are the modes of the Virasoro algebra. For the Virasoro field, we require that the vacuum vector $|0\rangle \in V$ (the lowest energy state in the vacuum module) satisfies $L_n |0\rangle = 0$ for $n \geq -1$. This ensures that the series expansion of the field is well behaved as $z \to 0$, as is demonstrated below.

We can verify the action of the field on the vacuum state

$$T(z)|0\rangle = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |0\rangle, \tag{1.13}$$

and see that if we take the limit $z \to 0$, that is we consider the asymptotic in-state, we have

$$\lim_{z \to 0} T(z)|0\rangle = \lim_{z \to 0} \sum_{n \in \mathbb{Z}} z^{-n-2} L_n |0\rangle = \lim_{z \to 0} \sum_{n \geq -2} z^{-n-2} L_n |0\rangle. \tag{1.14}$$

With the exception of $n = -2$ the remaining terms diverge in the limit $z \to 0$. The field being well defined at all times motivates the symmetries of the vacuum state. We have that for the above to be well defined, we set $L_n |0\rangle = 0$ for $n \geq -1$. Then the previous limit calculation becomes

$$\lim_{z \to 0} T(z) |0\rangle = L_{-2} |0\rangle . \tag{1.15}$$

The physical constraints on the vacuum are well motivated from the perspective of representation theory, which will be demonstrated in Section 1.4. Moreover, this technique of applying the limit as $z \to 0$ gives a way of associating a field to a vector in the vacuum module. We will develop this notion further later in this section.

There is an identity field $\mathbb{I}(z)$ with mode expansion

$$\mathbb{I}(z) = \sum_{n \in Z} \delta_{n,0} z^{-n}, \tag{1.16}$$

where $\delta_{0,0} = \mathbb{1}_V$, the identity endomorphism on V.

We can take derivatives of the fields with respect to the co-ordinate, giving

$$\partial_z \phi(z) = \partial_z \sum_{n \in \mathbb{Z}-h} \phi_n z^{-n-h} = \sum_{n \in \mathbb{Z}-h} \phi_n (-n-h) z^{-n-h-1}. \tag{1.17}$$

There is a product on the fields of a theory, known as the operator product expansion (OPE). Generally in a quantum field theory, we have the notion of a product of fields $\phi^i(z)\phi^j(w)$, whereby we can expand them as an asymptotic series in the distance of their arguments, as in

$$\phi^i(z)\phi^j(w) \sim \sum_k C_k^{ij}(z-w)\phi^k(w), \tag{1.18}$$

where the $C_k^{ij}(z-w)$ are structure constants and singular functions of the difference of co-ordinates, and the $\phi^k(w)$ are a set of local operators on the vacuum. In the case of conformal field theory, this expansion converges (see [7, 10] for a discussion of this property), and indeed we are able to write

$$\phi^i(z)\phi^j(w) = \sum_{k=-\infty}^{n} C_k^{ij} \frac{\phi^k(w)}{(z-w)^k}, \tag{1.19}$$

for a finite integer n, and where $C_k^{ij} \in \mathbb{C}$ are the structure constants of the algebra.

There is a distinguished field appearing in this expansion, namely $\phi^{k=0}(w) = \left(\phi^i \phi^j\right)(w)$, the normally-ordered product of $\phi^i(z)$ and $\phi^j(w)$.

The notion of normally-ordered product can be elevated to a bilinear operation on fields, taking in two fields, and returning a field as its output. The corresponding mode expansion is given by

$$\left(\phi^i \phi^j\right)(z) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}-h_i-h_j} : \phi^i_m \phi^j_{n-m} : z^{-n-h_i-h_j}, \tag{1.20}$$

where $h_i,\ h_j$, are the conformal dimensions of ϕ^i, and ϕ^j respectively, and

$$:\phi^i_m\phi^j_{n-m}:=\begin{cases}\phi^i_m\phi^j_{n-m}, & n-m>-h_j,\\ \phi^j_{n-m}\phi^i_m, & \text{otherwise.}\end{cases} \tag{1.21}$$

One may recognise $:\cdots:$ as the usual mode normal ordering prescription from quantum field theory. We remark that the cut-off point (in this case $-h_j$) for the normal ordering of modes may be any finite value. Any choice of cut-off only differs from any other choice by a finite number of terms.

The normally-ordered product $\left(\phi^i\phi^j\right)$ only generally commutes when $j=i$, and normally-ordered products are not associative. As such, we use the convention that the normally-ordered product of three or more fields is given by right nested, pairwise operations, that is,

$$\left(\phi^i\dots\phi^j\phi^k\right)(z)=\left(\phi^i\left(\dots\left(\phi^j\phi^k\right)\right)\right)(z). \tag{1.22}$$

Returning to the Virasoro algebra, the operator product expansion between the field $T(z)$ and $T(w)$ is given by

$$T(z)T(w)\sim\frac{c}{2}\frac{\mathbb{I}(w)}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{(z-w)}, \tag{1.23}$$

where $c\in\mathbb{C}$ is the central charge, $\mathbb{I}(w)$ is the identity field, and we have dropped the non-singular terms which is signified by the use of $\sim$ rather than equality. It is common to only consider the singular terms in the OPE as they contain all structural information about the resulting algebra of fields.

The space of fields forms a unital algebra (there is always an identity field) under the operator product expansion. A field theory is said to be generated by a set of fields $\{\phi^i\}$ under the OPE if the full space of fields is given by the generating fields, their derivatives, and normally ordered products of the fields and their derivatives. The Virasoro algebra as an algebra of fields is generated by the field $T(z)$.

To determine the OPE between derivatives of fields, we have that

$$\phi^i(z)\partial_w\phi^j(w)\sim\partial_w\left[\sum_{k=1}^{n}C^{ij}_k\frac{\phi^k(w)}{(z-w)^k}\right], \tag{1.24}$$

and similarly for $\partial_z\phi^i(z)\phi^j(w)$, that is, one simply takes the derivative of the fields appearing on the right-hand side of the OPE.

We can also exchange the order of fields in the OPE. By definition, we have that

$$\phi^j(w)\phi^i(z)\sim\sum_{k=1}^{n}C^{ji}_k\frac{\phi^k(z)}{(w-z)^k}. \tag{1.25}$$

We remark that this assumes the fields obey bosonic statistics. In general, fermionic fields are also allowed, whereby exchange of fields introduces a minus sign.

To compute the same OPE with a change of co-ordinates, i.e. $\phi^j(z)\phi^i(w)$, we consider that we can expand a field $\phi(z)$ about a point, as in

$$\phi(z)=\sum_{n=0}^{\infty}\partial^n\phi(w)\,(z-w)^n. \tag{1.26}$$

To determine the OPE between a field and a normally-ordered product, we make use of the so-called point splitting technique, whereby we split the normally-ordered product of two fields as

$$\phi^i(z)\left(\phi^j\phi^k\right)(w) = \oint_w \frac{dx}{2\pi i}\frac{1}{x-w}\left[\overline{\phi^i(z)\phi^j}(x)\phi^k(w) + \phi^j(x)\overline{\phi^i(z)\phi^k}(w)\right], \tag{1.27}$$

where the over-lined contractions denote the OPE of the involved fields. This technique will be demonstrated explicitly when considering the so-called Sugawara construction in Section 1.6.

The conformal transformation properties of fields in a conformal field theory appear via the OPE. A field $\phi(z)$ is called a scaling field if its OPE with the Virasoro field $T(z)$ is given by

$$T(z)\phi(w) \sim \ldots + \frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{1.28}$$

where h_ϕ is the conformal dimension of the field $\phi(z)$. The ellipsis represents possible terms appearing at poles of order three or greater.

Such a scaling field is called quasi-primary if the third-order pole of the OPE is zero. The definition of quasi-primary does not place any restrictions on the poles of order greater than three. As an example, we can see from

$$T(z)T(w) \sim \frac{c}{2}\frac{\mathbb{I}(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}, \tag{1.29}$$

that the Virasoro field is itself a quasi-primary field.

Finally, we say that a field $\phi(w)$ is primary if it is a scaling field, and all poles of order three or greater vanish, that is, we have exactly

$$T(z)\phi(w) \sim \frac{h_\phi \phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}. \tag{1.30}$$

We remark that when $c = 0$ the Virasoro field is a primary field.

We return to the discussion of associating fields to particular states in the vacuum module. Earlier we showed that by taking the $z \to 0$ limit of the action of the Virasoro state on the vacuum vector, we can associate the state $L_{-2}|0\rangle$ to the field $T(z)$. We can repeat this calculation for a general field $\phi^i(z)$, where the associated state is

$$\lim_{z\to 0}\phi^i(z)|0\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}-h_i}\phi^i_n z^{-n-h_i}|0\rangle = \sum_{n\le -h_i}\phi^i_n z^{-n-h_i}|0\rangle = \phi^i_{-h_i}|0\rangle. \tag{1.31}$$

In this case, to have a well defined in state at our equivalent of time $t = 0$, we require the modes ϕ^i_n for $n > -h_i$ annihilate the vacuum vector $|0\rangle$.

For derivatives of fields, we can continue in a similar fashion. We give the case where $\phi(z) = T(z)$, the Virasoro field, as an explicit example. The state associated to the n^{th} derivative of $T(z)$ is given by

$$\lim_{z\to 0}\partial^n T(z)|0\rangle = \lim_{z\to 0}\sum_{m\in\mathbb{Z}}\partial^n z^{-m-2}L_m|0\rangle = n!L_{-2-n}|0\rangle. \tag{1.32}$$

Furthermore, we can also perform the procedure for normally ordered products. Consider the action of the field $(TT)(z)$, we have

$$\lim_{z\to 0}(TT)(z)|0\rangle = \lim_{z\to 0}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-n-4} :L_m L_{n-m}: |0\rangle = L_{-2}L_{-2}|0\rangle . \tag{1.33}$$

Combining these results we determine the general formula for any field in the Virasoro field algebra, which is

$$\lim_{z\to 0}\frac{1}{(n_1-2)!\dots(n_k-2)!}\left(\partial^{n_k-2}T\dots\partial^{n_1-2}\right)(z)|0\rangle = L_{-n_k}\dots L_{-n_1}|0\rangle , \quad n_k\geq\dots\geq n_1\geq 2. \tag{1.34}$$

We also have that for the identity field

$$\lim_{z\to 0}\mathbb{I}(z)|0\rangle = \lim_{z\to 0}\sum_{n\in\mathbb{Z}}\delta_{n,0}z^{-n}|0\rangle = \delta_{0,0}|0\rangle = \mathbb{1}_V|0\rangle = |0\rangle , \tag{1.35}$$

that is, the identity field is the field associated to the vacuum state itself. As such, we can identify any state in the basis for the vacuum module with a corresponding field, and this extends by linearity to all fields and states. This relation is known as the state-field correspondence.

It is then natural to ask what is the state associated to a primary field $\phi(z)$? We had earlier that for a general field of conformal dimension h, the corresponding vector was given by

$$\lim_{z\to 0}\phi(z)|0\rangle = \phi_{-h}|0\rangle . \tag{1.36}$$

In fact, for $\phi(z)$ a primary field, the state $|v\rangle = \phi_{-h}|0\rangle$ defines a Virasoro highest-weight vector, of conformal (L_0) weight h. The vanishing of poles of order 2 or higher in the OPE for a primary field implies that the action of the Virasoro elements L_n vanish on the corresponding state for $n\geq 1$, which is exactly the highest-weight condition.

The fields associated to the states in such a module are then normally ordered products of the Virasoro field $T(z)$ with a single $\phi(z)$ field, and derivatives thereof. These fields are called the descendant fields of $\phi(z)$. The collection of $\phi(z)$ and its descendant fields generate the so-called conformal family of $\phi(z)$, denoted $[\phi]$.

From the calculational standpoint of quantum field theory, one is primarily interested in evaluating correlation functions, which are often understood as "overlap" functions, or transition probabilities from one state to another, written in the form

$$\langle\phi^1(z_1)\dots\phi^n(z_n)\rangle \tag{1.37}$$

for some set of fields $\phi^i(z)$ where the fields are ordered radially outwards, $|z_1|\geq\dots\geq|z_n|$ (the equivalent of time ordering in rotated co-ordinates), and $\langle\dots\rangle$ indicates that this is an evaluation of a vacuum module inner product. We will describe this inner product in Section 1.4.

A correlation function is often referred to by its "point" number, describing the number of fields which have been inserted. For example, a general two-point function for two quasi-primary fields $\phi^i(z)$ is

$$\langle\phi^1(z)\phi^2(w)\rangle = f(z,w), \tag{1.38}$$

where $f(z,w)$ is an as yet undetermined meromorphic function with possible poles at $z = w$. This expression is quite general, however, it is constrained by the action of the $\mathfrak{sl}(2)$ subalgebra spanned by $\{L_{-1}, L_0, L_1\} \in \mathfrak{Vir}$, which generate global conformal symmetries of the physical system.

Indeed, for such a two-point function, one can show that global conformal symmetry restricts the form of $f(z,w)$ to

$$\langle \phi^1(z)\phi^2(w)\rangle = f(z-w) = \frac{f^{12}\delta_{h_1,h_2}}{(z-w)^{h_1+h_2}}, \tag{1.39}$$

where f^{12} is a structure constant, and δ_{h_1,h_2} is the usual Kronecker delta, implying that the two-point function is only non-zero if the fields have the same conformal weight.

As an example, consider $T(z)$ which is a quasi-primary field, we have that

$$\langle T(z)T(w)\rangle = \left\langle \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right\rangle = \frac{c}{2}\frac{1}{(z-w)^4}. \tag{1.40}$$

The terms with single fields go to zero. This is a consequence of weight space orthogonality of the inner product discussed in Section 1.4.

Furthermore, for three-point functions of quasi-primary fields, conformal symmetry restricts the allowed functions to

$$\langle \phi^1(z_1)\phi^2(z_2)\phi^3(z_3)\rangle = \frac{C^{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_1+h_3-h_2}}. \tag{1.41}$$

where the C^{ijk} are the structure constants of the three-point functions. In fact, structure constants C^{ij}_k appearing in the OPE between quasi-primary fields $\phi^i(z)\phi^j(w)$, are given by $C^{ij}_k = \sum_\ell C^{ij\ell} f_{\ell k}$, where $f_{\ell k}$ is the inverse form of $f^{\ell k}$. Hence, the structure constants of the two- and three-point functions describe which fields ϕ^k appear in the OPE.

The important fact about these relations is that the two- and three-point functions between primary fields can be used with the knowledge that we have developed thus far, to determine the two- and three- point functions for any of the fields in the conformal family $[\phi]$. Higher-order correlation functions, such as four-point functions, can be re-expressed in terms of the three-point functions using the associativity condition of the OPE. This leads to an infinite hierarchy of constraint equations for the structure constants C^{ijk}.

In principle, this allows us to understand a conformal field theory, at a particular value of the central charge, as a collection of primary fields (highest-weight representations of $\mathfrak{Vir}$), and solving a conformal field theory is equivalent to determining the structure constants of the three-point functions. However, this is difficult in practise, as one may require very many (sometimes infinitely many) primary fields to describe a particular conformal field theory. Attempting to solve a theory in this way is known as the conformal bootstrap, whereby one uses only the conformal symmetry constraints to completely determine the theory.

A more complete description of these techniques can be found in any of the texts [5, 7, 8].

1.4 Unitarity and the Kac determinant

On the Verma modules of the Virasoro algebra $V_{h,c}$, with highest-weight vector $|v\rangle$, we can introduce a unique symmetric bilinear form called the Shapovalov form [12]

$$(\cdot\, , \cdot) : V_{h,c} \times V_{h,c} \to \mathbb{C}, \tag{1.42}$$

such that the highest-weight vector is normalised $(|v\rangle, |v\rangle) = 1$, and the form is contravariant with respect to the adjoint, so that for $x \in \mathfrak{Vir}$, and $|u\rangle,\ |w\rangle \in V_{h,c}$,

$$(x^\dagger |u\rangle, |w\rangle) = (|u\rangle, x|w\rangle). \tag{1.43}$$

The adjoint on the Virasoro algebra is given by

$$L_{-n}^\dagger = L_n, \quad c^\dagger = c. \tag{1.44}$$

We will make use of the so-called “bra-ket” notation, where we will identify $(|u\rangle, |w\rangle) := \langle u|w\rangle$, such that the highest-weight vector satisfies $\langle v|v\rangle = 1$, and that for $L_n \in \mathfrak{Vir}$ we have

$$\big(L_{-n}|v\rangle, L_{-n}|v\rangle\big) = \big(|v\rangle, L_n L_{-n}|v\rangle\big) = \langle v|L_n L_{-n}|v\rangle. \tag{1.45}$$

As the Verma modules are by definition cyclic, they are indecomposable. However, Verma modules over the Virasoro algebra are often reducible. A vector $|w\rangle \in V_{h,c}$ which satisfies the highest-weight condition

$$L_n |w\rangle = 0, \quad \forall n > 0, \tag{1.46}$$

is called a singular vector. We will label vectors which are singular but do not generate the module as proper singular vectors. A singular vector of a Verma module generates a Verma submodule $U_{h',c} \subseteq V_{h,c}$, and a proper singular vector generates a proper submodule $U_{h',c} \subset V_{h,c}$. A proper submodule is null with respect to the form introduced in (1.42). This is straightforward to demonstrate.

Suppose $|w\rangle \in V_{h,c}$ is a proper singular vector, and consider a general basis vector $L_{-m_n} \ldots L_{-m_1}|v\rangle \in V_{h,c}$. Taking the inner product, we have

$$\langle v| L_{m_1} \ldots L_{m_n} |w\rangle = \langle v| 0 = 0. \tag{1.47}$$

Hence, a proper singular vector generates a null submodule with respect to the Shapovalov form. A Verma module has a unique maximal submodule, labelled $J_{h,c}$ such that the quotient $L_{h,c} = V_{h,c}/J_{h,c}$ is the unique irreducible quotient of $V_{h,c}$. Indeed, all highest-weight representations are formed by quotients of Verma modules [13].

As an example of proper singular vectors, we consider the one-parameter family of Verma modules $V_{0,c}$. We denote the highest-weight vectors of these modules by $|0\rangle$, such that $c|0\rangle = c|0\rangle$ for $|0\rangle \in V_{0,c}$.

We claim that in all modules $V_{0,c}$ in the one-parameter family, the vector $L_{-1}|0\rangle$ is singular. To demonstrate, we apply the raising generator L_1

$$L_1 L_{-1}|0\rangle = L_{-1}L_1|0\rangle + [L_1, L_{-1}]|0\rangle = 2L_0|0\rangle = 0. \tag{1.48}$$

We note that it is sufficient to check that the action of L_1 and L_2 on a singular vector gives zero, as these modes generate the positive mode algebra under the Lie bracket. In the above case, the action of L_2 on the singular vector is simply zero by weight-space considerations. The modules $V_{0,c}$ are graded by L_0 eigenvalue, and $L_2(L_{-1}|0\rangle)$ maps into a trivial weight space. Thus, all Verma modules with $h=0$ have a proper submodule generated by the vector $L_{-1}|0\rangle$. We remark that the state-field interpretation of this result is that the derivative of the identity field is zero. Moreover, this demonstrates the requirement that $L_{-1}|0\rangle = 0$ on the vacuum module from the perspective of the representation theory.

An important tool in classifying the representations of the Virasoro algebra is the Kac determinant formula [14, 15]. Let us denote the Gram matrix of inner products at level n of the Verma module $V_{h,c}$ by $G^n_{h,c}$. The Kac determinant formula states that the determinant of the Gram matrix is given by the formula

$$\det(G^n_{h,c}) = K \prod_{\substack{r,s\geq 1,\\ rs\leq n}} (h - h_{r,s}(c))^{p(n-rs)}, \tag{1.49}$$

where K is an overall constant, $p(n)$ is the number of partitions of n, and we have the formula

$$h_{r,s}(t) = \frac{t}{4}\left(r^2-1\right) + \frac{1}{4t}(s^2-1) - \frac{1}{2}(rs-1), \tag{1.50}$$

where $t \in \mathbb{C}^\times$ parametrises the value of the central charge as

$$c = 13 - 6\left(t + \frac{1}{t}\right). \tag{1.51}$$

As we have already shown, the Shapovalov form vanishes on the proper submodules of $V_{h,c}$. This implies that the Kac determinant at level n vanishes if there is a proper singular vector at level n in $V_{h,c}$. As such, if $\det(G^n_{h,c}) \neq 0$ for all n, then the representation $V_{h,c}$ is irreducible. We say that a representation is unitary if the Shapovalov form is positive-definite on that representation, equivalently that we have $\det(G^n_{h,c}) > 0$ for all n.

The unitary representations are of particular interest as they are of highest importance to physical applications. The unitary representations contain no non-zero states with zero norm, and moreover, no states with negative norm. From a traditional physical perspective, this ensures that inner products of states have a meaningful probabilistic interpretation. However, there has been significant research into non-unitary conformal field theories, and we will return to this point later in the thesis.

Given their importance, it is natural to ask, for what values of h, c does one obtain unitary representations? We can gain some insight by considering the inner product at level one. In the Verma module $V_{h,c}$, the level one weight space is spanned by the vector $L_{-1}|v\rangle$. The norm of this state is given by

$$\langle v| L_1 L_{-1}|v\rangle = \langle v|[L_1, L_{-1}]|v\rangle = \langle v| 2L_0|v\rangle = 2h, \tag{1.52}$$

hence we require $h \geq 0$ for unitarity. We allow $h = 0$, as we are free to quotient by the singular vector $L_{-1}|0\rangle$, in which case, the resulting module may be unitary.

More generally, we can consider the following inner product at level n

$$\langle v| L_n L_{-n} |v\rangle = \langle v| [L_n, L_{-n}] |v\rangle = \langle v| 2nL_0 + \frac{c}{12} n(n^2-1) |v\rangle . \tag{1.53}$$

Here we require $c \geq 0$ for unitarity, otherwise the second term causes the inner product to be negative when we take n sufficiently large.

One can show the following fact through analysis of the determinant formula [13]. The Verma module $V_{h,c}$ is irreducible, i.e. $V_{h,c} = L_{h,c}$, for $c > 1,\ h > 0$. Furthermore, the irreducible representation $L_{h,c}$ is unitary for $c \geq 1$ and $h \geq 0$. We also have that $V_{h,1} = L_{h,1}$ if and only if $h \neq \frac{m^2}{4}$ for $m \in \mathbb{Z}$, and $V_{h,0} = L_{h,0}$ if and only if $h \neq \frac{m^2-1}{24}$ for $m \in \mathbb{Z}$.

The irreducible module $L_{0,0}$ is the trivial module, and is also unitary. Using the above two calculations, we see that in $V_{0,0}$, we would have to quotient out the submodules generated by $L_{-1}|0\rangle$ and $L_{-2}|0\rangle$. As L_{-1} and L_{-2} generate $\mathfrak{Vir}_-$, we arrive at the trivial module.

This leaves the question of unitarity in the region $0 \leq c < 1,\ h > 0$. In this region, the authors of [16,17] showed that it is possible to have unitary representations, but only at a discrete set of points in the parameter space. The allowed values of the central charge are

$$c(m) = 1 - \frac{6}{m(m+1)}, \quad m \geq 2. \tag{1.54}$$

For this parametrisation of $c(m)$, the formula for corresponding highest weights becomes

$$h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \quad 1 \leq r \leq m-1, \quad 1 \leq s \leq m. \tag{1.55}$$

The key point is that the Verma modules parametrised by the above are reducible (when $h = h_{r,s}$ the determinant vanishes), implying that although these constraints are necessary, one must quotient out the null submodules to obtain the unitary representations. The Verma modules $V_{h,c}$ with $c(m)$ given by (1.54), and highest weight given by (1.55), are labelled by the pair r,s, and have a singular vector at level rs, the product of the labels. We remark that this may not be the only singular vector in such a module. The corresponding irreducible modules, where one has quotiented by the maximal proper submodule, are the unitary representations.

It was proven in [18], that the modules parametrised above indeed all lead to unitary representations in the region $0 \leq c < 1,\ h > 0$. Their proof involved the so-called coset construction discussed in Section 1.7.

1.5 The unitary Virasoro minimal models and fusion

A classic example which demonstrates how these unitary representations are obtained is the construction of the level two singular vector Verma modules $V_{h,c}$ for $0 < c < 1$. Consider the vector

$$(L_{-2}+\alpha L_{-1}L_{-1})\,|v\rangle\,, \tag{1.56}$$

where $|v\rangle$ is the highest-weight vector. Supposing the vector is singular, we want to determine the corresponding constraints on the values of h, c, α.

We compute

$$L_1\,(L_{-2}+\alpha L_{-1}L_{-1})\,|v\rangle = 3L_{-1}\,|v\rangle+\alpha(2L_0L_{-1}+2L_{-1}L_0)\,|v\rangle = (3+2\alpha(2h+1))L_{-1}\,|v\rangle = 0, \tag{1.57}$$

which implies that $\alpha = \frac{-3}{2(2h+1)}$. Continuing for L_2, we have

$$L_2\,(L_{-2}+\alpha L_{-1}L_{-1})\,|v\rangle = \left(4L_0+\frac{c}{2}\right)|v\rangle+3\alpha L_1L_{-1}\,|v\rangle = \left(h(4+6\alpha)+\frac{c}{2}\right)|v\rangle = 0, \tag{1.58}$$

which implies that $c = \frac{2h(5-8h)}{2h+1}$. So for all such values of c, there is a singular vector at level two, given by (1.56) with the particular value of $\alpha(h)$. One can then verify that these formulas match (1.54) and (1.55) for $r=2,\ s=1$.

Under the state field correspondence, a vector of the form (1.56) corresponds to a descendant field in the conformal family $[\phi_{2,1}]$. When that vector is singular, it implies that the corresponding field is null. Correlation functions involving null fields are zero, in direct correspondence with the fact that proper singular vectors generate null submodules with respect to the form. This then leads to constraints on the two- and three-point functions of the theory. The null field condition implies that an n-point function satisfies

$$\left[\sum_{i=1}^{n}\left(\frac{h_i}{(z_i-z)^2}-\frac{\partial}{z_i-z}\right)-\frac{3\partial^2}{2(2h+1)}\right]\langle\phi(z)\phi^1(z_1)\ldots\phi^n(z_n)\rangle = 0. \tag{1.59}$$

This constraint is trivially satisfied for the two-point functions. However, for the three-point functions $\langle\phi^1(z)\phi^2(z)\phi^3(z)\rangle$, the differential equation leads to a constraint on the conformal weights of the fields, given by

$$2(2h_1+1)(h_1+2h_3-h_2) = 3(h_1-h_2+h_3)(h_1-h_2+h_3+1). \tag{1.60}$$

Recalling that the two- and three-point functions determine the structure constants of the OPE, this equation constrains the conformal weight of the allowed primary fields which can appear on the right hand side of products between primary fields. Moreover, since the conformal family of a primary field $[\phi]$ is equivalent to a Virasoro highest-weight module, the OPE leads to a notion of multiplying Virasoro modules together, with correlation functions describing how that product decomposes into Virasoro modules. This is known as the fusion product of modules.

For a field $[\phi_{2,1}]$ in a unitary minimal model we consider fusing with another general field $[\phi_{r,s}]$ of weight $h_{r,s}$ in the same minimal model. The above constraints imply that the fusion product can only possibly produce the following fields

$$[\phi_{2,1}]\times[\phi_{r,s}] = C_{(2,1),(r,s)}^{(r+1,s)}[\phi_{r+1,s}]+C_{(2,1),(r,s)}^{(r-1,s)}[\phi_{r-1,s}]. \tag{1.61}$$

We note that the corresponding structure constants C_{ij}^k may still be zero, however, all others can be ruled out using the null field constraints.

For two general unitary representations parametrised by (1.54) and (1.55), we have the following multiplication rules

$$[\phi_{r_1,s_1}] \times [\phi_{r_2,s_2}] = \sum_{\substack{k=1+|r_1-r_2| \\ k+p_1+p_2 \text{ odd}}}^{r_1+r_2-1} \sum_{\substack{\ell=1+|s_1-s_2| \\ \ell+s_1+s_2 \text{ odd}}}^{s_1+s_2-1} C_{(r_1,s_1),(r_2,s_2)}^{(k,\ell)} [\phi_{k,\ell}], \tag{1.62}$$

commonly referred to as fusion rules. The fusion rules describe the multiplication properties of modules within a theory. We remark that usually when one considers "multiplication" of modules over an algebra, the natural choice is the tensor product. However, the tensor product of modules does not preserve the value of the central charge. Rather, for the resulting module, the central charge is the sum of the central charges of the factors. As the value of the central charge effectively characterises the theory, it should be preserved under the product operation.

Generally speaking, the fusion product gives rise to a unital, commutative, associative algebra, called a fusion algebra, which is an algebra of modules over a conformal symmetry algebra for a given value of the central charge. The product on the algebra, referred to as the fusion rules, is written more abstractly as

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}^k [\phi_k], \tag{1.63}$$

where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the module multiplicities. We have that $N_{ij}^k = 0$ if and only if the three-point structure constants $C^{ijk} = 0$, and for the Virasoro unitary minimal models, $N_{ij}^k \in \{0,1\}$. We note that there is ambiguity in the literature as to whether the N_{ij}^k are the fusion rules, or whether the equations (1.63) are the fusion rules. We will use the convention that products of the form (1.63) are referred to as the fusion rules.

The identity in the fusion ring is given by the module of the identity field. Commutativity implies the $N_{ij}^k = N_{ji}^k$. Associativity gives that

$$\sum_k N_{ij}^k N_{k\ell}^m = \sum_k N_{j\ell}^k N_{ki}^m. \tag{1.64}$$

We say that a fusion algebra is rational if it is finite-dimensional. The corresponding conformal field theory is rational if its fusion algebra is rational, that is, the conformal field theory is made up of a finite number of conformal families. We then say that the minimal model with central charge c given by (1.54) is the set of fields on which the fusion algebra closes. In the case of the unitary Virasoro minimal models, the fusion algebras are rational. However, the non-unitary $N = 2$ superconformal minimal models, and admissible level $\widehat{\mathfrak{sl}}(2)$ minimal models we will encounter in Chapter 5 will not be rational.

One can associate a Grothendieck ring of power series in a finite number of variables to a fusion algebra by associating the irreducible modules with their character functions. For indecomposable modules, we associate them to the sums of characters of each of their composition factors. In this ring, we write the product as

$$\chi_i(q) \times \chi_j(q) = \sum_k N_{ij}^k \chi_k(q). \tag{1.65}$$

We remark that computing the product on the Grothendieck ring, sometimes referred to as Grothendieck fusion or character fusion, is generally significantly easier than computing fusion of modules. A celebrated algorithm for computing fusion products of modules is the Nahm-Gaberdiel-Kausch algorithm (NGK) [19–21] (see [22, 23] for an introduction and overview). This algorithm, while extremely powerful, is also computationally difficult. However, one can use character fusion as a guide to understand how to interpret the output of the NGK algorithm.

1.6 Conformal field theories with affine Lie algebra symmetries

A particularly relevant example of a conformal field theory is the conformal field theory associated to an affine Lie algebra. These theories, known as Wess-Zumino-Witten (WZW) models, arise from considering non-linear sigma models where the target space is a Lie group [24–26]. Their corresponding symmetry algebras are affine Lie algebras, which we will explain in detail shortly (see [5] for a thorough introduction to affine Lie algebras, and WZW models). There is a well-developed geometric understanding of the WZW models, however, here we will only be interested in their related symmetry algebras.

One process for constructing an affine Lie algebra is to begin with a semisimple Lie algebra $\mathfrak{g}$, and form the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. The affine Lie algebra is given as the central extension of the loop algebra by the level k, and inclusion of a derivation d. As a vector space the affine Lie algebra $\widehat{\mathfrak{g}}$ is given by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \oplus \mathbb{C}d. \tag{1.66}$$

A basis for the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is given by the modes $\{j_n^a = j^a \otimes t^n \mid a = 1, \ldots, \dim(\mathfrak{g}),\ n \in \mathbb{Z}\}$, where $\{j^a\}$ is a basis for the underlying semisimple Lie algebra $\mathfrak{g}$. The Lie bracket on the algebra is given by

$$\left[j_m^a, j_n^b\right] = \sum_c f^{ab}{}_c\, j_{m+n}^c + \kappa^{ab} k m \delta_{m+n,0}, \quad \left[d, j_n^a\right] = n j_n^a, \tag{1.67}$$

where $f^{ab}{}_c$ are the structure constants on $\mathfrak{g}$, and κ^{ab} is the Killing form on $\mathfrak{g}$. The corresponding fields are the so-called currents

$$J^a(z) = \sum_{n \in \mathbb{Z}} j_n^a\, z^{-n-1}, \tag{1.68}$$

which are conformal weight 1 fields. The corresponding OPE relations for this algebra are given by

$$J^a(z) J^b(w) \sim \frac{\kappa^{ab} k}{(z-w)^2} + \sum_c \frac{f^{ab}{}_c J^c(z)}{z-w}. \tag{1.69}$$

It is not obvious that an affine Lie algebra encodes conformal symmetry. Demonstrating this fact involves a construction known as the Sugawara construction [27]. Here, for completeness, we give a standard presentation of the construction for affine Lie algebras. We remark though, that the algebra need not be an affine Lie algebra to admit a Sugawara construction, rather one can still perform a well defined Sugawara construction in the absence of an invertible Killing form. One of the most famous examples of such a non-semisimple construction occurs when considering the conformal operator of the free boson theory, corresponding to the algebra $\widehat{\mathfrak{gl}}(1) \equiv \widehat{H}$. The algebra $\widehat{H}$ has a well defined Sugawara construction, despite the underlying algebra being abelian. We will consider non-semisimple Sugawara constructions in more detail in Chapter 4.

The goal is to construct an action of the Virasoro algebra from the currents of the affine Lie algebra. That is, we attempt to construct a field $T(z)$ of conformal weight 2, from an algebra of conformal weight 1 fields. As such, the field $T(z)$ should be a linear combination of normally-ordered products of currents, or derivatives of currents.

We begin with an ansatz for the Virasoro field of the form

$$T(z) = \lambda \sum_{a,b} \kappa_{ab} \left(J^a J^b\right)(z), \tag{1.70}$$

for some $\lambda \in \mathbb{C}$. Such an ansatz is further motivated by the form of the classical stress energy tensor, which is a product of classical fields in a Poisson algebra, here we naturally begin by considering its quantised counterpart.

Our goal is to determine the value of λ by enforcing that the fields $J^a(z)$ should be conformal weight 1 primary fields under the action of $T(z)$, and that the operator $T(z)$ should generate a Virasoro algebra.

We begin by computing the OPE $J^a(z)\left(J^b J^c\right)(w)$ using the point-splitting technique

$$\begin{aligned} J^a(z)\left(J^b J^c\right)(w) &\sim \oint_w \frac{dx}{2\pi i} \frac{1}{x-w} \left[\left(\frac{\kappa^{ab} k}{(z-x)^2} + f^{ab}{}_d \frac{J^d(x)}{z-x} \right) J^c(w) + J^b(x) \left(\frac{\kappa^{ac} k}{(z-w)^2} + f^{ac}{}_d \frac{J^d(x)}{z-w} \right) \right] \\ &\sim \oint_w \frac{dx}{2\pi i} \frac{1}{x-w} \left[\frac{\kappa^{ab} k J^c(w)}{(z-x)^2} + f^{ab}{}_d \frac{1}{z-x} \left(\frac{\kappa^{dc} k}{(x-w)^2} + f^{dc}{}_e \frac{J^e(w)}{z-w} + \left(J^d J^c\right)(w) \right) \right. \\ &\quad \left. + \left(\frac{\kappa^{ac} k J^b(x)}{(z-w)^2} + f^{ac}{}_d \frac{J^b(x) J^d(w)}{z-w} \right) \right] \\ &\sim \oint_w \frac{dx}{2\pi i} \frac{1}{x-w} \left[\frac{\kappa^{ab} k J^c(w)}{(z-x)^2} + f^{ab}{}_d \frac{1}{z-x} \left(\frac{\kappa^{dc} k}{(x-w)^2} + f^{dc}{}_e \frac{J^e(w)}{z-w} + \left(J^d J^c\right)(w) \right) \right] \\ &\quad + \frac{\kappa^{ac} k J^b(w)}{(z-w)^2} + f^{ac}{}_d \frac{\left(J^b J^d\right)(w)}{z-w}. \end{aligned} \tag{1.71}$$

Note, we have made use of the summation convention to reduce the notation. Multiplying through by κ_{bc}, as in the ansatz for $T(z)$, we simplify the resulting expressions using the identities

$$\kappa_{bc} f^{ab}{}_d \kappa^{dc} = f^{ab}{}_b = 0, \quad \kappa_{bc} f^{ab}{}_d f^{dc}{}_e = 2h^\vee \delta^a_e, \quad f^{ab}_d \left(J^d J^c\right) + f^{ac}_d \left(J^b J^d\right) = 0, \tag{1.72}$$

and the standard relation $\kappa_{bc}\kappa^{ab} = \delta^a_c$, where $h^\vee$ is the dual Coxeter number of the underlying semisimple Lie algebra. Integrating leaves us with

$$\kappa_{bc} J^a(z)\left(J^b J^c\right)(w) \sim 2(k + h^\vee) \frac{J^a(w)}{(z-w)^2}. \tag{1.73}$$

We can exchange the order and the variables of the OPE, using operator exchange and Taylor series expansion, which gives

$$\kappa_{bc}\left(J^bJ^c\right)(z)J^a(w)\sim 2(k+h^\vee)\left[\frac{J^a(w)}{(z-w)^2}+\frac{\partial J^a(w)}{z-w}\right], \tag{1.74}$$

implying that $\lambda=\frac{1}{2(k+h^\vee)}$.

As such, we have determined that

$$T(z)=\frac{1}{2(k+h^\vee)}\sum_{a,b}\kappa_{ab}\left(J^aJ^b\right), \tag{1.75}$$

or equivalently at the level of modes

$$L_n=\frac{1}{2(k+h^\vee)}\sum_{a,b}\kappa_{ab}\sum_{m\in\mathbb{Z}}:j^a_mj^b_{n-m}:. \tag{1.76}$$

We note that the commutation relations equivalent to the OPE $T(z)J^a(w)$ are

$$[L_n,j^a_m]=-mj^a_{n+m}, \tag{1.77}$$

and because of this we identify $L_0=-d$, the derivation on the algebra. Computing the OPE $T(z)T(w)$ proceeds in much the same way, by using point-splitting on $\left(J^bJ^c\right)(w)$, and then using the known OPE between $\left(J^bJ^c\right)(z)J^a(w)$. Computing the OPE and comparing to the Virasoro algebra defining OPE, we see that the Sugawara construction gives rise to a Virasoro algebra with

$$c=\frac{k\dim(\mathfrak{g})}{k+h^\vee}, \tag{1.78}$$

where $\dim(\mathfrak{g})$ is the dimension of the underlying Lie algebra.

The Sugawara construction of a Virasoro operator from a conformal symmetry algebra of currents is of great importance in understanding which algebras give rise to symmetries of conformal quantum field theories. We remarked at the beginning of the Sugawara construction that conformal weight considerations allow for $T(z)$ to contain derivatives of the form $\partial J^a(z)$. Sugawara constructions involving derivatives of the currents are also possible. In the literature, these are known as "improved" Sugawara operators [28–30], and are used in the Hamiltonian reduction procedure discussed in Section 1.8. One also encounters similar "twisted" Virasoro operators in free-field realisations of conformal algebras [31]. We will discuss free-field realisations further in Chapter 7.

1.7 The coset construction

There are many ways of constructing a conformal field theory from a particular algebra. In the last section, we saw that we could construct an action of the Virasoro algebra on an affine Lie algebra. In this section, we consider a generalisation of this procedure, motivated by using the conformal structure of the affine Lie algebras to expand the possible conformal field theories that we can describe in this way.

The Wess-Zumino-Witten models which give rise to affine Lie symmetries only have well-defined action functionals for positive integer values of the level k. The equation (1.78) then implies that the corresponding Virasoro central charge is bounded by $\mathrm{rank}(\mathfrak{g}) \leq c \leq \dim(\mathfrak{g})$, where $\mathfrak{g}$ is the semisimple Lie algebra underlying the affine Lie algebra. Hence, we seek a construction of conformal field theories for a wider range of c values (such as those for the unitary Virasoro minimal models), arising from the affine Lie algebras. One such construction arises from cosets of affine Lie algebras, and is known as the Goddard-Kent-Olive (GKO) construction [18, 32].

We begin by considering an affine Lie algebra $\widehat{\mathfrak{g}}$ at level k, and an affine Lie subalgebra $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ at level $n_e k$, where n_e is the embedding index of $\widehat{\mathfrak{h}}$. The commutation relations for these algebras are given by (1.67), where the superscript a runs over an index set of appropriate size. We can express the modes of $\widehat{\mathfrak{h}}$, denoted $\bar{j}^a_n$, as linear combinations of the generators of $\widehat{\mathfrak{g}}$, denoted j^b_n, that is

$$\bar{j}^a_n = \sum_b M^a_b J^b_n, \tag{1.79}$$

for some $M \in \mathbb{C}$. The coset algebra $\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}$ is given by the algebra of elements in $\widehat{\mathfrak{g}}$ that commute with the subalgebra $\widehat{\mathfrak{h}}$. We now work towards describing the coset algebra.

Using the Sugawara construction, we can construct Virasoro generators $L^{\widehat{\mathfrak{g}}}_n$ and $L^{\widehat{\mathfrak{h}}}_n$, acting on $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$ respectively. We can form another Virasoro operator, which we will refer to as the coset Virasoro operator, given by the difference of Virasoro operators

$$L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_n = L^{\widehat{\mathfrak{g}}}_n - L^{\widehat{\mathfrak{h}}}_n. \tag{1.80}$$

We can verify that this commutes with the subalgebra $\widehat{\mathfrak{h}}$, since

$$[L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_n, \bar{j}^a_m] = [L^{\widehat{\mathfrak{g}}}_n - L^{\widehat{\mathfrak{h}}}_n, \bar{j}^a_m] = [L^{\widehat{\mathfrak{g}}}_n, \bar{j}^a_m] - [L^{\widehat{\mathfrak{h}}}_n, \bar{j}^a_m] = m\bar{j}^a_m - m\bar{j}^a_m = 0. \tag{1.81}$$

Moreover, since $L^{\widehat{\mathfrak{h}}}_n$ is composed of normally ordered modes of the form $\bar{j}^a_m$, this result also implies that

$$[L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_n, L^{\widehat{\mathfrak{h}}}_m] = 0. \tag{1.82}$$

We can check that we have not accidentally constructed a trivial operator by verifying

$$[L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_n, L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_m] = [L^{\widehat{\mathfrak{g}}}_n, L^{\widehat{\mathfrak{g}}}_m] - [L^{\widehat{\mathfrak{h}}}_n, L^{\widehat{\mathfrak{h}}}_m] \neq 0. \tag{1.83}$$

By linearity, we have that the modes $L^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}_n$ do indeed span the Virasoro algebra, with central charge given by

$$c^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}} = c^{\widehat{\mathfrak{g}}} - c^{\widehat{\mathfrak{h}}}. \tag{1.84}$$

Applying (1.78), we have that

$$c^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}} = \frac{k \dim(\mathfrak{g})}{k + h^\vee_{\mathfrak{g}}} - \frac{n_e k \dim(\mathfrak{h})}{n_e k + h^\vee_{\mathfrak{h}}}. \tag{1.85}$$

As such, we have identified the subalgebra inside $\widehat{\mathfrak{g}}$ that commutes with $\widehat{\mathfrak{h}}$. Using the coset, we can construct Virasoro algebras with a larger range of c values than the original WZW models.

One of the most celebrated examples is the coset construction of the unitary minimal models of the Virasoro algebra. We begin by considering the coset

$$\frac{\widehat{\mathfrak{sl}}(2)_k \oplus \widehat{\mathfrak{sl}}(2)_\ell}{\widehat{\mathfrak{sl}}(2)_{k+\ell}}, \tag{1.86}$$

where $\widehat{\mathfrak{sl}}(2)_{k+\ell}$ is the algebra formed by the sum of elements in $\widehat{\mathfrak{g}}$. This is the so-called diagonal embedding, for which we have that $n_e = 1$.

Using the general results presented above, we know that the central charge of the resulting Virasoro algebra is given by

$$c^{\mathfrak{g}/\mathfrak{h}} = \frac{3k}{k+2} + \frac{3\ell}{\ell+2} - \frac{3(k+\ell)}{k+\ell+2}. \tag{1.87}$$

Rearranging gives

$$c^{\mathfrak{g}/\mathfrak{h}} = 1 - \frac{6\ell}{(k+2)(k+\ell+2)} + 2\frac{\ell-1}{\ell+2}. \tag{1.88}$$

We observe that if $\ell = 1$, this formula reduces exactly to (1.54), the central charge of the unitary minimal models with $c < 1$, where $k+2 = m$.

However, this observation regarding central charge values still lacks an understanding of its implications for the representation theory. Generally speaking, representations of the algebra $\widehat{\mathfrak{g}}$ should decompose into representations of the subalgebra $\widehat{\mathfrak{h}}$ via a branching rule. In particular for the coset, we have

$$V^{\widehat{\mathfrak{g}}}(\lambda) = \bigoplus_i V^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}(\eta_i) \otimes V^{\widehat{\mathfrak{h}}}(\mu_i), \tag{1.89}$$

where $V^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}(\eta_i)$ are representations of the coset algebra, the part in $\widehat{\mathfrak{g}}$ that commutes with $\widehat{\mathfrak{h}}$; and λ, η_i, μ_i are the corresponding weights of the representations.

The representations of $\widehat{\mathfrak{sl}}(2)$ which give rise to the unitary Virasoro minimal models via the coset are the irreducible highest-weight representations. These representations are characterised by the value of the level k, and of the eigenvalue of the element h_0 on the highest-weight vector, denoted λ^k, where h_0 is the zero mode corresponding to the generator of the Cartan subalgebra $h \in \mathfrak{sl}(2)$.

The irreducible highest-weight representations of $\widehat{\mathfrak{sl}}(2)$ at level $k \in \mathbb{Z}$ are denoted I_r^k for $1 \le r \le k+1$, and have highest weight $\lambda_r^k = r-1$. As an example, if we look at $k = 1$, the irreducible highest-weight representations are I_1^1, with highest weight 0, and I_2^1 with highest weight 1. The top spaces of these modules are the familiar finite dimensional highest-weight representations of $\mathfrak{sl}(2)$, with highest weight λ_r. The corresponding module for the affine Lie algebra then descends downward from the $\mathfrak{sl}(2)$ states forming the representation of $\widehat{\mathfrak{sl}}(2)$.

The branching rule resulting from the coset construction is then given by

$$I_r^k \otimes I_{r'}^1 = \bigoplus_{\substack{1\le s\le k+2 \\ r-s \text{ even if } r'=1 \\ r-s \text{ odd if } r'=2}} L_{h_{r,s},c} \otimes I_s^{k+1}. \tag{1.90}$$

Here the r,s weights for the representations of the subalgebra are also the labels that determine the highest weight $h_{r,s}$ of the resulting irreducible Virasoro module $L_{h_{r,s},c}$ over the coset algebra.

One way this claim can be verified is by using the corresponding statements for character functions of the modules involved. In the general case, for some $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$, one has that the character functions of modules for $\widehat{\mathfrak{g}}$ decompose as

$$\chi^k_{\widehat{\mathfrak{g}}}(\lambda;q,z) = \sum_i \chi^c_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}}(\eta_i;q) \times \chi^{n_e k}_{\widehat{\mathfrak{h}}}(\mu_i;q,z), \tag{1.91}$$

where the sum ranges over the allowed weights λ_j. We have introduced the character function for a generic affine Lie algebra module as $\chi^k_{\widehat{\mathfrak{g}}}(\lambda;q,z)$. As the weights of vectors in a module over the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ are specified by an L_0 eigenvalue and also a charge corresponding to the action of the Cartan subalgebra h_0, one needs to introduce another variable. A q-series alone is insufficient to describe the weight-space decomposition. The character functions of the coset modules are the branching functions of the characters for the algebras involved in the coset.

Finally, we remark that the coset construction of a given algebra is not unique. One is generally free to use whichever best suits the resulting analysis of the constructed coset. We will return to cosets in Chapter 5 where we introduce a different type of coset known as the Kazama-Suzuki coset. We will not give a discussion of the proof for the coset construction of the unitary Virasoro minimal models, see [5, 18].

1.8 Extended symmetry conformal field theories and W-algebras

Here we introduce extended symmetry algebras of conformal field theories, known as W-algebras. We begin by giving a historical picture of their discovery. In the paper [33], Zamolodchikov considered the problem of introducing an additional primary field $\phi(z)$ of conformal dimension h as a generating field to the Virasoro algebra. The question then was, for what values of h is the resulting algebra well defined? Well defined in the sense that the resulting field algebra closes on the space of all fields, and satisfies the OPE associativity condition. The initial investigation considered allowing fields with $h = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$.

Recalling that the OPE of a primary field with the Virasoro field is given by

$$T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}, \tag{1.92}$$

and the Virasoro OPE itself is fixed to be

$$T(z)T(w) \sim \frac{c}{2}\frac{\mathbb{I}(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.93}$$

the freedom which is restricted by implementing associativity is in the OPE $\phi(z)\phi(w)$.

As an example, the affine Lie algebras in addition to their Sugawara Virasoro field form an extended symmetry algebra, where the primary fields, the affine currents $J^a(z)$, are additional generating fields of conformal weight 1.

The case of most interest in the analysis introduces an additional primary field of conformal weight 3. In the paper [33], this field was referred to as $W(z)$. The resulting operator product expansion for the field $W(z)$ with itself, such that the algebra generated by $W(z)$ and $T(z)$ is associative, is given by

$$\begin{aligned} W(z)W(w) \sim & \frac{c}{3}\frac{\mathbb{I}(w)}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2}\left[\frac{32}{22+5c}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right] + \frac{1}{z-w}\left[\frac{16}{22+5c}\partial\Lambda + \frac{1}{15}\partial^3 T(w)\right], \end{aligned} \tag{1.94}$$

where

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z). \tag{1.95}$$

The field $\Lambda(z)$ is a quasi-primary field of conformal weight 4. The OPE $W(z)W(w)$ has two important features. First, the singular part of the OPE involves the field $\Lambda(z)$, which contains a normally-ordered product of fields. This implies that the corresponding Lie algebra of modes is technically not a Lie algebra as the Lie bracket does not close on the algebra itself. One can show that it closes on a completion of the universal enveloping algebra. Secondly, the structure constants, other than that appearing with the identity field, depend on the central charge of the algebra. Moreover, this dependence is singular as $c \to -\frac{22}{5}$.

This algebra would become the first (non-trivial) example of a W-algebra, named after the notation used by Zamolodchikov to describe the additional generator of conformal weight 3. We say non-trivial as the Virasoro algebra itself, with no additional generators is often taken to be the trivial example of a W-algebra.

There have been many definitions of W-algebras, as the research space has exploded with interest since their inception. A great deal of that interest has come from outside what can be considered "traditional" conformal field theory. Early working definitions took the approach of Zamolodchikov, simply supplementing the Virasoro algebra with additional primary generating fields of conformal weight ≥ 2. Using the notation $W(2,i,j,\dots,k)$ to denote the W-algebra with additional primary fields of weights $i,j,\dots,k$, many new examples were found this way, including a series $W_N = W(2,\dots,N)$ where we include one of each generator with weight $\leq N$, and its infinite limit W_∞ [34–37]. Another particularly interesting example is that of the triplet algebra $W(2,3,3,3)$ where one introduces three W fields, which amongst themselves have an additional $\mathfrak{sl}(2)$ type relation [34, 38]. W-algebras were also being realised as cosets of affine Lie algebras, see [39] for an introduction both W-algebras and their coset constructions.

However, the method that has become a standard way of constructing W-algebras is that of the Drinfeld-Sokolov reduction [40, 41], or alternatively quantum Hamiltonian reduction [39, 42, 43]. The reduction begins with an affine Lie algebra $\widehat{\mathfrak{g}}$, and considers an embedding of $\widehat{\mathfrak{sl}}(2) \hookrightarrow \widehat{\mathfrak{g}}$. One then imposes a constraint on the algebra, related to the nilpotent element $f(z) \in \widehat{\mathfrak{sl}}(2)$. This constraint is imposed as a so-called Becchi-Rouet-Stora-Tyutin (BRST) operator [30, 44–46], denoted $Q(z)$. The operator $Q(z)$ acts as a derivation on an appropriately chosen vector space to form a cochain complex, and the corresponding W-algebra is defined as the zeroth cohomology of this chain complex.

Although this construction is more complicated than that of Zamolodchikov's initial study, the understanding of a W-algebras as quantum Hamiltonian reductions of affine Lie algebras by a nilpotent element has lead to many new and interesting examples. A review of the field in the language of vertex operator algebras is given in [47]. Moreover, it provides a powerful framework for the analysis of W-algebras, and their representation theory. One particular upside is the geometric understanding of W-algebras afforded by this picture, which has brought many new perspectives to W-algebras and conformal field theory more generally.

As an example, if we take $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(n)$ and consider the principal embedding of $\widehat{\mathfrak{sl}}(2)$, we obtain exactly the W-algebra W_N for $n = N$. This motivates the choice of the Virasoro algebra as the trivial W-algebra, as it arises when one takes $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$, and one constrains $\widehat{\mathfrak{sl}}(2)$ by its nilpotent element $f(z)$, and then performing the Hamiltonian reduction. As such, the Virasoro algebra is in essence the simplest W-algebra arising through quantum Hamiltonian reduction.

The following publications have been incorporated as Chapter 2.

[2] J. Rasmussen, C. Raymond, *Higher-order Galilean contractions*,
Nuclear Physics **B 945**, 114680, (2019).
arXiv:1901.06069 [hep-th].

[1] J. Rasmussen, C. Raymond, *Galilean contractions of W-algebras*,
Nuclear Physics **B 922**, 435–479, (2017).
arXiv:1701.04437 [hep-th].

The material is presented with references. We note that some of the contents of [1] been used before for an RHD submission to The University of Queensland. We present that material here for completeness of this presentation, as it provides a framework that many of the following new results rely upon. We do not present any of the overlapping material as new or novel findings in this thesis.

Chapter 2

Higher-order Galilean algebras

2.1 Introduction to the Galilean contraction procedure

The first chapters of this thesis detail generalisations made to the process of Galilean contractions of symmetry algebras. The study we present is essentially an algebraic study of the possible structures that arise from generalised Galilean contraction procedures. However, the concrete examples of such algebras we choose to look at are motivated by finding new symmetry algebras for particular physical systems.

The relevant physical models are toy gravity models on a three-dimensional anti-de Sitter space time [48–50]. It was shown that the infinitesimal symmetries at null-infinity in the asymptotic limit as the spacetime becomes flat are described by an algebra known as the Bondi-Metzner-Sachs algebra $\mathfrak{bms}_3$ [51–53]. It was realised that this algebra generates conformal symmetry in these systems, and is in fact isomorphic to a non-trivial extension of the Virasoro algebra. This extended algebra is the symmetry algebra for a corresponding boundary conformal field theory, through a holographic correspondence [54, 55].

Concurrently with the above results, it was realised that the boundary symmetry algebra could be alternatively constructed through a contraction procedure, similar to an Inönü-Wigner contraction of a Lie algebras [56, 57]. The procedure, known as the Galilean contraction, involves a parameter dependent change of basis of two input algebras [49, 50, 54, 55]. As an example, for the case of two copies of Virasoro, namely $\mathfrak{Vir}$ and $\bar{\mathfrak{Vir}}$, we would introduce new generators

$$L^0_{n,\varepsilon} = L_n + \bar{L}_n, \quad L^1_{n,\varepsilon} = \varepsilon \left(L_n - \bar{L}_n \right), \quad \forall n \in \mathbb{Z}. \tag{2.1}$$

This change of basis is singular in the limit $\varepsilon \to 0$, yet the resulting algebra is well defined, and is known as the Galilean Virasoro algebra (of Galilean conformal algebra). In the general case, the resulting algebra is known as the Galilean algebra corresponding to the input algebras.

A great deal of physical understanding of Galilean algebras had been developed (see [58] for a comprehensive review of physical systems with $\mathfrak{bms}_3$ symmetries) in relation to toy quantum gravity models. However, quantum gravity is not the only area where Galilean structures arise. There are applications of Galilean conformal algebras in a number of fields, such as traditional quantum mechanics, and even hydrodynamics [59–64].

The study of Galilean algebras, particularly from the perspective of conformal field theory, had been taken case-by-case. That is, the Galilean contraction has been applied only to particular algebras with relevant physical properties of interest. There is a significant amount of research on the Galilean Virasoro algebra, as well as its extension by a affine current [65], some work on the superconformal algebras and their relevance to Galilean string theory [50, 66–68], and some analysis of the Galilean W-algebras, in particular W_3 and the so-called Bershadsky-Polyakov algebra [29, 69].

The Galilean W_3 algebra, and Galilean W-algebras more generally, are of particular interest, as they are linked via quantum gravity correspondences to Vasiliev higher-spin theories [71–75] (a comprehensive review of higher-spin theories is given in [76]). In such theories, one introduces a (possibly infinite) hierarchy of particles with increasing spins to the quantum gravity model, as a way of stabilising the theory. As such, the W_3 algebra is essentially the simplest non-trivial example that one can work with, and in [69, 77] the authors showed that the Galilean contraction of the algebra W_3 contains several interesting features that had not been seen in the other examples.

In [1, 4], we began a program of understanding the Galilean contraction from a more abstract point of view, attempting to understand what conformal symmetry algebras admitted a well-defined Galilean contraction, as well as attempting to understand their resulting structure. It was found that a key differentiator in the process of contraction was whether or not the algebra was linear in the following sense: did the operator product expansion of the generating fields contain only other generating fields and their derivatives, or did associativity require the introduction of normally-ordered products of fields (in the singular terms of the expansion).

The Galilean contraction was always well defined in the linear case, which corresponds to conformal symmetry algebras based on infinite dimensional Lie algebras. In the non-linear case however, of which the prime example are the many infinite families of W-algebras, a general proof of well definedness of the contraction procedure was not possible precisely because of the subtleties introduced by the non-linearity. We remark that for the corresponding infinite-dimensional Lie algebra, the Lie bracket on W-algebras generically closes in a completion of the universal enveloping algebra.

This chapter begins our program of investigating generalisations of the Galilean contraction procedure. Originally, the Galilean contraction procedure provided a framework for generating new (conformal) symmetry algebras from two copies of a known symmetry algebra. This construction had been of particular use in the physics literature, as it is relevant to the study of particular Chern-Simons theories on AdS-spacetime, and the corresponding boundary conformal field theories. As such, in [1, 4] we attempted to provide a rigorous mathematical understanding of the contraction procedure and its limitations.

Following that work, we considered natural generalisations of the contraction procedure, whereby one loosens the constraints on the input algebras. In this chapter we present an extension of the framework to allow for any number of input symmetry algebras. We denote the resulting algebras higher-order Galilean algebras.

We examine the applications of these new techniques to a number of examples including the Sugawara construction, and the W-algebra W_3. We again consider the link between these Galilean algebras and the related structures of Takiff algebras. Finally, we introduce a discussion of the graded structure on these algebras. The resulting higher-order Galilean algebras exhibit a truncated Z_N graded structure, from which we can make some simplifying remarks about their general structure.

2.2 Operator product algebras and the Galilean contraction

In our analysis of Galilean algebras, we choose to work with the field algebras of symmetries of a system, in the operator product algebra $\mathcal{A}$ (OPA) formalism. In the following, we restate some key ideas from the basics introduced in Chapter 1, in the language of OPAs. A complete discussion of the OPA formalism is given in [70].

An OPA is a $\mathbb{Z}_2$-graded vector space of fields $A(z) \in \mathcal{A}$, with mode expansions

$$A(z) = \sum_{n \in -\Delta_A + \mathbb{Z}} A_n z^{-n-\Delta_A}. \tag{2.2}$$

The vector space $\mathcal{A} = V_0 \oplus V_1$, where elements of V_0 (respectively V_1) are referred to as even (respectively odd).

We define a bilinear product on the fields of $\mathcal{A}$ by

$$[\cdot\cdot]_n : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (A,B) \mapsto [AB]_n, \; A,B \in \mathcal{A}, \; n \in \mathbb{Z}. \tag{2.3}$$

The algebra $\mathcal{A}$ has a distinguished element $\mathbb{I}$, the identity field, and an even linear map $\partial : \mathcal{A} \to \mathcal{A}$. The fields $A,B,C,\mathbb{I} \in \mathcal{A}$ and operator ∂ are required to satisfy

- $[\mathbb{I}A]_n = \delta_{n,0}A$.
- $[AB]_n = 0$ for n sufficiently large, $A(z), B(w) \in \mathcal{A}$.
- $\partial\,[AB]_n = [\partial AB]_n + [A\partial B]_n$, that is, ∂ is a derivation. When considered in the expression ∂AB, the operator ∂ is taken to be acting on its nearest partner only, i.e. $\partial AB = (\partial A)B$.
- Commutation of fields is determined by

$$[BA]_n = (-1)^{|A||B|} \sum_{\ell \geq n} \frac{(-1)^\ell}{(\ell-n)!} \partial^{\ell-n} [AB]_\ell, \quad \forall n \in \mathbb{Z}, \tag{2.4}$$

where $|A|$ is a parity operation, which gives 0 if A is even, and 1 if odd.

- Triple products of fields must satisfy the relations

$$[A[BC]_m]_n = (-1)^{|A||B|}[B[AC]_n]_m + \sum_{\ell \geq 0} \binom{n-1}{\ell-1} [[AB]_\ell C]_{m+n-\ell}, \quad \forall m,n \in \mathbb{Z}. \tag{2.5}$$

The algebra $\mathcal{A}$ may depend on a set of complex indeterminates known as central parameters. This concludes the definition of an OPA. In the following, we will continue by introducing some of the terminology of an OPA. It is also useful to re-frame the definition of an OPA in the light of its application in conformal field theory.

In this formalism, the OPE complete between two fields (including regular terms) is written

$$A(z)B(w) = \sum_{n=-\infty}^{\Delta_{A,B}} f^{AB}{}_{[AB]_n} \frac{[AB]_n(w)}{(z-w)^n}, \tag{2.6}$$

where, $[AB]_n$ is a placeholder for a field of conformal weight $\Delta_A + \Delta_B - n$ given by the product on $\mathcal{A}$, and $\Delta_{A,B} = \Delta_A + \Delta_B$. The $f^{AB}{}_{[AB]_n}$ are the corresponding structure constants which are in $\mathbb{C}$.

As discussed in the introduction, the non-trivial information in the OPE is contained in the singular terms. As such, it is conventional to ignore the non-singular terms. Hence we can rewrite the above as

$$A(z)B(w) \sim \sum_{n=1}^{\Delta_{A,B}} f^{AB}{}_{[AB]_n} \frac{[AB]_n(w)}{(z-w)^n}, \tag{2.7}$$

where the use of $\sim$ rather than equality explicitly denotes the suppression of all non-singular terms.

We make the assumption that the identity field $\mathbb{I}$ is the only field of conformal weight $\Delta = 0$. As such, the placeholder field $[AB]_{\Delta_A+\Delta_B}$ can be uniquely identified with $\mathbb{I}$, and we can write

$$A(z)B(w) \sim \frac{f^{AB}{}_{\mathbb{I}}\mathbb{I}(w)}{(z-w)^{\Delta_A+\Delta_B}} + \sum_{n=1}^{\Delta_{A,B}-1} f^{AB}{}_{[AB]_n} \frac{[AB]_n(w)}{(z-w)^n}, \tag{2.8}$$

The normally ordered product of two fields, denoted (AB) for $A,\ B \in \mathcal{A}$ is identified with $(AB) = [AB]_0$. A normally ordered product of fields is itself a field, and such a field is referred to as composite.

We say that a set of fields generates a given OPA if all fields in the OPA appear by taking OPEs, normally-ordered products, and derivatives of generating fields.

We say that an OPA $\mathcal{A}$ is conformal if it contains a distinct field $T(z)$ which generates a Virasoro operator product subalgebra of central charge c. The OPE of the subalgebra is

$$T(z)T(w) \sim \frac{c}{2}\frac{\mathbb{I}(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \tag{2.9}$$

In the introduction, we introduced scaling, and (quasi-)primary fields $A \in \mathcal{A}$. In this notation, a scaling field if its OPE with the Virasoro field satisfies

$$[TA]_2 = \Delta_A A, \qquad [TA]_1 = \partial A. \tag{2.10}$$

Moreover, a quasi-primary field is a scaling field for which the term $[TA]_3 = 0$, and a primary field has $[TA]_n = 0$ for all $n \geq 3$.

The space of fields for a conformal OPA has a basis given by the quasi-primary fields and their derivatives. This is a well-known fact in conformal field theory arising from $\mathfrak{sl}(2)$ representation theory, and a proof is given in [1]. We denote the quasi-primary basis for the space of fields $\mathcal{B}_{\mathcal{A}}$. In this basis, the OPE between any two fields (2.7) takes the explicit form

$$A(z)B(w) \sim \sum_{Q\in\mathcal{B}_{\mathcal{A}}} f^{AB}{}_{Q} \left(\sum_{n=0}^{\Delta_A+\Delta_B-\Delta_Q} \frac{\beta^{\Delta_Q;n}_{\Delta_A,\Delta_B} \partial^n Q(w)}{(z-w)^{\Delta_A+\Delta_B-\Delta_Q-n}} \right), \tag{2.11}$$

where $f^{AB}{}_{Q}$ are the structure constants, and

$$\beta^{\Delta_Q;n}_{\Delta_A,\Delta_B} = \frac{(\Delta_A - \Delta_B + \Delta_Q)_n}{n!(2\Delta_Q)_n}, \qquad (x)_n = \prod_{j=0}^{n-1}(x+j). \tag{2.12}$$

We make the observation that global conformal symmetry constrains every term in the OPE which is not the leading order quasi-primary field. As such, we are able to rewrite the OPE more compactly. Indeed, the OPE $A(z)B(w)$ may be written as follows,

$$A \times B \simeq \sum_{Q\in\mathcal{B}_{\mathcal{A}}} f^{AB}{}_{Q}\{Q\}, \tag{2.13}$$

where the notation $\{Q\}$ suppresses the corresponding conformal chain of derivatives appearing in the OPE poles, which is determined by the equations above. In the above relation, the summation is taken over all quasi-primary fields of conformal weight less than $\Delta_A + \Delta_B$. We also suppress the explicit dependence on the co-ordinates of the fields. We use the notation $\simeq$ to avoid any confusion with the use of $\sim$ in the usual OPE convention.

As an example, in this notation, we are able to rewrite the OPE for the Virasoro algebra as

$$T \times T \simeq \frac{c}{2}\{\mathbb{I}\} + 2\{T\}. \tag{2.14}$$

We would like to expand upon the fact that the quasi-primary basis of fields realises representation theory of $\mathfrak{sl}(2)$. A quasi-primary field is one for which the corresponding state vanishes under the action of L_1. Moreover, the derivatives of a quasi-primary field correspond to the action of L_{-1} on the corresponding state. All such states are eigenvalues of L_0. The corresponding family of fields forms a representation of the $\mathfrak{sl}(2)$ subalgebra within the Virasoro algebra, spanned by $\{L_{-1}, L_0, L_1\}$. This subalgebra generates the global conformal symmetry in the corresponding physical theory.

2.2.1 Galilean algebras and the contraction procedure

In this section, we introduce the Galilean contraction procedure. We begin by considering the algebra formed by two copies of an OPA $\mathcal{A}$, labelled $\mathcal{A} = \mathcal{A}_{(0)} \otimes \mathcal{A}_{(1)}$, assumed to be equivalent up to the value of their central parameter. The fields are labelled to indicate which copy of the underlying algebra they belong to, that is, $A_{(i)} \in \mathcal{A}_{(i)}$. Their OPEs are of the form (2.7), and their corresponding structure constants are the ${f^{AB}}_{C(i)}$'s.

We perform a parameter dependent change of basis, whereby we create the new fields

$$A_{0,\varepsilon} = A_{(0)} + A_{(1)}, \quad A_{1,\varepsilon} = \varepsilon \left(A_{(0)} - A_{(1)}\right), \quad \forall A \in \mathcal{A}/\{\mathbb{I}\}. \tag{2.15}$$

In this basis, fields $A_{i,\varepsilon}$ become generating fields for the algebra denoted $\mathcal{A}_\varepsilon$. We use subscript ε to signify that the new basis is dependent on the parameter ε. First we determine the OPE structure on the algebra $\mathcal{A}_\varepsilon$, which is given by

$$\begin{aligned}
A_{i,\varepsilon} \times B_{j,\varepsilon} &= \varepsilon^i \left(A_{(0)} + (-1)^i A_{(1)}\right) \times \varepsilon^j \left(B_{(0)} + (-1)^j B_{(1)}\right) \\
&= \varepsilon^{i+j} \left(A_{(0)} \times B_{(0)} + (-1)^{i+j} A_{(1)} \times B_{(1)}\right) \\
&= \varepsilon^{i+j} \left({f^{AB}}_{\mathbb{I}}\mathbb{I} + {f^{AB}}_C\{C_{(0)}\} + (-1)^{i+j}\left({f^{AB}}_{\mathbb{I}}\mathbb{I} + {f^{AB}}_C\{C_{(i)}\}\right)\right).
\end{aligned} \tag{2.16}$$

At this point, we reform into fields $C_{i,\varepsilon}$, depending on powers of ε. Moreover, as the identity field remains unique in this process, we form new structure constants to accompany the identity field $f_{i,\varepsilon} = \varepsilon^i (f_{(0)} + (-1)^i f_{(1)})$.

Following this, we take the limit $\varepsilon \to 0$. In this limit, any OPEs with unabsorbed factors of ε are sent to 0, and for the surviving fields and structure constants, we have that $A_{i,\varepsilon} \mapsto A_i$, and $f_{i,\varepsilon} \mapsto f_i$. This new set of fields generates the corresponding Galilean algebra $\mathcal{A}_G$.

We will use terminology that if the underlying algebra of symmetries is a Lie (super)algebra, the corresponding OPA is of Lie-type. The Galilean contraction is well defined on all OPAs of Lie type.

For OPAs of Lie type, the structure constants accompanying the identity field ${F^{AB}}_{\mathbb{I}}$ are fixed to be either constants in $\mathbb{C}$ or degree 1 polynomials in the central indeterminate c, with coefficients in $\mathbb{C}$. In the case when we have linear functions of c, the corresponding Galilean algebra has a set of central indeterminates which are determined by applying the change of basis to the central indeterminates of $\mathcal{A}$.

As an example, we present the case when we take $\mathcal{A} = \mathfrak{Vir}_{(0)} \otimes \mathfrak{Vir}_{(1)}$, that is, we Galilean contract two copies of the Virasoro algebra, with central charges $c_{(0)}$ and $c_{(1)}$ respectively. Performing the change of basis on the generating fields, we have

$$T_{0,\varepsilon} = T_{(0)} + T_{(1)}, \quad T_{1,\varepsilon} = \varepsilon \left(T_{(0)} - T_{(1)}\right). \tag{2.17}$$

The product is then given by

$$\begin{aligned}
T_{i,\varepsilon} \times T_{j,\varepsilon} &\simeq \varepsilon^{i+j} \left(\frac{c_{(0)}}{2}\{\mathbb{I}\} + 2\{T_{(0)}\} + (-1)^{i+j}\left[\frac{c_{(i)}}{2}\{\mathbb{I}\} + 2\{T_{(i)}\}\right]\right) \\
&\simeq \varepsilon^{i+j} \left(\frac{c_{(0)} + (-1)^{i+j} c_{(1)}}{2}\{\mathbb{I}\} + 2\left(\{T_{(0)}\} + (-1)^{i+j}\{T_{(1)}\}\right)\right).
\end{aligned} \tag{2.18}$$

Recombining generators into the $\mathcal{A}_\varepsilon$, and taking the contraction limit we have

$$T_i \times T_j \simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.19}$$

where $c_i = \varepsilon^i \left(c_{(0)} + (-1)^i c_{(1)}\right)$.

The resulting algebra is known as the Galilean Conformal algebra [54]. This algebra is a subalgebra of all Galilean algebras coming from a conformal operator product algebra. In fact, any the Galilean contraction of any conformal OPA will itself be conformal, with T_0 generating a Virasoro subalgebra of conformal charge c_0.

As a second example, we consider the Galilean contraction of the W-algebra W_3 [33]. As introduced in Section 1.8, the W_3 algebra introduces an additional primary field of conformal weight 3 to the Virasoro algebra, and constrains the OPE by imposing associativity.

The algebra W_3 is not of Lie type, as the structure constants accompanying fields other than the identity in the OPE are dependant on the central charge, and the OPE contains normally-ordered fields in the singular terms. We will refer to algebras with these properties as W-algebras, however, we remark that using the definition of W-algebra coming from [47] outlined in the introduction, some W-algebras arising from Hamiltonian reduction are of Lie type.

Explicitly, the OPE relations for the algebra W_3 in the notation introduced in this section are given by

$$\begin{aligned} T \times T &\simeq \frac{c}{2}\{\mathbb{I}\} + 2\{T\}, \quad T \times W \simeq 3\{W\}, \\ W \times W &\simeq \frac{c}{3}\{\mathbb{I}\} + 2\{T\} + \frac{32}{22+5c}\{\Lambda\}, \end{aligned} \tag{2.20}$$

where

$$\Lambda(z) = (TT) - \frac{3}{10}\partial^2 T \tag{2.21}$$

is a quasi-primary field of conformal weight 4. Not only do we see that the OPE contains a normally ordered product of fields in a singular term, we also have structure constants which are rational functions of the central charge. These properties are generic for W-algebras.

Using the framework developed above, it is not guaranteed that a corresponding Galilean algebra is well defined. However, it is possible to perform the contraction. A full description of the techniques required is given in [1]. One takes a series expansion of the structure constants in ε small, and uses in the inverse basis maps to the contraction basis on the normally-ordered fields. The resulting expressions are then combined, and the limit $\varepsilon \to 0$ is taken. The corresponding algebra is well defined, and is called the Galilean W_3 algebra of order-two, $(W_3)^2_G$.

The Galilean $(W_3)^2_G$ algebra is generated by fields T_0, T_1, W_0, W_1, has central charges c_0, c_1, and is defined by the OPEs

$$\begin{aligned}
T_i \times T_j &\simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
T_i \times W_j &\simeq \begin{cases} 3\{W_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
W_0 \times W_0 &\simeq \frac{c_0}{3}\{\mathbb{I}\} + 2\{T_0\} + \frac{64}{5c_1}\{\Lambda_{0,1}\} - \frac{32(44+5c_0)}{25c_1^2}\{\Lambda_{1,1}\}, \\
W_0 \times W_1 &\simeq \frac{c_1}{3}\{\mathbb{I}\} + 2\{T_1\} + \frac{32}{5c_1}\{\Lambda_{1,1}\},
\end{aligned} \tag{2.22}$$

where

$$\Lambda_{0,1} = (T_0T_1) - \frac{3}{10}\partial^2 T_1, \quad \Lambda_{1,1} = (T_1T_1). \tag{2.23}$$

The fields $\Lambda_{0,1}$ and $\Lambda_{1,1}$ are both quasi-primary with respect to the action of the Virasoro subalgebra generated by T_0. This algebra has been well studied in the literature, as it is the most accessible algebra not of Lie-type [1, 77–79].

2.3 Higher-order Galilean contractions

In this section, we introduce the first generalisation of the Galilean algebra, which will result in so-called higher-order Galilean algebras. Here we will show that the procedure can be generalised to allow input of any number of OPAs, equivalent up to the value of their central parameters.

2.3.1 Contraction prescription

For $N \in \mathbb{Z}_{>0}$, we begin by considering the N-fold tensor product

$$\mathcal{A}^{\otimes N} = \bigotimes_{i=0}^{N-1} \mathcal{A}_{(i)}. \tag{2.24}$$

Taking ε to be a parameter taking values in $\mathbb{C}^\times$, we make the following change of basis for the fields, and corresponding re-parameterisation of the central indeterminates

$$A_{i,\varepsilon} = \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} A_{(j)}, \qquad c_{i,\varepsilon} = \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} c_{(j)}, \qquad i = 0, \ldots, N-1, \tag{2.25}$$

where, as in the previous case, $A_{(j)}$ (respectively $c_{(j)}$) denotes the field $A \in \mathcal{A}_{(j)}$ (respectively the central parameter $c_{(j)}$) in the $(j+1)$th copy of the algebra $\mathcal{A}$, in the product $\mathcal{A}^{\otimes N}$. We also introduce ω, the principal Nth root of unity,

$$\omega = e^{2\pi \mathrm{i}/N}. \tag{2.26}$$

Although not introduced in the previous section, for $\varepsilon \neq 0$, the map

$$\mathcal{A}^{\otimes N} \to \mathcal{A}^{\otimes N}, \qquad (A_{(0)}, \ldots, A_{(N-1)}) \mapsto (A_{0,\varepsilon}, \ldots, A_{N-1,\varepsilon}), \tag{2.27}$$

(and similarly for the central parameters) is invertible, with

$$A_{(i)} = \frac{1}{N}\sum_{j=0}^{N-1} \omega^{-ij}\varepsilon^{-j}A_{j,\varepsilon}, \qquad i=0,\dots,N-1. \tag{2.28}$$

It is useful here to point out that when $N=2$, we have $\omega=-1$ and

$$A_{0,\varepsilon} = A_{(0)} + A_{(1)}, \qquad A_{1,\varepsilon} = \varepsilon\big(A_{(0)} - A_{(1)}\big), \tag{2.29}$$

with inverse maps

$$A_{(0)} = \tfrac{1}{2}\big(A_{0,\varepsilon} + \tfrac{1}{\varepsilon}A_{1,\varepsilon}\big), \qquad A_{(1)} = \tfrac{1}{2}\big(A_{0,\varepsilon} - \tfrac{1}{\varepsilon}A_{1,\varepsilon}\big). \tag{2.30}$$

These expressions are exactly those for the traditional Galilean contraction.

As in the traditional case, in the limit $\varepsilon \to 0$, the map (2.27) is singular for all $N>1$. This behaviour results in a new algebraic structure, where we map fields and parameters as

$$A_{i,\varepsilon} \mapsto A_i, \qquad c_{i,\varepsilon} \mapsto c_i, \tag{2.31}$$

to distinguish between algebras.

If the resulting algebra is a well-defined OPA, we refer to it as the N^{th}*-order Galilean OPA* and denote it $\mathcal{A}_G^N$. In particular, if $\mathcal{A}$ is an OPA of Lie-type (that is, the underlying algebra of modes is a Lie algebra), then all the corresponding higher-order Galilean contractions are indeed well defined.

For Lie-type algebras, it is straightforward to see that the corresponding OPE structure for a general N^{th}-order Galilean OPA will be given by

$$A_i \times B_j \simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + f^{AB}{}_{C}\{C_{i+j}\}, & \text{if } i+j<N, \\ 0, & \text{otherwise.} \end{cases} \tag{2.32}$$

Here we make use of the summation convention in the structure constants. We have explicitly separated the structure constant for the identity, and make the possible dependence on a central parameter c explicit (we choose c purely for the sake of similarity, it is not necessarily the Virasoro central charge).

2.3.2 Examples: Galilean Virasoro and affine algebras

Here we present the two most common examples of Lie-type algebras in conformal field theory, and their N^{th}-order Galilean contracted counterparts, as concrete examples. As they are both of Lie-type, the resulting OPEs for the contracted algebras are determined by equation (2.32).

Recall the Virasoro OPA $\mathfrak{Vir}$ of central charge c is generated by T, with OPE relation

$$T \times T \simeq \tfrac{c}{2}\{\mathbb{I}\} + 2\{T\}. \tag{2.33}$$

The Galilean Virasoro algebra of order N, $\mathfrak{Vir}_G^N$, is generated by the fields $T_0,\dots,T_{N-1}$, with central parameters $c_0,\dots,c_{N-1}$ and OPE relations

$$T_i \times T_j \simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & i+j<N, \\ 0, & i+j \geq N. \end{cases} \tag{2.34}$$

This yields an infinite family of extended Virasoro algebras, $\{\mathfrak{Vir}_G^N \,|\, N \in \mathbb{N}\}$. We identify $\mathfrak{Vir}_G^1 \simeq \mathfrak{Vir}$, and $\mathfrak{Vir}_G^2$ is the familiar Galilean Virasoro algebra from Section 2.1.

Our second example is that of an affine Lie algebra at level k. Affine Lie algebras as OPAs have a set of generating fields given by the currents J^a, $a \in I_s$, where I_s is an index set of size $\dim(\mathfrak{g})$, the dimension of the underlying semisimple Lie algebra. The OPE between currents can then be written in the form,

$$J^a \times J^b \simeq \kappa^{ab} k\{\mathbb{I}\} + f^{ab}{}_c\{J^c\}, \tag{2.35}$$

where $f^{ab}{}_c$ are structure constants coming from $\mathfrak{g}$, and κ the Killing form on $\mathfrak{g}$. Again, in this expression, we make use of the summation convention over the label c. Furthermore, the currents are fields of conformal dimension 1. This implies that at most, the terms on the right-hand side of the OPE have conformal dimension 1. We make note of this as the above expression need not feature braces appearing around the fields on the right hand side, as these are the only terms with appropriate conformal weight to appear in the singular part of the OPE.

As the corresponding OPA is of Lie type, its (higher-order) Galilean contractions are readily constructed. Applying the contraction procedure, we find that the corresponding Galilean affine Lie algebra of order N, $\widehat{\mathfrak{g}}_G^N$, is generated by currents $\{J_i^a \,|\, a = 1,\dots,\dim\mathfrak{g};\, i = 0,\dots,N-1\}$, with nontrivial OPEs given by

$$J_i^a \times J_j^b \simeq \begin{cases} \kappa^{ab} k_{i+j}\{\mathbb{I}\} + f^{ab}{}_c\{J_{i+j}^c\}, & i+j<N, \\ 0 & i+j \geq N. \end{cases} \tag{2.36}$$

2.4 General properties of higher-order Galilean contractions

2.4.1 Truncated graded structure of $\mathcal{A}_G^N$

We can introduce a linear operator $\mathrm{gr} : \mathcal{A}_G^N \to \mathbb{Z}_N$ such that

$$A_i \mapsto i, \quad c_i \mapsto i, \quad (A_i B_j) \mapsto i+j. \tag{2.37}$$

The action of gr is extended by linearity to Laurent polynomials of the central parameters. In particular, we have that $\mathrm{gr}(\partial) = 0$ and $\mathrm{gr}(\mathbb{I}) = 0$, so that

$$\mathrm{gr}\left(c_0 \partial A_3 + (A_2 B_1) + \frac{c_3 + c_1 c_2}{c_0^2}\mathbb{I}\right) = 3. \tag{2.38}$$

The algebra is then graded if the grading, as defined above, is compatible with the OPE structure of the algebra, in the sense that

$$\mathrm{gr}\left(A_i \times B_j\right) = i+j, \quad \forall A_i, B_j \in \mathcal{A}^G. \tag{2.39}$$

It is clear from the following calculation that algebras of Lie-type are graded. Consider the general OPE in $\mathcal{A}$ a Lie-type algebra,

$$A \times B = f^{AB}{}_{\mathbb{I}}(c)\{\mathbb{I}\} + \sum_{C} f^{AB}{}_{C}\{C\}, \tag{2.40}$$

where again we have made explicit the possible linear dependence on c (any central parameter) in the structure constant accompanying the identity. Moreover, the grading on a Galilean algebra (not necessarily of Lie type) is truncated at N, the number of input algebras.

We have already demonstrated that in the corresponding Galilean algebra, the OPE is generally of the form

$$A_i \times B_j = f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + \sum_{C} f^{AB}{}_{C}\{C_{i+j}\}, \tag{2.41}$$

and as such, obeys the grading requirement.

However, a priori it is not clear that Galilean W-algebras should be graded. The structure constants of W-algebras are generically algebraic functions of c, the central charge, and they do not accompany only the identity field. What can be said is that in all examples we have thus far calculated, the resulting algebra is indeed graded using the above definition. The graded structure of the contracted algebra implies constraints on the possible functions of c that can appear as structure constants of the uncontracted algebra. We are unable to make this more rigorous at this time, however, we will return to this idea in the discussion.

A useful implication of the graded structure for Lie-type OPAs is that OPE relations need only be calculated between the underlying $\mathcal{A}_0 \subset \mathcal{A}_G^N$ subalgebra (the zero-grade algebra), and the Galilean generating fields. Explicitly, the graded structure for a Lie-type OPA implies that

$$\mathrm{gr}\left(A_0 \times B_m\right) = \mathrm{gr}\left(A_i \times B_j\right) \iff m = i + j, \quad \forall 0 \leq m < N. \tag{2.42}$$

As such, by determining the action of the underlying $\mathcal{A}_0$ subalgebra on the generating fields of $\mathcal{A}_G^N$, we can deduce the full OPE structure of the algebra.

2.4.2 Relation to Takiff algebras

In this subsection, we present an alternative description of Galilean algebras, not from the perspective of arising from a contraction. The truncated graded structure on the Galilean algebra, along with the underlying subalgebra of grade 0 implies that we can develop an alternative realisation of the algebra in the following way. Consider the Virasoro OPA $\mathfrak{Vir}$, and take the tensor product with a polynomial ring in one variable,

$$\mathfrak{Vir}^{\infty} = \mathfrak{Vir} \otimes \mathbb{C}[x]. \tag{2.43}$$

where the generating field $T \in \mathfrak{Vir}$ is mapped to $T_n := T \otimes x^n \in \mathfrak{Vir}^{\infty}$, and similarly for the central parameters c_n. These polynomial extensions of Lie algebras were first introduced by Takiff in [80], and their applications to conformal symmetry algebras have been of interest in the mathematical physics literature [81–85].

The OPE on the algebra $\mathfrak{Vir}^{\infty}$ is the same as that for the higher order Galilean Virasoro algebra, but without the truncation condition. That is, we can understand this as the higher order contraction when we take the limit $N \to \infty$, which is also made clear by the choice of notation. That is, we have the identification $\mathfrak{Vir}^{\infty}_{G} \cong \mathfrak{Vir}^{\infty}$.

The corresponding Takiff algebra of order N is then given by considering the following quotient

$$\mathfrak{Vir}^{N}_{G} \cong \mathfrak{Vir}^{N} = \mathfrak{Vir} \otimes \mathbb{C}[x]/\langle x^{N} \rangle. \tag{2.44}$$

This quotient establishes the truncation condition on the product. As such, there is a clear isomorphism for Lie-type OPAs between their Takiff algebra of order N, and their corresponding N^{th}-order Galilean contracted algebra.

As a further example, these results carry clearly to the affine Lie algebras, $\hat{\mathfrak{g}}$, where

$$\widehat{\mathfrak{g}}^{\infty} \cong \widehat{\mathfrak{g}} \otimes \mathbb{C}[x], \tag{2.45}$$

and

$$\widehat{\mathfrak{g}}^{N} \cong \widehat{\mathfrak{g}} \otimes \mathbb{C}[x]/\langle x^{N} \rangle. \tag{2.46}$$

2.5 Higher-order Galilean Sugawara constructions

As mentioned in Section 1.6, the Sugawara construction describes a process for forming a Virasoro operator out of elements in the universal enveloping algebra for an affine Lie algebra, showing that field theories built on affine Lie symmetries exhibit conformal symmetry. We would like to show that this construction is also possible if the starting point is a Galilean affine Lie algebra, and that if the construction is well defined, it is compatible with the Galilean contraction procedure in the following sense. We would like that the diagram below be commutative, thus implying that performing a Sugawara construction on each of the N component algebras before taking the contraction, leads to the same algebra as if one had performed a Sugawara construction on the Galilean affine algebra.

$$\begin{array}{ccc} \widehat{\mathfrak{g}}^{\otimes N} & \xrightarrow{\text{Sug}^{\otimes N}} & \mathfrak{Vir}^{\otimes N} \\ \Big\downarrow{\scriptstyle \text{Gal}} & & \Big\downarrow{\scriptstyle \text{Gal}} \\ \widehat{\mathfrak{g}}^{N}_{G} & \xrightarrow{\text{Gal Sug}} & \mathfrak{Vir}^{N}_{G} \end{array}$$

Figure 2.1: *A diagram showing the two possible operations giving rise to a Galilean* $\widehat{\mathfrak{g}}^{N}_{G}$ *with a Sugawara constructed Galilean Virasoro subalgebra.*

In [1], we constructed a Sugawara operator for Galilean affine Lie algebras (of order two), and showed that the process commuted with the contraction procedure. Performing the Sugawara constructions before performing the contraction gave equivalent operators to performing the reverse operation. Indeed, we find that a similar result holds for the higher-order Galilean affine Lie algebras. To verify, we analyse each branch of the above diagram separately in the subsections that follow. The lower branch is considered in Section 2.5.1; the upper one in Section 2.5.2.

2.5.1 Galilean Sugawara construction

We begin by considering the branch where we have already performed the contraction. That is, we start with the algebra $\hat{\mathfrak{g}}_G^N$, generated by the fields $\{J_i^a \mid i=0,\dots,N-1;\ a\in I_s\}$, and with OPE given by (2.36). Using the knowledge that for affine Lie algebras, the Sugawara operator is a sum over normally ordered products of the fields (with appropriately constrained constants), we make the following ansatz for the generators T_i of the resulting $\mathfrak{Vir}_G^N$ algebra,

$$T_i = \sum_{r,s=0}^{N-1} \lambda_i^{r,s} \kappa_{ab} (J_r^a J_s^b), \qquad i=0,\dots,N-1, \tag{2.47}$$

where κ_{ab} are elements of the inverse Killing form on $\mathfrak{g}$. The coefficients $\lambda_i^{r,s}$ are then constrained by requiring that the currents be conformal weight 1 primary fields under the action of T_0, and that the grading on the algebra be respected. Explicitly, we require

$$T_i \times J_j^a \simeq \begin{cases} \{J_{i+j}^a\}, & i+j\in\{0,\dots,N-1\}, \\ 0, & i+j\geq N, \end{cases} \tag{2.48}$$

for all $J_j^a \in \widehat{\mathfrak{g}}_G^N$.

We begin by computing the OPE

$$\begin{aligned} J_j^a(z)T_i(w) \sim &\sum_{r,s=0}^{N-1} \frac{\lambda_i^{r,s}}{(z-w)^2}\left[k_{j+r}J_s^a(w)+k_{j+s}J_r^a(w)+2h^\vee J_{j+r+s}^a(w)\right] \\ &+\sum_{r,s=0}^{N-1} \frac{\lambda_i^{r,s}\kappa_{bc}}{z-w}\left[f^{ab}{}_d(J_{j+r}^d J_s^c)(w)+f^{ac}{}_d(J_r^b J_{j+s}^d)(w)\right], \end{aligned} \tag{2.49}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. The constant $h^\vee$ arises from the relation $\kappa_{bc}f^{ab}{}_d f^{dc}{}_e = 2h^\vee \delta_e^a$. We present the calculation here in the form of traditional OPEs as it is important to keep track of all terms in the OPE. To satisfy the constraints from (2.48), the first sum must equal $J_{i+j}^a(w)/(z-w)^2$. Furthermore, the second sum over normally-ordered products must vanish. The vanishing constraint implies that

$$\lambda_i^{r,s} = \begin{cases} \lambda_i^{\ell,N-1}, & r+s=N-1+\ell \quad (\ell=0,\dots,N-1), \\ 0, & r+s<N-1. \end{cases} \tag{2.50}$$

This leaves N independent coefficients undetermined, namely $\lambda_i^{0,N-1},\dots,\lambda_i^{N-1,N-1}$, for each $i\in\{0,\dots,N-1\}$.

The first-sum constraint requires then that

$$2\sum_{n=j}^{N-1}\sum_{\ell=0}^{n-j}\lambda_i^{\ell,N-1}k_{N-1-n+j+\ell}J_n^a+2Nh^\vee\lambda_i^{0,N-1}\delta_{j,0}J_{N-1}^a=\begin{cases}J_{i+j}^a, & i+j\le N-1,\\ 0, & i+j\ge N.\end{cases} \tag{2.51}$$

This can be rewritten as a lower-triangular system of linear equations for each i,

$$2\begin{pmatrix}k_{N-1} & & & & \\ k_{N-2} & k_{N-1} & & & \\ \vdots & \ddots & \ddots & & \\ k_1 & & \ddots & k_{N-1} & \\ k_0' & k_1 & \cdots & k_{N-2} & k_{N-1}\end{pmatrix}\begin{pmatrix}\lambda_i^{0,N-1}\\ \vdots\\ \\ \vdots\\ \lambda_i^{N-1,N-1}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}, \tag{2.52}$$

where we use the shorthand $k_0'=k_0+Nh^\vee$, and where the only non-zero component on the right-hand side is a 1 in position $i+1$.

Assuming that $k_{N-1}\neq 0$, we can reduce the problem to finding the inverse of a lower-triangular Toeplitz matrix of general form

$$A=\begin{pmatrix}1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & 1 & \\ a_{N-1} & \cdots & a_2 & a_1 & 1\end{pmatrix}, \tag{2.53}$$

where

$$a_m=\frac{k_{N-1-m}+Nh^\vee\delta_{m,N-1}}{k_{N-1}},\qquad m=1,\dots,N-1. \tag{2.54}$$

The inverse matrix is itself a lower-triangular Toeplitz matrix with 1's on the diagonal,

$$A^{-1}=\begin{pmatrix}b_0 & & & & \\ b_1 & b_0 & & & \\ b_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & b_0 & \\ b_{N-1} & \cdots & b_2 & b_1 & b_0\end{pmatrix},\qquad b_0=1, \tag{2.55}$$

We find that the nontrivial matrix elements are given by the following formula

$$b_n=\sum_{p\in(\mathbb{N}_0)^n}(-1)^{|p|}\frac{\delta_{\|p\|,n}|p|!}{p_1!\cdots p_n!}a_1^{p_1}\cdots a_n^{p_n}, \tag{2.56}$$

where

$$|p|=\sum_{i=1}^n p_i,\qquad \|p\|=\sum_{i=1}^n ip_i,\qquad p=(p_1,\dots,p_n). \tag{2.57}$$

We use $\mathbb{N}_0$ to denote the non-negative integers. The sum over tuples is the same as one would expect coming from a multinomial expansion.

This then determines the remaining coefficients, for which we have

$$\lambda_i^{\ell,N-1} = \begin{cases} 0, & \ell = 0,\dots,i-1, \\ \dfrac{b_{\ell-i}}{2k_{N-1}}, & \ell = i,\dots,N-1. \end{cases} \tag{2.58}$$

The resulting unique expression for T_i (2.47) is given by

$$T_i = \sum_{n=0}^{N-1-i} \frac{b_n}{2k_{N-1}} \sum_{t=0}^{N-1-i-n} \kappa_{ab}(J^a_{i+n+t}J^b_{N-1-t}). \tag{2.59}$$

When $N = 2$, we recover the Galilean Sugawara construction obtained in [1],

$$T_0 = \frac{\kappa_{ab}}{2k_1}\Big[(J^a_0J^b_1) + (J^a_1J^b_0)\Big] - \frac{k_0 + 2h^\vee}{2(k_1)^2}\kappa_{ab}(J^a_1J^b_1), \qquad T_1 = \frac{\kappa_{ab}}{2k_1}(J^a_1J^b_1). \tag{2.60}$$

For $N = 3$, we find the new expressions

$$\begin{aligned} T_0 &= \frac{\kappa_{ab}}{2k_2}\Big[(J^a_0J^b_2) + (J^a_1J^b_1) + (J^a_2J^b_0)\Big] - \frac{k_1\kappa_{ab}}{2(k_2)^2}\Big[(J^a_1J^b_2) + (J^a_2J^b_1)\Big] + \frac{(k_1)^2 - (k_0 + 3h^\vee)k_2}{2(k_2)^3}\kappa_{ab}(J^a_2J^b_2), \\ T_1 &= \frac{\kappa_{ab}}{2k_2}\Big[(J^a_1J^b_2) + (J^a_2J^b_1)\Big] - \frac{k_1\kappa_{ab}}{2(k_2)^2}(J^a_2J^b_2), \qquad T_2 = \frac{\kappa_{ab}}{2k_2}(J^a_2J^b_2). \end{aligned} \tag{2.61}$$

The fields T_i are quasi-primary under the action of T_0. Furthermore, for each $i = 0,\dots,N-1$, the coefficient of the identity field OPE $T_0 \times T_i$ determines the value of the central parameter c_i, since for $i < N$

$$T_0 \times T_i \simeq \frac{c_i}{2}\{\mathbb{I}\} + 2\{T_i\}. \tag{2.62}$$

Using (2.59), we find that the term accompanying the identity is given by

$$T_0 \times T_i \simeq \sum_{n=0}^{N-1-i} \frac{b_n}{2k_{N-1}} \sum_{t=0}^{N-1-i-n} \kappa_{ab}\kappa^{ab}k_{N-1+i+n}\{\mathbb{I}\} + \cdots, \tag{2.63}$$

where we have suppressed the remaining terms. Since $k_a = 0$ for $a \geq N$, the expression above is zero unless $n + i = 0$. Since both are non-negative, that is unless $n = i = 0$. In the case that $i = 0$, we have from $\kappa_{ab}\kappa^{ab} = \dim\mathfrak{g}$ that

$$c_0 = N\dim(\mathfrak{g}), \quad c_i = 0, \quad \text{for } 0 < i < N. \tag{2.64}$$

2.5.2 Sugawara before Galilean contraction

In this section we consider the alternate branch of the diagram in Figure 2.5, where we consider the contraction of an N-fold tensor product of affine Lie algebras, where on each individual factor, there is a T field coming from the Sugawara construction.

On each tensor factor of $\widehat{\mathfrak{g}}^N$, the field T given by the Sugawara construction is

$$T_{(i)} = \frac{\kappa_{ab}}{2(k_{(i)}+h^\vee)}(J^a_{(i)}J^b_{(i)}), \tag{2.65}$$

and it generates a Virasoro algebra of central charge

$$c_{(i)} = \frac{k_{(i)}\dim\mathfrak{g}}{k_{(i)}+h^\vee}, \qquad i=0,\ldots,N-1, \tag{2.66}$$

where $k_{(i)}$ is the level of the algebra $\widehat{\mathfrak{g}}_{(i)}$.

Applying the Galilean change of basis as in (2.25), we introduce new fields

$$\begin{aligned} T_{i,\varepsilon} = \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} T_{(j)} &= \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} \frac{\sum_{\ell,\ell'=0}^{N-1} \omega^{-j(\ell+\ell')}\varepsilon^{-\ell-\ell'}\kappa_{ab}(J^a_{\ell,\varepsilon}J^b_{\ell',\varepsilon})}{2N\big(\sum_{m=0}^{N-1}\omega^{-jm}\varepsilon^{-m}k_{m,\varepsilon}+Nh^\vee\big)} \\ &= \frac{1}{2Nk_{N-1,\varepsilon}} \sum_{j,\ell,\ell'=0}^{N-1} (\omega^j\varepsilon)^{N-1+i-\ell-\ell'} \frac{\kappa_{ab}(J^a_{\ell,\varepsilon}J^b_{\ell',\varepsilon})}{1+\sum_{m=1}^{N-1}a_{m,\varepsilon}(\omega^j\varepsilon)^m}, \end{aligned} \tag{2.67}$$

where

$$a_{m,\varepsilon} = \frac{k_{N-1-m,\varepsilon}+Nh^\vee\delta_{m,N-1}}{k_{N-1,\varepsilon}}, \qquad m=1,\ldots,N-1. \tag{2.68}$$

Here we will again make use of lower-triangular Toeplitz matrices. Such a matrix, as in (2.53), has a decomposition given by

$$A = I + a_1\eta + \cdots + a_{N-1}\eta^{N-1}, \tag{2.69}$$

where I is the $N\times N$ identity matrix, and η is the $N\times N$ matrix

$$\eta = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{2.70}$$

We can expand the currents appearing in the normally-ordered products on the right hand side of (2.67) using the inverse maps (2.28). Similarly, the coefficient $\frac{1}{2(k_{(i)}+h^\vee)}$ has a power series expansion given by the inverse matrix A^{-1}, which was outlined in (2.55)-(2.56). Combining these two expressions, we have that the Sugawara field $T_{i,\varepsilon}$ can be expanded into the form

$$\begin{aligned} T_{i,\varepsilon} &= \frac{1}{2Nk_{N-1,\varepsilon}} \sum_{j,\ell,\ell'=0}^{N-1} (\omega^j\varepsilon)^{N-1+i-\ell-\ell'} \kappa_{ab}(J^a_{\ell,\varepsilon}J^b_{\ell',\varepsilon}) \Big(\sum_{n=0}^{N-1} b_{n,\varepsilon}(\omega^j\varepsilon)^b + \mathcal{O}(\varepsilon^N) \Big) \\ &= \frac{1}{2Nk_{N-1,\varepsilon}} \sum_{\ell,\ell',n=0}^{N-1} b_{n,\varepsilon}\kappa_{ab}(J^a_{\ell,\varepsilon}J^b_{\ell',\varepsilon}) \sum_{j=0}^{N-1} (\omega^j\varepsilon)^{N-1+i-\ell-\ell'+n} + \mathcal{O}(\varepsilon^{i+1}), \end{aligned} \tag{2.71}$$

where

$$b_{0,\varepsilon} = 1, \qquad b_{n,\varepsilon} = \sum_{p\in(\mathbb{N}_0)^n} (-1)^{|p|} \frac{\delta_{\|p\|,n}|p|!}{p_1!\cdots p_n!} a_{1,\varepsilon}^{p_1}\cdots a_{n,\varepsilon}^{p_n}, \qquad n=1,\ldots,N-1. \tag{2.72}$$

We are left to check that the resulting expression is well defined in the contraction limit. Evaluating the summation over j yields a factor of the form

$$\sum_{j=0}^{N-1} \omega^{j(N-1+i-\ell-\ell'+n)} = \begin{cases} N, & N-1+i-\ell-\ell'+n \equiv 0 \pmod N, \\ 0, & N-1+i-\ell-\ell'+n \not\equiv 0 \pmod N, \end{cases} \tag{2.73}$$

since the sum is over a complete set of roots of unity. Moreover, we have that

$$N-1+i-\ell-\ell'+n > -N, \tag{2.74}$$

so it follows that coefficients in $T_{i,\varepsilon}$ containing ε^m for m negative are 0. The limit $\varepsilon \to 0$ is therefore well defined, and furthermore truncates the expansion, resulting in

$$T_i = \frac{1}{2k_{N-1}} \sum_{\ell,\ell',n=0}^{N-1} b_n \kappa_{ab} (J^a_\ell J^b_{\ell'}) \delta_{N-1+i-\ell-\ell'+n,0}, \tag{2.75}$$

which then matches exactly the expressions given in (2.59).

Similarly, for the central parameters, we evaluate

$$\begin{aligned} c_{i,\varepsilon} = \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} c_{(j)} &= \varepsilon^i \sum_{j=0}^{N-1} \omega^{ij} \frac{\sum_{\ell=0}^{N-1} \omega^{-j\ell} \varepsilon^{-\ell} k_{\ell,\varepsilon} \dim \mathfrak{g}}{\sum_{\ell'=0}^{N-1} \omega^{-j\ell'} \varepsilon^{-\ell'} k_{\ell',\varepsilon} + Nh^\vee} \\ &= \frac{\dim \mathfrak{g}}{k_{N-1,\varepsilon}} \sum_{\ell,n=0}^{N-1} b_{n,\varepsilon} k_{\ell,\varepsilon} \sum_{j=0}^{N-1} (\omega^j \varepsilon)^{N-1+i-\ell+n} + \mathcal{O}(\varepsilon^{i+1}), \end{aligned} \tag{2.76}$$

where we see that

$$c_i = \frac{\dim \mathfrak{g}}{k_{N-1}} \sum_{\ell,n=0}^{N-1} b_n k_\ell \delta_{N-1+i-\ell+n,0} = N \dim \mathfrak{g}\, \delta_{i,0}, \tag{2.77}$$

which also matches the results for the central charge from the previous section. These calculations then demonstrate that not only is a Sugawara construction possible for Galilean affine Lie algebras, but it is compatible with the contraction procedure in the sense of Figure 2.5.

2.6 Higher-order Galilean W_3 algebras

Here we present the higher-order Galilean contraction of the W-algebra W_3.

2.6.1 The W_3 algebra

For reference, we restate the OPE relations of the W_3 algebra here. The W_3 algebra [33] of central charge c is generated by two fields, T, W where T generates a Virasoro subalgebra, and W is a primary field of conformal weight 3 under the action of T. The defining OPE relations are given by

$$T \times T \simeq \tfrac{c}{2}\{\mathbb{I}\} + 2\{T\}, \qquad T \times W \simeq 3\{W\}, \qquad W \times W \simeq \tfrac{c}{3}\{\mathbb{I}\} + 2\{T\} + \tfrac{32}{22+5c}\{\Lambda^{2,2}\}, \tag{2.78}$$

where

$$\Lambda = (TT) - \tfrac{3}{10}\partial^2 T \tag{2.79}$$

is quasi-primary.

The order-two Galilean W_3 algebra was given in (2.22).

2.6.2 Higher-order W_3 algebras

Using the higher-order Galilean contraction procedure, we obtain an infinite family of Galilean algebras with W_3 symmetry. For any $N \in \mathbb{Z}_{>0}$, the higher-order Galilean algebra $(W_3)_G^N$ is generated by the fields $\{T_i, W_i \mid i = 0, \dots, N-1\}$, and has central parameters $\{c_i \,|\, i = 0, \dots, N-1\}$. This algebra is well defined for all N, which we will show in the following, using techniques similar to those employed in the Sugawara construction for Galilean algebras.

First, it is straightforward to show that the Lie-type OPE relations of W_3 lead to the non-zero relations

$$T_i \times T_j \simeq \tfrac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, \qquad T_i \times W_j \simeq 3\{W_{i+j}\}, \qquad i+j \in \{0, \dots, N-1\}, \tag{2.80}$$

while

$$T_i \times T_j \simeq T_i \times W_j \simeq W_i \times W_j \simeq 0, \qquad i+j \geq N. \tag{2.81}$$

It is left to show that the OPE $W_i \times W_j$ is well defined for general $N \in \mathbb{Z}_{>0}$, such that $i+j < N$. To determine $W_i \times W_j$ we begin by considering the corresponding OPE relation $W_{i,\varepsilon} \times W_{j,\varepsilon}$ in $W_3^{\otimes N}$, which leads to

$$\begin{aligned} W_{i,\varepsilon} \times W_{j,\varepsilon} &= \varepsilon^{i+j} \sum_{r,s=0}^{N-1} \omega^{ir+js} W_{(r)} \times W_{(s)} \\ &\simeq \varepsilon^{i+j} \sum_{r=0}^{N-1} \omega^{(i+j)r} \Big[\frac{c_{(r)}}{3}\{\mathbb{I}\} + 2\{T_{(r)}\} + \frac{32}{22+5c_{(r)}}\{\Lambda_{(r)}\} \Big] \\ &= \frac{c_{i+j,\varepsilon}}{3}\{\mathbb{I}\} + 2\{T_{i+j,\varepsilon}\} + \varepsilon^{i+j} \sum_{r=0}^{N-1} \frac{32}{22+5c_{(r)}} \omega^{(i+j)r}\{\Lambda_{(r)}\}. \end{aligned} \tag{2.82}$$

Here we apply the same technique of power series expansion of the co-efficient, along with applying the inverse basis maps to the fields $T_{(r)}$ in the normally ordered product $\Lambda_{(r)}$. These are the same techniques from the Sugawara construction in Section 2.5. Performing the expansion and combining the resulting terms gives

$$\begin{aligned} \sum_{r=0}^{N-1} \frac{32}{22+5c_{(r)}} (\omega^r \varepsilon)^{i+j} \Lambda_{(r)} = \frac{32}{5Nc_{N-1,\varepsilon}} \sum_{n,\ell,\ell'=0}^{N-1} b_{n,\varepsilon} \sum_{r=0}^{N-1} (\omega^r \varepsilon)^{N-1+i+j-\ell-\ell'+n} (T_{\ell,\varepsilon} T_{\ell',\varepsilon}) \\ - \frac{48}{25c_{N-1,\varepsilon}} \sum_{n,\ell=0}^{N-1} b_{n,\varepsilon} \sum_{r=0}^{N-1} (\omega^r \varepsilon)^{N-1+i+j-\ell+n} \partial^2 T_{\ell,\varepsilon} + \mathcal{O}(\varepsilon^{i+j+1}), \end{aligned} \tag{2.83}$$

where $b_{n,\varepsilon}$ (and b_n appearing in (2.85) below) are given as in (2.72), (respectively (2.56)), coming from the inverse Toeplitz matrix, however the coefficients used in the formula are now

$$a_{m,\varepsilon} = \frac{c_{N-1-m,\varepsilon} + \frac{22N}{5}\delta_{m,N-1}}{c_{N-1,\varepsilon}}, \qquad a_m = \frac{c_{N-1-m} + \frac{22N}{5}\delta_{m,N-1}}{c_{N-1}}, \qquad m = 1, \dots, N-1. \tag{2.84}$$

In the limit $\varepsilon \to 0$, this yields

$$\sum_{r=0}^{N-1} \frac{32}{22+5c_{(r)}} (\omega^r \varepsilon)^{i+j} \Lambda_{(r)} \to \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \sum_{t=0}^{N-1-i-j-n} (T_{i+j+n+t} T_{N-1-t}) - \frac{48N}{25c_{N-1}} \partial^2 T_{N-1} \delta_{i,0} \delta_{j,0}. \tag{2.85}$$

Observing that, for every pair $r,s \in \{0,\ldots,N-1\}$ such that $r+s \in \{N-1,\ldots,2N-2\}$,

$$\Lambda_{r,s} = (T_r T_s) - \frac{3}{10} \partial^2 T_{N-1} \delta_{r+s,N-1} \tag{2.86}$$

is a quasi-primary field with respect to T_0, we then conclude that, for $i+j \in \{0,\ldots,N-1\}$,

$$W_i \times W_j \simeq \frac{c_{i+j}}{3} \{\mathbb{I}\} + 2\{T_{i+j}\} + \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \sum_{t=0}^{N-1-i-j-n} \{\Lambda_{i+j+n+t,N-1-t}\}. \tag{2.87}$$

Using that $\Lambda_{r,s} = \Lambda_{s,r}$, which can be easily shown using the basis transformation maps, this can be written as

$$\begin{aligned} W_i \times W_j \simeq {} & \frac{c_{i+j}}{3} \{\mathbb{I}\} + 2\{T_{i+j}\} \\ & + \sum_{n=0}^{N-1-i-j} \frac{32b_n}{5c_{N-1}} \left(\sum_{t=0}^{\lfloor \frac{N-2-i-j-n}{2} \rfloor} 2\{\Lambda_{i+j+n+t,N-1-t}\} + \{\Lambda_{\frac{N-1+i+j+n}{2},\frac{N-1+i+j+n}{2}}\} \right), \end{aligned} \tag{2.88}$$

where the last term is present only if $\frac{N-1+i+j+n}{2}$ is integer.

Using these relations, we can say that in general, the higher-order Galilean W_3 algebra will be a graded algebra. The reasoning follows from understanding the grading on b_n. We have the formula

$$b_n = \sum_{p \in (\mathbb{N}_0)^n} (-1)^{|p|} \frac{\delta_{\|p\|,n} |p|!}{p_1! \cdots p_n!} a_1^{p_1} \cdots a_n^{p_n}, \tag{2.89}$$

where

$$a_m = \frac{c_{N-1-m} + \frac{22N}{5} \delta_{m,N-1}}{c_{N-1}}, \qquad m = 0,\ldots,N-1. \tag{2.90}$$

It is clear to read off that

$$\mathrm{gr}(a_m) = -m, \qquad m = 1,\ldots,N-1. \tag{2.91}$$

The a_m are the only terms contributing to the grading of b_n. For any given $p \in (\mathbb{N}_0)^n$, we have that

$$\mathrm{gr}(a_1^{p_1} \cdots a_n^{p_n}) = -(1^{p_1} + 2^{p_2} + \ldots + n^{p_n}) = -\|p\|, \tag{2.92}$$

and from the Kronecker delta, we have $\|p\| = n$, so

$$\mathrm{gr}(b_n) = -n, \qquad n = 0,\ldots,N-1. \tag{2.93}$$

It is then easy to verify that all terms appearing in the OPE (2.88) indeed have grade $i+j$. Hence, the higher-order Galilean W_3 algebra of order-N is a truncated graded algebra for all N.

As an example, we provide the OPE relations for the third-order Galilean W_3 algebra. The algebra $(W_3)_G^3$ is generated by the fields $T_0, T_1, T_2, W_0, W_1, W_2$, which satisfy (2.80)-(2.81) with $N = 3$, along with the OPEs as well as

$$W_0 \times W_0 \simeq \tfrac{c_0}{3}\{\mathbb{I}\} + 2\{T_0\} + \tfrac{64}{5c_2}\{\Lambda_{0,2}\} + \tfrac{32}{5c_2}\{\Lambda_{1,1}\} - \tfrac{64c_1}{5(c_2)^2}\{\Lambda_{1,2}\} - \tfrac{32[(66+5c_0)c_2 - 5(c_1)^2]}{25(c_2)^3}\{\Lambda_{2,2}\}, \tag{2.94}$$

$$W_0 \times W_1 \simeq \tfrac{c_1}{3}\{\mathbb{I}\} + 2\{T_1\} + \tfrac{64}{5c_2}\{\Lambda_{1,2}\} - \tfrac{32c_1}{5(c_2)^2}\{\Lambda_{2,2}\}, \tag{2.95}$$

$$W_0 \times W_2 \simeq W_1 \times W_1 \simeq \tfrac{c_2}{3}\{\mathbb{I}\} + 2\{T_2\} + \tfrac{32}{5c_2}\{\Lambda_{2,2}\}, \tag{2.96}$$

where

$$\Lambda_{0,2} = (T_0T_2) - \tfrac{3}{10}\partial^2 T_2, \qquad \Lambda_{1,1} = (T_1T_1) - \tfrac{3}{10}\partial^2 T_2, \qquad \Lambda_{1,2} = (T_1T_2), \qquad \Lambda_{2,2} = (T_2T_2), \tag{2.97}$$

are quasi-primary.

2.6.3 Renormalisation

In this section, we would like to make an observation about the structure constants appearing in higher order Galilean algebras. In particular, we would like to remark that for algebras which are not of Lie-type, and as such arise from the expansion technique, have "renormalisable" structure constants in the following sense. Consider the algebra $(W_3)_G^N$, with central parameters c_0 left free, and

$$c_i = (c)^i, \qquad i = 1, \dots, N-1, \tag{2.98}$$

for some $c \in \mathbb{C}^\times$, leaving only two independent central parameters: the central charge c_0 and c. That is, other than c_0 all remaining central charges are powers of some element of $c \in \mathbb{C}^\times$. The a_m coefficients in the expansion (2.84) then simplify to

$$a_m = c^{-m}\big(1 + \big[c_0 + \tfrac{22N}{5} - 1\big]\delta_{m,N-1}\big), \qquad m = 1, \dots, N-1. \tag{2.99}$$

This then simplifies the inverse of the matrix A in (2.69) to the expression

$$A^{-1} = I - c^{-1}\eta + \big[1 - c_0 - \tfrac{22N}{5}\big](c^{-1}\eta)^{N-1}, \tag{2.100}$$

so when we have $N > 2$, the b_n coefficients become

$$b_0 = 1, \qquad b_1 = -c^{-1}, \qquad b_n = 0 \;\; (1 < n < N-1), \qquad b_{N-1} = \big[1 - c_0 - \tfrac{22N}{5}\big]c^{-(N-1)}. \tag{2.101}$$

We can introduce renormalised generating fields, defined by

$$\widehat{T}_i = c^{-i}T_i, \qquad \widehat{W}_i = c^{-i}W_i, \qquad i = 0, \dots, N-1, \tag{2.102}$$

and correspondingly, the renormalised quasi-primary fields

$$\hat{\Lambda}^{2,2}_{r,s} = c^{-r-s}\Lambda^{2,2}_{r,s}. \tag{2.103}$$

Renormalising in this way leads to the new OPE relations for ($i+j \in \{0,\dots,N-1\}$) given by

$$\widehat{T}_i \times \widehat{T}_j \simeq \frac{c_0^{\delta_{i+j,0}}}{2}\{\mathbb{I}\} + 2\{\widehat{T}_{i+j}\}, \qquad \widehat{T}_i \times \widehat{W}_j \simeq 3\{\widehat{W}_{i+j}\}, \tag{2.104}$$

and

$$\begin{aligned}\widehat{W}_i \times \widehat{W}_j \simeq & \frac{c_0^{\delta_{i+j,0}}}{3}\{\mathbb{I}\} + 2\{\widehat{T}_{i+j}\} \\ & + \tfrac{32}{5}\sum_{n=0,1}\sum_{t=0}^{N-1-i-j-n}(-1)^n\{\hat{\Lambda}^{2,2}_{i+j+n+t,N-1-t}\} + \tfrac{32}{5}\left[1-c_0-\tfrac{22N}{5}\right]\{\hat{\Lambda}^{2,2}_{N-1,N-1}\}\delta_{i+j,0}.\end{aligned} \tag{2.105}$$

The central parameter c has thus been absorbed by a renormalisation of the algebra generators.

Naturally, because of the analogous construction, we can perform a similar renormalisation on the Galilean Sugawara construction given in Section 2.5. The currents are renormalised as

$$\widehat{J}^a_i = k^{-i}J^a_i, \qquad \widehat{T}_i = k^{-i}T_i, \qquad i = 0,\dots,N-1, \tag{2.106}$$

where $k_i = k^i$, $i = 1,\dots,N-1$, for some $k \in \mathbb{C}^\times$. This then leads to the following renormalised Galilean Virasoro fields

$$\widehat{T}_i = \tfrac{1}{2}\sum_{n=0,1}\sum_{t=0}^{N-1-i-n}\kappa_{ab}(\widehat{J}^a_{i+n+t}\widehat{J}^b_{N-1-t}) + \tfrac{1}{2}[1-k_0-Nh^\vee]\kappa_{ab}(\widehat{J}^a_{N-1}\widehat{J}^b_{N-1})\delta_{i,0}, \tag{2.107}$$

The non-zero OPE relations are given by (for $i+j \in \{0,\dots,N-1\}$)

$$\widehat{J}^a_i \times \widehat{J}^b_j \simeq \kappa^{ab}k_0^{\delta_{i+j,0}}\{\mathbb{I}\} + f^{ab}{}_c\{\widehat{J}^c_{i+j}\}, \quad \widehat{T}_i \times \widehat{J}^a_j \simeq \{\widehat{J}^a_{i+j}\}, \quad \widehat{T}_i \times \widehat{T}_j \simeq \tfrac{N\dim\mathfrak{g}}{2}\{\mathbb{I}\}\delta_{i+j,0} + 2\{\widehat{T}_{i+j}\}. \tag{2.108}$$

This concludes the discussion of higher-order Galilean algebras. In the next chapter, we will generalise this procedure, by changing the possible ε dependent basis transformations that we can make. In particular, we will consider the case of when the basis transformation is made up of a tensor product of smaller transformations, affecting only a part of the full algebra $\mathcal{A}^{\otimes N}$.

The following publication has been incorporated as Chapter 3.

E. Ragoucy, J. Rasmussen, C. Raymond, *Multi-graded Galilean conformal algebras*, Nucl. Phys. **B957** (2020), 115092, arXiv:2002.08637 [hep-th].

Chapter 3

Multi-graded Galilean algebras

3.1 Introduction

In this chapter, we extend the previous notion of higher-order Galilean algebras by relaxing the form of the basis change matrix. One can understand the basis change in (2.25) as acting on the ordered basis of generating fields $A_{(i)} \in \mathcal{A}^{\otimes N}$ with a Vandermonde matrix

$$\begin{pmatrix} A_{0,\varepsilon} \\ A_{1,\varepsilon} \\ \vdots \\ A_{N-1,\varepsilon} \end{pmatrix} = \begin{pmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \varepsilon\omega^0 & \varepsilon\omega^1 & \cdots & \varepsilon\omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{N-1}\omega^0 & \varepsilon^{N-1}\omega^{N-1} & \cdots & \varepsilon^{N-1}\omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} A_{(0)} \\ A_{(1)} \\ \vdots \\ A_{(N-1)} \end{pmatrix}, \tag{3.1}$$

and similarly for the central parameters. In this section, we show that the Vandermonde matrix which determines the change of basis can be replaced by a tensor product of Vandermonde matrices. The resulting algebra, called a multi-graded Galilean algebra, is still defined in the contraction limit $\varepsilon \to 0$. In fact, in this construction, one is able to assign a different contraction parameter to each tensor factor of the basis change matrix. As such, the multi-graded algebras can also be used to understand "partial" contractions, as well as exchanging the order of contractions on separate factors.

The resulting multi-graded algebras have a basis labelled by sequences related to a particular factorisation of $N = N_1 \dots N_k$, the number of underlying copies of the input algebra. The previously studied higher-order Galilean algebras relate to the factorisation $N = N \times 1$ in this framework. As well as introducing new infinite families of symmetry algebras, this new framework also describes the process of multiple contractions, that is, the Galilean contraction of Galilean algebras.

We investigate the so-called multi-graded structure of these algebras and their OPE relations, showing that the structure is graded by integer sequences, which are also truncated by the Galilean structure. We also discuss their relation to a a generalised family of polynomial Takiff algebras.

Furthermore, we consider a number of explicit examples relevant to conformal field theory. We present a Sugawara construction on multi-graded Galilean affine Lie algebras, and show that again it commutes with the contraction procedure. Finally, we demonstrate the multi-graded contraction for the W-algebra W_3.

3.2 Multi-graded Galilean algebras

3.2.1 Preliminary theory

In this section, we again begin by considering the tensor product of a number $N \in \mathbb{Z}_{>0}$ of operator product algebras $\mathcal{A}$, that is

$$\mathcal{A}^{\otimes N} = \bigotimes_{i=0}^{N-1} \mathcal{A}_{(i)}. \tag{3.2}$$

As before, $\mathcal{A}_{(i)}$ are copies of the same OPA $\mathcal{A}$ up to the value of their central parameters. We introduce the following notation

$$\mathbf{A}_* = \begin{pmatrix} A_{(0)} \\ \vdots \\ A_{(N-1)} \end{pmatrix}, \qquad \mathbf{c}_* = \begin{pmatrix} c_{(0)} \\ \vdots \\ c_{(N-1)} \end{pmatrix}, \tag{3.3}$$

where $A_{(i)}$ (respectively $c_{(i)}$) denotes the field $A \in \mathcal{A}_{(i)}$ (respectively the central parameter of $\mathcal{A}_{(i)}$). We also introduce the following matrices, for $\varepsilon \in \mathbb{C}^{\times}$.

$$U_N(\varepsilon,\omega) = D_N(\varepsilon)U_N(\omega), \qquad D_N(\varepsilon) = \operatorname{diag}(\varepsilon^0,\varepsilon^1,\ldots,\varepsilon^{N-1}), \tag{3.4}$$

where the matrix $U_N(\omega)$, i.e. explicitly without ε dependence, is given by

$$U_N(\omega) = \left(\omega^{ij}\right)_{0\le i,j\le N-1} = \begin{pmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{pmatrix}, \tag{3.5}$$

where, as before, we have taken $\omega = e^{2\pi i/N}$ to be a principal N^{th} root of unity. This is simply a small change of notation from that given in the previous section, which will better allow us to express the algebraic structure. Continuing, for $\varepsilon \neq 0$, the inverse matrices are given by

$$D_N^{-1}(\varepsilon) = D_N(\varepsilon^{-1}), \qquad U_N^{-1}(\omega) = \tfrac{1}{N}U_N(\omega^{-1}). \tag{3.6}$$

Using this, we can express the precontraction change of basis by

$$\mathbf{A}_\varepsilon = \begin{pmatrix} A_{0,\varepsilon} \\ \vdots \\ A_{N-1,\varepsilon} \end{pmatrix} = U_N(\varepsilon,\omega)\mathbf{A}_*, \qquad \mathbf{c}_\varepsilon = \begin{pmatrix} c_{0,\varepsilon} \\ \vdots \\ c_{N-1,\varepsilon} \end{pmatrix} = U_N(\varepsilon,\omega)\mathbf{c}_*, \tag{3.7}$$

which gives us the map

$$\mathcal{A}^{\otimes N} \to \mathcal{A}^{\otimes N}, \qquad \mathbf{A}_* \mapsto \mathbf{A}_\varepsilon, \qquad \mathbf{c}_* \mapsto \mathbf{c}_\varepsilon. \tag{3.8}$$

As before, this map is invertible for $\varepsilon \neq 0$ and singular in the limit $\varepsilon \to 0$ (unless we are in the trivial case of $N = 1$). If, in the limit $\varepsilon \to 0$ the resulting algebra is well defined, we call that algebra the Galilean algebra, denoted $\mathcal{A}_G^N$.

3.2.2 Introduction of grading sequences

We begin by fixing some $\sigma \in \mathbb{Z}_{>0}$. We then denote a sequence of $\mathbb{C}$-numbers of length σ by $\mathbf{S} = S_1, \ldots, S_\sigma$, where for example $\mathbf{0} = 0, \ldots, 0$ is the zero sequence. We can form linear combinations of sequences straightforwardly, by

$$\alpha\,\mathbf{i} + \beta\,\mathbf{j} = \alpha i_1 + \beta j_1, \ldots, \alpha i_\sigma + \beta j_\sigma, \qquad \alpha, \beta \in \mathbb{C}. \tag{3.9}$$

We can introduce multiplicative inverses. When every element of $\mathbf{S}$ is non-zero, we let $\mathbf{S}^{-1}$ denote the sequence of inverses $S_1^{-1}, \ldots, S_\sigma^{-1}$.

We can compare two sequences according to the rules

$$\mathbf{i} \leq \mathbf{j} \quad \text{if} \quad i_1 \leq j_1, \ldots, i_\sigma \leq j_\sigma; \qquad \mathbf{i} < \mathbf{j} \quad \text{if} \quad i_1 < j_1, \ldots, i_\sigma < j_\sigma. \tag{3.10}$$

A central object of our study will be the sequence $\mathbf{N} = N_1, \ldots, N_k$, where N is the number of input algebras, and $N_1 \ldots N_k$ is a factorisation of N.

We define the set of integer sequences I_N as

$$I_N = \{\mathbf{i} \in \mathbb{Z}^{\otimes \sigma} \,|\, \mathbf{0} \leq \mathbf{i} < \mathbf{N}\}. \tag{3.11}$$

We note that the set of integer sequences is not closed under the operation of inverse sequence, nor is that operation well defined on all elements of I_N.

The set I_N admits a canonical ordering where $\mathbf{i}$ appears before $\mathbf{j}$, or $\mathbf{i} \prec \mathbf{j}$ if and only if, for each m such that $i_m > j_m$, there exists $\ell < m$ such that $i_\ell < j_\ell$. That is, we order the sequences by comparing entries left-to-right. We remark that the output of this ordering is the same as the ordering of basis vectors when one considers the tensor product space $V = V_1 \otimes \ldots \otimes V_\sigma$ under the Kronecker product. We make this the canonical choice of ordering for exactly this reason. Each factor V_ℓ is an N_ℓ-dimensional vector space with ordered basis $\{e_1^\ell, \ldots, e_{N_\ell}^\ell\}$.

Explicitly, for an N-vector $\mathbf{v} \in V = V_1 \otimes \ldots \otimes V_\sigma$, the components $\{v_\mathbf{i} \mid \mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}\}$ in any given linear combination

$$\mathbf{v} = \sum_{\mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}} v_\mathbf{i} e_\mathbf{i} \in V, \tag{3.12}$$

are ordered according to the canonical ordering on I_N. The final sequence in any such ordering is given by $\mathbf{N} - \mathbf{1} = N_1 - 1, \ldots, N_k - 1$.

As an example, we have for for the sequence $\mathbf{N} = 2, 3$ the ordered components

$$v_{0,0},\, v_{0,1},\, v_{0,2},\, v_{1,0},\, v_{1,1},\, v_{1,2}. \tag{3.13}$$

We form the N-dimensional vectors

$$\mathbf{A}_* = \Big(A_{(\mathbf{i})}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}}, \qquad \mathbf{c}_* = \Big(c_{(\mathbf{i})}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}}, \tag{3.14}$$

whose entries are ordered by the canonical ordering on I_N. As per previous notation, $A_{(i)}$ refers to the field $A \in \mathcal{A}_{(i)}$, and likewise, $c_{(i)}$ is the central parameter of $\mathcal{A}_{(i)}$. Furthermore, we introduce the sequences

$$\omega = \omega_1,\ldots,\omega_\sigma, \qquad \omega_\ell = e^{2\pi \mathrm{i}/N_\ell}, \qquad \ell = 1,\ldots,\sigma, \tag{3.15}$$

and

$$\varepsilon = \varepsilon_1,\ldots,\varepsilon_\sigma, \qquad \varepsilon_\ell \in \mathbb{C}, \qquad \ell = 1,\ldots,\sigma. \tag{3.16}$$

We can then use this notation to define

$$U_{\mathbf{N}}(\varepsilon,\omega) = U_{N_1}(\varepsilon_1,\omega_1)\otimes\ldots\otimes U_{N_\sigma}(\varepsilon_\sigma,\omega_\sigma) = D_{\mathbf{N}}(\varepsilon)U_{\mathbf{N}}(\omega), \tag{3.17}$$

where

$$D_{\mathbf{N}}(\varepsilon) = D_{N_1}(\varepsilon_1)\otimes\ldots\otimes D_{N_\sigma}(\varepsilon_\sigma), \qquad U_{\mathbf{N}}(\omega) = U_{N_1}(\omega_1)\otimes\ldots\otimes U_{N_\sigma}(\omega_\sigma), \tag{3.18}$$

which simply generalises the previously introduced notions of D_N and U_N to the case where they are composed of a product of smaller matrices.

Now the contraction basis change map $\mathcal{A}^{\otimes N} \to \mathcal{A}_\varepsilon^{\otimes N}$, which is given by

$$\mathbf{A}_* \mapsto \mathbf{A}_\varepsilon = \Big(A_{\mathbf{i},\varepsilon}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}} = U_{\mathbf{N}}(\varepsilon,\omega)\mathbf{A}_*, \qquad \mathbf{c}_* \mapsto \mathbf{c}_\varepsilon = \Big(c_{\mathbf{i},\varepsilon}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}} = U_{\mathbf{N}}(\varepsilon,\omega)\mathbf{c}_*, \tag{3.19}$$

is invertible when $\varepsilon_1,\ldots,\varepsilon_\sigma \neq 0$, where it is given explicitly by

$$U_{\mathbf{N}}^{-1}(\varepsilon,\omega) = \tfrac{1}{N}U_{\mathbf{N}}(\omega^{-1})D_{\mathbf{N}}(\varepsilon^{-1}). \tag{3.20}$$

In particular, we have the following identity for $U_{\mathbf{N}}(\varepsilon,\omega)$ invertible, namely

$$\sum_{\mathbf{0}\preceq\mathbf{k}\prec\mathbf{N}} U_{\mathbf{N}}(\varepsilon,\omega)_{\mathbf{ik}}\,U_{\mathbf{N}}(\varepsilon,\omega)_{\mathbf{jk}}\,U_{\mathbf{N}}(\varepsilon^{-1},\omega^{-1})_{\mathbf{km}} = N\delta_{\mathbf{m},\mathbf{i}+\mathbf{j}}, \tag{3.21}$$

for $\mathbf{0}\preceq\mathbf{i},\mathbf{j},\mathbf{m}\prec\mathbf{N}$. This follows straightforwardly from the following result on sums of roots of unity,

$$\sum_{n=0}^{N_\ell-1} \omega_\ell^{nk} = N_\ell\delta_{k,0 \bmod N_\ell}, \qquad \ell = 1,\ldots,\sigma. \tag{3.22}$$

If in the limit $\varepsilon \to \mathbf{0}$, the resulting algebra is well defined, and the resulting algebra structure is that of a Galilean OPA (that is, we rule out trivial cases), we denote the resulting algebra by $\mathcal{A}_G^{\mathbf{N}}$.

As before, the OPE relations for Lie-type algebras are straightforwardly determined in general. For a Lie type algebra with product

$$A\times B \simeq f^{AB}{}_{\mathbb{I}}(c)\{\mathbb{I}\} + f^{AB}{}_C\{C\}, \tag{3.23}$$

where we assume a summation over fields C, the corresponding multi-graded OPA has product relations

$$A_{\mathbf{i}}\times B_{\mathbf{j}} \simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{\mathbf{i}+\mathbf{j}})\{\mathbb{I}\} + f^{AB})C\{C_{\mathbf{i}+\mathbf{j}}\}, & \text{if } \mathbf{i}+\mathbf{j}<\mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.24}$$

We continue by presenting the examples of multi-graded Galilean Virasoro and affine Lie algebras.

3.2.3 Example: Multi-graded Galilean Virasoro algebras

The multi-graded Galilean Virasoro algebra $\mathfrak{Vir}_G^{\mathbf{N}}$ for $N \in \mathbb{N}$ is generated by fields $T_{\mathbf{i}}$, labelled by the sequences $\mathbf{i}$ of length σ, for a given factorisation of $N = N_1 \dots N_\sigma$. The sequence $\mathbf{i}$ is bounded, in the sense of the canonical ordering on I_N, by $\mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}$. Similarly, the algebra has central parameters $\{c_{\mathbf{i}} \mid \mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}\}$. The OPE relations are given by

$$T_{\mathbf{i}} \times T_{\mathbf{j}} \simeq \begin{cases} \frac{c_{\mathbf{i}+\mathbf{j}}}{2}\{\mathbb{I}\} + 2\{T_{\mathbf{i}+\mathbf{j}}\}, & \text{if } \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.25}$$

As in the previous constructions, the field $T_{\mathbf{0}}$ generates a Virasoro subalgebra of central charge $c_{\mathbf{0}}$, and all other generating fields of the algebra are quasi-primary fields with respect to the Virasoro generator.

3.2.4 Example: Multi-graded Galilean affine Lie algebras

It is also straightforward to define the multi-graded affine Lie algebras. These algebras are generated by currents $\{J_{\mathbf{i}}^a \mid \mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}, a = 1, \dots, \dim(\mathfrak{g})\}$. The OPE on a multi-graded affine Lie algebra is given by

$$J_{\mathbf{i}}^a \times J_{\mathbf{j}}^b \simeq \begin{cases} \kappa^{ab} k_{\mathbf{i}+\mathbf{j}}\{\mathbb{I}\} + f^{ab}{}_c\{J_{\mathbf{i}+\mathbf{j}}^c\}, & \text{if } \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.26}$$

Similarly to the previous cases, the fields $J_{\mathbf{0}}^a$ generate a subalgebra isomorphic to $\widehat{\mathfrak{g}}$ at level $k_{\mathbf{0}}$.

3.3 General properties of multi-graded Galilean algebras

3.3.1 The grading on multi-graded Galilean algebras

We can extend our previous notion of a grading on a Galilean algebra to the multi-graded algebras in the following way. We extend the action of the grading operator to sequences $\mathrm{gr} : \mathcal{A}_G^{\mathbf{N}} \to I_N$, such that

$$A_{\mathbf{i}} \mapsto \mathbf{i}, \quad c_{\mathbf{i}} \mapsto \mathbf{i}, \quad \left(A_{\mathbf{i}} B_{\mathbf{j}}\right) \mapsto \mathbf{i}+\mathbf{j}. \tag{3.27}$$

With the action defined on the generating fields and central parameters, we can extend linearly and to Laurent polynomials of the central parameters as before.

As an example, an algebra based on the sequence $\mathbf{N} = 4,5$ contains the following combination of fields with the stated action of gr,

$$\mathrm{gr}\left(c_{0,0}\partial B_{3,1} - 35\,\frac{c_{1,1}c_{2,1} + 4c_{3,2}}{c_{1,4}}\left(A_{0,1}B_{1,2}\right)\right) = 3,1. \tag{3.28}$$

We then say that a given algebra is multi-graded if the grading is compatible with the OPE, in the sense that

$$\mathrm{gr}\left(A_{\mathbf{i}} \times B_{\mathbf{j}}\right) = \mathbf{i}+\mathbf{j}. \tag{3.29}$$

Similarly to the case of higher-order Galilean algebras, multi-graded algebras coming from tensor products of Lie-type algebras are indeed multi-graded in the sense outlined above. This follows directly from linearity of the OPE. Moreover, the grading on the algebra is truncated at $\mathbf{N}$ by the Galilean structure.

Also analogous to the higher order case, we cannot guarantee that the contraction of a non-linear OPA, such as algebras arising from products of copies of W_3 will necessarily be multi-graded in general. This is due to the contraction procedure, and thus the produced Galilean algebras, being sensitive to the structure constant functions of the central parameters. However, as will be seen in Section 3.5, the W_3 algebra will admit a grading by truncated sequences.

3.3.2 Permutation invariance

In all examples of multi-graded algebras that we have calculated, we see that the following holds for the OPE relations

$$A_{\mathbf{i}} \times B_{\mathbf{j}} \simeq A_{\mathbf{i}'} \times B_{\mathbf{j}'} \qquad \text{if} \qquad \mathbf{i}+\mathbf{j} = \mathbf{i}'+\mathbf{j}'. \tag{3.30}$$

For Lie type algebras, this follows form linearity, however it is not necessarily clear that such relations need to hold for the W-algebras. Together with the property of being multi-graded, this congruence implies that we need only determine the OPE relations between the underlying grade-0 subalgebra, and the generators of the Galilean algebra to completely determine the structure of the algebra.

Moreover, it also implies that equivalent factorisations of N lead to isomorphic multi-graded Galilean algebras. Explicitly, given a particular factorisation of $N = N_1 \dots N_\sigma$, which defines the sequence $\mathbf{N} = N_1, \dots, N_\sigma$, and its corresponding multi-graded Galilean algebra $\mathcal{A}_G^{\mathbf{N}}$, and given a permutation $\pi = \pi_1, \dots, \pi_\sigma$, a permutation of a length σ sequence which naturally defines an equivalent factorisation, then we have

$$\mathcal{A}_G^{\mathbf{N}} \cong \mathcal{A}_G^{\pi(\mathbf{N})}, \qquad \text{for } \pi(\mathbf{N}) = N_{\pi_1}, \dots, N_{\pi_\sigma}. \tag{3.31}$$

Furthermore, this leads to the interesting property that the algebra is also equivalent under exchange of order of contractions, that is, we can understand a multi-graded contraction to be equivalent to higher-order contractions of chunks of the algebra $\mathcal{A}^{\otimes N}$, where the size of each chunk is given by a factor of N in a particular factorisation. Explicitly, we have

$$\mathcal{A}_G^{N_1,N_2} \cong (\mathcal{A}_G^{N_1})_G^{N_2}, \tag{3.32}$$

and more generally,

$$(\dots((\mathcal{A}_G^{N_1})_G^{N_2})\dots)_G^{N_\sigma} \cong \mathcal{A}_G^{N_1,\dots,N_\sigma} \cong \mathcal{A}_G^{N_{\pi_1},\dots,N_{\pi_\sigma}} \cong (\dots((\mathcal{A}_G^{N_{\pi_1}})_G^{N_{\pi_2}})\dots)_G^{N_{\pi_\sigma}}. \tag{3.33}$$

This gives a systematic way of describing Galilean contractions of (higher-order) Galilean algebras.

3.3.3 Relation to multi-variable Takiff algebras

We consider taking the limit $\mathbf{N} \to \overline{\infty}$, where $\overline{\infty}$ denotes the infinity sequence in the sense that we let

$$N_\ell \to \infty, \; \forall\, N_\ell \text{ such that } \frac{N_\ell}{N_{\ell'}} \to R^{\ell}_{\ell'}, \quad R^{\ell}_{\ell'} \in \mathbb{R}, \quad \ell, \ell' = 1, \ldots, \sigma. \tag{3.34}$$

Taking $\widehat{\mathfrak{g}}^{\mathbf{N}}_G$ as an example, in the limit $\mathbf{N} \to \overline{\infty}$, the algebra becomes $\widehat{\mathfrak{g}}^{\overline{\infty}}_G$, which is generated by the currents $\{J^a_{\mathbf{i}} \mid \mathbf{i} \geq \mathbf{0},\ a = 1, \ldots, \dim(\mathfrak{g})\}$, with non-zero OPEs

$$J^a_{\mathbf{i}} \times J^b_{\mathbf{j}} \simeq \kappa^{ab} k_{\mathbf{i}+\mathbf{j}}\{\mathbb{I}\} + f^{ab}{}_c\{J^c_{\mathbf{i}+\mathbf{j}}\}. \tag{3.35}$$

This algebra is isomorphic to the tensor product

$$\widehat{\mathfrak{g}}^{\overline{\infty}}_G \cong \widehat{\mathfrak{g}} \otimes \mathbb{C}[x_1, \ldots, x_\sigma], \tag{3.36}$$

where $\mathbb{C}[x_1, \ldots, x_\sigma]$ is the polynomial ring in σ variables. As before, we realise the truncated structure of $\widehat{\mathfrak{g}}^{\mathbf{N}}_G$ by considering a quotient of this ring corresponding to the chosen factorisation of $N = N_1 \ldots N_\sigma$. That is, we have the isomorphism

$$\widehat{\mathfrak{g}}^{\mathbf{N}}_G \cong \widehat{\mathfrak{g}} \otimes \mathbb{C}[x_1, \ldots, x_\sigma] / \langle x_1^{N_1}, \ldots, x_\sigma^{N_\sigma} \rangle. \tag{3.37}$$

[82, 83, 85] Similarly, we have for the Virasoro algebra

$$\mathfrak{Vir}^{\overline{\infty}}_G \cong \mathfrak{Vir} \otimes \mathbb{C}[x_1, \ldots, x_\sigma], \quad \mathfrak{Vir}^{\mathbf{N}}_G \cong \mathfrak{Vir} \otimes \mathbb{C}[x_1, \ldots, x_\sigma] / \langle x_1^{N_1}, \ldots, x_\sigma^{N_\sigma} \rangle. \tag{3.38}$$

One need not truncate the sequence completely, that is, we may allow some positions of a sequence to be truncated, and others to run to infinity. Consider a subset of $\{1, \ldots, \sigma\}$ denoted $s = \{s_1, \ldots, s_\rho\}$, and take the limit $\mathbf{N} \to \overline{\infty}_s$ in the sense that

$$N_{s_1}, \ldots, N_{s_\rho} \to \infty, \text{ such that } \frac{N_{s_i}}{N_{s_j}} \to R^{s_i}_{s_j}, \quad R^{s_i}_{s_j} \in \mathbb{R},\ s_i, s_j \in s. \tag{3.39}$$

Such a construction gives rise to the semi-truncated algebras

$$\widehat{\mathfrak{g}}^{\overline{\infty}_s}_G \cong \widehat{\mathfrak{g}} \otimes C[x_1, \ldots, x_\sigma] / \langle x_{s_1}^{N_{s_1}}, \ldots, x_{s_\rho}^{N_{s_\rho}} \rangle, \quad \mathfrak{Vir}^{\overline{\infty}_s}_G \cong \mathfrak{Vir} \otimes C[x_1, \ldots, x_\sigma] / \langle x_{s_1}^{N_{s_1}}, \ldots, x_{s_\rho}^{N_{s_\rho}} \rangle. \tag{3.40}$$

We remark that the notion of a canonical ordering on the space of sequences breaks down in the case of semi-truncated algebras. However, their constructions as Takiff algebras are quite natural.

3.4 Multi-graded Galilean Sugawara construction

As in the case of the higher-order Galilean algebras, we want to demonstrate that multi-graded affine Lie algebras admit a Sugawara construction, and moreover, that the construction is compatible with the contraction procedure. As before, compatibility with the contraction procedure is understood to mean that the following diagram commutes.

$$\begin{array}{ccc}
\widehat{\mathfrak{g}}^{\otimes \mathbf{N}} & \xrightarrow{\mathrm{Sug}^{\otimes \mathbf{N}}} & \mathfrak{Vir}^{\otimes \mathbf{N}} \\
\Big\downarrow \mathrm{Gal} & & \Big\downarrow \mathrm{Gal} \\
\widehat{\mathfrak{g}}_G^{\mathbf{N}} & \xrightarrow{\mathrm{Gal\ Sug}} & \mathfrak{Vir}_G^{\mathbf{N}}
\end{array}$$

We will analyse the two branches separately, and show that the resulting algebras are equivalent.

We begin with the construction of a Galilean Virasoro action on the algebra $\widehat{\mathfrak{g}}_G^{\mathbf{N}}$. We make the following ansatz for the generators,

$$T_{\mathbf{i}} = \sum_{\mathbf{0} \preceq \mathbf{r},\mathbf{s} \prec \mathbf{N}} \lambda_{\mathbf{i}}^{\mathbf{r},\mathbf{s}} \kappa_{ab} \big(J_{\mathbf{r}}^a J_{\mathbf{s}}^b \big), \tag{3.41}$$

where κ_{ab} is the inverse Killing form on the semisimple Lie algebra $\mathfrak{g}$ underlying $\widehat{\mathfrak{g}}$. As in the previous case, the introduced unknowns $\lambda_{\mathbf{i}}^{\mathbf{r},\mathbf{s}}$ are determined by requiring that the currents $J_{\mathbf{i}}^a$ are primary fields of conformal weight 1 with respect to the operator $T_{\mathbf{0}}$, and that the $T_{\mathbf{i}}$ fields form a Galilean Virasoro algebra. This amounts to requiring that

$$T_{\mathbf{i}} \times J_{\mathbf{j}}^a \simeq \begin{cases} \{J_{\mathbf{i}+\mathbf{j}}^a\}, & \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.42}$$

As before, we compute

$$\begin{aligned} J_{\mathbf{j}}^a(z) T_{\mathbf{i}}(w) \sim{}& \frac{1}{(z-w)^2} \sum_{\mathbf{0} \preceq \mathbf{r},\mathbf{s} \prec \mathbf{N}} \lambda_{\mathbf{i}}^{\mathbf{r};\mathbf{s}} \big[k_{\mathbf{j}+\mathbf{r}} J_{\mathbf{s}}^a(w) + k_{\mathbf{j}+\mathbf{s}} J_{\mathbf{r}}^a(w) + 2h^\vee J_{\mathbf{j}+\mathbf{r}+\mathbf{s}}^a(w) \big] \\ &+ \frac{1}{z-w} \sum_{\mathbf{0} \preceq \mathbf{r},\mathbf{s} \prec \mathbf{N}} \lambda_{\mathbf{i}}^{\mathbf{r};\mathbf{s}} \kappa_{bc} \big[f^{ab}{}_d (J_{\mathbf{j}+\mathbf{r}}^d J_{\mathbf{s}}^c)(w) + f^{ac}{}_d (J_{\mathbf{r}}^b J_{\mathbf{j}+\mathbf{s}}^d)(w) \big], \end{aligned} \tag{3.43}$$

where again, the dual Coxeter number of $\mathfrak{g}$ arises through the identity

$$\kappa_{bc} f^{ab}{}_d f^{dc}{}_e = 2h^\vee \delta_e^a. \tag{3.44}$$

The OPE relations (3.42) imply that the poles involving normally-ordered products of fields must vanish. Moreover, they constrain the constants appearing with second-order poles. The first order pole constraint gives

$$\lambda_{\mathbf{i}}^{\mathbf{r};\mathbf{s}} = \begin{cases} \lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}}, & \mathbf{r}+\mathbf{s} = \mathbf{N}-\mathbf{1}+\mathbf{n}, \text{ for } (\mathbf{0} \preceq \mathbf{n} \prec \mathbf{N}), \\ 0, & \text{otherwise,} \end{cases} \tag{3.45}$$

where $\mathbf{1} = 1, \ldots, 1$, the sequence of ones. For each value of $\mathbf{i}$ where $\mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}$, this equation fixes all but the N coefficients $\lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}}$. The constraints coming from the second order pole then gives

$$2 \sum_{\mathbf{j} \preceq \mathbf{m} \prec \mathbf{N}} \sum_{\mathbf{0} \preceq \mathbf{n} \preceq \mathbf{m}-\mathbf{j}} \lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}} k_{\mathbf{N}-\mathbf{1}-\mathbf{m}+\mathbf{j}+\mathbf{n}} J_{\mathbf{m}}^a + 2Nh^\vee \lambda_{\mathbf{i}}^{\mathbf{0};\mathbf{N}-\mathbf{1}} \delta_{\mathbf{j},\mathbf{0}} J_{\mathbf{N}-\mathbf{1}}^a = \begin{cases} J_{\mathbf{i}+\mathbf{j}}^a, & \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.46}$$

As in the higher-order case, we notice that for each $\mathbf{i}$, the conditions (3.46) are nested so that we need only solve the system for $\mathbf{j} = \mathbf{0}$, which leads to

$$2 \sum_{\mathbf{0} \preceq \mathbf{m} \prec \mathbf{N}} \sum_{\mathbf{0} \preceq \mathbf{n} \preceq \mathbf{m}} \lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}} \big(k_{\mathbf{N}-\mathbf{1}-\mathbf{m}+\mathbf{n}} + Nh^\vee \delta_{\mathbf{n},\mathbf{0}} \delta_{\mathbf{m},\mathbf{N}-\mathbf{1}} \big) J_{\mathbf{m}}^a = J_{\mathbf{i}}^a. \tag{3.47}$$

Since the generating currents of the Galilean algebra are linearly independent, (3.47) gives rise to a lower-triangular system of linear equations in $\{\lambda_{\mathbf{i}}^{\mathbf{n},\mathbf{N}-\mathbf{1}} \mid \mathbf{0} \preceq \mathbf{n} \preceq \mathbf{N}-\mathbf{1}\}$. As in the case of higher-order contractions, we realise the system as a matrix of the coefficients

$$M_{\mathbf{m},\mathbf{n}} = \begin{cases} 2k'_{\mathbf{N}-\mathbf{1}-\mathbf{m}+\mathbf{n}}, & \mathbf{0} \preceq \mathbf{m}-\mathbf{n} \prec \mathbf{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.48}$$

where

$$k'_{\mathbf{m}} = k_{\mathbf{m}} + Nh^{\vee}\delta_{\mathbf{m},\mathbf{0}}, \qquad \mathbf{0} \preceq \mathbf{m} \prec \mathbf{N}. \tag{3.49}$$

As before, all of the diagonal entries of this matrix are given by $2k_{\mathbf{N}-\mathbf{1}}$, however, the internal structure is now quite different. The matrix M is block lower triangular, given by

$$M = \begin{pmatrix} M_1 & & & & \\ \vdots & \ddots & & & \\ M_{i_1} & \cdots & M_1 & & \\ \vdots & \ddots & \vdots & \ddots & \\ M_{N_1} & \cdots & M_{i_1} & \cdots & M_1 \end{pmatrix}, \tag{3.50}$$

where each $M_{i_1} \in \{M_1, \ldots, M_{N_1}\}$ is a lower-triangular matrix of size $\frac{N}{N_1} \times \frac{N}{N_1}$, with N_1 being the first factor in the decomposition of $N = N_1 \ldots N_\sigma$. The matrix M_{i_1} has the form

$$M_{i_1} = \begin{pmatrix} M_{i_1,1} & & & & \\ \vdots & \ddots & & & \\ M_{i_1,i_2} & \cdots & M_{i_1,1} & & \\ \vdots & \ddots & \vdots & \ddots & \\ M_{i_1,N_2} & \cdots & M_{i_1,i_2} & \cdots & M_{i_1,1} \end{pmatrix}, \tag{3.51}$$

where again each $M_{i_1,i_2} \in \{M_{i_1,1}, \ldots, M_{i_1,N_2}\}$ appearing is itself an $\frac{N}{N_1N_2} \times \frac{N}{N_1N_2}$ lower-triangular matrix, and so on. Thus, the innermost lower-triangular matrices appearing in this nested description of M are $N_\sigma \times N_\sigma$ Toeplitz matrices of the form

$$M_{i_1,\ldots,i_{\sigma-1}} = \begin{pmatrix} 2k_{i_1,\ldots,i_{\sigma-1},N_\sigma-1} & & & \\ \vdots & \ddots & & \\ 2k_{i_1,\ldots,i_{\sigma-1},N_\sigma-i_\sigma} & & & \\ \vdots & \ddots & \ddots & \\ 2k'_{i_1,\ldots,i_{\sigma-1},0} & \cdots & \cdots & 2k_{i_1,\ldots,i_{\sigma-1},N_\sigma-1} \end{pmatrix}. \tag{3.52}$$

For the sequence $\mathbf{N} = 3,2,3$, for example, we thus have

$$M = 2\begin{pmatrix}
k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{210} & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{202} & 0 & 0 & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{201} & k_{202} & 0 & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{200} & k_{201} & k_{202} & k_{210} & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{112} & 0 & 0 & 0 & 0 & 0 & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{111} & k_{112} & 0 & 0 & 0 & 0 & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{110} & k_{111} & k_{112} & 0 & 0 & 0 & k_{210} & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{102} & 0 & 0 & k_{112} & 0 & 0 & k_{202} & 0 & 0 & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{101} & k_{102} & 0 & k_{111} & k_{112} & 0 & k_{201} & k_{202} & 0 & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{100} & k_{101} & k_{102} & k_{110} & k_{111} & k_{112} & k_{200} & k_{201} & k_{202} & k_{210} & k_{211} & k_{212} & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{012} & 0 & 0 & 0 & 0 & 0 & k_{112} & 0 & 0 & 0 & 0 & 0 & k_{212} & 0 & 0 & 0 & 0 & 0 \\
k_{011} & k_{012} & 0 & 0 & 0 & 0 & k_{111} & k_{112} & 0 & 0 & 0 & 0 & k_{211} & k_{212} & 0 & 0 & 0 & 0 \\
k_{010} & k_{011} & k_{012} & 0 & 0 & 0 & k_{110} & k_{111} & k_{112} & 0 & 0 & 0 & k_{210} & k_{211} & k_{212} & 0 & 0 & 0 \\
k_{002} & 0 & 0 & k_{012} & 0 & 0 & k_{102} & 0 & 0 & k_{112} & 0 & 0 & k_{202} & 0 & 0 & k_{212} & 0 & 0 \\
k_{001} & k_{002} & 0 & k_{011} & k_{012} & 0 & k_{101} & k_{102} & 0 & k_{111} & k_{112} & 0 & k_{201} & k_{202} & 0 & k_{211} & k_{212} & 0 \\
k'_{000} & k_{001} & k_{002} & k_{010} & k_{011} & k_{012} & k_{100} & k_{101} & k_{102} & k_{110} & k_{111} & k_{112} & k_{200} & k_{201} & k_{202} & k_{210} & k_{211} & k_{212}
\end{pmatrix}, \tag{3.53}$$

written using the simplified notation $k_{i_1 i_2 i_3} = k_{i_1,i_2,i_3}$.

We remark that although the general description somewhat obfuscates the structure of the resulting matrix, we observe here that the resulting matrix has entries involving $k_{\mathbf{i}}$ in the ordered basis going from $M_{\mathbf{N},\mathbf{1}}$ to $M_{\mathbf{N},\mathbf{N}}$ and $M_{\mathbf{1},\mathbf{1}}$, and then the remaining entries are determined using the truncated $\mathbb{Z}^{\otimes\sigma}$ grading, simply by performing addition of the corresponding sequences. This fact follows from the grading on the algebra, and the use of the ordered basis.

The constants are then determined by finding the inverse matrix M^{-1}. The inverse matrix has a similar nested Toeplitz structure to M, and we again use the notation

$$M^{-1} = \Big(b_{\mathbf{m},\mathbf{n}}\Big)_{\mathbf{0}\preceq\mathbf{m},\mathbf{n}\prec\mathbf{N}}. \tag{3.54}$$

We note that all diagonal entries of this matrix will be $1/(2k_{\mathbf{N}-\mathbf{1}})$, and we have the equality

$$\lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}} = b_{\mathbf{n},\mathbf{i}}. \tag{3.55}$$

This completely determines the form of the Galilean Virasoro operators $T_{\mathbf{i}}$, which are given by

$$T_{\mathbf{i}} = \sum_{\mathbf{i}\preceq\mathbf{n}\prec\mathbf{N}} b_{\mathbf{n},\mathbf{i}} \sum_{\mathbf{0}\preceq\mathbf{t}\prec\mathbf{N}-\mathbf{n}} \kappa_{ab}\big(J^a_{\mathbf{n}+\mathbf{t}} J^b_{\mathbf{N}-\mathbf{1}-\mathbf{t}}\big). \tag{3.56}$$

It still remains to determine the values of the central parameters, which follow from computing the OPEs $T_{\mathbf{0}} \times T_{\mathbf{i}}$. These are calculated straight forwardly, and are given by

$$T_{\mathbf{0}}(z)T_{\mathbf{i}}(w) \sim \sum_{\mathbf{i}\preceq\mathbf{n}\prec\mathbf{N}} b_{\mathbf{n},\mathbf{i}} \sum_{\mathbf{0}\preceq\mathbf{t}\prec\mathbf{N}-\mathbf{n}} \frac{\kappa_{ab}\kappa^{ab} k_{\mathbf{N}-\mathbf{1}+\mathbf{n}}}{(z-w)^4} + \frac{2T_{\mathbf{i}}(w)}{(z-w)^2} + \frac{\partial T_{\mathbf{i}}(w)}{z-w}. \tag{3.57}$$

We see here that the fourth-order pole is zero unless $\mathbf{n} = \mathbf{0}$. That is, we only have a non-zero central parameter for $\mathbf{i} = \mathbf{0}$. The central parameters are

$$c_{\mathbf{0}} = N\dim(\mathfrak{g}), \quad c_{\mathbf{i}} = 0, \text{ for } \mathbf{i} \neq \mathbf{0}. \tag{3.58}$$

This matches closely the result in the case of higher-order algebras.

We are still left to show that the above construction is equivalent to beginning with N copies of an affine Lie algebra with a constructed Sugawara operator, and then performing a multi-graded contraction.

As such, we begin by stating the Sugawara operators for the factors $\mathcal{A}_{(\mathbf{i})}$. They each have a Virasoro field $T_{(\mathbf{i})}$, with central charge $c_{(\mathbf{i})}$ given by

$$T_{(\mathbf{i})} = \frac{\kappa_{ab}}{2(k_{(\mathbf{i})}+h^\vee)}\big(J^a_{(\mathbf{i})}J^b_{(\mathbf{i})}\big), \qquad c_{(\mathbf{i})} = \frac{k_{(\mathbf{i})}\dim\mathfrak{g}}{k_{(\mathbf{i})}+h^\vee}, \qquad \mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}. \tag{3.59}$$

We then form the vectors $\mathbf{T}_*$ and $\mathbf{c}_*$ as in (3.14), and perform the change of basis to the algebra $\mathcal{A}^{\mathbf{N}}_{\varepsilon}$ following (3.19), which gives

$$\mathbf{T}_\varepsilon = \Big(T_{\mathbf{i},\varepsilon}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}} = U_{\mathbf{N}}(\varepsilon,\omega)\mathbf{T}_*, \qquad \mathbf{c}_\varepsilon = \Big(c_{\mathbf{i},\varepsilon}\Big)_{\mathbf{0}\preceq\mathbf{i}\prec\mathbf{N}} = U_{\mathbf{N}}(\varepsilon,\omega)\mathbf{c}_*. \tag{3.60}$$

Expanding the form of the transformation matrices $U_{\mathbf{N}}(\varepsilon,\omega)$, as well as using the inverse Galilean basis change on the coefficients of $T_{(\mathbf{i})}$ involving $k_{(\mathbf{i})}$ we have

$$T_{\mathbf{i},\varepsilon} = \frac{\displaystyle\sum_{\mathbf{0}\preceq\mathbf{j},\mathbf{n},\mathbf{n}'\prec\mathbf{N}}\Big(\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{N_\ell-1+i_\ell-n_\ell-n'_\ell}\Big)\kappa_{ab}\big(J^a_{\mathbf{n},\varepsilon}J^b_{\mathbf{n}',\varepsilon}\big)}{\displaystyle 2Nk_{\mathbf{N-1},\varepsilon}\sum_{\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}}a_{\mathbf{m}}\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{m_\ell}}, \tag{3.61}$$

where we have made use of the shorthand

$$a_{\mathbf{m}} = \frac{k_{\mathbf{N-1-m},\varepsilon}+Nh^\vee\delta_{\mathbf{m},\mathbf{N-1}}}{k_{\mathbf{N-1},\varepsilon}}, \qquad \mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}. \tag{3.62}$$

Similarly to the higher-order case, we perform a power series expansion

$$\Big(\sum_{\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}}a_{\mathbf{m}}\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{m_\ell}\Big)^{-1} = \sum_{\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}}\hat{a}_{\mathbf{m}}\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{m_\ell} + \mathcal{O}(\varepsilon_1^{N_1},\ldots,\varepsilon_\sigma^{N_\sigma}), \tag{3.63}$$

which gives a multinomial expansion similar to the higher-order case, but here the labelling of the factors requires tuples of sequences. This greatly complicates finding an explicit expression, however, we can still analyse the result without having an explicit form. Continuing, since $a_{\mathbf{0}} = 1$ in the above equation, we have that the $\hat{a}_{\mathbf{m}}$, $\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}$ are well defined, and $\hat{a}_{\mathbf{0}} = 1$. Combining the power series for the coefficient with the expanded expression for the field, we have

$$T_{\mathbf{i},\varepsilon} = \frac{1}{2Nk_{\mathbf{N-1},\varepsilon}}\sum_{\mathbf{0}\preceq\mathbf{j},\mathbf{n},\mathbf{n}',\mathbf{m}\prec\mathbf{N}}\hat{a}_{\mathbf{m}}\Big(\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{N_\ell-1+i_\ell-n_\ell-n'_\ell+m_\ell}\Big)\kappa_{ab}\big(J^a_{\mathbf{n},\varepsilon}J^b_{\mathbf{n}',\varepsilon}\big) + \mathcal{O}(\varepsilon_1^{N_1},\ldots,\varepsilon_\sigma^{N_\sigma}). \tag{3.64}$$

We notice that for each $\ell\in\{1,\ldots,\sigma\}$, we can simplify the summation over sets of roots of unity in the following way

$$\sum_{j_\ell=0}^{N_\ell-1}\omega_\ell^{j_\ell(N_\ell-1+i_\ell-n_\ell-n'_\ell+m_\ell)} = \begin{cases} N_\ell, & N_\ell-1+i_\ell-n_\ell-n'_\ell+m_\ell\equiv 0 \pmod{N_\ell}, \\ 0, & N_\ell-1+i_\ell-n_\ell-n'_\ell+m_\ell\not\equiv 0 \pmod{N_\ell}. \end{cases} \tag{3.65}$$

This lifts to the expressions involving sequences as

$$\sum_{\mathbf{0}\preceq\mathbf{j}\prec\mathbf{N}}\prod_{\ell=1}^{\sigma}\omega_\ell^{j_\ell(N_\ell-1+i_\ell-n_\ell-n'_\ell+m_\ell)}=\begin{cases}N, & \mathbf{N}-\mathbf{1}+\mathbf{i}-\mathbf{n}-\mathbf{n}'+\mathbf{m}\equiv\mathbf{0}\pmod{\mathbf{N}},\\ 0, & \mathbf{N}-\mathbf{1}+\mathbf{i}-\mathbf{n}-\mathbf{n}'+\mathbf{m}\not\equiv\mathbf{0}\pmod{\mathbf{N}}.\end{cases}\tag{3.66}$$

Then, analogously to the higher-order case, since $\mathbf{N}-\mathbf{1}+\mathbf{i}-\mathbf{n}-\mathbf{n}'+\mathbf{m}>-\mathbf{N}$, we are not left with any negative powers of any of the ε_ℓ variables. This implies that the limit as $\varepsilon_\ell\to 0$ for all ℓ is well defined, and we do not have any divergences.

Performing the limit, we can give the form of the resulting Virasoro fields

$$\begin{aligned}T_{\mathbf{i},\varepsilon}\to T_{\mathbf{i}}&=\frac{1}{2k_{\mathbf{N}-\mathbf{1}}}\sum_{\mathbf{0}\preceq\mathbf{n},\mathbf{n}',\mathbf{m}\prec\mathbf{N}}\hat{a}_{\mathbf{m}}\kappa_{ab}\big(J^a_{\mathbf{n}}J^b_{\mathbf{n}'}\big)\delta_{\mathbf{N}-\mathbf{1}+\mathbf{i}-\mathbf{n}-\mathbf{n}'+\mathbf{m},\mathbf{0}}\\&=\sum_{\mathbf{i}\preceq\mathbf{n}\prec\mathbf{N}}\frac{\hat{a}_{\mathbf{n}-\mathbf{i}}}{2k_{\mathbf{N}-\mathbf{1}}}\sum_{\mathbf{0}\preceq\mathbf{t}\prec\mathbf{N}-\mathbf{n}}\kappa_{ab}\big(J^a_{\mathbf{n}+\mathbf{t}}J^b_{\mathbf{N}-\mathbf{1}-\mathbf{t}}\big).\end{aligned}\tag{3.67}$$

This matches the expressions for the alternative branch of the diagram when

$$b_{\mathbf{n},\mathbf{i}}=\frac{\hat{a}_{\mathbf{n}-\mathbf{i}}}{2k_{\mathbf{N}-\mathbf{1}}},\tag{3.68}$$

or equivalently, when we have

$$\lambda_{\mathbf{i}}^{\mathbf{n};\mathbf{N}-\mathbf{1}}=\frac{\hat{a}_{\mathbf{n}-\mathbf{i}}}{2k_{\mathbf{N}-\mathbf{1}}}.\tag{3.69}$$

It remains to determine the central parameters, which follow from essentially the same calculation, without the normally-ordered products of fields. In the precontraction basis, the central parameters are given by

$$\begin{aligned}c_{\mathbf{i},\varepsilon}&=\frac{\displaystyle\sum_{\mathbf{0}\preceq\mathbf{j},\mathbf{n}\prec\mathbf{N}}\Big(\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{N_\ell-1+i_\ell-n_\ell}\Big)k_{\mathbf{n},\varepsilon}\dim\mathfrak{g}}{\displaystyle k_{\mathbf{N}-\mathbf{1},\varepsilon}\sum_{\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}}a_{\mathbf{m}}\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{m_\ell}}\\&=\frac{\dim\mathfrak{g}}{k_{\mathbf{N}-\mathbf{1},\varepsilon}}\sum_{\mathbf{0}\preceq\mathbf{j},\mathbf{n},\mathbf{m}\prec\mathbf{N}}\hat{a}_{\mathbf{m}}k_{\mathbf{n},\varepsilon}\prod_{\ell=1}^{\sigma}(\varepsilon_\ell\omega_\ell^{j_\ell})^{N_\ell-1+i_\ell-n_\ell+m_\ell}+\mathcal{O}(\varepsilon_1^{N_1},\ldots,\varepsilon_\sigma^{N_\sigma}).\end{aligned}\tag{3.70}$$

Then, taking the limit $\varepsilon\to\mathbf{0}$, we find

$$c_{\mathbf{i},\varepsilon}\to c_{\mathbf{i}}=\frac{N\dim\mathfrak{g}}{k_{\mathbf{N}-\mathbf{1}}}\sum_{\mathbf{0}\preceq\mathbf{n},\mathbf{m}\prec\mathbf{N}}\hat{a}_{\mathbf{m}}k_{\mathbf{n}}\delta_{\mathbf{N}-\mathbf{1}+\mathbf{i}-\mathbf{n}+\mathbf{m},\mathbf{0}}=N\dim\mathfrak{g}\,\delta_{\mathbf{i},\mathbf{0}},\tag{3.71}$$

which matches the previous result.

Indeed, the relations in (3.69) follow from the similarity in structures of $M_{\mathbf{m},\mathbf{n}}$ in (3.48) and $a_{\mathbf{m}}$ in (3.62), again in essentially the same way that was outlined in the higher-order case. However, this here shows the commutativity of the two procedures in a slightly more general setting, that is, without an explicit description of the coefficients $b_{\mathbf{n}}$ or $\hat{a}_{\mathbf{m}}$. The coefficients are terms in the multinomial expansion of the sum

$$\Big(\sum_{\mathbf{0}\preceq\mathbf{m}\prec\mathbf{N}} a_{\mathbf{m}} \prod_{\ell=1}^{\sigma}(\varepsilon_\ell \omega_\ell^{j_\ell})^{m_\ell}\Big)^{-1}, \tag{3.72}$$

and as such are particularly difficult to express in general, however, they are straightforward to calculate in specific cases.

Here we provide a specific examples of the operators that arise. We consider the multi-graded affine Lie algebra $\widehat{\mathfrak{g}}_G^{2,3}$, that is, we have $N=6$. The set of sequences $\mathbf{0}\preceq\mathbf{i}\prec\mathbf{2},\mathbf{3}$ is given in canonical order by

$$I_{2,3} = \{0,0;\ 0,1;\ 0,2;\ 1,0;\ 1,1;\ 1,2\}. \tag{3.73}$$

The matrix M of coefficients in the expansion is given by

$$M = 2\begin{pmatrix} k_{12} & 0 & 0 & 0 & 0 & 0 \\ k_{11} & k_{12} & 0 & 0 & 0 & 0 \\ k_{10} & k_{11} & k_{12} & 0 & 0 & 0 \\ k_{02} & 0 & 0 & k_{12} & 0 & 0 \\ k_{01} & k_{02} & 0 & k_{11} & k_{12} & 0 \\ k'_{00} & k_{01} & k_{02} & k_{10} & k_{11} & k_{12} \end{pmatrix}. \tag{3.74}$$

Taking the inverse, and multiplying through the expansion, the resulting Virasoro fields are given by

$$\begin{aligned} T_{0,0} = \frac{\kappa_{ab}}{2k_{1,2}}\big[&(J^a_{0,0}J^b_{1,2}) + (J^a_{0,1}J^b_{1,1}) + (J^a_{0,2}J^b_{1,0}) + (J^a_{1,0}J^b_{0,2}) + (J^a_{1,1}J^b_{0,1}) + (J^a_{1,2}J^b_{0,0}) \\ &- \tfrac{k_{1,1}}{k_{1,2}}\big((J^a_{0,1}J^b_{1,2}) + (J^a_{0,2}J^b_{1,1}) + (J^a_{1,1}J^b_{0,2}) + (J^a_{1,2}J^b_{0,1})\big) + \tfrac{(k_{1,1})^2 - k_{1,0}k_{1,2}}{(k_{1,2})^2}\big((J^a_{0,2}J^b_{1,2}) + (J^a_{1,2}J^b_{0,2})\big) \\ &- \tfrac{k_{0,2}}{k_{1,2}}\big((J^a_{1,0}J^b_{1,2}) + (J^a_{1,1}J^b_{1,1}) + (J^a_{1,2}J^b_{1,0}\big) + \tfrac{2k_{0,2}k_{1,1} - k_{0,1}k_{1,2}}{(k_{1,2})^2}\big((J^a_{1,1}J^b_{1,2}) + (J^a_{1,2}J^b_{1,1})\big) \\ &- \tfrac{3k_{0,2}(k_{1,1})^2 - 2(k_{0,2}k_{1,0} + k_{0,1}k_{1,1})k_{1,2} + k'_{0,0}(k_{1,2})^2}{(k_{1,2})^3}(J^a_{1,2}J^b_{1,2})\big], \end{aligned} \tag{3.75}$$

$$\begin{aligned} T_{0,1} = \frac{\kappa_{ab}}{2k_{1,2}}\big[&(J^a_{0,1}J^b_{1,2}) + (J^a_{0,2}J^b_{1,1}) + (J^a_{1,1}J^b_{0,2}) + (J^a_{1,2}J^b_{0,1}) - \tfrac{k_{1,1}}{k_{1,2}}\big((J^a_{0,2}J^b_{1,2}) + (J^a_{1,2}J^b_{0,2})\big) \\ &- \tfrac{k_{0,2}}{k_{1,2}}\big((J^a_{1,1}J^b_{1,2}) + (J^a_{1,2}J^b_{1,1})\big) + \tfrac{2k_{0,2}k_{1,1} - k_{0,1}k_{1,2}}{(k_{1,2})^2}(J^a_{1,2}J^b_{1,2})\big], \end{aligned} \tag{3.76}$$

$$T_{0,2} = \frac{\kappa_{ab}}{2k_{1,2}}\big[(J^a_{0,2}J^b_{1,2}) + (J^a_{1,2}J^b_{0,2}) - \tfrac{k_{0,2}}{k_{1,2}}(J^a_{1,2}J^b_{1,2})\big], \tag{3.77}$$

$$T_{1,0} = \frac{\kappa_{ab}}{2k_{1,2}}\big[(J^a_{1,0}J^b_{1,2}) + (J^a_{1,1}J^b_{1,1}) + (J^a_{1,2}J^b_{1,0}) - \tfrac{k_{1,1}}{k_{1,2}}\big((J^a_{1,1}J^b_{1,2}) + (J^a_{1,2}J^b_{1,1})\big) + \tfrac{(k_{1,1})^2 - k_{1,0}k_{1,2}}{(k_{1,2})^2}(J^a_{1,2}J^b_{1,2})\big], \tag{3.78}$$

$$T_{1,1} = \frac{\kappa_{ab}}{2k_{1,2}}\big[(J^a_{1,1}J^b_{1,2}) + (J^a_{1,2}J^b_{1,1}) - \tfrac{k_{1,1}}{k_{1,2}}(J^a_{1,2}J^b_{1,2})\big], \tag{3.79}$$

$$T_{1,2} = \frac{\kappa_{ab}}{2k_{1,2}}(J^a_{1,2}J^b_{1,2}), \tag{3.80}$$

and the only non-zero central parameter is

$$c_{0,0} = 6\dim(\mathfrak{g}). \tag{3.81}$$

3.5 Multi-graded W_3 algebras

The final example we would like to consider is the multi-graded algebras coming from N copies of W_3. For a chosen factorisation $N = N_1 \dots N_\sigma$, and hence sequence $\mathbf{N} = N_1, \dots, N_\sigma$, the corresponding multi-graded Galilean W_3 algebra $(W_3)_G^{\mathbf{N}}$ is generated by the fields $\{T_{\mathbf{i}}, W_{\mathbf{i}} \mid \mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}\}$, and has central parameters $\{c_{\mathbf{i}} \mid \mathbf{0} \preceq \mathbf{i} \prec \mathbf{N}\}$, with the linear OPEs between generators given by

$$T_{\mathbf{i}} \times T_{\mathbf{j}} \simeq \begin{cases} \frac{c_{\mathbf{i}+\mathbf{j}}}{2}\{\mathbb{I}\} + 2\{T_{\mathbf{i}+\mathbf{j}}\}, & \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise}, \end{cases} \qquad T_{\mathbf{i}} \times W_{\mathbf{j}} \simeq \begin{cases} 3\{W_{\mathbf{i}+\mathbf{j}}\}, & \mathbf{i}+\mathbf{j} < \mathbf{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.82}$$

As in the higher-order case, it is highly non-trivial to compute the OPE $W_{\mathbf{i}} \times W_{\mathbf{j}}$. However, its well definedness follows from the same analysis that shows that the Sugawara operator is well defined in the contraction limit. When expanding the coefficients accompanying the normally-ordered fields as a power series, and combining this with the resulting expressions for the normally-ordered fields under the inverse basis map, we see that it is not possible to introduce any inverse powers of ε_ℓ. As such, we know that the resulting contraction limit is well defined, however, the terms arising as coefficients for the Galilean quasi-primary fields appearing in the OPE are again coefficients in a multinomial expansion. As such, we do not give a concrete formula for these coefficients in this case, however, the matrix M whose inverse encodes these coefficients is completely analogous to the case of Sugawara, along with the identification

$$c'_{\mathbf{0}} = c_{\mathbf{0}} + \tfrac{22N}{5}. \tag{3.83}$$

Following the prescription of the multi-graded Sugawara construction, it is then straightforward, albeit tedious, to determine the OPE structure between the $W_{\mathbf{i}}$ fields.

As an example, we present the OPEs for the algebra $(W_3)_G^{2,3}$. As in the Sugawara example, the canonically ordered basis of sequences is given by

$$I_{2,3} = \{0,0;\ 0,1;\ 0,2;\ 1,0;\ 1,1;\ 1,2\}. \tag{3.84}$$

We have generating fields $\{T_{\mathbf{i}}, W_{\mathbf{i}} \mid \forall i \in I_{2,3}\}$. The non-trivial OPEs are then given by

$$\begin{aligned} W_{0,0} \times W_{0,0} \simeq{}& \tfrac{c_{0,0}}{3}\{\mathbb{1}\} + 2\{T_{0,0}\} + \tfrac{64}{5c_{1,2}}\{\Lambda_{0,0;1,2} + \Lambda_{0,1;1,1} + \Lambda_{0,2;1,0}\} - \tfrac{64c_{1,1}}{5(c_{1,2})^2}\{\Lambda_{0,1;1,2} + \Lambda_{0,2;1,1}\} \\ &+ \tfrac{64[(c_{1,1})^2 - c_{1,0}c_{1,2}]}{5(c_{1,2})^3}\{\Lambda_{0,2;1,2}\} - \tfrac{32c_{0,2}}{5(c_{1,2})^2}\{2\Lambda_{1,0;1,2} + \Lambda_{1,1;1,1}\} + \tfrac{64[2c_{0,2}c_{1,1} - c_{0,1}c_{1,2}]}{5(c_{1,2})^3}\{\Lambda_{1,1;1,2}\} \\ &- \tfrac{32[3c_{0,2}(c_{1,1})^2 - 2(c_{0,1}c_{1,1} + c_{0,2}c_{1,0})c_{1,2} + c'_{0,0}(c_{1,2})^2]}{5(c_{1,2})^4}\{\Lambda_{1,2;1,2}\}, \end{aligned} \tag{3.85}$$

$$W_{0,0} \times W_{0,1} \simeq \tfrac{c_{0,1}}{3}\{\mathbb{1}\} + 2\{T_{0,1}\} + \tfrac{64}{5c_{1,2}}\{\Lambda_{0,1;1,2} + \Lambda_{0,2;1,1}\} - \tfrac{64c_{1,1}}{5(c_{1,2})^2}\{\Lambda_{0,2;1,2}\} - \tfrac{64c_{0,2}}{5(c_{1,2})^2}\{\Lambda_{1,1;1,2}\} + \tfrac{32[2c_{0,2}c_{1,1} - c_{0,1}c_{1,2}]}{5(c_{1,2})^3}\{\Lambda_{1,2;1,2}\}, \tag{3.86}$$

$$W_{0,0} \times W_{0,2} \simeq \tfrac{c_{0,2}}{3}\{\mathbb{1}\} + 2\{T_{0,2}\} + \tfrac{64}{5c_{1,2}}\{\Lambda_{0,2;1,2}\} - \tfrac{32c_{0,2}}{5(c_{1,2})^2}\{\Lambda_{1,2;1,2}\}, \tag{3.87}$$

$$W_{0,0} \times W_{1,0} \simeq \tfrac{c_{1,0}}{3}\{\mathbb{1}\} + 2\{T_{1,0}\} + \tfrac{32}{5c_{1,2}}\{2\Lambda_{1,0;1,2} + \Lambda_{1,1;1,1}\} - \tfrac{64c_{1,1}}{5(c_{1,2})^2}\{\Lambda_{1,1;1,2}\} + \tfrac{32[(c_{1,1})^2 - c_{1,0}c_{1,2}]}{5(c_{1,2})^3}\{\Lambda_{1,2;1,2}\}, \tag{3.88}$$

$$W_{0,0} \times W_{1,1} \simeq \tfrac{c_{1,1}}{3}\{\mathbb{1}\} + 2\{T_{1,1}\} + \tfrac{64}{5c_{1,2}}\{\Lambda_{1,1;1,2}\} - \tfrac{32c_{1,1}}{5(c_{1,2})^2}\{\Lambda_{1,2;1,2}\}, \tag{3.89}$$

$$W_{0,0} \times W_{1,2} \simeq \tfrac{c_{1,2}}{3}\{\mathbb{1}\} + 2\{T_{1,2}\} + \tfrac{32}{5c_{1,2}}\{\Lambda_{1,2;1,2}\}. \tag{3.90}$$

We have introduced the notation

$$\Lambda_{\mathbf{i};\mathbf{j}} = (T_{\mathbf{i}}T_{\mathbf{j}}) - \tfrac{3}{10}\partial^2 T_{\mathbf{N-1}}\delta_{\mathbf{i+j},\mathbf{N-1}} \tag{3.91}$$

for the quasi-primary fields (with respect to $T_{\mathbf{0}}$) appearing in the OPEs. The resulting algebra is indeed multi-graded, and as such the above set of OPEs along with the Lie-type relations, completely determine the structure. We do not have a proof for the multi-graded structure of $(W_3)_G^{\mathbf{N}}$ for general $\mathbf{N}$, as any such proof requires the concrete form of the multinomial expansion over sequences.

This concludes the discussion of multi-graded Galilean algebras. In the following section we continue our discussion of Galilean algebras by relaxing the condition that all input algebras need be the same, up to the value of their central parameters.

The following Chapter is based on work currently in preparation to appear as

[3] E. Ragoucy, J. Rasmussen, C. Raymond, *Asymmetric Galilean contractions*.

Chapter 4

Asymmetric Galilean algebras

4.1 Introduction

Here we continue the program of generalising the possible structures that can arise from a Galilean contraction of symmetry algebras for a conformal field theory. In this chapter, we will consider relaxing the condition that the algebras involved in the Galilean contraction need to be the same (up to value of their central charge). For conformal algebras, one would still need to contract together the Virasoro generators of each component algebra in order to have a sensible Galilean Virasoro action in the resulting algebra. This observation motivates our choice to consider the contraction of an algebra $\mathfrak{g}$ with an additional copy of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The remaining generators which are not in the subalgebra can then be rescaled by the contraction parameter. In the following, we will investigate the freedom of choice of subalgebra, as well as possible rescalings of the remaining generators.

4.2 Asymmetric contraction theory

We begin with $\mathfrak{g}$ an operator product algebra of Lie type (given the change in notation, we remark that $\mathfrak{g}$ is not necessarily an affine Lie algebra), and $\mathfrak{h} \subseteq \mathfrak{g}$ a subalgebra of Lie type. We remark that if we take $\mathfrak{h} = \mathfrak{g}$, we have that $\bar{\mathfrak{g}} = \mathbf{0}$, and the resulting contraction is the usual order-two Galilean contraction. The space generating fields of $\mathfrak{g}$ decomposes as vector spaces into $\mathfrak{g} = \mathfrak{h} \oplus \bar{\mathfrak{g}}$. We want to consider the possible structures which can arise when performing a Galilean contraction on $\mathfrak{g}$ with a copy of $\mathfrak{h}$, that is, a contraction of the algebra

$$\mathcal{A} = \mathfrak{g}_{(0)} \otimes \mathfrak{h}_{(1)}. \tag{4.1}$$

We begin the contraction by performing the ε-dependent change of basis $\mathcal{A} \to \mathcal{A}_\varepsilon$. The fields in the new basis are

$$A_j = \varepsilon^j \left(A_{(0)} + (-1)^j A_{(1)}\right), \quad X_m = \varepsilon^m X_{(0)}, \quad \forall\, A_{(0)} \in \mathfrak{h}_{(0)},\, A_{(1)} \in \mathfrak{h}_{(1)},\, X_{(0)} \in \bar{\mathfrak{g}}_{(0)}, \tag{4.2}$$

where $m \in \mathbb{R}$, and similarly, the general central elements (not necessarily the central charge) are given by $c_j = \varepsilon^j \left(c_{(0)} + (-1)^j c_{(1)}\right)$. The inverse basis maps are

$$A_{(0)} = \frac{1}{2}\left(A_0 + \varepsilon^{-1}A_1\right), \quad A_{(1)} = \frac{1}{2}\left(A_0 - \varepsilon^{-1}A_1\right), \quad X_{(0)} = \varepsilon^{-m}X_m. \tag{4.3}$$

As in the previous Galilean contractions, we are interested in the algebra arising in the limit $\varepsilon \to 0$. When the resulting algebra is well defined, we will refer to it as the asymmetric Galilean contracted algebra of $\mathfrak{g}$ with subalgebra $\mathfrak{h}$. We label this algebra as $\left(\mathfrak{g},\mathfrak{h}\right)_G$.

In the following we use notation where $A_i,\ B_i,\ C_i \in \mathfrak{h}$, and $X_m,\ Y_m \in \bar{\mathfrak{g}}$. The product structure on $\mathcal{A}_\varepsilon$ is determined using the product on $\mathcal{A}$ as follows. First, the product between fields in $\mathfrak{h}_\varepsilon$ is given by

$$A_{i,\varepsilon} \times B_{j,\varepsilon} \simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j,\varepsilon})\{\mathbb{I}\} + f^{AB}{}_C\ \{C_{i+j,\varepsilon}\}, & \text{if } i+j \leq 1, \\ \varepsilon^2\left(f^{AB}{}_{\mathbb{I}}(c_{0,\varepsilon})\{\mathbb{I}\} + f^{AB}{}_C\ \{C_{0,\varepsilon}\}\right), & \text{if } i=j=1. \end{cases} \tag{4.4}$$

These relations follow directly from $\mathfrak{g}$ and $\mathfrak{h}$ being algebras of Lie-type.

We are primarily interested in how the value of m in (4.2) changes the product structure of the resulting algebra. For $A_j \in \mathfrak{h}_\varepsilon$ and $X_m \in \bar{\mathfrak{g}}_\varepsilon$ we have that

$$\begin{aligned} A_{j,\varepsilon} \times X_{m,\varepsilon} &= \varepsilon^{m+j}\left(A_{(0)} + (-1)^j A_{(1)}\right) \times X_{(0)} \\ &= \varepsilon^{m+j} A_{(0)} \times X_{(0)} \\ &\simeq \varepsilon^{m+j}\left(f^{AX}{}_{\mathbb{I}}(c_{(0)})\{\mathbb{I}\} + f^{AX}{}_B\ \{B_{(0)}\} + f^{AX}{}_Y\ \{Y_{(0)}\}\right) \\ &\simeq \frac{1}{2}\varepsilon^{m+j}\left(f^{AX}{}_{\mathbb{I}}(c_{0,\varepsilon}) + \varepsilon^{-1} f^{AX}{}_{\mathbb{I}}(c_{1,\varepsilon})\right)\{\mathbb{I}\} + \frac{1}{2}\varepsilon^{m+j} f^{AX}{}_B\ \left(\{B_{0,\varepsilon}\} + \varepsilon^{-1}\{B_{1,\varepsilon}\}\right) \\ &\quad + \varepsilon^j f^{A,X}{}_Y\ \{Y_{m,\varepsilon}\}, \end{aligned} \tag{4.5}$$

where in the final step, we have applied the inverse transformations (4.3). Note that there are no remaining negative powers of ε, so this product is well defined in the contraction limit.

Finally, we calculate the product between fields in $\bar{\mathfrak{g}}$

$$\begin{aligned} X_{m,\varepsilon} \times Y_{m,\varepsilon} &= \varepsilon^{2m}\left(X_{(0)} \times Y_{(0)}\right) \\ &\simeq \varepsilon^{2m}\left(f^{XY}{}_B\ \{B_{(0)}\} + f^{XY}{}_Z\ \{Z_{(0)}\}\right) \\ &\simeq \varepsilon^{2m}\left(f^{XY}{}_{\mathbb{I}}(c_{(0)})\{\mathbb{I}\} + f^{XY}{}_B\ \{B_{(0)}\} + f^{XY}{}_Z\ \{Z_{(0)}\}\right) \\ &\simeq \frac{1}{2}\varepsilon^{2m}\left(f^{XY}{}_{\mathbb{I}}(c_{0,\varepsilon})\{\mathbb{I}\} + f^{XY}{}_B\ \{B_{0,\varepsilon}\}\right) + \frac{1}{2}\varepsilon^{2m-1}\left(f^{XY}{}_{\mathbb{I}}(c_{1,\varepsilon})\{\mathbb{I}\} + f^{XY}{}_B\ \{B_{1,\varepsilon}\}\right) \\ &\quad + \varepsilon^m f^{XY}{}_Z\ \{Z_{m,\varepsilon}\}, \end{aligned} \tag{4.6}$$

where $Z_{(0)} \in \bar{\mathfrak{g}}$. From these relations, we can determine the general form of products on an asymmetric Galilean algebra of Lie type.

The resulting Galilean algebras, given by taking the limit $\varepsilon \to 0$ fall into one of four types, depending on the value of m. The possible product structures are categorised below.

1. If $m > 1$, then the resulting asymmetric Galilean algebra is well defined, and the products are:

$$\begin{aligned}
A_i \times B_j &\simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + f^{AB}{}_{C}\ \{C_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
A_i \times X_m &\simeq \begin{cases} f^{AX}{}_{Y}\ \{Y_m\}, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
X_m \times Y_m &\simeq 0.
\end{aligned} \tag{4.7}$$

Moreover, for $0 < m < 1$ and $m \neq \frac{1}{2}$, we separate our analysis into two situations. If $0 < m < \frac{1}{2}$, then we must have $\bar{\mathfrak{g}} \times \bar{\mathfrak{g}} = 0$ or the product (4.6) is not defined in the contraction limit. If that product is zero, then the resulting structure is the same as that for $m > 1$. If $\frac{1}{2} < m < 1$, then any product (4.6) will vanish in the contraction limit, and we again have the same structure as when $m > 1$.

2. If $m = 1$, there are two possible cases. The first case is an asymmetric Galilean algebra with the same structure as the case $m > 1$. However, we may also have that $(\mathfrak{h} \times \bar{\mathfrak{g}}) \cap \mathfrak{h} \neq \{0\}$, for which the products are given by

$$\begin{aligned}
A_i \times B_j &\simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + f^{AB}{}_{C}\ \{C_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
A_i \times X_1 &\simeq \begin{cases} \frac{1}{2}\left(f^{AX}{}_{\mathbb{I}}(c_1)\{\mathbb{I}\} + f^{AX}{}_{B}\ \{B_1\}\right) + f^{AX}{}_{Y}\ \{Y_1\}, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
X_1 \times Y_1 &\simeq 0.
\end{aligned} \tag{4.8}$$

3. If $m = \frac{1}{2}$, the algebra is only well defined if the product $\mathfrak{h} \times \bar{\mathfrak{g}} \subseteq \bar{\mathfrak{g}}$, i.e. the product does not produce fields in $\mathfrak{h}$. In that case, the algebra is the same as in 1, unless $f^{XY}{}_{A} \neq 0$, in which case

$$\begin{aligned}
A_i \times B_j &\simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + f^{AB}{}_{C}\ \{C_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
A_i \times X_{1/2} &\simeq \begin{cases} f^{AX}{}_{Y}\ \{Y_{1/2}\}, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
X_{1/2} \times Y_{1/2} &\simeq \frac{1}{2} f^{X,Y}{}_{A}\ \{A_1\}.
\end{aligned} \tag{4.9}$$

4. If $m = 0$, the algebra is well defined only if $f^{XY}{}_{A} = 0$, and $f^{AX}{}_{\mathbb{I}} = f^{AX}{}_{B} = 0$, otherwise there is a divergence in the products by equations (4.5) and (4.6). This is equivalent to requiring that $\bar{\mathfrak{g}}$

forms an ideal. The OPEs on the resulting algebra are given by

$$
\begin{aligned}
A_i \times B_j &\simeq \begin{cases} f^{AB}{}_{\mathbb{I}}(c_{i+j})\{\mathbb{I}\} + f^{AB}{}_C\,\{C_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
A_i \times X_0 &\simeq \begin{cases} f^{AX}{}_Y\,\{Y_0\}, & \text{if } i=0 \text{ and } f^{AX}{}_Y \neq 0, \\ 0, & \text{otherwise,} \end{cases} \\
X_0 \times Y_0 &\simeq\ f^{X,Y}{}_Z\{Z_0\}, \qquad Z_0 \in \bar{\mathfrak{g}}_0.
\end{aligned} \tag{4.10}
$$

We remark that it is straightforward to verify that under the exchange of input algebras, explicitly given by $\mathcal{A} = \mathfrak{h}_{(0)} \oplus \mathfrak{g}_{(1)}$, the resulting contracted algebra is isomorphic to the above.

The case $m = \frac{1}{2}$ is in a sense the natural case. Indeed, it has been considered before in [86], when considering WZW models on non-compact Lie groups. It is the only rescaling of the subspace $\bar{\mathfrak{g}}$ such that products of generators in that space can map back into the subalgebra $\mathfrak{h}_1$. The resulting product structure is equivalent to that of a $\mathbb{Z}_2$-graded Lie algebra, where the even space is given by $\mathfrak{h}_G = \mathfrak{h}_{(0)} \oplus h_{(1)}$, and the odd space is $\bar{\mathfrak{g}}_{1/2}$. That is,

$$
\mathfrak{h}_G \times \mathfrak{h}_G \subseteq \mathfrak{h}_G, \quad \mathfrak{h}_G \times \bar{\mathfrak{g}}_m \subseteq \bar{\mathfrak{g}}_m, \quad \bar{\mathfrak{g}}_m \times \bar{\mathfrak{g}}_m \subseteq \mathfrak{h}_G. \tag{4.11}
$$

Relevant examples of Lie-type algebras with such a product structure are the Lie superalgebras, and the Cartan decomposition (with respect to the involution) of the real Lie algebra arising from symmetric spaces.

We remark that it is possible, when scaling generators in $\bar{\mathfrak{g}}$ individually, to map back into the order-2 Galilean subalgebra. That is, we no longer require a uniform scaling of the subspace. Suppose that the space of generating fields of $\bar{\mathfrak{g}}$ is two dimensional and the product behaves as $\bar{\mathfrak{g}} \times \bar{\mathfrak{g}} \subseteq \mathfrak{h}$. In that case, one is free to scale the generators individually by any value, say $m_1,\ m_2 \in \mathbb{R}$, such that $m_1 + m_2 = 1$, and still arrive at the same product structure. Examples where this inhomogeneous scaling is possible are given by the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$, and the $N = 2$ superconformal algebras, both considered in Section 4.3.

The exposition above covers the general theory of allowed structures. In the next sections we provide detailed examples relevant to the above description, and relevant to the literature.

4.3 Examples

In this section we present a range of examples. Some have been chosen to demonstrate the asymmetric Galilean contraction procedure, and others for their relevance to structures appearing in the literature.

4.3.1 The Galilean Lie algebra $\left(\widehat{\mathfrak{sl}}(2),\widehat{H}\right)_G$

As a first example, consider the operator product algebra $\widehat{\mathfrak{sl}}(2)$ at level k, generated by fields $\{e,\ h,\ f\}$. For brevity we have dropped the explicit dependence on the co-ordinate z. The operator product expansions between generating fields are given by

$$e\times f\simeq k\{\mathbb{I}\}+\{h\},\quad h\times e\simeq 2\{e\},\quad h\times f\simeq -2\{f\},\quad h\times h\simeq 2k\{\mathbb{I}\}. \tag{4.12}$$

We identify that this algebra contains a Heisenberg subalgebra $\widehat{H}\subset\widehat{\mathfrak{sl}}(2)$ generated by the field h. We perform the contraction procedure on the algebra $\mathcal{A}=\widehat{\mathfrak{sl}}(2)_{(0)}\oplus\widehat{H}_{(1)}$. The change of basis gives $\mathcal{A}_\varepsilon$, generated by the fields,

$$h_{j,\varepsilon}=\varepsilon^j\left(h_{(0)}+(-1)^jh_{(1)}\right),\quad e_{m,\varepsilon}=\varepsilon^m e_{(0)},\quad f_{m,\varepsilon}=\varepsilon^m f_{(0)},\quad j\in\{0,1\}, \tag{4.13}$$

and similarly, we have for the level $k_j=\varepsilon^j\left(k_{(0)}+(-1)^jk_{(1)}\right)$. Following the general description, the following product structures can arise in the limit $\varepsilon\to 0$,

$$\begin{aligned}
h_i\times h_j &\simeq \begin{cases} k_{i+j}\{\mathbb{I}\}, & \text{if } i+j\leq 1,\\ 0, & \text{otherwise,}\end{cases}\\
h_i\times e_m &\simeq \begin{cases} 2\{e_m\}, & \text{if } i=0,\\ 0, & \text{otherwise,}\end{cases}\\
h_i\times f_m &\simeq \begin{cases} -2\{f_m\}, & \text{if } i=0,\\ 0, & \text{otherwise,}\end{cases}\\
e_m\times f_m &\simeq \begin{cases} \frac{1}{2}\left(k_1\{\mathbb{I}\}+\{h_1\}\right), & \text{if } m=\frac{1}{2},\\ 0, & \text{for } m>0,\ m\neq\frac{1}{2}.\end{cases}
\end{aligned} \tag{4.14}$$

Note that because $\widehat{\mathfrak{sl}}(2)$ is semisimple, we cannot choose any subspace $\bar{\mathfrak{g}}$ such that it forms an ideal. Thus, the algebra is not well defined when $m=0$. We remark that this algebra is an example of when the subspace $\bar{\mathfrak{g}}$ need not be uniformly scaled. Rather, one can scale $e_{m_1}=\varepsilon^{m_1}e_{(0)}$ and $f_{m_2}=\varepsilon^{m_2}f_{(0)}$ by any real numbers such that $m_1+m_2=1$. It is natural to ask whether this property generalises to the root space decompositions of other simple Lie algebras. We have not analysed this in detail, however, we can remark that care needs to be taken when rescaling generators related to neighbouring roots say in the Chevalley basis. Non-trivial adjoint action between $\mathfrak{sl}(2)$ triples will constrain the allowed structures. The case of $\left(\widehat{\mathfrak{sl}}(2),\widehat{H}\right)_G$ is free of these complications.

4.3.2 The Galilean Lie algebra $\left(\widehat{\mathfrak{sl}}(2),\widehat{\mathfrak{b}}\right)_G$

Continuing from the above example, we can also consider the contraction of $\widehat{\mathfrak{sl}}(2)$ at level k with the Borel subalgebra, generated by the fields e,h. In this case, we form the fields

$$e_j=\varepsilon^j\left(e_{(0)}+(-1)^je_{(1)}\right),\quad h_{j,\varepsilon}=\varepsilon^j\left(h_{(0)}+(-1)^jh_{(1)}\right),\quad f_m=\varepsilon^m f_{(0)},\quad j\in\{0,1\}. \tag{4.15}$$

The non-trivial products on the contracted algebra are given by

$$
\begin{aligned}
h_i \times h_j &\simeq \begin{cases} k_{i+j}\{\mathbb{I}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
h_i \times e_j &\simeq \begin{cases} 2\{e_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
h_i \times f_m &\simeq \begin{cases} -\{f_1\}, & \text{if } i+m = 1, \\ 0, & \text{if } i+m > 1, \end{cases} \\
e_i \times f_m &\simeq \begin{cases} \frac{1}{2}\left(k_1\{\mathbb{I}\} + \{h_1\}\right), & \text{if } i+m = 1, \\ 0, & \text{if } i+m > 1. \end{cases}
\end{aligned} \tag{4.16}
$$

Again, as $\widehat{\mathfrak{sl}}(2)$ is semisimple, we do not have a well defined contracted algebra for $m = 0$. However, we have well defined contracted algebras for $m = 1$, and $m > 1$.

4.3.3 The asymmetric Galilean Virasoro algebra $\left((\mathfrak{Vir})^2_G, \mathfrak{Vir}\right)_G$

To elucidate the case of $m = 0$, we consider contracting a Galilean Virasoro algebra $(\mathfrak{Vir})^2_G$ with a Virasoro algebra $\mathfrak{Vir}$. The Galilean Virasoro algebra is generated by the fields $\{T_0,\ T_1\}$, and has OPEs given by

$$
T_i \times T_j \simeq \begin{cases} \frac{c}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}
$$

From these relations we see that the Galilean Virasoro algebra has a Virasoro subalgebra generated by T_0, and the field T_1 generates an abelian ideal.

The algebra $(\mathfrak{Vir})^2_G \otimes \mathfrak{Vir}$ is generated by the fields $\{T_0^{(0)}, T_1^{(0)}, T^{(1)}\}$, where we have instead used the superscript indices to indicate which algebra the fields are from. We use (0) to denote the Galilean Virasoro, and (1) to denote the additional Virasoro algebra. We form the new linear combinations

$$
T_{0,\varepsilon} = T_0^{(0)} + T^{(1)}, \quad T_{1,\varepsilon} = \varepsilon\left(T_0^{(0)} - T^{(1)}\right), \quad \bar{T}_{m,\varepsilon} = \varepsilon^m T_1^{(0)}. \tag{4.18}
$$

Along with the redefined fields, we also define new central parameters

$$
c_{0,\varepsilon} = c_0^{(0)} + c^{(1)}, \quad c_{1,\varepsilon} = \varepsilon\left(c_0^{(0)} - c^{(1)}\right), \quad c_{m,\varepsilon} = \varepsilon^m c_1^{(0)}. \tag{4.19}
$$

The products between these fields become

$$
\begin{aligned}
T_{i,\varepsilon} \times T_{j,\varepsilon} &\simeq \begin{cases} \frac{c_{i+j,\varepsilon}}{2}\{\mathbb{I}\} + 2\{T_{i+j,\varepsilon}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
T_{0,\varepsilon} \times \bar{T}_{m,\varepsilon} &\simeq \frac{\bar{c}_m}{2}\{\mathbb{I}\} + 2\{\bar{T}_m\}, \\
T_{1,\varepsilon} \times \bar{T}_{m,\varepsilon} &\simeq \varepsilon^1\left(\frac{\bar{c}_m}{2}\{\mathbb{I}\} + 2\{\bar{T}_m\}\right).
\end{aligned} \tag{4.20}
$$

In the contraction limit, the resulting contracted algebra is generated by the fields $T_0,\ T_1,\ \bar{T}_m$, with non-zero products,

$$T_i \times T_j \simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$T_0 \times \bar{T}_m \simeq \frac{\bar{c}_m}{2}\{\mathbb{I}\} + 2\{\bar{T}_m\}. \tag{4.21}$$

This result matches the general form from the classification in Section 4.1. Unlike the previous examples, the contracted algebra in this example is also defined if $m = 0$. However, we remark that because the product $T_1^{(0)} \times T_1^{(0)} \simeq 0$, the product $\bar{T}_m \times \bar{T}_m$ is trivial when $m = 0$.

The resulting algebra has a Virasoro subalgebra, as expected for any conformal Galilean algebra. However, it also features two commuting quasi-primary Galilean partner fields T_1 and $\bar{T}_m$, both of grade 1 with respect to the Galilean algebra grading. Grading in this sense follows exactly from that introduced in Section 2.4.

4.3.4 The asymmetric $N = 2$ superconformal algebra $\left(SCA_2, \mathfrak{Vir}_1\right)_G$

Here we consider the $N = 2$ superconformal operator product algebra. This algebra is generated by four fields $\{T,\ J,\ G^+,\ G^-\}$, where the fields $T,\ J$, are bosonic and $G^\pm$ are fermionic. The defining OPE relations are

$$T \times T \simeq \frac{c}{2}\{\mathbb{I}\} + 2\{T\}, \quad T \times J \simeq \{J\}, \quad J \times J \simeq \frac{c}{3}\{\mathbb{I}\},$$
$$T \times G^\pm \simeq \frac{3}{2}\{G^\pm\}, \quad J \times G^\pm \simeq \pm\{G^\pm\},$$
$$G^\pm \times G^\mp \simeq 2\frac{c}{3}\{\mathbb{I}\} \pm 2\{J\} + 2\{T\}. \tag{4.22}$$

We see that the bosonic fields of the algebra form a subalgebra, denoted $\mathfrak{Vir}_1$, as it is the Virasoro algebra extended by a conformal weight one current. We contract the full $N = 2$ superconformal algebra with an additional copy of the bosonic subalgebra $\mathfrak{Vir}_1$, whereby the pre-contraction basis becomes

$$T_{0,\varepsilon} = T_{(0)} + T_{(1)}, \quad T_{1,\varepsilon} = \varepsilon\left(T_{(0)} - T_{(1)}\right),$$
$$J_{0,\varepsilon} = J_{(0)} + J_{(1)}, \quad J_{1,\varepsilon} = \varepsilon\left(J_{(0)} - J_{(1)}\right),$$
$$G^+_{m,\varepsilon} = \varepsilon^m G^+_{(0)}, \quad G^-_{m,\varepsilon} = \varepsilon^m G^-_{(0)}. \tag{4.23}$$

The products on the resulting contracted algebra then become

$$
\begin{aligned}
T_i \times T_j &\simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
T_i \times J_j &\simeq \begin{cases} \{J_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\
T_i \times G^{\pm}_m &\simeq \begin{cases} \frac{3}{2}\{G_m\}, & \text{if } i = 0, \\ 0, & \text{otherwise,} \end{cases} \\
J_i \times G^{\pm}_m &\simeq \begin{cases} \pm\{G_m\}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.24}
$$

Particularly interesting behaviour occurs in the product

$$
\begin{aligned}
G^{+}_{m,\varepsilon} \times G^{-}_{m,\varepsilon} &= \varepsilon^{2m} G^{+}_{(0)} \times G^{-}_{(0)} \\
&\simeq \varepsilon^{2m} \left(2\frac{c_{(0)}}{3}\{\mathbb{I}\} \pm 2\{J_{(0)}\} + 2\{T_{(0)}\} \right) \\
&\simeq \varepsilon^{2m} \left(\frac{c_1}{3}\{\mathbb{I}\} \pm \{J_0\} + \{T_0\} \right) + \varepsilon^{2m-1} \left(\frac{c_1}{3}\{\mathbb{I}\} \pm \{J_1\} + \{T_1\} \right),
\end{aligned} \tag{4.25}
$$

where we see that if $m = 0$, the product is not defined. This is clear since the fermionic generators do not form an ideal. If $m = \frac{1}{2}$, we have that the product in the contraction limit is given by

$$
G^{\pm}_{1/2} \times G^{\mp}_{1/2} \simeq \frac{c_1}{3}\{\mathbb{I}\} \pm \{J_1\} + \{T_1\}. \tag{4.26}
$$

Finally, if $m > \frac{1}{2}$ then we have that the product $G^{+} \times G^{-}$ is trivial.

We remark that this algebra forms another example of when the generators of the $\bar{\mathfrak{g}}$ fermionic subspace can be individually rescaled. This fact has already been noticed in the string theory literature [67, 68], where two particular choices of m_1, m_2 have been studied. The individual rescaling for the superconformal algebra is a result of the underlying $\mathbb{Z}_2$ grading. It would be interesting to investigate similar generalisations for more general superalgebras, such as the $N = 4$ superconformal algebras, where one has significantly more possibilities in the products between fermions.

4.3.5 The asymmetric $\left(W_3, \mathfrak{Vir}\right)_G$ algebra

As a final example, we give the possible cases for contracting the W_3 algebra, with its $\mathfrak{Vir}$ subalgebra. This is an algebra we had considered previously in [2]. The algebra W_3 is generated by fields $\{T,\ W\}$ with relations

$$
\begin{aligned}
T \times T &\simeq \frac{c}{2}\{\mathbb{I}\} + 2\{T\}, \quad T \times W \simeq 3\{W\}, \\
W \times W &\simeq \frac{c}{3}\{\mathbb{I}\} + 2\{T\} + \frac{32}{22+5c}\{\Lambda\},
\end{aligned} \tag{4.27}
$$

where we have introduced the quasi-primary field

$$\Lambda = (TT) - \frac{3}{10}\partial^2 T. \tag{4.28}$$

To asymmetrically contract, we form the combinations

$$T_{0,\varepsilon} = T_{(0)} + T_{(1)}, \quad T_{1,\varepsilon} = \varepsilon\left(T_{(0)} - T_{(1)}\right), \quad W_{m,\varepsilon} = \varepsilon^m W_{(0)}. \tag{4.29}$$

Evaluating the new product structure, the Lie-type relations are straightforwardly determined, giving

$$\begin{aligned} T_i \times T_j &\simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\ T_i \times W_m &\simeq \begin{cases} \{3W_m\}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.30}$$

For the final product we have

$$\begin{aligned} W_{m,\varepsilon} \times W_{m,\varepsilon} &= \varepsilon^{2m} W_{(0)} \times W_{(0)} \\ &\simeq \varepsilon^{2m}\left(\frac{c_{(0)}}{3}\{\mathbb{I}\} + 2\{T_{(0)}\} + \frac{32}{22+5c_{(0)}}\{\Lambda_{(0)}\}\right) \\ &\simeq \varepsilon^{2m}\left(\frac{1}{6}(c_0 + \frac{1}{\varepsilon}c_1)\{\mathbb{I}\} + \{T_0\} + \frac{1}{\varepsilon}\{T_1\} + \frac{32}{22+5c_{(0)}}\{\Lambda_{(0)}\}\right). \end{aligned} \tag{4.31}$$

Expanding the coefficient as

$$\frac{32}{22+5c_{(0)}} = \frac{64}{5c_1}\varepsilon - \frac{(44+5c_0)}{25c_1}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{4.32}$$

and the field as

$$\Lambda_{(0)} = \frac{1}{4}\left((T_0T_0) + \frac{2}{\varepsilon}(T_0T_1) + \frac{1}{\varepsilon^2}(T_1T_1)\right) - \frac{3}{10}\partial^2\frac{1}{2}(T_0 + T_1). \tag{4.33}$$

Combining the expansions, and taking the limit, we find that the products of the resulting contracted algebra are

$$\begin{aligned} T_i \times T_j &\simeq \begin{cases} \frac{c_{i+j}}{2}\{\mathbb{I}\} + 2\{T_{i+j}\}, & \text{if } i+j \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\ T_0 \times W_m &\simeq 3\{W_m\}, \\ W_m \times W_m &\simeq \begin{cases} \frac{c_1}{6}\{\mathbb{I}\} + \{T_1\} + \frac{16}{5c_1}\{\Lambda_{1,1}\}, & \text{if } m = 1/2, \\ 0 & \text{for } m > \frac{1}{2}, \end{cases} \end{aligned} \tag{4.34}$$

where we have introduced the field

$$\Lambda_{1,1} = (T_1T_1). \tag{4.35}$$

The algebra is not defined in the contraction limit for $m < \frac{1}{2}$.

4.4 The asymmetric Sugawara construction

Motivated by constructing conformal field theories from Galilean algebras, we want to determine whether it is possible to construct a Virasoro operator on an asymmetrically contracted Galilean affine Lie algebra. Furthermore, we want to determine whether this process is consistent with first constructing Virasoro operators for the affine Lie algebra and its corresponding subalgebra, and then performing the contraction.

We begin by considering the semisimple affine Lie algebra $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}} \subseteq \widehat{\mathfrak{g}}$ a subalgebra. We note, that although $\widehat{\mathfrak{h}}$ needs to be a subalgebra, it need not be semisimple.

It was shown through the series of papers [87–90] that Sugawara constructions for non-semisimple Lie algebras are possible. One requires a non-degenerate invariant symmetric bilinear form, which we denote Ω^{ab}. As the algebra is no longer semisimple, Ω is not given by the Killing form. Such a form Ω can be found for the large class of non-semisimple Lie algebras given by the so-called double extension construction of [89]. The classic example of such a non-semisimple Sugawara construction is the construction of the free boson Virasoro operator with $c = 1$ (see [5] for a discussion).

The Sugawara operator for each simple part of $\widehat{\mathfrak{g}}$ is given by

$$T^{\widehat{\mathfrak{g}}} = \frac{1}{2(k_{(0)} + h^{\vee}_{\mathfrak{g}})} \left[\kappa_{ab} \big(J^a_{(0)} J^b_{(0)} \big) + \kappa_{\alpha\beta} \big(J^{\alpha}_{(0)} J^{\beta}_{(0)} \big) \right], \tag{4.36}$$

where we have used Latin indices to label currents in the subalgebra $\mathfrak{h}$, and Greek indices to label currents in $\bar{\mathfrak{g}}$, κ with lower indices is the inverse Killing form on $\mathfrak{g}$, and we make use of the summation convention. The Sugawara operator on a semisimple $\widehat{\mathfrak{g}}$ is simply the linear combination of each simple operator, and as such we will treat the simple case only to improve readability.

We then consider this operator under the inverse basis change maps in (4.3), along with the series expansion of the leading coefficient.

The series expansion gives

$$\frac{1}{2(k_{(0)} + h^{\vee}_{\mathfrak{g}})} \mapsto \frac{1}{k_1}\varepsilon - \frac{k_0 + 2h^{\vee}_{\mathfrak{g}}}{k_1^2}\varepsilon^2 + \mathcal{O}(\varepsilon^3), \tag{4.37}$$

and the fields become

$$\kappa_{ab} \big(J^a_{(0)} J^b_{(0)} \big) \mapsto \frac{\kappa_{ab}}{4} \Big(\big(J^a_0 + \varepsilon^{-1} J^a_1 \big) \big(J^b_0 + \varepsilon^{-1} J^b_1 \big) \Big), \quad \kappa_{\alpha\beta} \big(J^{\alpha}_{(0)} J^{\beta}_{(0)} \big) \mapsto \kappa_{\alpha\beta} \varepsilon^{-2m} \big(J^{\alpha}_m J^{\beta}_m \big). \tag{4.38}$$

It is clear from combining the above expressions that when $m > \frac{1}{2}$, a contraction is not possible, as the ε^{-2m} term will diverge in the contraction limit.

We focus on the case when $m = \frac{1}{2}$, as the above form for the Sugawara operator only holds for when $\widehat{\mathfrak{g}}$ is semisimple, and this is incompatible with the asymmetric contraction conditions at $m = 0$. The Sugawara operator on the copy of the subalgebra $\mathfrak{h}_{(1)}$ is given in its general form [88, 89] by

$$T^{\mathfrak{h}} = \Omega_{ab} \big(J^a J^b \big), \tag{4.39}$$

where Ω_{ab} is the inverse of the form on $\mathfrak{h}$. If we suppose that $\mathfrak{h}$ is itself simple, then $\Omega_{ab} = \frac{1}{2(k_{(1)}+h^\vee_{\mathfrak{h}})}\kappa_{ab}$. In this case, the resulting operators become

$$
\begin{aligned}
T_0 &= \frac{2}{k_1}\kappa_{ab}\left[\left(J_0^a J_1^b\right)+\left(J_1^a J_0^b\right)\right]+\frac{1}{k_1}\kappa_{\alpha\beta}\left(J_m^\alpha J_m^\beta\right)-\frac{2(k_0+h^\vee_{\mathfrak{g}}+h^\vee_{\mathfrak{h}})}{k_1^2}\kappa_{ab}\left(J_1^a J_1^b\right),\\
T_1 &= \frac{2}{k_1}\kappa_{ab}\left(J_1^a J_1^b\right),
\end{aligned} \tag{4.40}
$$

with central parameters $c_0 = \dim(\mathfrak{g})+\dim(\mathfrak{h})$ and $c_1 = 0$. In the case that $\mathfrak{h}$ is semisimple, we have a sum of Sugawara operators, and the computation proceeds similarly.

In the case that the subalgebra $\mathfrak{h}$ is not semisimple, then the Sugawara operator is still given by (4.39), however, the form of Ω_{ab} is different. We still have that the individual Sugawara constructions are well defined. As such, we can form Virasoro operators on the two algebras, $T_{(0)}$ and $T_{(1)}$, and subsequently contract by forming the combinations

$$
T_{0,\varepsilon} = T_{(0)} + T_{(1)}, \quad T_{1,\varepsilon} = \varepsilon\left(T_{(0)} - T_{(1)}\right). \tag{4.41}
$$

In the contraction limit, these fields generate a Galilean Virasoro algebra action on the contracted algebra. Moreover, the resulting Galilean Virasoro algebra will have

$$
c_0 = \dim(\mathfrak{g})+\dim(\mathfrak{h}), \quad c_1 = 0, \tag{4.42}
$$

which follows directly from

$$
c_{(0)} = \frac{k_{(0)}\dim(\mathfrak{g})}{k_{(0)}+h^\vee_{\mathfrak{g}}}, \quad c_{(1)} = \dim(\mathfrak{h}), \tag{4.43}
$$

where we can series expand $k_{(0)}$ as a function of ε. For $\mathfrak{h}$ non-semisimple, $c_{(1)} = \dim(\mathfrak{h})$ is known from [89].

We return to the case $m = 0$. In this case, a Sugawara construction on $\widehat{\mathfrak{g}}$ is only possible if the subspace $\bar{\mathfrak{g}}$ forms an ideal. This implies the algebra $\widehat{\mathfrak{g}}$ is not semisimple. Sugawara constructions are still possible for non-semisimple Lie algebras. However, there is so much freedom in the form Ω for non-semisimple algebras that we are unable to give a general form of the resulting Sugawara operator, in the case that the Sugawara construction is performed before contraction. However, we can say that when there is a well defined Sugawara construction on the algebras $\mathfrak{g}$ and $\mathfrak{h}$, then Virasoro operators constructed as in (4.41) will indeed generate a Galilean Virasoro action on the asymmetrically contracted algebra $\left(\mathfrak{g},\mathfrak{h}\right)_G$. Furthermore, using the results of [89], we know that the central parameters of that Galilean Virasoro algebra will be given by $c_0 = \dim(\mathfrak{g})+\dim(\mathfrak{h})$, and $c_1 = 0$. Due to the freedom in Ω, we cannot verify the commutativity of the Sugawara construction and the Galilean contraction, as that requires an explicit form of the Sugawara operator. We return to discuss this point further in the conclusion.

4.4.1 The Sugawara construction for $\left(\widehat{\mathfrak{sl}}(2),\widehat{H}\right)_G$

Here we provide the example of contracting the affine Lie algebra $\widehat{\mathfrak{sl}}(2,\mathbb{C})$ with a $\widehat{H}$ subalgebra, the example coming from Section 4.3. For $m=\frac{1}{2}$, the resulting asymmetric algebra is generated by the fields $\{e_{1/2},f_{1/2},h_0,h_1\}$. The non-trivial OPE relations are

$$\begin{aligned} h_i\times h_j &\simeq 2k_{i+j}\{\mathbb{I}\}, \quad i+j\leq 1,\\ h_0\times e_{1/2}\simeq 2\{e_{1/2}\}, &\qquad h_0\times f_{1/2}\simeq -2\{f_{1/2}\},\\ e_{1/2}\times f_{1/2} &\simeq \frac{k_1}{2}\{\mathbb{I}\}+\frac{1}{2}\{h_1\}. \end{aligned} \tag{4.44}$$

The Sugawara operators are given by constructing the Sugawara operators in the pre-contraction algebras, and then performing the expansion procedure, are given by

$$\begin{aligned} T_0 &= \frac{1}{2k_1}\left[\left(h_0h_1\right)+2\left(e_{1/2}f_{1/2}\right)\right]-\frac{2+k_0}{4k_1^2}\left(h_1h_1\right),\\ T_1 &= \frac{1}{k_1}\left(h_1h_1\right), \end{aligned} \tag{4.45}$$

which form a Galilean Virasoro algebra of order two, with central charge $c_0=4$ and $c_1=0$.

Constructing the Sugawara operator for $m=\frac{1}{2}$ in the contracted algebra (that is, performing Sugawara after contraction), is a known result, given in [86], and as such we will not restate the result here beyond saying that the two processes are commutative in the usual sense. We will however remark again to reiterate, that despite the asymmetric contraction being well defined for $m\geq 0$, subject to conditions, only in the case that $m=\frac{1}{2}$, do we have an interesting, general Sugawara construction.

This concludes the results of the research into extended Galilean algebras. A thorough discussion of the results of our work on the Galilean contraction, and possible avenues for further research, are discussed in Chapter 7.

Chapter 5

Introduction to $N = 2$ superconformal cosets

5.1 Introduction

In the second half of this thesis, we investigate modules known as staggered modules for the $N = 2$ superconformal algebras. Staggered modules are reducible yet indecomposable modules, characterised by the non-diagonalisability of one of the zero mode generators of the algebra. In the case of the staggered modules investigated here, the Virasoro zero mode L_0 is no longer diagonalisable on the module.

The $N = 2$ superconformal algebras are a family of infinite dimensional Lie superalgebras. First introduced in [91], they describe the infinitesimal symmetries of a fermionic string on the world sheet in string theory. The $N = 2$ superalgebras were introduced as an extension of the Neveu-Schwarz $N = 1$ superconformal algebra, as a possible gauge theory description of the quark colour confinement problem.

Here we are most interested in understanding their algebraic construction as a coset (commutant) algebra. In the papers [92, 93], Kazama and Suzuki introduced a coset construction similar to that of GKO (see Section 1.7) to describe supersymmetric string theories. Similar to the GKO setting, the Kazama-Suzuki coset describing the $N = 2$ superconformal algebras is not unique in the following sense. A Hermitian symmetric space is given by a coset of Lie groups (with some additional geometric data). One can construct the generators of the $N = 2$ superconformal algebra from the currents of the affine algebras corresponding to the coset Lie groups. The central charge of the resulting $N = 2$ superconformal algebra is a function of the levels of the affine Lie algebras in the coset.

The original work of Kazama-Suzuki was a construction of $N = 1$ superconformal models as a coset. However, under particular conditions, the symmetries of the coset expanded, and the resulting superalgebra possessed $N = 2$ superconformal symmetry. A complete understanding of this additional symmetry mechanism, along with a classification result, was given in [94].

A great deal of work has gone into the study of $N = 2$ Kazama-Suzuki cosets, and in particular, the Heisenberg coset

$$SCA_2 = \frac{\widehat{\mathfrak{sl}}(2) \oplus \mathfrak{bc}}{\widehat{H}}, \tag{5.1}$$

where the $N = 2$ superconformal algebras are constructed from a coset of the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ with respect to a Heisenberg algebra $\widehat{H}$ (equivalently, the algebra of the free bosonic field). The additional algebra is the so-called $\mathfrak{bc}$-ghost system, or the fermionic ghost superalgebra [5]. Specifics of these algebras are discussed later in this chapter, however, we would like to remark that this particular construction was instrumental in classifying the highest-weight modules of the $N = 2$ superconformal algebras [95–98].

More recently, there has been instrumental results on the general properties of Heisenberg algebra cosets [99]. Using these results, the authors of [100] were able to provide a concrete map between the minimal models of $\widehat{\mathfrak{sl}}(2)$ at admissible level, and the minimal models of $N = 2$ superconformal algebras. In this work, they describe in detail the action of the coset on the so-called relaxed highest-weight modules (see [101] for a discussion of relaxed highest-weight modules) of $\widehat{\mathfrak{sl}}(2)$, and discuss the resulting families of $N = 2$ modules.

In this work we are primarily interested in so-called staggered modules. These are reducible yet indecomposable modules characterised by a non-diagonalisable action of a generator of the zero-mode subalgebra, most often the zero mode of the Virasoro algebra L_0. The terminology "staggered module" was first introduced in [102], studying reducible but indecomposable modules for the Virasoro algebra. Staggered modules have been widely studied in the setting of conformal field theory, for example, for the Virasoro algebra [103] (a classification was given in [104]), the triplet algebra [20, 105], the affine Lie algebras $\widehat{\mathfrak{sl}}(2)$ [106–108] and $\widehat{\mathfrak{osp}}(1|2)$ [109], the Heisenberg algebra [110], and $N = 1$ superconformal algebras [111, 112]. Moreover, staggered modules and have also been shown to arise in systems in statistical physics [113, 114].

Staggered modules are intimately tied to the study of logarithmic conformal field theory [115–117] (see [105, 118] for overviews of the field), named after the appearance of logarithmic divergences in the OPEs of fields in the theories. Such CFTs are fundamentally non-unitary theories. Early examples of physical systems exhibiting logarithmic behaviour comes from the $c = 0$ conformal field theory related to lattice models of percolation [119, 120], and the $c = -2$ systems related to so-called critical dense polymers [121, 122].

More concrete mathematical understanding of the mechanisms behind the divergences came from the series of papers [20, 123, 124], where the authors investigated how staggered modules arise through fusion. Logarithmic conformal field theory has been extensively studied, leading to the so called logarithmic minimal models [113, 125, 126] which are logarithmic extensions of the Virasoro minimal models related to the triplet algebras of [127], as well as the Wess-Zumino-Witten models at fractional values of the level k [107, 118, 128]. However, the field of logarithmic conformal field theory is still at the leading edge of research in conformal field theory, and representation theory in mathematical physics.

The Verlinde formula [129], which provides a formula for the structure coefficients of a given Grothendieck (character) ring, returns strange results in the logarithmic setting. As discussed in Section 1.5, the fusion co-efficients are interpreted as multiplicities of the modules that appear, however in the logarithmic case for example, applying the Verlinde formula can lead to negative co-efficients. A detailed description of how the Verlinde formula is to be understood in the logarithmic setting is given in [118]. Moreover, character fusion is generically not equivalent to module fusion, rather it provides a guide. One method for calculating module fusion is an algorithm known as the Nahm-Gaberdiel-Kausch (NGK) algorithm [19–21] (see [23] for an introduction), and while extremely powerful, it is still difficult to calculate with concretely. We also briefly mention here that there is a well developed theory of tensor categories [130–132] which runs parallel to the understanding of fusion through the NGK algorithm (see [133] for a review). A good description of the relation between these two approaches is given in [23].

In this chapter, we introduce the theory required to present the results of our exploratory investigation into staggered modules over the $N = 2$ superconformal algebras. Recent results on the coset construction of the $N = 2$ superconformal algebras have opened the possibility of investigating $N = 2$ staggered modules, using the corresponding staggered modules over the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$. The presence of staggered modules in the non-unitary minimal models of $\widehat{\mathfrak{sl}}(2)$ has been demonstrated explicitly for specific values of the level k, namely $k = -\frac{1}{2}$ and $k = -\frac{4}{3}$, and moreover is conjectured to occur in general [107, 109]. Here we demonstrate via the coset given in (5.1) that staggered modules also arise in the non-unitary minimal models of the $N = 2$ superconformal algebras.

We start by introducing the notion of a vertex operator superalgebra (VOSA), and its modules. Although this formalism is not heavily used in our results, it provides a unifying mathematical framework for understanding conformal field theory, and many of the results that are key to our work here rely upon it.

Following this, we introduce the algebras involved in the coset. Our presentation here generally follows the notation and structure of [100], where the authors introduced this coset for studying the (non-)unitary minimal models of the $N = 2$ superconformal algebras. We introduce the $N = 2$ superconformal algebras, their highest-weight representation theory, and their conjugation and spectral flow morphisms. We continue by introducing other algebras required in the construction, namely the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$, the Heisenberg algebra $\widehat{H}$, and the $\mathfrak{bc}$-ghost system. We discuss the modules and automorphisms of these algebras relevant to the coset construction, and to our results which follow in Chapter 6. Finally, we introduce to the coset itself, the related branching rules for the representations, and the action of the algebra automorphisms on the branching rules. In this introduction, we focus on key points relevant to the presentation of our results. A comprehensive description of the coset construction for both unitary and non-unitary $N=2$ minimal models as is given in [100].

5.1.1 A brief introduction to vertex operator superalgebras

In this chapter, it is convenient to briefly introduce the notion of a vertex operator (super)algebra. Vertex operator superalgebras provide a rigorous mathematical framework that unifies the symmetry algebra and the space of fields. The link between vertex operator superalgebras and the previously used operator product algebras will be clear from the way the definitions are presented.

We begin by stating the definition of both a vertex operator superalgebra, and a module thereof, following [134]. The data of a vertex operator superalgebra is a $\mathbb{Z}_2$-graded vector space $V=V^0\oplus V^1$, where

$$V^0=\bigoplus_{n\in\mathbb{Z}_{\geq 0}} V_n, \qquad V^1=\bigoplus_{n\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} V_n, \tag{5.2}$$

that is V is also $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded; two distinguished elements $\mathbb{1}\in V_0$ and $\omega\in V_2$; and a map

$$Y: V\to \mathrm{End}(V)[[z,z^{-1}]], \quad a\mapsto \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad \forall a\in V, \tag{5.3}$$

called the field map, which associates each element of V to an operator-valued formal series in the variable z. The data $(V,Y,\mathbf{1},\omega)$ satisfy the following axioms:

- $Y(a,z)=0$ iff $a=0$.
- **Vacuum**: $Y(\mathbf{1},z)=\mathrm{id}_V$ where id_V is the identity endomorphism.
- **Virasoro**: The field corresponding to the vector ω is given by

$$Y(\omega,z)=\sum_{n\in\mathbb{Z}}\omega_n z^{-n-1}=\sum_{n\in\mathbb{Z}} L_{-n}z^{-n-2}, \tag{5.4}$$

  where the modes L_n satisfy

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{1}{12}n(n^2-1)\delta_{n+m,0}c\mathbb{1}, \quad c\in\mathbb{C}, \tag{5.5}$$

  where c is the central charge (often called the rank of V in the mathematics literature).

- **Derivative**: The mode L_{-1} acts as

$$Y(L_{-1}a,z)=\frac{d}{dz}Y(a,z), \quad \forall a\in V. \tag{5.6}$$

- **Grading**: The action of L_0 grades the vector space V

$$L_0|_{V_n}=n\,\mathrm{id}|_{V_n}. \tag{5.7}$$

- **Jacobi/Borcherds**: The following associativity condition holds for any $m,n \in \mathbb{Z}$,

$$\begin{aligned}&\operatorname{Res}_{z-w}\big(Y(Y(a,z-w)b,w)\iota_{w,z-w}((z-w)^m z^n)\big)\\ &\qquad = \operatorname{Res}_z\big(Y(a,z)Y(b,w)\iota_{z,w}(z-w)^m z^n\big) - (-1)^{|a||b|}\operatorname{Res}_z\big(Y(b,w)Y(a,z)\iota_{w,z}(z-w)^m z^n\big),\end{aligned} \tag{5.8}$$

where Res_z is the usual z^{-1} co-efficient extraction operator; for a rational function with possible poles at $z=w,\ z=0,\ w=0$, the expression $\iota_{z,w}f(z,w)$ denotes the power series expansion of $f(z,w)$ in the domain $|z|>|w|$; and $|a|=i$ if $a \in V^i$.

We say that an element $a \in V$ is homogeneous of degree $n = \deg(a) \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ if $a \in V_n$. The definition for a vertex operator algebra (VOA) is taken exactly as above, except V^1 is trivial. The notion of vertex operator subalgebra is defined in the obvious way. A vertex oeprator superalgebra ideal is a subspace $U \subseteq V$ such that

$$Y(a,z)U \subseteq U[[z,z^{-1}]] \quad \forall a \in V. \tag{5.9}$$

A vertex operator superalgebra is simple if its only ideals are V and the trivial ideal. If U is an ideal such that $\mathbf{1} \notin U$, and $\omega \notin U$, then the quotient V/U admits a natural VOSA structure.

One can see from the definition that a vertex operator (super)algebra merges the underlying algebra of symmetries, its action on the vacuum representation, and the state-field correspondence into a single framework. Moreover, as we now identify what were the generators of the Lie algebra, specifically with endomorphisms on the vacuum module, the central charge is no longer an element of the algebra, rather a parameter which determines the vertex operator (super)algebra. In the commutation relations, this parameter accompanies the identity endomorphism, which is related to the vacuum vector under the field map. In this way, we have eliminated the overlap in notation.

As an example, the Virasoro vertex operator algebra takes the vacuum representation of highest weight zero and central charge $c \in \mathbb{C}$, and $L_{-1}|0\rangle = 0$, as the vector space V. The highest-weight vector $|0\rangle$ is the vacuum vector of the theory, and $L_{-2}|0\rangle$ is the conformal vector. The vertex operator algebra arising from this vector space is known as the universal vertex operator algebra. is It is well known that for particular values of the central charge, the vacuum module is reducible, implying that the corresponding vertex operator algebra for those values of c is not simple. As such, the corresponding simple VOA is exactly the one formed by taking the corresponding simple quotient of the vacuum module as the vector space V.

Correspondingly, a module over a vertex operator superalgebra $(V,Y,\mathbf{1},\omega)$ is a vector space $M = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_{\geq 0}} M_n$, and a linear map

$$Y_M(\cdot,z) : V \to \operatorname{End}(M)[[z,z^{-1}]], \qquad a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \tag{5.10}$$

satisfying

- For every homogeneous $a \in V$, $a_n M_m \subset M_{m+\deg(a)-n-1}$.
- **Vacuum**: $Y_M(\mathbf{1}, z) = \mathbb{1}_M$
- **Virasoro**: $Y_m(\omega, z) = L_n z^{-n-2}$ where the L_n satisfy the Virasoro commutation relations with central charge c.
- **Derivative**: $Y_M(L_{-1}a, z) = \frac{d}{dz} Y_M(a, z)$ for all $a \in V$.
- **Jacobi/Borcherds**: We require that the identity

$$\begin{aligned}\operatorname{Res}_{z-w}&\big(Y_m(Y(a,z-w)b,w)\iota_{w,z-w}((z-w)^m z^n)\big)\\ &= \operatorname{Res}_z\left(Y_M(a,z)Y_M(b,w)\iota_{z,w}(z-w)^m z^n\right)\\ &\quad -(-1)^{|a||b|}\operatorname{Res}_z\left(Y_M(b,w)Y_M(a,z)\iota_{w,z}(z-w)^m z^n\right),\end{aligned} \tag{5.11}$$

  holds for all $m, n \in \mathbb{Z}$.

In essence, this definition is exactly what one would expect for the definition of a module. We have defined an action of the (super)algebra on a vector space, and it is compatible with the product relations. Moreover, the action of the Virasoro (sub)algebra is preserved.

Although we will not be making use of the relations presented in the definition of vertex operator algebra directly, it is particularly useful to understand the framework. Moreover, it underpins many of the fundamental results which we use in this chapter. The ideas of unitarity of representations and rationality of minimal models introduced in Chapter 1 all carry into this framework.

5.2 Introduction to the $N = 2$ superconformal algebras

We start by introducing the $N = 2$ superconformal algebras. These are a family of infinite-dimensional Lie superalgebras, generated by the modes of two bosons and two fermions. We will focus on two superalgebras within the family, namely the Neveu-Schwarz algebra NS= $\langle L_n, J_m, G_r^+, G_s^-, c \mid n,\ m \in \mathbb{Z},\ r,\ s \in \mathbb{Z} + \frac{1}{2}\rangle$, and the Ramond algebra R= $\langle L_n, J_m, G_r^+, G_s^-, c \mid n,\ m \in \mathbb{Z},\ r,\ s \in \mathbb{Z}\rangle$. Both superalgebras have defining commutation relations given by

$$\begin{gathered}[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\mathbb{1},\\ \{G_r^\pm, G_s^\mp\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}\mathbb{1}\\ [L_m, J_n] = -nJ_{n+m}, \quad [J_m, J_n] = \frac{c}{3}m\delta_{n+m,0}\mathbb{1}\\ \left[L_m, G_r^\pm\right] = \left(\frac{m}{2} - r\right)G_{m+r}^\pm, \quad \left[J_m, G_r^\pm\right] = \pm G_{m+r}^\pm.\end{gathered} \tag{5.12}$$

Correspondingly, the $N = 2$ superconformal algebras are vertex operator superalgebras, generated by two bosonic fields, namely $J(z),\ T(z)$ of conformal weight 1 and 2 respectively, and two fermionic fields $G^\pm(z)$ each of conformal weight $\frac{3}{2}$. The field $T(z)$ generates a Virasoro subalgebra, and the other generating fields are primary. The defining non-trivial operator product expansions for the universal $N = 2$ superconformal algebra are given by

$$\begin{gathered} J(z)J(w) \sim \frac{(c/3)\mathbb{1}}{(z-w)^2}, \quad T(z)T(w) \sim \frac{(c/2)\mathbb{1}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\ T(z)J(w) \sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad T(z)G^{\pm}(w) \sim \frac{(3/2)G^{\pm}}{(z-w)} + \frac{\partial G^{\pm}(w)}{z-w}, \\ G^{\pm}(z)G^{\mp}(w) \sim \frac{(2/3)c\mathbb{1}}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial J(w)}{z-w}. \end{gathered} \tag{5.13}$$

We will focus on the minimal model $N = 2$ algebras $M(u,v)$, labelled by u, v, which occur when

$$c = 3\left(1 - 2\frac{v}{u}\right), \quad u \in \mathbb{Z}_{\geq 2}, \quad v \in \mathbb{Z}_{\geq 1}, \quad \gcd(u,v) = 1. \tag{5.14}$$

For these values of c, the universal vertex operator superalgebra is not simple, and $M(u,v)$ denotes the corresponding unique irreducible quotient. Our aim is to investigate the representation theory of the algebras $M(u,v)$. The minimal models are unitary when $v = 1$. Our results will focus on the case when $v \geq 2$, as we are interested in non-unitary theories. Unitary theories do not contain staggered modules.

5.2.1 Representation theory of the $N = 2$ superconformal algebras

We begin by introducing the highest-weight modules of the $N = 2$ superconformal algebras. Both the Neveu-Schwarz and Ramond algebras admit a triangular decomposition

$$SCA_2 = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \tag{5.15}$$

where, for the Neveu-Schwarz algebra we have

$$\begin{aligned} \mathfrak{g}_- &= \text{span}\{G_n^-, G_n^+, L_n, J_n \mid n < 0\}, \\ \mathfrak{g}_+ &= \text{span}\{G_n^-, G_n^+, L_n, J_n \mid n > 0\}, \\ \mathfrak{g}_0 &= \text{span}\{L_0,\ J_0,\ \mathbb{1}\}; \end{aligned} \tag{5.16}$$

and for the Ramond algebra, we have

$$\begin{aligned} \mathfrak{g}_- &= \text{span}\{G_m^-, G_n^+, L_n, J_n \mid m \leq 0, n < 0\} \\ \mathfrak{g}_+ &= \text{span}\{G_m^+, G_n^-, L_n, J_n \mid m \geq 0, n > 0\}, \\ \mathfrak{g}_0 &= \text{span}\{L_0,\ J_0,\ \mathbb{1}\}. \end{aligned} \tag{5.17}$$

We then develop the Neveu-Schwarz Verma modules in the standard way. Consider the 1-dimensional module $\mathbb{C}_{j,\Delta}^{\text{NS},\pm}$, $j, \Delta \in \mathbb{C}$, of the subalgebra $\mathfrak{g}_+ \oplus \mathfrak{g}_0$ where $L_0, J_0, \mathbb{1}$ act as $\Delta, j, 1$ respectively, and all other actions are trivial. One can induce from $\mathbb{C}_{j,\Delta}^{\text{NS},\pm}$ to a module over the full $N = 2$ superconformal algebra, which we denote $V_{j,\Delta}^{\text{NS},\pm}$. This is the Verma module of weight (j,Δ). The Verma module has a unique maximal submodule. We let $L_{j,\Delta}^{\text{NS},\pm}$ denote the irreducible quotient of $V_{j,\Delta}^{\text{NS},+}$ by its maximal submodule. The Verma modules decompose into weight spaces labelled by (J_0, L_0) eigenvalues.

A Neveu-Schwarz highest-weight vector is one which is annihilated by the action of $\mathfrak{g}_+$, and is a simultaneous eigenvector of $L_0, J_0, \mathbb{1}$. Any vector which generates a highest-weight submodule (not necessarily maximal) must necessarily vanish under the action of $\mathfrak{g}_+$, and must be a simultaneous eigenvector of $L_0, J_0, \mathbb{1}$. That is, it satisfies the highest-weight condition (without necessarily being the vector with highest weight). Any such vector which satisfies the highest-weight conditions, is called a *singular vector*. As in the case of Virasoro, the proper singular vectors generate proper submodules, and moreover, proper submodules are again null with respect to the Shapovalov form on $V_{j,\Delta}$.

As modules over a superalgebra, the Verma modules are $\mathbb{Z}_2$ graded, that is they decompose into an even and an odd subspace $V = V_0 \oplus V_1$. The parity of such a module is determined by whether the highest-weight vector lies in the space V_0 or V_1. If it is an even vector, then the module has even parity denoted by superscript $+$, otherwise the module has odd parity, and is denoted by superscript $-$. The fermionic generators map between these subspaces, and the bosonic generators preserve them. There is a parity reversal functor $\Pi : V \to V$, such that $\Pi(V_i) \mapsto V_{1-i}$.

Construction of Verma modules for the Ramond algebra proceeds similarly, however we now have $G_0^+ \in \mathfrak{g}_+$ and $G_0^- \in \mathfrak{g}_-$. Continuing, we take the 1-dimensional representation $\mathbb{C}_{j,\Delta}^{\mathrm{R},\pm}$ over the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_+$, where J_0, L_0, $\mathbb{1}$ act as j, Δ, 1 respectively, and all other actions are trivial. We then induce from this to a module over the full Ramond algebra, denoted $V_{j,\Delta}^{\mathrm{R},\pm}$, the Verma module of weight (j,Δ). Similarly to the Neveu-Schwarz modules, we denote the irreducible quotient of the Verma module by its unique maximal submodule by $L_{j,\Delta}^{\mathrm{R},\pm}$. A Ramond highest-weight vector is a vector v which is annihilated by all positive modes and G_0^+, and is a simultaneous eigenvector of $L_0, J_0, \mathbb{1}$.

The Verma modules over both superalgebras have a bilinear form given by the Shapovalov form [12] (see Section 2.3.2 for an introduction), whereby the highest-weight vector $|v\rangle$ is normalised to $\langle v|v\rangle = 1$, and the adjoints of the modes are

$$L_n^\dagger = L_{-n}, \quad J_n^\dagger = J_{-n}, \quad \left(G_r^\pm\right)^\dagger = G_{-r}^\mp, \quad \mathbb{1}^\dagger = \mathbb{1}. \tag{5.18}$$

In the paper [135] the authors give formulas for the determinant of the Gram matrix of inner products on Verma modules for the Neveu-Schwarz algebra. We have that the determinant to level n in the module is given by

$$\det(V_{j,\Delta,c}) = \prod_{1 \le rs \le 2n} (f_{r,s})^{P_{\mathrm{NS}}(n-rs/2,m)} \times \prod_{k \in \mathbb{Z}+\frac{1}{2}} (g_k)^{\tilde{P}_{\mathrm{NS}}(n-|k|,m-\mathrm{sgn}(k);k)}, \tag{5.19}$$

where $P_{\mathrm{NS}}(n,m)$ and $\tilde{P}_{\mathrm{NS}}(n,m;k)$ are partition functions which we will not require when determining zeroes (their explicit form is given in [135]), and the functions $f_{r,s}$ and g_k are defined as

$$f_{r,s}(j,\Delta,t) = \frac{(st-2r)^2}{4t^2} - j^2 - \frac{4\Delta}{t} - \frac{1}{t^2}, \tag{5.20}$$

and

$$g_k(j,\Delta,t) = 2\Delta - 2kj - \frac{2}{t}\left(k^2 - \frac{1}{4}\right). \tag{5.21}$$

When $f_{r,s} = 0$, there is an uncharged singular vector at level $\frac{rs}{2}$ in the Verma module. Similarly, when $g_k = 0$ there is a charged singular vector at level $|k|$, with J_0-weight $\text{sign}(k)$ relative to the highest-weight vector.

A surprising result is that all singular vectors of Verma modules appear with relative charge (J_0-eigenvalue) $(-2), -1, 0, 1$. The singular vectors with relative charge -2 only occur in Ramond sector modules. We remark that the corresponding formulas for the Ramond sector follow directly from the spectral flow identifications introduced at the end of this section.

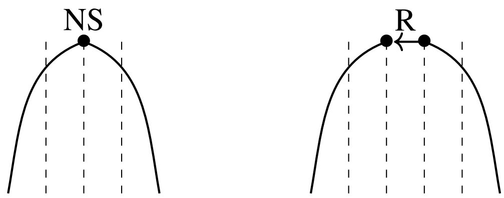


Figure 5.1: *The figure shows the extremal diagram (the shape of the outer-most states) for a generic Verma module over the Neveu-Schwarz (NS) and Ramond (R) module. The vertical dotted lines indicate the values for the charge where singular vectors can appear. The fermionic generators squaring to zero implies the modules have a parabola-like shape in weight space.*

We note that unlike many other common examples in representation theory, the submodules of SCA_2 Verma modules are not necessarily Verma modules. To be a Verma module, one must allow for free left action of the universal enveloping algebra of the lowering generators $U(\mathfrak{g}_-)$, however, fermionic generators square to zero, implying that charged singular vectors cannot generate Verma submodules [135].

Moreover, the Verma modules of the $N = 2$ superconformal algebras admit subsingular vectors. Suppose a vector $x \in V$ is singular, and generates a submodule $U \subset V$. Then a subsingular vector $w \in V$ is a singular vector in the quotient V/U, which is not singular in the module V. Subsingular vectors appear generically in the representation theory of the $N = 2$ superconformal algebras.

Finally, we remark that it is also possible to have more than one linearly independent singular vector in the same weight space. This phenomenon was outlined clearly in [95], where the authors describe the requirements for unitarity of the $N = 2$ minimal models. Although this will not affect our results, as we will deal with modules composed from irreducible representations, we make this remark to outline the significant differences between $N = 2$ superconformal and Virasoro representation theory.

Highest-weight modules over the Neveu-Schwarz algebra have a 1-dimensional space of L_0-eigenvectors with minimal eigenvalue (we refer to these states as ground states). For the Ramond algebra, generically this space is 2-dimensional, and given by $\text{span}\{v,\ G_0^- v\}$. However, since

$$G_0^+ G_0^- v = \left(2L_0 - \frac{c}{12}\right) v, \tag{5.22}$$

if $\Delta = \frac{c}{24}$, then the vector $G_0^- v$ is a singular vector. This implies the corresponding irreducible module $L^{\mathrm{R},\pm}_{j,\frac{c}{24}}$ will have a 1-dimensional space of ground states.

5.2.2 Automorphisms of $N=2$ superconformal algebras

The $N=2$ superconformal algebras admit a number of well studied automorphisms [136]. Here we will be interested in two in particular, namely the conjugation automorphism γ and the spectral flow family of automorphisms labelled by σ^ℓ for $\ell \in \mathbb{Z}$.

The conjugation automorphism acts on the generators as

$$\gamma(L_n) = L_n, \quad \gamma(J_n) = -J_n, \quad \gamma(G_r^\pm) = G_r^\mp. \tag{5.23}$$

The effect of conjugation is to reverse the charge of the fermionic generators.

The spectral flow automorphisms act on the generators as

$$\sigma^\ell(L_n) = L_n - \ell J_n + \frac{1}{6}\ell^2 \delta_{n,0} c\mathbb{1}, \quad \sigma^\ell(J_n) = J_n + \frac{\ell}{3}\delta_{n,0} c\mathbb{1}, \quad \sigma^\ell(G_r^\pm) = G^\pm_{r\mp\ell}. \tag{5.24}$$

Both automorphisms leave the identity element unchanged.

We note that the spectral flow for $\ell \in \mathbb{C}$ defines an isomorphism between superalgebras in the family of $N=2$ superconformal algebras. However, it is only an automorphism if $\ell \in \mathbb{Z}$ [136, 137].

Also of interest is when $\ell \in \mathbb{Z} + \frac{1}{2}$, where it provides a map between the Neveu-Schwarz and Ramond algebras. We note also that the spectral flow is not an automorphism of the corresponding vertex operator superalgebra, only the algebra considered as an infinite dimensional Lie superalgebra (or equivalently, as a vertex superalgebra). This is because the map does not preserve the conformal vector $L_{-2}|0\rangle$ in the vertex operator superalgebra setting. However, for the simplicity of our presentation, we will use the term automorphism also when $\ell \in \frac{1}{2}\mathbb{Z}$.

The automorphisms σ^ℓ and γ allow for identifications of modules in the following way. The action of an automorphism on the modules of an algebra is referred to as twisting, and the corresponding modules are referred to as twisted modules. Twisted modules are related to their untwisted counterparts in the following way. Suppose we have a some vector space isomorphism ξ from a given module M to a twisted module $\xi(M)$. The twisted module is endowed with the action of $\mathfrak{g}$ such that

$$x\xi(|v\rangle) = \xi(\omega^{-1}(x)|v\rangle), \qquad \text{for } x \in \mathfrak{g},\ v \in M, \tag{5.25}$$

where ω is an automorphism of $\mathfrak{g}$. In the remainder of the text, we will not distinguish between the vector space isomorphism ξ and the algebra automorphism ω. Rather, as an example, to show the action of the spectral flow, we have

$$x\sigma^\ell(|v\rangle) = \sigma^\ell\big(\sigma^{-\ell}(x)|v\rangle\big), \quad x \in \mathfrak{g}. \tag{5.26}$$

Using these equations, it is possible to derive relations between irreducible highest-weight modules. For conjugation,

$$\gamma(L^{\mathrm{NS},\pm}_{j,\Delta}) \cong L^{\mathrm{NS},\pm}_{-j;\Delta}, \quad \gamma\left(L^{\mathrm{R},\pm}_{j;\Delta}\right) \cong \begin{cases} L^{\mathrm{R},\pm}_{-j;\Delta}, & \text{if } \Delta = \frac{c}{24}, \\ L^{\mathrm{R},\mp}_{-j+1,\Delta}, & \text{otherwise,} \end{cases} \tag{5.27}$$

and for the spectral flow

$$\begin{aligned} \sigma^{1/2}\left(L^{\mathrm{NS},\pm}_{j,\Delta}\right) &\cong L^{\mathrm{R},\pm}_{j+c/6,\Delta+j/2+c/24}, \\ \sigma^{1/2}\left(L^{\mathrm{R},\pm}_{j,\Delta}\right) &\cong \begin{cases} L^{\mathrm{NS},\pm}_{j+c/6,j/2+c/12}, & \text{if } \Delta = \frac{c}{24}, \\ L^{\mathrm{NS},\mp}_{j-1+c/6,\Delta+(j-1)/2+c/24}, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.28}$$

5.3 The coset construction of $N=2$ minimal models

In this section we outline the elements required to understand the coset construction of the $N=2$ superconformal minimal models $M(u,v)$. We introduce the remaining algebras required for the coset, the coset construction itself, the resulting branching rules, and the corresponding module dictionary which interprets the output. We generally follow the presentation and notation of [100].

5.3.1 The affine Lie algebra $\widehat{\mathfrak{sl}}(2)$

Here we introduce the vertex operator algebra corresponding to the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$. These algebras, and their related families of minimal models are key in understanding the construction of the $N=2$ superconformal minimal models at admissible level. Moreover, the representation theory of these algebras dictates the possible families of representations arising for the superconformal algebras through the coset construction.

The family of vertex operator algebras based on the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ at level k are generated by three fields, $e(z),\ f(z),\ h(z)$, with operator product expansions

$$h(z)e(w) \sim \frac{2e(w)}{z-w}, \quad h(z)h(w) \sim \frac{2k\mathbb{1}}{(z-w)^2}, \quad h(z)f(w) \sim \frac{-2f(w)}{z-w}, \quad e(z)f(w) \sim \frac{k\mathbb{1}}{(z-w)^2} + \frac{h(w)}{z-w}, \tag{5.29}$$

where the level $k \in \mathbb{C}\backslash\{-2\}$, and $e(z)e(w) \sim 0$, $f(z)f(w) \sim 0$. We omit the case $k=-2$, known as the critical level, because at that level we no longer have a Sugawara construction. The underlying Lie algebra of modes has non-zero commutation relations

$$\begin{aligned} [h_m, e_n] &= 2e_{m+n}, & [h_m, f_n] &= -2f_{m+n}, \\ [h_m, h_n] &= 2mk\delta_{n+m,0}\mathbb{1}, & [e_m, f_n] &= h_{m+n} + mk\delta_{n+m,0}\mathbb{1}. \end{aligned} \tag{5.30}$$

The algebra $\widehat{\mathfrak{sl}}(2)$ also admits conjugation and spectral flow automorphisms, denoted γ_{aff} and $\sigma^{\ell}_{\text{aff}}$, which have the following action on the generators

$$\gamma_{\text{aff}}(e_n)=f_n,\quad \gamma_{\text{aff}}(f_n)=e_n,\quad \gamma_{\text{aff}}(h_n)=-h_n,\quad \gamma_{\text{aff}}(L_0^{\text{aff}})=L_0^{\text{aff}},$$
$$\sigma^{\ell}_{\text{aff}}(e_n)=e_{n-\ell},\quad \sigma^{\ell}_{\text{aff}}(f_n)=f_{n+\ell},\quad \sigma^{\ell}_{\text{aff}}(h_n)=h_n+\ell\delta_{n,0}k\mathbb{1},\quad \sigma^{\ell}_{\text{aff}}(L_0^{\text{aff}})=L_0^{\text{aff}}+\frac{1}{2}\ell h_0+\frac{1}{4}\ell^2k\mathbb{1}. \tag{5.31}$$

We will make use of the notation "aff" to distinguish objects which relate to $\widehat{\mathfrak{sl}}(2)$ from those which relate to the $N=2$ superconformal algebras, whenever there is any ambiguity.

We can naturally construct a Virasoro field via the Sugawara construction, as discussed in Section 1.6 (recalling that $h^{\vee}=2$ for $\mathfrak{sl}(2)$), which is given by

$$T^{\text{aff}}(z)=\frac{1}{2(k+2)}\left[\frac{1}{2}(hh)(z)+(ef)(z)+(fe)(z)\right]. \tag{5.32}$$

The modes of the field $T^{\text{aff}}(z)$ generate a Virasoro algebra with central charge

$$c=\frac{3k}{k+2}, \tag{5.33}$$

as we have seen in the introduction. We can introduce the following convenient reparameterisation, in terms of $t=k+2$, so we have that the central charge becomes

$$c=3\left(1-\frac{2}{t}\right). \tag{5.34}$$

By definition, the fields $e(z),h(z),f(z)$ are conformal weight 1 primary fields with respect to the action of $T^{\text{aff}}(z)$. As in the case of $N=2$, we will introduce $u,v\in\mathbb{Z}$ such that $u\geq 2$, $v\geq 1$, and $\gcd(u,v)=1$; and we set $t=\frac{u}{v}$.

The values of t satisfying the parametrisation in terms of $u,\ v$, are referred to as admissible levels. For exactly these values, the universal vertex operator algebra based on $\widehat{\mathfrak{sl}}(2)$ (the conformal vector is given by the corresponding mode of the Sugawara operator) is not simple. The representations of the corresponding simple quotients of these algebras are the minimal models. Following notation in [100], we denote the minimal model $A_1(u,v)$. Explicitly, the admissible levels and central charges are

$$k=-2+\frac{u}{v},\quad c=3\left(1-2\frac{v}{u}\right),\quad u\in\mathbb{Z}_{\geq 2},v\in\mathbb{Z}_{\geq 1},\gcd(u,v)=1. \tag{5.35}$$

When $v=1$, the level is "integral". For these values of the level, one obtains the Wess-Zumino-Witten models. When $v\geq 2$, the resulting level is fractional, and the corresponding models are no longer unitary. There is a classification of the irreducible representations for these values of the level [101,140]. In this discussion, we will draw on two particular types of modules for the $A_1(u,v)$ minimal models, which we will denote L-type, and $D^{\pm}$-type.

In order to discuss these modules, it is convenient to introduce the following parameters

$$\lambda_{r,s}=r-1-st,\quad \Delta^{\text{aff}}_{r,s}=\frac{(r-st)^2-1}{4t}, \tag{5.36}$$

which describe the h_0-eigenvalue and L_0^{aff}-eigenvalue of the highest-weight vectors of the $A_1(u,v)$ modules respectively. The module families of interest to our discussion here are then the following.

- L-type: The L-type modules are irreducible highest-weight modules labelled by $L_{r,0}$, where $1 \leq r \leq u-1$. The highest-weight vector of these modules have $(L_0^{\mathrm{aff}}, h_0)$ eigenvalues given by $(\Delta_{r,0}^{\mathrm{aff}}, \lambda_{r,0})$. These can be conceptualised by understanding that the space of vectors with maximal L_0-eigenvalue form a finite dimensional representation of the algebra $\mathfrak{sl}(2)$ of weight $\lambda_{r,0}$.

- D^+-type: The D^+-type modules, labelled $D_{r,s}^+$ are again irreducible highest-weight modules, with highest weight $(\Delta_{r,s}^{\mathrm{aff}}, \lambda_{r,s})$ for $1 \leq r \leq u-1$, $1 \leq s \leq v-1$. Again the space of vectors with minimal L_0^{aff} weight forms a representation of $\mathfrak{sl}(2)$, however, as the highest weight of this representation $\lambda_{r,s}$ is no longer integer, this representation is now infinite dimensional.

- D^--type: The D^--type modules, labelled $D_{r,s}^-$ are irreducible modules, however they are no longer highest weight. Rather, they are conjugate modules to the D^+-type modules, that is $\gamma_{\mathrm{aff}}(D_{r,s}^{\pm}) \cong D_{r,s}^{\mp}$. As such, the space of states with minimal L_0^{aff} eigenvalue forms a lowest weight representation of $\mathfrak{sl}(2)$ with lowest weight $-\lambda_{r,s}$, where $1 \leq r \leq u-1$, $1 \leq s \leq v-1$. It is clear by the action of conjugation that the maximal L_0 eigenvalue of such modules is $\Delta_{r,s}^{\mathrm{aff}}$.

We remark that there are several other families of modules, which are necessary to completely describe the representation theory of the $A_1(u,v)$ minimal models, however, we will not require these modules for this work. A thorough description of such modules can be found in [101].

5.3.2 The coset construction of the $N=2$ algebras

The key tool of our investigation is the Kazama-Suzuki coset (commutant) construction of the $N=2$ superconformal algebra. The particular coset of interest is

$$\frac{A_1(u,v) \otimes \mathfrak{bc}}{\widehat{H}} = M(u,v), \tag{5.37}$$

where $A_1(u,v)$ and $M(u,v)$ are the minimal models of $\widehat{\mathfrak{sl}}(2)$ and the $N=2$ superconformal algebras respectively, $\widehat{H}$ is a simple Heisenberg vertex operator algebra, and $\mathfrak{bc}$ is a simple vertex operator superalgebra known as the $\mathfrak{bc}$-ghost system. We introduce these additional algebras and their relevant representations briefly below for completeness.

We also remark that in the vertex operator algebra setting, the notion of coset has been made rigorous through the commutant of vertex operator algebras [139]. Given a vertex operator algebra V, and a vertex operator subalgebra $U \subset V$, the commutant of U in V, denoted $\mathrm{Com}(U,V)$ is the subalgebra of V which commutes with the subalgebra U. The above coset is equivalently given by $\mathrm{Com}(\widehat{H}, A_1(u,v) \otimes \mathfrak{bc})$.

We continue by briefly introducing the other algebras relevant to the coset construction. The Heisenberg algebra is the infinite dimensional Lie algebra related to the abelian algebra $\mathfrak{gl}(1)$. It is also commonly referred to as the free boson algebra, as its corresponding field theory is that of a free bosonic field. It is generated by the modes $\{a_n \mid n \in \mathbb{Z}\}$, and has commutation relations

$$[a_m, a_n] = m\delta_{m+n,0}\mathbb{1}. \tag{5.38}$$

As a field theory, it is generated by a single field $a(z)$ with OPE given by

$$a(z)a(w) \sim \frac{\mathbb{1}}{(z-w)^2}. \tag{5.39}$$

The algebra becomes a vertex operator algebra once we introduce the Virasoro operator coming from the Sugawara construction

$$T^{\widehat{H}}(z) = \frac{1}{2}(aa)(z), \tag{5.40}$$

which generates a Virasoro algebra of central charge $c = 1$.

The Verma modules of the Heisenberg algebra are the Fock spaces (or Fock modules) F_p, for $p \in \mathbb{C}$, which are generated by a highest-weight vector of weight p. It is a well known fact that the Heisenberg algebra Fock spaces are irreducible.

The $\mathfrak{bc}$-ghost system is an infinite dimensional Lie superalgebra generated by the modes $\{b_n, c_n \mid n \in \mathbb{Z}+\frac{1}{2}\}$ in the Neveu-Schwarz sector, and $\{b_n, c_n \mid n \in \mathbb{Z}\}$ in the Ramond sector, with non-zero commutation relations

$$\{b_n, c_n\} = \delta_{m+n,0}\mathbb{1}. \tag{5.41}$$

Correspondingly the superalgebra is generated by the fields $b(z), c(z)$ with non-zero OPE relations

$$b(z)c(w) \sim c(z)b(w) \sim \frac{\mathbb{1}}{z-w}. \tag{5.42}$$

This algebra becomes a vertex operator superalgebra with Virasoro operator given by

$$T^{\mathfrak{bc}}(z) = \frac{1}{2}\left[(\partial bc) - (b\partial c)\right], \tag{5.43}$$

with central charge $c = 1$.

The highest-weight representations of the $\mathfrak{bc}$-ghost system are $\mathbb{Z}_2$-graded modules (as they are fermionic), and fall into four isomorphism classes. The representations are labelled N_i for $i \in 0,1,2,3$, where N_0 and $N_2 = \Pi(N_0)$ are Neveu-Schwarz modules (Π is the parity reversal functor, exchanging the even and odd subspaces); and N_1 and $N_3 = \Pi(N_1)$ are Ramond modules. All N_i modules are irreducible.

Each of these algebras have both spectral flow and conjugation automorphisms. For the Heisenberg algebra, conjugation is denoted $\gamma_{\widehat{H}}$, and has the action

$$\gamma_{\widehat{H}}(a_n) = -a_n, \quad \gamma_{\widehat{H}}(L_n^{\widehat{H}}) = L_n^{\widehat{H}}. \tag{5.44}$$

Spectral flow is denoted $\sigma^m_{\widehat{H}}$, for $m \in \mathbb{C}$, and has the action

$$\sigma^m_{\widehat{H}}(a_n) = a_n - m\delta_{n,0}\mathbb{1}, \quad \sigma^m_{\widehat{H}}(L^{\widehat{H}}_n) = L^{\widehat{H}}_n - ma_n + \frac{1}{2}m^2\delta_{n,0}\mathbb{1}. \tag{5.45}$$

For the $\mathfrak{bc}$-ghost system, we denote the automorphisms $\gamma_{\mathfrak{bc}}$ and $\sigma^\ell_{\mathfrak{bc}}$ for $\ell \in \frac{1}{2}\mathbb{Z}$, with the action

$$\gamma_{\mathfrak{bc}}(b_n) = c_n, \quad \gamma_{\mathfrak{bc}}(c_n) = b_n, \quad \gamma_{\mathfrak{bc}}(L^{\mathfrak{bc}}_n) = L^{\mathfrak{bc}}_n, \tag{5.46}$$

and

$$\sigma^\ell_{\mathfrak{bc}}(b_n) = b_{n-\ell}, \quad \sigma^\ell_{\mathfrak{bc}}(c_n) = c_{n+\ell}, \quad \sigma^\ell_{\mathfrak{bc}}(L^{\mathfrak{bc}}_n) = L^{\mathfrak{bc}}_n - \ell Q_n + \frac{1}{2}\ell^2\delta_{n,0}\mathbb{1}, \tag{5.47}$$

where

$$Q(z) = \big(bc\big)(z) = \sum_{n\in\mathbb{Z}} Q_n z^{-n-1}. \tag{5.48}$$

For both algebras, the automorphisms have trivial action on the element $\mathbb{1}$.

A complete discussion of the coset construction and related embedding is given in [100]. Here we are primarily interested in the corresponding statement for representations. The coset implies a branching rule of representations, namely that

$$(M_\lambda \otimes N_i)\!\downarrow \cong \bigoplus_{p\in\lambda+i+2\mathbb{Z}} F_p \otimes {}^{[i]}C^M_p, \tag{5.49}$$

which states that for M_λ a representation of $A_1(u,v)$, and N_i a representation of $\mathfrak{bc}$, their restriction decomposes as a tensor product of Fock space representations F_p of the Heisenberg algebra, and representations ${}^{[i]}C^M_p$ of the $N=2$ superconformal minimal model $M(u,v)$, where (u,v) are the same as those labelling the model $A_1(u,v)$.

There are a number of symmetries between ${}^{[i]}C^M_p$ modules which can be deduced from the branching rules. We begin by noting that all algebras involved in the coset have conjugation and spectral flow automorphisms. As such, there is a corresponding notion of conjugation and spectral flow of their modules, as well as for the branching rule.

It is an important result from [99] that the automorphisms act on the representations appearing in the branching rules as

$$\gamma_{\text{aff}} \otimes \gamma_{\mathfrak{bc}} = \gamma_{\widehat{H}} \otimes \gamma_{N=2}, \quad \sigma^\ell_{\text{aff}} \otimes \sigma^m_{\mathfrak{bc}} = \sigma^{\ell k+2m}_{\widehat{H}} \otimes \sigma^\ell_{N=2}, \quad \ell \in \mathbb{Z},\ m \in \frac{1}{2}\mathbb{Z}. \tag{5.50}$$

Using these relations, the authors of [100] derive the following useful identifications between modules appearing in the branching rules

$${}^{[i]}C^{\sigma^\ell(M)}_p \cong {}^{[i+2]}C^M_{p-\ell t}, \quad \sigma^\ell_{N=2}({}^{[i]}C^M_p) \cong {}^{[i-2]}C^M_{p-2\ell}, \quad {}^{[i]}C^{\gamma(M)}_p \cong \gamma_{N=2}({}^{[-i]}C^M_{-p}), \tag{5.51}$$

where we recall that $t = \frac{u}{v}$.

5.3.3 Module dictionary

In [100] the authors established a dictionary for translating highest-weight $M(u,v)$ modules appearing in the branching rules, namely ${}^{[i]}C_p^M$, into modules of the form $L^{\bullet;\pm}_{\Delta;j}$ introduced in Section 5.2. Here we will restate some of those results for the completeness of our discussion. We focus on the case when the parameters $u,\ v$ are such that $\gcd(u,v)=1$ and $u,v\geq 2$, that is, we focus only on admissible values of the central charge c leading to non-unitary minimal models. The branching rules for the L- and $D^{\pm}$-type $A_1(u,v)$ modules are as follows,

$$(L_{r,0}\otimes N_i)\downarrow \cong \bigoplus_{i+\lambda_{r,0}+2\mathbb{Z}} F_p\otimes {}^{[i]}C^L_{p;r,0}, \tag{5.52}$$

$$(D^+_{r,s}\otimes N_i)\downarrow \cong \bigoplus_{i+\lambda_{r,s}+2\mathbb{Z}} F_p\otimes {}^{[i]}C^D_{p;r,s}. \tag{5.53}$$

We reiterate that the D^--type $A_1(u,v)$ modules are conjugate to D^+-type, and the action of conjugation on the branching rules was given in (5.50).

With the branching rules made explicit, we can now state the dictionary. We remark that dictionaries are only presented for $i=0,1$, where i labels the $\mathfrak{bc}$-ghost system module, as the cases when $i=2,3$ are obtained from these respectively by applying the parity reversal operator. At the level of $N=2$ superconformal modules $L^{\bullet;\pm}_{j;h}$, this amounts to changing the superscript sign. We remark that the notation for L-type $A_1(u,v)$ modules is distinguished from $M(u,v)$ modules by the superscripts on $M(u,v)$ modules indicating their algebra and parity.

The dictionary for L-type $A_1(u,v)$ modules is

$${}^{[0]}C^L_{p;r,0}\cong L^{\mathrm{NS},\bullet}_{j,\Delta},\ \ p\in\lambda_{r,0}+2\mathbb{Z},\qquad \begin{cases} \bullet=-, j=\frac{p}{t}+1,\ \Delta=\Delta^{N=2}_{p;r,0}-\frac{p+r}{2}, & p\leq -r-1,\\ \bullet=+, j=\frac{p}{t},\ \Delta=\Delta^{N=2}_{p;r,0}, & 1-r\leq p\leq r-1,\\ \bullet=-, j=\frac{p}{t}-1,\ \Delta=\Delta^{N=2}_{p;r,0}+\frac{p-r}{2}, & p\geq r+1,\end{cases}$$

$${}^{[1]}C^L_{p;r,0}\cong L^{\mathrm{R},\bullet}_{j,\Delta},\ \ p\in 1+\lambda_{r,0}+2\mathbb{Z},\qquad \begin{cases} \bullet=+,\ j=\frac{p}{t}+\frac{3}{2},\ \Delta=\Delta^{N=2}_{p;r,0}-\frac{p+r}{2}+\frac{1}{8}, & p\leq -r-2,\\ \bullet=-,\ j=\frac{p}{t}+\frac{1}{2},\ \Delta=\Delta^{N=2}_{p;r,0}+\frac{1}{8}, & -r\leq p\leq r-2,\\ \bullet=+,\ j=\frac{p}{t}-\frac{1}{2},\ \Delta=\Delta^{N=2}_{p;r,0}+\frac{p-r}{2}+\frac{1}{8}, & p\geq r,\end{cases}$$

and for D-type modules

$${}^{[0]}C^D_{p;r,s}\cong L^{\mathrm{NS},\bullet}_{j,\Delta},\ \ p\in\lambda_{r,s}+2\mathbb{Z},\qquad \begin{cases} \bullet=+, j=\frac{p}{t},\ \Delta=\Delta^{N=2}_{p;r,0}, & p\leq\lambda_{r,s},\\ \bullet=-, j=\frac{p}{t}-1,\ \Delta=\Delta^{N=2}_{p;r,0}+-\frac{p-\lambda_{r,s}-1}{2}, & p\geq\lambda_{r,s}+2,\end{cases}$$

$${}^{[1]}C^D_{p;r,s}\cong L^{\mathrm{R},\bullet}_{j,\Delta},\ \ p\in 1+\lambda_{r,s}+2\mathbb{Z},\qquad \begin{cases} \bullet=-,\ j=\frac{p}{t}+\frac{1}{2},\ \Delta=\Delta^{N=2}_{p;r,0}+\frac{1}{8}, & -r\leq p\leq r-2,\\ \bullet=+,\ j=\frac{p}{t}-\frac{1}{2},\ \Delta=\Delta^{N=2}_{p;r,0}+\frac{p-r}{2}+\frac{1}{8}, & p\geq r,\end{cases}$$

where we have made use of the formula

$$\Delta^{\mathrm{N=2}}_{p;r,s} = \frac{(r-st)^2-1}{4t} - \frac{p^2}{4t}, \tag{5.54}$$

which gives the L_0-eigenvalue of the highest-weight vector for the $N=2$ module, coming from the branching rule.

This concludes our introduction to the coset construction. In the following Chapter, we introduce staggered modules over the $N=2$ superconformal algebras arising via the coset, and present the results of our investigation into these modules.

The work in this chapter is being prepared for print in the paper

C. Raymond, D. Ridout, J. Rasmussen, *Staggered modules of the* $N = 2$ *superconformal algebras*, in preparation.

Chapter 6

Staggered modules over the $N = 2$ superconformal algebras

In this chapter, we begin our discussion of the reducible yet indecomposable modules known as staggered modules, and present the results of our investigation into such modules for the $N = 2$ superconformal minimal models $M(u,v)$. We begin by introducing staggered modules, discussing their structure, we give their branching rules under the coset, and we describe the action of the algebra automorphisms on these modules using the results of [99]. Following this, we begin our presentation of the results.

First, we give formulas which describe the weight space action of the spectral flow on the irreducible component modules of the staggered modules. This gives insight into the action of the spectral flow, as well as the generic structure of the staggered modules for the $M(u,v)$ minimal models. We follow this with explicit examples, whereby we determine the structure for two staggered modules in the minimal model $M(2,3)$, one for the Neveu-Schwarz algebra, and one for the Ramond.

Using observations made from the action of the spectral flow on the weights of the component modules, as well as the explicit examples, we derive some general symmetries of the staggered module families appearing in the branching rule. We show that these families have related structures under the spectral flow, and as such, it is sufficient to determine the structure for a representative of each family. We describe how to calculate the action of spectral flow on the parameters which characterise the structure of a staggered module, and hence, determine the structure for the full branching rule family. Furthermore, we derive a Kac table symmetry in the module labels allowing for identifications between modules, and demonstrate explicitly that with these general symmetries, we have determined the full spectrum of staggered modules arising from $A_1(2,3)$ staggered modules via the coset.

Finally, we investigate the so-called vacuum staggered module, for all admissible values of the central charge leading to non-unitary models. We determine the values of the structure parameters for this module for all admissible c. Moreover, we investigate the modules that arise from spectral flow by $\frac{1}{2}$ in either direction, giving an explicit realisation of the spectral flow preserving the structure parameters.

6.1 Introduction to staggered modules

We are primarily interested in the staggered modules over the $N = 2$ superconformal algebras $M(u,v)$. We can use the coset construction to realise $M(u,v)$ staggered modules from $\widehat{\mathfrak{sl}}(2)$ staggered modules. These modules are reducible yet indecomposable, and are characterised by a non-diagonalisable action of L_0 (the operator contains a Jordan block). Moreover, these representations are non-unitary. Staggered modules for the corresponding $\widehat{\mathfrak{sl}}(2)$ minimal models $A_1(u,v)$ have been explicitly constructed for $(u,v) = (2,3)$ and $(3,2)$ [107]. Moreover, they are conjectured to exist generically for all admissible levels k through character fusion [109].

The $A_1(u,v)$ staggered modules relevant to the work in this thesis have a generic form given by the Loewy diagram given in Figure 6.1. The arrows on the diagram indicate the action of the universal enveloping algebra, and we will refer to the action of the algebra as "gluing" the component modules together. The modules in a Loewy diagram are irreducible modules. The component modules are naturally submodules of the staggered module.

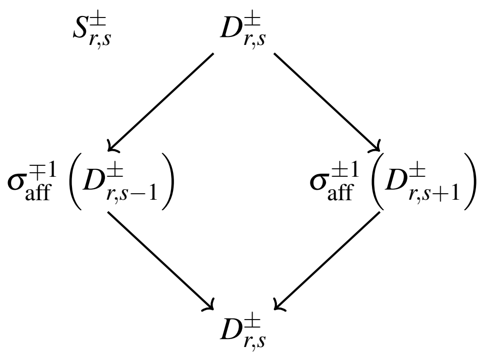


Figure 6.1: *The figure shows the Loewy diagram for the staggered modules $S^{\pm}_{r,s}$ of the $A_1(u,v)$ minimal models, where $1 \le r \le u-1$ and $0 \le s \le v-1$. The individual component modules are the $D^{\pm}$-type irreducible modules, and their spectral flows. For these modules we have $D^{\pm}_{r,0} \equiv L_{r,0}$. The label $s \pm 1$ may fall outside the allowed domain, in which case we make use of the identifications $D^{\pm}_{r,-1} = D^{\mp}_{r,1}$ and $D^{\pm}_{r,v} = \sigma^{\pm 1}\left(D^{\pm}_{u-r,1}\right)$.*

A staggered module can also be understood in terms of a socle series. The socle of a module M, is the maximal completely reducible submodule of M. The we use this notion to define an ascending series of submodules $\mathrm{Soc}^i(M)$ for $i > 0$, known as the socle series of M, where $\mathrm{Soc}^i(M)$ is the unique submodule of M such that $\mathrm{Soc}^i(M)/\mathrm{Soc}^{i-1}(M)$ is the socle of $M/\mathrm{Soc}^{i-1}(M)$. When $\mathrm{Soc}^i(M)$ is completely reducible, we refer to that module as the head of the series.

The component modules of a Loewy diagram are irreducible modules. For the diagram in Figure 6.1, we have $\mathrm{Soc}^0(S^{\pm}_{r,s}) = D^{\pm}_{r,s}$. With the socle series of a general staggered module in mind, we will use the terms “head” and “socle” to differentiate between the two equivalent component modules $D^{\pm}_{r,s}$, and the equivalent irreducible components in $N = 2$ staggered modules. Furthermore, the other modules will be referred to as the left- and right modules, depending on their weights. We take the convention that the conformal weight Δ increases down the page, and the charge, j ($N = 2$) or λ ($\widehat{\mathfrak{sl}}(2)$), increases left-to-right across the page.

We apply results from recent work in [99], which establishing a Schur-Weyl duality for cosets of Heisenberg algebras to determine the branching rules for the $A_1(u,v)$ staggered modules. One such result is that if the input $A_1(u,v)$ representation M_λ is an irreducible representation, then the resulting family of $M(u,v)$ modules ${}^{[i]}C^M_p$ appearing in the branching rule (5.53) will be irreducible. Moreover, this holds for indecomposable modules and their irreducible components, such as staggered modules.

This result implies that the four-component staggered modules $S^{\pm}_{r,s}$ in $A_1(u,v)$ given in the figure above, give rise to an infinite family of four-component staggered $M(u,v)$ modules. Moreover, the individual component modules of the $M(u,v)$ staggered modules will be irreducible highest-weight representations of the $N = 2$ superconformal algebras. We label the corresponding $M(u,v)$ staggered modules ${}^{[i]}P_{p;,r,s}$, and their branching rules are then given by

$$(S_{r,s} \otimes N_i)\downarrow \cong \bigoplus_{p \in \lambda_{r,s}+i+2\mathbb{Z}} F_p \otimes {}^{[i]}P_{p;,r,s} \tag{6.1}$$

for $1 \leq r \leq u-1$ and $0 \leq s \leq v-1$. The corresponding Loewy diagram of the module ${}^{[i]}P_{p;,r,s}$ is given in general in the Figure 6.2.

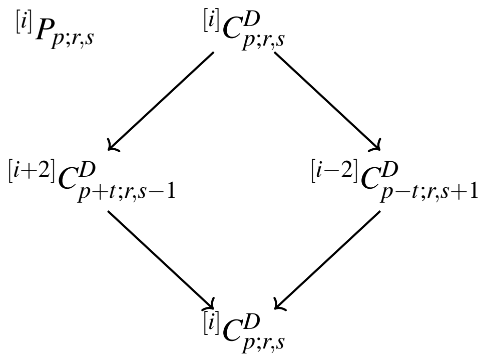


Figure 6.2: *The Loewy diagram for a general staggered module ${}^{[i]}P_{p;r,s}$, displaying the component modules in branching rule notation. These are exactly the modules which arise by applying the branching rules component-wise to $S^{\pm}_{r,s}$.*

We require the following identifications, calculated in [100], to ensure the branching rules are compatible with the module dictionary,

$${}^{[i]}C^D_{p;r,-1} = {}^{[i+2]}C^D_{p+t;u-r,v-2}, \quad {}^{[i]}C^D_{p;r,0} = {}^{[i]}C^L_{p;r,0}, \quad {}^{[i]}C^D_{p;r,v} = {}^{[i]}C^D_{p-1;u-r,1}. \tag{6.2}$$

Following insight from the study of staggered modules over the Virasoro algebra [104] and affine Lie algebras [107], we introduce the notion of β-parameters. The Loewy diagram of a staggered module is insufficient to completely describe the structure of the module. Staggered modules of the previously mentioned algebras have associated invariants, known (particularly for Virasoro) as β-parameters, which, in addition to the Loewy diagram, characterise the module structure. The β-parameters are defined by gluing action of the algebra on the component modules. We will use notation $|H\rangle$, $|L\rangle$, $|R\rangle$, $|S\rangle$ for the highest-weight vectors of the head, left, right, and socle component modules respectively, when considered part of the staggered module. We remark that these vectors are not all highest-weight vectors in the staggered module. We will use notation $|j;\Delta\rangle$ when referring to the true highest-weight vector of the corresponding irreducible component modules, considered separately from the staggered module.

The non-diagonal action of L_0 is given by

$$L_0\,|H\rangle = \Delta_H\,|H\rangle + |S\rangle\,, \tag{6.3}$$

where Δ_H is the conformal weight of the head module. However, this action does not define $|H\rangle$ uniquely in the staggered module. We are free to add some multiple of $|S\rangle$ to $|H\rangle$ without changing the above property. One can think of this as a gauge freedom. This gauge freedom implies that we cannot normalise the vector $|H\rangle$ with respect to a form on the staggered module. To determine the structure on the module, we need to characterise the action of the algebra which glues the components together. The work of [104, 107, 138] motivates the following definitions to describe the structure.

We begin with the left and right component highest-weight vectors $|L\rangle$ and $|R\rangle$. We are able to normalise the Shapovalov form on the left and right states, such that $\langle R|R\rangle = 1 = \langle L|L\rangle$. This choice fixes the form on the indecomposable submodules generated by $|L\rangle$ and $|R\rangle$. We remark that the intersection of these submodules is the socle, and the form agrees on the socle submodule.

We define the socle vector $|S\rangle$ relative to the left and right vectors by setting

$$U_R\,|R\rangle = |S\rangle\,, \quad U_L\,|L\rangle = |S\rangle\,, \tag{6.4}$$

where U_L and U_R are elements of the universal enveloping algebra of the $N=2$ superconformal algebra. We choose U_L and U_R such that $U_R\,|j_R;\Delta_R\rangle = 0$ and $U_L\,|j_L;\Delta_L\rangle = 0$ in the corresponding irreducible component modules, that is, such that they are quotiented singular vectors. There is a freedom of choice in normalisation of these vectors.

The β-parameters are then defined by the equations

$$U_R^\dagger\,|H\rangle = \beta_R\,|R\rangle\,, \quad U_L^\dagger\,|H\rangle = \beta_L\,|L\rangle\,, \tag{6.5}$$

where $U_R^\dagger$ and $U_L^\dagger$ are elements of the enveloping algebra adjoint to U_R and U_L. Similarly to the left- and right-module cases, the action of these elements in the corresponding irreducible highest-weight module of the head is $U_R^\dagger\,|j_H;\Delta_H\rangle = 0$ and $U_L^\dagger\,|j_H;\Delta_H\rangle = 0$.

Equivalently, in terms of the introduced form, we have

$$\langle R| U_R^\dagger |H\rangle = \beta_R, \quad \langle L| U_L^\dagger |H\rangle = \beta_L. \tag{6.6}$$

It is straightforward to check that the β-parameters are independent of a choice of gauge. This arises from the socle module being null with respect to the form on the submodules generated by $|R\rangle$ and $|L\rangle$, i.e. since U_R and U_L must be singular. However, the explicit value of β will depend on the chosen normalisations of the logarithmic partner vectors, that is, a choice of U_R, U_L and the action of L_0 on $|H\rangle$.

One has not determined the structure of the staggered module until one has also determined its β-parameters. The β-parameters are fixed by relations coming from quotiented singular vectors of the head module, which lift to relations on the staggered module, and the action of the algebra on the module. From a physical perspective, the β's are coupling parameters between modules. As an example, for staggered modules over the Virasoro algebra (and in several other examples), they arise in the correlation functions of corresponding physical models [115, 117, 119, 120, 138], however, no such physical theories are currently known for the $N = 2$ algebras.

We remark that at this stage we have followed the prescription set out by others for determining the structure of staggered modules, and applied it to the $N = 2$ superconformal algebras. We have not proven that the isomorphism classes of $N = 2$ staggered modules are completely determined by the values of their β-invariants. An equivalent result exists for Virasoro staggered modules in [104]. We will return to this point in the Conclusion Chapter 7.

6.1.1 The action of automorphisms on staggered modules

Using the results of [99], the action of automorphisms on the staggered module structure maintains the Loewy diagrams, and simply affects each of the component modules. It will of course affect the elements of the universal enveloping algebra (previously introduced as U_L, U_R) which glue the module together, however, the component submodules must be preserved. Here we consider the action of the spectral flow on the staggered module ${}^{[i]}P_{p;r,s}$, that is, we wish to determine $\sigma^\ell \left({}^{[i]}P_{p;r,s}\right)$, for $\ell \in \frac{1}{2}\mathbb{Z}$. The result is given in Figure 6.3.

We can deduce several things from the diagram above. We know that the weight support of the family of modules ${}^{[i]}P_{p;r,s}$ is $p \in \lambda_{r,s} + i + 2\mathbb{Z}$. Recalling that the value of i is taken modulo 4, we can conclude that the family of staggered modules produced in the branching rule is preserved by the spectral flow, up to parity. Furthermore, if we take ℓ even then parity is also preserved. We note however, that the parity of the modules does not affect the staggered module structure. As such, we have that the spectral flow does indeed preserve the family of produced staggered modules. Moreover, if we allow for $\ell \in \frac{1}{2}\mathbb{Z}$ we map between Ramond- and Neveu-Schwarz sector modules.

We can apply the module dictionary to the modules appearing in the branching rules to understand the

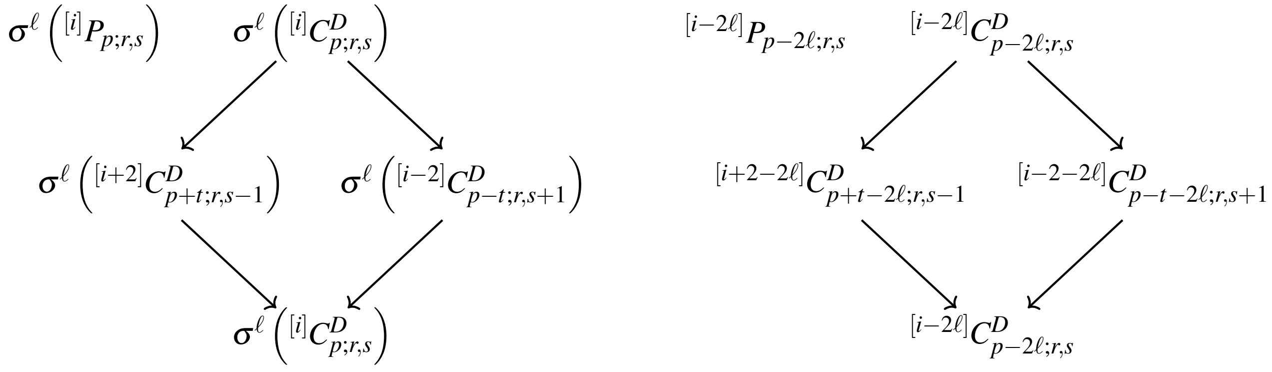


Figure 6.3: *The left Loewy diagram gives the explicit action of the spectral flow on the staggered module* $\sigma^\ell\left({}^{[i]}P_{p;r,s}\right)$. *The result is the Loewy diagram where each component module has been spectrally flowed. The right figure is the result of applying the equation* (5.51) *and determining the related modules in a form compatible with the module dictionary.*

action of the spectral flow in weight space. The action is visualised in the following diagram.

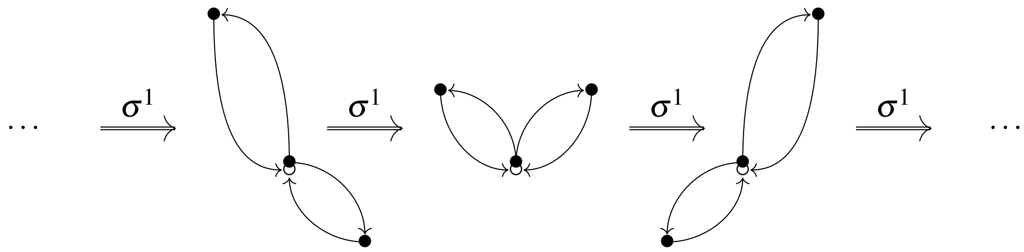


Figure 6.4: *The figure shows the action of spectral flow on the relative weights of the component modules. The highest-weight vectors are represented by $\bullet$, with the exception of the socle, which is represented by $\circ$. The arrows demonstrate the action of the algebra gluing the component modules.*

If we ignore any overall change in weights (j,h), we can see that the net effect of the spectral flow is to shift the left and right modules either up or down in opposite directions to each other.

We can deduce formulas for the action of the spectral flow on the weights of the component modules, and in particular, relations describing the distances between their weights under flow. We introduce the notation $j_{HL}, j_{HR}, \Delta_{HL}, \Delta_{HR}$ to denote the difference in charge and conformal weight between the head and left/right component modules, i.e. $j_{HL} = j_H - j_L$. As an example, the module ${}^{[0]}P_{\lambda_{1,0};1,0}$ analysed in Section 6.2 has distances

$$j_{HL} = -1, \quad j_{HR} = 1, \quad \Delta_{HL} = -\frac{1}{2}, \quad \Delta_{HR} = \frac{1}{2}. \tag{6.7}$$

Given that this is a concrete example, the distances are fixed values. However, we expect the resulting distances to be functions of the flow parameter ℓ when we consider spectral flows of modules.

Recalling the general structure of the module ${}^{[i]}P_{p;r,s}$, and the action of the spectral flow on the component modules, we can apply the module dictionary to the resulting modules for which p depends on ℓ. The relations (6.2) imply that we need to consider three separate cases. If $s \neq 0, v-1$ then no simplifying relations are required. However in the other cases, either the left or the right module is adjusted by (6.2) to apply the module dictionary. Finally, if $s = 1$, then three of the $N = 2$ component modules come from D-type $A_1(u,v)$ modules, and one comes from an L-type $A_1(u,v)$ module, so we distinguish this case also.

In the general case $s \neq 0, 1, v-1$, the distance functions are

$$j_{HL} = -1, \quad j_{HR} = 1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, \quad \Delta_{HR} = \frac{1}{2} + \ell, \tag{6.8}$$

which demonstrates exactly the behaviour in Figure 6.4.

In the cases $s \in \{0, 1, v-1\}$ the analysis proceeds similarly, however one needs to take care to identify the left/right modules appearing according to the relations (6.2), and to take care with the bounds on the formulas for the weights in the module dictionary, when applying spectral flow. A full table of the possible outcomes is as follows.

- For $s = 0$,

$$\begin{aligned}
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell + r, && j_{HR} = 2, \quad \Delta_{HR} = 1 + 2\ell - r, && r \leq \ell, \\
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell + r, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell, && 0 \leq \ell \leq r, \\
&j_{HL} = -2, \quad \Delta_{HL} = -1 - 2\ell + r, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell, && \ell < 0.
\end{aligned} \tag{6.9}$$

- For $s = 1$,

$$\begin{aligned}
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell, && \ell < 0, \\
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell, && 0 \leq \ell \leq r-1, \\
&j_{HL} = -2, \quad \Delta_{HL} = -1 - 2\ell + r, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell, && r-1 < \ell.
\end{aligned} \tag{6.10}$$

- For $s = v-1$,

$$\begin{aligned}
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, && j_{HR} = 2, \quad \Delta_{HR} = 1 - 2\ell + 2(u-r), && \ell < r-u, \\
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, && j_{HR} = 1, \quad \Delta_{HR} = \frac{1}{2} + \ell + u - r, && r-u \leq \ell \leq 0, \\
&j_{HL} = -1, \quad \Delta_{HL} = -\frac{1}{2} - \ell, && j_{HR} = 2, \quad \Delta_{HR} = 1 + 2\ell + u - r, && \ell > 0.
\end{aligned} \tag{6.11}$$

In the special case when $s = 1$ and $v = 2$, we have that distances j_{HL}, Δ_{HL} are given by the formulae for $s = 1$, and the distances j_{HR}, Δ_{HR} are given by the formulae for $s = v-1$.

We see that the charge distances j_{HL} or j_{HR} are fixed to either ± 1 or ± 2, and that the conformal weight distances Δ_{HL} and Δ_{HL} become functionf of ℓ. What is particularly interesting is that the relative charge becomes ± 2 for sufficiently large flow parameter ℓ, when $s = 0,1,\ v-1$. The determinant formula (5.21) states clearly that in Neveu-Schwarz modules, charged singular vectors only occur for relative charge ± 1. As such, we should not expect values of ± 2 for Neveu-Schwarz modules.

This peculiarity suggests that rather than the highest-weight vectors of the component modules being identified with singular vectors, they are instead identified with subsingular vectors. Since the component modules are irreducible, the subsingular vectors have also been quotiented, implying that this subsingular embedding does not pose any problems. However it is remarkable that this behaviour is so common in the $N = 2$ staggered modules.

6.2 Staggered modules of the minimal model $M(2,3)$

We begin our presentation of explicit examples of staggered modules for the $N = 2$ superconformal algebras with those in the minimal model $M(2,3)$. For these values of the parameters, the superconformal algebras have $c = -6$. The corresponding minimal model $A_1(2,3)$ has representations labelled by $r = 1$, and $s = 0,1,2$. It has two distinct staggered module families, represented by $S^{\pm}_{1,0}$ and $S^{\pm}_{1,1}$. The corresponding $A_1(u,v)$ staggered module family is the spectral flow orbit of the representative. There is a third staggered module family represented by $S^{\pm}_{1,2}$, however, this family is equivalent to $S^{\pm}_{1,0}$ under spectral flow. The resulting $M(2,3)$ staggered module families are ${}^{[i]}P_{p;1,0}$, ${}^{[i]}P_{p;1,1}$, and ${}^{[i]}P_{p;1,2}$, where $p \in \lambda_{r,s} + i + 2\mathbb{Z}$ and $\lambda_{r,s} = r - 1 - st$.

6.2.1 The module ${}^{[0]}P_{\lambda_{1,0};1,0}$ in $M(2,3)$

Thus we are left to describe the module families ${}^{[i]}P_{p;1,0}$ and ${}^{[i]}P_{p;1,1}$. We will begin by taking particular examples, the first of which will be the module ${}^{[0]}P_{0;1,0}$, which has its Loewy diagram presented in Figure 6.2.1, along with a visualisation of the highest-weight vectors in weight space and the gluing action of the algebra. The weights of the vectors are listed in parentheses, and we use the notation $|H\rangle$ for the highest-weight vector of the head module, and so on, for all of left, right, and socle.

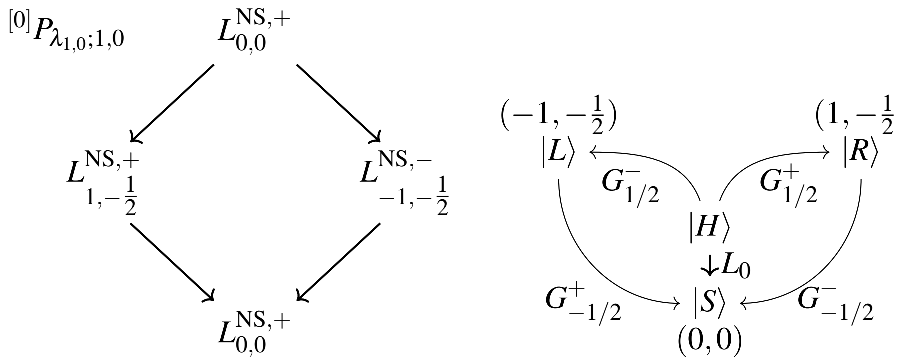


Figure 6.5: *The Loewy diagram and weight space embedding structure for the module ${}^{[0]}P_{\lambda_{1,0};1,0}$. It will be our convention when displaying Loewy diagrams that the component modules align with Figure 6.2. The weight space diagrams display the correct relative positions of vectors.*

Applying the determinant formula for the Verma modules of weights $(j,\Delta)=(0,0),(-1,-\frac{1}{2}),(1,\frac{1}{2})$, we have the following expressions for the charged singular vectors,

$$G^{\pm}_{-1/2}\left|0;0\right\rangle, \quad G^{+}_{-1/2}\left|-1,-\tfrac{1}{2}\right\rangle, \quad G^{-}_{-1/2}\left|1,-\tfrac{1}{2}\right\rangle. \tag{6.12}$$

We then make the following choices for the algebra action gluing the modules

$$L_0\left|H\right\rangle=\left|S\right\rangle, \quad G^{+}_{1/2}\left|H\right\rangle=\beta_R\left|R\right\rangle, \quad G^{-}_{1/2}\left|H\right\rangle=\beta_L\left|L\right\rangle, \quad G^{+}_{-1/2}\left|L\right\rangle=\left|S\right\rangle, \quad G^{-}_{-1/2}\left|R\right\rangle=\left|S\right\rangle, \tag{6.13}$$

where $\beta_\bullet$ are the constants which define the structure of the module.

The values of the β-parameters are then determined by using constraints on the module, arising from additional quotiented singular vectors and other symmetries. These relations must be compatible with the staggered structure, in the sense that a quotiented singular vector relation in the head component module must lift to be a quotiented relation in the full staggered module.

For the module ${}^{[0]}P_{\lambda_{1,0};1,0}$, the head component module is the irreducible module $L^{\text{NS},+}_{0,0}$. The quotiented singular vectors of the irreducible module imply $G^{\pm}_{-1/2}\left|0;0\right\rangle=0$. Thus, we must have relations of the form

$$G^{+}_{-1/2}\left|H\right\rangle=\left(\alpha_R L_{-1}+\gamma_R J_{-1}\right)\left|R\right\rangle, \quad G^{-}_{-1/2}\left|H\right\rangle=\left(\alpha_L L_{-1}+\gamma_L J_{-1}\right)\left|L\right\rangle, \tag{6.14}$$

in the staggered module, where $\alpha_\bullet,\ \gamma_\bullet\in\mathbb{C}$ are additional unknowns which are determined by the action of the raising generators. The expressions on the right-hand side are general linear combinations of vectors in ${}^{[0]}P_{\lambda_{1,0};1,0}$ with appropriate weight.

We proceed by determining the action of the generators of the raising subalgebra on these relations. A detailed example of one of the calculations follows. Applying L_1 to $G^{+}_{-1/2}\left|H\right\rangle$, we have

$$\begin{aligned}
L_1 G^{+}_{-1/2}\left|H\right\rangle &= L_1\left(\alpha_R L_{-1}+\gamma_R J_{-1}\right)\left|R\right\rangle, \\
\left[L_1,G^{+}_{-1/2}\right]\left|H\right\rangle &= \left(\alpha_R\left[L_1,L_{-1}\right]+\gamma_R\left[L_1,J_{-1}\right]\right)\left|R\right\rangle, \\
G^{+}_{1/2}\left|H\right\rangle &= \left(2\alpha_R L_0+\gamma_R J_0\right)\left|R\right\rangle, \\
\beta_R\left|R\right\rangle &= \left(-\alpha_R+\gamma_R\right)\left|R\right\rangle.
\end{aligned} \tag{6.15}$$

Similarly, the vectors $J_1G^{+}_{-1/2}\left|H\right\rangle$, $J_1G^{-}_{-1/2}\left|H\right\rangle$, and $L_1G^{-}_{-1/2}\left|H\right\rangle$, lead to the relations

$$\beta_R=\alpha_R-2\gamma_R, \quad \beta_L=\alpha_L+2\gamma_L, \quad \beta_L=-\alpha_L-\gamma_L, \tag{6.16}$$

respectively. Our final set of relations comes from considering the action of $\{G^{+}_{-1/2},G^{-}_{1/2}\}=2L_0-J_0$ and $\{G^{-}_{-1/2},G^{+}_{1/2}\}=2L_0+J_0$ on $\left|H\right\rangle$. As an example, we consider

$$\begin{aligned}
\{G^{+}_{-1/2},G^{-}_{1/2}\}\left|H\right\rangle=\left(2L_0-J_0\right)\left|H\right\rangle &= \left(G^{+}_{-1/2}G^{-}_{1/2}+G^{-}_{1/2}G^{+}_{-1/2}\right)\left|H\right\rangle, \\
2\left|S\right\rangle &= G^{+}_{-1/2}\beta_L\left|L\right\rangle+G^{-}_{1/2}\left(\alpha_R L_{-1}+\gamma_R J_{-1}\right)\left|R\right\rangle, \\
2\left|S\right\rangle &= \beta_L\left|S\right\rangle+\alpha_R\left[G^{-}_{1/2},L_{-1}\right]\left|R\right\rangle+\gamma_R\left[G^{-}_{-1/2},J_{-1}\right]\left|R\right\rangle, \\
\Longrightarrow\ 2 &= \beta_L+\alpha_R+\gamma_R.
\end{aligned} \tag{6.17}$$

Similarly, for the remaining relation, we have

$$2 = \beta_R + \alpha_L - \gamma_L. \tag{6.18}$$

Combining these results, we arrive at a determined set of linear equations with solution

$$\alpha_R = \alpha_L = \frac{3}{2}, \quad \beta_R = \beta_L = -\frac{1}{2}, \quad \gamma_R = 1, \quad \gamma_L = -1. \tag{6.19}$$

The values of $\beta_{\mathrm{L/R}}$, along with the Loewy diagram, then determine the structure of the module.

6.2.2 The module ${}^{[1]}P_{\lambda_{1,1}+1;1,1}$ in $M(2,3)$

For the family ${}^{[i]}P_{p;1,1}$, we choose to explore the Ramond sector module ${}^{[1]}P_{\lambda_{1,1}+1;1,1}$, so as to give a concrete example of a Ramond sector staggered module. The Loewy diagram for this module is presented in Figure 6.2.2, alongside the weight space diagram. This module is particularly curious, as all the component modules have $\Delta = c/24$, implying that they are all single ground state Ramond sector modules.

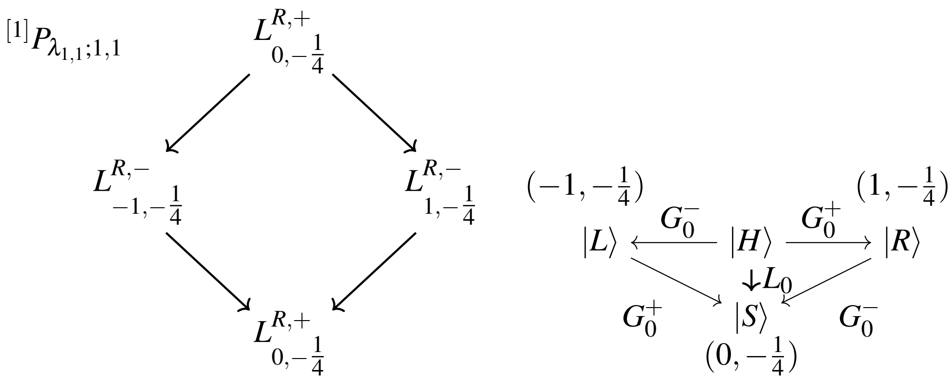


Figure 6.6: *The Loewy diagram and weight space embedding structure for the module* ${}^{[1]}P_{\lambda_{1,1};1,1}$.

In this module, the we choose the defining relations to be

$$L_0 |H\rangle = -\frac{1}{4}|H\rangle + |S\rangle, \quad G_0^+ |H\rangle = \beta_R |R\rangle, \quad G_0^- |H\rangle = \beta_L |L\rangle, \quad G_0^+ |L\rangle = |S\rangle, \quad G_0^- |R\rangle = |S\rangle. \tag{6.20}$$

As $\Delta = \frac{c}{24}$ for all component modules, the relevant singular vectors for this staggered module are the $G_0^- |j,\Delta\rangle$. The uncharged singular vectors appear deep in the module (beginning at level 6), and as such, simply determining these vectors is computationally challenging.

We are able to determine one relation between the β-parameters using the charged singular vector relations at top level. We consider the following relation

$$\left(G_0^+ G_0^- + G_0^- G_0^+\right)|H\rangle = \left(2L_0 + \frac{1}{2}\right)|H\rangle, \tag{6.21}$$

where we continue

$$\begin{aligned}\left(G_0^+G_0^- + G_0^-G_0^+\right)|H\rangle &= G_0^+G_0^-|H\rangle + G_0^-G_0^+|H\rangle\,,\\ &= G_0^+\beta_L|L\rangle + G_0^-\beta_R|R\rangle\,,\\ &= (\beta_L+\beta_R)|S\rangle\,,\end{aligned} \tag{6.22}$$

and

$$\left(2L_0+\frac{1}{2}\right)|H\rangle = 2\left(-\frac{1}{4}|H\rangle + |S\rangle\right) + \frac{1}{2}|H\rangle = 2|S\rangle\,, \tag{6.23}$$

arriving at

$$\beta_L+\beta_R = 2. \tag{6.24}$$

The remaining constraint on the β-parameters comes from the observation that $\gamma_{N=2}\left({}^{[1]}P_{\lambda_{1,1};1,1}\right) = {}^{[1]}P_{\lambda_{1,1};1,1}$, that is, this staggered module is conjugation invariant. This implies that $\beta_L = \beta_R$, since under the action of conjugation, the definitions of these two parameters are simply exchanged. Solving this then gives

$$\beta_R = \beta_L = 1. \tag{6.25}$$

We remark that the staggered modules considered above have the property that the socle submodule is a null submodule. The norm of $|S\rangle = G_0^-|R\rangle$ is given by

$$\langle S|S\rangle = \langle R|G_0^+G_0^-|R\rangle = \langle R|\,2L_0 - \frac{c}{12}|R\rangle = 0. \tag{6.26}$$

In the previous example of a Neveu-Schwarz module the socle was also a null submodule. It is straight forward to show that for $G^-_{-1/2}|R\rangle = |S\rangle$, then

$$\langle S|S\rangle = \langle R|G^+_{1/2}G^-_{-1/2}|R\rangle = -\langle R|\{G^+_{1/2},G^-_{-1/2}\}|R\rangle = 0, \tag{6.27}$$

since $\{G^+_{1/2}G^-_{-1/2}\} = 2L_0+J_0$, and we have that $L_0|R\rangle = -1/2|R\rangle$ and $J_0|R\rangle = |R\rangle$ in the previous example. This feature occurs generically in staggered modules, not just for staggered modules over the $N=2$ algebras. It is expected, as the socle is defined in relation to the right module as $U_R|R\rangle = |S\rangle$, where the vector $U_R|R\rangle$ is singular in the right module. As such, applying any raising generator gives 0. We include this remark to explicitly elucidate staggered module structure.

With this result, we have determined the structure of a Ramond and a Neveu-Schwarz staggered module in the minimal model $M(2,3)$. Moreover, these modules are not related by spectral flow. However, there remains an infinite number of staggered modules in each of the spectral flow orbits of these examples, for which we require a similar analysis in order to fully understand the staggered module content of the minimal model $M(2,3)$. This motivates us to look for general symmetries of the staggered module families that allow us to simplify the analysis.

6.3 General symmetries of $M(u,v)$ staggered modules

Motivated by the previous examples, in this section we derive some general symmetries of the staggered modules. In particular, we focus on understanding the action of spectral flow on the module structure, and we attempt to identify Kac table symmetries which reduces the number of independent branching rule families one needs to consider.

6.3.1 Spectral flow symmetries of the $M(u,v)$ staggered modules

In Section 6.1, we introduced the action of the spectral flow on staggered modules, and showed that it reduces to understanding the spectral flow of the component modules, and then modifying the gluing relations accordingly. Moreover, we showed that the families of staggered modules produced in the branching rule were all related by spectral flow.

In order to fully understand the complete family of staggered modules, we need to understand the action of spectral flow on the β-parameters. We know the corresponding minimal model $A_1(2,3)$ has two independent (with respect to the action of spectral flow) staggered modules, as such, it seems natural to expect a relation between staggered module families appearing in the branching rules of these two $A_1(2,3)$ modules.

As introduced earlier, the β-parameters are defined by the relations

$$U_R^{\dagger}\left|H\right\rangle = \beta_R\left|R\right\rangle, \quad U_L^{\dagger}\left|H\right\rangle = \beta_L\left|L\right\rangle, \tag{6.28}$$

where U_R, U_L are expressions in the universal enveloping algebra $U(\mathfrak{g})$ of either the Neveu-Schwarz or Ramond algebra, and $\beta_L, \beta_R \in \mathbb{C}$.

If we consider these relations under the action of the spectral flow (without loss of generality, we consider the right module), we see that

$$\sigma^{\ell}\left(U_R\left|H\right\rangle\right) = \sigma^{\ell}\left(\beta_R\left|R\right\rangle\right) = \beta_R\sigma^{\ell}\left(\left|R\right\rangle\right). \tag{6.29}$$

We see that the value of the β-parameter is preserved, however, it is no longer guaranteed (in fact it can only occur incredibly rarely) that $\sigma^{\ell}\left(\left|H\right\rangle\right)$ is the highest-weight vector of the corresponding flowed head module. Nor that the expression $\sigma^{\ell}\left(U_R\left|H\right\rangle\right)$ describes the gluing of highest-weight vectors of the head and right modules. What is true however is that the vectors in a given component module must be preserved under the spectral flow, implying that the same parameter β_R describes gluing between non-highest-weight vectors in the flowed module.

Making the comparison

$$U_R\sigma^{\ell}(\left|H\right\rangle) = \sigma^{\ell}(\sigma^{-\ell}(U_R)\left|H\right\rangle), \tag{6.30}$$

we can determine concrete expressions for the vectors which are glued with parameter β_R in the flowed module. We can then apply the appropriate raising generators to these vectors in order to raise the relation to a relation between highest-weight vectors. This results in a new parameter $\beta'_R \propto \beta_r$, where the constant of proportionality is determined by the action of the algebra on the irreducible component module.

One property of the spectral flow is that it will preserve the extremal vectors of a given module, that is, the vectors that form the boundary of the module in weight space. Precisely, the extremal states of an $N=2$ superconformal Verma module are the highest-weight vector, and the states

$$|x^-(n)\rangle = \prod_{k=0}^{n} G^-_{-k-1/2}|v\rangle, \quad |x^+_n\rangle = \prod_{k=0}^{n} G^+_{-k-1/2}|v\rangle, \quad n \geq 0, \tag{6.31}$$

in the Neveu-Schwarz sector, and

$$|x^-(n)\rangle = \prod_{k=0}^{n} G^-_{-k}|v\rangle, \quad |x^+_n\rangle = \prod_{k=0}^{n} G^+_{-k-1}|v\rangle, \quad n \geq 0, \tag{6.32}$$

in the Ramond sector. The product produces strings of generators with increasing mode index to the right. It is clear from the equations above that the weight spaces of the extremal vectors are one-dimensional. As the highest-weight vector is an extremal vector, under the action of the spectral flow a highest-weight vector is always mapped to an extremal vector under flow [137]. It may occur that an extremal vector is (sub)singular in the Verma module, in which case, there is no corresponding vector in the irreducible module. In that case, we simply adjust the bounds of the products, as all singular vectors occur with relative charge $\{(-2),-1,0,1\}$, as discussed in Section 5.2.

Consider the spectral flow of an irreducible module highest-weight vector $\sigma^\ell(|v\rangle)$ under the action of a set of raising generators. Given the action of spectral flow on extremal vectors, we can consider the following without loss of generality:

$$\prod_{k=1}^{n} G^+_{k-1/2}\sigma^\ell(|v\rangle) = \sigma^\ell\left(\prod_{k=1}^{n} G^+_{k-\ell-1/2}|v\rangle\right). \tag{6.33}$$

If $\ell \geq n$, the action of the raising operators is non-zero, and is simply the image of $\prod_{k=0}^{n} G^+_{k-\ell+1/2}|v\rangle$. If $n > \ell$, then the resulting vector must be zero. When $\ell = n$, we act on an extremal vector with the largest possible raising operator, such that the action is non-zero and the resulting vector is extremal. The resulting vector must be the highest weight vector of the flowed module. Moreover, this is determines exactly the combination one needs to act by to calculate the constant of proportionality for the β-parameters. When one flows with $\ell < 0$, we have the same argument involving G^-_r modes, and this carries to the Ramond sector in the obvious way. Moreover, the adjustment for the case involving a quotiented extremal vector, whereby one removes a generator from the product, carries over as well.

The implication of these results is that it is sufficient to determine the β-parameters for a single module in a branching rule family.

6.3.2 Kac table symmetries of the $M(u,v)$ models

We also want to understand the symmetries of the r,s labels for the ${}^{[i]}P_{p;r,s}$ modules. These are referred to as Kac table symmetries, as they can be used to label axes on a grid called a Kac table which encodes the highest weights of the modules for those values of r, s. In the minimal model $M(2,3)$ we already see some indicative behaviour, whereby ${}^{[i]}P_{p;1,0}$ and ${}^{[i]}P_{p;1,2}$ are equivalent up to spectral flow. That is, for a specific value of p, the corresponding staggered module ${}^{[i]}P_{p;1,0}$ is equal (up to parity) to the staggered module ${}^{[i]}P_{p';1,2}$, for an appropriate p'. We will use notation that ${}^{[i_1]}P_{p_1;r_1,s_1} \equiv {}^{[i_2]}P_{p_2;r_2,s_2}$ to denote this equivalence.

In the minimal model $M(3,2)$, there are 4 possible staggered modules, namely ${}^{[i]}P_{p;r,s}$ for $r = 1,2$ and $s = 0,1$. We see further evidence of general Kac table symmetries in $M(3,2)$, since ${}^{[i]}P_{p;1,0} \equiv {}^{[i]}P_{p;2,1}$ and ${}^{[i]}P_{p;2,0} \equiv {}^{[i]}P_{p;1,1}$. All of these taken together is suggestive of the general result

$$ {}^{[i]}P_{p;r,0} \equiv {}^{[i]}P_{p;u-r,v-1}. \tag{6.34} $$

What is clear is that two different modules ${}^{[i_1]}C_{p_1;r_1,s_1}$ and ${}^{[i_2]}C_{p_2;r_2,s_2}$ can label the same $L_{j,\Delta}$ module. We also know that because of the action of the spectral flow, we may take one module in the family produced by ${}^{[i]}P_{p;r,0}$, that is, choose a particular value for p, i, and then see if it is possible for that module to arise for certain values of parameters in ${}^{[i]}P_{p;u-r,v-1}$.

We choose the case ${}^{[0]}P_{\lambda_{r,0};r,0}$, noting that $\lambda_{r,0} = r-1$, for which the Loewy diagram is given in Figure 6.7.

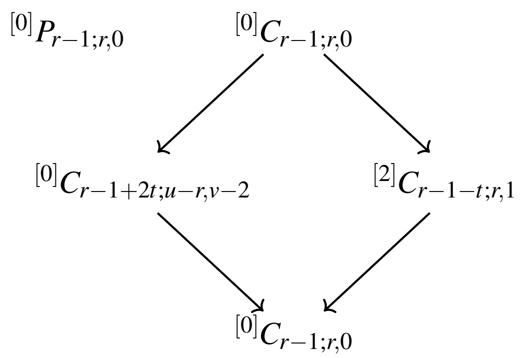


Figure 6.7: *We have used the symmetry* ${}^{[2]}C_{r-1+t;r,-1} = {}^{[0]}C_{r-1+2t;u-r,v-2}$ *to simplify the component modules.*

Then we search for values of p', i such that ${}^{[i]}P_{p';u-r,v-1}$ produces the same staggered module, up to parity, in the sense that the weights of the component modules are the same. We begin with the head module ${}^{[0]}C_{r-1;r,0}$. This module has $p \le r-1$, applying the module dictionary gives $L_{j;\Delta}$ with $j = \frac{r-1}{t}$ and $\Delta = \frac{r-1}{2t}$.

Correspondingly, the Loewy diagram for ${}^{[i]}P_{p';u-r,v-1}$ is given in Figure 6.8.

We require that for suitable choice of p', i the module ${}^{[i]}C_{p';u-r,v-1}$ is the same as ${}^{[0]}C_{r-1;r,0}$. We know $p' \in \lambda_{u-r,v-1} + i + 2\mathbb{Z}$, where $\lambda_{u-r,v-1} = u - r - 1 - (v-1)t = -r - 1 + t$. We begin by checking the parameter space for $p' \ge \lambda_{u-r,v-1} + 2$. Applying the module dictionary, this implies that the

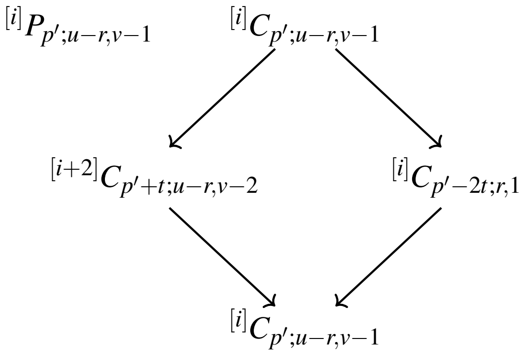


Figure 6.8: *We have used the symmetry* ${}^{[i+2]}C_{p'+t;u-r,v} = {}^{[i]}C_{p'-2t;r,1}$ *to simplify the component modules.*

corresponding module has charge $j = \frac{p'}{t} - 1$, and conformal dimension

$$\Delta = \frac{((u-r)-(v-1)t)^2 - 1}{4t} + \frac{p'^2}{4t} + \frac{p' - \lambda_{u-r,v-1} - 1}{2}. \tag{6.35}$$

Setting the charges equal, we find that $\frac{p'}{t} - 1 = \frac{r-1}{t} \implies p' = r - 1 + t$, which satisfies the inequality for p' stated in the module dictionary. Substituting this value of p' into the equation for Δ we find

$$\Delta = \frac{r-1}{2t}. \tag{6.36}$$

Hence, if we take $p' = r - 1 + t$, then the module $[i]C_{r-1+t;u-r,v-1} = {}^{[0]}C_{r-1;r,0}$, up to parity. Checking the module dictionary, we see that taking $i = 2$ sets the parities equal, and does not affect the weight support of p'. It remains to show that fixing $p' = r - 1 + t$ is in the weight support for p'. The weight support is given by

$$p' \in \lambda_{u-r,v-1} + 2\mathbb{Z} = -r - 1 + t + 2\mathbb{Z}. \tag{6.37}$$

We check that

$$r - 1 + t \in -r - 1 + t + 2\mathbb{Z}, \quad \text{or} \quad 2r - 1 + t \in -1 + t + 2\mathbb{Z}, \tag{6.38}$$

and since $r \in \mathbb{Z}$, this is satisfied. Hence, it is possible to find the head module of ${}^{[0]}P_{r-1;r,0}$ in the family ${}^{[i]}P_{r-1+t;u-r,v-1}$.

We are left to check that the left and right modules also match for these values of the parameters. To demonstrate, we give the Loewy diagram for the module ${}^{[0]}P_{r-1;r,0}$ alongside the Loewy diagram for ${}^{[2]}P_{r-1+t;u-r,v-1}$ in Figure 6.9.

We see clearly in the diagram above that taking $p' = r - 1 + t$ sets the remaining modules equal to each other. This shows that for a given module in the family ${}^{[i]}P_{p;,r,0}$, we can find the same module in ${}^{[i]}P_{p',u-r,v-1}$. As spectral flow is independent of the values of r, s, these two families must then be equivalent under the action of spectral flow.

This demonstrates explicitly a Kac table symmetry for the minimal models $M(u,v)$. So far this is the only such symmetry we have derived concretely. We will show in an explicit example that two of the three possible families of modules in the minimal model $M(2,3)$ are equivalent.

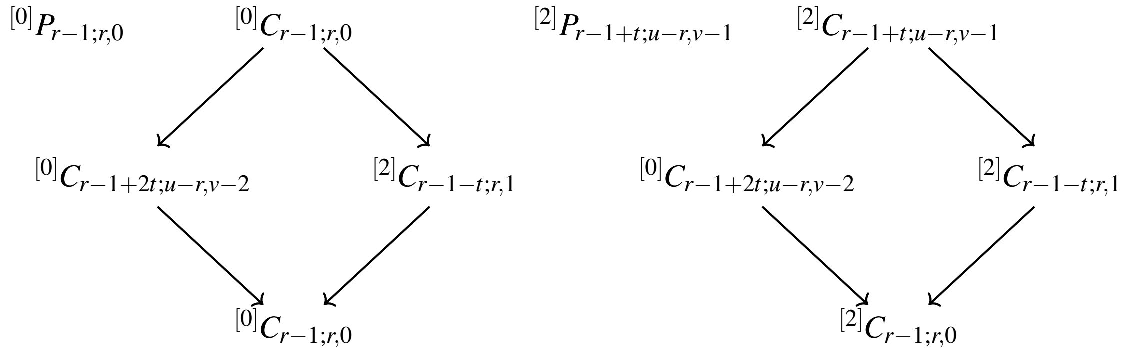


Figure 6.9: *A comparison of the two modules when $p' = p + t = r - 1 + t$.*

6.3.3 Symmetries of staggered modules in $M(2,3)$

The first explicit examples of staggered modules we considered were the modules ${}^{[0]}P_{\lambda_{1,0};1,0}$ and ${}^{[1]}P_{\lambda_{1,1}+1;1,1}$ in $M(2,3)$. There is an additional family of staggered modules ${}^{[i]}P_{p;1,2}$ for $p \in \lambda_{1,2} + i + 2\mathbb{Z}$. With the tools developed in this section, we can show that the module families ${}^{[i]}P_{p;1,0}$ and ${}^{[i]}P_{p;1,2}$ are related by spectral flow, and are thus equivalent. The modules ${}^{[0]}P_{\lambda_{1,0};1,0}$ and ${}^{[0]}P_{\lambda_{1,2};1,2}$ have Loewy diagrams given in Figure 6.10.

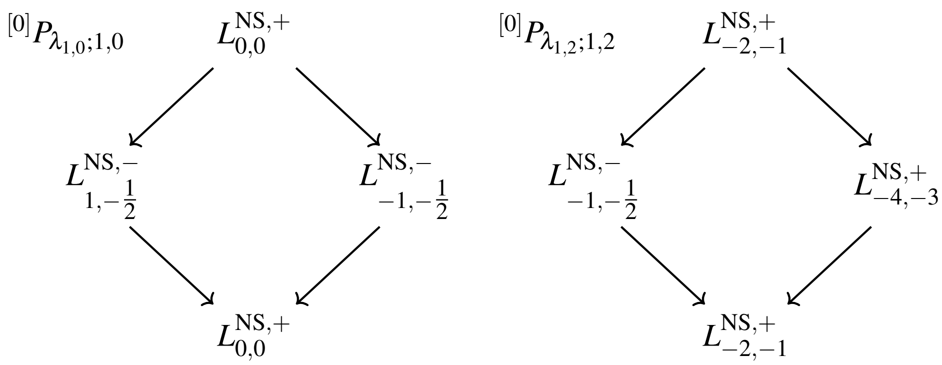


Figure 6.10: *Comparison of the Loewy diagrams showing the component modules and their embeddings for ${}^{[0]}P_{\lambda_{1,0};1,0}$ and ${}^{[0]}P_{\lambda_{1,2};1,2}$. We have applied the module dictionary to the ${}^{[i]}C_{p;r,s}$ modules to determine the corresponding irreducible components.*

We can then determine through direct calculation, and verify numerically using a Mathematica program that implements both the spectral flow and the module dictionary, that indeed $\sigma^1({}^{[0]}P_{\lambda_{1,0};1,0}) = {}^{[0]}P_{\lambda_{1,2};1,2}$. As these two modules lie in the same spectral flow orbit, and the flow is independent of r, s labels, we may conclude that the two families of staggered modules are equivalent.

This mirrors the staggered modules of $A_1(2,3)$, whereby the modules $S_{1,0}$ and $S_{1,2}$ have been shown to be equivalent up to spectral flow. The effect of spectral flow on the coset implies that the corresponding $M(2,3)$ families related to those modules should also be related by spectral flow.

6.4 The module ${}^{[0]}P_{\lambda_{1,0};1,0}$ in $M(u,v)$

In Section 6.2, we looked at the specific example of the module ${}^{[0]}P_{\lambda_{1,0};1,0} \in M(2,3)$. In this section we will consider the staggered modules which have the same component module conformal weights, however, we allow the value of the central charge to be any admissible value. That is, we consider the module ${}^{[0]}P_{\lambda_{1,0};1,0} \in M(u,v)$. For this particular staggered module, the head module is the irreducible module $L^{+,\mathrm{NS}}_{0;0}$. For all admissible values of the central charge the left and right modules are $L^{-,\mathrm{NS}}_{\pm 1,-1/2}$, i.e. the weights of the component modules are fixed for all values of the central charge.

This can be seen directly by looking at the determinant formula, where for $(j,\Delta) = (0,0)$, the charged singular vectors appear at charge ± 1, and conformal weight $\frac{1}{2}$. Moreover, the left and right modules exhibit the same behaviour, the charged singular vector where gluing occurs in the staggered module is unchanged for all admissible c.

This implies that the irreducible components of the module ${}^{[0]}P_{\lambda_{1,0};1,0}$ have the same weights for all admissible values of c. This reinforces the notion that the Loewy diagram by itself is not enough to characterise the structure of a staggered module. We naturally expect that the β-parameters will be c, or equivalently t, dependent. As a way of shortening notation, and because of the weights of the head module, we will refer to this module as the vacuum staggered module.

Proceeding as in the earlier analysis, we define the gluing relations on the module to be

$$G^+_{1/2}|H\rangle = \beta_R|R\rangle\,, \quad G^-_{1/2}|H\rangle = \beta_L|L\rangle\,, \quad L_0|H\rangle = |S\rangle\,, \quad G^+_{-1/2}|L\rangle = |S\rangle\,, \quad G^-_{-1/2}|R\rangle = |S\rangle\,, \tag{6.39}$$

noting that these have not changed from the first example.

We also have that the uncharged singular vectors in the left/right modules appear in the same weight spaces as the initial example, so again we introduce

$$G^+_{-1/2}|H\rangle = (\alpha_R L_{-1} + \gamma_R J_{-1})|R\rangle\,, \quad G^-_{-1/2}|H\rangle = (\alpha_L L_{-1} + \gamma_L J_{-1})|L\rangle\,, \tag{6.40}$$

however, in this treatment, we expect all of the introduced parameters to be functions of t. Our goal is to determine the action of the raising generators on these relations, so as before, we check the action of L_1, J_1 on the above constraints. As an example, we have

$$\begin{aligned}
J_1 G^+_{-1/2}|H\rangle &= J_1(\alpha_R L_{-1} + \gamma_R J_{-1})|R\rangle\,, \\
\left[J_1, G^+_{-1/2}\right]|H\rangle &= \alpha_R[J_1, L_{-1}]|R\rangle + \gamma_R[J_1, J_{-1}]|R\rangle\,, \\
G^+_{1/2}|H\rangle &= \alpha_R J_0|R\rangle + \gamma_R\frac{c}{3}|R\rangle\,, \\
\beta_R|R\rangle &= \alpha_R|R\rangle + \gamma_R\frac{c}{3}|R\rangle\,.
\end{aligned} \tag{6.41}$$

Recalling that $c = 3\left(1 - \frac{2}{t}\right)$, this is an example of how t dependence is introduced into the parameter values. We also derive constraints coming from the action of $\{G^\pm_{-1/2}, G^\mp_{1/2}\} = 2L_0 \mp J_0$ as in the first example.

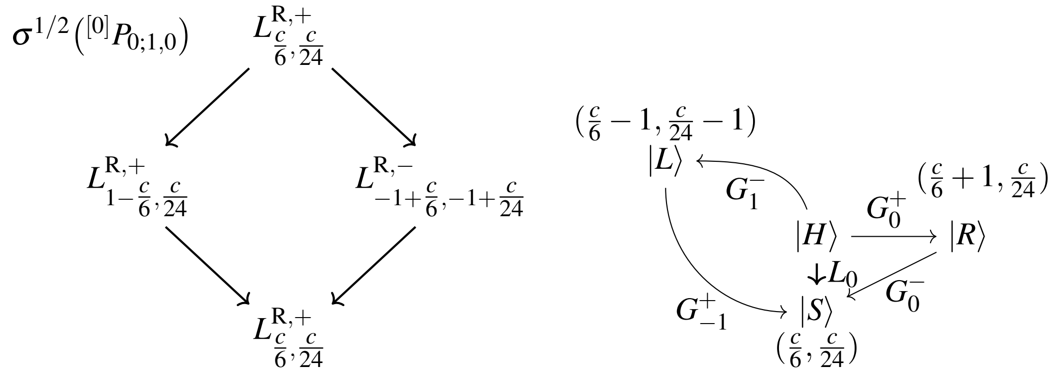
The resulting set of equations has a solution given by

$$\alpha_R = \alpha_L = \frac{1}{t}, \quad \beta_R = \beta_L = \frac{t-1}{t}, \quad \gamma_R = 1, \quad \gamma_L = -1. \tag{6.42}$$

As expected, we see that the β-parameters are indeed functions of t. Combining this result with our previous results, we have in principle determined the structure of the staggered module family ${}^{[i]}P_{p,1,0}$ (and equivalently ${}^{[i]}P_{p+t,u-1,v-1}$) in all possible minimal models, in particular, we have also determined the structure for the corresponding family in $M(3,2)$. In order to better understand this module, as well as provide explicit examples for some of our previous results, we will continue by looking at the spectral flows of this module.

6.4.1 Spectral flow on ${}^{[0]}P_{\lambda_{1,0};1,0}$

We begin by looking at the module $\sigma^{1/2}\left({}^{[0]}P_{0;1,0}\right)$, where we have used that $\lambda_{1,0} = 0$. The resulting module is a Ramond sector module, where the Loewy diagram and weight space diagram are given below. Note that since we are treating the central charge as general and the spectral flow is dependant on c, the labels on the Loewy diagram feature general values of c.



In this module, the gluing and definition of the β-parameters is given by

$$G_0^+ |H\rangle = \beta_R |R\rangle, \quad G_1^- |H\rangle = \beta_L |L\rangle, \quad G_0^- |R\rangle = |S\rangle, \quad G_{-1}^+ |L\rangle = |S\rangle, \quad L_0 |H\rangle = \frac{c}{24} |H\rangle + |S\rangle. \tag{6.43}$$

Note that the right, head, and socle modules all have $\Delta = \frac{c}{24}$. It is important here to remark that this module has a particularly nice feature, in that it is one of the few times where the spectral flow of highest-weight vectors are again the highest-weight vectors. We noted earlier in Section 6.3 that the spectral flow needed only to preserve the property of being extremal, but it can occur, (for sufficiently small values of ℓ) that indeed highest-weight vectors are mapped to highest-weight vectors. The next example will demonstrate when this is no longer the case.

As in the vacuum staggered modules, the left and right component modules here also feature uncharged singular vectors which have been quotiented. These singular vectors lead to the constraint equations

$$\begin{aligned} G^+_{-1}\,|H\rangle &= (\alpha_R L_{-1} + \gamma_R J_{-1})\,|R\rangle\,, \\ G^-_0\,|H\rangle &= (\alpha_L L_{-1} + \gamma_L J_{-1})\,|L\rangle\,, \end{aligned} \tag{6.44}$$

where again the relevant relations are obtained by acting as before with $\{G^+_0,\ G^-_1,\ J_1,\ L_1\}$ on these states.

Moreover, we again consider the combinations of two fermionic generators when one acts in a gluing direction, and the other acts in a singular direction. For this particular module, we have the relations

$$\{G^+_0, G^-_0\} = 2L_0 - \frac{c}{12}, \quad \& \quad \{G^+_{-1}, G^-_1\} = 2L_0 - 2J_0 + \frac{c}{4}. \tag{6.45}$$

As an example, we compute

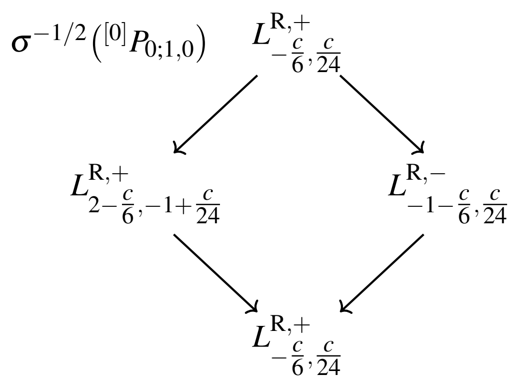
$$\begin{aligned} \left(2L_0 - 2J_0 + \frac{c}{4}\right)|H\rangle &= G^+_{-1}G^-_1\,|H\rangle + G^-_1 G^+_{-1}\,|H\rangle\,, \\ \left(\frac{c}{12} - \frac{c}{3} + \frac{c}{4}\right)|H\rangle + 2\,|S\rangle &= G^+_{-1}\beta_L\,|L\rangle + G^-_1\,(\alpha_R L_{-1} + \gamma_R J_{-1})\,|R\rangle\,, \\ 2\,|S\rangle &= \beta_L\,|S\rangle + \alpha_R\,|S\rangle + \gamma_R\,|S\rangle\,. \end{aligned} \tag{6.46}$$

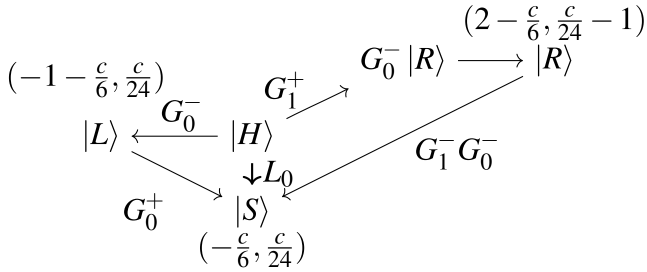
The resulting set of equations again has a solution for all admissible $c(t)$ given by

$$\alpha_R = \alpha_L = \frac{1}{t}, \quad \beta_R = \beta_L = \frac{t-1}{t}, \quad \gamma_R = 1 - \frac{1}{2t}, \quad \gamma_L = -1 - \frac{1}{2t}. \tag{6.47}$$

We see that $\beta_{\mathrm{L/R}}$ are unchanged under the action of $\sigma^{1/2}$. From the perspective of our general results of the flow, this is expected, as there is no modifying factor arising from the spectral flow.

To investigate the action of spectral flow further, we can also consider the module $\sigma^{-1/2}\big({}^{[0]}P_{0;1,0}\big)$. The Loewy diagram and weight space diagram are given below. Here, it is the right module which no





longer has $\Delta = \frac{c}{24}$. The gluing of the staggered module is given by

$$G^-_0\,|H\rangle = \beta_L\,|L\rangle\,, \quad G^+_0\,|L\rangle = |S\rangle\,, \quad G^+_1\,|H\rangle = \tilde{\beta}_R G^-_0\,|R\rangle\,, \quad G^-_{-1}G^-_0\,|R\rangle = |S\rangle\,, \quad L_0\,|H\rangle = \frac{c}{24}\,|H\rangle + |S\rangle\,. \tag{6.48}$$

Note, for this module, we choose to define $\tilde{\beta}_R$ by $G_1^+ |H\rangle = \tilde{\beta}_R G_0^- |R\rangle$, rather than $G_0^+ G_1^+ |H\rangle = \beta_R |R\rangle$. The redefinition takes into account exactly the expression that one needs to go from the glued vector to the highest-weight vector (action by G_0^+). Defining in this way, we expect that the β-parameters will be exactly the same functions as in the previous example. For completeness, the redefinition introduces a scaling factor in the following way,

$$G_0^+ G_0^- |-\tfrac{c}{6}+2; \tfrac{c}{24}-1\rangle = \left(2L_0 - \frac{c}{12}\right) |-\tfrac{c}{6}+2; \tfrac{c}{24}-1\rangle = -2 |-\tfrac{c}{6}+2; \tfrac{c}{24}-1\rangle . \tag{6.49}$$

Following the definition of β_R, the function t we find here and for the previous examples will have an overall factor of -2.

As before, the β-values for the module are fixed using relations from the uncharged singular vectors

$$G_0^+ |H\rangle = \left(\alpha_R L_{-1} G_0^- + \gamma_R J_{-1} G_0^-\right) |R\rangle , \quad G_{-1}^- |H\rangle = \left(\alpha_L L_{-1} + \gamma_L J_{-1}\right) |L\rangle . \tag{6.50}$$

Note that the vector $G_{-1}^- |-\frac{c}{6}; \frac{c}{24}\rangle$ is not singular in the Verma module generated by $|-\frac{c}{6}; \frac{c}{24}\rangle$. It is straight forward to verify that $G_0^+ G_{-1}^- |-\frac{c}{6}; \frac{c}{24}\rangle = (2L_{-1} + J_{-1}) |-\frac{c}{6}; \frac{c}{24}\rangle$. However, the vector $(2L_{-1} + J_{-1}) |-\frac{c}{6}; \frac{c}{24}\rangle$ becomes singular in the quotient $V_{-c/6;c/24} \ / \ G_0^- |-\frac{c}{6}; \frac{c}{24}\rangle$. In this quotient, $G_{-1}^- |-\frac{c}{6}; \frac{c}{24}\rangle$ is then a descendant of $(2L_{-1} + J_{-1}) |-\frac{c}{6}; \frac{c}{24}\rangle$.

The uncharged singular vector relations are solved in exactly the same way as the previous example. We have constraints on the parameters arising from applying the relations

$$\{G_0^+, G_0^-\} = 2L_0 - \frac{c}{12}, \quad \& \quad \{G_1^+, G_{-1}^-\} = 2L_0 + 2J_0 + \frac{c}{4}. \tag{6.51}$$

The solutions for the parameter functions are

$$\alpha_L = \alpha_R = \frac{1}{t}, \quad \beta_L = \tilde{\beta}_R = \frac{t-1}{t}, \quad \gamma_L = -1 + \frac{1}{2t}, \quad \gamma_R = 1 + \frac{1}{2t}. \tag{6.52}$$

We remark again that we only see the same function for $\tilde{\beta}_R$ because we have scaled out the shift introduced by the spectral flow.

This concludes the explicit examples we have to show, and with it, the results of this introductory exploration of staggered modules for the $N = 2$ superconformal algebras. In the conclusions chapter, we will discuss natural directions for further work, as well as offering perspective with some of the long term goals of this program of research.

Chapter 7

Conclusion

7.1 Generalised Galilean algebras

We finalise this thesis with concluding thoughts and future ideas regarding the work presented within, beginning with our work on Galilean algebras. In Chapters 2-4, we continued a program of research which introduced a consistent mathematical framework for performing Galilean contractions of (conformal) symmetry algebras. Originally, the Galilean contraction procedure took as input two (equivalent up to choice of central extension value) symmetry algebras, and produced a new symmetry algebra. Our first result was to generalise the procedure to allow for input of any number of symmetry algebras, equivalent up to value of their central extensions. This generalisation gives rise to the so-called higher-order Galilean algebras. The higher-order Galilean algebra have a truncated $\mathbb{Z}_N$-graded structure.

We continued by observing that the change of basis matrix which implements the contraction procedure was of Vandermonde type. We generalise the higher-order Galilean contraction by considering basis changes implemented by tensor products of Vandermonde matrices related to the higher order Galilean contraction. This generalisation resulted in Galilean algebras with truncated $\mathbb{Z}_{\mathbf{N}}^{\sigma}$-graded structure, for a given sequence $\mathbf{N}$ which encodes a factorisation of N, the number of input algebras. The analysis of these so-called multi-graded algebras also developed a framework for understanding multiple iterated contractions, that is, understanding the process of Galilean contractions of Galilean algebras. For example, these results describe the Galilean contraction of higher-order Galilean algebras.

Finally, we looked at the case of contracting an algebra asymmetrically with one of its subalgebras. This choice relaxes the condition that the input algebras of the Galilean contraction need be the same. Although some of these algebras had been studied previously, see [86], the systematic description presented here is new. We see however, that the previously studied case, coming from studies of Wess-Zumino-Witten models based on (non)-compact Lie groups in [86], is still the most natural choice.

Our construction begins with a symmetry algebra $\mathfrak{g}$, and a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and consider the Galilean contraction of $\mathfrak{g}$ with a copy of the subalgebra $\mathfrak{h}$. The remaining subspace $\bar{\mathfrak{g}}$, which is not contracted with the copy of $\mathfrak{h}$, can then be rescaled by the contraction parameter ε in a number of ways. Rescaling the subspace $\bar{\mathfrak{g}}$ by ε^m for $m = \frac{1}{2}$, provides the richest possible family of examples of physical interest.

Our construction of asymmetric contractions considered in a sense the simplest asymmetric algebras with a notion of Galilean contraction. A natural extension of our work is to consider two algebras $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(1)}$, which share a subalgebra $\mathfrak{h}$. One can then consider the Galilean contraction of the equivalent subalgebras $\mathfrak{h}_{(0)}$ and $\mathfrak{h}_{(1)}$, where the remaining subspaces $\bar{\mathfrak{g}}_{(0)}$ and $\bar{\mathfrak{g}}_{(1)}$ can be individually rescaled by the contraction parameter.

Along with each of the generalisations to the Galilean contraction procedure, we provided a number of examples. These examples are focused on algebras relevant to conformal field theory and the wider mathematical physics literature. We would like to remark here that there are alternative procedures similar to that of the order-two Galilean contraction, yielding similar algebras, see [62, 64, 152]. Several of these results are for the case where the underlying symmetry algebra is a finite dimensional Lie algebra. We have chosen to focus on infinite dimensional symmetry algebras, in the form of operator product algebras, as we are specifically interested in the applications to conformal field theory. However, the results that we have presented are almost universally applicable to finite dimensional algebras with minimal modification, as the OPE on an operator product algebra is equivalent to the Lie bracket.

In the rest of this chapter, we would like to outline some of the open questions which remain, and the ideas which were not not covered in the main text. Our analysis has been almost entirely focused on the product structures of the resulting Galilean algebras within the framework of the various generalisations. In doing so, we have developed many new infinite families of such algebras. However, we have not provided any analysis of the representation theory of these algebras. We note that there is existing work, both in the mathematics and physics literature [141–143], on the representation theory of the Galilean Virasoro algebra of order two. Indeed, some of these results should carry over to many of the new algebras developed here, as they are results which hold for the underlying Galilean Virasoro algebra; a subalgebra of all conformal Galilean algebras. As the representation theory of the algebras $\widehat{\mathfrak{sl}}(2)$ and $\mathfrak{Vir}$ is relatively well understood (compared to their Galilean counterparts), their Galilean contracted algebras are natural choices for future study.

A particularly fruitful method of understanding conformal field theories has been that of free-field realisations. This is a procedure whereby one constructs a particular conformal symmetry algebra using normally-ordered products fields of “simpler” algebras, namely those based on free-field theories [31]. This is similar in spirit to the Sugawara construction of the Virasoro algebra, whereby one constructs an action of the Virasoro algebra on a current algebra. There are several advantages to using free-field realisations. Algebras of free fields allow for easier computation with the OPE. Moreover, generically speaking the free-field algebras have relatively well-understood and well-behaved representation theory, leading to lifting of results from the free-field theory to the constructed theory. However, one must generally take care as the constructed representations are not always faithful. Some results for free-field realisations of Galilean algebras exist [143, 144], however a more complete understanding of the free-field realisations for general Galilean algebras would be of great use in developing the corresponding representation theory.

Throughout this thesis, we have posed the question of whether a given procedure or construction is compatible (in a commutative sense) with the Galilean contraction procedure. It is not clear to what extent free-field realisations are compatible with the Galilean contraction. As an example, a crucial component of many free-field realisations is the bosonic $\beta\gamma$-ghost system. The system is generated by two bosonic fields, $\beta(z)$, $\gamma(z)$, with conformal weight $\lambda \in \mathbb{R}$ and $1-\lambda$ respectively. The non-trivial OPE relations are

$$\beta(z)\gamma(w) \sim \frac{\mathbb{I}}{z-w}. \tag{7.1}$$

The resulting order-two Galilean contracted algebra is generated by fields $\beta_i(z),\ \gamma_i(z)$ for $i=0,1$. The products between these fields are given by

$$\beta_i(z)\gamma_j(w) \sim \begin{cases} \frac{2\mathbb{I}}{z-w}, & \text{if } i=j=0, \\ 0, & \text{otherwise.} \end{cases} \tag{7.2}$$

The grade-0 part of the contracted algebra has a shifted structure constant (which can be scaled away), and any products with the grade-1 part are trivial. Hence, the resulting algebra is simply a trivial extension of the original algebra. As such, it is not clear to what extent (if any) a notion of compatibility with the contraction is beneficial to our understanding of Galilean free-field contractions. Further evidence in this direction is that in the papers [143, 144], the authors used uncontracted free-field algebras in their constructions.

Throughout this thesis, we have ensured that Galilean affine Lie algebras admit a Sugawara construction, and moreover, we have ensured compatibility between the Sugawara construction and the contraction procedure. The importance of a Sugawara operator in the theory is that it demonstrates an action of the Virasoro algebra, and hence, conformal symmetry. This is particularly relevant from the perspective of physical applications. For all the generalised Galilean contractions considered in this thesis, we found that the central charge of the resulting Galilean Virasoro algebra was given by the dimension of the underlying (semi)simple Lie algebra. This fits precisely with the results of [89] for Sugawara constructions for non-semisimple algebras. The authors provide a framework, known as a double-extension construction, and show that all algebras which can be realised as a double extension have a well defined Sugawara construction, and moreover, the central charge is given by the dimension of the underlying algebras.

It is a natural next step to verify that Galilean affine Lie algebras arise as double extensions of appropriately chosen Lie algebras. Demonstrating this was unfortunately outside the scope of this work. A description in terms of double extensions may make possible the determination of closed form expressions for the Sugawara operators in the asymmetric Sugawara construction at $m = 0$. Such a description restricts the general form of the invariant metric Ω, and these restrictions may be sufficient to give closed form expressions for the Sugawara operators.

We would like to develop a general theory of Galilean W-algebras (in the sense of algebras not of Lie-type). We recall that difficulties arise in the contraction procedure for algebras with functions of the central charge as structure constants. The chosen method of series expanding the functions in ε-small requires a case-by-case analysis, as one requires knowledge of the structure constants. The structure constants of the W-algebras are not known in general. As such, our results are limited in scope, considering only the most accessible example of W_3 for the higher-order, multi-graded, and asymmetric Galilean contractions.

The general theory of W-algebras has been rapidly developed, with many new techniques being used to better understand their structure (see [47] for an overview). Quantum Hamiltonian reduction, which we discussed in Section 1.8, is of particular interest. For a given affine Lie algebra $\widehat{\mathfrak{g}}$, quantum Hamiltonian reduction associates a W-algebra to each embedding of $\widehat{\mathfrak{sl}}(2) \hookrightarrow \widehat{\mathfrak{g}}$. As such, if we can show that the Galilean contraction procedure is compatible with the quantum Hamiltonian reduction procedure, we will have a general description of Galilean W-algebras which avoids the difficulties related to the structure constants.

Despite the shortfalls of our current understanding of contracting W-algebras, it does raise an interesting question: is it possible to use the structure of the Galilean W-algebra to determine the structure constants of the original algebra? This of course assumes that the corresponding Galilean W-algebra exists, which is still an open problem. However, the graded structure of the resulting Galilean algebra places strong constraints on the structure constant functions. If we assume that a particular Galilean contraction is well defined, and that the resulting algebra is graded, then we can use that information to deduce certain properties of the structure constants for the original (uncontracted) W-algebras.

Such a result is of interest, since despite having unified frameworks for constructing W-algebras, such as quantum Hamiltonian reduction, and the Casimir W-algebra constructions (see [39] for an introduction), determining the structure constants, and thus the OPEs of a given W-algebra is still a highly non-trivial exercise. Traditionally, this was done following Zamolodchikov's original prescription, where one enforces associativity to constrain the OPE (and thus the structure constants). Deducing the structure constants from the graded Galilean algebras may provide a fruitful way of obtaining concrete descriptions of uncontracted W-algebras (this idea was somewhat explored in [1]).

Finally, we would like to address the idea of cosets of Galilean algebras. We are naturally interested in seeing if a coset construction can also be used to study Galilean algebras. Some work has already been done in this direction [86], where the author considers non-compact Lie groups that lead to asymmetric Galilean algebras at $m = \frac{1}{2}$. In that paper, the author manages to show that it is indeed possible to construct a Virasoro operator in the resulting coset theory, and that it follows from a Sugawara construction for non-semisimple algebras [89]. Moreover, the author remarks that a coset construction of a modified Virasoro operator is also possible. However, we were able to show in a number of examples that one cannot construct a Galilean Virasoro algebra $\mathfrak{Vir}_G^2$ in the coset theory. That is, we could not construct an accompanying T_1 field. An interpretation of these results is still lacking.

What is of particular interest in a coset construction is the central charge of the resulting Virasoro algebra. As we have consistently seen, the Galilean Virasoro algebras coming from the Sugawara construction have integer central charges (in particular, a multiple of the dimension of the underlying semisimple $\mathfrak{g}$). In [86] the author demonstrates that the resulting Virasoro algebra appearing in the coset can have rational central charge, implying that such a coset theory does give rise to a wider range of Virasoro models.

7.2 The study of $N=2$ staggered modules

In Chapters 5 and 6 we have investigated the first examples of particular indecomposable yet reducible modules over the $N=2$ superconformal algebras, known as staggered modules. These modules have been constructed using recent results [99, 100] on the coset construction (commutant construction) which relates $\widehat{\mathfrak{sl}}(2)$ and $N=2$ minimal models at admissible levels (and correspondingly admissible central charges).

Staggered modules arise in non-unitary $N=2$ minimal models $M(u,v)$, which are exactly those models corresponding to $\widehat{\mathfrak{sl}}(2)$ minimal models $A_1(u,v)$ for $v \neq 1$. The minimal models $A_1(3,2)$ and $A_1(2,3)$ are known to contain staggered modules, and such modules conjectured to exist in the models $A_1(u,v)$ for all admissible u,v. The work of [100] determined branching rules for $A_1(u,v)$ staggered modules, and we investigated the resulting families of $M(u,v)$ staggered modules which arose.

We began by presenting a general discussion of staggered modules, outlining the required general theory. Following this, we demonstrated explicitly the effect of the automorphisms of the $N=2$ algebra on the staggered modules. This led to the key observation that the families of staggered modules appearing in the branching rules are related by the spectral flow automorphism.

Following this, we constructed explicit examples of $N=2$ staggered modules for both the Neveu-Schwarz and Ramond algebras in the minimal model $M(2,3)$. We gave their Loewy diagrams, described the gluing of component modules, and explicitly determined their structure by calculating the values of their associated β-parameters.

Observations from these first examples motivated the search for more general symmetries of the minimal models. We used these observations to demonstrate two key general facts. First, there is a Kac table symmetry, for fixed u,v, giving an equivalence of staggered module families appearing in the branching rule. Modules with r,s label $r,0$ are equivalent to those with label $u-r, v-1$. In particular, in the minimal model $M(2,3)$ this demonstrated that the families ${}^{[i]}P_{p;1,0}$ and ${}^{[i]}P_{p;1,2}$ are equivalent families of staggered modules, a fact which is also true for their related $A_1(2,3)$ counterparts.

Furthermore, we showed that the value of the β-parameters for a given staggered module family are related under the action of the spectral flow. We demonstrated that the spectral flow only affects the β-values by a constant term which is directly calculable given the value of ℓ of the spectral flow. Since all the modules in a given branching rule family are related by spectral flow, we conclude that all the modules in a given family are structurally related. These results imply that in order to understand the structure of a particular staggered module family, one need only determine the structure of a representative of that family.

We continued by demonstrating that all component modules of the vacuum staggered module (with head module highest weight $(0,0)$) have the same highest weights for all admissible levels of the central charge. Moreover, as these modules are labelled by $r=1, s=0$, such a vacuum staggered module should appear as a member of all non-unitary minimal models. We use the language that such a module should appear, as we have not explicitly proven existence of these modules in general. We determine the value of the β-parameters for this vacuum staggered module for all admissible levels of the central charge. This determines the corresponding module family appearing in the minimal model $M(3,2)$, as well as its Kac symmetric counterpart.

Finally, we gave an analysis of the Ramond sector modules that arise from the action of σ^{ℓ} for $\ell = \pm\frac{1}{2}$ on the vacuum staggered modules for admissible c. We see that these Ramond modules have the same values of the β-parameters, up to a constant factor, which exactly matches the prediction provided by the general results of the preceding discussion on spectral flow.

We would like to reiterate that our description of the structure of modules in terms of their β-parameters was motivated by the work of [104, 107, 138]. We follow the standard prescription in the literature. However, we have not provided a proof that the parameters characterise the isomorphism classes of modules. In [104], the authors defined linear functionals on the space of isomorphism classes of staggered modules over the Virasoro algebra, and determined that the β's parametrise that space. This then rigorously proves the fact that the structure of a staggered module is characterised (up to isomorphism) by its β-parameters and structural diagram. We hope to demonstrate a similar result for the rank-two $N=2$ superconformal staggered modules in the future.

As this is the first investigation of staggered modules for the $N=2$ algebras, there are many questions that remain. Our results here have determined the structure of the staggered module families in the minimal model $M(2,3)$ which arise via the branching rule from the staggered modules in $A_1(2,3)$. However, we cannot yet conclude that these families exhaust all possible staggered modules in $M(2,3)$. That result has been conjectured by [100, 109]. However, we are in a position to begin calculating fusion of staggered modules in $M(2,3)$, to attempt to prove the conjectured fusion rules coming from the Grothendieck rings.

We would like to have a similar description of the minimal model $M(3,2)$. Its counterpart minimal model $A_1(3,2)$ is the only other $\widehat{\mathfrak{sl}}(2)$ minimal model for which staggered modules have been explicitly constructed. We have already determined the structure of the module families ${}^{[i]}P_{p;1,0}$ and ${}^{[i]}P_{p;2,1}$ using our general results on the vacuum staggered module, and the Kac table symmetries. Moreover, we know that ${}^{[i]}P_{p;1,1}$ and ${}^{[i]}P_{p;2,0}$ are also related by Kac table symmetries. Hence, we need only determine the structure of one representative module from either of these families.

The difficulty in doing this comes with finding a suitable representative module. In the examples we have presented here, the singular vector relations which fix the value of the β-parameters are close enough to the highest-weight vectors that the corresponding calculations are feasible. An early investigation of the modules appearing in the families ${}^{[i]}P_{p;1,1}$ and ${}^{[i]}P_{p;2,0}$ has only yielded cases where these relations are too deep in the module to be useful for calculation, even with the aid of a computational package like Mathematica.

There are two main long-term goals of this project. First is the classification of rank-two staggered modules for the $N = 2$ superconformal algebras. Ideally, this would be a result similar to that of [104] where the authors classified the rank-two staggered modules for the Virasoro algebra up to isomorphism. Logarithmic modules have been shown to be fundamental to a complete understanding of conformal field theories [118]. Moreover, conformal field theories with $N = 2$ supersymmetry (albeit in four dimensions) have lead to new discoveries, whereby those models have fundamental algebraic invariants given by certain vertex operator algebras [145–147]. There may be useful understanding to be gained in the two-dimensional superconformal setting, which can then be applied to these so-called 4D-2D correspondences.

The second long term goal is related to the search for an appropriate categorical understanding of modules in conformal field theory. Much research [148–151] has focused on understanding the modular tensor categories formed by the representations of vertex operator (super)algebras. It has become clear that the understanding of module fusion coming from rational conformal field theory (where the fusion ring is finite dimensional) needs to be expanded. However, verifying fusion rules calculationally is a notoriously difficult problem. The corresponding categorical framework for studying fusion has proven to be very powerful.

It has been conjectured [109] that for the $A_1(u,v)$ minimal models, the staggered modules are projective modules in an appropriately enlarged module category, equipped with a fusion product. Enlarged in the sense that we do not consider only highest-weight modules, but also relaxed highest-weight modules, where the definition of highest-weight vector is relaxed so that $e_0 \in \widehat{\mathfrak{sl}}(2)$ is not an annihilator; as well as modules arising from spectral flows of the aforementioned types. Such a category is in essence a fusion category of all weight modules with finite dimensional weight spaces. The conditions on the category come from the Grothendieck fusion rules for $A_1(u,v)$ [100].

It is then conjectured that the $S^{\pm}_{r,s}$ modules form the projective covers of the L- and $D^{\pm}$-type modules, as well as an additional family of modules referred to as $E^{\pm}$-type, which are reducible but indecomposable modules whose quotients are $D^{\pm}$-type. The $N = 2$ superconformal algebras have an upside in that the fermionic generators square to zero. This implies that under spectral flow, highest-weight modules map to highest-weight modules, unlike in $\widehat{\mathfrak{sl}}(2)$. This property still holds if one relaxes the highest-weight annihilation conditions, for example letting G^{+}_{0} act without annihilating in the Ramond sector. With the improved understanding of the coset construction, and its implications for the representation theory, it could be a more straightforward task to answer the problem of projectivity of staggered modules for $N = 2$ superconformal minimal models. One can then attempt to use the "reverse" Kazama-Suzuki coset construction to translate that result back to $A_1(u,v)$ minimal models, using the work of [98, 153]

There is also the question of physical descriptions of these modules. Similar results showed that staggered modules appear in certain Virasoro models, for example percolation at $c = 0$ [115, 120] and so-called critical dense polymers at $c = -2$ [121]. It is natural to ask if there are similar systems coming from statistical physics or string theory which demonstrate staggered $N = 2$ modules "in the wild"? One particular downside of these modules from a physical perspective is that they are non-unitary. While research into non-unitary quantum systems has flourished in the mathematics community, these ideas have naturally been slower to catch on in the physics community, as unitarity is a key component of our understanding of quantum theory.

Finally, we can also attempt to consider more general staggered structures. Thus far, we have looked at only those staggered modules in the $M(u,v)$ minimal models which arise through the coset construction from $A_1(u,v)$ minimal models. Such modules are made of four component modules, glued such that L_0 has a rank two Jordan block. While the coset construction provides a valuable calculational framework for determining our results, we could also consider staggered modules of both the $N = 2$ superconformal and $\widehat{\mathfrak{sl}}(2)$ algebras more generally. For example, there are alternative constructions of staggered modules over $\widehat{\mathfrak{sl}}(2)$ [108], related to so-called Kac modules. Similarly, we can consider $N = 2$ staggered modules more generally, not restricting to modules arising via the coset. Rather one could consider what possible gluings of highest-weight representations lead to indecomposable yet reducible modules, with a non-diagonal action of a zero-mode generator. This allows one to ask several new questions. Is it possible to have non-diagonalisability in the generator J_0? Are Jordan blocks of rank greater than 2 possible? Logarithmic modules with higher-rank Jordan blocks have been seen to appear for the Virasoro algebra [122, 154]. The downside of a more general approach is that it is very difficult to prove existence of modules. A key benefit of the coset construction of $M(u,v)$ minimal models is the known existence of the related $A_1(u,v)$ staggered modules for $(u,v) = (3,2), (2,3)$.

Many interesting open questions remain for both the construction of Galilean conformal field theories and their representations, as well as the logarithmic representations of the $N = 2$ superconformal algebras. We hope to answer some of these in the future.

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