Oscillating Bianchi IX Universe in Hořava-Lifshitz Gravity

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Abstract

We study a vacuum Bianchi IX universe in the context of Hořava-Lifshitz (HL) gravity. In particular, we focus on the classical dynamics of the universe and analyze how anisotropy changes the history of the universe. If the initial anisotropy is large, we find the universe which ends up with a big crunch after oscillations if a cosmological constant Λ is zero or negative. For $\Lambda > 0$, we find a variety of histories of the universe, that is de Sitter expanding universe after oscillations in addition to the oscillating solution and the previous big crunch solution. This fate of the universe shows sensitive dependence of initial conditions, which is one of the typical properties of a chaotic system. If the initial anisotropy is near the upper bound, we find the universe starting from a big bang and ending up with a big crunch for $\Lambda \leq 0$, while de Sitter expanding universe starting from a big bang for $\Lambda > 0$.

1 Introduction

Among attempts to construct a complete quantum gravitational theory, Hořava-Lifshitz (HL) gravity has been attracted much interest as a candidate for such a theory over the past years. HL gravity is characterized by its power-counting renormalizablity, which is brought about by a Lifshitz-like anisotropic scaling as $t \to \ell^z t, \vec{x} \to \ell \vec{x}$, with the dynamical critical exponent $z = 3$ in the ultra-violet (UV) limit [1]. In order to recover general relativity (or the Lorentz invariance) in our world, one expects that the constant λ converges to unity in the infrared (IR) limit in the renormalization flow. Although it has been argued that there exist some fundamental problems in HL gravity, some extensions are proposed to remedy these difficulties. It is intriguing issue whether or not HL gravity can be a complete theory of quantum gravity.

As pointed out by earlier works, a big bang initial singularity may be avoided in the framework of HL cosmology due to the higher order terms in the spatial curvature R_{ij} in the action[2]. In this context, many researchers have studied the dynamics of the Friedmann-Lemaitre-Robertson-Walker (FLRW) universe in HL gravity. Although we have also shown a singularity avoidance in HL cosmology[3], the following question may arise: Is this singularity avoidance generic? Is such a non-singular spacetime stable against anisotropic and/or inhomogeneous perturbations? In order to answer for these questions, we have to study more generic spacetime than the FLRW universe.

Therefore it is important to study whether or not non-singular universes in the present HL cosmology still exist with anisotropy and/or inhomogeneity. In the present paper, we shall investigate the possibility of the singularity avoidance in homogeneous but anisotropic Bianchi IX universe. Since we are interested in a singularity avoidance, we focus on an oscillating universe and analyze how anisotropy changes the history of the universe.

2 Bianchi IX universe in Hořava-Lifshitz gravity

First we introduce our Lagrangian of HL gravity, by which we will discuss the Bianchi IX universe. The basic variables in HL gravity are the lapse function, N, the shift vector, N_i, and the spatial metric, g_{ij}. These variables are subject to the action [1, 4]

$$S_{\rm HL} = \frac{1}{2\kappa^2} \int dt d^3x \sqrt{g} N \left(L_K - V_{\rm HL}[g_{ij}]\right) \tag{2.1}$$

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where $\kappa^2 = 1/M_{\rm PL}^2$ ($M_{\rm PL}$: the Planck mass) and the kinetic term is given by

$$L_K = K_{\ ij} K^{\ ij} - \lambda K^2 \tag{2.2}$$

with

$$K_{\ ij} := \frac{1}{2N}\left(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i\right) \tag{2.3}$$

$$K := g^{ij} K_{\ ij} \tag{2.4}$$

being the extrinsic curvature and its trace. The potential term $V_{\rm HL}$ will be defined shortly. In GR we have $\lambda = 1$, only for which the kinetic term is invariant under general coordinate transformations. In HL gravity, however, Lorentz symmetry is broken in exchange for renormalizability and the theory is invariant under the foliation-preserving diffeomorphism transformations,

$$t \to \bar{t}(t), \quad x^i \to \bar{x}^i(t, x^j). \tag{2.5}$$

As implied by the symmetry (2.5), it is most natural to consider the projectable version of HL gravity, for which the lapse function depends only on t: $N = N(t)$ [1]. Since the Hamiltonian constraint is derived from the variation with respect to the lapse function, in the projectable version of the theory, the resultant constraint equation is not imposed locally at each point in space, but rather is an integration over the whole space. In the cosmological setting, the projectability condition results in an additional dust-like component in the Friedmann equation [5].

The most generic form of the potential $V_{\rm HL}$ is given by [4]

$$\begin{aligned} V_{\rm HL} \ = \ & 2\Lambda + g_1 R + \kappa^2\left(g_2 R^2 + g_3 R^i{}_j R^j{}_i\right) + \kappa^3 g_4 \epsilon^{ijk} R_{i\ell} \nabla_j R^\ell{}_k \\ & + \kappa^4\left(g_5 R^3 + g_6 R\, R^i{}_j R^j{}_i + g_7 R^i{}_j R^j{}_k R^k{}_i + g_8 R \Delta R + g_9 \nabla_i R_{jk} \nabla^i R^{jk}\right), \end{aligned} \tag{2.6}$$

where Λ is a cosmological constant, $R^i{}_j$ and R are the Ricci and scalar curvatures of the 3-metric g_{ij}, respectively, and g_i's ($i = 1, ..., 9$) are the dimensionless coupling constants. By a suitable rescaling of time we set $g_1 = -1$. We also adopt the unit of $\kappa^2 = 1$ ($M_{\rm PL} = 1$) throughout the paper.

Let us consider a Bianchi IX spacetime, which metric is written as

$$ds^2 = -dt^2 + \frac{a^2}{4} e^{2\beta_{ij}} \omega^i \omega^j\,, \tag{2.7}$$

where the invariant basis ω^i is given by

$$\begin{aligned} \omega^1 &= -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2\,, \\ \omega^2 &= \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2\,, \\ \omega^3 &= \cos x^1 dx^2 + dx^3\,. \end{aligned} \tag{2.8}$$

A typical scale of length of the universe is given by a, which reduces to the usual scale factor in the case of the FLRW universe. We shall call it a scale factor in Bianchi IX model as well. The traceless tensor β_{ij} measures the anisotropy of the universe. The spacelike sections of the Bianchi IX is isomorphic to a three-sphere S^3, and a closed FLRW model is a special case of the above metric in the isotropic limit ($\beta_{ij} \to 0$).

For a vacuum spacetime, without loss of generality, we can assume that β_{ij} is diagonalized as

$$\beta_{ij} = {\rm diag}\left(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+\right)\,. \tag{2.9}$$

The basic equations describing the dynamics of Bianchi IX spacetime in HL gravity are now given by the followings:

$$H^2 = \frac{2}{3(3\lambda-1)}\left[\sigma^2 + \frac{64}{a^6} V(a, \beta_\pm) + \frac{8C}{a^3}\right], \tag{2.10}$$

$$\dot{H} + 3H^2 = \frac{8}{3(3\lambda-1)}\left[\frac{8}{a^5}\frac{\partial V}{\partial a} + \frac{3C}{a^3}\right], \tag{2.11}$$

$$\dot{\beta}_\pm = \sigma_\pm \tag{2.12}$$

$$\dot{\sigma}_\pm + 3H\sigma_\pm + \frac{32}{3a^6}\frac{\partial V}{\partial \beta_\pm} = 0\,, \tag{2.13}$$

where

$$\sigma^2 := \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} = 3\left(\sigma_+^2 + \sigma_-^2\right) . \tag{2.14}$$

$(\sigma_{\alpha\beta}) = \mathrm{diag}(\sigma_+ + \sqrt{3}\sigma_-, \sigma_+ - \sqrt{3}\sigma_-, -2\sigma_+)$ is the shear tensor of a timelike normal vector perpendicular to the homogeneous three-space, and σ is its magnitude. It may be convenient to introduce the dimensionless shear by

$$\Sigma_\pm = \frac{\sigma_\pm}{H} \quad \text{and} \quad \Sigma = \frac{\sigma}{H} , \tag{2.15}$$

which measure the relative anisotropies to the expansion rate H. We also introduce the phase variable φ defined by

$$\varphi := \arctan\left(\frac{\sigma_-}{\sigma_+}\right) , \tag{2.16}$$

which parameterizes the direction of the anisotropic expansion. The constant C arises from the projectability condition and it could be "dark matter" [5], but here we assume $C = 0$ just for simplicity. The potential V, which depends on a as well as $\beta_\pm$, is defined in [6].

We have performed numerical calculations to investigate the stability of oscillating FLRW universe against anisotropy. As a result, we have found five types of the fate of the universe.

(A) Anisotropic oscillation : This oscillating solution shows only small deviation from the isotropic FLRWuniverse. The scale factor a (and then the volume) oscillates very regularly.

(B) Big crunch after oscillations : An initially oscillating universe eventually collapses into a big crunch (a = 0) after many oscillations because of increase of the anisotropy. This is a singular solution.

(C) From big bang to big crunch : The spacetime starts from a big bang and ends up with a big crunch. This type of solution cannot avoid singularity.

(D) de Sitter expansion after oscillation : An initially oscillating universe eventually evolves into an exponentially expanding de Sitter universe because of a cosmological constant.

(E) de Sitter expansion from big bang : the spacetime evolves into de Sitter phase without oscillation. The large anisotropy makes a jump from the oscillating phase to de Sitter phase in the beginning.

We summarize the results in Figure 1 (the coupling constants are chosen as $\lambda = 1, g_1 = 1, \Lambda = 3/10, g_3 = 1, g_5 = -3/100, g_9 = 1/100 and g_2 = g_4 = g_6 = g_7 = 0$).

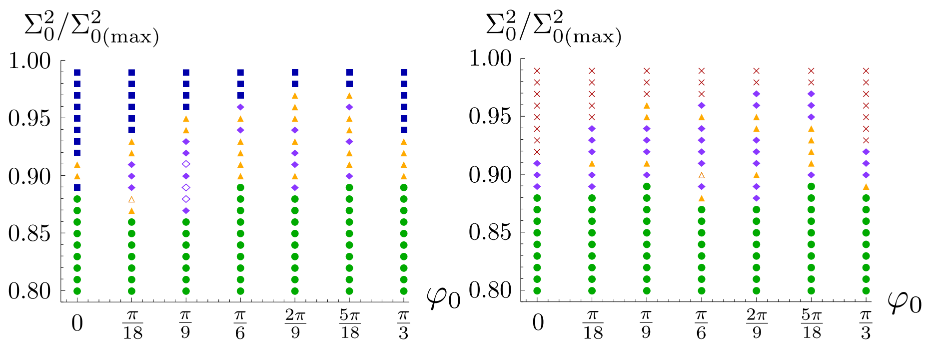

Figure 1: The fate of Bianchi IX universe in terms of initial anisotropy Σ_0^2 and φ_0. We judge the fate of the universe at $t = 100t_{\rm PL}$ ($t_{\rm PL}$: the Planck time). The histories (A), (B), (C), (D) and (E) are represented by a filled green circle, yellow triangle, red cross, purple diamond and blue square, respectively. The empty yellow triangle and purple diamond are classified to the histories (B) and (D), respectively, but those oscillating periods are longer than $100t_{\rm PL}$. We have set $a_0 = 1.0758$. $\dot{a}_0$ is fixed by the constraint equation [(a) $\dot{a}_0 > 0$ and (b) $\dot{a}_0 < 0$].

3 Conclusion

If the initial anisotropy is large, we find the universe which ends up with a big crunch after oscillations if a cosmological constant Λ is zero or negative. For $\Lambda > 0$, we find a variety of histories of the universe, that is de Sitter expanding universe after oscillations in addition to the oscillating solution and the previous big crunch solution. This fate of the universe shows sensitive dependence of initial conditions, which is one of the typical properties of a chaotic system. If the initial anisotropy is near the upper bound, we find the universe starting from a big bang and ending up with a big crunch for $\Lambda \leq 0$, while de Sitter expanding universe starting from a big bang for $\Lambda > 0$.

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