The Sixth International Workshop

Group Analysis of Differential Equations and Integrable Systems

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This book includes papers of participants of the Sixth International Workshop "Group Analysis of Differential Equations and Integrable Systems". The topics covered by the papers range from theoretical developments of group analysis of differential equations, theory of Lie algebras, noncommutative geometry and integrability to their applications in various fields including fluid mechanics, classical mechanics, Hamiltonian mechanics, continuum mechanics, mathematical biology and financial mathematics. The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis, integrability and their applications.

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International Workshop

Group Analysis of Differential Equations and Integrable Systems

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Contents

Preface

The Sixth International Workshop “Group Analysis of Differential Equations and Integrable Systems” (GADEIS-VI) was conducted at Protaras, Cyprus, during the period June 17–21, 2012. There were fifty three participants from twenty one countries (Austria, Canada, Cyprus, Czech Republic, Egypt, France, Greece, Ireland, Italy, Norway, Poland, Romania, Russia, Serbia, South Africa, Spain, Thailand, the Netherlands, Ukraine, the United Kingdom and the United States of America) and thirty eight lectures were presented. The topics of the Workshop ranged from theoretical developments of group analysis of differential equations, theory of Lie algebras, noncommutative geometry and integrability to their applications in various fields including fluid mechanics, classical mechanics, Hamiltonian mechanics, continuum mechanics, mathematical biology and financial mathematics. Twenty two papers are presented in this book of proceedings.

The Series of Workshops is a joint initiative by the Department of Mathematics and Statistics, University of Cyprus, and the Department of Applied Research of the Institute of Mathematics, National Academy of Sciences, Ukraine. The Workshops evolved from close collaboration among Cypriot and Ukrainian scientists. The first three meetings were held at the Athalassa campus of the University of Cyprus (October 27, 2005, September 25–28, 2006, and October 4–5, 2007). The fourth (October 26–30, 2008), the fifth (June 6–10, 2010) and the sixth meetings were held at the coastal resort of Protaras.

We would like to thank all the authors who have published papers in the Proceedings. All of the papers have been reviewed by one or two independent referees. We express our appreciation of the care taken by the referees and thank them for making constructive suggestions for improvement to most of the papers. The importance of peer review in the maintenance of high standards of scientific research can never be overstated.

The Editors

Sixth Workshop "Group Analysis of Differential Equations and Integrable Systems", 2012, 6–16

Lie Group Method for Solving a Problem of a Heat Mass Transfer

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The problem of heat and mass transfer in non-Newtonian power law, two-dimensional, laminar, boundary layer flow of a viscous incompressible fluid over an inclined plate is considered. By employing the Lie group method to this system, its symmetries are determined. Using the Lie reduction method the analytic solutions of the given equations are found. Dimensionless velocity, temperature and concentration profiles are studied and presented graphically for different physical parameters and the power law exponents.

1 Introduction

The flow of non-Newtonian fluids, including the power-law model has many applications in food processing, polymer, petrol-chemical, geothermal, rubber, paint and biological industries, as well as many engineering problems such as cooling of nuclear reactors, the boundary layer control in aerodynamics, and crystal growth. Lie symmetries provide a constructive method of reducing systems of partial differential equations into system of ordinary differential equations. A number of problems in science and engineering are solved using similarity analysis (see Ibragimov [5], Olver [7], and Seshadri and Na [8]). Many physical applications are illustrated by Abd-el-Malek et al. [1–4]. In 2006 Sivasankaran et al. [9], have applied the Lie group analysis to study the same problem but without considering the power-law fluid in the momentum equation, the heat generation in the energy equation, and the thermo-phoretic velocity in the diffusion equation. In this work, we construct analytical solutions and present qualitative discussion for the laminar boundary-layer flow of non-Newtonian power law fluids using Lie group methods.

2 The mathematical formulation

We study the system proposed by Olajuwon in [6]. It consists of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (1)$$

the momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\nu\frac{\partial}{\partial y}\Big(-\frac{\partial u}{\partial y}\Big)^{n}+g\beta(T-T_{\infty})\cos\alpha+g\beta^{*}(C-C_{\infty})\cos\alpha, \quad (2)$$

the energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_{\rho}}\frac{\partial^{2}T}{\partial y^{2}}+\frac{Q}{\rho c}(T-T_{\infty}), \quad (3)$$

the diffusion equation

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^{2}C}{\partial y^{2}}-\frac{\partial}{\partial y}\big(V_{T}(C-C_{\infty})\big), \quad (4)$$

and the associated boundary conditions

$$\begin{aligned} &u=v=0, \quad T=T_{w}, \quad C=C_{w} \quad \text{at} \quad y=0,\\ &u=0, \quad T=T_{\infty}, \quad C=C_{\infty} \quad \text{as} \quad y\to\infty. \end{aligned} \quad (5)$$

Here u and v are velocity components; x and y are space coordinates; T, T_w and T_∞ are the temperature of the fluid inside the boundary layer, the plate and the fluid temperature in the free stream, respectively. The plate is maintained at a temperature T_w, and the free stream air is at a temperature T_∞, where $T_\infty > T_w$ to a cold surface. C, C_w, and C_∞ are the concentration of the fluid inside the boundary layer, beside the plate, and the fluid concentration in the free stream, respectively; ν is the kinematic viscosity of the fluid; g is the acceleration due to gravity; β is the coefficient of thermal expansion; β^* is the coefficient of expansion with concentration; k is the thermal conductivity of fluid; ρ is the density of the fluid; c_ρ is the specific heat of the fluid; Q is the heat generation constant; D is the diffusion coefficient; α is the angle of inclination, n is the non-Newtonian parameter (power index) and the thermophoretic velocity V_T can be written as

$$V_{T}=-\frac{k\nu}{T_{r}}\frac{\partial T}{\partial y},$$

where T_r is some reference temperature and ν is the thermophoretic coefficient.

The non-dimensional variables are

$$\begin{aligned} &\bar{x}=\frac{xU_{\infty}}{\nu}, \quad \bar{y}=\frac{yU_{\infty}}{\nu}, \quad \bar{u}=\frac{u}{U_{\infty}}, \quad \bar{v}=\frac{v}{V_{\infty}},\\ &\theta(x,y)=\frac{T(x,y)-T_{\infty}}{T_{w}-T_{\infty}}, \quad \phi(x,y)=\frac{C(x,y)-C_{\infty}}{C_{w}-C_{\infty}}. \end{aligned}$$

The stream function formulation is $\bar{u}=\psi_y \quad \bar{v}=-\psi_x$. Here and below the subscripts of the functions ϕ, ψ and θ denote partial derivatives with respect to the corresponding variables.

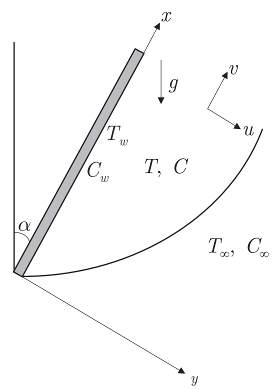

Figure 1. The physical problem.

Using this assumption, equation (1) becomes an identity and the remaining governing differential equations (2)–(4), after dropping the bars, transform to

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} + \frac{nU_\infty^{2n-2}}{\nu^{n-1}}\left(-\psi_{yy}\right)^{n-1}\psi_{yyy} - (\mathrm{Gr}\,\theta + \mathrm{Gc}\,\phi)\cos\alpha = 0, \qquad (6)$$

where Gr and Gc are constants, namely, $\mathrm{Gr} = \nu g\beta(T_w - T_\infty)U_\infty^{-3}$ is the thermal Grashof number, and $\mathrm{Gc} = \nu g\beta^*(C_w - C_\infty)U_\infty^{-3}$ is the solute Grashof number;

$$\psi_y\theta_x - \psi_x\theta_y - \frac{1}{\mathrm{Pr}}\,\theta_{yy} - \mathrm{He}\,\theta = 0, \qquad (7)$$

where $\mathrm{He} = Q\nu/(\rho c_p U_\infty^2)$ is the heat generation parameter, and $\mathrm{Pr} = \nu\rho c_p/k$ is the Prandtl number;

$$\psi_y\theta_x - \psi_x\theta_y - \frac{1}{\mathrm{Sc}}\,\phi_{yy} + \tau\left(\phi\,\theta_{yy} + \theta_y\phi_y\right) = 0, \qquad (8)$$

where $\mathrm{Sc} = \nu/D$ is the Schmidt number, $\tau = -k(T_w - T_\infty)/T_r$ is the thermophoretic parameter, and $k = v^{n-1}U_\infty^{2-2n}$. The corresponding boundary and initial conditions (5) become

$$\begin{aligned} &\psi_x = \psi_y = 0, \quad \theta = 1, \quad \phi = 1 \quad \text{at} \quad y = 0,\\ &\psi_y = 0, \quad \theta = 0, \quad \phi = 0 \quad \text{as} \quad y \to \infty. \end{aligned} \qquad (9)$$

3 Solution of the problem using Lie symmetry group method

In this section the symmetry group of equations (6)–(8) and the boundary conditions (9) are calculated using Lie method. Under this transformation, the two independent variables reduce to one and the equations (6)–(8) are transformed into ordinary differential equations (ODEs), where the independent variable is called similarity variable.

Firstly we briefly discuss how to determine Lie point symmetry generators admitted by equations (6)–(8), since the procedure is well known (see, e.g., [5,7]). Consider the one-parameter Lie group of infinitesimal transformations in the space of independent and dependent variables $(x, y, \theta, \phi, \psi)$ given by

$$x^* = x + \varepsilon\,\xi^1 + O(\varepsilon^2), \quad y^* = y + \varepsilon\,\xi^2 + O(\varepsilon^2),$$
$$\theta^* = \theta + \varepsilon\,\eta^1 + O(\varepsilon^2), \quad \varphi^* = \varphi + \varepsilon\,\eta^2 + O(\varepsilon^2), \quad \psi^* = \psi + \varepsilon\,\eta^3 + O(\varepsilon^2).$$

Here $\xi^i = \xi^i(x, y, \theta, \varphi, \psi)$, $\eta^j = \eta^j(x, y, \theta, \varphi, \psi)$ with $i = 1, 2$ and $j = 1, 2, 3$; ε is the Lie group parameter. A system of partial differential equations (6)–(8) is said to admit a symmetry generated by the vector field

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta^1 \frac{\partial}{\partial \theta} + \eta^2 \frac{\partial}{\partial \varphi} + \eta^3 \frac{\partial}{\partial \psi}, \tag{10}$$

if the transformation $(x, y, \theta, \varphi, \psi) \to (x^*, y^*, \theta^*, \varphi^*, \psi^*)$ leaves this system invariant.

A vector field X given by (10) is said to be a Lie point symmetry for equations (6)–(8) if the action of the third prolongation $X^{(3)}$ of the operator X

$$X^{(3)} = X + \xi^1_x \frac{\partial}{\partial \theta_x} + \xi^1_y \frac{\partial}{\partial \theta_y} + \xi^2_x \frac{\partial}{\partial \varphi_x} + \xi^2_y \frac{\partial}{\partial \varphi_y} + \xi^3_x \frac{\partial}{\partial \psi_x} + \xi^3_y \frac{\partial}{\partial \psi_y}$$
$$+ \xi^1_{yy} \frac{\partial}{\partial \theta_{yy}} + \xi^2_{yy} \frac{\partial}{\partial \varphi_{yy}} + \xi^3_{xy} \frac{\partial}{\partial \psi_{xy}} + \xi^3_{yy} \frac{\partial}{\partial \psi_{yy}} + \xi^3_{yyy} \frac{\partial}{\partial \psi_{yyy}}$$

on the left hand sides of equations (6)–(8) results to differential functions identically vanishing on the manifold defined by the system (6)–(8) in the prolonged space. The formulas for calculation of the coefficients appearing in the operator $X^{(3)}$ can be found, e.g., in [7]. After some cumbersome calculations we get

$$\xi^1_x = \xi^1_y = \xi^1_\theta = \xi^1_\varphi = \xi^1_\psi = 0, \quad \xi^2_y = \xi^2_\theta = \xi^2_\varphi = \xi^2_\psi = 0,$$
$$\eta^3_x = \eta^3_y = \eta^3_\theta = \eta^3_\varphi = \eta^3_\psi = 0, \quad \eta^1 = \eta^2 = 0.$$

Solving this system we obtain

$$\xi^1 = c_1, \quad \xi^2 = F(x), \quad \eta^1 = 0, \quad \eta^2 = 0, \quad \eta^3 = c_2,$$

where c_1 and c_2 are arbitrary constants, $F(x)$ is an arbitrary smooth function of the variable x. Therefore, system (6)–(8) admits the symmetry generator

$$X = c_1 \frac{\partial}{\partial x} + F(x) \frac{\partial}{\partial y} + c_2 \frac{\partial}{\partial \psi}.$$

If $c_1 \neq 0$, then without loss of generality we can set $c_1 = 1$ and $F(x) = 0$ using the adjoint action of the corresponding algebra.

To get the transformation of the independent and dependent variables which reduces system (6)–(8) to the system of ODEs we should solve the following characteristic system

$$\frac{dx}{1} = \frac{dy}{0} = \frac{d\theta}{0} = \frac{d\varphi}{0} = \frac{d\psi}{c_2}.$$

Its general solution is

$$\eta(x,y) = y, \quad \psi(\eta) = Cx + F_1(\eta), \quad \varphi(\eta) = F_2(\eta), \quad \theta(\eta) = F_3(\eta),$$
$$v = -\psi_x = -C, \quad u = \psi_y = F_1'(\eta),$$

where $C = c_2$, $F_j(\eta)$, $j = 1, 2, 3$, are arbitrary functions. The corresponding boundary conditions are

$$F_3(\eta) = 1, \quad F_2(\eta) = 1, \quad F_1'(\eta) = 0 \quad \text{at} \quad \eta = 0,$$
$$F_3(\eta) = 0, \quad F_2(\eta) = 0, \quad F_1'(\eta) = 0 \quad \text{as} \quad \eta \to \infty.$$

Then equation (7) becomes

$$F_3'' + C \Pr F_3' + \Pr \mathrm{He}\, F_3 = 0.$$

Since practically $|\,\mathrm{He}\,| < 1$, then the solution of this differential equation is

$$\theta = F_3(\eta) = e^{-r\eta}, \quad C\sqrt{\Pr} \geq \sqrt{\mathrm{He}},$$

where $r = \frac{1}{2} C \Pr + \frac{1}{2}\sqrt{(C\Pr)^2 - 4\Pr \mathrm{He}}$.

Equation (8) reduces to the ODE

$$F_2'' + C\, \mathrm{Sc}\, F_2' - \tau\, \mathrm{Sc}(F_2 F_3'' + F_2' F_3') = 0,$$

whose solution is

$$\varphi = F_2(\eta) = \frac{c_3 \int \exp(-\mathrm{Sc}\, \tau e^{-r\eta} + C\, \mathrm{Sc}\, \eta) d\eta + c_4}{\exp(-\mathrm{Sc}\, \tau e^{-r\eta} + C\, \mathrm{Sc}\, \eta)},$$

where c_3 and c_4 are arbitrary constants. We take $e^{-r\eta} \approx 1 - r\eta$ and apply boundary conditions to give

$$\varphi = F_2(\eta) = e^{-m\eta}, \quad \text{where} \quad m = (r\tau + C)\, \mathrm{Sc}\,.$$

Substituting the obtained expressions into equation (6) we get the ODE

$$\frac{n(U_\infty^{2n-2})}{v^{n-1}} \frac{d(F_1'')^n}{d\eta} + CF_1'' + \mathrm{Gr} \cos\alpha e^{-r\eta} + \mathrm{Gc}\cos\alpha e^{-m\eta} = 0. \tag{11}$$

Exact solutions of equation (11) can found for $n = 1$. In this case it becomes

$$F_1''' + CF_1'' + \mathrm{Gr}\cos\alpha e^{-r\eta} + \mathrm{Gc}\cos\alpha e^{-m\eta} = 0,$$

whose solution for $m, r \neq C$ can be written in the form

$$u = F_1'(\eta) = \frac{\mathrm{Gr}\cos\alpha}{r(C-r)}\left(e^{-r\eta} - e^{-C\eta}\right) + \frac{\mathrm{Gc}\cos\alpha}{m(C-m)}\left(e^{-m\eta} - e^{-C\eta}\right).$$

For $n \neq 1$ we have solved (11) numerically.

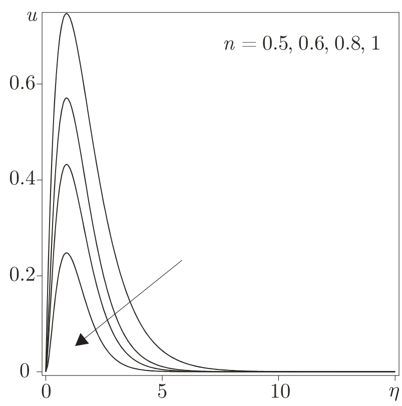

Figure 2. Effect of the power index $n \leq 1$ on the velocity profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $v = 1$, $\tau = 1$, $U_\infty = 1$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$.

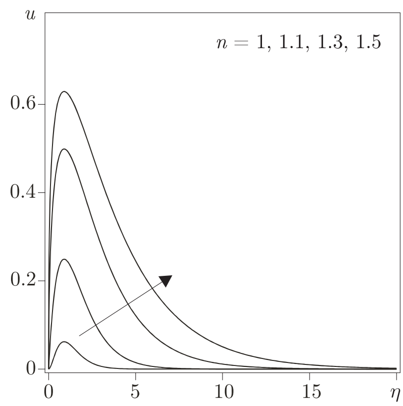

Figure 3. Effect of the power index $n \geq 1$ on the velocity profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $v = 1$, $\tau = 1$, $U_\infty = 1$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1$, $\alpha = 60°$.

4 Illustration of the results

In this section, we explore the properties of the characteristics of the flow that obtained using Lie method. The effect of a various parameters such as the non-Newtonian parameter n, the Prandtl number Pr, the heat generation parameter He, and the Schmidt number Sc on the velocity components, temperature and concentration profiles are studied.

4.1. Effect of the power index *n* on the velocity components. Figs. 2 and 3 illustrate the effect of n on the dimensionless velocity components profiles across the boundary layer. Since for $n = 1; 1.1; 1.5$ the corresponding values of $\eta = 7; 9; 16$, it is clear that the thickness of velocity boundary layer increases with the increase of the non-Newtonian parameter n.

4.2. Effect of the Prandtl number Pr.

◇ *On the temperature.* Fig. 4 displays the effects of Pr on the temperature profiles. Physically speaking, Pr is an important parameter in heat transfer processes as it characterizes the ratio of thicknesses of the viscous and thermal boundary layers. Increasing the value of Pr causes the fluid temperature and its boundary layer thickness to decrease significantly as seen from Fig. 4. This decrease in temperature produces a net reduction of the thermal buoyancy effect in the momentum equation which results in less induced flow along the plate and consequently, the fluid velocity decreases.

◇ *On the concentration.* Fig. 5 displays the effects of Pr on the concentration profiles. It is observed that the concentration distribution inside the boundary layer also decreases.

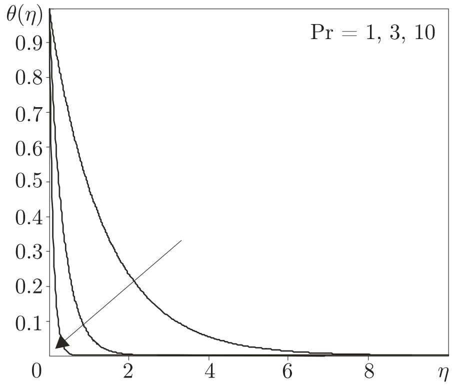

Figure 4. Effect of Pr on the temperature profiles across the boundary layer for $C = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60^\circ$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

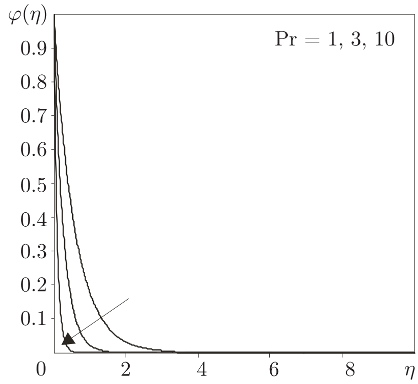

Figure 5. Effect of Pr on the concentration profiles across the boundary layer for $C = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60^\circ$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

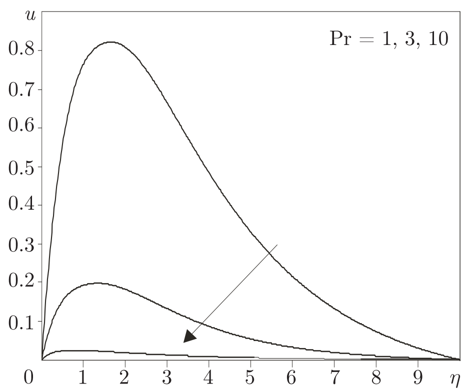

Figure 6. Effect of Pr on the velocity profiles across the boundary layer for $C = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60^\circ$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

⋄ *On the velocity.* Fig. 6 shows the effect of the Prandtl number Pr on the velocity profiles across the boundary layer. We observe that the velocity decreases monotonically with the increase of Pr.

4.3. Effect of the heat generation parameter He.

⋄ *On the temperature.* Fig. 7 shows the temperature profile for various values of He. It is observed that the fluid temperature increases with increase of He. This is expected since heat generation causes the thermal boundary layer to become thicker and the fluid become warmer. Also, it is clear that the temperature approaches zero faster for small values of He.

⋄ *On the concentration.* Fig. 8 shows the concentration profile for various values of He. It is observed that the fluid concentration decreases with the increase of He and it goes to minimum faster with a big value of He.

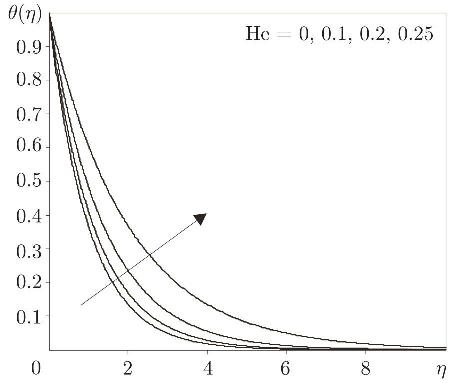

Figure 7. Effect of He on the temperature profiles across the boundary layer $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $\upsilon = 1$, $\tau = 1$, $n = 0.8$.

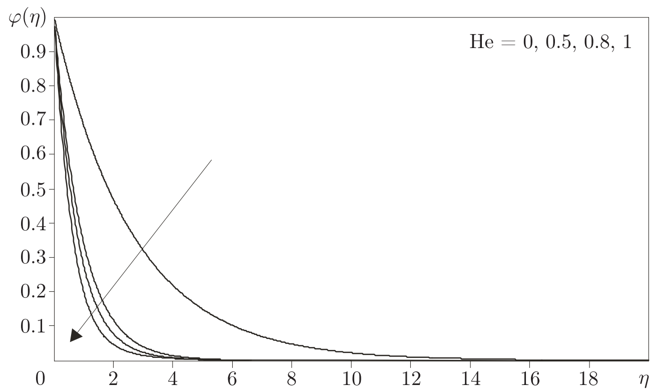

Figure 8. Effect of He on the dimensionless concentration profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $\upsilon = 1$, $\tau = 1$, $n = 0.8$.

4.4. Effect of the Schmidt number Sc. Fig. 9 illustrates the influence of Sc on the concentration profiles. By analogy with the Prandtl number Pr, the Schmidt number Sc is an important parameter in mass transfer processes as it characterizes the ratio of thicknesses of the viscous and concentration boundary layers. Its effect on the species concentration boundary-layer thickness has similarities to the Pr effect on the thermal boundary-layer thickness. That is, increases in the values of Sc cause the species concentration boundary layer thickness to decrease significantly.

4.5. Effect of the Grashof number Gc. Fig. 10 illustrates the influence of Gc on the velocity profiles across the boundary layer. Its effect on the velocity across the boundary-layer thickness is to increase the velocity by increasing the values of Gc. It is clear that the maximum value of the velocity at different values of Gc occurs at about $\eta = 1.5$.

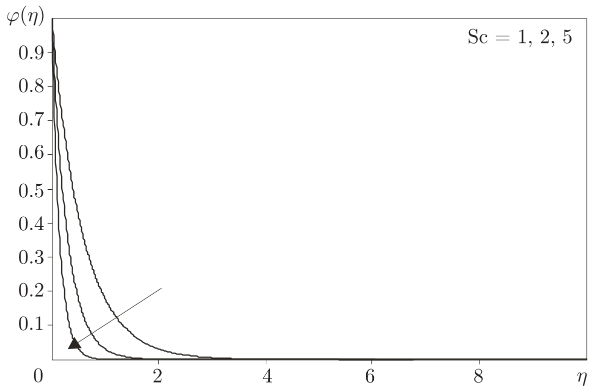

Figure 9. Effect of Sc on the dimensionless concentration profiles across the boundary layer $C = 1$, $\mathrm{Pr} = 1.0$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

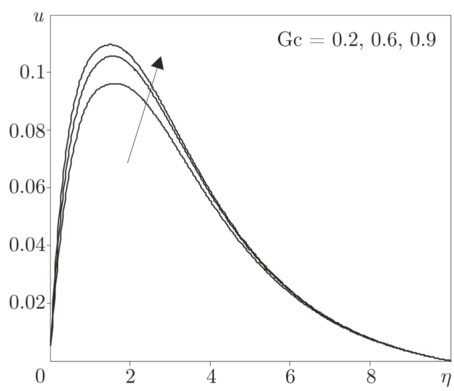

Figure 10. Effect of Gc on the velocity profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

4.6. Effect of the thermal Grashof number Gr. Fig. 11 illustrates the influence of Gr on the velocity profiles across the boundary layer. Its effect on the velocity across the boundary-layer thickness is to increase the velocity by increasing the values of Gr. It is clear that the maximum value of the velocity at different values of Gr occurs at about $\eta = 1.5$.

4.7. Effect of the inclination angle. Fig. 12 illustrates the effect of the inclination angle α on the velocity profiles across the boundary layer. The velocity is reduced significantly by the deviation of the plate from the vertical direction, i.e. by the increase of the angle.

4.8. Effect of the parameter C. Parameter C represents the component of velocity normal to surface of the plate in opposite direction, i.e., $C = -v$. Fig. 13 shows that increase of C causes the decrease of temperature across the boundary layer. Also the thermal boundary layer is decreased by the increase of C.

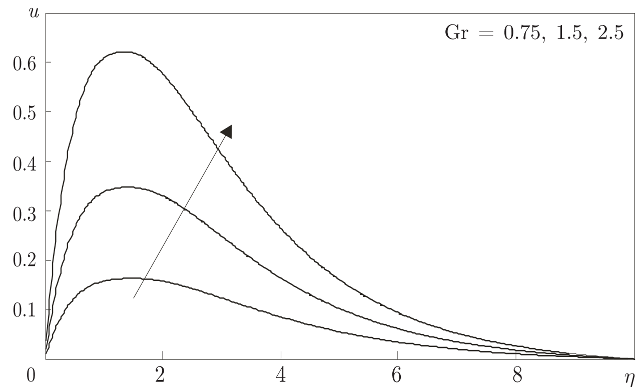

Figure 11. Effect of Gr on the velocity profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

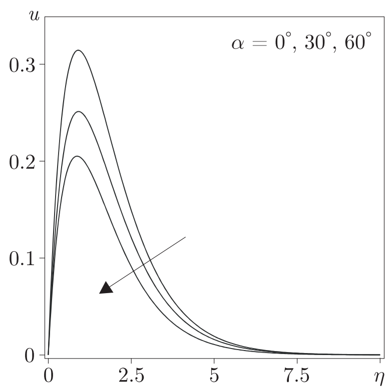

Figure 12. Effect of the inclination angle α on the temperature profiles across the boundary layer for $C = 1$, $\mathrm{Pr} = 1$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

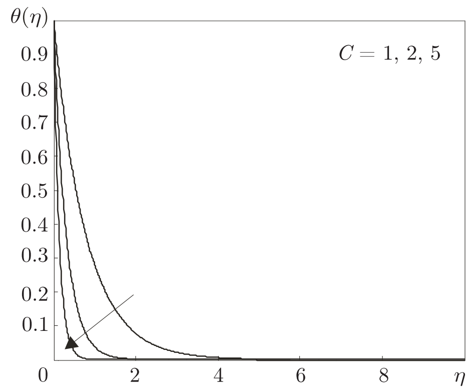

Figure 13. Effect of the constant C on the temperature profiles across the boundary layer for $\mathrm{Pr} = 2$, $\mathrm{He} = 0.2$, $\mathrm{Sc} = 0.22$, $\mathrm{Gr} = 0.9$, $\mathrm{Gc} = 1.0$, $\alpha = 60°$, $U_\infty = 1$, $v = 1$, $\tau = 1$, $n = 0.8$.

5 Conclusion and discussion

Lie group method is proved to be a useful approach for solving the two-dimensional boundary-layer flow of non-Newtonian power-law fluids and obtaining the velocity profiles for different cases. Through the application of the Lie method, we succeeded to study the effect of different physical and geometrical parameters such as: the Prandtl number Pr, the heat generation parameter He, the Schmidt number Sc, the solute Grashof number Gc, the thermal Grashof number Gr, the power law index n, the component of velocity normal to the surface $C = -v$, and the inclination angle α on the temperature, the concentration, and the component of velocity that is in direction of the plate, across the boundary layer thickness.

Increasing the value of Pr causes the fluid temperature and its boundary layer thickness to decrease significantly and consequently, the fluid velocity decreases. In addition, the concentration distribution inside the boundary layer also decreases. The fluid temperature increases with increase in the He, and the rate at which the temperature goes to zero is fast with a low value of He. The fluid concentration goes to maximum faster with a high value of He. Increasing values of Sc cause the species concentration boundary layer thickness to decrease significantly. The velocity component u increases with the increase of both Gc and Gr and decreases with the increase of α and C. The power law index n has two different effects on u, for the case of $n \leq 1$, the velocity u increases with the increase of n, while for $n \geq 1$, the velocity u decreases with the increase of n.

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Reduction operators of the linear rod equation

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We study reduction operators (called also nonclassical or conditional symmetries) of the (1+1)-dimensional linear rod equation. In particular, we prove and illustrate a new theorem on linear reduction operators of linear partial differential equations.

1 Introduction

For linear partial differential equations, there exist well-developed classical methods of their analytical solution, which, in particular, includes the separation of variables, different integral transforms, Fourier series and their generalizations. At the same time, the study of symmetry properties of such equations is important, first of all, for the development of methods of symmetry analysis itself.

In this paper we consider the (1+1)-dimensional constant-coefficient linear rod equation $u_{tt} + \lambda u_{xxxx} = 0$, where $\lambda > 0$, for unknown function u of the two independent variables t and x. This equation describes transverse vibrations of elastic rods. It is a special case of the Euler–Bernoulli beam equations, corresponding to constant values of parameters. Lie symmetries and the general equivalence problem for the class of Euler–Bernoulli beam equations were studied in [5, 6, 11]. By simple scaling of t or x, without loss of generality we can set $\lambda = 1$, i.e., it is sufficient to consider the equation

$$u_{tt} + u_{xxxx} = 0. \tag{1}$$

Some simple exact solutions of this equation are presented in [9, Section 9.2.2].[1] The maximal Lie invariance algebra of equation (1) is

$$\mathfrak{g} = \langle \partial_t,\ \partial_x,\ 2t\partial_t + x\partial_x,\ u\partial_u,\ h(t,x)\partial_u \rangle,$$

where $h = h(t,x)$ is an arbitrary solution of equation (1).

We study reduction operators (called also nonclassical or conditional symmetries) of the (1+1)-dimensional linear rod equation (1). First, in Section 2 we

[1] See also http://eqworld.ipmnet.ru/en/solutions/lpde/lpde501.pdf.

prove a theorem on linear reduction operators of general linear partial differential equations. This is why the notation in this section is different from the other part of the paper. The consideration of the next two sections illustrates both the statement and the proof of the theorem. The description of singular reduction operators of (1) in Section 3 is exhaustive. In contrast to this, only particular classes of regular reduction operators of (1) are found in Section 4. Possible generalizations of results obtained in the paper are discussed in the conclusion. We list interesting symmetry properties of equation (1) and additionally indicate the relation between the (1+1)-dimensional linear rod equation (1) and the (1+1)-dimensional free Schrödinger equation.

2 Linear reduction operators of linear equation

In order to present a theoretical background on reduction operators, based on [1–4, 10, 12], we first consider a general rth order differential equation $\mathcal{L}$ of the form $L(x, u_{(r)}) = 0$ for the unknown function u of the independent variables $x = (x_1, \dots, x_n)$. Here, $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to x of order not greater than r, including u as the derivative of order zero. Any vector field Q in the foliated space of the n independent variables x and the single dependent variable u takes the form

$$Q = \xi^i(x, u)\partial_i + \eta(x, u)\partial_u,$$

where the coefficients ξ^i and η are smooth functions of x and u. The first-order differential function $Q[u] = \eta - \xi^i u_i$ is called the characteristic of Q.

Here and in what follows the index i runs from 1 to n, and we use the summation convention for repeated indices, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N} \cup \{0\}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and δ_i is the multi-index whose ith entry equals 1 and whose other entries are zero. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. The variable u_α of the rth order jet space $J^r = J^r(x|u)$ corresponds to the derivative $\partial^{|\alpha|}u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, and $u_i \equiv u_{\delta_i}$. All considerations are in the local smooth setting. Then the equation $\mathcal{L}$ can be viewed as an algebraic equation in the jet space J^r and is identified with the manifold of its solutions in J^r:

$$\mathcal{L} = \{(x, u_{(r)}) \in J^r \mid L(x, u_{(r)}) = 0\}.$$

We use the same symbol $\mathcal{L}$ for this manifold and write $\mathcal{Q}_{(r)}$ for the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u] = 0$ in J^r, i.e.,

$$\mathcal{Q}_{(r)} = \{(x, u_{(r)}) \in J^r \mid D_1^{\alpha_1} \cdots D_n^{\alpha_n} Q[u] = 0, \ \ \alpha_i \in \mathbb{N} \cup \{0\}, \ \ |\alpha| < r\},$$

where $D_i = \partial_{x_i} + u_{\alpha+\delta_i}\partial_{u_\alpha}$ is the operator of total differentiation with respect to the variable x_i.

Definition 1. The differential equation $\mathcal{L}$ is called *conditionally invariant* with respect to the vector field Q if the relation $Q_{(r)}L(x,u_{(r)})|_{\mathcal{L}\cap\mathcal{Q}_{(r)}}=0$ holds. This relation is called the *conditional invariance criterion* [1–3, 12]. Then Q is called a *conditional symmetry* (or Q-conditional symmetry, or nonclassical symmetry, etc.) operator of the equation $\mathcal{L}$.

In this definition, $Q_{(r)}$ denotes the standard rth prolongation of Q [7, 8]:

$$Q_{(r)}=Q+\sum_{0<|\alpha|\leqslant r}\eta^\alpha\partial_{u_\alpha},\quad\text{where}\quad \eta^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}Q[u]+\xi^iu_{\alpha+\delta_i}.$$

The equation $\mathcal{L}$ is conditionally invariant with respect to the vector field Q if and only if an ansatz constructed with Q reduces $\mathcal{L}$ to a differential equation with $n-1$ independent variables [12]. Thus, we will briefly call a conditional symmetry operator of the equation $\mathcal{L}$ a *reduction operator* of this equation.

Reduction operators $\widetilde{Q}$ and Q are called *equivalent*, $\widetilde{Q}\sim Q$, if they differ by a multiplier which is a nonvanishing function of x and u: $\widetilde{Q}=\lambda Q$, where $\lambda=\lambda(x,u)\neq0$. Reduction operators Q and $\widetilde{Q}$ are called equivalent with respect to a group G of point transformations if there exists $g\in G$ for which the operators Q and $g_*\widetilde{Q}$ are equivalent, where g_* is the mapping induced by g on the set of vector fields.

Now consider an rth order linear differential equation $\mathcal{L}$ of the form

$$L[u]:=\sum_{|\alpha|\leqslant r}a^\alpha(x)u_\alpha=0$$

for the unknown function u of the independent variables $x=(x_1,\dots,x_n)$, where some coefficient a^α with $|\alpha|=r$ does not vanish.

Among Lie symmetries of linear differential equations, a distinguished role is played by symmetries associated with first-order linear differential operators acting on $u=u(x)$. If $n\geqslant2$ and $r\geqslant2$ or $n=1$ and $r\geqslant3$, the system of determining equations SDE($\mathcal{L}$) for the coefficients of vector fields from the maximal Lie invariance algebra $\mathfrak{g}^{\max}$ of $\mathcal{L}$ necessarily implies the equations $\xi^i_u=0$ and $\eta_{uu}=0$. In other words, any of such vector fields can be represented as

$$Q=\xi^i(x)\partial_i+(\eta^1(x)u+\eta^0(x))\partial_u,\tag{2}$$

and the system SDE($\mathcal{L}$) additionally gives that η^0 is an arbitrary solution of $\mathcal{L}$. The vector fields $\eta^0(x)\partial_u$, where η^0 runs through the set of solutions of the equation $\mathcal{L}$, form an ideal of the algebra $\mathfrak{g}^{\max}$ and generate point symmetries that are associated with the linear superposition principle. Up to the equivalence in $\mathfrak{g}^{\max}$ that is generated by adjoint actions of elements from the ideal, we can assume $\eta^0=0$ in (2) if at least one of the coefficients ξ^i or η^1 does not vanish.

The purpose of the further consideration in this section is to extend the last claim to reduction operators of the form (2), which will be called *linear reduction* *operators*. Note that general conditions when a linear differential equation admits only reduction operators which are equivalent to linear ones are not known.

Additionally recall that a vector field Q is called *(weakly) singular* for the differential equation $\mathcal{L}$: $L[u]=0$ if there exists a differential function $\tilde L=\tilde L[u]$ of an order less than r and a nonvanishing differential function $\lambda=\lambda[u]$ of an order not greater than r such that $L|_{\mathcal{Q}_{(r)}}=\lambda\tilde L|_{\mathcal{Q}_{(r)}}$. Otherwise Q is called a *(weakly) regular* vector field for $\mathcal{L}$. A vector field Q is *ultra-singular* for the equation $\mathcal{L}$ if this equation is satisfied by any solution of the characteristic equation $Q[u]:=\eta-\xi^iu_i=0$. See [1, 4] for theoretical background on singular reduction operators.

Theorem 1. *Let a linear partial differential equation $\mathcal{L}$ possess a reduction operator Q of the form* (2)*. Then the coefficient η^0 is represented as $\eta^0=\xi^i\zeta^0_i-\eta^1\zeta^0$, where $\zeta^0=\zeta^0(x)$ is a solution of $\mathcal{L}$. Hence, up to equivalence generated by action of the Lie symmetry group of $\mathcal{L}$ on the set of reduction operators of $\mathcal{L}$, the coefficient η^0 can be set equal to zero. Any vector field of the form $\xi^i\partial_i+(\eta^1u+\xi^i\zeta_i-\eta^1\zeta)\partial_u$, where $\zeta=\zeta(x)$ is an arbitrary solution of $\mathcal{L}$, is a reduction operator of $\mathcal{L}$.*

Proof. Since Q is a reduction operator, at least one of the coefficients ξ^i does not vanish. Consider the vector field $\hat Q=\xi^i(x)\partial_i+\eta^1(x)u\partial_u$. Let $X^1(x)$, …, $X^{n-1}(x)$ be functionally independent solutions of the equation $\xi^iv_i=0$, let $X^n(x)$ be a particular solution of the equation $\xi^iv_i=1$ and let $U(x)$ be a nonvanishing solution of the equation $\xi^iv_i+\eta^1v=0$. We introduce the notation $X=(X^1,\dots,X^n)$. Then the components of X and the function $U(x)u$ are functionally independent in total as functions of (x,u). This means that the change of variables $\mathcal{T}$: $\tilde x=X(x)$, $\tilde u=U(x)u$ is well defined.

We carry out this change of variables and represent all objects and relations in the new variables $(\tilde x,\tilde u)$. Thus, the vector field $\hat Q$ coincides with the generator of shifts with respect to the variable $\tilde x_n$, $\hat Q=\partial_{\tilde x_n}$, and hence $Q=\partial_{\tilde x_n}+\tilde\eta^0(\tilde x)\partial_{\tilde u}$, where $\tilde\eta^0(\tilde x)=U(x)\eta^0(x)$. Then the characteristic equation associated with the vector field Q in the new variables is $\tilde u_{\tilde x_n}=\tilde\eta^0$. The change of variables $\mathcal{T}$ also preserves the linearity of the equation $\mathcal{L}$, which takes the form

$$\tilde L[\tilde u]=\sum_{|\alpha|\leqslant r}\tilde a^\alpha(\tilde x)\tilde u_\alpha=0, \tag{3}$$

where each coefficient $\tilde a^\alpha$ are expressed in terms of the coefficients $a^{\alpha'}$, $|\alpha'|\geqslant|\alpha|$, and derivatives of X^i and U. The variable $\tilde u_\alpha$ of the jet space J^r corresponds to the derivative $\partial^{|\alpha|}\tilde u/\partial\tilde x_1^{\alpha_1}\dots\partial\tilde x_n^{\alpha_n}$. Up to nonvanishing multiplier, a coefficient $\tilde a^{\alpha^0}$, where $|\alpha^0|=r$, can be assumed to be identically equal to 1.

We denote an antiderivative of $\tilde\eta^0$ with respect to $\tilde x_n$ by $\tilde\zeta^0$,

$$\tilde\eta^0=\tilde\zeta^0_{\tilde x_n}.$$

We separately consider two cases depending on whether or not the reduction operator Q is ultra-singular for $\mathcal{L}$, and show that in each of these cases there exists an antiderivative $\tilde{\zeta}^0$ of $\tilde{\eta}^0$ satisfying the representation (3) of the equation $\mathcal{L}$ in the new variables, $\tilde{L}[\tilde{\zeta}^0]=0$.

Suppose that the reduction operator Q is ultra-singular for $\mathcal{L}$. As the property of ultra-singularity is not affected by changes of variables, this means that the representation $\tilde{L}[\tilde{u}]=0$ of the equation $\mathcal{L}$ in the new variables is satisfied by any solution of the characteristic equation $\tilde{u}_{\tilde{x}_n}=\tilde{\eta}^0$, i.e.,

$$\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha\tilde{\eta}^0_{\alpha-\delta_n}+\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde{a}^\alpha\tilde{u}_\alpha=0,$$

where the derivatives $\tilde{u}_\alpha$ with $\alpha_n=0$ are not constrained. Splitting with respect to them, we obtain the system of equations $\tilde{a}^\alpha=0$ for α running the set of multi-indices with $|\alpha|\leqslant r$ and $\alpha_n=0$ and an equation for the coefficient $\tilde{\eta}^0$,

$$\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha\tilde{\eta}^0_{\alpha-\delta_n}:=\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha\tilde{\zeta}^0_\alpha=0.$$

So, the summation in equation (3) is in fact for the values of the multi-index α with $\alpha_n\neq 0$ and hence the function $\tilde{\zeta}^0$ satisfies this equation.

Suppose that the reduction operator Q is not ultra-singular for $\mathcal{L}$. As the rth prolongation of Q is given by $Q_{(r)}=\partial_{\tilde{x}_n}+\sum_{|\alpha|\leqslant r}\tilde{\eta}^0_\alpha(\tilde{x})\partial_{\tilde{u}_\alpha}$, the conditional invariance criterion implies for this case that

$$Q_{(r)}\tilde{L}[\tilde{u}]=\sum_{|\alpha|\leqslant r}(\tilde{a}^\alpha_{\tilde{x}_n}\tilde{u}_\alpha+\tilde{a}^\alpha\tilde{\eta}^0_\alpha)=0 \tag{4}$$

for all points of the jet space J^r where $\tilde{L}[\tilde{u}]=0$ and $\tilde{u}_{\alpha'}=\tilde{\eta}^0_{\alpha'-\delta_n}$ with $|\alpha'|\leqslant r$ and $\alpha_n>0$. As $\tilde{a}^{\alpha^0}=1$, the differential function $Q_{(r)}\tilde{L}[\tilde{u}]$ does not depend on the derivative $\tilde{u}_{\alpha^0}$. Hence the constraint $\tilde{L}[\tilde{u}]=0$ is not essential in the course of confining to the manifold $\mathcal{L}\cap\mathcal{Q}_{(r)}$. The derivatives $\tilde{u}_\alpha$ with $\alpha_n=0$ are not constrained. Splitting with respect to them in (4) gives the system of equations $\tilde{a}^\alpha_{\tilde{x}_n}=0$ for α running the set of multi-indices with $|\alpha|\leqslant r$ and $\alpha_n=0$ as a necessary condition for the equation $\mathcal{L}$ to admit the reduction operator Q. Then on the manifold $\mathcal{Q}_{(r)}$ we get

$$\begin{aligned}
Q_{(r)}\tilde{L}[\tilde{u}]&=\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde{a}^\alpha_{\tilde{x}_n}\tilde{u}_\alpha+\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha_{\tilde{x}_n}\tilde{u}_\alpha+\sum_{|\alpha|\leqslant r}\tilde{a}^\alpha\tilde{\eta}^0_\alpha\\
&=\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha_{\tilde{x}_n}\tilde{\eta}^0_{\alpha-\delta_n}+\sum_{|\alpha|\leqslant r}\tilde{a}^\alpha\tilde{\eta}^0_\alpha\\
&=\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde{a}^\alpha_{\tilde{x}_n}\tilde{\zeta}^0_\alpha+\sum_{|\alpha|\leqslant r,\alpha_n\neq 0}\tilde{a}^\alpha_{\tilde{x}_n}\tilde{\zeta}^0_\alpha+\sum_{|\alpha|\leqslant r}\tilde{a}^\alpha\tilde{\zeta}^0_{\alpha+\delta_n}\\
&=\left(\sum_{|\alpha|\leqslant r}\tilde{a}^\alpha\tilde{\zeta}^0_\alpha\right)_{\tilde{x}_n}=0.
\end{aligned}$$

The integration of the last equality with respect to $\tilde{x}_n$ gives that the function $\tilde{\zeta}^0=\tilde{\zeta}^0(x)$ satisfies the inhomogeneous linear equation

$$\tilde{L}[\tilde{\zeta}^0]:=\sum_{|\alpha|\leqslant r}\tilde{a}^\alpha\tilde{\zeta}^0_\alpha=g(x_1,\dots,x_{n-1}) \tag{5}$$

for some smooth function $g=g(x_1,\dots,x_{n-1})$. As in this case the reduction operator Q is not ultra-singular for $\mathcal{L}$, there exists the multi-index α with $|\alpha|\leqslant r$ and $\alpha_n=0$ such that $\tilde{a}^\alpha\neq 0$. Hence equation (5) has a particular solution h that does not depend on $\tilde{x}_n$, $h=h(x_1,\dots,x_{n-1})$.[2] The function $\tilde{\zeta}^0-h$ is also an antiderivative of $\tilde{\eta}^0$ with respect to $\tilde{x}_n$ and, at the same time, it satisfies the corresponding homogeneous linear equation, $\tilde{L}[\tilde{\zeta}^0-h]=0$. Therefore, without loss of generality we can assume that the antiderivative $\tilde{\zeta}^0$ itself is a solution of equation (3), $\tilde{L}[\tilde{\zeta}^0]=0$.

We carry out the inverse change of the variables in the equality $\tilde{\eta}^0=\tilde{\zeta}^0_{\tilde{x}_n}=\hat{Q}\tilde{\zeta}^0$ and introduce the function $\zeta^0=\tilde{\zeta}^0/U$, which satisfies the equation $\mathcal{L}$ in the old variables (x,u). We have $U\eta^0=\xi^i(U\zeta^0)_i=U\xi^i\zeta^0_i+(\xi^iU_i)\zeta^0=U(\xi^i\zeta^0_i-\eta^1\zeta^0)$, i.e., $\eta^0=\xi^i\zeta^0_i-\eta^1\zeta^0$. Here we use that $\xi^iU_i=-\eta^1U$. The mapping generated by the point symmetry transformation $\bar{x}=x$, $\bar{u}=u-\zeta^0(x)$ of $\mathcal{L}$ on the set of reduction operators of $\mathcal{L}$ maps the vector field Q to the vector field $\hat{Q}$, for which the coefficient η^0 is zero. This means that $\hat{Q}$ is a reduction operator of $\mathcal{L}$. Applying the similar mapping generated by the point symmetry transformation $\bar{x}=x$, $\bar{u}=u+\zeta(x)$ with an arbitrary solution $\zeta=\zeta(x)$ of $\mathcal{L}$, we obtain that any vector field of the form $\xi^i\partial_i+(\eta^1u+\xi^i\zeta_i-\eta^1\zeta)\partial_u$ is a reduction operator of $\mathcal{L}$. □

An ansatz constructed for the unknown function u with the vector field Q is

$$u=\frac{1}{U(x)}\varphi(\omega_1,\dots,\omega_{n-1})+\zeta^0(x),$$

where φ is the invariant dependent variable, $\omega_1=X^1(x)$, …, $\omega_{n-1}=X^{n-1}(x)$ are invariant independent variables, and we use the notation from the proof of the theorem. The corresponding reduced equation is

$$\sum_{|\alpha|\leqslant r,\alpha_n=0}\tilde{a}^\alpha(\omega_1,\dots,\omega_{n-1})\frac{\partial^{|\alpha|}\varphi}{\partial\omega_1^{\alpha_1}\dots\partial\omega_{n-1}^{\alpha_{n-1}}}=0.$$

It is obvious that the form of the reduced equation does not depend on the parameter-function $\zeta^0(x)$. The substitution of an arbitrary solution of $\mathcal{L}$ instead of $\zeta^0(x)$ gives the same reduced equation.

[2]If $n>2$, then for the guaranteed existence of such a classical solution we suppose that all functions are analytical. In the case $n=2$ or for specific linear equations the requested smoothness of functions can be lowered.

3 Singular reduction operators of the rod equation

For the linear rod equation (1), i.e., $\mathcal{L}$: $u_{tt} + u_{xxxx} = 0$, the general form of reduction operators is

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

where the coefficients τ, ξ and η are smooth functions of (t,x,u) with $(\tau,\xi) \neq (0,0)$. Similarly to the evolution equations, a vector field Q is singular for the linear rod equation (1) if and only if the coefficient τ identically vanishes. Note that vector fields that are weakly singular for this equation are also strongly singular for it. Then $\xi \neq 0$ and hence up to usual equivalence of reduction operators we can set $\xi = 1$. In other words, for the exhaustive study of singular reduction operators of the linear rod equation (1) it suffices to consider vector fields of the form

$$Q = \partial_x + \eta(t,x,u)\partial_u.$$

The manifold $\mathcal{L} \cap \mathcal{Q}_{(4)}$ is defined by the equations

$$u_x = \eta, \quad u_{xx} = \eta_x + \eta\eta_u, \quad u_{xxx} = (\partial_x + \eta\partial_u)^2\eta, \quad u_{xxxx} = (\partial_x + \eta\partial_u)^3\eta,$$
$$u_{tt} = -u_{xxxx} = -(\partial_x + \eta\partial_u)^3\eta.$$

Hence the conditional invariance criterion implies that

$$\eta_{tt} + 2\eta_{tu}u_t + \eta_{uu}u_t^2 - \eta_u(\partial_x + \eta\partial_u)^3\eta + (\partial_x + \eta\partial_u)^4\eta = 0.$$

Collecting coefficients of different powers of the unconstrained derivative u_t and splitting with respect to it, we derive the system of three determining equations for the coefficient η:

$$\eta_{uu} = 0, \quad \eta_{tu} = 0, \quad \eta_{tt} - \eta_u(\partial_x + \eta\partial_u)^3\eta + (\partial_x + \eta\partial_u)^4\eta = 0.$$

Thus, in contrast to a (1+1)-dimensional evolution equation, where there is a single determining equation for the coefficient η of singular reduction operators and this equation is reduced, in a certain sense, to the evolution equation under consideration, finding singular reduction operators of the linear rod equation is not a no-go problem. The equations $\eta_{uu} = 0$ and $\eta_{tu} = 0$ give the expression

$$\eta = \eta^1(x)u + \eta^0(t,x)$$

for the coefficient η, where $\eta^1 = \eta^1(x)$ and $\eta^0 = \eta^0(t,x)$ are smooth functions of their variables. Theorem 1 implies that, up to equivalence generated by the maximal Lie symmetry group $G^{\max}$ of the linear rod equation on the set of reduction operators of this equation, we can set $\eta^0 = 0$. We also show this directly.

After substituting the expression for η into the last determining equation, we can additionally split with respect to u to obtain

$$\partial_x(\partial_x+\eta^1)^3\eta^1=0,\quad \eta^0_{tt}-\eta^1\eta^{03}+\eta^{04}=0,$$

where the functions η^{03} and η^{04} are defined by the recurrent relation $\eta^{00}:=\eta^0$ and $\eta^{0k}=\eta^{0,k-1}_x+\eta^0(\partial_x+\eta^1)^{k-1}\eta^1$, $k=1,2,3,4$. We make the differential substitution

$$\eta^1=\frac{\theta_x}{\theta},\quad \eta^0=\zeta_x-\frac{\theta_x}{\theta}\zeta,$$

where $\theta=\theta(x)$ and $\zeta=\zeta(t,x)$ are the new unknown functions. It is possible to show by induction that

$$\eta^{0k}=\frac{\partial^{k+1}\zeta}{\partial x^{k+1}}-\frac{\zeta}{\theta}\frac{d^{k+1}\theta}{dx^{k+1}},\quad k=1,2,\dots .$$

Hence the differential substitution reduces the system for η^1 and η^0 to a system for θ and ζ,

$$\left(\frac{\theta_{xxxx}}{\theta}\right)_x=0,\quad \zeta_{ttx}-\frac{\theta_x}{\theta}\zeta_{tt}-\frac{\theta_x}{\theta}\zeta_{xxxx}+\frac{\theta_x\theta_{xxxx}}{\theta^2}\zeta+\zeta_{xxxxx}-\frac{\theta_{xxxxx}}{\theta}\zeta=0.$$

Integrating once the first equation, we get the constant-coefficient linear ordinary differential equation $\theta_{xxxx}=\kappa\theta$, where κ is the integration constant. The second equation can be represented as

$$\left(\frac{\zeta_{tt}+\zeta_{xxxx}}{\theta}\right)_x-\left(\frac{\theta_{xxxx}}{\theta}\right)_x\zeta=0,\quad \text{hence}\quad \left(\frac{\zeta_{tt}+\zeta_{xxxx}}{\theta}\right)_x=0.$$

The integration of the last equation with respect to x results in the equation $\zeta_{tt}+\zeta_{xxxx}=\rho(t)\theta$, where ρ is a smooth function of t. The function ζ is defined up to the transformation $\tilde\zeta=\zeta+\sigma\theta$, where σ is an arbitrary smooth function of t. This transformation allows us to set $\rho=0$. Indeed, $\tilde\zeta_{tt}+\tilde\zeta_{xxxx}=\rho\theta+\sigma_{tt}\theta+\sigma\kappa\theta=0$ if $\sigma_{tt}+\kappa\sigma=-\rho$. In other words, we can assume that the function ζ satisfies the linear rod equation (1). Then the mapping generated by the point symmetry transformation $\bar t=t$, $\bar x=x$, $\bar u=u-\zeta(t,x)$ of equation (1) on the set of reduction operators of this equation maps the vector field Q to the vector field of the same form, where $\zeta=0$ and hence $\eta^0=0$.

Proposition 1. *Up to equivalence generated by symmetry transformations of linear superposition, the set of singular reduction operators of the linear rod equation* (1) *is exhausted by the vector fields of the form*

$$Q_{\rm s}=\partial_x+\frac{\theta_x}{\theta}u\partial_u,$$

where the function $\theta=\theta(x)$ satisfies the ordinary differential equation $\theta_{xxxx}=\kappa\theta$ for some constant κ.

An ansatz constructed with the reduction operator Q is $u=\theta(x)\varphi(\omega)$, where $\omega=t$ is the invariant independent variable and φ is the invariant dependent variable. The corresponding reduced equation is $\varphi_{\omega\omega}+\kappa\varphi=0$. As an interpretation, we can say that the reduction operator $Q_{\rm s}$ is related to separation of variables in the linear rod equation (1). It is obvious that the reduction operator $Q_{\rm s}$ is equivalent to a Lie symmetry operator only if $\theta_x/\theta=\mathrm{const}$.

4 Regular reduction operators of the rod equation

Consider regular reduction operators of the linear rod equation (1), for which the coefficient τ does not vanish. Up to usual equivalence of reduction operators we can set $\tau=1$, i.e., it suffices to consider vector fields of the form

$$Q=\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u.$$

Essential among the equations defining the manifold $\mathcal{L}\cap\mathcal{Q}_{(4)}$ are the equations

$$\begin{aligned}&u_t=\eta-\xi u_x,\quad u_{tx}=\eta_x+\eta u_x-\xi_x u_x-\xi_u u_x^2-\xi u_{xx},\\ &u_{tt}=-u_{xxxx}=\eta_t+\eta_u(\eta-\xi u_x)-(\xi_t+\xi_u(\eta-\xi u_x))u_x\\ &\qquad -\xi(\eta_x+\eta u_x-\xi_x u_x-\xi_u u_x^2-\xi u_{xx}).\end{aligned}$$

Collecting coefficients of $u_{xx}u_{xxx}$ in the condition following from the conditional invariance criterion, we obtain the equation $\xi_u=0$. Other terms with u_{xxx} give the equations $\eta_{uu}=0$ and $\eta_{xu}=\frac32\xi_{xx}$. Therefore, we have

$$\xi=\xi(t,x),\quad \eta=\eta^1(t,x)u+\eta^0(t,x),\quad \text{where}\quad \eta^1:=\frac32\xi_x+\gamma(t)$$

with a smooth function $\gamma=\gamma(t)$. The other determining equations reduce to

$$2\xi_t\xi+5\xi_{xxx}+4\xi^2\xi_x=0, \tag{6}$$
$$\xi_{tt}+\xi_{xxxx}+2(\eta^1\xi)_t+2\xi_t\xi_x-4\eta^1_{xxx}+8\xi\xi_x\eta^1-4\xi\xi_x{}^2=0, \tag{7}$$
$$\eta^1_{tt}+\eta^1_{xxxx}+2\eta^1\eta^1_t-2\xi_t\eta^1_x+4\xi_x(\eta^1_t+\eta^1\eta^1-\xi\eta^1_x)=0, \tag{8}$$
$$\eta^0_{tt}+\eta^0_{xxxx}+2\eta^0\eta^1_t-2\xi_t\eta^0_x+4\xi_x(\eta^0_t+\eta^1\eta^0-\xi\eta^0_x)=0, \tag{9}$$

where every appearance of η^1 should be replaced by $\frac32\xi_x+\gamma(t)$.

Similarly to singular reduction operators, Theorem 1 again implies that, up to equivalence generated by the maximal Lie symmetry group $G^{\max}$ of the linear rod equation on the set of reduction operators of this equation, we can set $\eta^0=0$. We show that the direct proof of this fact is not trivial. Indeed, let the function ζ be defined by the relation $\eta^0=\zeta_t+\xi\zeta_x-\eta^1\zeta$. As it is a first-order quasi-linear partial differential equation with respect to ζ, such a function ζ exists. We use this relation to substitute for η^0 into equation (9). Taking into account equations (6)–(8) and $\eta_{xu}=\frac32\xi_{xx}$, we derive the following equation for the function ζ:

$$(\partial_t+\xi\partial_x-\eta^1+4\xi_x)(\zeta_{tt}+\zeta_{xxxx})=0,$$

i.e., $\zeta_{tt}+\zeta_{xxxx}=h(t,x)$, where the function $h=h(t,x)$ satisfies the equation

$$h_t+\xi h_x+(-\eta^1+4\xi_x)h=0.$$

The function $h=h(t,x)$ can be set to zero. Indeed, the function ζ is defined up to summand that is a solution of the equation $g_t+\xi g_x-\eta^1 g=0$. Any such solution is represented as $g=g^0(t,x)\varphi(\omega)$, where g^0 is a fixed solution of the same equation, φ ia an arbitrary function of ω, and $\omega=\omega(t,x)$ is a nonconstant solution of the equation $\omega_t+\xi\omega_x=0$. Then $\chi={\omega_x}^4$ satisfies the equation

$$\chi_t+\xi\chi_x+4\xi_x\chi=0.$$

Therefore, the function h possesses the representation $h=g^0{\omega_x}^4\psi(\omega)$ for some smooth function ψ of ω. The above determining equations imply that the vector field $\partial_t+\xi\partial_x+\eta^1u\partial_u$ is a reduction operator for the equation $u_{tt}+u_{xxxx}=0$. Hence we have

$$g_{tt}+g_{xxxx}=g^0{\omega_x}^4\varphi_{\omega\omega\omega\omega}+\cdots=g^0{\omega_x}^4(\varphi_{\omega\omega\omega\omega}+\cdots),$$

where the expression in the brackets depends merely on ω and the dots denote terms including derivatives of φ of orders less than four. This means that the ansatz $g=g^0(t,x)\varphi(\omega)$ reduces the equation $g_{tt}+g_{xxxx}=h$ to the ordinary differential equation $\varphi_{\omega\omega\omega\omega}+\cdots=\psi$, which definitely has a solution $\varphi^0=\varphi^0(\omega)$. Subtracting the corresponding function $g=g^0\varphi^0$ from the function ζ, we annihilate the function h.

Therefore, without loss of generality we can assume that the function ζ satisfies the initial equation (1). Then the mapping generated by the point symmetry transformation $\bar t=t$, $\bar x=x$, $\bar u=u-\zeta(t,x)$ of (1) on the set of reduction operators of (1) maps the vector field Q to the vector field of the same form, where $\zeta=0$ and hence $\eta^0=0$.

As a result, the study of regular reduction operators of the linear rod equation (1) reduces to the solution of the overdetermined system of nonlinear differential equations (6)–(8) for the functions $\xi=\xi(t,x)$ and $\gamma=\gamma(t)$. (Recall that $\eta^1:=\frac{3}{2}\xi_x+\gamma(t)$.) This solution appears an unexpectedly complicated problem. Hence we have considered particular cases of regular reduction operators by imposing additional constraints on the functions ξ and γ. Thus, cumbersome and tricky computations with `Maple` show that any regular reduction operator of (1) with $\gamma=0$ is equivalent to a Lie symmetry operator of this equation. The same result is true under the assumption $\xi_{xx}=0$ and $\xi\neq0$. There are no regular reduction operators with $\xi_t=0$ and $\xi_x\neq0$.

Suppose that $\xi=0$. Then equations (6) and (7) are identically satisfied and the coefficient η^1 is represented as $\eta^1=\gamma(t)$. Equation (8) implies the single ordinary differential equation $\gamma_{tt}+2\gamma\gamma_t=0$ for the function γ, which is once integrated to $\gamma_t+\gamma^2=-\kappa$, where κ is the integration constant. Hence the function γ admits the representation $\gamma=\varphi_t/\varphi$, where the function $\varphi=\varphi(t)$ is a solution of the linear ordinary differential equation $\varphi_{tt} + \kappa\varphi = 0$. The corresponding reduction operator

$$Q_\mathrm{r} = \partial_t + \frac{\varphi_t}{\varphi} u\partial_u,$$

results in the ansatz $u = \varphi(t)\theta(\omega)$, where $\omega = x$ is the invariant independent variable and θ is the invariant dependent variable. The corresponding reduced equation is $\theta_{\omega\omega\omega\omega} = \kappa\theta$. Therefore, similarly to the singular reduction operator Q_s from Proposition 1 the regular reduction operator Q_r is related to separation of variables in the linear rod equation (1). This operator can be considered as a regular counterpart of the operator Q_s. The reduction operator Q_r is equivalent to a Lie symmetry operator only if $\varphi_t/\varphi = \mathrm{const}$.

5 Conclusion

In spite of the rod equation (1) is linear and has only obvious Lie symmetries, it is interesting from the symmetry point of view since it possesses a number of nontrivial properties related to the field of symmetry analysis. We list five of these properties:

- Equation (1) possesses both regular and singular nonclassical symmetries which are inequivalent to Lie symmetries and associated with separation of variables.
- A potential system of the rod equation (1) coincides with the (1+1)-dimensional free Schrödinger equation. Hence equation (1) possesses purely potential and nonclassical potential symmetries.
- A function is a solution of the rod equation (1) if and only if it is the real (resp. imagine) part of a solution of the (1+1)-dimensional free Schrödinger equation. This allows us to construct new families of exact solutions of (1) in an easy way.
- Equation (1) has a nonlocal recursion operator whose action on local symmetries (which necessarily are affine in derivatives of u) gives nontrivial local symmetries of higher order. As a result, for arbitrary fixed order, excluding order two, this equation possesses local symmetries of this order which do not belong to the enveloping algebras of local symmetries of lower orders.
- As the linear differential operator associated with (1) is formally self-adjoint, the space of cosymmetries and the space of characteristics of local symmetries coincides. This implies that equation (1) has conservation laws of arbitrarily high order.

A detail discussion of these properties will be a subject of a forthcoming paper. In the present paper, we have studied the first property and below we briefly present the next two properties.

The linear differential operator $L := \partial_t^2 + \partial_x^4$ associated with equation (1) is factorized to the product of the free Schrödinger operator and its formal adjoint:

$$L = (i\partial_t + \partial_x^2)(-i\partial_t + \partial_x^2).$$

This indicates that the solution of (1) is closely connected with the solution of the free (1+1)-dimensional Schrödinger equation

$$i\psi_t + \psi_{xx} = 0. \tag{10}$$

To make this connection explicit, we consider the potential system constructed for equation (1) with the conservation law having the characteristic 1:

$$v_x = u_t, \quad v_t = -u_{xxx}. \tag{11}$$

The second equation of (11) is in conserved form that allows us to introduce the potential w satisfying the conditions

$$w_x = v, \quad w_t = -u_{xx}. \tag{12}$$

Excluding v from the joint system of (11) and (12), we obtain the system

$$u_t = w_{xx}, \quad w_t = -u_{xx}. \tag{13}$$

The maximal Lie invariance algebra of system (13) is

$$\begin{aligned}\mathfrak{g}_1 = \langle &\partial_t,\, \partial_x,\, 2t\partial_t + x\partial_x,\, w\partial_u - u\partial_w,\, 2t\partial_x + xw\partial_u - xu\partial_w,\\ &4t^2\partial_x + 4tx\partial_x + \big(x^2w - 2tu\big)\partial_u - \big(x^2u + 2tw\big)\partial_w,\\ &u\partial_u + w\partial_w,\, \beta(t,x)\partial_u + \gamma(t,x)\partial_w\rangle,\end{aligned} \tag{14}$$

where $(\beta(t,x), \gamma(t,x))$ is an arbitrary solution of system (13).

System (13) implies that the complex-valued function $\psi = w + iu$ of the variables t and x satisfies equation (10) and the function w is a solution of equation (1). Finally, we have the following simple assertion.

Proposition 2. *The function $u = u(t,x)$ is a solution of equation* (1) *if and only if it is the real* (*resp. imagine*) *part of a solution of the (1+1)-dimensional free Schrödinger equation $i\psi_t + \psi_{xx} = 0$.*

A fixed solution of equation (1) corresponds to a set of solutions of equation (10) which differ by summands of the form $C_1x + C_0$, where C_0 and C_1 are arbitrary real constants. As wide families of exact solutions of equation (10) are already known, Proposition 2 gives the simplest way of finding exact solutions for equation (1).

In fact, the main result of the paper is Theorem 1 on single linear reduction operators of general linear partial differential equations. The next step is to extend this assertion to multidimensional reduction modules that are generated by linear vector fields.

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Lotka–Volterra Systems Associated with Graphs

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For each connected graph we associate a family of Lotka–Volterra systems. In particular we examine a class of Lotka–Volterra systems associated with complex simple Lie algebras. In the case of ADE type Lie algebras we present two approaches to constructing these systems.

1 Introduction

For each graph there exists a Lotka–Volterra system which is constructed in a simple way. We may restrict our attention to the case of connected graphs. When the graph is not connected the corresponding Lotka–Volterra system breaks up into smaller systems associated to the components of the graph. Equivalently, the Poisson structure is a direct product of Poisson structures of smaller dimension. With some minor exceptions, the Poisson structure of the systems we consider are quadratic with entries which are homogeneous with non-zero coefficients ± 1. We begin with the well-known case of Dynkin diagrams and we construct a family of Lotka–Volterra systems associated with complex simple Lie algebras.

The Volterra model, also known as KM system is a well-known integrable system defined by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}), \qquad i = 1, 2, \ldots, n, \tag{1}$$

where $x_0 = x_{n+1} = 0$. It was studied originally by Volterra in [15] to describe population evolution in a hierarchical system of competing species. It was first solved by Kac and van Moerbeke in [10], using a discrete version of inverse scattering due to Flaschka [7]. In [13] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (1) can be considered as a finite-dimensional approximation of the Korteweg–de Vries equation. The Volterra system is associated with a simple Lie algebra of type A_n. Bogoyavlenskij generalized this system for each simple Lie algebra and showed that the corresponding systems are also integrable. See [1, 2] for more details.

The Hamiltonian description of system (1) can be found in [6] and [4]. The Lax pair in [4] is given by

$$\dot{L} = [B, L],$$

where

$$L = \begin{pmatrix} x_1 & 0 & \sqrt{x_1x_2} & 0 & \dots & & 0 \\ 0 & x_1{+}x_2 & 0 & \sqrt{x_2x_3} & & & \vdots \\ \sqrt{x_1x_2} & 0 & x_2{+}x_3 & & \ddots & & \\ 0 & \sqrt{x_2x_3} & & & & & \\ \vdots & & \dots & & & & \sqrt{x_{n-1}x_n} \\ & & & & & x_{n-1}{+}x_n & 0 \\ & & & & \sqrt{x_{n-1}x_n} & 0 & x_n \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{\sqrt{x_1x_2}}{2} & 0 & \dots & & 0 \\ 0 & 0 & 0 & \frac{\sqrt{x_2x_3}}{2} & & & \vdots \\ -\frac{\sqrt{x_1x_2}}{2} & 0 & 0 & & \ddots & & \\ 0 & -\frac{\sqrt{x_2x_3}}{2} & & & & & \\ \vdots & & \dots & & & & \frac{\sqrt{x_{n-1}x_n}}{2} \\ & & & & & 0 & 0 \\ & & & & -\frac{\sqrt{x_{n-1}x_n}}{2} & 0 & 0 \end{pmatrix}.$$

Due to the Lax pair, it follows that the functions $H_i = \frac{1}{i}\operatorname{tr} L^i$ are constants of motion. Following [4] we define the following quadratic Poisson bracket,

$$\{x_i, x_{i+1}\} = x_i x_{i+1},$$

and all other brackets equal to zero. For this bracket $\det(L)$ is a Casimir and the eigenvalues of L are in involution. Of course, the functions H_i are also in involution. Taking the function $\sum_{i=1}^n x_i$ as the Hamiltonian we obtain equations (1). This bracket can be realized from the second Poisson bracket of the Toda lattice by setting the momentum variables equal to zero [6].

There is another Lax pair where L is in the nilpotent subalgebra corresponding to the negative roots. The Lax pair is of the form $\dot{L} = [L, B]$, where

$$L = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ x_1 & 0 & 1 & \ddots & & \vdots \\ 0 & x_2 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & x_n & 0 \end{pmatrix}, \tag{2}$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ x_1x_2 & 0 & 0 & \ddots & & \vdots \\ \vdots & x_2x_3 & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & x_{n-1}x_n & 0 & 0 \end{pmatrix}.$$

Finally, there is also a symmetric version due to Moser:

$$L = \begin{pmatrix} 0 & u_1 & 0 & \cdots & \cdots & 0 \\ u_1 & 0 & u_2 & \ddots & & \vdots \\ 0 & u_2 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & u_n \\ 0 & \cdots & \cdots & 0 & u_n & 0 \end{pmatrix}, \tag{3}$$

and

$$B = \begin{pmatrix} 0 & 0 & u_1u_2 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ u_1u_2 & 0 & 0 & \ddots & u_2u_3 & \vdots \\ \vdots & u_2u_3 & \ddots & \ddots & & u_{n-1}u_n \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & u_{n-1}u_n & 0 & 0 \end{pmatrix}.$$

The change of variables $x_i = 2u_i^2$ gives equations (1). The existence of these three Lax pairs implies that the open KM-system is Liouville integrable.

It is evident from the form of L in the various Lax pairs, that the position of the variables x_i correspond to the simple root vectors of a root system of type A_n. On the other hand the position of the variables in the matrix B is at the position corresponding to the sum of two simple roots α_i and α_j. In this paper we will generalize this construction for each complex simple Lie algebra.

2 Lotka–Volterra systems

The KM-system is a special case of the so called Lotka–Volterra systems. The most general form of the equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \qquad i = 1, 2, \ldots, n.$$

We may assume that there are no linear terms ($\varepsilon_i = 0$). We also assume that the matrix $A = (a_{ij})$ is skew-symmetric. The associated Poisson bracket for the Lotka–Volterra system is defined by

$$\{x_i, x_j\} = a_{ij} x_i x_j, \qquad i, j = 1, 2, \ldots, n. \tag{4}$$

The system is Hamiltonian with Hamiltonian function

$$H = x_1 + x_2 + \cdots + x_n.$$

Hamilton's equations take the form $\dot{x}_i = \{x_i, H\}$.

The Poisson tensor (4) is Poisson isomorphic to the constant Poisson structure defined by the constant matrix A, see [8]. If $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ is a vector in the kernel of A then the function

$$f = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

is a Casimir. This type of integral can be traced back to Volterra [15]; see also [3, 8, 14].

3 Simple LV-systems

3.1 Complex simple Lie algebras

Cartan matrices appear in the classification of simple Lie algebras over the complex numbers. A Cartan matrix is associated to each such Lie algebra. It is an $\ell \times \ell$ square matrix where ℓ is the rank of the Lie algebra. The Cartan matrix encodes all the properties of the simple Lie algebra it represents. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ a basis of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. The elements of the Cartan matrix C are given by

$$c_{ij} := 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \tag{5}$$

where the inner product is induced by the Killing form. The $\ell \times \ell$-matrix C is invertible and it is called the *Cartan matrix* of $\mathfrak{g}$. The detailed machinery for constructing the Cartan matrix from the root system can be found, e.g., in [9, p. 55] or [11, p. 111]. In the following example, we give full details for the case of $sl(4, \mathbb{C})$ which is of type A_3.

Example 1. Let E be the hyperplane of $\mathbb{R}^4$ for which the coordinates sum to 0 (i.e., vectors are orthogonal to $(1, 1, 1, 1)$). Let Δ be the set of vectors in E of length $\sqrt{2}$ with integer coordinates. There are 12 such vectors in all. We use the standard inner product in $\mathbb{R}^4$ and the standard orthonormal basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. Then, it is easy to see that $\Delta = \{\epsilon_i - \epsilon_j \mid i \neq j\}$. The vectors

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \epsilon_2 - \epsilon_3, \qquad \alpha_3 = \epsilon_3 - \epsilon_4$$

form a basis of the root system in the sense that each vector in Δ is a linear combination of these three vectors with integer coefficients, either all nonnegative or all nonpositive. For example, $\epsilon_1 - \epsilon_3 = \alpha_1 + \alpha_2$, $\epsilon_2 - \epsilon_4 = \alpha_2 + \alpha_3$ and $\epsilon_1 - \epsilon_4 = \alpha_1 + \alpha_2 + \alpha_3$. Therefore $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, and the set of positive roots Δ^+ is given by

$$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

Define the matrix C using (5). It is clear that $c_{ii} = 2$ and

$$c_{i,i+1} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})} = -1, \qquad i = 1, 2.$$

Similar calculations lead to the following form of the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The complex simple Lie algebras are classified as

$$A_l, \quad B_l, \quad C_l, \quad D_l, \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2.$$

Traditionally, A_l, B_l, C_l, D_l are called the classical Lie algebras while E_6, E_7, E_8, F_4, G_2 are called the exceptional Lie algebras. Moreover, for any Cartan matrix there exists just one complex simple Lie algebra up to isomorphism which gives rise to it. The classification is due to Killing and Cartan around 1890.

Simple Lie algebras over $\mathbb{C}$ are classified by using the associated Dynkin diagram. It is a graph whose vertices correspond to the elements of Π. Each pair of vertices α_i, α_j are connected by

$$m_{ij} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

edges, where

$$m_{ij} \in \{0, 1, 2, 3\}.$$

To a given Dynkin diagram Γ with n nodes, we associate the *Coxeter adjacency matrix* which is the $n \times n$ matrix $A = 2I - C$, where C is the Cartan matrix.

3.2 First approach: from the root system

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a basis of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. Let $X_{\alpha_1}, \dots, X_{\alpha_n}$ be the corresponding root vectors in $\mathfrak{g}$. Define

$$L = \sum_{\alpha_i \in \Pi} x_i X_{\alpha_i}.$$

To find the matrix B we use the following procedure. For each i, j we form $[X_{\alpha_i}, X_{\alpha_j}]$. If $\alpha_i+\alpha_j$ is a root then we include a term of the form $\pm x_i x_j [X_{\alpha_i}, X_{\alpha_j}]$ in B. By making suitable choices for the $\pm$ signs it is possible to construct a consistent Lax pair. Then we define the system using the Lax equation

$$\dot{L} = [L, B].$$

For a root system of type A_n we obtain the KM system.

3.3 Second approach: from the Dynkin diagram

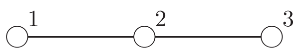
If a system is of type ADE we can define the system in the following alternative way. Consider the Dynkin diagram of $\mathfrak{g}$ and define a Lotka–Volterra system by the equations

$$\dot{x}_i = x_i \sum_{j=1}^{\ell} m_{ij} x_j,$$

where the skew-symmetric matrix m_{ij} for $i < j$ is defined to be $m_{ij} = 1$ if vertex i is connected with vertex j and 0 otherwise. For $i > j$ the term m_{ij} is defined by skew-symmetry. Note that if we replace one of the m_{ij} for $i < j$ from $+1$ to -1 we may end up with an inequivalent system. In our definition, the upper part of the matrix (m_{ij}) consists only of 0 and 1. However, it is possible to define for each connected graph 2^m systems, where m is the number of edges, by assigning the ± 1 sign to each edge. Of course, some of these systems will be isomorphic. One more observation: there are several inequivalent ways to label a graph and therefore the association between graphs and Lotka–Volterra systems is not always a bijection. The number of distinct labellings of a given unlabeled simple graph G on n vertices is known to be

$$\frac{n!}{|\mathrm{aut}\,(G)|}.$$

Example 2. Consider a Dynkin diagram with graph A_3.

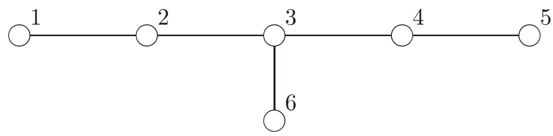


We label the vertices from left to right. To define $\dot{x}_1$ we note that vertex 1 is joined only with vertex 2. Therefore we include a term $x_1 x_2$. We define $m_{13} = 0$ since vertex 1 is not connected with vertex 3. Similarly we define $m_{23} = 1$ since vertex 2 is connected with vertex 3. Therefore we obtain the KM system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2, \\ \dot{x}_2 &= -x_1 x_2 + x_2 x_3, \\ \dot{x}_3 &= -x_2 x_3. \end{aligned} \tag{6}$$

This system is integrable since the function $F = x_1 x_3$ is a Casimir. Taking into account the Hamiltonian $x_1 + x_2 + x_3$ we have Liouville integrability.

Example 3. (E_6 system)

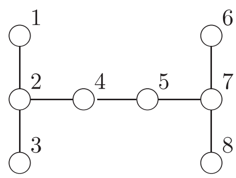


$$\begin{aligned}
\dot{x}_1 &= x_1x_2, & \dot{x}_2 &= x_2(-x_1+x_3),\\
\dot{x}_3 &= x_3(-x_2+x_4+x_5), & \dot{x}_4 &= -x_3x_4,\\
\dot{x}_5 &= x_5(-x_3+x_6), & \dot{x}_6 &= -x_5x_6.
\end{aligned} \tag{7}$$

The associated Poisson structure is symplectic. Therefore to prove integrability one needs another two constants of motion besides the Hamiltonian.

This method can be used not only for Dynkin diagrams corresponding to simple Lie algebras but also for an arbitrary graph.

Example 4. This graph has an associated Lotka–Volterra system.



It is given by

$$\begin{aligned}
\dot{x}_1 &= x_1x_2, & \dot{x}_2 &= -x_1x_2+x_2x_3+x_2x_4,\\
\dot{x}_3 &= -x_2x_3, & \dot{x}_4 &= -x_2x_4+x_4x_5,\\
\dot{x}_5 &= -x_4x_5+x_5x_7, & \dot{x}_6 &= x_6x_7,\\
\dot{x}_7 &= -x_5x_7-x_6x_7+x_7x_8, & \dot{x}_8 &= -x_7x_8.
\end{aligned} \tag{8}$$

This system has two Casimirs $F_1 = x_1x_3$ and $F_2 = x_6x_8$.

Example 5. The periodic KM-system is given by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}), \qquad i = 1, 2, \dots, n, \tag{9}$$

with periodic condition $x_{i+n} = x_i$ for all i. It is associated to a Dynkin diagram of affine type $A^{(1)}_{n-1}$.

We examine in detail the case $n = 4$.

One Lax pair is a generalization of Moser's

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ a_4 & 0 & a_3 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 - a_4a_3 & 0 \\ 0 & 0 & 0 & -a_1a_4 + a_2a_3 \\ -a_1a_2 + a_4a_3 & 0 & 0 & 0 \\ 0 & a_1a_4 - a_2a_3 & 0 & 0 \end{pmatrix}.$$

The Lax pair is equivalent to the following equations of motion:

$$\dot{a_1} = a_1a_2^2 - a_1a_4^2, \qquad \dot{a_2} = -a_2a_1^2 + a_2a_3^2,$$

$$\dot{a_3} = a_3a_4^2 - a_3a_2^2, \qquad \dot{a_4} = a_4a_1^2 - a_4a_3^2.$$

Using the substitution $x_i = a_i^2$ and a scaling we obtain the equations for the periodic KM-system

$$\dot{x_1} = x_1x_2 - x_1x_4, \qquad \dot{x_2} = -x_1x_2 + x_2x_3,$$

$$\dot{x_3} = x_4x_3 - x_2x_3, \qquad \dot{x_4} = x_1x_4 - x_4x_3.$$

Using the Poisson tensor

$$\pi = x_1x_2\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - x_1x_4\frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_1} + x_2x_3\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_3x_4\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$$

and the Hamiltonian $H = x_1 + x_2 + x_3 + x_4$ we have a Hamiltonian formulation of the system. The Poisson tensor is of rank 2. It has two Casimirs $F_1 = x_1x_3$ and $F_2 = x_2x_4$. Therefore the system is integrable.

An alternative Lax pair is the following

$$L = \begin{pmatrix} 0 & 1 & 0 & x_4 \\ x_1 & 0 & 1 & 0 \\ 0 & x_2 & 0 & 1 \\ 1 & 0 & x_3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & x_3x_4 & 0 \\ 0 & 0 & 0 & x_1x_4 \\ x_1x_2 & 0 & 0 & 0 \\ 0 & x_2x_3 & 0 & 0 \end{pmatrix}.$$

Example 6. (D_4 system) By examining the Dynkin diagram of the simple Lie algebra of type D_4 we obtain the system

$$\begin{aligned} \dot{x}_1 &= x_1x_2, & \dot{x}_2 &= -x_1x_2 + x_2x_3 + x_2x_4, \\ \dot{x}_3 &= -x_2x_3, & \dot{x}_4 &= -x_2x_4. \end{aligned} \tag{10}$$

One can obtain the same equations in the following way. Define the matrix L using the root vectors of a Lie algebra of type D_4

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & x_4 & -x_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2x_4 & -x_2x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_1x_2 & 0 & 0 \end{pmatrix}.$$

Then the Lax equation $\dot{L} = [L, B]$ is equivalent to (10). We note that

$$H_k = \frac{1}{k} \operatorname{tr} L^k, \quad k = 1, 2, \ldots$$

are integrals of motion for the system. In fact

$$4H_2 = x_1 + x_2 + x_3 + x_4,$$

$$4H_4 = \operatorname{tr} L^4 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4.$$

There are also two Casimirs $F_1 = x_1x_4$ and $F_2 = x_1x_3$. It turns out that $\det(L) = (F_1 + F_2)^2$. We have

$$H_2^2 - 4H_4 = 8(x_1x_3 + x_1x_4) = 8(F_1 + F_2).$$

We can find the Casimirs by computing the kernel of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The two eigenvectors with eigenvalue 0 are $(1, 0, 0, 1)$ and $(1, 0, 1, 0)$. We obtain the two Casimirs $F_1 = x_1^1 x_2^0 x_3^0 x_4^1 = x_1x_4$ and $F_2 = x_1^1 x_2^0 x_3^1 x_4^0 = x_1x_3$.

There is also a periodic version of D_n with some obvious modifications in the Lax pair. For $n = 4$ the affine $D_4^{(1)}$ system is given by

$$\begin{aligned} &\dot{x}_0 = x_0x_2, \\ &\dot{x}_1 = x_1x_2, \qquad \dot{x}_2 = -x_0x_2 - x_1x_2 + x_2x_3 + x_2x_4, \\ &\dot{x}_3 = -x_2x_3, \quad \dot{x}_4 = -x_2x_4. \end{aligned} \tag{11}$$

The eigenvectors of the coefficient matrix corresponding to the eigenvalue 0 are $(0, 1, 0, 1, 0)$, $(1, 0, 0, 0, 1)$ and $(1, 0, 0, 1, 0)$. Therefore the Poisson tensor has three Casimirs x_0x_3, x_0x_4 and x_1x_3. It is therefore integrable.

Example 7. It is possible to consider graphs which are not simple. For example the graph associated with the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2, & \dot{x}_2 &= x_2(x_3 - x_1), \\ \dot{x}_3 &= x_3(x_4 - x_2), & \dot{x}_4 &= -x_4(x_3 + x_4) \end{aligned} \tag{12}$$

has a loop at vertex 4. This system is an open version of a B_n system considered by Bogoyavlensky in [1, 2]. The Hamiltonian formulation of these systems, Lax pairs and master symmetries were considered by Kouzaris in [12]. There is also a Lax pair in [5]. The system in our example has two integrals of motion, one of degree 2 and one of degree 4. The quadratic integral is

$$F_1 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_4.$$

The fourth degree invariant is

$$\begin{aligned} F_2 = {} & x_1^4 + x_2^4 + x_3^4 + 4x_1^2x_2x_3 + 6x_1^2x_2^2 + 4x_1x_2x_3x_4 + 4x_3^2x_4^2 \\ & + 4x_3x_4x_2^2 + 4x_1x_2^3 + 4x_3^3x_4 + 4x_1^3x_2 + 8x_3^2x_2x_4 \\ & + 8x_1x_3x_2^2 + 4x_1x_2x_3^2 + 4x_2^3x_3 + 4x_2x_3^3 + 6x_2^2x_3^2. \end{aligned}$$

3.4 Third approach: Lie algebra decomposition

An alternative method to define the systems is the following. Let $\tilde{A} = 2I - C$ be the Coxeter adjacency matrix. Decompose $\tilde{A} = A + B$ where $A = (a_{ij})$ is the skew-symmetric part of $\tilde{A}$ and B its lower triangular part. Define the Lotka–Volterra system using the formula

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_i x_j, \qquad i = 1, 2, \dots, n.$$

This method can be used to define Lotka–Volterra systems for any complex simple Lie algebra (including B_n, C_n, G_2 and F_4). Alternatively, we may use the approach of subsection (3.3). When there are multiple edges we define m_{ij} for $i < j$ to be the number of edges from i to j.

Example 8. Consider a Lie algebra of type B_3. The Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Since

$$2I - C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix},$$

we may define a B_3 Lotka–Volterra system as follows:

$$\begin{aligned}\dot{x}_1 &= x_1x_2,\\ \dot{x}_2 &= -x_1x_2 + 2x_2x_3,\\ \dot{x}_3 &= -2x_2x_3.\end{aligned} \tag{13}$$

The Casimir for this system is $F = x_1^2x_3$. Note that a 3-dimensional Lotka–Volterra system of the type we are considering, i.e., defined with a skew-symmetric matrix is always Liouville integrable.

4 The Bogoyavlenskij lattices

Bogoyavlenskij in [3] has generalized the KM-system in the following way

$$\dot{x}_i = x_i\left(\sum_{j=1}^{p} x_{i+j} - \sum_{j=1}^{p} x_{i-j}\right)$$

with periodic condition $x_{n+i} = x_i$. We will denote this system with $B(n,p)$. All the results in this section, except the bihamiltonian pair are from [3]. The system has a Lax pair of the form

$$\dot{L} = [L, A],$$

where $L = X + \lambda M$, $A = b - \lambda M^{p+1}$. The matrix X has the form $x_{i,i-p} = x_i$ for $p+1 \le i \le n$ and $x_{i,i+n-p} = x_i$ for $1 \le i \le p$. The matrix M is defined by $m_{i,i+1} = m_{n,1} = 1$. The matrix b is diagonal with entries $b_{ii} = -(x_i + x_{i+1} + \cdots + x_{i+p})$.

Example 9. Let us consider the system $B(6,2)$, i.e., $n = 6$, $p = 2$. The equations of motion become

$$\begin{aligned}\dot{x}_1 &= x_1(x_2 + x_3 - x_5 - x_6), & \dot{x}_2 &= x_2(x_3 + x_4 - x_1 - x_6),\\ \dot{x}_3 &= x_3(x_4 + x_5 - x_2 - x_1), & \dot{x}_4 &= x_4(x_5 + x_6 - x_3 - x_2),\\ \dot{x}_5 &= x_5(x_6 + x_1 - x_4 - x_3), & \dot{x}_6 &= x_6(x_1 + x_2 - x_5 - x_4).\end{aligned} \tag{14}$$

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & x_1 & 0\\ 0 & 0 & 0 & 0 & 0 & x_2\\ x_3 & 0 & 0 & 0 & 0 & 0\\ 0 & x_4 & 0 & 0 & 0 & 0\\ 0 & 0 & x_5 & 0 & 0 & 0\\ 0 & 0 & 0 & x_6 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & \lambda & 0 & 0 & x_1 & 0\\ 0 & 0 & \lambda & 0 & 0 & x_2\\ x_3 & 0 & 0 & \lambda & 0 & 0\\ 0 & x_4 & 0 & 0 & \lambda & 0\\ 0 & 0 & x_5 & 0 & 0 & \lambda\\ \lambda & 0 & 0 & x_6 & 0 & 0 \end{pmatrix}.$$

Let $p(x) = \det(L - xI)$ be the characteristic polynomial of L. Then the coefficient of x^3 is of the form $H\lambda^2 + F_2$, where $H = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ is the Hamiltonian, and $F_2 = x_1x_3x_5 + x_2x_4x_6$. On the other hand the constant term of $p(x)$ has the form $F_3\lambda^2 + F_4$, where $F_3 = x_1x_2x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_5x_6$ and $F_4 = x_1x_2x_3x_4x_5x_6$.

By examining the eigenvectors of the coefficient matrix we can see that the functions $C_1 = x_2x_5$, $C_2 = x_1x_4$, $C_3 = x_2x_4x_6$, and $C_4 = x_1x_3x_5$ are all Casimirs. Therefore we have a rank 2 Poisson bracket and the system is clearly integrable. It is easy to see that the functions F_2, F_3, F_4 can be expressed as functions of C_1, C_2, C_3, C_4.

Now restrict this system on the invariant submanifold $x_5 = x_6 = 0$. We obtain the system

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 + x_3), & \dot{x}_2 &= x_2(x_3 + x_4 - x_1), \\ \dot{x}_3 &= x_3(x_4 - x_2 - x_1), & \dot{x}_4 &= x_4(-x_3 - x_2). \end{aligned}$$

This system is integrable. It has two Casimirs $F_1 = x_1x_4 = C_2$ and $F_2 = \dfrac{x_1x_3}{x_2} = C_4/C_1$.

Example 10. Similarly, the system $B(5,2)$ has a single Casimir $x_1x_2x_3x_4x_5$. The additional integral is

$$F = x_1x_2x_4 + x_1x_3x_4 + x_1x_3x_5 + x_2x_3x_5 + x_2x_4x_5.$$

The system is Hamiltonian using the quadratic bracket (4) and the Hamiltonian function

$$H = x_1 + x_2 + x_3 + x_4 + x_5.$$

Denote this quadratic bracket by π_2. Define the Poisson tensor π_0 as follows

$$\pi_0 = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that π_0 is compatible with π_2 and that we have a bihamiltonian pair

$$\pi_2 dH = \pi_0 dF.$$

The function H is the Casimir of bracket π_0.

More generally if $n = 2p + 1$ then we can define a (skew-symmetric) tensor π_0 with non-zero entries $\pi_0[i, i+n-p-1] = -1$ for $1 \le i \le p+1$ and $\pi_0[i, i+n-p] = 1$

for $1 \le i \le p$. The tensors π_2 and π_0 are compatible and they form a bihamiltonian pair.

Restricting on the submanifold $x_5 = 0$ we obtain the system

$$\dot{x}_1 = x_1(x_2 + x_3 - x_4), \qquad \dot{x}_2 = x_2(x_3 + x_4 - x_1),$$
$$\dot{x}_3 = x_3(x_4 - x_2 - x_1), \qquad \dot{x}_4 = x_4(x_1 - x_3 - x_2).$$

This system is integrable with second integral given by $x_1x_4(x_2 + x_3)$, i.e., the restriction of F on the submanifold.

Example 11. Restricting the $B(7, 2)$ on the submanifold $x_4 = x_6 = x_7 = 0$ and renaming $x_5 \to x_4$ results in the following system

$$\dot{x}_1 = x_1(x_2 + x_3), \qquad \dot{x}_2 = x_2(x_3 - x_1),$$
$$\dot{x}_3 = x_3(x_4 - x_2 - x_1), \qquad \dot{x}_4 = -x_4x_3.$$

The additional integral is $F = x_4(x_1 + x_2)$.

Example 12. Restricting the $B(7, 3)$ on the submanifold $x_5 = x_6 = x_7 = 0$ results in the following system

$$\dot{x}_1 = x_1(x_2 + x_3 + x_4), \qquad \dot{x}_2 = x_2(x_3 + x_4 - x_1),$$
$$\dot{x}_3 = x_3(x_4 - x_2 - x_1), \qquad \dot{x}_4 = x_4(-x_1 - x_2 - x_3).$$

The Poisson matrix in this example is symplectic. The system is integrable since it has two constants of motion

$$F_1 = \frac{(x_1 + x_2)x_4}{x_3} \quad \text{and} \quad F_2 = \frac{(x_3 + x_4)x_1}{x_2}.$$

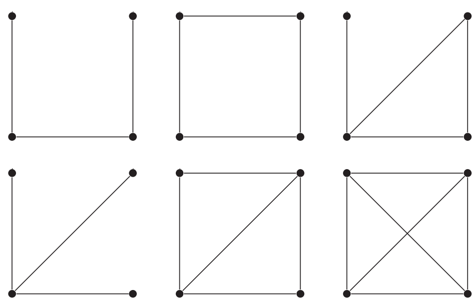
Note that

$$F_3 = \frac{(x_1 + x_2 + x_3)(x_2 + x_3 + x_4)}{x_2 + x_3}$$

is also a first integral.

5 Connected graphs on 4 vertices

It is well-known that there are 6 connected simple graphs on four vertices. They are given in the following figure.



It seems that most of the systems associated to a connected simple graph on four vertices, constructed using our approach, are Liouville integrable.

- The first graph corresponds to the open KM-system which is integrable, see (2).
- For the second graph (the square) we can associate three Lotka–Volterra systems each one corresponding to the different ways to label the graph. One system is

  $$\dot{x}_1 = x_1(x_2 + x_3), \qquad \dot{x}_2 = x_2(x_4 - x_1),$$
  $$\dot{x}_3 = x_3(x_4 - x_1), \qquad \dot{x}_4 = x_4(-x_2 - x_3).$$

  It is integrable with Casimirs $F_1 = \dfrac{x_3}{x_2}$ and $F_2 = x_1 x_4$.

  A second case is

  $$\dot{x}_1 = x_1(x_3 + x_4), \qquad \dot{x}_2 = x_2(x_3 + x_4),$$
  $$\dot{x}_3 = x_3(-x_1 - x_2), \qquad \dot{x}_4 = x_4(-x_1 - x_2).$$

  It is integrable with Casimirs $F_1 = \dfrac{x_2}{x_1}$ and $F_2 = \dfrac{x_4}{x_3}$.

  Finally, there is also

  $$\dot{x}_1 = x_1(x_2 + x_4), \qquad \dot{x}_2 = x_2(x_3 - x_1),$$
  $$\dot{x}_3 = x_3(x_4 - x_2), \qquad \dot{x}_4 = x_4(-x_1 - x_3).$$

  In this case the Poisson structure is symplectic and we need a second integral to establish integrability.
- One integrable case of the third graph was treated in Example 11.
- The fourth graph is a Dynkin diagram of type D_4. One example which has a Lax pair was considered in Example 6.

  Another example which is obtained by changing the order of the labels is

  $$\dot{x}_1 = x_1(x_2 + x_3 + x_4), \qquad \dot{x}_2 = -x_1 x_2,$$
  $$\dot{x}_3 = -x_1 x_3, \qquad \dot{x}_4 = -x_1 x_4.$$

  It has two Casimirs $F_1 = \dfrac{x_4}{x_2}$ and $F_2 = \dfrac{x_3}{x_2}$.
- One case of Graph 5 is treated in Example 9.
- One case of Graph 6 is treated in Example 12.

Remark. There are three non-isomorphic trees on 5 vertices. We have seen three Lotka–Volterra systems associated to these graphs. The KM-system, the D_5 system and the affine $D_4^{(1)}$ periodic system. The integrability of all systems associated with a connected simple graph on 5 vertices is an interesting problem to consider.

Acknowledgments

My interest in Lotka–Volterra systems originated during my visit to Lisbon in 1999. I would like to thank Rui Fernandes for the invitation. The symmetric Moser type Lax pair similar to (10) is due to Fernandes. I also thank Charalampos Evripidou and Stelios Charalambides for pointing out that the systems defined in this paper depend on the labeling of the graphs. A major part of this paper was inspired by the works of Oleg Bogoyavlenskij, especially by reference [3].

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Poisson Brackets with Prescribed Casimirs. II

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We continue our investigation of the problem of constructing Poisson brackets having as Casimirs a given set of functions. In this paper we consider the case of odd-dimensional manifolds. We present an application which involves a class of Lotka–Volterra systems associated with affine Lie algebras.

1 Introduction

We consider the following problem. Given $f_1, \dots, f_k$ smooth functions on an n-dimensional manifold M, functionally independent almost everywhere, to find a Poisson bracket on M whose set of Casimir functions consists of the given ones. The case of an even-dimensional manifold was examined in the first part of this paper [5]. In this paper we consider the case of odd dimension. Detailed proofs can be found in [6]. The main purpose of this paper is to present a new application of these results to the construction of a new class of Lotka–Volterra systems from Toda systems.

First we recall the results in [5] (see also [6]). Suppose dim $M = 2n$. Let $f_1, \dots, f_{2n-2k}$ be smooth functions on M, functionally independent almost everywhere on M, ω_0 an almost symplectic form on M, Λ_0 its associated bivector field and $X_{f_i} = \Lambda_0^{\#}(df_i)$ the Hamiltonian vector fields of f_i, $i = 1, \dots, 2n-2k$, with respect to Λ_0, such that

$$f = \left\langle df_1 \wedge \cdots \wedge df_{2n-2k}, \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle = \left\langle \frac{\omega_0^{n-k}}{(n-k)!}, X_{f_1} \wedge \cdots \wedge X_{f_{2n-2k}} \right\rangle \neq 0$$

on an open and dense subset $\mathcal{U}$ of M. Consider the $(2n-2)$-form

$$\Phi = -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k} \qquad (1)$$

on M. In (1), σ is a section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$, D° being the annihilator of the distribution D generated by the Hamiltonian vector fields X_{f_i},

$i = 1, \dots, 2n-2k$. Moreover, σ satisfies the equation

$$2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma), \tag{2}$$

where δ is the operator $\delta = *d*$ and $*$ is the standard star operator [10]. The function g is defined to be $g = i_{\Lambda_0}\sigma$. The form Φ corresponds to a Poisson tensor field Λ on M with orbits of dimension at most $2k$ for which $f_1, \dots, f_{2n-2k}$ are Casimirs. Precisely, $\Lambda = \Lambda_0^{\#}(\sigma)$ and the associated bracket of Λ on $C^\infty(M)$ is given, for any $h_1, h_2 \in C^\infty(M)$, by

$$\begin{aligned}&\{h_1, h_2\}\Omega = \\ &\qquad -\frac{1}{f} dh_1 \wedge dh_2 \wedge \Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \dots \wedge df_{2n-2k}.\end{aligned} \tag{3}$$

Conversely, if Λ is a Poisson tensor on (M, ω_0) of rank at most $2k$ on an open and dense subset $\mathcal{U}$ of M, then there are $2n-2k$ functionally independent smooth functions $f_1, \dots, f_{2n-2k}$ on $\mathcal{U}$ and a suitable 2-form σ on M such that $\Psi_\Lambda = -i_\Lambda\Omega$ and $\{\cdot,\cdot\}$ is of the form (1) and (3), respectively.

In this paper we consider the case where M is an odd-dimensional manifold, i.e., $m = 2n+1$, and we establish a similar formula for the Poisson brackets on $C^\infty(M)$ with the prescribed properties. For this construction, we assume that M is equipped with a suitable almost cosymplectic structure (ϑ_0, Θ_0) and with the volume form $\Omega = \vartheta_0 \wedge \dfrac{\Theta_0^n}{n!}$. In [6], using these results, we showed how to obtain the A_n Volterra quadratic Poisson bracket starting from the A_n Lie–Poisson bracket of the periodic Toda lattice. In this paper we illustrate the same procedure for the case of B_n and we arrive to a construction, using a new method, of a family of Volterra quadratic Poisson structures having the same Casimirs. The algorithm can of course be generalized to any type of complex simple Lie algebra.

2 On odd-dimensional manifolds

Let M be a $(2n+1)$-dimensional manifold. We remark that any Poisson tensor Λ on M admitting $f_1, \dots, f_{2n+1-2k} \in C^\infty(M)$ as Casimir functions can be viewed as a Poisson tensor on $M' = M \times \mathbb{R}$ admitting $f_1, \dots, f_{2n+1-2k}$ and $f_{2n+2-2k}(x, s) = s$ (s being the canonical coordinate on the factor $\mathbb{R}$) as Casimir functions, and conversely. Thus, the problem of constructing Poisson brackets on $C^\infty(M)$ having as center the space of functions generated by $(f_1, \dots, f_{2n+1-2k})$ is equivalent to that of constructing Poisson brackets on $C^\infty(M')$ having as center the space of functions generated by $(f_1, \dots, f_{2n+1-2k}, s)$. In what follows we establish a formula analogous to (3) for Poisson brackets on odd-dimensional manifolds. In the construction we use the notion of *almost cosymplectic* structures on M, see, e.g., [11, 12].

We consider $(M, f_1, \dots, f_{2n+1-2k})$, with $f_1, \dots, f_{2n+1-2k}$ functionally independent almost everywhere on M, and an almost cosymplectic structure (ϑ_0, Θ_0)

on M whose associated nondegenerate almost Jacobi structure (Λ_0, E_0) verifies the condition

$$f = \left\langle df_1 \wedge \cdots \wedge df_{2n+1-2k},\ E_0 \wedge \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle \neq 0 \tag{4}$$

on an open and dense subset $\mathcal{U}$ of M. Let $\omega_0' = \Theta_0 + ds \wedge \vartheta_0$ and $\Lambda_0' = \Lambda_0 + \frac{\partial}{\partial s} \wedge E_0$ be the associated tensors on $M' = M \times \mathbb{R}$. Since, for any $m = 1, \dots, n+1$,

$$\begin{aligned} \frac{{\omega_0'}^m}{m!} &= \frac{\Theta_0^m}{m!} + ds \wedge \vartheta_0 \wedge \frac{\Theta_0^{m-1}}{(m-1)!}, \\ \frac{{\Lambda_0'}^m}{m!} &= \frac{\Lambda_0^m}{m!} + \frac{\partial}{\partial s} \wedge E_0 \wedge \frac{\Lambda_0^{m-1}}{(m-1)!}, \end{aligned} \tag{5}$$

it is clear that

$$\begin{aligned} &\left\langle df_1 \wedge \cdots \wedge df_{2n+1-2k} \wedge ds,\ \frac{{\Lambda_0'}^{n+1-k}}{(n+1-k)!} \right\rangle \\ &= \left\langle df_1 \wedge \cdots \wedge df_{2n+1-2k} \wedge ds,\ \frac{\Lambda_0^{n+1-k}}{(n+1-k)!} + \frac{\partial}{\partial s} \wedge E_0 \wedge \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle \\ &= \left\langle df_1 \wedge \cdots \wedge df_{2n+1-2k} \wedge ds,\ \frac{\partial}{\partial s} \wedge E_0 \wedge \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle = -f \neq 0 \end{aligned} \tag{6}$$

on the open and dense subset $\mathcal{U}' = \mathcal{U} \times \mathbb{R}$ of M'. Furthermore, we view any bivector field Λ on $(M, \vartheta_0, \Theta_0)$, having as Casimirs the given functions, as a bivector field on (M', ω_0') having $f_1, \dots, f_{2n+1-2k}$ and $f_{2n+2-2k}(x, s) = s$ as Casimirs. Let D'° be the annihilator of the distribution $D' = \langle X'_{f_1}, \dots, X'_{f_{2n+2-2k}} \rangle$ on M' generated by the Hamiltonian vector fields $X'_{f_i} = \Lambda_0'^{\#}(df_i) = \Lambda_0^{\#}(df_i) - \langle df_i, E_0 \rangle \frac{\partial}{\partial s}$, $i = 1, \dots, 2n+1-2k$, and $X'_{f_{2n+2-2k}} = \Lambda_0'^{\#}(ds) = E_0$ of $f_1, \dots, f_{2n+1-2k}$ and $f_{2n+2-2k}(x, s) = s$ with respect to Λ_0'. Then, according to the results for the case of even-dimensional manifolds, there exists a unique 2-form σ' on M', which is a section of $\bigwedge^2 D'^{\circ}$ of maximal rank $2k$ on $\mathcal{U}' = \mathcal{U} \times \mathbb{R}$, such that $\Lambda = \Lambda_0'^{\#}(\sigma')$. Moreover, since Λ is independent of s and without a term of type $X \wedge \frac{\partial}{\partial s}$, σ' must be of the type

$$\sigma' = \sigma + \tau \wedge ds, \tag{7}$$

where σ and τ are, respectively, a 2-form and a 1-form on M having the following additional properties:

i) σ is a section $\bigwedge^2 \langle E_0 \rangle^{\circ}$, i.e., σ is a semi-basic 2-form on M with respect to (Λ_0, E_0);

ii) τ is a section of $D^\circ = \langle X_{f_1}, \dots, X_{f_{2n+1-2k}}, E_0\rangle^\circ$, where $X_{f_i} = \Lambda_0^\#(df_i)$, i.e., τ is a semi-basic 1-form on (M, Λ_0, E_0) which is also semi-basic with respect to $X_{f_1}, \dots, X_{f_{2n+1-2k}}$;

iii) for any f_i, $i = 1, \dots, 2n+1-2k$, $\sigma(X_{f_i}, \cdot) + \langle df_i, E_0\rangle\tau = 0$.

Consequently, Λ is written, in a unique way, as $\Lambda = \Lambda_0^\#(\sigma) + \Lambda_0^\#(\tau) \wedge E_0$.

Summarizing, we may formulate the next proposition.

Proposition 1. *Under the above notations and assumptions, a bivector field Λ on $(M, \vartheta_0, \Theta_0)$, of rank at most $2k$, has as unique Casimirs the functions $f_1, \dots, f_{2n+1-2k}$ if and only if its corresponding pair of forms (σ, τ) satisfies the properties (i)–(iii) and $(\operatorname{rank}\sigma, \operatorname{rank}\tau) = (2k, 0)$ or $(2k, 1)$ or $(2k-2, 1)$ on $\mathcal{U}$.*

On the other hand, it follows from (3) that the bracket $\{\cdot,\cdot\}$ of Λ on $C^\infty(M)$ is calculated, for any $h_1, h_2 \in C^\infty(M)$, viewed as elements of $C^\infty(M')$, by the formula

$$\{h_1, h_2\}\Omega' \overset{(6)}{=} \frac{1}{f} dh_1 \wedge dh_2 \wedge \left(\sigma' + \frac{g'}{k-1}\omega_0'\right) \wedge \frac{{\omega_0'}^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n+1-2k} \wedge ds,$$

where $\Omega' = \dfrac{{\omega_0'}^{n+1}}{(n+1)!}$ and $g' = i_{\Lambda_0'}\sigma'$. But, $\Omega' \overset{(5)}{=} -\Omega \wedge ds$, $\Omega = \vartheta_0 \wedge \dfrac{\Theta_0^n}{n!}$ being a volume form on M, and $g' = i_{\Lambda_0'}\sigma' = i_{\Lambda_0 + \partial/\partial s \wedge E_0}(\sigma + \tau \wedge ds) = i_{\Lambda_0}\sigma = g$. Thus, taking into account (5) and (7), we have

$$\{h_1, h_2\}\Omega \wedge ds = -\frac{1}{f} dh_1 \wedge dh_2 \wedge \left(\sigma + \frac{g}{k-1}\Theta_0\right) \wedge \frac{\Theta_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n+1-2k} \wedge ds,$$

which is equivalent to

$$\{h_1, h_2\}\Omega = -\frac{1}{f} dh_1 \wedge dh_2 \wedge \left(\sigma + \frac{g}{k-1}\Theta_0\right) \wedge \frac{\Theta_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n+1-2k}.$$

However, according to (2), $\{\cdot,\cdot\}$ is a Poisson bracket on $C^\infty(M) \subset C^\infty(M')$ if and only if

$$2\sigma' \wedge \delta'(\sigma') = \delta'(\sigma' \wedge \sigma'), \tag{8}$$

where $\delta' = *' d *'$ is the codifferential on $\Omega(M')$ of (M', ω_0') defined by the corresponding star operator $*'\colon \Omega^p(M') \to \Omega^{2n+2-p}(M')$. We want to translate (8) to a condition on (σ, τ). Let $\Omega^p_{\mathrm{sb}}(M)$ be the space of semi-basic p-forms on (M, Λ_0, E_0),

$*$ the isomorphism between $\Omega^p_{\mathrm{sb}}(M)$ and $\Omega^{2n-p}_{\mathrm{sb}}(M)$ given, for any $\varphi \in \Omega^p_{\mathrm{sb}}(M)$, by

$$*\varphi = (-1)^{(p-1)p/2} i_{\Lambda_0^{\#}(\varphi)} \frac{\Theta_0^n}{n!},$$

$d_{sp} \colon \Omega^p_{\mathrm{sb}}(M) \to \Omega^{p+1}_{\mathrm{sb}}(M)$ the operator which corresponds to each semi-basic form φ the semi-basic part of its differential $d\varphi$, and $\delta = * d_{\mathrm{sb}} *$ the associated "codifferential" operator on $\Omega_{\mathrm{sb}}(M) = \oplus_{p\in\mathbb{Z}} \Omega^p_{\mathrm{sb}}(M)$. By a straightforward, but long, computation, we show that (8) is equivalent to the system

$$\begin{aligned} &2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma), \\ &\delta(\sigma \wedge \tau) + \delta(\sigma) \wedge \tau - \sigma \wedge \delta(\tau) = (i_{\Lambda_0^{\#}(d\vartheta_0)}\sigma)\sigma - \tfrac{1}{2} i_{\Lambda_0^{\#}(d\vartheta_0)}(\sigma \wedge \sigma). \end{aligned} \tag{9}$$

Hence, we deduce:

Proposition 2. *Under the above assumptions and notations,*

$$\Lambda = \Lambda_0^{\#}(\sigma) + \Lambda_0^{\#}(\tau) \wedge E_0$$

defines a Poisson structure on $(M, \vartheta_0, \Theta_0)$ *if and only if* (σ, τ) *satisfies* (9).

Concluding, we obtain the following result.

Theorem 1. *Let* $f_1, \ldots, f_{2n+1-2k}$ *be smooth functions on a* $(2n+1)$*-dimensional smooth manifold* M *which are functionally independent almost everywhere,* (ϑ_0, Θ_0) *an almost cosymplectic structure on* M *such that* (4) *holds on an open and dense subset* $\mathcal{U}$ *of* M, $\Omega = \vartheta_0 \wedge \dfrac{\Theta_0^n}{n!}$ *the corresponding volume form on* M, *and* (σ, τ) *an element of* $\Omega^2_{\mathrm{sb}}(M) \times \Omega^1_{\mathrm{sb}}(M)$, *with* $(\operatorname{rank}\sigma, \operatorname{rank}\tau) = (2k, 0)$ *or* $(2k, 1)$, *or* $(2k-2, 1)$ *on* $\mathcal{U}$, *that has the properties (ii)–(iii) and satisfies* (9)*. Then, the bracket* $\{\cdot,\cdot\}$ *on* $C^\infty(M)$ *given, for any* $h_1, h_2 \in C^\infty(M)$*, by*

$$\begin{aligned} &\{h_1, h_2\}\Omega = \\ &\qquad -\frac{1}{f} dh_1 \wedge dh_2 \wedge \Big(\sigma + \frac{g}{k-1}\Theta_0\Big) \wedge \frac{\Theta_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n+1-2k}, \end{aligned} \tag{10}$$

where f *is given by* (4) *and* $g = i_{\Lambda_0}\sigma$*, defines a Poisson structure* Λ *on* M, $\Lambda = \Lambda_0^{\#}(\sigma) + \Lambda_0^{\#}(\tau) \wedge E_0$*, with symplectic leaves of dimension at most* $2k$ *for which* $f_1, \ldots, f_{2n+1-2k}$ *are Casimirs. The converse is also true.*

We close this Section by considering the following question due to F. Magri.

Question: Using the presented theory, we construct a Poisson structure Λ on a m-dimensional manifold M, of rank at most $2k$ on an open and dense subset $\mathcal{U}$ of M, which has as Casimirs a family $(f_1, \ldots, f_{m-2k})$ of given functions on M, functionally independent almost everywhere, and whose bracket is given, in the even case, by (3) and, in the odd case, by (10). From every point $x \in \mathcal{U}$ passes a symplectic leave S of Λ which has a symplectic form ω_S. What is the relationship between ω_S and σ?

In the case where $m = 2n$, we considered a nondegenerate 2-form ω_0 on M such that, at each point $x \in \mathcal{U}$, T_xM is written as a sum of symplectic subspaces relatively to ω_{0_x}. Precisely,

$$T_xM = D_x \oplus \mathrm{orth}_{\omega_{0_x}} D_x,$$

where D_x is the fibre of D at x and $\mathrm{orth}_{\omega_{0_x}} D_x$ is the orthogonal of D_x with respect to the symplectic form ω_{0_x} on T_xM, [10]. Also,

$$T_xM = D_x \oplus \Lambda_{0_x}^{\#}(D_x^\circ) \quad \text{and} \quad T_x^*M = D_x^\circ \oplus \langle df_1, \dots, df_{2n-2k}\rangle_x, \tag{11}$$

where D_x° is the fibre of D° at x and $\langle df_1, \dots, df_{2n-2k}\rangle_x$ is the fibre at x of the Pfaffian system generated by df_i, $i = 1, \dots, 2(n-k)$. From the last decomposition and the fact that $\Lambda = \Lambda_0^{\#}(\sigma)$, with σ section of $\bigwedge^2 D^\circ$, we deduce that

$$\mathrm{Im}\Lambda^{\#} = \Lambda_0^{\#}(D^\circ) = \mathrm{orth}_{\omega_0} D$$

on $\mathcal{U}$. Hence

$$TS = \mathrm{orth}_{\omega_0} D \quad \text{and} \quad T^*S = D^\circ.$$

Consequently, ω_S is a section of $\bigwedge^2 D^\circ$. On the other hand, taking into account (11), we have

$$\bigwedge^2 T^*M = \Big(\bigwedge^2 D^\circ\Big) \oplus (D^\circ \wedge \langle df_1, \dots, df_{2n-2k}\rangle) \oplus \Big(\bigwedge^2 \langle df_1, \dots, df_{2n-2k}\rangle\Big)$$

on $\mathcal{U}$. Consequently, the 2-form ω_0 can be written as a sum of type

$$\omega_0 = \omega_0^{D^\circ} + \sum \tau_i \wedge df_i + \sum g_{ij} df_i \wedge df_j,$$

where $\omega_0^{D^\circ}$ is the part of ω_0 which is a section of $\bigwedge^2 D^\circ$ on $\mathcal{U}$, τ_i are sections of D° on $\mathcal{U}$ and g_{ij} are smooth functions on M. Due to the above expression of ω_0, formula (1) takes the form

$$\Phi = -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0^{D^\circ}\Big) \wedge \frac{(\omega_0^{D^\circ})^{k-2}}{(k-2)!} \wedge df_1 \wedge \dots \wedge df_{2n-2k}.$$

The 2-form $-\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0^{D^\circ}\Big)$ is the symplectic form on S. By setting

$$\omega_S = -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0^{D^\circ}\Big),$$

after a long computation, we prove that $\Lambda = \Lambda_0^{\#}(\sigma) = \Lambda^{\#}(\omega_S)$.

With a similar manner we can prove that, in the case where $m = 2n + 1$,

$$\omega_S = -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\Theta_0^{D^\circ}\Big).$$

3 From Toda to Volterra

In this section, by applying Theorem 1 and by imitating the procedure of constructing A_n-type Volterra lattices from the A_n-type Toda lattices presented in [5, 6], we construct a family Poisson structures associated to B_n-type Volterra lattices from the Lie–Poisson structure Λ_T of B_n-type Toda lattice that share the same Casimir with Λ_T.

3.1 Periodic B_n-Toda lattices

The periodic B_n-Toda lattice of n particles is the system of ordinary differential equations on $\mathbb{R}^{2n}$ which is Hamiltonian with respect to the canonical Poisson structure on $\mathbb{R}^{2n}$ and with Hamiltonian function

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{q_n} + e^{-q_1 - q_2}. \tag{12}$$

In Flaschka's coordinate system $(a_0, a_1, \dots, a_n, b_1, \dots, b_n)$ defined by

$$\begin{aligned} a_i &= \tfrac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad i = 1, 2, \dots, n-1, \\ a_n &= \tfrac{1}{2} e^{\frac{1}{2} q_n}, \\ a_0 &= \tfrac{1}{2} e^{-\frac{1}{2}(q_1 + q_2)}, \\ b_i &= -\tfrac{1}{2} p_i, \quad i = 1, 2, \dots, n, \end{aligned} \tag{13}$$

the Hamiltonian system (12) takes the form

$$\begin{aligned} \dot a_i &= a_i (b_{i+1} - b_i), \quad i = 1, 2, \dots, n-1, \\ \dot a_n &= -a_n b_n, \\ \dot a_0 &= a_0 (b_1 + b_2), \\ \dot b_i &= 2(a_i^2 - a_{i-1}^2), \quad i \neq 2, \\ \dot b_2 &= 2(a_2^2 - a_1^2 - a_0^2). \end{aligned} \tag{14}$$

It can be written as a Lax pair $\dot L = [B, L]$, where L is the matrix

$$L = \begin{pmatrix} b_1 & a_1 & & & & & a_0 & 0 \\ a_1 & \ddots & \ddots & & & & & -a_0 \\ & \ddots & \ddots & a_{n-1} & & & & \\ & & a_{n-1} & b_n & a_n & & & \\ & & & a_n & 0 & -a_n & & \\ & & & & -a_n & -b_n & \ddots & \\ & a_0 & & & & \ddots & \ddots & -a_1 \\ & & -a_0 & & & & -a_1 & -b_1 \end{pmatrix}. \tag{15}$$

Moreover, in the new variables $(a_0,\ldots,a_n,b_1,\ldots,b_n)$, the canonical bracket on $\mathbb{R}^{2n}$ is transformed into the Lie–Poisson bracket Λ_T on $\mathbb{R}^{2n+1}$ given by

$$\Lambda_T = a_0\frac{\partial}{\partial a_0}\wedge\left(\frac{\partial}{\partial b_1}+\frac{\partial}{\partial b_2}\right)+\sum_{i=1}^{n-1}a_i\frac{\partial}{\partial a_i}\wedge\left(\frac{\partial}{\partial b_{i+1}}-\frac{\partial}{\partial b_i}\right)-a_n\frac{\partial}{\partial a_n}\wedge\frac{\partial}{\partial b_n}$$

with respect to which the system (14) is Hamiltonian with Hamiltonian function $h=\frac{1}{2}\operatorname{Trace}L^2$. The rank of Λ_T is $2n$ on an open and dense subset $\mathcal{U}$ of $\mathbb{R}^{2n+1}$ and it has a single Casimir

$$F = a_0a_1a_2^2\cdots a_{n-1}^2a_n^2.$$

3.2 From Toda to Volterra lattices of B_n-type

In the following, using the procedure illustrated in Section 2 and Theorem 1, we construct another Poisson structure Λ on $\mathbb{R}^{2n+1}$ having the same Casimir invariant with Λ_T, the function F. To make our construction more clear, we work in the specific case where $n=4$.

We start with the Poisson tensor field

$$\Lambda_T = a_0\frac{\partial}{\partial a_0}\wedge\left(\frac{\partial}{\partial b_1}+\frac{\partial}{\partial b_2}\right)+\sum_{i=1}^{3}a_i\frac{\partial}{\partial a_i}\wedge\left(\frac{\partial}{\partial b_{i+1}}-\frac{\partial}{\partial b_i}\right)-a_4\frac{\partial}{\partial a_4}\wedge\frac{\partial}{\partial b_4}$$

on $\mathbb{R}^9$ and the function $F=a_0a_1a_2^2a_3^2a_4^2$. The tensor Λ_T can be viewed as a Poisson tensor field on $\mathbb{R}^{10}$ admitting F and b_0 (b_0 being the canonical coordinate on the extra factor $\mathbb{R}$) as Casimirs. On $\mathbb{R}^{10}$ we consider the canonical Poisson structure

$$\Lambda_0 = \frac{\partial}{\partial a_0}\wedge\frac{\partial}{\partial b_0}+\sum_{i=1}^{4}\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial b_i}$$

and the distribution D generated by the Hamiltonian vector fields $X_F=\Lambda_0^{\#}(dF)$ and $X_{b_0}=\Lambda_0^{\#}(db_0)$, where

$$X_F = a_1a_2^2a_3^2a_4^2\frac{\partial}{\partial b_0}+a_0a_2^2a_3^2a_4^2\frac{\partial}{\partial b_1}+2a_0a_1a_2a_3^2a_4^2\frac{\partial}{\partial b_2} + 2a_0a_1a_2^2a_3a_4^2\frac{\partial}{\partial b_3}+2a_0a_1a_2^2a_3^2a_4\frac{\partial}{\partial b_4}$$

and $X_{b_0}=-\frac{\partial}{\partial a_0}$, whose annihilator D° is

$$D^\circ = \left\{\sum_{i=0}^{4}(\alpha_i da_i+\beta_i db_i)\in\Omega^1(\mathbb{R}^{10}) \;\middle|\; \alpha_0=0 \text{ and } \beta_0a_1a_2^2a_3^2a_4^2 + a_0\beta_1a_2^2a_3^2a_4^2+2a_0a_1a_2\beta_2a_3^2a_4^2+2a_0a_1a_2^2a_3\beta_3a_4^2+2a_0a_1a_2^2a_3^2a_4\beta_4=0\right\}.$$

The family of 1-forms $(da_1,\ da_2,\ da_3,\ da_4,\ \sigma_0,\ \sigma_1,\ \sigma_2,\ \sigma_3)$, where

$$\begin{aligned}\sigma_0 &= 2a_0db_0 - a_2db_2,\\ \sigma_1 &= 2a_1db_1 - a_2db_2,\\ \sigma_2 &= -a_0db_0 - a_1db_1 + 2a_2db_2 - a_3db_3,\\ \sigma_3 &= -a_2db_2 + 2a_3db_3 - a_4db_4,\end{aligned}$$

provides, at every point (a,b) of $\mathbb{R}\times\mathcal{U}$, a basis of $D^\circ_{(a,b)}$. The section of maximal rank σ_T of $\bigwedge^2 D^\circ \to \mathbb{R}\times\mathcal{U}$ which corresponds to Λ_T, via the isomorphism $\Lambda_0^\#$, is written in this basis as

$$\begin{aligned}\sigma_T &= \tfrac{1}{2}da_1\wedge(\sigma_0-\sigma_1) + \tfrac{1}{2}da_2\wedge(\sigma_0+\sigma_1)\\ &\quad + (da_3+da_4)\wedge\left(\tfrac{1}{2}\sigma_0 + \tfrac{1}{2}\sigma_1 + \sigma_2 + \sigma_3\right).\end{aligned}$$

Now, we consider on $\mathbb{R}^{10}$ the 2-form

$$\begin{aligned}\sigma &= da_1\wedge da_2 + da_3\wedge da_4 + \sigma_0\wedge\sigma_1 + \sigma_2\wedge\sigma_3\\ &= da_1\wedge da_2 + da_3\wedge da_4 + 4a_0a_1db_0\wedge db_1\\ &\quad - a_0a_2db_0\wedge db_2 - 2a_0a_3db_0\wedge db_3 + a_0a_4db_0\wedge db_4\\ &\quad + 3a_1a_2db_1\wedge db_2 - 2a_1a_3db_1\wedge db_3 + a_1a_4db_1\wedge db_4\\ &\quad + 3a_2a_3db_2\wedge db_3 - 2a_2a_4db_2\wedge db_4 + a_3a_4db_3\wedge db_4,\end{aligned}$$

which, by construction, is a section of $\bigwedge^2 D^\circ \to \mathbb{R}\times\mathcal{U}$ and satisfies (2) (it can be checked after a long computation). Thus, its image via $\Lambda_0^\#$ defines a Poisson structure Λ on $\mathbb{R}^{10}$ having as Casimirs the functions F and b_0. We have

$$\begin{aligned}\Lambda &= \frac{\partial}{\partial b_1}\wedge\frac{\partial}{\partial b_2} + \frac{\partial}{\partial b_3}\wedge\frac{\partial}{\partial b_4} + 4a_0a_1\frac{\partial}{\partial a_0}\wedge\frac{\partial}{\partial a_1} - a_0a_2\frac{\partial}{\partial a_0}\wedge\frac{\partial}{\partial a_2}\\ &\quad - 2a_0a_3\frac{\partial}{\partial a_0}\wedge\frac{\partial}{\partial a_3} + a_0a_4\frac{\partial}{\partial a_0}\wedge\frac{\partial}{\partial a_4} + 3a_1a_2\frac{\partial}{\partial a_1}\wedge\frac{\partial}{\partial a_2}\\ &\quad - 2a_1a_3\frac{\partial}{\partial a_1}\wedge\frac{\partial}{\partial a_3} + a_1a_4\frac{\partial}{\partial a_1}\wedge\frac{\partial}{\partial a_4} + 3a_2a_3\frac{\partial}{\partial a_2}\wedge\frac{\partial}{\partial a_3}\\ &\quad - 2a_2a_4\frac{\partial}{\partial a_2}\wedge\frac{\partial}{\partial a_4} + a_3a_4\frac{\partial}{\partial a_3}\wedge\frac{\partial}{\partial a_4}.\end{aligned}$$

Consequently, Λ can be viewed as a Poisson structure on $\mathbb{R}^9$ ($\mathbb{R}^9 = \{(a,b)\in\mathbb{R}^{10} \mid b_0 = 0\}$) with F as Casimir. We remark that $(\mathbb{R}^9,\Lambda) = (\mathbb{R}^5,\Lambda_V)\times(\mathbb{R}^4,\Lambda')$ is the product of two Poisson manifolds whose the Poisson structures have, respectively, the following form:

$$\Lambda_V = \begin{pmatrix} 0 & 4a_0a_1 & -a_0a_2 & -2a_0a_3 & a_0a_4\\ -4a_0a_1 & 0 & 3a_1a_2 & -2a_1a_3 & a_1a_4\\ a_0a_2 & -3a_1a_2 & 0 & 3a_2a_3 & -2a_2a_4\\ 2a_0a_3 & 2a_1a_3 & -3a_2a_3 & 0 & a_3a_4\\ -a_0a_4 & -a_1a_4 & 2a_2a_4 & -a_3a_4 & 0\end{pmatrix},$$

$$\Lambda' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The Poisson tensor Λ_V is a quadratic Poisson tensor on $\mathbb{R}^5$ that has F as unique Casimir function and whose the Hamiltonian vector field $X_H = \Lambda_V^{\#}(dH)$, $H = a_0 + a_1 + a_2 + a_3 + a_4$, has the form of a Lotka–Volterra system (see, [3, 4, 7])

$$\begin{aligned} \dot{a_0} &= a_0(4a_1 - a_2 - 2a_3 + a_4), \qquad \dot{a_1} = a_1(-4a_0 + 3a_2 - 2a_3 + a_4), \\ \dot{a_2} &= a_2(a_0 - 3a_1 + 3a_3 - 2a_4), \qquad \dot{a_3} = a_3(2a_0 + 2a_1 - 3a_2 + a_4), \\ \dot{a_4} &= a_4(-a_0 - a_1 + 2a_2 - a_3). \end{aligned}$$

The above result agrees in principle with the philosophy of [4] that Volterra lattices are obtained from the Toda lattices by restriction to the a variables.

4 Another method of constructing Λ_V

In this section we develop a new method of constructing the Poisson structure Λ_V based on the properties of the Cartan matrix of the corresponding affine Lie algebra $B_4^{(1)}$. It is a simple application of our results which connect dynamical systems of Lotka–Volterra type with complex simple Lie algebras and which will presented in brief. A detailed presentation will be given in a forthcoming paper.

We consider the complex simple Lie algebra B_4 and its associated affine Lie algebra $B_4^{(1)}$ [1, 8]. Let (v_1, v_2, v_3, v_4) be the set of simple roots of the B_4 root system given by

$$v_1 = (1, -1, 0, 0), \quad v_2 = (0, 1, -1, 0), \quad v_3 = (0, 0, 1, -1), \quad v_4 = (0, 0, 0, 1),$$

and $v_0 = (-1, -1, 0, 0)$ its minimal negative root. It is well known [1] that the vectors of the family $(v_0, v_1, v_2, v_3, v_4)$ satisfy the linear relation

$$v_0 + v_1 + 2v_2 + 2v_3 + 2v_4 = 0 \tag{16}$$

and that the vector $u = (1, 1, 2, 2, 2)$ of the coefficients of (16) generates the kernel of the Cartan matrix C of $B_4^{(1)}$. The elements of $C = (c_{ij})$ are constructed from the system of roots $(v_0, v_1, v_2, v_3, v_4)$ via the formula

$$c_{ij} = 2\frac{(v_i, v_j)}{(v_i, v_i)}, \qquad i, j = 0, \dots, 4,$$

where $(\cdot, \cdot)$ denotes the usual inner product of $\mathbb{R}^4$. Hence, we can easily calculate that

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}.$$

We note that, since $\ker(C) = \langle(1,1,2,2,2)\rangle$, all the rows of C, which are vectors in $\mathbb{R}^5$, lie in the hyperplane M of $\mathbb{R}^5$ that is orthogonal to the vector $u = (1,1,2,2,2)$.

We consider a coordinate system $(a_0, a_1, a_2, a_3, a_4)$ on $\mathbb{R}^5$, we denote by l_i the vector corresponding to the i-row, $i = 0,\dots,4$, of C, and to each one we associate a vector field X_i on $\mathbb{R}^5$ as follows:

$$
\begin{aligned}
l_0 &= (2,0,-1,0,0) &&\to\ X_0 = 2a_0\frac{\partial}{\partial a_0} - a_2\frac{\partial}{\partial a_2},\\
l_1 &= (0,2,-1,0,0) &&\to\ X_1 = 2a_1\frac{\partial}{\partial a_1} - a_2\frac{\partial}{\partial a_2},\\
l_2 &= (-1,-1,2,-1,0) &&\to\ X_2 = -a_0\frac{\partial}{\partial a_0} - a_1\frac{\partial}{\partial a_1} + 2a_2\frac{\partial}{\partial a_2} - a_3\frac{\partial}{\partial a_3},\\
l_3 &= (0,0,-1,2,-1) &&\to\ X_3 = -a_2\frac{\partial}{\partial a_2} + 2a_3\frac{\partial}{\partial a_3} - a_4\frac{\partial}{\partial a_4},\\
l_4 &= (0,0,0,-2,2) &&\to\ X_4 = -2a_3\frac{\partial}{\partial a_3} + 2a_4\frac{\partial}{\partial a_4}.
\end{aligned}
$$

Note that $X_4 = -X_0 - X_1 - 2X_2 - 2X_3$, which means that $\operatorname{rank}(X_0,\dots,X_4) = 4$. Also, we have $[X_i, X_j] = 0$, for any $i,j = 0,\dots,4$.

Now, using any four from the five vectors fields X_i, $i = 0,\dots,4$, we can construct various quadratic Poisson structures on $\mathbb{R}^5$ of rank 4 having $F = a_0a_1a_2^2a_3^2a_4^2$ as unique Casimir. (We remark that the vector of exponents of F coincides with the generator of $\ker(C)$.) For example, we can consider the bivector fields

$$
X_0 \wedge X_2 + X_1 \wedge X_3,\ X_0 \wedge X_3 + X_1 \wedge X_2,\ X_0 \wedge (X_0 + X_1 - X_2) + X_2 \wedge X_3
$$

and check their referred properties. In this framework, the Poisson tensor Λ_V of the previous Section is written as

$$
\Lambda_V = X_0 \wedge X_1 + X_2 \wedge X_3.
$$

It is natural to ask the following question: *Which ones of the resulting Poisson tensors on $\mathbb{R}^5$ are Poisson isomorphic? Is there a rule giving such isomorphism?*

Since the Poisson structures that we are seeking are exterior products of linear combinations of $X_0,\dots,X_4$, which are in one-to-one correspondence with the rows of the Cartan matrix C, it is natural to assume that the required Poisson maps will be related with the transformations of $\mathbb{R}^5$ which map the rows of C to a linear combination of these rows. Any such transformation of $\mathbb{R}^5$ preserves, by construction, the hyperplane M of $\mathbb{R}^5$ and its transpose preserves the orthogonal space $\langle u\rangle$ of M. In other words, if $A = (\alpha_{ij})$ is the matrix of a such transformation, for any vector $v \in M$, we have

$$
(Av, u) = 0 \ \Leftrightarrow\ (v, {}^t\!Au) = 0 \ \Leftrightarrow\ {}^t\!Au = ku,\ k \in \mathbb{R}. \tag{17}
$$

Let us study in detail the two Poisson tensors $\Lambda_V = X_0 \wedge X_1 + X_2 \wedge X_3$ and $\Lambda'_V = X_0 \wedge X_2 + X_1 \wedge X_3$. A transformation between the two tensors will be associated with a transformation of the corresponding rows of the Cartan matrix. So, we seek $\phi\colon \mathbb{R}^5 \to \mathbb{R}^5$,

$$\phi(a) = (\phi_0(a), \phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a)),$$

such that

$$\phi_* X_0 = X_1, \quad \phi_* X_1 = X_3, \quad \phi_* X_2 = X_0, \quad \phi_* X_3 = X_2. \tag{18}$$

As a result we will have $\phi_* \Lambda_V = \Lambda'_V$.

Let us determine first, the linear transformations of $\mathbb{R}^5$ which map $l_0 \to l_1$, $l_1 \to l_3$, $l_2 \to l_0$ and $l_3 \to l_2$. Such a transformation will keep l_4 within the plane M, since $l_4 = -l_0 - l_1 - 2l_2 - 2l_3$. We denote by $A = (\alpha_{ij})$ its matrix. Then, the equations $Al_0 = l_1$, $Al_1 = l_3$, $Al_2 = l_0$ and $Al_3 = l_2$ define, respectively, the following systems of linear equations:

$$\begin{aligned} &2\alpha_{00} - \alpha_{02} = 0, && 2\alpha_{01} - \alpha_{02} = 0,\\ &2\alpha_{10} - \alpha_{12} = 2, && 2\alpha_{11} - \alpha_{12} = 0,\\ &2\alpha_{20} - \alpha_{22} = -1, && 2\alpha_{21} - \alpha_{22} = -1,\\ &2\alpha_{30} - \alpha_{32} = 0, && 2\alpha_{31} - \alpha_{32} = 2,\\ &2\alpha_{40} - \alpha_{42} = 0; && 2\alpha_{41} - \alpha_{42} = -1; \end{aligned}$$

$$\begin{aligned} &-\alpha_{00} - \alpha_{01} + 2\alpha_{02} - \alpha_{03} = 2, && -\alpha_{02} + 2\alpha_{03} - \alpha_{04} = -1,\\ &-\alpha_{10} - \alpha_{11} + 2\alpha_{12} - \alpha_{13} = 0, && -\alpha_{12} + 2\alpha_{13} - \alpha_{14} = -1,\\ &-\alpha_{20} - \alpha_{21} + 2\alpha_{22} - \alpha_{23} = -1, && -\alpha_{22} + 2\alpha_{23} - \alpha_{24} = 2,\\ &-\alpha_{30} - \alpha_{31} + 2\alpha_{32} - \alpha_{33} = 0, && -\alpha_{32} + 2\alpha_{33} - \alpha_{34} = -1,\\ &-\alpha_{40} - \alpha_{41} + 2\alpha_{42} - \alpha_{43} = 0; && -\alpha_{42} + 2\alpha_{43} - \alpha_{44} = -1. \end{aligned}$$

Choosing the fist equation from each system and solving we obtain the solution:

$$\alpha_{00} = \kappa, \quad \alpha_{01} = \kappa, \quad \alpha_{02} = 2\kappa, \quad \alpha_{03} = 2\kappa - 2, \quad \alpha_{04} = 2\kappa - 3, \quad \kappa \in \mathbb{R}.$$

Repeating in the same way we obtain the following form for the matrix A:

$$A = \begin{pmatrix} \kappa & \kappa & 2\kappa & 2\kappa - 2 & 2\kappa - 3\\ \lambda + 1 & \lambda & 2\lambda & 2\lambda - 1 & 2\lambda - 1\\ \mu & \mu & 2\mu + 1 & 2\mu + 3 & 2\mu + 3\\ \nu & \nu + 1 & 2\nu & 2\nu - 1 & 2\nu - 1\\ \xi & \xi - \frac{1}{2} & 2\xi & 2\xi + \frac{1}{2} & 2\xi + 1 \end{pmatrix}, \quad \kappa, \lambda, \mu, \nu, \xi \in \mathbb{R}.$$

Thereafter, we choose κ, λ, μ, ν and ξ in way that ${}^t\!Au = u$. One such choice is $\kappa = \lambda = \mu = \nu = \xi = 0$. Then,

$$A = \begin{pmatrix} 0 & 0 & 0 & -2 & -3\\ 1 & 0 & 0 & -1 & -1\\ 0 & 0 & 1 & 3 & 3\\ 0 & 1 & 0 & -1 & -1\\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

We now return to the system (18). We denote by $T\phi = \left(\dfrac{\partial \phi_i}{\partial a_j}\right)$ the Jacobian matrix of the map ϕ which represents the tangent map ϕ_* of ϕ. The equation $\phi_*(X_0(a)) = X_1(\phi(a))$ is equivalent to the system of partial differential equations

$$2\alpha_0 \frac{\partial \phi_0}{\partial a_0} - \alpha_2 \frac{\partial \phi_0}{\partial a_2} = 0, \qquad 2\alpha_0 \frac{\partial \phi_1}{\partial a_0} - \alpha_2 \frac{\partial \phi_1}{\partial a_2} = 2\phi_1,$$
$$2\alpha_0 \frac{\partial \phi_2}{\partial a_0} - \alpha_2 \frac{\partial \phi_2}{\partial a_2} = -\phi_2, \qquad 2\alpha_0 \frac{\partial \phi_3}{\partial a_0} - \alpha_2 \frac{\partial \phi_3}{\partial a_2} = 0,$$
$$2\alpha_0 \frac{\partial \phi_4}{\partial a_0} - \alpha_2 \frac{\partial \phi_4}{\partial a_2} = 0.$$

Similarly, using the other equations of (18), we obtain similar systems of partial differential equations. Selecting the first equation from each system, we obtain

$$2\alpha_0 \frac{\partial \phi_0}{\partial a_0} - \alpha_2 \frac{\partial \phi_0}{\partial a_2} = 0,$$
$$2\alpha_1 \frac{\partial \phi_0}{\partial a_1} - \alpha_2 \frac{\partial \phi_0}{\partial a_2} = 0,$$
$$-\alpha_0 \frac{\partial \phi_0}{\partial a_0} - \alpha_1 \frac{\partial \phi_0}{\partial a_1} + 2\alpha_2 \frac{\partial \phi_0}{\partial a_2} - \alpha_3 \frac{\partial \phi_0}{\partial a_3} = 2\phi_0,$$
$$-\alpha_2 \frac{\partial \phi_0}{\partial a_2} + 2\alpha_3 \frac{\partial \phi_0}{\partial a_3} - \alpha_4 \frac{\partial \phi_0}{\partial a_4} = -\phi_0.$$

The solution of this system is similar to the earlier ones. We have

$$\alpha_0 \frac{\partial \phi_0}{\partial a_0} = \kappa, \quad \alpha_1 \frac{\partial \phi_1}{\partial a_1} = \kappa, \quad \alpha_2 \frac{\partial \phi_0}{\partial a_2} = 2\kappa,$$
$$\alpha_3 \frac{\partial \phi_0}{\partial a_3} = 2\kappa - 2\phi_0, \quad \alpha_4 \frac{\partial \phi_0}{\partial a_4} = 2\kappa - 3\phi_0, \quad \kappa \in \mathbb{R}.$$

Taking $\kappa = 0$ we obtain the solution

$$\phi_0(a) = a_3^{-2} a_4^{-3}.$$

We observe that

$$\phi_0(a) = a_0^0 a_1^0 a_2^0 a_3^{-2} a_4^{-3},$$

i.e., the exponents of this expression are precisely the entries of the first row of the matrix A. Working in a similar fashion we compute the other unknowns ϕ_i, $i = 1, \ldots, 4$. Hence we have

$$\phi_0(a) = a_3^{-2} a_4^{-3}, \qquad \phi_1(a) = a_0 a_3^{-1} a_4^{-1}, \quad \phi_2(a) = a_2 a_3^3 a_4^3,$$
$$\phi_3(a) = a_1 a_3^{-1} a_4^{-1}, \quad \phi_4(a) = a_1^{-\frac{1}{2}} a_3^{\frac{1}{2}} a_4.$$

A simple check shows that

$$\phi_* \Lambda_V = \Lambda'_V,$$

where

$$\Lambda_V(a) = X_0 \wedge X_1 + X_2 \wedge X_3$$
$$= \begin{pmatrix} 0 & 4a_0a_1 & -a_0a_2 & -2a_0a_3 & a_0a_4 \\ -4a_0a_1 & 0 & 3a_1a_2 & -2a_1a_3 & a_1a_4 \\ a_0a_2 & -3a_1a_2 & 0 & 3a_2a_3 & -2a_2a_4 \\ 2a_0a_3 & 2a_1a_3 & -3a_2a_3 & 0 & a_3a_4 \\ -a_0a_4 & -a_1a_4 & 2a_2a_4 & -a_3a_4 & 0 \end{pmatrix},$$

and

$$\Lambda'_V(\phi(a)) = X_0 \wedge X_2 + X_1 \wedge X_3$$
$$= \begin{pmatrix} 0 & -2\phi_0\phi_1 & 3\phi_0\phi_2 & -2\phi_0\phi_3 & 0 \\ 2\phi_0\phi_1 & 0 & -3\phi_1\phi_2 & 4\phi_1\phi_3 & -2\phi_1\phi_4 \\ -3\phi_0\phi_2 & 3\phi_1\phi_2 & 0 & -\phi_2\phi_3 & \phi_2\phi_4 \\ 2\phi_0\phi_3 & -4\phi_1\phi_3 & \phi_2\phi_3 & 0 & 0 \\ 0 & 2\phi_1\phi_4 & -\phi_2\phi_4 & 0 & 0 \end{pmatrix}.$$

Furthermore, the transformation ϕ preserves the Casimir F:

$$(\phi^*F)(a) = F(\phi(a)) = \phi_0\phi_1\phi_2^2\phi_3^2\phi_4^2 = a_0a_1a_2^2a_3^2a_4^2 = F(a).$$

The Hamiltonian vector fields of the structures Λ_V and Λ'_V with Hamiltonians $H_V(a) = a_0 + a_1 + a_2 + a_3 + a_4$ and $H'_V(\phi) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4$, respectively, are Lotka–Volterra systems.

Finally, by introducing the transformation $y_i = \ln a_i$, $i = 0, \ldots, a_4$, [7] which is a global orientation-preserving diffeomorphism inside the positive orthant, we remark that Λ_V is isomorphic to the constant Poisson structure on $\mathbb{R}^5$ represented by the matrix

$$\Sigma_V = \begin{pmatrix} 0 & 4 & -1 & -2 & 1 \\ -4 & 0 & 3 & -2 & 1 \\ -1 & -3 & 0 & 3 & -2 \\ 2 & 2 & -3 & 0 & 1 \\ -1 & -1 & 2 & -1 & 0 \end{pmatrix},$$

whose kernel is generated by the vector $u = (1, 1, 2, 2, 2)$ of the exponents of F. This observation is a well known result for quadratic Poisson structures with a degenerate matrix of constant coefficients [7].

5 Conclusion and open questions

In this paper we use two different methods to generate Lotka–Volterra systems associated with affine Lie algebras. In the first method, we use the construction of Poisson brackets from a prescribed set of Casimirs to construct some Lotka–Volterra systems associated to complex simple Lie algebras of type B_n. They are obtained from the well-known Poisson structures of the B_n-periodic Toda lattices of Bogoyavlensky [2]. In the second method, we use the properties of the Cartan matrix of the corresponding affine Lie algebra $B_n^{(1)}$ to construct a family of quadratic Poisson brackets of Volterra type and hence a family of Lotka–Volterra systems. We have illustrated with the example of B_4 but the procedure can be applied for any $n \geq 2$ and to any affine Lie algebra. We conclude with a list of questions and open problems.

- We have started with a set of functions $f_1, \dots, f_{m-2k}$ and we have a construction which gives some Poisson brackets with the prescribed Casimirs. Does this exhaust all possibilities? In other words, is any Poisson bracket which has the functions $f_1, \dots, f_{m-2k}$ as Casimirs obtained by this procedure?

- Are the Lotka–Volterra obtained from the periodic Toda lattices (which are known to be integrable) also Liouville integrable? For example is the system defined by the tensor Λ_V and the Hamiltonian $H_V = a_0 + \cdots + a_4$ Liouville integrable?

- Why is the kernel of the Cartan matrix the same as the kernel of Σ_V?

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The Miller–Weller Equation: Complete Group Classification and Conservation Laws

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We consider a quasilinear equation which arises in financial mathematics: the equation introduced by Miller and Weller [*J. Econom. Dynam. Control* **19** (1995), 279–302]. The equation is studied under the prism of the theory of modern group analysis. Specifically the complete group classification is performed and the cases that can be mapped either to the heat equation or to some type of Burgers equation are indicated. Finally the nonlinear self-adjointness of the equation is investigated; with the help of a formal Lagrangian for the equation and, using its symmetries, conservations laws are constructed.

1 Introduction

In the Introduction of his now famous paper of 1973 Merton [22] remarked that 'since options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned'. The observation was not unreasonable for at that time trading in options was a minor part of most serious portfolios, including those of speculators[1]. The transactional costs involved with options *vis-a-vis* the actual stock were themselves a deterrent. By way of curious irony in the Conclusion of their also now famous and contemporaneous paper Black and Scholes [4] made the very telling point that their results could be extended to many other situations and, in a sense, that virtually every financial instrument could be regarded in terms of an option. Merton acknowledged the applicability of the approach of Black and

[1] One recalls the regular weekly column, Speculators's Diary, in the Sydney Bulletin of the times dealt in options maybe once or twice a year and then as a somewhat daring activity.

Scholes due to the specific assumptions they had made in their model. Subsequent research has shown that a number of these assumptions can be relaxed without affecting the viability of the model.

A number of research papers is devoted to models in financial mathematics and, in particular, their resolution via symmetry methods. Most of these equations have been shown to be related via a coordinate transformation to the classical heat equation [10, 25]. If such a connection exists then, evidently, all the known properties and solutions of the heat equation can be transformed and be applied to the equation under study. Furthermore the derivation of some similarity solutions in some instances is very important for applications. In this paper we examine a quasilinear equation, the equation of Miller and Weller [23]:

$$u_t + \frac{\sigma^2}{2} u_{xx} + (\alpha x + \beta u) u_x + \delta u = \gamma x. \tag{1}$$

There are various approaches to the analysis and resolution of a differential equation. In this paper we are concerned with the Lie point symmetry analysis as it is applied to certain equations that arise in financial mathematics [10]: We employ the methods of Modern Group Analysis to classify and obtain the Lie algebra of symmetries of the equations. Classification in this context means to find specific subsets in the parametric space of the equation, constants and free functions that give symmetries additional to those of the most general case. Because those special cases have a Lie algebra of higher dimension than the one of the general case, their investigation might be more profitable.

The Lie algebraic approach to the solution of evolution equations that arise in financial mathematics requires, in general, the existence of a sufficient number of Lie point symmetries. As a consequence we employ the following procedure (other methods include those found in [17, 18]): If an equation (or system of equations; the generalisation should be obvious) is well-endowed with Lie point symmetries, then one should seek to determine if the Lie algebra is isomorphic to that of the heat equation or to any well-studied equation having the Lie remarkable attribute. If this be the case, then a suitable transformation can be derived from the two different representations of the same Lie algebra. If that be not the case, then one should pursue other standard techniques, symmetry-based or not. In this respect it is perhaps ironic that the first treatments of the Black–Scholes equation using the Lie symmetry technique [12, 14] happened to be a treatment of an equation of both maximal symmetry and linearisable to the heat equation. The route to the solution in [12] was quite different and less straightforward from the ones applied in [10, 20] in subsequent years.

Finally the given model is investigated under the prism of self-adjointness [15, 16]. For the cases that a kind of self-adjointness exists we give the corresponding Lagrangian and the conservations laws that can be derived from the symmetries of the equation. Furthermore we use those conservation laws to reduce the equation and obtain a specific solution. For all the results presented in the present paper the symbolic package SYM for Mathematica [3, 7–9] was extensively used.

2 The Miller–Weller equation

Adopting the approach of Miller and Weller [23] on continuous-time stochastic saddlepoint-systems we consider the following model for rational expectations. There is an economic fundamental the value, x_t, of which follows a diffusion process. There is also an asset the price, y_t, of which is a forecast of rational expectations of future fundamentals (properly discounted). According to the model of Miller and Weller [23] the fundamental and asset prices are connected as follows

$$\begin{aligned} dx_t &= \alpha(x_t - x^*)dt + \beta(y_t - y^*)dt + \sigma dW_t \quad \text{and} \\ y_t &= E\left[\int_t^\infty -\gamma(x_s - x^*)e^{-\delta(s-t)}ds \mid \mathcal{F}_t\right] + y^*. \end{aligned}$$

The stars denote equilibrium states, which without loss of generality are assumed to be equal to zero. The uncertainty is assumed to be introduced into the model by a (possibly vector-valued) Wiener process W_t and the rational expectations of the fundamental are taken over the information set $\mathcal{F}_t = \sigma(W_s, s \leq t)$, the natural filtration generated by the Wiener process. The constant $\delta > 0$ is a discount factor. The divergence from equilibrium of the asset is assumed to have some feedback effect upon the dynamics of the fundamental. According to Miller and Weller [23] a number of classic models may be cast into this form. An example is Blanchard's model [5] which relates stockmarket prices to the level of real activity in the economy. Another model related to the above is the model of Krugman for target zones [19]. In this case $\beta = 0$ and the asset price is not assumed to have any feedback on the dynamics of the fundamental.

In a recent work [28] it has been shown that the original model of Miller and Weller is equivalent to an infinite-horizon forward-backward stochastic differential equation (FBSDE) of the form

$$\begin{aligned} dx_t &= (\alpha x_t + \beta y_t)dt + (\sigma_1 x_t + \sigma_2 y_t)dW_t, \\ dy_t &= (\gamma x_t + \delta y_t)dt + z_t dW_t, \end{aligned}$$

where z_t is a stochastic process that has to be chosen so that the original system is well posed from the point of view of measurability of the solutions with respect to the filtration generated by the Wiener process.

One way to look for solutions of such systems is to assume that the backward variable y (the asset) is provided by some function of t and the forward variable x, that is $y_t = f(x_t, t)$. We now apply Itô's rule on $f(x_t, t)$ to find dY_t. Then, when we match this expression with the expression for dy_t provided by the FBSDE system, we find that, in order to obtain compatibility, equation (1) should hold with some properly chosen final condition.

We refer to equation (1) as the general case of the Miller–Weller equation.

3 Analysis of the Miller–Weller equation

Before we proceed, we wish to highlight the major differences of the Miller–Weller equation as compared to other well-studied models in financial mathematics. In the Black–Scholes equation the price of the option, $u(t, x)$, depends upon the value of the underlying asset, x, and time, t. The asset, say the stock, is assumed to follow a geometric Brownian motion and is lognormally distributed. In order to model successfully other financial assets, such as bonds and interest rates, other processes are assumed. This is mainly because the fluctuation of these random variables is considerably more 'stable' than the inherent high volatility of, say, a stock. Vasicek assumed that the interest rate, r, follows a mean-reverting (Ornstein–Uhlenbeck) process [26]. The arguments behind such an assumption lie in the very nature of the financial asset. (If the (short-term) interest rate is large, then, on average, it should move down and, if the interest rate is small, then it is more probable that it rises.) In contrast to the lognormal process there is no reason for r to remain positive. This leads to a process that is known as the square root model for the short-term interest rate as in the Cox–Ingersoll–Ross (CIR) model for nonnegative interest rates [6]. Lastly the equation due to Longstaff [21] is obtained by assuming a process that is again of mean reversion but with two square roots in the stochastic differential equation (hence the name, double square-root model).

Interestingly, if we consider a generalisation of all the processes mentioned above [13],

$$\mathrm{d}x = (\nu - \mu\ x)\ \mathrm{d}t + \sigma\ x^{\gamma}\ \mathrm{d}B,$$

then the partial differential equation for zero-coupon bond pricing takes the form

$$u_t + \tfrac{1}{2}\sigma^2 x^{2\gamma} u_{xx} + (\nu - \mu x - \lambda\sigma x^{\gamma})\, u_x - xu = 0 \tag{2}$$

with the terminal condition $u(x, T) = 1$.

For specific values of the parameters, γ and λ, we recover, essentially, the Vasicek equation ($\gamma = 0$), the CIR equation ($\gamma = 1/2$ and $\lambda = 0$) and the Longstaff equation ($\gamma = 1/2$). From a symmetry point of view, equation (2) possesses solely the 'obvious' symmetries, ∂_t, $u\partial_u$ and $f\partial_u$, where f is a solution of the equation itself. Additional symmetries are obtained only for the particular values $\gamma = 0$, $1/2$, $3/2$ and 2 [25]. It is evident that the Miller–Weller equation does not fit in the class of partial differential equations as reflected in (2). Equation (2) is essentially a linear partial differential equation and therefore, if there exist values for the parameters that would allow it to exhibit more (five is a fair number) Lie point symmetries, then these cases would be easily reduced to the heat equation. Indeed for all those models that have been proposed in the literature and are special cases of (2) the transformation to the heat equation has been given [10]. The Miller–Weller equation is nonlinear provided $\beta \neq 0$. Obviously for all cases mentioned in what follows, where $\beta = 0$, there exist point transformations that link the equations to the heat equation. The focus of the present work lies in the nonlinear cases.

3.1 Group classification of the Miller–Weller equation

It is evident that for $\sigma = 0$ equation (1) is a quasilinear first-order PDE that can be handled adequately by the available analytical tools for first-order pdes. For this reason we exclude this case from our classification and we assume in what follows that $\sigma \neq 0$. Also, for $\beta = 0$, the equation becomes linear and its symmetries are isomorphic to the Lie point symmetries of the heat equation. Hence, by following the same logic as in [10], we can easily obtain a point transformation that links those two equations, see also [17] for an effective criterion that encompasses the class of parabolic partial differential equations

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u.$$

The analysis of equation (1) gives two (distinct) cases, $\alpha \neq \delta$ and $\alpha = \delta$, see Table 3.1 that follows.

The Miller–Weller equation for $\beta \neq 0$ and $\alpha = \delta$ has the Lie algebra $sl(2,\mathbb{R}) \oplus_s 2A_1$ as can be inferred by the Table of Lie Brackets:[2]

$[\cdot,\cdot]$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$\mathfrak{X}_4$	$\mathfrak{X}_5$
$\mathfrak{X}_1$	0	$\mathfrak{X}_2$	$-\mathfrak{X}_3$	$2\mathfrak{X}_4$	$-2\mathfrak{X}_5$
$\mathfrak{X}_2$	$-\mathfrak{X}_2$	0	0	0	$-2\mathfrak{X}_3$
$\mathfrak{X}_3$	$\mathfrak{X}_3$	0	0	$-2\mathfrak{X}_2$	0
$\mathfrak{X}_4$	$-2\mathfrak{X}_4$	0	$2\mathfrak{X}_2$	0	$4\mathfrak{X}_1$
$\mathfrak{X}_5$	$2\mathfrak{X}_5$	$2\mathfrak{X}_3$	0	$-4\mathfrak{X}_1$	0

The complete classification of the nondimensional generalised Burgers equation (NDGB)

$$u_t + uu_x + F(t)u_{xx} = 0 \tag{3}$$

was performed in [11,27].

The Miller–Weller equation for $\beta \neq 0$ and $\alpha = \delta$ has the same Lie algebra as the classical Burgers equation ($F(t) = \text{const.}$ in (3)). For each of the three subcases, following the same reasoning as in [10], a suitable transformation can be found that links the two equations. Namely, for Case 4 the point transformation,

$$u = \frac{1}{\beta}\left[4e^{\Omega_2^+ t}U(X,T)\Omega_2^+ - (\alpha + \Omega_2^+)x\right],$$
$$X = xe^{\Omega_2^+ t}, \qquad T = 2e^{2\Omega_2^+ t},$$

[2] $sl(2,\mathbb{R})$ is formed by the span of the elements $\mathfrak{X}_1$, $\mathfrak{X}_4$, $\mathfrak{X}_5$ and $2A_1$ from the rest. The given table can be obtained from all the subcases of $\alpha = \delta$ after a suitable change of basis.

Table 1. Group classification of the class $u_t + \frac{\sigma^2}{2}u_{xx} + (\alpha x + \beta u)u_x + \delta u = \gamma x,\ \beta \neq 0.$

N	Case	Symmetries
0	$\forall \alpha, \beta, \gamma, \delta, \sigma$	$\mathfrak{X}_1 = \partial_t$
		$\alpha \neq \delta$
1	$\Omega_1 > 0$	$\mathfrak{X}_{2,3} = 2e^{\frac{1}{2}(\alpha-\delta\pm\Omega_1^+)t}\partial_x - \frac{1}{\beta}(\alpha+\delta\mp\Omega_1^+)e^{\frac{1}{2}(\alpha-\delta\pm\Omega_1^+)t}\partial_u$
2	$\Omega_1 < 0$	$\mathfrak{X}_2 = 2e^{\frac{1}{2}(\alpha-\delta)t}\cos\frac{\Omega_1^+ t}{2}\partial_x$ $-\frac{1}{\beta}\left((\alpha+\delta)\cos\frac{\Omega_1^+ t}{2} + \Omega_1^+\sin\frac{\Omega_1^+ t}{2}\right)e^{\frac{1}{2}(\alpha-\delta)t}\partial_u$ $\mathfrak{X}_3 = 2e^{\frac{1}{2}(\alpha-\delta)t}\sin\frac{\Omega_1^+ t}{2}\partial_x$ $-\frac{1}{\beta}\left((\alpha+\delta)\sin\frac{\Omega_1^+ t}{2} - \Omega_1^+\cos\frac{\Omega_1^+ t}{2}\right)e^{\frac{1}{2}(\alpha-\delta)t}\partial_u$
3	$\Omega_1 = 0$	$\mathfrak{X}_2 = 2e^{\frac{1}{2}(\alpha-\delta)t}\partial_x - \frac{(\alpha+\delta)}{\beta}e^{\frac{1}{2}(\alpha-\delta)t}\partial_u$ $\mathfrak{X}_3 = 2e^{\frac{1}{2}(\alpha-\delta)t}t\partial_x - \frac{2-(\alpha+\delta)t}{\beta}e^{\frac{1}{2}(\alpha-\delta)t}\partial_u$
		$\alpha = \delta$
4	$\Omega_2 > 0$	$\mathfrak{X}_{2,3} = e^{\pm\Omega_2^+ t}\partial_x - \frac{\alpha\mp\Omega_2^+}{\beta}e^{\pm\Omega_2^+ t}\partial_u,$ $\mathfrak{X}_{4,5} = \Omega_2^+ e^{\pm 2\Omega_2^+ t}x\partial_x \pm e^{\pm 2\Omega_2^+ t}\partial_t - \frac{\beta u + 2(\alpha\mp\Omega_2^+)x}{\beta}\Omega_2^+ e^{\pm 2t\Omega_2^+}\partial_u$
5	$\Omega_2 < 0$	$\mathfrak{X}_2 = \cos(\Omega_2^+ t)\partial_x - \frac{1}{\beta}\left(\alpha\cos(\Omega_2^+ t) + \Omega_2^+\sin(\Omega_2^+ t)\right)\partial_u,$ $\mathfrak{X}_3 = \sin(\Omega_2^+ t)\partial_x - \frac{1}{\beta}\left(\alpha\sin(\Omega_2^+ t) - \Omega_2^+\cos(\Omega_2^+ t)\right)\partial_u$ $\mathfrak{X}_4 = \Omega_2^+\cos\left(2\Omega_2^+ t\right)x\partial_x + \sin\left(2\Omega_2^+ t\right)\partial_t$ $-\frac{\Omega_2^+}{\beta}((\beta u + 2\alpha x)\cos\left(2\Omega_2^+ t\right) + 2\Omega_2^+\sin\left(2\Omega_2^+ t\right)x)\partial_u$ $\mathfrak{X}_5 = \Omega_2^+\sin\left(2\Omega_2^+ t\right)x\partial_x - \cos\left(2\Omega_2^+ t\right)\partial_t$ $-\frac{\Omega_2^+}{\beta}((\beta u + 2\alpha x)\sin\left(2\Omega_2^+ t\right) - 2\Omega_2^+\cos\left(2\Omega_2^+ t\right)x)\partial_u$
6	$\Omega_2 = 0$	$\mathfrak{X}_2 = \partial_x - \frac{\alpha}{\beta}\partial_u,$ $\mathfrak{X}_3 = t\partial_x + \frac{1-\alpha t}{\beta}\partial_u,$ $\mathfrak{X}_4 = x\partial_x + 2t\partial_t - (u + 2\frac{\alpha}{\beta}x)\partial_u$ $\mathfrak{X}_5 = tx\partial_x + t^2\partial_t + \frac{x-2\alpha xt-\beta tu}{\beta}\partial_u$

$\Omega_1 = 4\beta\gamma + (\alpha+\delta)^2$, $\Omega_1^+ = \sqrt{|4\beta\gamma + (\alpha+\delta)^2|}$, $\Omega_2 = \beta\gamma + \alpha^2$, $\Omega_2^+ = \sqrt{|\beta\gamma + \alpha^2|}$.

yields the classical Burgers equation with $F(t) = \sigma^2/(8\Omega_2^+)$; for Case 5 the point transformation,

$$u = \frac{1}{\beta}\left[\frac{\Omega_2^+ U(X,T)}{\cos(\Omega_2^+ t)} - (\alpha + \Omega_2^+ \tan(\Omega_2^+ t))x\right],$$
$$X = x\sec(\Omega_2^+ t), \qquad T = \tan(\Omega_2^+ t),$$

yields the classical Burgers equation with $F(t) = \sigma^2/(2\Omega_2^+)$ and for Case (6) the transformation $u = (U - \alpha x)/\beta$ gives the classical Burgers equation with $F(t) = \sigma^2/2$. These results come in complete accordance with the established fact that, if an equation from the class

$$u_t = u_{xx} + F(t, x, u, u_x)$$

admits a five-dimensional Lie symmetry algebra, then it can be mapped to the Burgers equation [29].

Accordingly, for the case $\alpha \neq \delta$, the Lie algebras for each subcase are

$[\cdot,\cdot]$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$
$\mathfrak{X}_1$	0	$\frac{1}{2}\left(\alpha - \delta + \Omega_1^+\right)\mathfrak{X}_2$	$\frac{1}{2}\left(\alpha - \delta - \Omega_1^+\right)\mathfrak{X}_3$
$\mathfrak{X}_2$	$-\frac{1}{2}\left(\alpha - \delta + \Omega_1^+\right)\mathfrak{X}_2$	0	0
$\mathfrak{X}_3$	$-\frac{1}{2}\left(\alpha - \delta - \Omega_1^+\right)\mathfrak{X}_3$	0	0

when $\Omega_1^+ > 0$ and the Lie algebra is $E(1,1)$ ($A_{3,4}$ in the standard classification scheme),

$[\cdot,\cdot]$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$
$\mathfrak{X}_1$	0	$\frac{1}{2}(\alpha - \delta)\mathfrak{X}_2 - \frac{1}{2}\Omega_1^+\mathfrak{X}_3$	$\frac{1}{2}\Omega_1^+\mathfrak{X}_2 + \frac{1}{2}(\alpha - \delta)\mathfrak{X}_3$
$\mathfrak{X}_2$	$-\frac{1}{2}(\alpha - \delta)\mathfrak{X}_2 + \frac{1}{2}\Omega_1^+\mathfrak{X}_3$	0	0
$\mathfrak{X}_3$	$-\frac{1}{2}\Omega_1^+\mathfrak{X}_2 - \frac{1}{2}(\alpha - \delta)\mathfrak{X}_3$	0	0

when $\Omega_1 < 0$ and the Lie algebra is $E(1,1)$ ($A_{3,4}$ in the standard classification scheme) and

$[\cdot,\cdot]$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$
$\mathfrak{X}_1$	0	$\frac{1}{2}(\alpha - \delta)\mathfrak{X}_2$	$\mathfrak{X}_2 + \frac{1}{2}(\alpha - \delta)\mathfrak{X}_3$
$\mathfrak{X}_2$	$-\frac{1}{2}(\alpha - \delta)\mathfrak{X}_2$	0	0
$\mathfrak{X}_3$	$-\mathfrak{X}_2 - \frac{1}{2}(\alpha - \delta)\mathfrak{X}_3$	0	0

for $\Omega_1 = 0$ with the Lie algebra $A_{3,2}$.

For Case 1 the point transformation,

$$u = \frac{1}{2\beta}\left[4e^{\frac{\alpha-\delta+\Omega_1^+}{2}t}\Omega_1^+ U(X,T) - x\left(\alpha + \delta + \Omega_1^+\right)\right],$$
$$X = \frac{1}{2}e^{\frac{\delta-\alpha+\Omega_1^+}{2}t}x, \qquad T = e^{\Omega_1^+ t},$$

yields the NDGB equation with $F(t) = \frac{\sigma^2}{8\Omega_1^+} t^{\frac{\delta-\alpha}{\Omega_1^+}}$; for Case 2 the point transformation,

$$u = \frac{1}{2\beta}\left[2\Omega_1^+ e^{\frac{\alpha-\delta}{2}t} \sec\left(\frac{\Omega_1^+}{2}t\right) U(X,T) - x\left(\alpha+\delta+\Omega_1 \tan\left(\frac{\Omega_1^+}{2}t\right)\right)\right],$$
$$X = \frac{1}{2} e^{-\frac{\alpha-\delta}{2}t} \sec\left(\frac{\Omega_1^+}{2}t\right) x, \qquad T = \tan\left(\frac{\Omega_1^+}{2}t\right),$$

yields the NDGB equation with $F(t) = \frac{\sigma^2}{4\Omega_1^+}\exp(2\frac{\delta-\alpha}{\Omega_1^+}\tan^{-1} t)$ and, finally, for Case 3 the point transformation,

$$u = \frac{1}{2\beta}\left[4e^{\frac{1}{2}(\alpha-\delta)t} U(X,T) - (\alpha+\delta)x\right], \qquad X = \frac{e^{\frac{1}{2}(\delta-\alpha)t}x}{2}, \qquad T = t,$$

yields the NDGB equation with $F(t) = \frac{\sigma^2}{8} e^{(\delta-\alpha)t}$.

3.2 Conservation laws of the Miller–Weller equation

We turn our attention to find conservation laws for the Miller–Weller equation. This is accomplished by following the method proposed by Ibragimov [15,16]. Let the formal Lagrangian be

$$\upsilon(x,t)\left(-\gamma x + \delta u + u_t + (x\alpha + \beta u)u_x + \tfrac{1}{2}\sigma^2 u_{xx}\right). \tag{4}$$

The equation adjoint to (1) is

$$\upsilon_t = \tfrac{1}{2}\sigma^2 \upsilon_{xx} - (\alpha x + \beta u)\upsilon_x + (\delta - \alpha)\upsilon. \tag{5}$$

Without any other knowledge about the adjoint equation, (4) is a Lagrangian of the system consisting of the equations (1) and (5). To be also a Lagrangian for equation (1), equation (1) must be self-adjoint. We check under which conditions this is true.

Obviously, by direct substitution of υ by u into equation (5), one can see that equation (1), or some special case of it, is not strictly self-adjoint.

We turn to quasiself-adjointness by assuming that $\upsilon = \Phi(u)$; after substituting υ into (5) and eliminating u_t using (1) we arrive at an identity that is a multivariable polynomial with respect to u_x and u_{xx}. Equation of the coefficients of this polynomial to zero gives the following system:

$$\Phi'' = 0, \qquad \Phi' = 0, \qquad (u\delta - x\gamma)\Phi' - (\alpha-\delta)\Phi(u) = 0. \tag{6}$$

System (6) has a nonzero solution if and only if $\alpha = \delta$. For that case $\Phi(u) = c$, where c is a constant that, without loss of generality, can be taken equal to one. The formal Lagrangian for this case is the equation itself. In fact it is easy to put the equation into the conserved form

$$\left(-\frac{\gamma}{2}x^2 + \alpha x u(x,t) + \frac{\beta}{2}u(x,t)^2 + \frac{1}{2}\sigma^2 u_x\right)_x + u_t.$$

On the other hand, for equation (1) to be nonlinear self-adjoint, we assume that $v = \Phi(x,t,u(x,t))$. Similarly, by using this assumption, we arrive at the system:

$$\begin{aligned} &\Phi_{uu} = 0, \qquad \Phi_{xu} = 0, \qquad \Phi_u = 0, \\ &(\delta-\alpha)\Phi - x\gamma\Phi_u + u\delta\Phi_u - \Phi_t - (x\alpha + u\beta)\Phi_x + \tfrac{1}{2}\sigma^2\Phi_{xx} = 0. \end{aligned} \tag{7}$$

The nontrivial solution of the system is $\Phi = ce^{(\delta-\alpha)t}$. In summary the formal Lagrangian of equation (1) is

$$e^{(\delta-\alpha)t}\left(-\gamma x + \delta u(x,t) + u_t + (x\alpha + \beta u(x,t))u_x + \tfrac{1}{2}\sigma^2 u_{xx}\right).$$

By using the symmetries found in the previous Section we can look for nontrivial conservation vectors for the cases of Table 3.1. The only nontrivial conservation vector found in this way is for $\alpha \neq \delta$ and the use of symmetry $\mathfrak{X}_1$

$$\left(-\tfrac{1}{2}e^{(\delta-\alpha)t}\left(2(x\alpha+\beta u)u_t + \sigma^2 u_{xt}\right), -e^{(\delta-\alpha)t}u_t\right).$$

We use this conservation law to find a solution of the equation. We assume that

$$2(x\alpha+\beta u)u_t + \sigma^2 u_{xt} = g(t), \qquad e^{(\delta-\alpha)t}u_t = h(x)$$

and find that a solution of the equation is

$$u(x,t) = c_1\frac{e^{(\alpha-\delta)t}}{\alpha-\delta} - \frac{\alpha}{\beta}x, \tag{8}$$

when $\alpha\delta + \beta\gamma = 0$.

These results come in accordance to the well-established framework in [24] in which the complete treatment of second-order evolution equations was given using the direct method [1, 2].

4 Discussion

In this paper we have considered the Miller–Weller equation which arises in financial mathematics. Although the equation is nonlinear, its complete group classification reveals a deep connection to the heat equation for $\beta = 0$ and the nondimensional generalised Burgers equation for $\beta \neq 0$.

The central theme of the paper is the use of the Lie theory of continuous groups for the resolution of differential equations. The provenance of these equations is oftentimes unimportant, but in this instance we have chosen a class of partial differential equations of considerable application in Finance. The methods based upon symmetry are algorithmic and a symbolic package may be used, in this instance it was SYM for `Mathematica`.

We do emphasise that we wished to illustrate two approaches to the solution of the Miller–Weller equation, namely a similarity solution and a solution through a conservation law. We were not concerned with the solution of (1) in the realistic context of finance for which one needs to consider the terminal condition. Typically this would be of the form $u(x,\, T) = 1$ (the actual value of the number is immaterial) which means that we have the dual conditions, $t = T$ and $u = 1$, to be satisfied by any solution of the equation, (1), itself. To find that solution one needs to take a combination of the symmetries of the equation, (1), which is compatible with these conditions. Then one may proceed either through the approach of the conservation law or by a similarity reduction. In the case that $\alpha \neq \delta, \Omega_1 > 0$ the solution is

$$u(t,\, x) = \frac{2\Omega_1^+ \exp \frac{(\alpha-\delta)(t-T)+\Omega_1^+(t+T)}{2} + 2\gamma x \left(\exp\left(\Omega_1^+ t\right) - \exp\left(\Omega_1^+ T\right)\right)}{\left(\Omega_1^+ - \alpha - \delta\right) \exp\left(\Omega_1^+ T\right) + \left(\Omega_1^+ + \alpha + \delta\right) \exp\left(\Omega_1^+ t\right)}.$$

Consequently even in the case of reduced symmetry the use of the Lie theory of continuous groups leads us to a solution.

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Differential Invariants for the Korteweg–de Vries Equation

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Differential invariants for the maximal Lie invariance group of the Korteweg–de Vries equation are computed using the moving frame method and compared with existing results. Closed forms of differential invariants of any order are presented for two sets of normalization conditions. Minimal bases of differential invariants associated with the chosen normalization conditions are given.

1 Introduction

Invariants and differential invariants are important objects associated with transformation groups. They play a role for finding invariant, partially invariant and differentially invariant solutions [11, 14, 19], in computer vision [16], for the construction of invariant discretization schemes [3, 4, 7, 12, 13, 15, 21] and in the study of invariant parameterization schemes [1, 2, 20].

There are two main ways to construct differential invariants for Lie group actions. The notation we use follows the book [14] and the papers [5, 8, 15–18]. Let G be a (pseudo)group of transformations acting on the space of variables (x, u), where $x = (x^1, \dots, x^p)$ is the tuple of independent variables and $u = (u^1, \dots, u^q)$ is the tuple of dependent variables. Let $\mathfrak{g}$ be the Lie algebra of vector fields that is associated with G.

The first way for the computation of differential invariants uses the infinitesimal method [6, 11, 14, 19]. The criterion for a function I defined on a subset of the corresponding nth-order jet space to be a differential invariant of the maximal Lie invariance group G is that the condition

$$\mathrm{pr}^{(n)}\mathbf{v}(I) = 0, \qquad (1)$$

holds for any vector field $\mathbf{v} \in \mathfrak{g}$. In equation (1), the vector field $\mathbf{v}$ is of the form $\mathbf{v} = \xi^i(x, u)\partial_{x^i} + \phi_\alpha(x, u)\partial_{u^\alpha}$ (the summation over double indices is applied), and $\mathrm{pr}^{(n)}\mathbf{v}$ denotes the standard nth prolongation of $\mathbf{v}$. In the framework of the infinitesimal method, the differential invariants I are computed by solving the system of quasilinear first-order partial differential equations of the form (1), where the vector field $\mathbf{v}$ runs through a generating set of $\mathfrak{g}$.

The second possibility for computing differential invariants uses moving frames [5, 8, 9]. The main advantage of the moving frame method is that it avoids the integration of differential equations, which is necessary in the infinitesimal approach. At the same time, using moving frames allows one to invoke the powerful recurrence relations, which can be helpful in studying the structure of the algebra of differential invariants.

In this paper, we study differential invariants for the maximal Lie invariance group of the Korteweg–de Vries (KdV) equation. This problem was already considered in [5,17] and in [6] within the framework of the moving frame and infinitesimal approaches, respectively. Thus, on one hand it is instructive to compare and review the results available in the literature. On the other hand, we extend these results in the present paper. In particular, we explicitly present functional bases of differential invariants of arbitrary order for the aforementioned group.

The further organization of the paper is the following. In Section 2 we restate the maximal Lie invariance group of the KdV equation. Section 3 collects some results related to a moving frame for the maximal Lie invariance group of the KdV equation as presented in [5]. We also introduce an alternative moving frame in this section. Section 4 contains our main results, which are a complete list of functionally independent differential invariants for the maximal Lie invariance group of KdV equation of *any* order as well as the description of a basis of differential invariants for the new normalization introduced in Section 3. Section 5 contains some remarks related to the results of the paper.

2 Lie symmetries of the KdV equation

The KdV equation is undoubtedly one of the most important partial differential equations in mathematical physics. It describes the motion of long shallow-water waves in a channel. Here we will use it in the following dimensionless form:

$$u_t + uu_x + u_{xxx} = 0. \qquad (2)$$

The KdV equation is completely integrable using inverse scattering [10]. The coefficients of each vector field $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ generating a one-parameter Lie symmetry group of the KdV equation satisfy the system of determining equations

$$\tau_x = \tau_u = \xi_u = \eta_t = \eta_x = 0, \quad \eta = \xi_t - \tfrac{2}{3}u\tau_t, \quad \eta_u = -\tfrac{2}{3}\tau_t = -2\xi_x \qquad (3)$$

with the general solution

$$\tau = 3c_4t + c_1, \quad \xi = c_4x + c_3t + c_2, \quad \eta = -2c_4u + c_3,$$

where $c_1, \dots, c_4$ are arbitrary constants. Hence the maximal Lie invariance algebra $\mathfrak{g}$ of (2) is spanned by the four vector fields

$$\partial_t, \quad \partial_x, \quad t\partial_x + \partial_u, \quad 3t\partial_t + x\partial_x - 2u\partial_u. \qquad (4)$$

Associated with these basis elements are the one-parameter symmetry groups of (i) time translations, (ii) space translations, (iii) Galilean boosts and (iv) scalings. The most general Lie symmetry transformation of the KdV equation can be constructed using these elementary one-parameter groups:

$$T = e^{3\varepsilon_4}(t+\varepsilon_2), \quad X = e^{\varepsilon_4}(x+\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_3 t), \quad U = e^{-2\varepsilon_4}(u+\varepsilon_3), \tag{5}$$

where $\varepsilon_1,\dots,\varepsilon_4 \in \mathbb{R}$ are continuous group parameters. The KdV equation also admits a discrete point symmetry, given by simultaneous changes of the signs of the variables t and x.

The prolongation of the general element Q of the algebra $\mathfrak{g}$ has

$$\eta^\alpha = -(3\alpha_1+\alpha_2+2)c_4u_\alpha - \alpha_1c_3u_{\alpha_1-1,\alpha_2+1},$$

as the coefficient of ∂_{u_α}, where $\alpha=(\alpha_1,\alpha_2)$ is a multiindex, $\alpha_1,\alpha_2\in\mathbb{N}\cup\{0\}$, and $u_\alpha = \partial^{\alpha_1+\alpha_2}u/\partial t^{\alpha_1}\partial x^{\alpha_2}$ as usual.

Using the chain rule, from the above transformation formula (5) one obtains the expressions for the transformed derivative operators,

$$\mathrm{D}_T = e^{-3\varepsilon_4}(\mathrm{D}_t-\varepsilon_3\mathrm{D}_x), \quad \mathrm{D}_X = e^{-\varepsilon_4}\mathrm{D}_x.$$

In [5] these operators were used for listing some of the lower order transformed partial derivatives of u. However, in order to obtain a closed formula for a functional basis of differential invariants of *arbitrary* order for the KdV equation, it is useful to attempt to derive a closed-form expression for the transformed derivatives of u. Such an expression is

$$\begin{aligned} U_\alpha &= e^{-(3\alpha_1+\alpha_2+2)\varepsilon_4}(\mathrm{D}_t-\varepsilon_3\mathrm{D}_x)^{\alpha_1}\mathrm{D}_x^{\alpha_2}u \\ &= e^{-(3\alpha_1+\alpha_2+2)\varepsilon_4}\sum_{k=0}^{\alpha_1}(-\varepsilon_3)^k\binom{\alpha_1}{k}u_{\alpha_1-k,\alpha_2+k}. \end{aligned} \tag{6}$$

In particular, the expressions for U_T and U_X are

$$U_T = e^{-5\varepsilon_4}(u_t-\varepsilon_3u_x), \quad U_X = e^{-3\varepsilon_4}u_x.$$

3 A moving frame for the KdV equation

As the maximal Lie invariance group of the KdV equation is finite-dimensional, we only review the construction of moving frames for finite-dimensional group actions here. Details on the moving frame construction for Lie pseudogroups can be found, e.g., in [5, 18].

Definition 1. Let there be given a Lie group G acting on a manifold M. A *right moving frame* is a mapping $\rho\colon M\to G$ that satisfies the property $\rho(g\cdot z)=\rho(z)g^{-1}$ for any $g\in G$ and $z\in M$.

The theorem on moving frames, see e.g. [9, 16, 17], guarantees the existence of a moving frame in the neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z. Moving frames are constructed using a procedure called normalization, which is based on the selection of a submanifold (the cross-section) that intersects the group orbits only once and transversally.

There exist infinitely many possibilities to construct a moving frame. The single moving frames differ in the choice of the respective cross-sections. The moving frame constructed in [5] rests on the normalization conditions

$$T = 0, \quad X = 0, \quad U = 0, \quad U_T = 1, \tag{7}$$

i.e., it is defined on the first jet space J^1. It is necessary to construct the moving frame on the first jet space, as the maximal Lie invariance group of the KdV equation does not act freely on the space M, spanned by t, x and u. The action of G first becomes free when prolonged to J^1, which is then the proper space to construct the moving frame $\rho^{(1)}\colon J^1 \to G$ on. Solving the above algebraic system (7) for the group parameters $\varepsilon_1, \dots, \varepsilon_4$ yields the moving frame $\rho^{(1)}$

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{5}\ln(u_t + uu_x), \tag{8}$$

which is well defined provided that $u_t + uu_x > 0$. This moving frame becomes singular when $u_t + uu_x = 0$. The latter condition is equivalent, on the manifold of the KdV equation, to the condition that $u_{xxx} = 0$ and implies, together with the KdV equation, that $u_{xx} = 0$.

Another possible normalization, leading to an alternative moving frame, is the following:

$$T = 0, \quad X = 0, \quad U = 0, \quad U_X = 1.$$

Solving the normalization conditions gives the associated moving frame

$$\varepsilon_1 = -t, \quad \varepsilon_2 = -x, \quad \varepsilon_3 = -u, \quad \varepsilon_4 = \frac{1}{3}\ln u_x, \tag{9}$$

which is well defined provided that $u_x > 0$.

Note that for $u_t + uu_x < 0$ (resp. $u_x < 0$) one can replace the condition $U_T = 1$ by $U_T = -1$ (resp. $U_X = 1$ by $U_X = -1$).

4 Differential invariants for the KdV equation

The above moving frames can now be used to construct differential invariants using the *method of invariantization* [5, 16, 17].

Definition 2. The *invariantization* of a function $f\colon M \to \mathbb{R}$ is the function defined by

$$\iota(f) = f(\rho(z) \cdot z).$$

We first construct the set of all functionally independent differential invariants for the maximal Lie invariance group of the KdV equation using the moving frame (8). An exhaustive list of differential invariants of any order was not given in [5]. Such a list is obtained by plugging the moving frame (8) into the transformed derivatives (6). This yields

$$I_\alpha = \iota(U_\alpha) = (u_t + uu_x)^{-(3\alpha_1+\alpha_2+2)/5} \sum_{k=0}^{\alpha_1} \binom{\alpha_1}{k} u^k u_{\alpha_1-k,\alpha_2+k}, \tag{10}$$

where $\alpha_1 > 1$ or $\alpha_2 > 0$. Invariantizing t, x, u and u_t, one recovers the normalization conditions (7) and the associated differential invariants are dubbed *phantom invariants*. The corresponding invariantized form of the KdV equation is $1 + I_{03} = 0$.

Using the alternative moving frame (9), invariantization of (6) leads to the following set of functionally independent differential invariants of the maximal Lie invariance group of the KdV equation,

$$I_\alpha = \iota(U_\alpha) = u_x^{-(3\alpha_1+\alpha_2+2)/3} \sum_{k=0}^{\alpha_1} \binom{\alpha_1}{k} u^k u_{\alpha_1-k,\alpha_2+k}, \tag{11}$$

where $\alpha_1 > 0$ or $\alpha_2 > 1$, and $H^1 = \iota(t) = 0$, $H^2 = \iota(x) = 0$, $I_{00} = \iota(u) = 0$ and $I_{01} = \iota(u_x) = 1$ exhaust the set phantom invariants for this moving frame. Then the invariantization of the KdV equation yields the invariant form $I_{10} + I_{03} = 0$. The advantage of the form (11) of differential invariants compared to the form (10), which follows from the normalization (7) chosen in [5], is that these invariants are singular only on the subset $u_x = 0$, which is contained in the subset $u_{xx} = 0$ on which the invariants (10) are singular (again, when restrict to the KdV equation).

In principle, by computing the form of differential invariants of any order we have already solved the problem to exhaustively describe all the differential invariants for the maximal Lie invariance group of the KdV equation. On the other hand, it is instructive to study the structure of the algebra of differential invariants in some more detail.

In particular, an interesting open problem in the theory of differential invariants is to find minimal generating set of differential invariants in an algorithmic way. This is the set of differential invariants that is sufficient to generate all differential invariants by means of acting on the generating invariants with the operators of invariant differentiation and taking combinations of the basis invariants with these invariant derivatives. Often the computation of the *syzygies* among the differential invariants is a crucial step to prove the minimality of a given generating set. The two operators of invariantization for the maximal Lie invariance group G of the KdV equation follow from the invariantization of the operators of total differentiation D_t and D_x and they are

$$\mathrm{D}_t^{\mathrm{i}} = \iota(\mathrm{D}_t) = (u_t + uu_x)^{-3/5}(\mathrm{D}_t + u\mathrm{D}_x),$$
$$\mathrm{D}_x^{\mathrm{i}} = \iota(\mathrm{D}_x) = (u_t + uu_x)^{-1/5}\mathrm{D}_x.$$

In [5] it was claimed that the invariants

$$I_{01} = \frac{u_x}{(u_t + uu_x)^{3/5}}, \quad I_{20} = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}$$

form a generating set of the algebra of differential invariants for the KdV equation. While this is certainly true, this set is not minimal. In [17] it was shown that the differential invariant I_{01} is in fact sufficient to generate the entire algebra of differential invariants for the KdV equation. The crucial step missed in finding the minimal generating set in [5] was the use of the commutator formula for the operators of invariant differentiation $\mathrm{D}^{\mathrm{i}}_t$ and $\mathrm{D}^{\mathrm{i}}_x$, which is

$$[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x] = \tfrac{3}{5}(I_{11} + I_{01}^2)\mathrm{D}^{\mathrm{i}}_t - \tfrac{1}{5}(I_{20} + 6I_{01})\mathrm{D}^{\mathrm{i}}_x. \tag{12}$$

From the recurrence relation

$$\mathrm{D}^{\mathrm{i}}_t I_{01} = -\tfrac{3}{5}I_{01}^2 + I_{11} - \tfrac{3}{5}I_{01}I_{20}$$

one can solve for I_{11} in terms of I_{01} and I_{20}. Applying the commutation relation (12) to the invariant I_{01} then allows solving for I_{20} solely in terms of I_{01}, which explicitly gives

$$I_{20} = \frac{[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x]I_{01} - \frac{3}{5}(\mathrm{D}^{\mathrm{i}}_t I_{01} + \frac{8}{5}I_{01}^2)\mathrm{D}^{\mathrm{i}}_t I_{01} + \frac{6}{5}I_{01}\mathrm{D}^{\mathrm{i}}_x I_{01}}{\frac{9}{25}I_{01}\mathrm{D}^{\mathrm{i}}_t I_{01} - \frac{4}{5}\mathrm{D}^{\mathrm{i}}_x I_{01}},$$

which shows that I_{01} is indeed the minimal generating set of the algebra of differential invariants for the KdV equation.

We now repeat the computation of a basis of differential invariants for the moving frame (9). The associated operators of invariant differentiation for this moving frame are the same that were constructed in [6] within the framework of the infinitesimal approach,

$$\mathrm{D}^{\mathrm{i}}_t = u_x^{-1}(\mathrm{D}_t + u\mathrm{D}_x), \quad \mathrm{D}^{\mathrm{i}}_x = u_x^{-1/3}\mathrm{D}_x.$$

The computation of corresponding recurrence relations differs from that given in [5, 17] only in minor details. Identifying $c_3 = \xi_t$ and $c_4 = \frac{1}{3}\tau_t$, we obtain the invariantized forms

$$\hat\tau = \iota(\tau), \quad \hat\xi = \iota(\xi), \quad \hat\eta = \hat\eta^{00} = \iota(\eta),$$
$$\hat\eta^\alpha = \iota(\eta^\alpha) = -\frac{3\alpha_1 + \alpha_2 + 2}{3}I_\alpha\hat\tau^1 - \alpha_1 I_{\alpha_1-1,\alpha_2+1}\hat\eta, \quad \alpha_1 + \alpha_2 > 0,$$

and the first three forms $\hat\tau$, $\hat\xi$ and $\hat\eta$ jointly with $\hat\tau^1 = \iota(\tau^1)$ make up, in view of the invariantized counterpart of the determining equations (3), a basis of the invariantized Maurer–Cartan forms of the algebra $\mathfrak{g}$. The recurrence formulas for the normalized differential invariants are

$$\mathrm{d_h}H^1 = \omega^1 + \hat\tau, \quad \mathrm{d_h}H^2 = \omega^2 + \hat\xi, \quad \mathrm{d_h}I_\alpha = I_{\alpha_1+1,\alpha_2}\omega^1 + I_{\alpha_1,\alpha_2+1}\omega^2 + \hat\eta^\alpha,$$

where the form $\omega^1 = \iota(\mathrm{d}x)$ and $\omega^2 = \iota(\mathrm{d}x)$ constitute the associated invariantized horizontal co-frame, $\mathrm{d_h}$ is the horizontal differential and so $\mathrm{d_h}F = (\mathrm{D}^{\mathrm{i}}_t F)\omega^1 + (\mathrm{D}^{\mathrm{i}}_x F)\omega^2$. We take into account that $H^1 = 0$, $H^2 = 0$, $I_{00} = 0$ and $I_{01} = 1$ and solve the corresponding recurrence formulas with respect to the basis invariantized Maurer–Cartan forms,

$$\hat{\tau} = -\omega^1, \quad \hat{\xi} = -\omega^2, \quad \hat{\eta} = -I_{10}\omega^1 - \omega^2, \quad \hat{\tau}^1 = I_{11}\omega^1 + I_{02}\omega^2.$$

Then splitting of the other recurrence formulas yields

$$\mathrm{D}^{\mathrm{i}}_t I_\alpha = I_{\alpha_1+1,\alpha_2} - \frac{3\alpha_1 + \alpha_2 + 2}{3} I_{11} I_\alpha + \alpha_1 I_{10} I_{\alpha_1-1,\alpha_2+1},$$
$$\mathrm{D}^{\mathrm{i}}_x I_\alpha = I_{\alpha_1,\alpha_2+1} - \frac{3\alpha_1 + \alpha_2 + 2}{3} I_{02} I_\alpha + \alpha_1 I_{\alpha_1-1,\alpha_2+1},$$

where $\alpha_1 > 0$ or $\alpha_2 > 1$.

It is obvious from the above split recurrence formulas for that the whole set of differential invariants of the maximal Lie symmetry group of the KdV equation is generated by the two lowest-order normalized invariants

$$I_{10} = u_x^{-5/3}(u_t + uu_x), \quad I_{02} = u_x^{-4/3} u_{xx}.$$

At the same time, the differential invariant I_{02} is expressed in terms of invariant derivatives of I_{10} and hence a basis associated with the moving frame (9) consists of the single element I_{10}. Indeed, we have

$$[\mathrm{D}^{\mathrm{i}}_x, \mathrm{D}^{\mathrm{i}}_t] = -I_{02}\mathrm{D}^{\mathrm{i}}_t + (1 + \tfrac{1}{3} I_{11})\mathrm{D}^{\mathrm{i}}_x = -I_{02}\mathrm{D}^{\mathrm{i}}_t + (\tfrac{1}{3}(\mathrm{D}^{\mathrm{i}}_x I_{10}) + \tfrac{5}{9} I_{10} I_{02} + \tfrac{2}{3})\mathrm{D}^{\mathrm{i}}_x$$

as $I_{11} = \mathrm{D}^{\mathrm{i}}_x I_{10} + \frac{5}{3} I_{10} I_{02} - 1$. Applying the commutation relation for $\mathrm{D}^{\mathrm{i}}_x$ and $\mathrm{D}^{\mathrm{i}}_t$ to I_{10} and solving the obtained equation with respect to I_{20}, we derive the requested expression,

$$I_{20} = \frac{[\mathrm{D}^{\mathrm{i}}_t, \mathrm{D}^{\mathrm{i}}_x] I_{10} - \frac{1}{3}(\mathrm{D}^{\mathrm{i}}_x I_{10} + 2)\mathrm{D}^{\mathrm{i}}_x I_{10}}{\frac{5}{9} I_{10} \mathrm{D}^{\mathrm{i}}_x I_{10} - \mathrm{D}^{\mathrm{i}}_t I_{10}}.$$

5 Conclusion

The present paper is devoted to the construction of differential invariants for the maximal Lie invariance group of the KdV equation. We illustrate by examples that it is worthwhile to examine different possibilities for choosing the normalization conditions, which is a cornerstone for the moving frame computation. This is an important investigation as the form of differential invariants obtained depends strongly on the set of normalization equations chosen. In the present case of the maximal Lie invariance group of the KdV equation, using $U_X = 1$ as a normalization condition instead of the condition $U_T = 1$ chosen in [5] leads to the normalized differential invariants (11) which have a simpler form than the normalized differential invariants (10) associated with the latter condition. The same claim is true concerning the corresponding operators of invariant differentiation, recurrence formulas, etc. Moreover, the differential invariants (11) are singular only on a proper subset of the set of solutions of the KdV equation for which the differential invariants (10) are singular. The invariantized form of the KdV equation is more appropriate using the normalization condition $U_X = 1$. In contrast to the condition $U_T = 1$, this condition also naturally leads to the separation of differential invariants which involve only derivatives of u with respect to x that may be essential as the KdV equation is an evolution equation.

We also show that for Lie groups of rather simple structure, it is possible to construct functional bases of differential invariants of arbitrary order in an explicit and closed form like (10) and (11). This observation was first presented in [2] for an infinite-dimensional Lie pseudogroup. Such a closed-form expression is beneficial as it is generally simpler than the form of differential invariants obtained when acting with operators of invariant differentiation on basis differential invariants. It is difficult to conceive finding similar expressions for arbitrary order within the framework of the infinitesimal method in a reasonable way.

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Quad-Graphs and Discriminantly Separable Polynomials of Type $\mathcal{P}_3^2$

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We present a relationship between discriminantly separable polynomials and quad-graphs. We start from a classification of strongly discriminantly separable polynomials in three variables of degree two in each variable. We provide their geometric interpretation, connecting discriminantly separable polynomials with the equations of pencils of conics written in the Darboux coordinates. Then we give a construction of integrable quad-graphs associated with representatives of strongly discriminantly separable polynomials.

1 Introduction

1.1 Discriminantly separable polynomials

The theory of *discriminantly separable polynomials* has been introduced in [5], where a new approach to the Kowalevski integration procedure has been suggested. In [5] a family of discriminantly separable polynomials has been constructed starting from the equations of pencils of conics $\mathcal{F}(w, x_1, x_2) = 0$, where w, x_1 and x_2 are the pencil parameter and the Darboux coordinates respectively. We recall some of the details: given two conics C_1 and C_2 in general position by their tangential equations

$$\begin{aligned} &C_1\colon\ a_0w_1^2 + a_2w_2^2 + a_4w_3^2 + 2a_3w_2w_3 + 2a_5w_1w_3 + 2a_1w_1w_2 = 0;\\ &C_2\colon\ w_2^2 - 4w_1w_3 = 0. \end{aligned} \tag{1}$$

Then the conics of this general pencil $C(s) := C_1 + sC_2$ share four common tangents. The coordinate equations of the conics of the pencil are

$$F(s, z_1, z_2, z_3) := \det M(s, z_1, z_2, z_3) = 0,$$

where the matrix M is

$$M(s,z_1,z_2,z_3)=\begin{bmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & a_0 & a_1 & a_5-2s \\ z_2 & a_1 & a_2+s & a_3 \\ z_3 & a_5-2s & a_3 & a_4 \end{bmatrix}.$$

The point equation of the pencil $C(s)$ is then of the form of the quadratic polynomial in s

$$F := H + Ks + Ls^2 = 0,$$

where H, K and L are quadratic expressions in (z_1, z_2, z_3).

Next we introduce a new system of coordinates in the plane, the Darboux coordinates (see [4]). Given the plane with the standard coordinates (z_1, z_2, z_3), we rationally parametrize by $(1, \ell, \ell^2)$ the conic C_2 from (1). The tangent line to the conic C_2 through a point of the conic with the parameter ℓ_0 is given by the equation

$$t_{C_2}(\ell_0)\colon\ z_1\ell_0^2 - 2z_2\ell_0 + z_3 = 0.$$

For a given point P outside the conic in the plane with coordinates $P = (\hat{z}_1, \hat{z}_2, \hat{z}_3)$, there are two corresponding solutions x_1 and x_2 of the equation quadratic in ℓ

$$\hat{z}_1\ell^2 - 2\hat{z}_2\ell + \hat{z}_3 = 0.$$

Each of the solutions corresponds to a tangent to the conic C_2 from the point P. We will call the pair (x_1, x_2) the Darboux coordinates of the point P. One finds immediately the converse formulae

$$\hat{z}_1 = 1, \quad \hat{z}_2 = \frac{x_1 + x_2}{2}, \quad \hat{z}_3 = x_1x_2.$$

Changing the variables in the polynomial F from the projective coordinates $(z_1 : z_2 : z_3)$ to the Darboux coordinates, we rewrite its equation F in the form

$$F(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2).$$

The key algebraic property of the pencil equation written in this form, as a quadratic equation in each of three variables s, x_1, and x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*

$$\mathcal{D}_w(\mathcal{F})(x_1, x_2) = P(x_1)P(x_2),$$
$$\mathcal{D}_{x_i}(\mathcal{F})(w, x_j) = J(w)P(x_j), \quad (i, j) = \text{c. p.}(1, 2),$$

where J and P are polynomials of degree 3 and 4 respectively, and the elliptic curves

$$\Gamma_1\colon\ y^2 = P(x), \quad \Gamma_2\colon\ y^2 = J(s)$$

are isomorphic (see Proposition 1 of [5]).

Now we recall the definition of discriminantly separable polynomials. With $\mathcal{P}_m^n$ we denote polynomials of m variables degree n in each variable.

Definition 1. A polynomial $F(x_1, \dots, x_n)$ is *discriminantly separable* if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is *symmetrically discriminantly separable* if

$$f_2 = f_3 = \cdots = f_n,$$

while it is *strongly discriminantly separable* if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

It is *weakly discriminantly separable* if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \dots, n$

$$\mathcal{D}_{x_i} F(x_1, \dots, \hat{x}_i, \dots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

Here we do not get into detail about the famous Kowalevski case as the direct motivation for introducing this class of polynomials. We just emphasize that *Kowalevski fundamental equation* (see [5, 7, 8]) is an instance of symetrically discriminantly separable polynomial. Based on that observation we developed a class of integrable systems and called them *Kowalevski-type systems*, basically by replacing the Kowalevski fundamental equation by other discriminantly separable polynomials (see [6]).

In the sequel, we consider the polynomials of type $\mathcal{P}_3^2$. In [5] it has been shown which classes of the discriminantly separable polinomials are equivalent with the strongly discriminantly separable polynomials, modulo the following gauge transformations

$$x_i \mapsto \frac{ax_i + b}{cx_i + d}, \quad i = 1, 2, 3. \tag{2}$$

Namely, using the transformations (2) and starting from the strongly discriminantly separable polynomials one can get other discriminantly separable polynomials if related elliptic curves are isomorphic. Thus we restrict our classification to strongly discriminantly separable polynomials of type $\mathcal{P}_3^2$.

1.2 Quad-graphs

In order to present connections between discriminantly separable polynomials and the theory of integrable systems on quad-graphs from [1, 2], we recall first some of the definitions from the works of Adler, Bobenko, and Suris.

Basic building blocks of the systems on quad-graphs are the equations on the quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0, \tag{3}$$

where Q is a multiaffine polynomial. Equations of type (3) are called *quad-equations*. The field variables x_i are assigned to four vertices of a quadrilateral, and the parametars α and β are assigned to the edges of a quadrilateral assuming that opposite edges carry the same parameter. The equation (3) can be solved for each variable and the solution is a rational function of the other three variables. A solution (x_1, x_2, x_3, x_4) of the equation (3) is *singular* with respect to x_i if it also satisfies the equation $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$.

Following [2] we consider the idea of integrability as the consistency. We assign six quad-equations to the faces of coordinate cube. The system is said to be *3D-consistent* if three values for x_{123} obtained from equations on right, back and top faces coincide for arbitrary initial data x, x_1, x_2, and x_3. Then applying discriminant-like operators introduced in [2]

$$\delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2hh_{xx}, \tag{4}$$

one can make descent from the faces to the edges and then to the vertices of the cube: from a multiaffine polynomial $Q(x_1, x_2, x_3, x_4)$ to a biquadratic polynomial $h(x_i, x_j) := \delta_{x_k,x_l}(Q(x_i, x_j, x_k, x_l))$ and further to a polynomial $P(x_i) = \delta_{x_j}(h(x_i, x_j))$ of degree up to four. The subscripts on the right sides of the formulas in (4) denote the partial derivatives.

By using the relative invariants of polynomials under the fractional linear transformations authors in [2] derive the following formulae that express Q through biquadratic polynomials of three edges

$$\frac{2Q_{x_1}}{Q} = \frac{h^{12}_{x_1} h^{34} - h^{14}_{x_1} h^{23} + h^{23} h^{34}_{x_3} - h^{23}_{x_3} h^{34}}{h^{12} h^{34} - h^{14} h^{23}}. \tag{5}$$

2 Classification of strongly discriminantly separable polynomials of type $\mathcal{P}_3^2$

Let

$$\mathcal{F}(x_1, x_2, x_3) = \sum_{i,j,k=0}^{2} a_{ijk} x_1^i x_2^j x_3^k$$

be a strongly discriminantly separable polynomial with

$$\mathcal{D}_{x_i}\mathcal{F}(x_j, x_k) = P(x_j)P(x_k), \quad (i, j, k) = \text{c. p.}(1, 2, 3). \tag{6}$$

Here by $\mathcal{D}_{x_i}\mathcal{F}(x_j, x_k)$ we denote the discriminant of $\mathcal{F}$ considered as a quadratic polynomial in x_i.

This classification is heavily based upon the classification of pencils of conics. In the case of general position, the conics of a pencil intersect in four distinct points, and we code such situation with $(1, 1, 1, 1)$, see Fig. 1. It corresponds to the case in which polynomial P has four simple zeros (case **(A)**). In this case, the family of strongly discriminantly separable polynomials coincides with the family constructed in [5] from a general pencil of conics. This family, as it has been indicated in [5], corresponds to the two-valued Buchstaber–Novikov group associated with a cubic curve $\Gamma_2\colon\ y^2 = J(s)$. The other cases within the classification with nonzero polynomial P correspond to the situations for which:

(B) the polynomial P has two simple zeros and one double zero, we code it $(1,1,2)$, and the conics of the corresponding pencil intersect in two simple points, and they have a common tangent in the third point of intersection, see Fig. 2;

(C) the polynomial P has two double zeros, code $(2,2)$, and the conics of the corresponding pencil intersect in two points, having a common tangent in each of the points of intersection, see Fig. 3;

(D) the polynomial P has one simple zero and one triple zero, we code it $(1,3)$. The conics of the corresponding pencil intersect in one simple point, and they have another common point of tangency of third order, see Fig. 4;

(E) the polynomial P has one quadriple zero, code (4), and the conics of the corresponding pencil intersect in one point, having tangency of fourth order there, see Fig. 5.

Theorem 1. *The strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3)$ satisfying* (6) *modulo fractional linear transformations are exhausted by the following list depending on distribution of roots of a non-zero polynomial $P(x)$:*

(A) *four simple zeros, for the canonical form $P_A(x) = (k^2x^2 - 1)(x^2 - 1)$,*

$$\mathcal{F}_A = \tfrac{1}{2}\left(-k^2x_1^2 - k^2x_2^2 + 1 + k^2x_1^2x_2^2\right)x_3^2 + \left(1 - k^2\right)x_1x_2x_3 \\ + \tfrac{1}{2}\left(x_1^2 + x_2^2 - k^2x_1^2x_2^2 - 1\right),$$

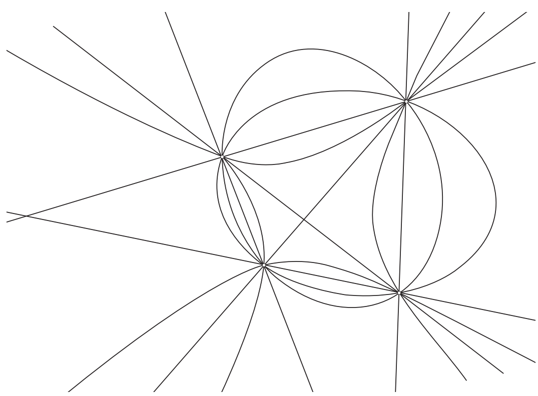

Figure 1. Pencil with four simple points.

(B) *two simple zeros and one double zero, for the canonical form* $P_B(x) = x^2 - e^2$ *with* $e \neq 0$,

$$\mathcal{F}_B = x_1x_2x_3 + \frac{e}{2}\left(x_1^2 + x_2^2 + x_3^2 - e^2\right),$$

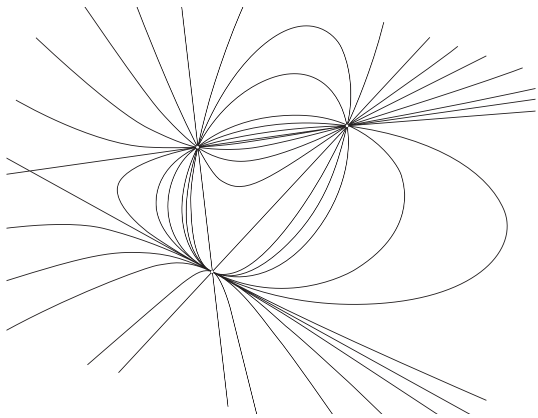

Figure 2. Pencil with one double and two simple points.

(C) *two double zeros, for the canonical form* $P_C(x) = x^2$,

$$\mathcal{F}_{C1} = \lambda x_1^2x_3^2 + \mu x_1x_2x_3 + \nu x_2^2, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{C2} = \lambda x_1^2x_2^2x_3^2 + \mu x_1x_2x_3 + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

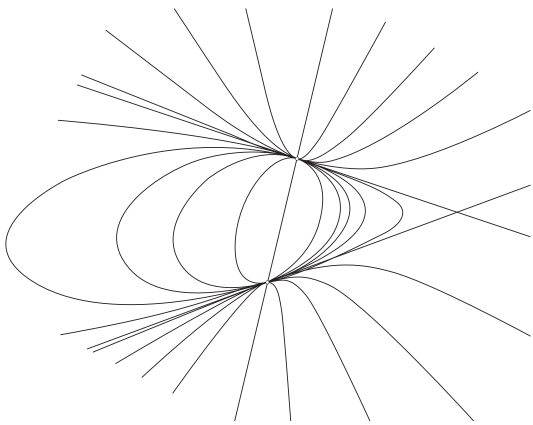

Figure 3. Pencil with two double points.

(D) *one simple and one triple zero, for the canonical form* $P_D(x) = x$,

$$\mathcal{F}_D = -\tfrac{1}{2}(x_1x_2 + x_2x_3 + x_1x_3) + \tfrac{1}{4}\left(x_1^2 + x_2^2 + x_3^2\right),$$

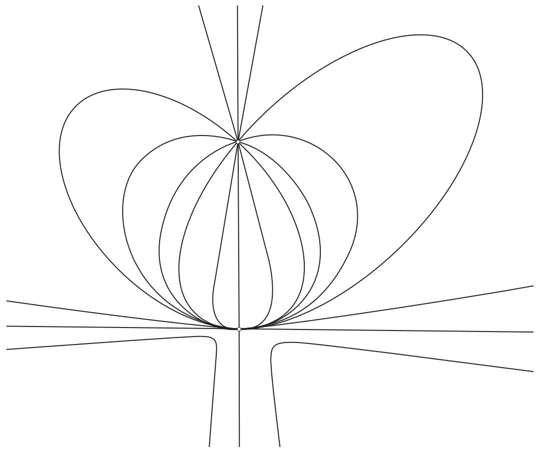

Figure 4. Pencil with one simple and one triple point.

(E) *one quadruple zero, for the canonical form* $P_E(x) = 1$,

$$\mathcal{F}_{E1} = \lambda(x_1 + x_2 + x_3)^2 + \mu(x_1 + x_2 + x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E2} = \lambda(x_2 + x_3 - x_1)^2 + \mu(x_2 + x_3 - x_1) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E3} = \lambda(x_1 + x_3 - x_2)^2 + \mu(x_1 + x_3 - x_2) + \nu, \quad \mu^2 - 4\lambda\nu = 1,$$

$$\mathcal{F}_{E4} = \lambda(x_1 + x_2 - x_3)^2 + \mu(x_1 + x_2 - x_3) + \nu, \quad \mu^2 - 4\lambda\nu = 1.$$

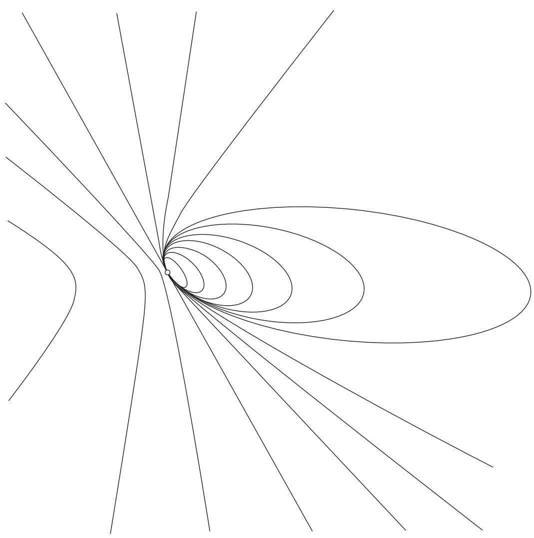

Figure 5. Pencil with one quadruple point.

Comparing with [3] we get the following theorem.

Theorem 2. *If a polynomial P has four simple zeros, all strongly discriminantly separable polynomials $\mathcal{F}$ satisfying* (6) *are equivalent to the two-valued groups*

$$(x_1x_2 + x_2x_3 + x_1x_3)(4 + g_3x_1x_2x_3) = \left(x_1 + x_2 + x_3 - \tfrac{1}{4}g_2x_1x_2x_3\right)^2.$$

3 From discriminant separability to integrable quad-graphs

The classification of integrable quad-equations with complex fields x under some nondegeneracy assumptions of the polynomial Q is presented in [2]. A biquadratic polynomial $h(x, y)$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to fractional linear transformations is divisible by a factor of the form $x - c$ or $y - c$, with $c =$ const. A multiaffine function $Q(x_1, x_2, x_3, x_4)$ is said to be of *type Q* if all four of its accompanying biquadratic polynomials h^{jk} are nondegenerate. Otherwise, it is of *type H*. Previous notions were introduced in [2].

The requirement that the discriminants of $h(x_1, x_2)$ do not depend on α, see [1, 2], will be satisfied if as a biquadratic polynomials $h(x_1, x_2)$ we take

$$\hat{h}(x_1, x_2) := \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}},$$

where $\mathcal{F}$ is strongly discriminantly separable polynomial and P is a polynomial that shows up in a discriminant factorization from our classification. Now with formulae (5) we can calculate quad-equations corresponding to each representative of strongly discriminantly separable polynomials.

Theorem 3. *Quad-equations of type Q that correspond to the biquadratic polynomials*

$$\hat{h}(x_1, x_2; \alpha) = \frac{\mathcal{F}_I(x_1, x_2, \alpha)}{\sqrt{P_I(\alpha)}}, \quad I = A, B, C, D, E,$$

are given in the following list:

$$\begin{aligned}\hat{Q}_A = {} & \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1}\sqrt{\frac{\alpha^2 - 1}{k^2\alpha^2 - 1}}\sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}\left(k^2x_1x_2x_3x_4 + 1\right) \\ & + \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}(x_1x_2 + x_3x_4) + \sqrt{\frac{\beta^2 - 1}{k^2\beta^2 - 1}}(x_1x_4 + x_2x_3) \\ & + \frac{\beta\sqrt{P_A(\alpha)} + \alpha\sqrt{P_A(\beta)}}{k^2\alpha^2\beta^2 - 1}(x_1x_3 + x_2x_4) = 0,\end{aligned}$$

$$\begin{aligned}\hat{Q}_B = {} & \frac{1}{e}\left(\alpha\sqrt{\beta^2 - e^2} + \beta\sqrt{\alpha^2 - e^2}\right)(x_1x_3 + x_2x_4) \\ & + \sqrt{\beta^2 - e^2}(x_1x_4 + x_2x_3) + \sqrt{\alpha^2 - e^2}(x_1x_2 + x_3x_4)\end{aligned}$$

$$- \frac{1}{e}\sqrt{\beta^2 - e^2}\sqrt{\alpha^2 - e^2}\left(\alpha\sqrt{\beta^2 - e^2} + \beta\sqrt{\alpha^2 - e^2}\right) = 0,$$

$$\hat{Q}_C = \left(\alpha - \frac{1}{\alpha}\right)(x_1x_2 + x_3x_4) + \left(\beta - \frac{1}{\beta}\right)(x_1x_4 + x_2x_3)$$
$$- \left(\alpha\beta - \frac{1}{\alpha\beta}\right)(x_1x_3 + x_2x_4) = 0,$$

$$\hat{Q}_D = \sqrt{\alpha}(x_1 - x_4)(x_2 - x_3) + \sqrt{\beta}(x_1 - x_2)(x_4 - x_3)$$
$$- \sqrt{\alpha}\sqrt{\beta}\left(\sqrt{\alpha} + \sqrt{\beta}\right)(x_1 + x_2 + x_3 + x_4)$$
$$+ \sqrt{\alpha}\sqrt{\beta}\left(\sqrt{\alpha} + \sqrt{\beta}\right)(\alpha + \sqrt{\alpha\beta} + \beta) = 0,$$

$$\hat{Q}_E = \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \alpha\beta(\alpha + \beta) = 0.$$

The list of the quad-equations from the previous theorem corresponds to the list from [1] with some transformations of the coefficients α and β.

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Construction of Lumps with Nontrivial Interaction

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We develop a method based upon the singular manifold method that yields an iterative and analytic procedure to construct solutions for a Bogoyavlenskii–Kadomtsev–Petviashvili equation. This method allows us to construct a rich collection of lump solutions with a nontrivial evolution behavior

1 Introduction

In recent years it has been proven in several papers [1, 7, 9] that the Kadomtsev–Petiviashvili I (KPI) equation contains a whole variety of smooth rationally decaying "lump" configurations associated with higher-order pole meromorphic eigenfunctions. These configurations have an interesting dynamics and the lumps may scatter in a nontrivial way. Furthermore algorithmic methods, based upon the Painlevé property, have been developed in order to construct lump-type solutions for different equations such as the (2+1)-dimensional nonlinear Schrödinger equation (NLS) [6, 10], the Kadomtsev–Petiviashvili I equation and the Generalized Dispersive Long Wave equation (GDLW) [5].

The present contribution is related to the construction of lump solutions for the (2+1)-dimensional equation [12]

$$(4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx} u_y)_x + \sigma u_{yyy} = 0, \quad \sigma = \pm 1, \tag{1}$$

which represents a modification of the Calogero–Bogoyavlenskii–Schiff equation (CBS) [2, 3, 8]:

$$4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx} u_y = 0.$$

Equation (1) has often been called the Bogoyavlenskii–Kadomtsev–Petviashvili equation (KP-B) [4]. As in the case of the KP equation, there are two versions of (1), depending upon the sign of σ. Here we restrict ourselves to the minus sign. Therefore we consider the equation

$$(4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx} u_y)_x - u_{yyy} = 0$$

or the equivalent system

$$4u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx} u_y = \omega_{yy}, \quad u_y = \omega_x. \tag{2}$$

We refer to (2) as to KP-BI in what follows.

In Section 2 we summarize the results that the singular manifold method provides for KP-BI. These results are not essentially new because they were obtained by the author in [4] for the KP-BII version of the equation. Section 3 is devoted to the construction of rational solitons.

2 The singular manifold method for KP-BI

It has been proven that (2) has the Painlevé property [12]. Therefore the singular manifold method can be applied to it. In this section we adapt previous results obtained in [4] for KP-BII to KP-BI. This is why we only present the main results with no detailed explanation since these have been shown in our earlier paper.

2.1 The singular manifold method

This method [11] requires the truncation of the Painlevé series for the fields u and ω of (2) in the following form:

$$u^{[1]} = u^{[0]} + \frac{\phi_x^{[0]}}{\phi^{[0]}}, \quad \omega^{[1]} = \omega^{[0]} + \frac{\phi_y^{[0]}}{\phi^{[0]}}, \tag{3}$$

where $\phi^{[0]}(x,y,t)$ is the singular manifold and $u^{[i]}$, $\omega^{[i]}$ $(i = 0,1)$ are solutions of (2). This means that (3) can be considered as an auto-Bäcklund transformation. The substitution of (3) into (2) yields a polynomial in negative powers of $\phi^{[0]}$ that can be handled with `Maple`. As a result (see [4]) we can express the seed solutions $u^{[0]}$ and $\omega^{[0]}$ in terms of the singular manifold,

$$u_x^{[0]} = \tfrac{1}{4}\left(-v_x - \tfrac{1}{2}v^2 - z_y + \tfrac{1}{2}z_x\right), \quad u_y^{[0]} = \omega_x^{[0]} = \tfrac{1}{4}\left(-r - 2v_y + 2z_x z_y\right), \tag{4}$$

where v, r and z are related to the singular manifold ϕ by the notation

$$v = \frac{\phi_{xx}^{[0]}}{\phi_x^{[0]}}, \quad r = \frac{\phi_t^{[0]}}{\phi_x^{[0]}}, \quad z_x = \frac{\phi_y^{[0]}}{\phi_x^{[0]}}. \tag{5}$$

Furthermore the singular manifold $\phi^{[0]}$ satisfies the equation

$$s_y + r_x - z_{yy} - z_x z_{xy} - 2z_y z_{xx} = 0, \tag{6}$$

where $s = v_x - \frac{1}{2}v^2$ is the Schwartzian derivative.

2.2 Lax pair

Equations (4) can be linearized due to the following definition of the functions $\psi^{[0]}(x,y,t)$ and $\chi^{[0]}(x,y,t)$:

$$v = \frac{\psi_x^{[0]}}{\psi^{[0]}} + \frac{\chi_x^{[0]}}{\chi^{[0]}}, \quad z_x = i\left(\frac{\psi_x^{[0]}}{\psi^{[0]}} - \frac{\chi_x^{[0]}}{\chi^{[0]}}\right). \tag{7}$$

When one combines (4), (5) and (6), the following Lax pair arises:

$$\begin{aligned} \psi^{[0]}_{xx} &= -i\psi^{[0]}_{y} - 2u^{[0]}_{x}\psi^{[0]}, \\ \psi^{[0]}_{t} &= 2i\psi^{[0]}_{yy} - 4u^{[0]}_{y}\psi^{[0]}_{x} + \left(2u^{[0]}_{xy} + 2i\omega^{[0]}_{y}\right)\psi^{[0]} \end{aligned} \tag{8}$$

together with its complex conjugate

$$\begin{aligned} \chi^{[0]}_{xx} &= i\chi^{[0]}_{y} - 2u^{[0]}_{x}\chi^{[0]}, \\ \chi^{[0]}_{t} &= -2i\chi^{[0]}_{yy} - 4u^{[0]}_{y}\chi^{[0]}_{x} + \left(2u^{[0]}_{xy} - 2i\omega^{[0]}_{y}\right)\chi^{[0]}. \end{aligned} \tag{9}$$

In terms of $\chi^{[0]}$ and $\psi^{[0]}$ the derivatives of $\phi^{[0]}$ are

$$\begin{aligned} v &= \frac{\phi^{[0]}_{xx}}{\phi^{[0]}_{x}} = \frac{\psi^{[0]}_{x}}{\psi^{[0]}} + \frac{\chi^{[0]}_{x}}{\chi^{[0]}} \quad \Rightarrow \quad \phi^{[0]}_{x} = \psi^{[0]}\chi^{[0]}, \\ r &= \frac{\phi^{[0]}_{t}}{\phi^{[0]}_{x}} = -4u^{[0]}_{y} + 2\frac{\psi^{[0]}_{y}}{\psi^{[0]}}\frac{\psi^{[0]}_{x}}{\psi^{[0]}} + 2\frac{\psi^{[0]}_{x}}{\psi^{[0]}}\frac{\psi^{[0]}_{y}}{\psi^{[0]}} - 2\frac{\psi^{[0]}_{xy}}{\psi^{[0]}} - 2\frac{\chi^{[0]}_{xy}}{\chi^{[0]}}, \\ z_x &= \frac{\phi^{[0]}_{y}}{\phi^{[0]}_{x}} = i\left(\frac{\psi^{[0]}_{x}}{\psi^{[0]}} - \frac{\chi^{[0]}_{x}}{\chi^{[0]}}\right), \end{aligned}$$

which allow us to write $d\phi^{[0]}$ as

$$\begin{aligned} d\phi^{[0]} &= \psi^{[0]}\chi^{[0]}\,dx + i\left(\chi^{[0]}\psi^{[0]}_{x} - \psi^{[0]}\chi^{[0]}_{x}\right)dy \\ &\quad + \left(-4u^{[0]}_{y}\psi^{[0]}\chi^{[0]} + 2\psi^{[0]}_{y}\chi^{[0]}_{x} + 2\psi^{[0]}_{x}\chi^{[0]}_{y} - 2\chi^{[0]}\psi^{[0]}_{xy} - 2\psi^{[0]}\chi^{[0]}_{xy}\right)dt. \end{aligned} \tag{10}$$

It is easy to check that the condition of the exact derivative in (10) is satisfied by the Lax pairs (8) and (9).

2.3 Darboux transformations

Let $(\psi^{[0]}_1, \chi^{[0]}_1)$ and $(\psi^{[0]}_2, \chi^{[0]}_2)$ be two pairs of eigenfunctions of the Lax pair (8)–(9) corresponding to the seed solution $u^{[0]}$ and $\omega^{[0]}$,

$$\begin{aligned} \psi^{[0]}_{j,xx} &= -i\psi^{[0]}_{j,y} - 2u^{[0]}_{x}\psi^{[0]}_{j}, \\ \psi^{[0]}_{j,t} &= 2i\psi^{[0]}_{j,yy} - 4u^{[0]}_{y}\psi^{[0]}_{j,x} + (2u^{[0]}_{xy} + 2i\omega^{[0]}_{y})\psi^{[0]}_{j}, \end{aligned} \tag{11}$$

$$\begin{aligned} \chi^{[0]}_{j,xx} &= i\chi^{[0]}_{j,y} - 2u^{[0]}_{x}\chi^{[0]}_{j}, \\ \chi^{[0]}_{j,t} &= -2i\chi^{[0]}_{j,yy} - 4u^{[0]}_{y}\chi^{[0]}_{j,x} + (2u^{[0]}_{xy} - 2i\omega^{[0]}_{y})\chi^{[0]}_{j}, \end{aligned} \tag{12}$$

where $j = 1, 2$. These Lax pairs can be considered as nonlinear equations between the fields and the eigenfunctions [4]. This means that the Painlevé expansion of the fields

$$u^{[1]} = u^{[0]} + \frac{\phi^{[0]}_{1,x}}{\phi^{[0]}_{1}}, \quad \omega^{[1]} = \omega^{[0]} + \frac{\phi^{[0]}_{1,y}}{\phi^{[0]}_{1}} \tag{13}$$

should be accompanied by an expansion of the eigenfunctions and the singular manifold itself. These expansions are

$$\begin{aligned}&\psi_2^{[1]} = \psi_2^{[0]} - \psi_1^{[0]}\frac{\Omega_{1,2}}{\phi_1^{[0]}}, \quad \chi_2^{[1]} = \chi_2^{[0]} - \chi_1^{[0]}\frac{\Omega_{2,1}}{\phi_1^{[0]}},\\ &\phi_2^{[1]} = \phi_2^{[0]} - \frac{\Omega_{1,2}\Omega_{2,1}}{\phi_1^{[0]}}.\end{aligned} \tag{14}$$

The substitution of (14) into (11)–(12) yields

$$\begin{aligned}d\Omega_{i,j} &= \psi_j^{[0]}\chi_i^{[0]}\,dx + i\left(\chi_i^{[0]}\psi_{j,x}^{[0]} - \psi_j^{[0]}\chi_{i,x}^{[0]}\right)dy\\ &- \left(4u_y^{[0]}\psi_i^{[0]}\chi_j^{[0]} - 2\psi_{j,y}^{[0]}\chi_{i,x}^{[0]} - 2\psi_{j,x}^{[0]}\chi_{i,y}^{[0]} + 2\chi_i^{[0]}\psi_{j,xy}^{[0]} + 2\psi_j^{[0]}\chi_{i,xy}^{[0]}\right)dt.\end{aligned} \tag{15}$$

Direct comparison of (10) and (15) affords $\phi_j^{[0]} = \Omega_{j,j}$. Therefore the knowledge of the two seed eigenfunctions $(\psi_j^{[0]}, \chi_j^{[0]})$, $j = 1, 2$, allows us to compute the matrix entries $\Omega_{i,j}$ and this yields the Darboux transformation (13)–(14).

2.4 Iteration: τ-functions

According to the above results $\phi_2^{[1]}$ is a singular manifold for the iterated fields $u^{[1]}$ and $\omega^{[1]}$. Therefore the Painlevé expansion for these iterated fields can be written as

$$u^{[2]} = u^{[1]} + \frac{\phi_{2,x}^{[1]}}{\phi_2^{[1]}}, \quad \omega^{[2]} = \omega^{[1]} + \frac{\phi_{2,y}^{[1]}}{\phi_2^{[1]}},$$

which when combined with (13) is

$$u^{[2]} = u^{[0]} + \frac{(\tau_{1,2})_x}{\tau_{1,2}}, \quad \omega^{[2]} = \omega^{[0]} + \frac{(\tau_{1,2})_y}{\tau_{1,2}},$$

where $\tau_{1,2} = \phi_2^{[1]}\phi_1^{[0]}$. According to (13) this implies that

$$\tau_{1,2} = \phi_2^{[0]}\phi_1^{[0]} - \Omega_{1,2}\Omega_{2,1} = \det(\Omega_{i,j}). \tag{16}$$

3 Lumps

The iteration method described above can be started from the most trivial initial solution $u^{[0]} = \omega^{[0]} = 0$. In this case the Lax pair is

$$\begin{aligned}&\psi_{j,xx}^{[0]} = -i\psi_{j,y}^{[0]}, \quad \psi_{j,t}^{[0]} = 2i\psi_{j,yy}^{[0]},\\ &\chi_{j,xx}^{[0]} = i\chi_{j,y}^{[0]}, \quad \chi_{j,t}^{[0]} = -2i\chi_{j,yy}^{[0]}.\end{aligned} \tag{17}$$

It is trivial to prove that equations (17) have the solutions

$$\begin{aligned}\psi_1^{[0]} &= P_m(x,y,t;k)\exp[Q_0(x,y,t;k)], \quad \psi_2^{[0]} = \left(\chi_1^{[0]}\right)^*,\\ \chi_1^{[0]} &= P_n(x,y,t;k)\exp[-Q_0(x,y,t;k)], \quad \chi_2^{[0]} = \left(\psi_1^{[0]}\right)^*,\end{aligned} \tag{18}$$

where m and n are arbitrary integers and k is an arbitrary complex constant. We have

$$\begin{aligned}&Q_0(x,y,t;k) = kx + ik^2y + 2ik^4t, \quad \text{i.e.}\\ &(Q_0(x,y,t;k))^* = k^*x - i(k^*)^2y - 2i(k^*)^4t\end{aligned}$$

and $P_j(x,y,t;k)$ is defined by

$$\begin{aligned}&P_j(x,y,t;k)\exp[Q_0(x,y,t;k)]\\ &\quad = \frac{\partial^j}{\partial k^j}\left(P_{j-1}(x,y,t;k)\exp[Q_0(x,y,t;k)]\right), \qquad P_0 = 1.\end{aligned} \tag{19}$$

These solutions are characterized by two integers, n and m, that provide a rich collection of different solutions corresponding to the same wave number, k. Thus, in our opinion, all of them should be considered as one-soliton solutions despite the different behaviours shown by the solutions corresponding to the different combinations of n and m. We now present some of these cases.

3.1 Lump $(0,0)$: $n = 0,\ m = 0$

The eigenfunctions (18) are

$$\begin{aligned}\psi_1^{[0]} &= \exp[Q_0(x,y,t;k)], \quad \psi_2^{[0]} = \exp[-Q_0^*(x,y,t;k)],\\ \chi_1^{[0]} &= \exp[-Q_0(x,y,t;k)], \quad \chi_2^{[0]} = \exp[Q_0^*(x,y,t;k)].\end{aligned}$$

The matrix elements (15) can be integrated to give

$$\begin{aligned}\phi_1^{[0]} &= \Omega_{1,1} = x + 2iky - 8ik^3t, \quad \phi_2^{[0]} = \Omega_{2,2} = x - 2ik^*y + 8i\,(k^*)^3\,t,\\ \Omega_{1,2} &= -\frac{1}{k+k^*}\exp[-Q_0(x,y,t;k)]\exp[-Q_0^*(x,y,t;k)],\\ \Omega_{2,1} &= \frac{1}{k+k^*}\exp[Q_0(x,y,t;k)]\exp[Q_0^*(x,y,t;k)].\end{aligned}$$

Therefore the τ-function (16) is the positive defined expression

$$\tau_{1,2} = X_1^2 + Y_1^2 + \frac{1}{4a_0^2},$$

where $k = a_0 + ib_0$,

$$X_1 = x - 2b_0y + 8b_0\left(3a_0^2 - b_0^2\right)t, \quad Y_1 = 2a_0\left(y + 4(3b_0^2 - a_0^2)t\right).$$

The profile of this solution is shown in Fig. 1. It represents a lump (static in the variables X_1 and Y_1) of height $8a_0^2$.

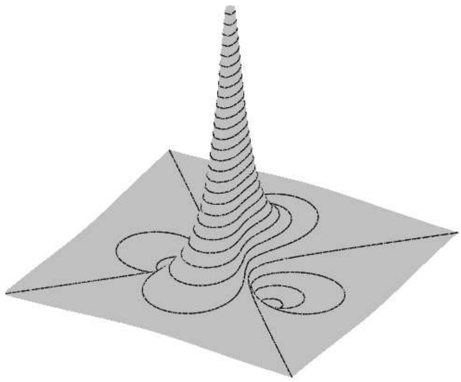

Figure 1. Lump of the $(0,0)$ type.

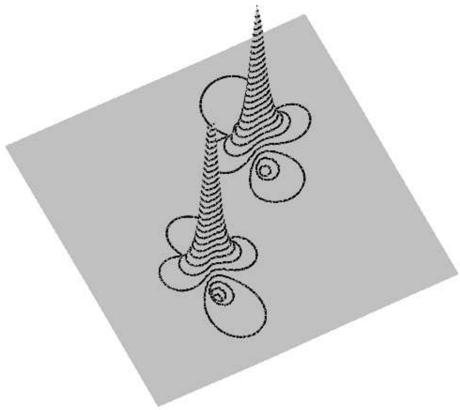

Figure 2. Lump of the $(1,0)$ type.

3.2 Lump $(1,0)$: $n=1,\ m=0$

The eigenfunctions (18) are

$$\psi_1^{[0]} = P_1(x,y,t;k)\exp[Q_0(x,y,t;k)], \quad \psi_2^{[0]} = \exp[-Q_0^*(x,y,t;k)],$$
$$\chi_1^{[0]} = \exp[-Q_0(x,y,t;k)], \quad \chi_2^{[0]} = \{P_1(x,y,t;k)\}^*\exp[Q_0^*(x,y,t;k)],$$

where according to (19) we have

$$P_1(x,y,t;k) = x + 2iky - 8ik^3t.$$

In this case the matrix elements (15) are

$$\begin{aligned}\phi_1^{[0]} &= \frac{1}{2}x^2 + iy + 2ixyk - 2y^2k^2 - 12itk^2 - 8ixtk^3 + 16ytk^4 - 32t^2k^6\\ &= \frac{X_1^2 - Y_1^2}{2} + i\frac{2a_0X_1Y_1 + Y_1 - 16a_0^3t}{2a_0}, \quad \phi_2^{[0]} = \left(\phi_1^{[0]}\right)^*,\\ \Omega_{1,2} &= -\frac{1}{2a_0}\exp[-Q_0(x,y,t;k)]\exp[-Q_0^*(x,y,t;k)],\\ \Omega_{2,1} &= \frac{1 - 2a_0X_1 + 2a_0^2\left(X_1^2 + Y_1^2\right)}{4a_0^3}\exp[Q_0(x,y,t;k)]\exp[Q_0^*(x,y,t;k)].\end{aligned}$$

Therefore the τ-function (16) is the positive defined expression

$$\begin{aligned}\tau_{1,2} = {}&\left(\frac{X_1^2 - Y_1^2}{2}\right)^2 + \left(\frac{2a_0X_1Y_1 + Y_1 - 16a_0^3t}{2a_0}\right)^2\\ &+ \left(\frac{X_1 - \frac{1}{2a_0}}{2a_0}\right)^2 + \left(\frac{Y_1}{2a_0}\right)^2 + \frac{1}{16a_0^4}.\end{aligned} \tag{20}$$

The profile of this solution is shown in Fig. 2. If we wish to show its behaviour when $t \to \pm\infty$, we need to look along the lines

$$X_1 = \hat{X}_1 + c_1 t^{1/2}, \quad Y_1 = \hat{Y}_1 + c_2 t^{1/2}$$

such that (20) is different from 0 when $t \to \pm\infty$.

• For $t < 0$ the possibilities are $c_1 = \pm 2a_0\sqrt{-2}$, $c_2 = -c_1$ which yields two lumps approaching with opposite velocities along the lines

$$X_1 = \hat{X}_1 \pm 2a_0(-2t)^{1/2}, \quad Y_1 = \hat{Y}_1 \pm 2a_0(-2t)^{1/2}$$

and the limit of $\tau_{1,2}$ along these lines is

$$\tau_{1,2} = \left(\hat{X}_1 + \frac{1}{4a_0}\right)^2 + \left(\hat{Y}_1 - \frac{1}{4a_0}\right)^2 + \frac{1}{4a_0^4}.$$

• For $t > 0$ the possibilities are $c_1 = \pm 2a_0\sqrt{2}, \quad c_2 = c_1$ which yields two lumps with opposite velocities along the lines

$$X_1 = \hat{X}_1 \pm 2a_0(2t)^{1/2}, \quad Y_1 = \hat{Y}_1 \pm 2a_0(2t)^{1/2}$$

and the limit of $\tau_{1,2}$ along these lines is

$$\tau_{1,2} = \left(\hat{X}_1 + \frac{1}{4a_0}\right)^2 + \left(\hat{Y}_1 + \frac{1}{4a_0}\right)^2 + \frac{1}{4a_0^4}.$$

3.3 Lump (1, 1): $n = 1$, $m = 1$

The eigenfunctions (18) are

$$\begin{aligned}
\psi_1^{[0]} &= P_1(x, y, t; k) \exp[Q_0(x, y, t; k)], \\
\psi_2^{[0]} &= \{P_1(x, y, t; k)\}^* \exp[-Q_0^*(x, y, t; k)], \\
\chi_1^{[0]} &= P_1(x, y, t; k) \exp[-Q_0(x, y, t; k)], \\
\chi_2^{[0]} &= \{P_1(x, y, t; k)\}^* \exp[Q_0^*(x, y, t; k)].
\end{aligned}$$

This yields the following matrix elements according to (15):

$$\begin{aligned}
\phi_1^{[0]} &= \frac{X_1\left(X_1^2 - 3Y_1^2\right)}{3} + i\left(X_1^2 Y_1 - \frac{Y_1^3}{3} + 8a_0 t\right), \quad \phi_2^{[0]} = \left(\phi_1^{[0]}\right)^*, \\
\Omega_{1,2} &= -\frac{1 + 2a_0 X_1 + 2a_0^2\left(X_1^2 + Y_1^2\right)}{4a_0^3} \exp[-Q_0(x, y, t; k)] \exp[-Q_0^*(x, y, t; k)], \\
\Omega_{2,1} &= \frac{1 - 2a_0 X_1 + 2a_0^2\left(X_1^2 + Y_1^2\right)}{4a_0^3} \exp[Q_0(x, y, t; k)] \exp[Q_0^*(x, y, t; k)].
\end{aligned}$$

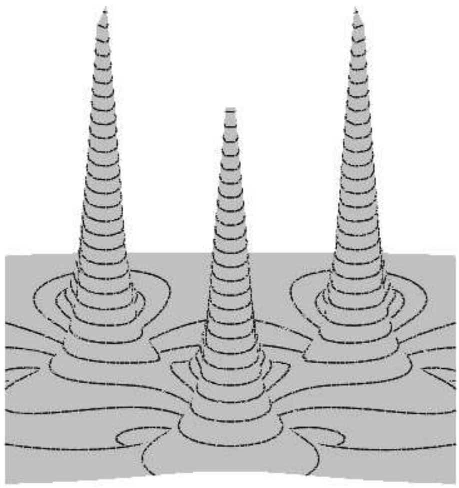

Figure 3. Lump of the $(1,1)$ type.

Therefore the τ-function (16) is the positive defined expression

$$\tau_{1,2} = \left(\frac{1}{3}X_1\left(X_1^2 - 3Y_1^2\right)\right)^2 + \left(X_1^2Y_1 - \frac{1}{3}Y_1^3 + 8a_0t\right)^2 + \left(\frac{X_1^2 + Y_1^2}{2a_0}\right)^2 + \left(\frac{Y_1}{2a_0^2}\right)^2 + \frac{1}{16a_0^2}.$$

The profile of this solution is shown in Fig. 3.

The asymptotic behaviour of this solution can be obtained by considering the transformation

$$X_1 = \hat{X}_1 + c_1t^{\frac{1}{3}}, \quad Y_1 = \hat{Y}_1 + c_2t^{\frac{1}{3}}.$$

There are three possible solutions for c_i. For all of them $\tau_{1,2}$ is

$$\tau_{1,2} \to \hat{X}_1^2 + \hat{Y}_1^2 + \frac{1}{4a_0^2}.$$

- $c_1 = 0$, $c_2 = 2(3a_0)^{\frac{1}{3}}$. This corresponds to a lump moving along the line

$$X_1 = \hat{X}_1, \quad Y_1 = \hat{Y}_1 + 2(3a_0t)^{\frac{1}{3}}.$$

- $c_1 = -\sqrt{3}\,(-3a_0)^{\frac{1}{3}}$, $c_2 = (-3a_0)^{\frac{1}{3}}$. This corresponds to a lump moving along the line

$$X_1 = \hat{X}_1 - \sqrt{3}\,(-3a_0t)^{\frac{1}{3}}, \quad Y_1 = \hat{Y}_1 + (-3a_0t)^{\frac{1}{3}}.$$

- $c_1 = \sqrt{3}\,(-3a_0)^{\frac{1}{3}}$, $c_2 = 2(3a_0)^{\frac{1}{3}}$. This corresponds to a lump moving along the line

$$X_1 = \hat{X}_1 + \sqrt{3}\,(-3a_0t)^{\frac{1}{3}}, \quad Y_1 = \hat{Y}_1 + (-3a_0t)^{\frac{1}{3}}.$$

4 Conclusions

The singular manifold method allows us to derive an iterative method to construct lump solutions characterized by two integers the different combinations of which yield rich possibilities of nontrivial self-interactions between components of the solution.

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On the Equation for the Power Moment Generating Function of the Boltzmann Equation. Group Classification with respect to a Source Function

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Lie symmetries of the spatially homogeneous and isotropic Boltzmann equation with sources were first studied by Nonnenmacher (1984). In fact, he considered the associated equations for the generating function of the power moments of the unknown distribution function. However, it was not taken into account that this equation is still a nonlocal partial differential equation. In the present paper their Lie symmetries are studied using the original approach developed by Grigoriev and Meleshko (1986) for group analysis of equations with nonlocal operators, which allows us to correct Nonnenmacher's results. The group classification with respect to sources is carried out for the equations under consideration using the algebraic method.

1 Introduction

The Boltzmann kinetic equation is the basis of the classical kinetic theory of rarefied gases. Considerable interest in the study of the Boltzmann equation was always the search for exact (invariant) solutions directly associated with the fundamental properties of the equation. After the studies of the class of the local Maxwellians [3, 4, 11] new classes of invariant solutions were constructed in the 1960s in [13–15]. A decade later the BKW-solution was almost simultaneously derived in [1] and in [10]. Contrary to the Maxwellians, the Boltzmann collision integral does not vanish for this solution. The discovery of the BKW-solution stimulated a great splash of studies of exact solutions of various kinetic equations. However, the progress at that time was really limited to the construction of BKW-type solutions for different simplified models of the Boltzmann equation[1].

[1] See [5] for the review.

The Boltzmann equation is an integro-differential equation. Whereas the classical group analysis method has been developed for studying partial differential equations, the main obstacle for applying group analysis to integro-differential equations comes from presence of nonlocal integral operators. The direct group analysis for equations with nonlocal operators was worked out and successfully used in [6, 7, 12]. In particular, a complete group classification of the spatially homogeneous and isotropic Boltzmann equation without sources was obtained in [7, 8].

One of the alternative approaches for studying solutions of the Boltzmann equation, by transition to an equation for a moment generating function, was first considered in [10]. The BKW-solution was obtained there. In [16], such an approach was applied to the spatially homogeneous and isotropic Boltzmann equation with sources. The author of [16] used the group analysis method for studying solutions of the equation for the generating function. However, it was not taken into account that this equation is still a nonlocal one. In the present paper we use our method [7, 8] to amend the results of [16]. A group classification of the equation for a moment generating function with respect to a source function is obtained.

2 General equations

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form [1]

$$\varphi_t(y,t) + \varphi(y,t)\varphi(0,t) = \int_0^1 \varphi(ys,t)\varphi(y(1-s),t)\, ds + \hat{q}(y,t). \tag{1}$$

Here the function $\varphi(y,t)$ is related with the Fourier transform $\tilde{\varphi}(k,t)$ of the isotropic distribution function $f(v,t)$ by the formulae

$$\varphi(k^2/2,t) = \tilde{\varphi}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)f(v,t)\,dv.$$

The function $\hat{q}(y,t)$ is defined by the Fourier transform of the source function $q(v,t)$ in a similar way:

$$\hat{q}(k^2/2,t) = \tilde{\hat{q}}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)q(v,t)\,dv.$$

The inverse Fourier transform of $\tilde{\varphi}(k,t)$ gives the distribution function

$$f(v,t) = \frac{4\pi}{v}\int_0^\infty k\sin(kv)\tilde{\varphi}(k,t)\,dk.$$

Normalized moments of the distribution function are introduced by the formulae

$$M_n(t) = \frac{4\pi}{(2n+1)!!}\int_0^\infty f(v,t)v^{2n+2}dv, \qquad n = 0, 1, 2\dots . \tag{2}$$

Following [2], one can obtain a system of equations for the moments (2) from (1). It is sufficient to substitute the expansions in power series

$$\varphi(y,t) = \sum_{n=0}^{\infty}(-1)^n M_n(t)\frac{y^n}{n!}, \qquad \hat{q}(y,t) = \sum_{n=0}^{\infty}(-1)^n q_n(t)\frac{y^n}{n!},$$

into (1), where

$$q_n(t) = \frac{1}{(2n+1)!!}4\pi\int_0^{\infty} q(v,t)v^{2n+2}dv, \qquad n = 0,1,2,\ldots,$$

are the normalized moments of the source function. As a result, one derives the moment system considered in [16]:

$$\frac{dM_n(t)}{dt} + M_n(t)M_0(t) = \frac{1}{n+1}\sum_{k=0}^{n} M_k(t)M_{n-k}(t) + q_n(t). \tag{3}$$

For $q(v,t) = 0$ this system was derived in [10] in a very complicated way.

Let us define moment generation functions for the distribution function $f(v,t)$ and for the source function $q(v,t)$:

$$G(\omega,t) = \sum_{n=0}^{\infty}\omega^n M_n(t), \qquad S(\omega,t) = \sum_{n=0}^{\infty}\omega^n q_n(t).$$

Multiplying equations (3) by ω^n, and summing over all n, one obtains for $G(\omega,t)$ the equation

$$\frac{\partial^2(\omega G)}{\partial t\partial\omega} + M_0(t)\frac{\partial(\omega G)}{\partial\omega} = G^2 + \frac{\partial(\omega S)}{\partial\omega}. \tag{4}$$

Here the obvious relations are used

$$\sum_{n=0}^{\infty}(n+1)\omega^n M_n(t) = \frac{\partial(\omega G)}{\partial\omega}, \quad \sum_{n=0}^{\infty}(n+1)\omega^n q_n(t) = \frac{\partial(\omega S)}{\partial\omega},$$

$$\sum_{n=0}^{\infty}\omega^n\sum_{k=0}^{n} M_k(t)M_{n-k}(t) = G^2.$$

In contrast to the case of homogeneous relaxation with $q(v,t) = 0$, the gas density $M_0(t) \equiv \varphi(0,t)$ is not constant. From equation (3) for $n = 0$ one can obtain

$$M_0(t) = \int_0^t q_0(t')dt' + M_0(0).$$

Notice also that

$$M_0(t) = G(t,0). \tag{5}$$

This is the reason why equation (4) has a nonlocal term. This fact was not taken into account in [16] in the process of finding an admitted Lie group. The lack of this condition can lead to incorrect admitted Lie groups. In the present paper this omission is corrected.

3 Admitted Lie algebra of the equation for the generating function

Equation (4) is conveniently rewritten in the form

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \tag{6}$$

where $u(0) = u(t,0)$. Here $\omega = x$, $G = u$ and $(\omega S)_\omega = g$.

As mentioned, because of the presence of the term $u(0)$, equation (6) is not a partial differential equation. Therefore, the classical group analysis method cannot be applied to this equation. A method that can be used for such equations with nonlocal terms was developed in [6, 7, 12]. In this section the latter method is applied for finding an admitted Lie group of equation (6).

Admitted generators are sought in the form

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \zeta(t,x,u)\partial_u.$$

According to the algorithm [6, 7, 12], the determining equation for equation (6) is

$$x\psi_{tx} + \psi_t + u(0)(x\psi_x + \psi) - 2\psi u + \psi(0)(xu)_x = 0, \tag{7}$$

where

$$\psi(t,x) = \zeta(t,x,u(t,x)) - u_t(t,x)\tau(t,x,u(t,x)) - u_x(t,x)\xi(t,x,u(t,x)),$$
$$\psi(0) = \psi(t,0).$$

After substituting the derivatives u_{tx}, u_{txx} and u_{ttx} found from equation (6) and its derivatives with respect to x and t into (7), one obtains the determining equation

$$\begin{aligned}
&\zeta_{tx}x^2 + \zeta_t x + \zeta_u g x + \zeta_u u^2 x + g\xi + u^2\xi - 2ux\zeta + ux\zeta(0)\\
&- x\left(g_t\tau + g_x\xi + g(\tau_t + \xi_x)\right) - \tau_t u^2 x - \xi_x u^2 x - xu_x(0)(u_x x + u)\xi(0)\\
&+ u(0)(\zeta_x x^2 - \zeta_u u x - u\xi + x\zeta + x\xi_x u + x\tau_t u) - \tau_x u_{tt}x^2 - x^2 u_x u_{tt}\tau_u\\
&- u_t u_{xx}\xi_u x^2 - u_{xx}\xi_t x^2 + u_t u_x x\big(\zeta_{uu}x - \tau_{tu}x + \tau_u u(0)x - \xi_{xu}x + \xi_u\big)\\
&+ u_t\big(\xi_x x + \zeta_{xu}x^2 - \xi - \tau_{tx}x^2 - 2\tau_u x(g + u^2) + u(0)x(2\tau_u u - x\tau_x)\big)\\
&+ u_t^2 x(\tau_u - x\tau_{xu}) + xu_t(0)(\tau - \tau(0))(u_x x + u) + u_x^2 x^2(\xi_u u(0) - \xi_{tu})\\
&+ xu_x\big(x(\tau_t u(0) + \zeta(0) + \zeta_{tu}) - \xi_{tx}x - \xi_t - 2\xi_u g - 2\xi_u u^2 + 2\xi_u u u(0)\big)\\
&- u_t^2 u_x \tau_{uu} x^2 - u_t u_x^2 \xi_{uu} x^2 = 0.
\end{aligned} \tag{8}$$

Here

$$\tau(0) = \tau(t,0,u(t,0)), \quad \xi(0) = \xi(t,0,u(t,0)), \quad \zeta(0) = \zeta(t,0,u(t,0)),$$
$$u_t(0) = u_t(t,0), \quad u_x(0) = u_x(t,0).$$

Differentiating the determining equation (8) with respect to u_{tt}, u_{xx}, and then with respect to u_t and u_x, one gets $\tau_u = 0$, $\tau_x = 0$, $\xi_u = 0$, $\xi_t = 0$. Therefore,

$$\tau = \tau(t), \quad \xi = \xi(x),$$

and hence $\tau(0) = \tau$. Differentiating the determining equation with respect to u_t, and then u_x, one finds $\zeta_{uu} = 0$, i.e.,

$$\zeta(t, x, u) = u\zeta_1(t, x) + \zeta_0(t, x).$$

The coefficient with $u_x u_x(0)$ in the determining equation (8) gives $\xi(0) = 0$. Continuing splitting the determining equation (8) with respect to u_t and then with respect to u_x, one finds

$$\zeta_1(t, x) = -x^{-1}\xi(x) + \zeta_{10}(t).$$

Hence $\zeta(0) = \zeta(t, 0) = u(0)(\zeta_{10}(t) - \xi'(0)) + \zeta_0(t, 0)$. The coefficient with $u_x u(0)$ leads to the condition

$$\zeta_{10} = -\tau_t + \xi'(0).$$

Differentiating the determining equation with respect to u twice, one has

$$\xi_x = 2\frac{\xi}{x} - \xi'(0).$$

The general solution of this equation is

$$\xi = x(c_1 x + c_0).$$

Equating the coefficient with u_x to zero, one derives $\tau_{tt}(t) = \zeta_0(t, 0)$. The coefficient with $u(0)$ in the determining equation (8) gives $x\zeta_{0x} + \zeta_0 = 0$. This equation has a unique solution which is nonsingular at $x = 0$,

$$\zeta_0(t, x) = 0.$$

Therefore, $\zeta_0(t, 0) = 0$ and

$$\tau = c_2 t + c_3.$$

The remaining part of the determining equation (8) becomes

$$g_t(c_2 t + c_3) + x g_x(c_1 x + c_0) = -2g(c_1 x + c_2). \tag{9}$$

Thus, each admitted generator has the form

$$X = c_0 X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3,$$

where

$$X_0 = x\partial_x, \quad X_1 = x(x\partial_x - u\partial_u), \quad X_2 = t\partial_t - u\partial_u, \quad X_3 = \partial_t. \tag{10}$$

The values of the constants c_0, c_1, c_2 and c_3 and relations between them depend on the function $g(t,x)$.

The trivial case of the function

$$g = 0$$

satisfies equation (9), and corresponds to the case of the spatially homogeneous and isotropic Boltzmann equation without a source term. In this case, the complete group classification of the Boltzmann equation was carried out in [7, 8] using its Fourier image (1) with $\hat{q}(y,t) = 0$. The four-dimensional Lie algebra $L^4 = \{Y_1, Y_2, Y_3, Y_4\}$ spanned by the generators

$$Y_0 = y\partial_y, \quad Y_1 = y\varphi\partial_\varphi, \quad Y_2 = t\partial_t - \varphi\partial_\varphi, \quad Y_3 = \partial_t \tag{11}$$

defines the complete admitted Lie group G^4 of (1). There are direct relations between the generators (10) and (11).

Indeed, since the functions $\varphi(y,t)$ and $u(x,t)$ are related through the moments $M_n(t)$, $n = 0, 1, 2, \ldots$, it is sufficient to check that the transformations of moments defined through these functions coincide.

Consider the transformations corresponding to the generators Y_0 and X_0,

$$\begin{aligned} Y_0 = y\partial_y\colon &\quad \bar{t} = t, \quad \bar{y} = ye^a, \quad \bar{\varphi} = \varphi;\\ X_0 = x\partial_x\colon &\quad \bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u. \end{aligned}$$

The transformed functions are

$$\bar{\varphi}(\bar{y},\bar{t}) = \varphi(\bar{y}e^{-a},\bar{t}), \quad \bar{u}(\bar{x},\bar{t}) = u(\bar{x}e^{-a},\bar{t}).$$

The transformations of moments are, respectively:

$$\begin{aligned} \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} = (-1)^n \frac{\partial^n \varphi(\bar{y}e^{-a},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0}\\ &= (-1)^n e^{-na}\frac{\partial^n \varphi}{\partial y^n}(0,\bar{t}) = e^{-na} M_n(\bar{t});\\ \bar{M}_n(\bar{t}) &= n!\frac{\partial^n u(\bar{x}e^{-a},\bar{t})}{\partial \bar{x}^n}_{|\bar{x}=0} = e^{-na} n! \frac{\partial^n u}{\partial x^n}(0,\bar{t}) = e^{-na} M_n(\bar{t}). \end{aligned}$$

Hence, one can see that the transformations of moments defined through the functions $\varphi(y,t)$ and $u(x,t)$ coincide.

The transformations corresponding to the generators Y_1 and X_1,

$$\begin{aligned} Y_1 = y\varphi\partial_\varphi\colon &\qquad \bar{t} = t, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{ya};\\ X_1 = x(x\partial_x - u\partial_u)\colon &\qquad \bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u, \end{aligned}$$

act on the functions $\varphi(y,t)$ and $u(x,t)$ and their moments in the following way:

$$\bar{\varphi}(\bar{y},\bar{t}) = e^{\bar{y}a}\varphi(\bar{y},\bar{t}), \quad \bar{u}(\bar{x},\bar{t}) = \frac{1}{1+a\bar{x}}u\left(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}\right),$$

$$\begin{aligned}
\bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} = (-1)^n \frac{\partial^n (e^{\bar{y}a}\varphi(\bar{y},\bar{t}))}{\partial \bar{y}^n}_{|\bar{y}=0} \\
&= (-1)^n \left(\left(\frac{\partial}{\partial y} + a\right)^n \varphi\right)(0,\bar{t});
\end{aligned}$$

$$\bar{M}_n(\bar{t}) = n! \frac{\partial^n \bar{u}(\bar{x},\bar{t})}{\partial \bar{x}^n}_{|\bar{x}=0} = n! \frac{\partial^n}{\partial \bar{x}^n}\left(\frac{1}{1+a\bar{x}}u\left(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}\right)\right)_{|\bar{x}=0},$$

respectively. Using computer symbolic calculations with `REDUCE` [9] one can check that these transformations of moments also coincide.

The vector fields Y_2 and X_2 generate the following transformations:

$$\begin{aligned}
&Y_2 = t\partial_t - \varphi\partial_\varphi: \quad \bar{t} = te^a, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{-a}; \\
&X_2 = t\partial_t - u\partial_u: \quad \bar{t} = te^a, \quad \bar{x} = x, \quad \bar{u} = ue^{-a},
\end{aligned}$$

which map the functions $\varphi(y,t)$ and $u(x,t)$ to the functions $\bar{\varphi}(\bar{y},\bar{t}) = e^{-a}\varphi(\bar{y},\bar{t}e^{-a})$ and $\bar{u}(\bar{x},\bar{t}) = e^{-a}u(\bar{x},\bar{t}e^{-a})$, respectively. The transformations of moments are

$$\begin{aligned}
\bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y},\bar{t})}{\partial \bar{y}^n}_{|\bar{y}=0} = (-1)^n e^{-a} \frac{\partial^n \varphi(\bar{y},\bar{t}e^{-a})}{\partial \bar{y}^n}_{|\bar{y}=0} \\
&= (-1)^n e^{-a} \frac{\partial^n \varphi}{\partial y^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a};
\end{aligned}$$

$$\begin{aligned}
\bar{M}_n(\bar{t}) &= n! \frac{\partial^n \bar{u}(\bar{x},\bar{t})}{\partial \bar{x}^n}_{|\bar{x}=0} = n! e^{-a} \frac{\partial^n u(\bar{x},\bar{t}e^{-a})}{\partial \bar{x}^n}_{|\bar{x}=0} \\
&= n! e^{-a} \frac{\partial^n u}{\partial x^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a}.
\end{aligned}$$

The case where the transformations of moments corresponding to the generators $Y_3 = \partial_t$ and $X_3 = \partial_t$ coincide is trivial. These direct relations between the Lie algebras confirm correctness of our calculations.

4 Comparison with results of [16]

Let us formulate the results of [16] using the variables of the present paper. The admitted generator obtained in [16] has the form

$$Z_g = \tau(t)\left(\partial_t - M_0(t)u\partial_u\right) + \alpha u\partial_u + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x, \tag{12}$$

where

$$m_0(t) = \int_0^t M_0(t')\,dt', \qquad \tau(t) = \left(\beta - \alpha\int_0^t e^{-m_0(t')}\,dt'\right)e^{m_0(t)},$$

α, β, γ and δ are constants. The function $g(t,x)$ has to satisfy the equation

$$\tau(t)\frac{\partial g}{\partial t} + x(x(\gamma-\delta)-\gamma)\frac{\partial g}{\partial x} = -2\left(x(\gamma-\delta) + M_0(t)\tau(t) - \alpha\right) g. \tag{13}$$

Since $M_0(t)$ is unknown, comparison of our results is only possible for $g = 0$. Moreover, in contrast to equation (9), the source function $g(t,x)$ in (13) as a solution of equation (13) depends on the function $M_0(t)$, whereas the function $M_0(t)$ also depends on the source function. This makes equation (13) nonlocal and very complicated.

Comparing the operator Z_g for $g = 0$ with (10), one obtains that the part related with the constants γ and δ coincides with the result of the present paper, whereas the part related with the constants α and β is completely different. Indeed, in this case equation (13) is satisfied identically, $M_0(t) = M_0(0)$, and for

$$M_0(0) \neq 0\colon \quad m_0(t) = tM_0(0), \quad \tau(t) = \beta e^{tM_0(0)} + \frac{\alpha}{M_0(0)}\left(1 - e^{tM_0(0)}\right);$$

$$M_0(0) = 0\colon \quad m_0(t) = 0, \quad \tau(t) = \beta - \alpha t.$$

The admitted generator (13) becomes

$$M_0(0) \neq 0\colon \quad Z_0 = \left(\beta - \frac{\alpha}{M_0(0)}\right) e^{tM_0(0)} \left(\partial_t - M_0(0)u\partial_u\right) + \frac{\alpha}{M_0(0)}\partial_t + (\gamma-\delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x;$$

$$M_0(0) = 0\colon \quad Z_0 = \beta\partial_t - \alpha(t\partial_t - u\partial_u) + (\gamma-\delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x.$$

One can see that the above results coincide with [16] only for $M_0(0) = 0$. The case $M_0(0) = 0$ corresponds to a gas with zero density which is not realistic. For $M_0(0) \neq 0$, the coefficient with the exponent $e^{tM_0(0)}$ plays a crucial role. This coefficient only vanishes for

$$\alpha = M_0(0)\beta. \tag{14}$$

In this case the admitted Lie algebra found in [16] is a proper subalgebra of the Lie algebra defined by the generators (10). Thus, all invariant solutions with $(\alpha,\beta,\gamma,\delta) = (M_0(0)\beta,\beta,\gamma,\delta)$ considered in [16] are particular cases of invariant solutions obtained in [6, 7]. In particular, the well-known BKW-solution is an invariant solution with respect to the generator $Y_{\mathrm{BKW}} = c(Y_1 - Y_0) + Y_3$. In the Lie algebra (10), this solution is related with the generator $X_{\mathrm{BKW}} = c(X_1 - X_0) + X_3$. Other classes of invariant solutions studied in [16] correspond to (14) with the particular choice of $\beta = 0$.

5 On equivalence transformations of the equation for the generating function

For the group classification, one needs to know equivalence transformations. Let us find some of them using the generators (10) and considering their transformations of the left hand side of equation (6)

$$Lu = xu_{tx} + u_t - u^2 + u(0)(xu_x + u).$$

The transformations corresponding to the generator $X_0 = x\partial_x$ map a function $u(t,x)$ into the function

$$\bar{u}(\bar{t},\bar{x}) = u(\bar{t},\bar{x}e^{-a}),$$

where a is the group parameter. Hence $\bar{L}\bar{u} = Lu$. One can check that the Lie group of transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = g$$

is an equivalence Lie group of equation (6).

Similarly, one derives that the transformations corresponding to the generator $X_3 = \partial_t$ define the equivalence Lie group:

$$\bar{t} = t + a, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{g} = g.$$

The transformations corresponding to the generator $X_2 = t\partial_t - u\partial_u$ map a function $u(t,x)$ into the function

$$\bar{u}(\bar{t},\bar{x}) = e^{-a}u(\bar{t}e^{-a},\bar{x}).$$

Hence $\bar{L}\bar{u} = e^{-2a}Lu$. One can conclude that the transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = ge^{-2a}$$

compose an equivalence Lie group of equation (6).

The transformations corresponding to the generator $X_1 = x(x\partial_x - u\partial_u)$ map a function $u(t,x)$ into the function

$$\bar{u}(\bar{t},\bar{x}) = \frac{1}{1+a\bar{x}}u\left(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}\right).$$

Hence $\bar{L}\bar{u} = (1-ax)^2 Lu$ and the transformations

$$\bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u, \quad \bar{g} = (1-ax)^2 g$$

compose an equivalence Lie group of transformations.

Thus, it has been shown that the Lie group corresponding to the generators

$$X_0^{\mathrm{e}} = x\partial_x, \quad X_1^{\mathrm{e}} = x(x\partial_x - u\partial_u - 2g\partial_g), \quad X_2^{\mathrm{e}} = t\partial_t - u\partial_u - 2g\partial_g, \quad X_3^{\mathrm{e}} = \partial_t$$

is an equivalence Lie group of equation (6).

There are also two involutions corresponding to the changes

$$E_1\colon\ \bar{x} = -x; \qquad E_2\colon\ \bar{t} = -t, \quad \bar{u} = -u.$$

6 Group classification

Group classification of equation (6) is carried out up to the equivalence transformations considered above.

Equation (9) can be rewritten in the form

$$c_0h_0 + c_1h_1 + c_2h_2 + c_3h_3 = 0, \tag{15}$$

where

$$h_0 = xg_x, \quad h_1 = x(xg_x + 2g), \quad h_2 = tg_t + 2g, \quad h_3 = g_t. \tag{16}$$

One of the methods for analyzing relations between the constants c_0, c_1, c_2 and c_3 is employing the algorithm developed for the gas dynamics equations [17]: one analyzes the vector space $\mathrm{Span}(V)$, where the set V consists of the vectors

$$v = (h_0, h_1, h_2, h_3)$$

with t and x are varied. This algorithm allows one to study all possible admitted Lie algebras of equation (6) without omission. Unfortunately, it is difficult to implement.

In [20][2] an algebraic algorithm for group classification was applied, which essentially reduces this study to a simpler problem. Here we follow this algorithm[3]. Observe here that because of the nonlinearity of the equivalence transformations corresponding to the generator X_1, it is difficult to select out equivalent cases with respect to these transformations, whereas the algebraic algorithm does not have such complication.

First we study the Lie algebra L_4 composed by the generators X_0, X_1, X_2 and X_3. The commutator table is

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

The inner automorphisms are

$$\begin{aligned} A_0 &: \quad \hat{x}_1 = x_1e^a, \\ A_1 &: \quad \hat{x}_1 = x_1 + ax_0, \\ A_2 &: \quad \hat{x}_3 = x_3e^a, \\ A_3 &: \quad \hat{x}_3 = x_3 + ax_2, \end{aligned}$$

where only the changed coordinates are presented.

[2]See also references therein.

[3]The authors thank the anonymous referee for pointing to the possibility of applying to the analysis of equation (15) the algorithm considered in [20].

Second, one can notice that the results of using the equivalence transformations corresponding to the generators $X_0^e, X_1^e, X_2^e, X_3^e$ are similar to changing coordinates of a generator X with regarding to the basis change. These changes are similar to the inner automorphisms. Indeed, the coefficients of the generator X are changed according to the relation [17]:

$$X = (X\bar{t})\partial_{\bar{t}} + (X\bar{x})\partial_{\bar{x}} + (X\bar{u})\partial_{\bar{u}}.$$

Any generator X can be expressed as a linear combination of the basis generators:

$$\hat{x}_0\hat{X}_0 + \hat{x}_1\hat{X}_1 + \hat{x}_2\hat{X}_2 + \hat{x}_3\hat{X}_3 = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3, \tag{17}$$

where

$$\hat{X}_0 = \bar{x}\partial_{\bar{x}}, \quad \hat{X}_1 = \bar{x}(\bar{x}\partial_{\bar{x}} - \bar{u}\partial_{\bar{u}}), \quad \hat{X}_2 = \bar{t}\partial_{\bar{t}} - \bar{u}\partial_{\bar{u}}, \quad \hat{X}_3 = \partial_{\bar{t}}.$$

Using the invariance of a generator with respect to a change of the variables, the basis generators X_i $(i = 0, 1, 2, 3)$ and $\hat{X}_j$ $(j = 0, 1, 2, 3)$ in corresponding equivalence transformations are related as follows

$$\begin{aligned}
X_0^{\mathrm{e}}:& \quad X_0 = \hat{X}_0, \quad X_1 = e^{-a}\hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;\\
X_1^{\mathrm{e}}:& \quad X_0 = \hat{X}_0 + a\hat{X}_1, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3;\\
X_2^{\mathrm{e}}:& \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = e^{a}\hat{X}_3;\\
X_3^{\mathrm{e}}:& \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2 - a\hat{X}_3, \quad X_3 = \hat{X}_3.
\end{aligned}$$

Substituting these relations into the identity (17), one obtains that the coordinates of the generator X in the basis X_0, X_1, X_2, X_3 and in the basis $\hat{X}_0, \hat{X}_1, \hat{X}_2, \hat{X}_3$ are related similar to the changes defined by the inner automorphisms.

This observation allows us to use an optimal system of subalgebras of the Lie algebra L_4 for studying equation (15). Construction of such optimal system is not difficult. Moreover, it may be simplified if one notices that $L_4 = F_1 \bigoplus F_2$, where $F_1 = \{X_0, X_1\}$ and $F_2 = \{X_2, X_3\}$ are ideals of the Lie algebra L_4. This decomposition gives possibility to apply a two-step algorithm [18, 19]. The result of construction of an optimal system of subalgebras is presented in Table 1.

Notice also that in constructing the optimal system of subalgebras we also used transformations corresponding to the involutions E_1 and E_2:

$$E_1: \ \hat{x}_1 = -x_1; \qquad E_2: \ \hat{x}_3 = -x_3.$$

To obtain functions $g(t, x)$ using the optimal system of subalgebras one needs to substitute the constants c_i corresponding to the basis generators of a subalgebra into equation (15), and solve the system of equations thus obtained. The result of group classification is presented in Table 2, where α, $\beta \neq 1$,$\gamma \neq -2$ and k are constant, and the function Φ is an arbitrary function of its argument.

Table 1. Optimal system of subalgebras.

	Basis		Basis
1	$X_0,\ X_1,\ X_2,\ X_3$	11	$X_2+\alpha X_0,\ X_1$
2	$X_2+\alpha X_0,\ X_1,\ X_3$	12	$X_0+X_3,\ X_1$
3	$X_0,\ X_1,\ X_3$	13	$X_3,\ X_1$
4	$X_0,\ X_1,\ X_2$	14	$X_0,\ X_1$
5	$X_0,\ X_2,\ X_3$	15	$X_2+\alpha X_0$
6	$X_2,\ X_3$	16	X_2+X_1
7	$X_2-X_0,\ X_1+\ X_3$	17	X_3+X_0
8	$\alpha X_2-2X_0,\ X_3$	18	X_3+X_1
9	$X_1+X_2,\ X_3$	19	X_0
10	$X_0,\ X_2$	20	X_1
		21	X_3

Table 2. Group classification.

	$g(t,x)$	Generators		$g(t,x)$	Generators
1	0	$X_0,\ X_1,\ X_2,\ X_3$	9	kx^{-2}	$X_1,\ X_3$
2	kx^{-2}	$X_2+X_0,\ X_3,\ X_1$	10	$t^{-2}\Phi(xt^{-\alpha})$	$X_2+\alpha X_0$
3	$kx^2(xt+1)^{-4}$	$X_2-X_0,\ X_1+X_3$	11	$x^{-2}e^{2x^{-1}}\Phi(te^{x^{-1}})$	X_2+X_1
4	kx^{γ}	$\gamma X_2-2X_0,\ X_3$	12	$\Phi(xe^{-t})$	X_3+X_0
5	$kx^{-2}e^{2x^{-1}}$	$X_2+X_1,\ X_3$	13	$x^{-2}\Phi(t+x^{-1})$	X_3+X_1
6	kt^{-2}	$X_0,\ X_2$	14	$\Phi(t)$	X_0
7	$kx^{-2}t^{2(\beta-1)}$	$X_2+\beta X_0,\ X_1$	15	$x^{-2}\Phi(t)$	X_1
8	$kx^{-2}e^{2t}$	$X_3+X_0,\ X_1$	16	$\Phi(x)$	X_3

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Towards an Axiomatic Noncommutative Geometry of Quantum Space and Time

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By exploring a possible physical realisation of the geometric concept of noncommutative tangent bundle, we outline an axiomatic quantum picture of space as topological manifold and time as a count of its reconfiguration events.

This is a physics-oriented companion to the brief communication [4] and its formalisation [6]; we analyse a possible physical meaning of the notions, structures, and logic in a class of noncommutative geometries considered therein, see also [8, 12]. We now try to formalise the geometry of such intuitive ideas as space and time, aiming to recognise further such phenomena of Nature as the mass and gravity (here, dark matter and vacuum energy) and Hubble's law. Affine Lie algebras, the definition of real line $\mathbb{R}$ via 0, 1, addition, and bisection, and Voronoï diagrams play key rôles here.

This text itself is a part of the essay [7]: let us first describe the configuration of physical vacuum, while in the second half of [7] we study the admissible means and rules of coding sub-atomic particles and explore the ways of the particles' (trans)formations, reactions, and decays. In particular, we then address the mass endowment mechanism, generation of (anti)matter and annihilation, CP-symmetry violation, three lepton-neutrino matchings, spin, helicity and chirality, electric charge and electromagnetism, as well as the algorithms underlying the weak and strong processes. The goal which we set for a semantic analysis of the postulates and their implications within the algebra and calculus in [4, 6] is the construction of an elementary, toy model unifying the four fundamental interactions. We agree that the potential of our topological and combinatorial picture to explain or propose possible (dis)verifying experiments is not still the required ability for a model to predict.

In fact, we only discuss a possible physical sense of axioms and operations or deductions which are admissible in the chosen setup. Still, our synthesis may be not a unique way to relate this mathematical formalism to Nature.

Remark 1. We attempt to identify and describe the physics which not necessarily *is*. We now sketch the processes and motivate their laws which could be dominant in the early Universe only. Alternatively, it may happen that these processes are realised nowadays (or are presently registered as signals which were emitted from afar in our remote past) only under very restrictive hypotheses about the local space-time geometry, e.g., near a black hole or near its singularity. Nevertheless, we develop the formalism in a hope that it does render the quantum structure of the Universe at Planck scale.

Our main message is this: It may be that at the Planck scale, the geometry of this world is disappointingly simple[1] because

- it does not refer to the diffeo-structure of the visible space, i.e., to its locally vector space organisation with velocity along piecewise-smooth trajectories and their length, and with smooth transition functions between charts in the atlas for that manifold; instead, the events occur in the Universe, which does not amount to the visible space, by using its much more rough homeo-structure of topological manifold with continuous transition functions, whereas the incidence relations between points along continuous paths replace the obsolete notions of length and speed;

- the Universe consists of *naught* but the homeo-class space itself and the information which it carries or is able to carry; the fact of existence, behaviour, and known forms of the interaction between particles refer to the locally available information (in particular, stored in a single point by using a local modification of the topology); the presence of gauge degrees of freedom at each point of the diffeo-class space is the manifestation of its own homeo-structure; the gauge transformations are performed pointwise, either entirely independently at different points or in no more than a (piecewise) continuous way, whence an attempt to bind Nature with their differentiability –in order to introduce the gauge connections by taking derivatives of arbitrary functions– is an *ad hoc* assumption of the objects' description.

Indeed, let us notice that the idea of a connection in a principal fibre bundle appeals to the (existence of) structures, and to their values outside that point, which therefore requires the existence of other points. Moreover, the gauge freedom of any kind allows, in its basic formulation [10], pointwise-uncorrelated gauge transformations of the field of matter. Consequently, the postulate that *length* is defined and hence the distance between space-time points and between fields can be measured, giving rise to the construction of derivatives, and the postulate that the pointwise-uncorrelated values glue to a (piecewise-)smooth local section are the act of will, i.e., an *ad hoc* assumption in a description to which Nature is indifferent (e.g., see [9, Eq. (4,1)]). We remark separately that the operations which are recognised as gauge transformations but which stem from the presence and structure of space itself (e.g., local homeomorphisms of topological spaces) are at least but also no more than continuous, see sec. 3.1 on p. 120. We conclude

[1] This is likely if we recall how Science gradually cast away various essences such as the phlogiston (though as an abstract principle it was useful for the technology of steam engine), æther (to which we owe the radio and knot theory, recalling that W. Thomson's vortex rings preceded Bohr's planetary model of atom), or the long-range gravity force (which remains helpful for navigation in the Solar system).

that local processes can not be governed by smooth gauge theory (or only by it; in particular, gauge connection fields did not exist at the moment of the Big Bang).

If this is indeed so, the rules of Nature's behaviour are the arithmetic – comparison or addition of topologically-defined integer numbers – and the associative algebra of gluing or splitting words written consecutively in its alphabet(s), and the coding of such topological objects as walks and cycles or knots.

1 The two avatars

This Universe is a topological space. In the beginning, its topology was trivial: $\mathcal{T}_0 = \{\varnothing, \text{Universe}\}$. Nowadays it is not Hausdorff so that there are points which we can not tell one from another. Modifications of the topology $\mathcal{T}$ are also possible.[2]

Within the regions of *vacuum* where the topology allows one to distinguish between points, the Universe is endowed simultaneously with two structures: one is the homeo-class structure of topological manifold with *continuous* transition functions between coordinates (those form continuous nets on the charts); the other is the diffeo-structure of smooth manifold such that the transition functions are *smooth* and local coordinates form the smooth nets.

The Universe co-exists in its homeo and diffeo avatars. The homeo structure is the *quantum world*; it carries the information about the geometry and about the types, formation, and actual existence and states of the particles. The laws of fundamental interactions between particles retrieve and process that information, thus determining the processes that run at the quantum level. Each particle or any other object in the quantum, homeo-class geometry has a continuous world-line.

Our conscience percepts the Universe and events in it at the macroscopic level using the smooth, diffeo-class geometry, that is, by understanding of the local charts as domains in a vector space over $\mathbb{R}$ with the usual arithmetic of vectors. The notion of *length* is defined in the macroscopic world.[3] This notion allows us to measure local macroscopic distances by using rigid rods and also measure local time intervals between events by employing the postulate of invariant light speed, that is, by using derivatives of the former equipment. With the help of rigid rods and light, we introduce the macroscopic notions of instant velocity and define the nominal concept of a smooth trajectory, not referring it to any material object but only to the local properties of the smooth macroscopic space-time. The transition between macroscopic charts with smooth coordinates in the diffeo-class space-time are governed by the Lorentz transformations. The topology of macroscopic space-time outside particles and black hole singularities is induced by the (indefinite) metric in inertial reference frames.

[2]We claim that exclusions of sets from the list $\mathcal{T}$ and re-inclusions of the information about such sets provide the mass endowment mechanism and formation of the black holes' singularities.

[3]By convention, a length scale is *macroscopic* if the typical distances considerably exceed the electric-charge diameter of proton, which is approximately $1\ \mathsf{fm} = 10^{-15}\ \mathsf{m}$.

The tautological mapping from the quantum, homeo-class world to the macroscopic, diffeo-class realisation of the Universe is continuous but not a homeomorphism; by construction, it can not be an isometry. Under this tautological mapping, the information which is realised by the quantum geometry takes the shape of particles in the macroscopic world; however, a part of this information is lost along the way due to the introduction of length (more precisely, of Lorentz' interval): there are topologically-nontrivial quantum objects – in fact, a whole dimension – which acquire zero visible size in all reference frames.

On top of that, because the composition of (1) local homeomorphisms from the standard domains to the charts of the homeo-class topological manifold with (2) the tautological mapping is only continuous at all points of the Universe, the images of points and point particles in its visible, diffeo-class realisation are observed by us as if they are in a perpetual inexplicable "motion". Namely, as it often happens with legal documents, no other rights may be derived from the statement that the composition is continuous: the pledge is to take points from nested sets in the atlas $\mathcal{T}$ to near-by points as we see them, but the continuous mapping does not presume that we, upon our own initiative, shall apply our notion of length to some continuous curves connecting those images. In effect, inertial trajectories of material objects in the quantum world are continuous but nowhere differentiable.[4] The visible world-lines are at most $(c, 1)$-Lipschitz, where c is the speed of light and the power 1 states that no material object is allowed to run out of the light cone of its future.

Example 1. Suppose that we know (setting aside all the subtleties related to the act of measurement) that a quantum object – e.g., a marked point of it where all the mass or all the charge is contained – is located now at a given point and moreover, it does not move with respect to other points according to the incidence relations between points in a Hausdorff topology. Nevertheless, we may not know its visible instant velocity because that notion refers to the limit procedure in a vector space and hence is not applicable.

Example 2. Likewise, choose an inertial reference frame and consider a situation when a domain in quantum space is homeomorphic to a domain within a crystal structure [2, 3]. Suppose that a vertex of the lattice decides to visit its neighbour and thus goes along the edge connecting them. Not only its visible initial location was non-constant in time and the initial instant velocity undefined, but this will remain so at all points of the continuous, nowhere-differentiable image of the trajectory along the edge; the journey will end at an unpredictable location of the endpoint with undefined terminal velocity. We conclude, referring again to the Red–White antagonism, that it is impossible to tell which of the two worlds, topological homeo or smooth diffeo, is straight and which is shaking.

[4] This creates the classical antagonism between Red and White: namely, Norm is firm, straight, and always all right while Shake is indecisive and trembling. Only we cannot tell who is who in this Universe.

For the same reason, provided that we postulate a point particle's *visible* instant velocity (irrespective of its actually on-going displacement with respect to the topologically-neighboured points in the quantum world), we may not determine at which point of the visible macroscopic space it is located. (Let us remark that the above examples and reasonings are not applicable to the propagation of light which can not stay at rest with respect to the incidence relations.) The balance of resolution for the location of a material quantum object in the smooth space at a given time and for its momentum is determined by the Heisenberg uncertainty principle.[5] Simultaneously, the propagation of a point particle from a given point to a given endpoint along an *a priori* unknown continuous trajectory in the visible, diffeo-class world is the cornerstone of the concept of Feynman's path integral.

We now propose to abandon the futile attempts to measure or approximate the undefined notions but study the interactions between quantum objects by referring their laws to the homeo-class geometry of the Universe. Let us remember that the difficulties and uncertainties which we gain – when measuring length and calculating derivatives such as the velocity – in a description of the quantum processes do not stem from their true nature. Integrating empirically the laws of its evolution, the Universe stays, and will stay forever indifferent to the fact that we can not grasp all its details at once, since we ourself first proclaimed our intention to take proportions with respect to the standard metre instead of inspecting the topological invariants of phenomena.

Corollary 1. *The processes in the quantum,* homeo*-class* (*locally*) *Hausdorff topological manifold without length can not be adequately described by* (*the geometry of partial*) *differential equations* (*c.f.* [5, 11])*. On the other hand, the construction of* σ*-algebras associates the* measure *in its true sense with sets of points but not with distances between points; consequently,* integral *equations could be more relevant.*

2 The time phenomenon

There are at least two ways to understand what the *time* is in context of a paradoxal observation by our conscience that everything in this world is staying perpetually in the *present*.

A realist approach to the notion of time postulates the existence of a full-right uncompactified dimension with a reasonable topology of the resulting space-time. One then operates with the count of time by using the invariant Lorentz interval, light cones of the past and future, and world-lines of material objects. An inconvenience of this approach is that, in order to maintain the everlasting presence, the visible world must unceasingly glide along the time direction, i.e., to keep in the same place, it takes all the running one can do. Note that under

[5] Note that we may not track the behaviour of "empty" points of the quantum space if they are not referred to by any material object located there; consequently, we do not attempt to introduce a "temperature" of the vacuum.

Lorentz' transformations the local observer's time can be bent towards another observer's space and *vice versa* but in earnest the time can not be swapped with any spatial direction.

The concept of a (3+1)-dimensional smooth or topological manifold into which the time is incorporated *a priori* contains the following logical difficulty. An infinitely-stretching absolutely empty, flat Minkowski space-time $\big(\mathbb{R}\times\mathbb{R}^3, (+---)\big)$ without a single object in it would exist *forever*. In our opinion, there is no time at all in that empty world: the cups, tea, and bread-and-butter always remain the same, so it is always six o'clock. Nobody counts to the Time hence time does not count.

Definition 1. We accept that the time is a count of reconfiguration events in this Universe; such events are, for example,[6] the reconfigurations of geometry (i.e., an act of modification in the topology) or the operation of an algorithm that transfers information over points, creating an event of output statement by processing the local configuration of the Universe in its input (such is the propagation of light).

Thus, events create time. Events which do not reconfigure the Universe (e.g., a correct statement that for a topologically-admissible arc connecting point $\underline{0}$ to point $\underline{1}$ there is a null path running from $\underline{0}$ to $\underline{1}$ and then back again along the same arc) *do not* express the count of time (although a *verification* of such statement by using light signals does take and hence creates time).

The notions of recorded past and expected future are derived from the relation of order in the count of events by an appointed observer; let us remember that an opinion of another observer about the order of events could be different.

Example 3. Consider the reconfiguration of the Universe produced by a trip of Chapeau Rouge from point $\underline{0}$ to point $\underline{1}$ along a continuous arc connecting them in a coordinate chart of the homeo-realisation of the Universe. This amounts to the input information that the two endpoints, the arc, and Chapeau Rouge exist, that the available choice of topology confirms that the path is continuous, and to the work of the algorithm the negates the already passed points and thus prescribes the admissible direction to go further.

In absence of length and in absence of any devices at the observer's disposal, the time is discrete: it is counted by the events (1) Chapeau Rouge is at the starting point; (2) Chapeau Rouge has reached the endpoint.

The observer can grind the time scale by recursively installing the intermediate checkpoints somewhere in between the points which are already marked; this is done by using the incidence relation for points on the continuous path and does not refer to the notion of length (in fact, it refers to the definition of real numbers

[6] Were the Universe truly smooth, Poisson, and possess the Hamiltonian functional, then the time by definition would be taking the Poisson bracket with such master-functional of the current state; a weakened and much more likely formulation is the generation of time by events of evaluating binary operations at locally defined Hamiltonian functionals that correspond to separate particles, c.f. [6].

by using 0, 1, addition, and bisection). The limiting procedure makes the count of time continuous.

It is the postulate of invariant light speed which endows the Universe with its local smooth structure ("twice earlier $\Longleftrightarrow$ twice closer"). The light automaton is programmed to choose the next point by processing the information about earlier visited points and creates an event of specific type; the principle is that all observers accept its performance identically. By using the bisection method, we first mark the midpoint $\underline{{}^1\!/_2}$ on the chosen curve and replicate the automaton $\underline{0} \to \underline{1}$ to the automata $\underline{0} \to \underline{{}^1\!/_2}$ and $\underline{{}^1\!/_2} \to \underline{1}$; then we declare that the old automaton counts the unit step of time and each of the new automata counts one half. The recursive process and the limiting procedure create the smooth structure of space-time for a given observer.[7]

The inconvenience is that this smooth structure is not applicable to material objects which are known to travel slower than light; in order to monitor the steady progress of Chapeau Rouge on her way from $\underline{0}$ to $\underline{1}$, one must use as many light signals as there are checkpoints installed along the path. Even if the *energy* emitted by the new, "shorter-range" automata drops at the moment of each replication, the total energy which one has to spend in the continuous limit is either null or infinite; the first option is useless because it does not communicate any information to the observer; the second option is not impossible if the Universe is infinite and the observer agrees to waste a finite fraction of this world, still it is impractical.

In the next section we introduce a possible local topological structure of the homeo-class realisations of the Universe. If one feels it necessary to multiply dimensions, then we advise to let a macroscopic observer view the construction from a chosen inertial frame; the time direction is then locally decoupled to the real line $\mathbb{R}$. At the end of the next section, the structure of the macroscopic images of domains under the tautological mapping from homeo to diffeo is then recalculated to all other inertial reference frames by using Lorentz' transformations. Let us only remark that the "smooth time" parameter is introduced in the diffeo-class world in order to legalise the limiting procedures such as the correlation of arbitrarily small length and the speed of light in the local vector-space organisation of space.

3 Local configuration of quantum space

Now we introduce the local topological configuration of empty quantum space, that is, *vacuum* away from the singularities of black holes. In technical terms, we define the admissible local structure by taking "as is" or via self-similar continuous

[7] To use light as the pacemaker of the clock, we ought to describe first what a photon is; to do that we operate in [7] with the homeo-class geometry. We also notice that *before* the emission of the first photon in history of the Universe (or before creation of any other massless particles which travel with the same invariant speed c), the time had been counted by using events of other origin; we argue that such events were the reconfigurations in the topology $\mathcal{T}$.

limit the lattice of affine Lie algebra (primarily using the root systems A_3, B_3, or C_3 of simple complex Lie algebras) and thus consider a truncated Kaluza–Klein model without the Minkowski dimension of Newtonian time but instead, with a topology brought in by hand (though it is equivalent to a standard one for each continuous limit); we then analyse the origins of Hubble's law.

To describe a domain in quantum, or *quantised*, space we first consider Euclidean space $\mathbb{E}^3$ containing the affine lattice generated by the irreducible root system A_3, B_3, or C_3 (see Remark 6 on p. 121). Let us denote by $\vec{\mathrm{x}}_i \equiv \vec{\mathrm{x}}_i^{+1}$ the generators of the lattice at hand and by $\vec{\mathrm{x}}_i^{-1}$ their inverses (so that the paths $\underrightarrow{\vec{\mathrm{x}}_i \cdot \vec{\mathrm{x}}_i^{-1}} = \underrightarrow{\vec{\mathrm{x}}_i^{-1} \cdot \vec{\mathrm{x}}_i} = 1$ end at the point where they start). Each lattice determines the tiling of space $\mathbb{E}^3$, its vertices and edges constituting the 1-skeleton of the CW-complex with trivial topology (see Remark 8 on p. 125). We let a finite domain in $\mathbb{E}^3$ with a given configuration of vertices and the adjacency table of the lattice be the *spatial component* of a prototype domain in the discrete quantum space.

Second, we take the product $\mathbb{E}^3 \times \mathbb{E}^2$ of space with a two-plane into which we place the circle $\mathbb{S}^1$ passing around the origin. Viewing the circle as an oriented one-dimensional topological manifold, we create an extra, compactified dimension in the local quantum geometry. Namely, to each vertex of the prototype domain we attach the *tadpole* $\mathbb{S}^1$, i.e., the edge that starts and ends at the same point and loops in the extra dimension (outside the old $\mathbb{E}^3$). By convention, we denote by $\mathbb{S}^1 \equiv \mathbb{S}^{+1}$ the tadpole walked counterclockwise with respect to the standard orientation of $\mathbb{E}^2$ and by $\mathbb{S}^{-1}$ the reverse, clockwise cycle.

Remark 2. Under the tautological mapping of the quantum world to the diffeo-class visible world, the tadpoles are assigned zero length because the distance between their start- and endpoints vanishes for each of them.[8] We conclude that the entire compactified dimension is invisible to us; this is why the tautological mapping between the homeo- and diffeo-realisations of the Universe is not bijective: it compresses one extra dimension at each point to a null vector.[9]

Namely, passing to the additive notation $\big((\vec{\mathrm{x}}_i, \vec{\mathrm{d}}), \pm\big)$ instead of the multiplicative alphabet $\big((\vec{\mathrm{x}}_i^{\pm 1}, \mathbb{S}^{\pm 1}), \cdot\big)$, that is, viewing the letters as vectors in Euclidean space but not as the shift operators and introducing the null vector $\vec{\mathrm{d}}$, we recover the standard description of the affine basis for the Kac–Moody algebra at hand; clearly, the length of the null vector equals zero. Recall further that the circle $\mathbb{S}^1$ is the total space in a double cover over the real projective line $\mathbb{RP}^1 \simeq \mathbb{S}^1/{\sim}$; one full rotation $\mathbb{S}^{+1}$ corresponds[10] to running along the projective line twice: $\mathbb{S}^1 = \overrightarrow{\mathrm{tt}}$

[8] We note that the notion of length *is* applicable to the generators $\mathbb{S}^{\pm 1}$ of such null vectors but it is not applicable to the edges $\vec{\mathrm{x}}_i^{\pm 1}$ in space: their length is undefined because the homeomorphisms from domains in $\mathbb{E}^3$ containing the lattice to the spatial counterparts of the prototype domains are not fixed but can change with time.

[9] We also notice that the generators $\mathbb{S}^{\pm 1}$ encode nontrivial walks in quantum space but produce no visible path which would leave a single point in the macroscopic world; we postulate that the contours $\mathbb{S}^{\pm 1}$ determine the electric charge $\pm e$, see [7].

[10] We introduce a separate notation $\vec{\mathrm{t}}$ for one rotation along the projective line anticipating

and $\mathbb{S}^{-1} = \bigl(\overrightarrow{\mathrm{t}\mathrm{t}}\bigr)^{-1}$; the double cover over $\mathbb{RP}^1$ is then responsible for the familiar coefficient '2' in front of the null vector $\vec{\mathrm{d}}$.

Remark 3. The vertices of the CW-complex are the *quanta* of space; there is a deep logical motivation for their existence. Namely, by assembling to one vertex a continuum of physical points within a domain which is dual to the set of neighbouring vertices in the lattice, Nature replaces the *continuous* adjacency table between points to a *finite*, lattice-dependent table so that there are only finitely many neighbours of each vertex and hence a finite local configuration of information channels.

In conclusion, space is continuous but the Universe operates with quantum phenomena in it, thus achieving a great economy in the information processing.

Remark 4. The tadpole $\mathbb{S}^1$ at a vertex of quantum space is an indexed union $\bigcup_{i\in\mathcal{I}} \mathbb{S}^1_i$ of tadpoles referred by i to an indexing set $\mathcal{I}$ of points in the quantum domain which is marked by the vertex. Typically, this set is at least countable, $\mathcal{I} \supseteq \mathbb{Z}$; one could view it as an enumerated set of binary approximations for points in that domain (here we use the auxiliary metric in $\mathbb{E}^3$); we emphasize that by choosing the indexing set in this way we endow it with *order*, c.f. [7].

This convention allows us to handle infinitely many tadpoles attached to an everywhere dense set in the quantum domain by ascribing a different statistics to a unique tadpole which is attached to the vertex which marks that domain.

Note 1. We postulate that the spatial edges $\vec{\mathrm{x}}_i^{\pm 1}$ of the lattice are *fermionic* so that no such edge can be walked twice in the same direction by one path; a path can run twice along the same edge only in the opposite directions. Note that different paths can go independently in the same direction along a common edge; we also notice that a path can run many times through a vertex, approaching it each time by a different edge in its adjacency table.

Unlike it is with spatial edges, the tadpoles $\mathbb{S}^{\pm 1}$ attached to the vertices are *bosonic* so that paths can rotate on these caroussels any finite number of times in any direction (which does not really matter because the overall difference $\sharp\mathbb{S}^1 - \sharp\mathbb{S}^{-1}$ of positive and negative rotation numbers is constrained by the value of electric charge of the particle encoded by the path). However, let us remember that in earnest we are dealing with ordered infinite sets $\{\mathbb{S}^{\pm 1}_m,\ m \in \mathbb{Z} \subseteq \mathcal{I}\}$ of *fermionic* tadpoles brought to the marker of a quantum domain; in the continuous limit of a quantum tiling, this set spreads over the domain — one fermionic tadpole per each indexed point.

So, let us recall that the part of a lattice in a domain of $\mathbb{E}^3$, with tadpole attached to each vertex, is *discrete*. We say that the original fermionic lattice with bosonic tadpoles is the *quantum space*; in what follows we formalise the geometry of elementary particles in terms of the *alphabet* $\mathfrak{A} = \bigl((\vec{\mathrm{x}}_i^{\pm 1}, \mathbb{S}^{\pm 1}), \cdot\bigr)$.

its future application in the description of "building blocks" for strong interaction and also its possible use in the study of the quantum Hall effect.

The standard bisection technique (see sec. 2) allows us to convert the discrete tiling to its continuous limit in which the topology is inherited from the adjacency table of the affine lattice (the neighbourhoods in the Voronoï diagram are the duals of adjacent vertices' configuration in the spatial, $\mathbb{E}^3$-tiling component of the CW-complex); the limit topology is locally equivalent to the product topology for $\mathbb{S}^1$ and Euclidean space $\mathbb{E}^3$; the orientation field for $\mathbb{S}^1$ over $\mathbb{E}^3$ is continuous.

Definition 2. The self-similar limit of the discrete structure in a domain of quantum space is a *domain* in the homeo-class realisation of the Universe.

Remark 5. The introduction of a continuous field of fermionic lattice generators $\vec{\mathrm{x}}_i^{\pm 1}$ and fermionic loops $\mathbb{S}^{\pm 1}$ or $\vec{\mathrm{t}}^{\pm 1}$ over *each* point of continuous space, which we have performed here in full detail, is the homeo-class analog of the *noncommutative tangent bundle* over the smooth visible realisation of the Universe, see [6].

Quantum space is discontinuous; in sec. 2 we argued that a verification of the continuity for its self-similar limit requires the expense of infinite energy whenever one attempts to monitor a steady motion of a material object travelling slower than the speed of light and for that purpose encodes the object's path by the alphabet $\mathfrak{A}_\infty = \bigl((\frac{1}{2^n}\vec{\mathrm{x}}_i^{\pm 1}, \mathbb{S}^{\pm 1};\ n \in \mathbb{N} \cup \{0\}), \cdot\bigr)$. However, a motivation why the limit should nevertheless be studied – and is more than a mathematical formality – is as follows. Namely, a *continuous* coding of points in space by using binary arithmetic permits us to consider continuous paths – in particular, closed contours, – not referring them to a specific lattice. Indeed, our ability to describe and handle such contours does not imply that any material object is actually transported along those paths; hence energy is not spent but the drawn figures, and homotopies of these images in space, do encode information: a generic continuous path is an infinitely-long cyclic word written by using infinitely-short letters of the alphabet $\mathfrak{A}_\infty$. The massless chargeless contours propagate freely in homeo-class domains until a very rare event of their disruption and weak interaction with other material objects. However, this is only a part of the story.

3.1 The $U(1) \times SU(2)$-picture

First, let us notice that there is no marked origin in the affine lattice and therefore it acts on itself by finite shifts. Note further that this action is topological: it appeals to the incidence relations between vertices but not to the smooth, local vector-space organisation of $\mathbb{E}^3$.

Having placed the affine lattice in $\mathbb{E}^3$, one could – by an act of will to which Nature is indifferent – extend the algebra of finite shifts to the space of homeomorphisms of $\mathbb{E}^3$, i.e., the local action of space upon itself by a continuous field of translations. Moreover, by compactifying space to $\mathbb{E}^3 \cup \{\mathrm{pt}\} \simeq \mathbb{S}^3$, one extends this action to homeomorphisms of the three-sphere. By yet another misleading isomorphism $\mathbb{S}^3 \simeq SU(2)$ – which is given, e.g., by the Pauli matrices – one is tempted to conclude that

1. the complex field $\mathbb{C}$ is immanent to static geometry of the Universe, and

2. the freedom of appointing for reference point any vertex in the affine lattice, now realised as a set of points inside $SU(2)$, means the introduction of the $SU(2)$-principal fibre bundle over the space-time.

Yet even more: though the pseudogroup of local homeomorphisms of space states that the field of pointwise-defined shifts is continuous, it is postulated that this deformation field is smooth, hence there exist derivatives of local sections for the principal fibre bundle. This pile of *ad hoc* conventions delimits the smooth complex $SU(2)$-gauge theory of weak interaction [10].

Likewise, each tadpole's circle $\mathbb{S}^1$ carries the gauge freedom of marking a starting (hence, end-) point on it and also it can be subjected to an arbitrary homeomorphism (not necessarily a diffeomorphism), which leaves the tadpoles $\mathbb{S}^{\pm 1}$ intact. The choice of marked points is made *pointwise* at vertices of the lattice (or at all points of continuous space if we deal with the limit) — without any idea of smoothness superimposed to continuity. Now we note another misleading isomorphism $\mathbb{S}^1 \simeq U(1)$, which also tempts one to introduce complex numbers in the static quantum geometry.

Summarising, we see that electroweak phenomena could be quantum space in disguise.

3.2 Hubble's law

Second, let us recall that there are exactly three canonical tilings of Euclidean space.

Remark 6. The three irreducible tilings of space are determined by simple root systems A_3, B_3, and C_3. They have equal legal rights in the geometric construction, still we believe that the tetrahedral tiling which corresponds to A_3 dominates over the two others whenever one is concerned with the symmetry and stability of particles whose contours are encoded by words written in these root systems' alphabets (see [7]). Thus, more symmetric particles are more stable.

We recall that the adjacency tables for vertices are different for the three irreducible lattices in $\mathbb{E}^3$ so that the local configurations of information channels between points in the continuous limits are also different; the three continuous versions of one space differ by the algorithms of processing locally available information.[11] However, the limit topologies are equivalent in a sense that a continuous path in one picture stays continuous in any of the other two; the transliteration of a continuous path then amounts to a *second order phase transition*

[11]Recall that the two gentlemen of Verona could embark and sail to Milan with the morning tide; alternatively, they could take a train, fly in an airplane, or go by car. Their route organisation would have been different in these four cases, yet the starting- and end-points coincide; the four paths are integrated by rotation of screw or wheel.

when the object stays identically the same but the underlying crystal structure changes.[12]

In the sequel, we prefer to operate locally on the affine lattice A_3 yet we allow a formal union of the three irreducible alphabets in the fibre of the noncommutative tangent bundle over each point of space. We view the irreducibility as the mechanism which holds space from slicing to lower-dimensional components; because of that, we shall not consider the reducible cases $A_1 \oplus A_1 \oplus A_1$, $A_1 \oplus A_2$, $A_1 \oplus B_2$, and $A_1 \oplus G_2$. We also emphasize that we always preview a possibility of taking the continuous limit in mathematical reasonings but we let the space be quantised by the edges of the graph, i.e., by the 1-skeleton of the CW-complex.

Viewing the world as it is (e.g., compared with the multiplicative structures in [6]), we have to admit that a perfectly ordered life inside a Kac–Moody algebra is an inachievable ideal. In practice, the 1-skeleton of the CW-complex experiences an everlasting reconstruction; this is why up to this moment we have not described the attachment algorithm or transition mappings between overlapping quantum domains; they just attach as graphs and the verity is that the CW-complex is globally defined — it *is* space in which the Universe exists.

A possible mechanism of the perpetual modification in the graph's local topology (but not in the triviality of topology of the CW-complex) is that Natura abhorret a vacuo. In its zeal to shake off its quantum discontinuity, Nature does attempt to perform the infinite bisection and construct the complete real line $\mathbb{R}$ by using binary arithmetic. Let us recall that such recursive procedure replicates one unit-time light automaton to two automata plugged consecutively, one after another. But Nature unceasingly replicates each light automaton with its two copies that are *identical* to their sample. This leads to the observed proper elongation of space.[13]

Namely, within each fixed half-time[14] on-average one half of the actually available edges split in two new edges. (We remark that this division does not happen with the *contracted* edges, see next section; on the same grounds it is the edges but not vertices that split, for the latter could in fact be a superposition of many vertices according to the record of past modifications in local topology.) Each event of edge splitting creates a new vertex – the midpoint – and fills in the adjacency table for it, connecting it by one edge with all vertices in the cells delimited by the splitting edge in their faces.

[12]We expect that the transliterations – from one alphabet to another – of cyclic words encoding the contours whose meaning is a chargeless spin-$\frac{1}{2}\hbar$ particle explain the known neutrino oscillation.

[13]Notice that a release of energy in the course of edge decontractions (see sec. 4) is compensated with a simultaneous increase of the volume of continuous space so that the energy density remains constant; then, a part of this energy is being absorbed by the black holes or is radiated to the spatial infinity. A simultaneous release of ever-growing amount of energy per unit-time at the outer periphery of decontracting Universe (see next section) does not burn the objects inside it but it instead cools down to the present 2.725° K of the cosmic microwave background radiation.

[14]This time interval is counted by the local billiard clock – the edge itself – that sends light signals to and fro the edge; this parameter can vary as the Universe grows older.

Also, a tadpole is attached to the new vertex within the compactified dimension. We recall that the vertices label quantum domains in space so that the graph's adjacency table configures the domains' neighbours. We now note that the process of spontaneous edges' splittings roots in the conventional round-up $[n-\frac{1}{2}, n+\frac{1}{2}) \mapsto n$: the edge's midpoint is referred to *one* of the edge's endpoints — hence, the midpoint's fermionic tadpole is communicated to that endpoint. The splitting goes as follows: in terms of the order in $\mathbb{Q}$ along the splitting edge $(n-1, n)$, its midpoint $\{n - \frac{1}{2}\}$ detaches from $\{n\}$, proclaiming the existence of a separate quantum domain $[n-\frac{3}{4}, n-\frac{1}{4})$ which becomes adjacent with $[n-\frac{3}{2}, n-\frac{3}{4})$ and $[n - \frac{1}{4}, n + \frac{1}{2})$; the round-up demarkation reproduces twofold at $\{n - \frac{3}{4}\}$ and $\{n - \frac{1}{4}\}$. The marker $\{n - \frac{1}{2}\}$ of the new domain grabs –and endows with order the set of– fermionic tadpoles attached to all the points $\{n - \frac{3}{4} + \frac{m}{2^\ell}\}$ which (by the values of $m, \ell \in \mathbb{N} \cup \{0\}$) locally get into the bounds $[n - \frac{3}{4}, n - \frac{1}{4})$.

But because the light automata remain the same for the first and second fragments of the edge, each of them counts the propagation of light signal along each new edge as a unit-time event. Consequently, not only the Universe grows at its periphery, but a trip between distant objects all across the Universe takes more and more time.

Corollary 2 (The Hubble law)**.** *The Doppler-shift-measured velocity v at which distant material objects, locally staying at rest with respect to physical points, i.e., with respect to the incidence relations between points in quantum space, recede from each other is directly proportional to the proper distance D between these objects:*

$$v = H_0 \cdot D,$$

here H_0 is the Hubble constant (*now it approximately equals* $74.3 \pm 2.1 (\mathsf{km/s})/\mathsf{Mpc}$).

Notice that the picture is uniform with respect to all observers associated with such objects anywhere in space.[15]

Thus, Hubble's law testifies steady self-generation of space due to which Cosmos obeys the principle "twice farther, twice faster" at sufficiently large scale. We conclude that we do hear the process of space expansion in the form of the cosmic microwave background radiation; we thus predict that the 1.873 mm-signal can not be altogether shielded by any macroscopic medium.

4 The mass

The crucial idea in our description of the geometry of vacuum – which is not yet inhabited by any particles – is that contractions of edges in the graph are allowed (but highly not recommended unless possible consequences are fully understood).

[15] A relative motion of the Milky Way with respect to the underlying quantum space structure is observed by the detection of Doppler's shift in the relic radiation at certain direction and its antipode.

We emphasize that this does not require stretching, pulling, compressing, or any other forms of physical activity — an ordinary accountant with pencil, eraser, and access to the book $\mathcal{T}$ with topology of the Universe can accomplish more epic deeds in the course of one reaction than Heracles did in his entire life.

Definition 3. The *contraction* of an edge is a declaration that its endpoints merge and there remains nothing in between (i.e., a tadpole is not formed); the respective ordered sets of fermionic tadpoles $\mathbb{S}$ attached to the merging endpoints unite, preserving the tadpoles' directions and their ordering (so that a path in positive or negative direction along either bosonic or fermionic understanding for the old tadpoles becomes the respectively directed path on the new one). The *decontraction* of a previously contracted edge is its restoration in between its endpoints which become no longer coinciding, and the splitting of the indexed ordered tadpole sets between the two endpoints.

Remark 7. A contraction of edges in the 1-skeleton of the CW-complex can force the formation of tadpoles from remaining edges: for example, two contractions which distinguish between Left and Right could lead to the CP-symmetry violation in weak processes (see [7]).

Let us also notice that the on-going splitting of edges, which is responsible for the Hubble law, is a random decontraction process spread over quantum space.

Let us inspect how the concept of Riemann curvature tensor works in the non-commutative setup when one transports an edge in the CW-complex's 1-skeleton along a contour starting at a vertex formed by contracting an edge (see Fig. 1 on the cubic lattice). Namely, let the edge ab be contracted; consider the lattice ele-

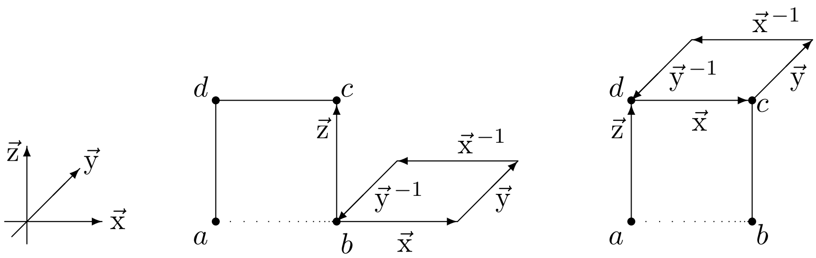

Figure 1. The curvature mechanism.

ment $\vec{z}$. First, transport its starting point a along the contour $\overrightarrow{\vec{x}\vec{y}\vec{x}^{-1}\vec{y}^{-1}}$ and *then* step along $\vec{z}$; the walk's endpoint is c. However, by walking the route $\overrightarrow{\vec{z}\vec{x}\vec{y}\vec{x}^{-1}\vec{y}^{-1}}$ and thus transporting the endpoint along the chosen contour, one reaches the vertex d instead of c. By definition, a path connecting c to d is the value $R_a(\vec{x},\vec{y})\,\vec{z}$ at $\vec{z}$ of the quantum curvature operator for the path determined by the ordered pair $(\vec{x},\vec{y})$ at the point a.

Note 2. In this text we postulate, not deriving the mass-energy balance equation $E = mc^2$ from the underlying geometric mechanism [14], that a *presence* of a contracted edge is seen as mass, whereas a time-generating event of reconfiguration absorbs energy – creating mass – by contracting an edge and releases that energy at the endpoints in the course of its decontraction; this is the mass-energy correlation mechanism. (Note only that one may not measure the stored energy as "force$\times$distance" and thus introduce a stress of the lattice because length is undefined on it).

Corollary 3. *In absence of visible matter and energy, the vacuum can be curved and cause the force of gravity.*

We expect that the *dark matter* is the configuration field of space contractions along its subsets; in fact, massive but invisible dark matter is not matter at all.

Let us define the *entropy* S of a given contraction of edges as minus the number of vertices — which themselves are not attached to the contracting edges but which neighbour via an edge with vertices that merge. Basically, this entropy of a contractions' configuration in space is the topologically-dual to *area* (here, the number of faces for the nearest surface that encapsulates the contracted edges).

For example, a contraction of one edge of the square lattice in $\mathbb{E}^2$ produces a snowflake so that $S = -6$, making $2 \times (-6)$ for two distant contractions. However, let us notice that the entropy of two consecutive edges for that tiling equals -8 and is equal to -7 for a corner. In view of this, a reconfiguration of contractions in a finite volume of the graph could be a thermodynamical process and gravity force could have the entropic origin, c.f. [13].

We conjecture that initially, the entire space of the Universe was contracted to one point so that every tiling of it by the 1-skeleton of the CW-complex was either that vertex or the vertex with tadpole attached to it. Simultaneously, we expect that space is topologically trivial so that all its possible tilings may not contain extra edges which would create shortcuts between distant cells.

Remark 8. Numeric experiments [1] reveal the following property which the CW-complex in $\mathbb{E}^3$ gains in the course of bisection – i.e., making the tiling finer – under an extra *ad hoc* assumption that the cells can be glued, orientation-preserving, along prescribed pairs of faces not necessarily to their true neighbours but possibly to sufficiently remote cells. This creates a possibility to obtain a topologically nontrivial CW-complex which nominally fills in $\mathbb{E}^3$ but such that the local density of genus can be positive. The probability of reconfigurations was postulated to drop exponentially with increase of the $\mathbb{N}$-valued distance between cells.

Then the fundamental solution of the usual heat equation – or the square mean deviation of random walks – was calculated by using a natural convention that the dissipating medium (e.g., smoke) or the random walks' endpoints spread freely through the faces of reconfigured tiling. The effective dimension was then determined from the rapidity of dissipation, and the modelling was repeated a suitable number of times.

Numeric experiment has shown that, as the sides of elementary domains become smaller but the effective distance, at which the probability of faces' reattachment drops exp(1) times, is kept constant, the effective dimension of $(3+1)$-dimensional combination of space and time drops from four to *exactly two* in the continuous limit.

In the paper [7] we try to view Physics as text whose *meaning* is Nature. We focus on its alphabet, glossary, grammar rules, and a possible location where the text is retrieved from, edited, and then stored back to. We know that the text of Nature is incredibly interesting; in our efforts to read it, we have not yet advanced much in learning its grammar, and still more feebly we perceive the overall plot.

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A Comparison of Definitions for the Schouten Bracket on Jet Spaces

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The Schouten bracket (or antibracket) plays a central role in the Poisson formalism and the Batalin–Vilkovisky quantization of gauge systems. There are several (in)equivalent ways to realize this concept on jet spaces. In this paper, we compare the definitions, examining in what ways they agree or disagree and how they relate to the case of usual manifolds.

1 Introduction

The Schouten bracket is a natural generalization of the commutator of vector fields to the fields of multivectors. It was introduced by J.A. Schouten [23, 24], who with A. Nijenhuis [18] established its main properties. Later it was observed by A. Lichnerowicz [16,17], that the bracket provides a way to check if a bivector π on a manifold determines a Poisson bracket via the formula $[\![\pi, \pi]\!] = 0$, which was the first intrinsically coordinate-free method to see this and established the use of the bracket in the Poisson formalism. Moreover, this makes the bracket instrumental in the definition of the Poisson(–Lichnerowich) cohomology on a Poisson manifold.

Historically, the bracket on jet space [7] seems to have been researched in two distinct areas of mathematics and physics, which have been separate for a long time. The first branch is the quantization of gauge systems; here the bracket is known as the *antibracket*. It occurs for example in the seminal papers on the BRST and BV formalism, [3, 4, 25] and [1, 2] respectively, where it is used to create a nilpotent operator $[\![\Omega, \cdot\,]\!]$ providing a resolution of the space of observables. Other occurrences of the bracket in this context are [5, 9, 28] and [27], the last of which contains some geometrical interpretation of the bracket.

In the Poisson formalism on jet spaces it was understood in [8] that the bracket plays a similar role for recognizing Poisson brackets as on usual manifolds. Concepts such as Hamiltonian operators and the relation of the bracket with the Yang–Baxter equation are developed in [8] and [21, 22]; for a review, see [6]. A version of the bracket that can be restricted to equations was developed in [10]. Later, a different, recursive way of defining the bracket, that we will discuss in this paper, was shown in [14].

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Generalizations to the $\mathbb{Z}_2$-setup and the purely non-commutative setting of the entire theory have been discussed in [13, 20] and more recently [11]; for a review, see [12].

The realization that the brackets in these areas of mathematics and physics coincide is not an obvious one. Accordingly, a number of seemingly distinct ways of defining the bracket has been developed, of which the equivalence is not always immediate and sometimes a subtle issue. This paper aims to examine four of those definitions, of which three will turn out to be equivalent when care is taken.

The paper is structured as follows. We first recall in Section 2 the notions of horizontal jet spaces and variational multivectors; at this point it will become clear why the definition of the bracket for usual manifold fails in the case of jet spaces. In Section 3 we first define the Schouten bracket as an odd Poisson bracket; then, after giving some examples of the bracket acting on two multivectors, we show that this definition is equivalent to the recursive one introduced in [14]. Using the recursive definition we shall prove the Jacobi identity for the bracket, which yields a third definition for the bracket, in terms of graded vector fields and their commutators.

We use the following notation, in most cases matching that from [12]. Let $\pi\colon E \to M$ be a vector bundle of rank m over a smooth real oriented manifold of dimension n; in this paper we assume all maps to be smooth. x^i are the coordinates, with indices $i, j, k, \ldots$, along the base manifold; q^α are the fiber coordinates with indices $\alpha, \beta, \gamma, \ldots$. We take the infinite jet space $\pi_\infty\colon J^\infty(\pi) \to M$ associated with this bundle; a point from the jet space is then $\theta = (x^i, q^\alpha, q^\alpha_{x^i}, q^\alpha_{x^i x^j}, \ldots, q^\alpha_\sigma, \ldots) \in J^\infty(\pi)$, where σ is a multi-index. If $s \in \Gamma(\pi)$ is a section of π we denote with $j^\infty(s)$ its infinite jet, which is a section $j^\infty(s) \in \Gamma(\pi_\infty)$. Its value at $x \in M$ is

$$j^\infty_x(s) = \left(x^i, s^\alpha(x), \frac{\partial s^\alpha}{\partial x^i}(x), \ldots, \frac{\partial^{|\sigma|} s^\alpha}{\partial x^\sigma}(x), \ldots \right) \in J^\infty(\pi).$$

The evolutionary vector fields, which we will call *vectors*, are then $\partial^{(q)}_\varphi = \sum_{|\sigma|\geq 0} \sum_{\alpha=1}^m D_\sigma(\varphi^\alpha) \frac{\partial}{\partial q^\alpha_\sigma}$, where $D_\sigma = D_{x^{i_1}} \circ \cdots \circ D_{x^{i_k}}$ are (compositions of) the total derivatives. Here $\varphi \in \varkappa(\pi) := \Gamma(\pi^*_\infty(\pi)) = \Gamma(\pi) \otimes_{C^\infty(M)} \mathcal{F}(\pi)$, where $\mathcal{F}(\pi)$ is the ring of smooth functions on the jet space. The *covectors* are then $p \in \widehat{\varkappa}(\pi) := \widehat{\varkappa(\pi)} := \mathrm{Hom}_{\mathcal{F}(\pi)}(\varkappa(\pi), \overline{\Lambda}(\pi))$, i.e., linear functions that map vectors to the space of top-level horizontal forms on jet space. We will denote the coupling between covectors and vectors with $\langle p, \varphi \rangle \in \overline{\Lambda}(\pi)$. The horizontal cohomology, i.e., $\overline{\Lambda}(\pi)$ modulo the image of the horizontal exterior differential $\overline{\mathrm{d}}$, is denoted by $\overline{H}^n(\pi)$; the equivalence class of $\omega \in \overline{\Lambda}(\pi)$ is denoted by $\int \omega \in \overline{H}^n(\pi)$. We will assume that the sections are such that integration by parts is allowed and does not result in any boundary terms; for example, the base manifold is compact, or the sections all have compact support, or decay sufficiently fast towards infinity. Lastly, the variational derivative with respect to q^α is $\frac{\delta}{\delta q^\alpha} = \sum_{|\sigma|\geq 0} (-)^\sigma D_\sigma \frac{\partial}{\partial q^\alpha}$, while the Euler operator is $\delta = \int \mathrm{d}_{\mathcal{C}} \cdot$, where $\mathrm{d}_{\mathcal{C}}$ is the Cartan differential.

For a more detailed exposition of these matters, see for example [12, 14, 15, 19].

2 Preliminaries

Let ξ be a vector bundle over $J^\infty(\pi)$, and suppose s_1 and s_2 are two sections of this bundle. We say that they are *horizontally equivalent* [10] at a point $\theta \in J^\infty(\pi)$ if $D_\sigma(s_1^\alpha) = D_\sigma(s_2^\alpha)$ at θ for all multi-indices σ and fiber-indices α. Denote the equivalence class by $[s]_\theta$. The set

$$\overline{J^\infty_\pi}(\xi) := \{[s]_\theta \mid s \in \Gamma(\xi),\, \theta \in J^\infty(\pi)\}$$

is called the *horizontal jet bundle* of ξ. It is clearly a bundle over $J^\infty(\pi)$, whose elements above θ are determined by all the derivatives $s^\alpha_\sigma := D_\sigma(s^\alpha)$ for all multi-indices σ and fiber-indices α.

Now suppose ζ is a bundle over M. Let us consider the induced vector bundle $\pi^*_\infty(\zeta)$ over $J^\infty(\pi)$, and the horizontal jet bundle $\overline{J^\infty_\pi}(\pi^*_\infty(\zeta))$.

Proposition 1. *As bundles over $J^\infty(\pi)$, the horizontal jet bundle $\overline{J^\infty_\pi}(\pi^*_\infty(\zeta))$ and $\pi^*_\infty(J^\infty(\zeta))$ are equivalent.*

Proof. The pullback bundle $\pi^*_\infty(J^\infty(\zeta))$ is as a set equal to

$$\pi^*_\infty(J^\infty(\zeta)) = \{(j^\infty_x(q), j^\infty_x(u)) \in J^\infty(\pi) \times J^\infty(\zeta) \mid x \in M\}.$$

On the other hand, consider an element $[s]_\theta \in \overline{J^\infty_\pi}(\pi^*_\infty(\zeta))$. Thus, s is a section $s \in \Gamma(\pi^*_\infty(\zeta))$. By Borel's theorem, an arbitrary element over $x \in M$ from $J^\infty(\pi)$ can be written as $j^\infty_x(q)$ for some $q \in \Gamma(\pi)$. Now define a section $u \in \Gamma(\zeta)$ by $u := j^\infty(q)^* s = s \circ j^\infty(q)$, i.e., $u(x) = s(j^\infty_x(q))$. Then by the definition of the total derivative, we have

$$\frac{\partial u^a}{\partial x^i}(x) = \frac{\partial}{\partial x^i}\big(s^a \circ j^\infty(q)\big)(x) = \big(D_{x^i} s^a\big)\big(j^\infty_x(q)\big),$$

that is, the partial derivatives of u and the total derivatives of s coincide. This shows that if we define a map by

$$[s]_{j^\infty_x(q)} \mapsto (j^\infty_x(q), j^\infty_x(u)) \in \pi^*_\infty(J^\infty(\zeta)),$$

where u is the section associated to s and q as outlined above, then this map is well-defined and smooth. Moreover, since the partial derivatives of a section u at x and the total derivatives of a section s at $j^\infty_x(q)$ completely define the equivalence classes $j^\infty_x(u)$ and $[s]_{j^\infty_x(q)}$ respectively, this map is also a bijection. Lastly, it is clear that as a bundle morphism over $J^\infty(\pi)$, it preserves fibers. □

When ζ is a bundle over M instead of over $J^\infty(\pi)$, and there is no confusion possible, we will abbreviate $\overline{J^\infty_\pi}(\pi^*_\infty(\zeta))$ with $\overline{J^\infty_\pi}(\zeta)$.

This identification endows the horizontal jet space $\overline{J^\infty_\pi}(\zeta)$ with the Cartan connection – namely the pullback connection on $\pi^*_\infty(J^\infty(\zeta))$. Therefore there exist total derivatives D_i on the horizontal jet space $\overline{J^\infty_\pi}(\zeta)$; in coordinates these are just, denoting the fiber coordinate of ζ with u, the operators

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha,\sigma} q^\alpha_{\sigma+1_i} \frac{\partial}{\partial q^\alpha_\sigma} + \sum_{\beta,\tau} u^\beta_{\tau+1_i} \frac{\partial}{\partial u^\beta_\tau}.$$

Thus, instead of the horizontal derivatives $D_\sigma(u^\alpha)$ of sections there are now the fiber coordinates u^α_σ, which have no derivatives along the fiber coordinates: $\frac{\partial}{\partial q^\alpha_\sigma} u^\beta_\tau = 0$.

Now consider the bundle $\widehat{\pi} : E^* \otimes \Lambda^n(M) \to M$. Then $\pi^*_\infty(\widehat{\pi}) = \pi^*_\infty(E^*) \otimes \overline{\Lambda}(\pi)$, so that $\widehat{\varkappa}(\pi) = \Gamma(\pi^*_\infty(\widehat{\pi}))$. Thus, the formalism described above is applicable to covectors, so we either take p to be an element from $\widehat{\varkappa}(\pi)$, an actual covector, or $p \in \overline{J^\infty_\pi}(\widehat{\pi})$.

At this point we take the fibers of the bundle $\widehat{\pi}$ and of $\pi^*_\infty(\widehat{\pi})$, and reverse their parity, $\Pi\colon p \mapsto b$, while we keep the entire underlying jet space intact [26]. The result is the horizontal jet space $\overline{J^\infty_\pi}(\Pi\widehat{\pi})$ with odd fibers over x. An element θ from this space has coordinates

$$\theta = (x^i, q^\alpha, q^\alpha_{x^i}, \ldots, q^\alpha_\sigma, \ldots; b_\alpha, b_{\alpha,x^i}, \ldots, b_{\alpha,\sigma}, \ldots).$$

The coupling $\langle p, \varphi\rangle = \sum_\alpha p_\alpha \varphi^\alpha \,\mathrm{dVol}(M)$ extends tautologically to the odd b's, as do the total derivatives: $D_\sigma b_\alpha = b_{\alpha,\sigma}$.

Definition 1. Let $k \in \mathbb{N} \cup \{0\}$. A *variational k-vector*, or a *variational multivector*, is an element of $\overline{H}^n(\pi^*_\infty(\Pi\widehat{\pi}))$, having a density that is k-linear in the odd b's or their derivatives (i.e., it is a homogeneous polynomials of degree k in $b_{\alpha,\sigma}$). If ξ is a k-vector we will call $k =: \deg(\xi)$ its *degree.* Note that by partial integration, any such k-vector ξ can be written as

$$\xi(b) = \int \langle b, A(b, \ldots, b)\rangle$$

for some total totally skew-symmetric total differential operator A that takes $k-1$ arguments, takes values in $\varkappa(\pi)$, and is skew-adjoint in each of its arguments (e.g., in the case of a 2-vector, $\int\langle b^1, A(b^2)\rangle = \int\langle b^2, A(b^1)\rangle$).

Note that this does not imply that *every* density is, or has to be, a homogeneous polynomial of degree k; for example,

$$\int bb_x \,\mathrm{dVol}(M) = \int (bb_x + D_x(bb_x b_{xx}))\,\mathrm{dVol}(M).$$

To *evaluate* such a k-vector on k covectors $p^1, \ldots, p^k$, we proceed as follows: we put each covector in each possible slot, keeping track of the minus sign associated to the permutation, and normalize by the volume of the symmetric group:

$$\xi(p^1, \ldots, p^k) = \frac{1}{k!} \sum_{s \in S_k} (-)^s \,\xi(p^{s(1)}, \ldots, p^{s(k)}), \tag{1}$$

i.e., in the coordinate expression of (the representative of) ξ we replace the i-th b that we come across with $p^{s(i)}$ (moving from left to right), and sum over all permutations $s \in S_k$. Thus, under this evaluation k-vectors are k-linear total differential skew-symmetric functions on k covectors, landing in the horizontal cohomology of the jet space.

Remark 1. Contrary to the case of usual manifolds M, where the space of k-vectors is isomorphic to $\bigwedge^k TM$, the space of *variational* k-vectors does *not* split in such a fashion. As a result, the two formulas

$$[\![X, Y \wedge Z]\!] = [\![X, Y]\!] \wedge Z + (-)^{(\deg(X)-1)\deg(Y)} Y \wedge [\![X, Z]\!]$$

for multivectors X, Y and Z, and

$$[\![X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_\ell]\!] \tag{2}$$
$$= \sum_{\substack{i \le i \le k \\ 1 \le j \le \ell}} (-)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge Y_\ell$$

for vector fields X_i and Y_j, no longer hold. Both of these formulas provide a way of defining the bracket on usual, smooth manifolds (together with $[\![X, f]\!] = X(f)$ for vector fields X and functions $f \in C^\infty(M)$, and $[\![X, Y]\!] = [X, Y]$ for vector fields X and Y).

To sketch an argument why the space of variational k-vectors does not split in this way, take for example a 0-vector $\omega = \int f \,\mathrm{dVol}(M)$ and a 1-vector, which we can write as $\eta = \int \langle b, \varphi \rangle$ for some $\varphi \in \varkappa(\pi)$. How would we define the wedge product $\omega \wedge \eta$? Both of the factors contain a volume form and if we just put them together using the wedge product we get 0, so this approach does not work.

Suppose then we set in this case $\omega \wedge \eta = \int f \langle b, \varphi \rangle$. Now the problem is that f is not uniquely determined by ω and φ is not uniquely determined by η; both are fixed only up to $\bar{\mathrm{d}}$-exact terms. For example,

$$\omega = \int f \,\mathrm{dVol}(M) = \int (f + D_i(g)) \,\mathrm{dVol}(M),$$

but

$$\int f \langle b, \varphi \rangle \neq \int f \langle b, \varphi \rangle + \int D_i(g) \langle b, \varphi \rangle,$$

because the second term is in general not identically zero.

Similarly, we have $\eta = \int \langle b, \varphi \rangle = \int \big(\langle b, \varphi \rangle + \bar{\mathrm{d}}(\alpha(b)) \big)$, for any linear map α mapping b into $(n-1)$-forms. In the same way as above, this trivial term stops being trivial whenever we multiply it on the left with the density of a 0-vector, say. The difficulty persists for multivectors of any degree k and so there is no reasonable wedge product or splitting.

3 Definitions of the bracket

3.1 Odd Poisson bracket

Definition 2. Let ξ and η be k and ℓ-vectors respectively. The *variational Schouten bracket* $[\![\xi,\eta]\!]$ of ξ and η is the $(k+\ell-1)$-vector defined by[1]

$$[\![\xi,\eta]\!] = \int \sum_\alpha \left[\frac{\overrightarrow{\delta\xi}}{\delta q^\alpha} \frac{\overleftarrow{\delta\eta}}{\delta b_\alpha} - \frac{\overrightarrow{\delta\xi}}{\delta b_\alpha} \frac{\overleftarrow{\delta\eta}}{\delta q^\alpha} \right] \tag{3}$$

in which one easily recognizes a Poisson bracket. Since there are now the anticommuting coordinates b_α, we indicate with the arrows above the variational derivatives whether we mean a left or a right derivative (i.e., if we push the variation δb_α or δq^α through to the left or to the right). The fact that this is a $(k+\ell-1)$-vector comes from the variational derivatives $\frac{\delta}{\delta b}$ occuring in the expression: if η takes k arguments then $\frac{\delta\eta}{\delta b}$ takes $k-1$ arguments.

We will use the following two lemmas to calculate Examples 1 through 4.

Lemma 1. *Suppose $\xi = \langle b, A(b,\ldots,b)\rangle$ is a k-vector. Then*

$$\frac{\overleftarrow{\delta\xi}}{\delta b_\alpha} = kA(b,\ldots,b)^\alpha. \tag{4}$$

Proof. We calculate

$$\begin{aligned} \delta b_\alpha \frac{\overleftarrow{\delta\xi}}{\delta b_\alpha} &= \overleftarrow{\delta_b}\xi = \overleftarrow{\delta_b}\langle b, A(b,\ldots,b)\rangle \\ &= \langle \delta b, A(b,\ldots,b)\rangle + \sum_{n=1}^{k-1} \langle b, A(b,\ldots,\delta b,\ldots,b)\rangle. \end{aligned}$$

Now δb anticommutes with the b left to it, and A is antisymmetric in all of its arguments, so we can switch δb with the b on its left, giving two cancelling minus signs. Doing this multiple times, we obtain

$$= \langle \delta b, A(b,\ldots,b)\rangle + \sum_{j=1}^{k-1} \langle b, A(\delta b, b,\ldots,b)\rangle.$$

[1] To be precise, if $\xi = \int f(b,\ldots,b)\,\mathrm{dVol}(M)$ and $\eta = \int g(b,\ldots,b)\,\mathrm{dVol}(M)$, where f and g are both homogeneous polynomials in $b_{\alpha,\sigma}$ of degree k and ℓ respectively, then the bracket is given by

$$[\![\xi,\eta]\!] = \int \sum_\alpha \left(\frac{\overrightarrow{\delta f}}{\delta q^\alpha} \frac{\overleftarrow{\delta g}}{\delta b_\alpha} - \frac{\overrightarrow{\delta f}}{\delta b_\alpha} \frac{\overleftarrow{\delta g}}{\delta q^\alpha} \right) \mathrm{dVol}(M),$$

which does not depend on the representatives f and g because $\delta \circ \bar{\mathrm{d}} = 0$. This notation, although correct, does not seem to be used in the literature.

Next we first switch δb with the b to its left, and then use the fact that A is skew-symmetric in its first argument, again giving two cancelling minus signs:

$$
\begin{aligned}
&= \langle \delta b, A(b,\dots,b)\rangle + (k-1)\langle \delta b, A(b,\dots,b)\rangle \\
&= k\langle \delta b, A(b,\dots,b)\rangle \\
&= k\,\delta b_\alpha A(b,\dots,b)^\alpha .
\end{aligned}
$$

The result follows by comparing the coefficients of δb^α. □

Lemma 2. *Let ξ and η be k and ℓ-vectors respectively, so that $\xi = \int \langle b, A(b,\dots,b)\rangle$ and $\eta = \int \langle b, B(b,\dots,b)\rangle$ respectively. Then*

$$
[\![\xi,\eta]\!] = \int \left[(-)^{k(\ell-1)} \ell \partial^{(q)}_{B(b,\dots,b)} \xi - (-)^{k-1} k \partial^{(q)}_{A(b,\dots,b)} \eta \right]. \tag{5}
$$

Proof. In the second term of the definition of the Schouten bracket, we first reverse the arrow on the b-derivative, giving a sign $(-)^{k-1}$. In the first term, we swap the two factors $(\overrightarrow{\delta\xi}/\delta q^\alpha)(\overleftarrow{\delta\eta}/\delta b_\alpha)$. For this we have to move the $\ell-1$ b's of $\overleftarrow{\delta\eta}/\delta b_\alpha$ through the k b's of $\overrightarrow{\delta\xi}/\delta q^\alpha$, giving a sign $(-)^{k(\ell-1)}$. Thus

$$
\begin{aligned}
[\![\xi,\eta]\!] &= \int \left[\frac{\overrightarrow{\delta\xi}}{\delta q^\alpha} \frac{\overleftarrow{\delta\eta}}{\delta b_\alpha} - \frac{\overrightarrow{\delta\xi}}{\delta b_\alpha} \frac{\overleftarrow{\delta\eta}}{\delta q^\alpha} \right] \\
&= \int \left[(-)^{k(\ell-1)} \frac{\overleftarrow{\delta\eta}}{\delta b_\alpha} \frac{\overrightarrow{\delta\xi}}{\delta q^\alpha} - (-)^{k-1} \frac{\overleftarrow{\delta\xi}}{\delta b_\alpha} \frac{\overleftarrow{\delta\eta}}{\delta q^\alpha} \right] \\
&= \int \left[(-)^{k(\ell-1)} \ell D_\sigma B(b)^\alpha \frac{\partial\xi}{\partial q^\alpha_\sigma} - (-)^{k-1} k D_\sigma A(b)^\alpha \frac{\partial\eta}{\partial q^\alpha_\sigma} \right] \\
&= \int \left[(-)^{k(\ell-1)} \ell \partial^{(q)}_{B(b)} \xi - (-)^{k-1} k \partial^{(q)}_{A(b)} \eta \right].
\end{aligned}
$$
□

Example 1. Take a one-vector $\varphi \in \varkappa(\pi)$, i.e., $\xi = \langle b, \varphi\rangle$, and let $\mathcal{H} \in \overline{H}^n(\pi)$ be a 0-vector. Then

$$
[\![\mathcal{H},\varphi]\!] = \int \partial^{(q)}_\varphi \mathcal{H},
$$

i.e., the Schouten bracket calculates the velocity of $\mathcal{H}$ along $\partial^{(q)}_\varphi$.

Example 2. Suppose ξ and η are two one-vectors, i.e., $\xi = \int \langle b, \varphi_1\rangle$ and $\eta = \int \langle b, \varphi_2\rangle$ for some $\varphi_1, \varphi_2 \in \varkappa(\pi)$. Then

$$
\begin{aligned}
[\![\xi,\eta]\!] &= \int \left(\partial^{(q)}_{\varphi_2}(\xi) - \partial^{(q)}_{\varphi_1}(\eta) \right) = \int \left(\partial^{(q)}_{\varphi_2} \langle b, \varphi_1\rangle - \partial^{(q)}_{\varphi_1} \langle b, \varphi_2\rangle \right) \\
&= \int \left(\langle b, \partial^{(q)}_{\varphi_2} \varphi_1\rangle - \langle b, \partial^{(q)}_{\varphi_1} \varphi_2\rangle \right)
\end{aligned}
$$

which holds because b does not depend on the jet coordinates q^α, whence

$$= \int \langle b, [\varphi_2, \varphi_1]\rangle = -\int \langle b, [\varphi_1, \varphi_2]\rangle.$$

Thus, in this case the variational Schouten bracket just calculates the ordinary commutator of evolutionary vector fields, up to a minus sign (cf. equation (2)).

Example 3. Suppose the base and fiber are both $\mathbb{R}$, and let $\xi = \int bb_x \,\mathrm{d}x$ be a (nontrivial) two-vector and $\eta = \int bx^3 q_{xx}\,\mathrm{d}x$ be a one-vector. Then

$$[\![\xi, \eta]\!] = 0 + \int 2\partial^{(q)}_{b_x}(bx^3 q_{xx})\,\mathrm{d}x = 2\int D_x^2(b_x)bx^3\,\mathrm{d}x = 2\int x^3 b_{xxx} b\,\mathrm{d}x.$$

We shall return to this example on p. 136 (see Example 5).

Example 4. In this final example, let $\xi = \int bb_x\,\mathrm{d}x$ again and $\eta = \int q_x bb_x\,\mathrm{d}x$; then

$$[\![\xi, \eta]\!] = 0 + 2\int \partial^{(q)}_{b_x}(q_x bb_x)\,\mathrm{d}x = 2\int D_x(b_x)\cdot bb_x\,\mathrm{d}x = 2\int bb_x b_{xx}\,\mathrm{d}x.$$

Notice the factor 2 standing in front of the answers in the last two examples; it will become important in the next section.

3.2 A recursive definition

The second way of defining the bracket, due to I. Krasil'shchik and A. Verbovetsky [14], is done in terms of the *insertion operator*: let ξ be a k-vector, and let $p \in \widehat{\varkappa}(\pi)$ or $p \in \overline{J^\infty_\pi}(\widehat{\pi})$ (i.e., p can be either an actual covector or an element from the corresponding horizontal jet space). Denote by $\xi(p)$ or $\iota_p(\xi)$ the $(k-1)$-vector that one obtains by putting p in the rightmost slot of ξ:

$$\xi(p)(b) = \iota_p(\xi)(b) = \xi(\underbrace{b, \ldots, b}_{k-1}, p) = \frac{1}{k}\sum_{j=1}^{k}(-)^{k-j}\xi(b, \ldots, b, p, b, \ldots, b), \qquad (6)$$

where p is in the j-th slot. Note that if we were to insert $k-1$ additional elements of $\widehat{\varkappa}(\pi)$ in this expression in this way, we recover formula (1).

Lemma 3. *If $\xi = \int \langle b, A(b, \ldots, b)\rangle$ is a k-vector, then*

$$\frac{\overleftarrow{\delta}\xi(p)}{\delta b_\alpha} = \frac{k-1}{k}\frac{\overleftarrow{\delta}\xi}{\delta b_\alpha}(p) \quad \textit{and} \quad \frac{\overrightarrow{\delta}\xi(p)}{\delta b_\alpha} = -\frac{k-1}{k}\frac{\overrightarrow{\delta}\xi}{\delta b_\alpha}(p). \qquad (7)$$

Proof. $\xi(p)$ is a $(k-1)$-vector, so $\overleftarrow{\delta}\xi(p)/\delta b_\alpha = (k-1)A(b, \ldots, b, p)^\alpha$ by Lemma 1. However, ξ is a k-vector, so $(\overleftarrow{\delta}\xi/\delta b_\alpha)(p) = (kA(b, \ldots, b)^\alpha)(p) = kA(b, \ldots, b, p)^\alpha$, from which the first equality of the lemma follows. The second equality is established by reversing the arrow of the derivative, using the first equality, and restoring the arrow to its original direction again; this results in the extra minus sign in this equality. ∎

On the other hand, if $p \in \overline{J^\infty_\pi}(\widehat{\pi})$ then $\delta\xi(p)/\delta q^\alpha = (\delta\xi/\delta q^\alpha)(p)$. Indeed, we have $\partial p_{\beta,\tau}/\partial q^\alpha_\sigma = 0$, and if f is one of the densities of a k-vector, then the total derivative D_{x^i} and the insertion operator ι_p commute. For example,

$$\iota_p(D_{x^i} b_{\alpha,\sigma}) = \iota_p(b_{\alpha,\sigma+1_i}) = p_{\alpha,\sigma+1_i}$$

and

$$D_{x^i}(\iota_p(b_{\alpha,\sigma})) = D_{x^i}(p_{\alpha,\sigma}) = p_{\alpha,\sigma+1_i}.$$

Thus, from the formula $\frac{\delta}{\delta q^\alpha} = \sum_{|\sigma|>0}(-)^\sigma D_\sigma \frac{\partial}{\partial q^\alpha_\sigma}$ for the variational derivative it follows that $\delta\xi(p)/\delta q^\alpha = (\delta\xi/\delta q^\alpha)(p)$.

Theorem 1. *Let ξ and η be k and ℓ-vectors, respectively, and $p \in \overline{J^\infty_\pi}(\widehat{\pi})$. Then*

$$[\![\xi, \eta]\!](p) = \frac{\ell}{k+\ell-1}[\![\xi, \eta(p)]\!] + (-)^{\ell-1}\frac{k}{k+\ell-1}[\![\xi(p), \eta]\!]. \tag{8}$$

Proof. We relate the two sides of the equation by letting p range over the slots as in equation (6). In this calculation we will for brevity omit the fiber indices α.

Consider the first term of the left hand side, $\left(\frac{\overrightarrow{\delta\xi}}{\delta q}\frac{\overleftarrow{\delta\eta}}{\delta b}\right)(p)$. If we were to take the sum as in equation (6), we would obtain an expression containing $k+\ell-1$ slots; in some cases p is in one of the $\ell-1$ slots of $\overleftarrow{\delta\eta}/\delta b$ and in the other cases it is in one of the k slots of $\overrightarrow{\delta\xi}/\delta q$. All of these terms carry the normalizing factor $1/(k+\ell-1)$. Now we notice the following:

- Each term in which p is in a slot coming from $\overleftarrow{\delta\eta}/\delta b$ has a matching term in the expansion of $\frac{\overrightarrow{\delta\xi}}{\delta q}\,\iota_p\left(\frac{\overleftarrow{\delta\eta}}{\delta b}\right)$ according to (6), except that there each term would carry a factor $1/(\ell-1)$, because now p only has access to the $\ell-1$ slots of $\overleftarrow{\delta\eta}/\delta b$.
- Similarly, each term of the left hand side of (8) in which p is in one of the slots of $\overrightarrow{\delta\xi}/\delta q$ has a matching term in the expansion of $\iota_p\left(\frac{\overrightarrow{\delta\xi}}{\delta q}\right)\frac{\overleftarrow{\delta\eta}}{\delta b}$, but there they carry a factor $1/k$.
- Moreover, in that case they also carry the sign $(-)^{\ell-1}$, which comes from the fact that here p had to pass over the $\ell-1$ slots of $\overleftarrow{\delta\eta}/\delta b$.

Gathering these remarks, we find

$$\begin{aligned}\left(\frac{\overrightarrow{\delta\xi}}{\delta q}\frac{\overleftarrow{\delta\eta}}{\delta b}\right)(p) &= \frac{\ell-1}{k+\ell-1}\frac{\overrightarrow{\delta\xi}}{\delta q}\,\iota_p\left(\frac{\overleftarrow{\delta\eta}}{\delta b}\right) + (-)^{\ell-1}\frac{k}{k+\ell-1}\iota_p\left(\frac{\overrightarrow{\delta\xi}}{\delta q}\right)\frac{\overleftarrow{\delta\eta}}{\delta b}\\ &= \frac{\ell}{k+\ell-1}\frac{\overrightarrow{\delta\xi}}{\delta q}\frac{\overleftarrow{\delta\eta(p)}}{\delta b} + (-)^{\ell-1}\frac{k}{k+\ell-1}\frac{\overrightarrow{\delta\xi(p)}}{\delta q}\frac{\overleftarrow{\delta\eta}}{\delta b},\end{aligned}$$

where we have used the first equation of Lemma 3 in the first term.

Now we consider the second term of the left hand side of (8), and use a similar reasoning:

$$\left(\frac{\overrightarrow{\delta\xi}}{\delta b}\frac{\overleftarrow{\delta\eta}}{\delta q}\right)(p) = \frac{\ell}{k+\ell-1}\frac{\overrightarrow{\delta\xi}}{\delta b}\,\iota_p\left(\frac{\overleftarrow{\delta\eta}}{\delta q}\right) + (-)^{\ell}\frac{k-1}{k+\ell-1}\iota_p\left(\frac{\overrightarrow{\delta\xi}}{\delta b}\right)\frac{\overleftarrow{\delta\eta}}{\delta q}$$
$$= \frac{\ell}{k+\ell-1}\frac{\overrightarrow{\delta\xi}}{\delta b}\frac{\overleftarrow{\delta\eta(p)}}{\delta q} + (-)^{\ell+1}\frac{k}{k+\ell-1}\frac{\overrightarrow{\delta\xi(p)}}{\delta b}\frac{\overleftarrow{\delta\eta}}{\delta q},$$

where now the second equation of Lemma 3 has been used. Subtracting the results of these two calculations, we obtain exactly the right hand side of equation (8). □

Thus, by recursively reducing the degrees of the arguments of the bracket, formula (8) expresses the value of the bracket of a k-vector and an ℓ-vector on $k+\ell-1$ covectors. We can interpret it as a second definition of the Schouten bracket, provided that we also set

$$[\![\mathcal{H},\varphi]\!] = \int \partial^{(q)}_{\varphi}\mathcal{H} = \int \langle \delta\mathcal{H},\varphi\rangle$$

for 1-vectors φ and 0-vectors $\mathcal{H} \in \overline{H}^n(\pi)$. Theorem 1 then says that this definition is equivalent to Definition 2. However, let us notice the following:

Remark 2. There are numerical factors in front of the two terms of the right hand side; these are absent in [14]. For example, the bracket of a 2-vector ξ and a 0-vector $\mathcal{H}$ is $[\![\mathcal{H},\xi]\!](p) = 2\xi(\delta\mathcal{H},p)$ according to both Definition 2 and Theorem 1; note the factor 2.

Remark 3. Secondly, it is important that the p that is inserted in (8) is *not* an actual covector, but that $p \in \overline{J^\infty_\pi}(\widehat{\pi})$. Otherwise, unwanted terms like $\partial^{(q)}_{\varphi}(p)$ occur in the final steps, and equivalence with Definition 2 is spoiled. Thus one takes two multivectors, inserts elements from the horizontal jet space according to the formula, and only plugs in the (derivatives of) actual covectors at the end of the day. This remark is again absent from [14].

Example 5. Let us re-calculate Example 3 using this formula. So, let $\xi = \int bb_x\,\mathrm{d}x$ and $\eta = \int bx^3q_{xx}\,\mathrm{d}x$, and let $p^1, p^2 \in \overline{J^\infty_\pi}(\widehat{\pi})$. Then

$$[\![\xi,\eta]\!](p^1,p^2) = [\![\xi,\eta]\!](p^2)(p^1) = \frac{1}{2}[\![\xi,\eta(p^2)]\!](p^1) + \frac{2}{2}[\![\xi(p^2),\eta]\!](p^1)$$
$$= -2\cdot\frac{1}{2}\cdot[\![\xi(p^1),\eta(p^2)]\!] + 1\cdot[\![\xi(p^2),\eta(p^1)]\!] + 1\cdot[\![\xi(p^1,p^2),\eta]\!]$$
$$= \int\left[(-)^2\,\partial^{(q)}_{p^1_x}(p^2x^3q_{xx}) - \partial^{(q)}_{p^2_x}(p^1x^3q_{xx}) + \tfrac{1}{2}\partial^{(q)}_{x^3q_{xx}}(p^1p^2_x - p^2p^1_x)\right]\mathrm{d}x$$
$$= \int\left[x^3p^1_{xxx}p^2 - (p^1 \rightleftarrows p^2)\right]\mathrm{d}x.$$

(Keeping track of the coefficients and signs is a good exercise.) This is precisely what one gets after evaluating the result of Example 3 on p^1 and p^2.

Theorem 1 allows us to reduce the Jacobi identity for the Schouten bracket to that of the commutator of one-vectors.

Proposition 2. *Let r, s and t be the degrees of the variational multivectors ξ, η and ζ, respectively. The Schouten bracket satisfies the graded Jacobi identity*:

$$\begin{aligned}&(-)^{(r-1)(t-1)}[\![\xi,[\![\eta,\zeta]\!]]\!]\\ &\quad+(-)^{(r-1)(s-1)}[\![\eta,[\![\zeta,\xi]\!]]\!]\\ &\quad+(-)^{(s-1)(t-1)}[\![\zeta,[\![\xi,\eta]\!]]\!]=0.\end{aligned} \tag{9}$$

Proof. We proceed by induction using Theorem 1. When the degrees of the three vectors do not exceed 1, the statement follows from the reductions of the Schouten bracket to known structures, as in Examples 1 and 2. Now let the degrees be arbitrary natural numbers. Denote by I_1, I_2 and I_3 the respective terms of the left hand side of (9). Then for any $p\in\overline{J^\infty_\pi}(\widehat{\pi})$ we have that

$$\begin{aligned}I_1(p)&=(-)^{(r-1)(t-1)}\,[\![\xi,[\![\eta,\zeta]\!]]\!](p)\\ &=\frac{(-)^{(r-1)(t-1)}}{r+s+t-2}\left((s+t-1)\,[\![\xi,[\![\eta,\zeta]\!](p)]\!]+r(-)^{s+t-2}\,[\![\xi(p),[\![\eta,\zeta]\!]]\!]\right)\\ &=\frac{(-)^{(r-1)(t-1)}}{r+s+t-2}\Big(t\,[\![\xi,[\![\eta,\zeta(p)]\!]]\!]+s(-)^{t-1}\,[\![\xi,[\![\eta(p),\zeta]\!]]\!]\\ &\qquad\qquad+r(-)^{s+t-2}\,[\![\xi(p),[\![\eta,\zeta]\!]]\!]\Big)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}I_2(p)&=\frac{(-)^{(r-1)(s-1)}}{r+s+t-2}\Big(r\,[\![\eta,[\![\zeta,\xi(p)]\!]]\!]+t(-)^{r-1}\,[\![\eta,[\![\zeta(p),\xi]\!]]\!]\\ &\qquad\qquad+s(-)^{r+t-2}\,[\![\eta(p),[\![\zeta,\xi]\!]]\!]\Big),\\ I_3(p)&=\frac{(-)^{(s-1)(t-1)}}{r+s+t-2}\Big(s\,[\![\zeta,[\![\xi,\eta(p)]\!]]\!]+r(-)^{s-1}\,[\![\zeta,[\![\xi(p),\eta]\!]]\!]\\ &\qquad\qquad+t(-)^{r+s-2}\,[\![\zeta(p),[\![\xi,\eta]\!]]\!]\Big)\,.\end{aligned}$$

For notational convenience, let us set $I_1(p)+I_2(p)+I_3(p)=:I/(r+s+t-2)$. Next we rearrange the terms in I:

$$\begin{aligned}I&=(-)^{r-1}t\Big\{(-)^{(r-1)(t-2)}\,[\![\xi,[\![\eta,\zeta(p)]\!]]\!]+(-)^{(r-1)(s-1)}\,[\![\eta,[\![\zeta(p),\xi]\!]]\!]\\ &\quad+(-)^{(s-1)(t-2)}\,[\![\zeta(p),[\![\xi,\eta]\!]]\!]\Big\}+(-)^{t-1}s\Big\{(-)^{(r-1)(t-1)}\,[\![\xi,[\![\eta(p),\zeta]\!]]\!]\\ &\quad+(-)^{(r-1)(s-2)}\,[\![\eta(p),[\![\zeta,\xi]\!]]\!]+(-)^{(s-2)(t-1)}\,[\![\zeta,[\![\xi,\eta(p)]\!]]\!]\Big\}\\ &\quad+(-)^{s-1}r\Big\{(-)^{(r-2)(t-1)}\,[\![\xi(p),[\![\eta,\zeta]\!]]\!]+(-)^{(r-2)(s-1)}\,[\![\eta,[\![\zeta,\xi(p)]\!]]\!]\\ &\quad+(-)^{(s-1)(t-1)}\,[\![\zeta,[\![\xi(p),\eta]\!]]\!]\Big\},\end{aligned}$$

i.e., we obtain the Jacobi identity for ξ, η and $\zeta(p)$; for ξ, $\eta(p)$ and ζ; and for $\xi(p)$, η and ζ (each times some unimportant factors). Thus we see that if we know that the identity holds for $(r-1,s,t)$, $(r,s-1,t)$ and $(r,s,t-1)$, then it holds for (r,s,t). □

3.3 Graded vector fields

Proposition 3. *If ξ and η are k and ℓ-vectors respectively, then their Schouten bracket is equal to*

$$[\![\xi,\eta]\!] = \int Q^{\xi}(\eta) = \int (\xi)\overleftarrow{Q}^{\eta}, \tag{10}$$

where for any k-vector ξ, the graded evolutionary vector field Q^{ξ} is defined by

$$Q^{\xi} := \partial^{(q)}_{-\overrightarrow{\delta\xi}/\delta b} + \partial^{(b)}_{\overrightarrow{\delta\xi}/\delta q}. \tag{11}$$

Proof. This is readily seen from the equalities

$$\int \partial^{(b)}_{\overrightarrow{\delta\xi}/\delta q}(\eta) = \int \sum_{\alpha,\sigma} D_\sigma\left(\frac{\overrightarrow{\delta\xi}}{\delta q^\alpha}\right)\frac{\partial\eta}{\partial b_{\alpha,\sigma}} = \int \sum_{\alpha,\sigma} \frac{\overrightarrow{\delta\xi}}{\delta q^\alpha}(-)^\sigma D_\sigma \frac{\partial\eta}{\partial b_{\alpha,\sigma}}$$
$$= \int \sum_{\alpha} \frac{\overrightarrow{\delta\xi}}{\delta q^\alpha}\frac{\overleftarrow{\delta\eta}}{\delta b_\alpha}$$

which is the first term of the Schouten bracket $[\![\xi,\eta]\!]$. The second term of (11) is done similarly. □

As a consequence of Proposition 3, the Schouten bracket is a derivation: if η is a product of k factors, then $[\![\xi,\eta]\!] = \int Q^{\xi}(\eta)$ has k terms, where in the i-th term, Q^{ξ} acts on the i-th factor while leaving the others alone. However, while the bracket is a derivation in both of its arguments separately, it is *not* a bi-derivation (i.e., a derivation in both arguments simultaneously), as in equation (2). To see why this is so, take a multivector η and let us suppose for simplicity that it has a density that consists of a single term containing ℓ coordinates, which can be either q's or b's: $\eta = \prod_{i=1}^{\ell} a_i$, for a set of letters a_i. Then the i-th term of $[\![\xi,\eta]\!] = \int Q^{\xi}(\eta)$ is a sign which is not important for the present purpose, times $a_1 \cdots Q^{\xi}(a_i)\cdots a_\ell$.

Now suppose that $\xi = \prod_{j=1}^{k} c_j$ for some set of letters c_j, and note that $Q^{\xi}(a_i) = (\xi)\overleftarrow{Q}^{a_i}$ + trivial terms. Let us call the trivial term ω for the moment. Then we see that

$$a_1 \cdots Q^{\xi}(a_i)\cdots a_\ell = a_1 \cdots (\xi)\overleftarrow{Q}^{a_i}\cdots a_\ell + a_1\cdots\omega\cdots a_\ell.$$

Here the first term expands to what it should be in order for the bracket to be a bi-derivation, namely a sum consisting of terms of the form

$$a_1\cdots c_1\cdots(c_j)\overleftarrow{Q}^{a_i}\cdots c_k\cdots a_\ell$$

times possible minus signs. The second term, however, is generally no longer trivial, so that it does not vanish. Therefore the bracket is not in general a bi-derivation.

Theorem 2. *The Schouten bracket is related to the graded commutator of graded vector fields as follows:*

$$\int Q^{[\![\xi,\eta]\!]} f = \int [Q^\xi, Q^\eta] f \tag{12}$$

for any smooth function f on the horizontal jet space $\overline{J^\infty_\pi}(\widehat{\pi})$.

Proof. From the definition of the graded commutator and equation (11) we infer that

$$\begin{aligned}\int [Q^\xi, Q^\eta] f &= \int Q^\xi(Q^\eta(f)) - (-)^{(k-1)(\ell-1)} \int Q^\eta(Q^\xi(f)) \\ &= [\![\xi, [\![\eta, f]\!]]\!] - (-)^{(k-1)(\ell-1)} [\![\eta, [\![\xi, f]\!]]\!] = [\![[\![\xi, \eta]\!], f]\!] \\ &= \int Q^{[\![\xi,\eta]\!]} f.\end{aligned}$$

where we used the Jacobi identity in the third line. □

This provides a third way of defining the Schouten bracket, equivalent to the previous two. Since the only fact that is used in this proof is that the Schouten bracket satisfies the graded Jacobi identity (Proposition 2), Theorem 2 is actually equivalent to the Jacobi identity for the Schouten bracket. It is also possible to prove Theorem 2 directly (see [12, p. 84], by inspecting both sides of equation (12); in that case the Jacobi identity may be proved as a consequence of Theorem 2.

As a bonus, we see that if P is a Poisson bi-vector, i.e., $[\![P, P]\!] = 0$, then Q^P is a differential, $(Q^P)^2 = 0$. This gives rise to the Poisson(–Lichnerowicz) cohomology groups H^k_P.

4 Conclusion

The research into the generalization of the Schouten bracket to jet spaces has historically been split in two directions. In the Poisson formalism, it is related to notions such as Poisson cohomology, integrability and the Yang–Baxter equation; while in the quantization of gauge system it is used in the BV-formalism to create a differential $D = [\![\Omega, \cdot\,]\!]$, also leading to cohomology groups. Although the definition of the bracket on usual manifolds by the formula that expresses it as a bi-derivation no longer works, there are several other ways of defining the bracket, which are equivalent if care is taken.

We finally recall that these definitions of the Schouten bracket also exist and remain coinciding in the $\mathbb{Z}_2$-graded setup $J^\infty\big((\pi_0|\pi_1)\big) \to M^{n_0|n_1}$, and in the setup of purely non-commutative manifolds and non-commutative bundles (see [13, 20] and lastly [12], which contains details and discussion, and generalizes the topic of this paper to the non-commutative world).

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On Differential Operators of Infinite Order in Sequence Spaces

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We consider differential operators of infinite order with constant coefficients in the sequence space s. Necessary and sufficient conditions for these operators to be equivalent to the usual derivative are given.

1 Introduction

Given two differential operators T and S in a space H, an operator X is an “operateur de transmutation” from T to S if X is an isomorphism from H onto H such that $SX = XT$. Obviously this notion depends on T, S and the space H. It was introduced in 1938 by Delsarte [2], T and S being differential operators of second order and H a space of functions of one variable defined for $x \geqslant 0$. From 1950 onwards Lions studied several generalizations and applications [13–15].

If T and S are differential operators of order $m > 2$ with infinitely differentiable coefficients and H is a space of infinitely differentiable functions on $\mathbb{R}$, then there do not exist, in general, “operateurs de transmutation”. In 1957, Fage presented a theorem of transmutation for certain classes of functions of real variables [4, 5].

The situation is, on the contrary, very simple taking T and S differential operators without singularities in the complex domain $\mathbb{C}$ and H the space of entire functions of one real variable. Delsarte and Lions proved in 1957 that in this case, provided the operators are of the same order, there is always an “operateur de transmutation” [3].

In the beginning of the 1960s the term “equivalence of operators” appeared. Two operators T and S are said to be equivalent if there is an “operateur of transmutation” between them. The subject was intensively studied, mainly by USSR mathematicians. Some of the relevant works are [1, 6–12, 22, 23].

A few years ago we started to consider the problem of the equivalence of differential operators taking the space H to be a sequence space and substituting the usual derivative by the general Gelfand–Leontev derivative. In this general context we have not found closed results; the dependence on the matrix defining the sequence space and the steps involved in the Gelfand–Leontev derivative seem to point to many different possibilities. Some particular cases are given in [18, 19]. For related problems, see [16, 17, 20, 21].

Nowadays we are interested in differential operators of infinite order. Nagnibida and Oliinyk [22] studied the equivalence of differential operators of infinite order in the spaces of analytic functions, on a disc and on the whole complex plane, giving a very neat result. We deal with the same problem (stated more precisely below) in the more general setting of sequence spaces.

2 Terminology

Denote by $\lambda^1(A)$ the sequence space given by the matrix $A = (a_n^k)$, $n, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $a_n^k > 0$ and $a_n^k \leqslant a_n^{k+1}$ for all k and n, that is

$$\lambda^1(A) = \left\{ f(x) = \sum_{n=0}^{\infty} \xi_n x^n \;\middle|\; \xi_n \in \mathbb{C},\ \|f\|_k = \sum_{n=0}^{\infty} |\xi_n| a_n^k < \infty\ \ \forall k \in \mathbb{N}_0 \right\}.$$

A sequence space $\lambda^1(A)$ is called an infinite power series space $\Lambda_\infty(\alpha)$ if $(a_n^k) = (e^{k\alpha_n})$, where $\{\alpha_n\}$ is an increasing sequence of positive numbers going to infinity. Among such spaces, the space $\mathcal{H}(\mathbb{C})$ of entire functions and the space s of rapidly decreasing sequences are well known. If $A = (e^{-\alpha_n/k})$, the sequence space is called a finite series power space $\Lambda_1(\alpha)$. The best-known example among finite series power spaces is the space of analytical functions on a disc.

An operator T is a differential operator of infinite order, with constant coefficients, if $T = \sum_{n=0}^{\infty} \varphi_n D^n$ with $\varphi_n \in \mathbb{C}$. Let us call it $\varphi(D)$.

An operator T from $\lambda^1(A)$ to $\lambda^1(B)$ is continuous if and only if

$$\forall\ k \in \mathbb{N}_0\ \ \exists\ C(k) > 0,\ \exists\ N(k) \in \mathbb{N}_0\colon \quad \|Te_n\|_k \leqslant C(k)\|e_n\|_{N(k)}\ \ \forall\ n,$$

where $\{e_n\}$ is the canonical basis of the sequence space.

3 Statement of the problem

Consider a sequence space $\lambda^1(A)$ with its natural topology and a differential operator $\varphi(D)$ of infinite order with constant coefficients in the sequence space, and study necessary and sufficient conditions for the equivalence of the operators $\varphi(D)$ and D^n, where n is a fixed natural number.

As we pointed out above this problem was solved by Nagnibida and Oliinyk for spaces of analytic functions [22]. They proved the following:

Theorem 1. *The operator $\varphi(D) = \sum_{n=0}^{\infty} \varphi_n D^n$ is equivalent to the operator D^n in the space of analytic functions on a disc of radius R if and only if $\varphi(D) = \sum_{k=0}^{n} \varphi_k D^k$ and $|\varphi_n| = 1$.*

Theorem 2. *The operator $\varphi(D) = \sum_{n=0}^{\infty} \varphi_n D^n$ is equivalent to the operator D^n in the space on entire functions if and only if $\varphi(D) = \sum_{k=0}^{n} \varphi_k D^k$ and $\varphi_n \neq 0$.*

We consider an infinite power series space, namely, the sequence space

$$s = \left\{ f(x) = \sum_{n=0}^{\infty} \xi_n x^n \;\middle|\; \xi_n \in \mathbb{C},\ \|f\|_k = |\xi_0| + \sum_{n=1}^{\infty} |\xi_n| n^k < \infty \ \forall\, k \in \mathbb{N}_0 \right\}$$

and the derivative D and study the equivalence of differential operators of infinite order and the operator D^n, where n is a fixed natural number.

4 Main result

Assume that $n = 1$. We give the complete solution to the problem in this case and are working to generalize the result for any n but, at the moment, there are some technical difficulties that we hope to overcome in the near future.

Theorem 3. *The operator $\varphi(D) = \sum_{n=0}^{\infty} \varphi_n D^n$ is equivalent to the operator D in the space s if and only if $\varphi(D) = \varphi_0 I + \varphi_1 D$, $\varphi_1 \neq 0$.*

The following statements are used in the proof of the theorem:

1) If $\varphi(D) = \sum_{n=0}^{\infty} \varphi_n D^n$ is equivalent to D then $\varphi(D) = p_1(D) e^{aD}$, where $p_1(D) = \beta_0 I + \beta_1 D$ with $\beta_1 \neq 0$. The proof is similar to [22].

2) If $\varphi(D) = p_1(D) e^{aD}$ is equivalent to D, then there exists an isomorphism T satisfying the relation $T p_1(D) e^{aD} = DT$, which implies the differential equation

$$p_1(\lambda) e^{a\lambda} T e^{\lambda z} = \frac{d}{dz}(T e^{\lambda z})$$

for any $\lambda \in \mathbb{C}$. Hence $T(e^{\lambda z}) = B(\lambda) e^{h(\lambda) z}$, where $h(\lambda) = p_1(\lambda) e^{a\lambda}$.

3) The k-norm of the elements $e^{\lambda z}$ in s is given by

$$\|e^{\lambda z}\|_k = \big(1 + P_k(|\lambda|)\big) e^{|\lambda|},$$

where P_k is a polynomial of degree k and its coefficients are natural numbers. Therefore

$$\|T e^{\lambda z}\|_k = |B(\lambda)| \big(1 + P_k(|h(\lambda)|)\big) e^{|h(\lambda)|}.$$

Proof. Assume that $p_1(D) e^{aD}$ is equivalent to D. The point now is to get that $a = 0$.

The isomorphism T verifies

$$\forall\, r \in \mathbb{N}_0 \ \exists\, C(r) > 0,\ \exists\, N(r) \in \mathbb{N}_0 \colon \quad \|T e^{\lambda z}\|_r \leqslant C(r) \|e^{\lambda z}\|_{N(r)} \ \forall\, \lambda \in \mathbb{C},$$

which, in the case under consideration, means

$$|B(\lambda)| \big(1 + P_r(|h(\lambda)|)\big) e^{|h(\lambda)|} \leqslant C(r) \big(1 + P_{N(r)}(|\lambda|)\big) e^{|\lambda|} \ \forall\, \lambda \in \mathbb{C}.$$

Taking $\lambda = \bar{a}k$ with $k \in \mathbb{N}_0$, we have

$$|B(\bar{a}k)|\big(1 + P_r(|h(\bar{a}k)|)\big)e^{|h(\bar{a}k)|} \leqslant C(r)\big(1 + P_{N(r)}(|\bar{a}k|)\big)e^{|\bar{a}k|} \;\; \forall\, k \in \mathbb{N}_0.$$

Write

$$\begin{aligned} &|B(\bar{a}k)|\big(1 + P_{\frac{r}{2}}(|h(\bar{a}k)|)\big)e^{|h(\bar{a}k)|} \\ &\leqslant C(r)\big(1 + P_{N(r)}(|\bar{a}k|)\big)e^{|\bar{a}k|}\frac{\big(1 + P_{\frac{r}{2}}(|h(\bar{a}k)|)\big)}{\big(1 + P_r(|h(\bar{a}k)|)\big)} \;\; \forall\, k \in \mathbb{N}_0, \forall\, r \in \mathbb{N}_0. \end{aligned}$$

The dominant part, when k goes to $+\infty$, of the right-hand part of the previous formula is $e^{|\overline{a}|(1-\frac{r}{2}|a|)k}$, that, obviously, goes to zero for $r > 2/|a|$. Then it follows from

$$|B(\bar{a}k)|\big(1 + P_{\frac{r}{2}}(|h(\bar{a}k)|)\big)e^{|h(\bar{a}k)|} = \|Te^{\bar{a}kz}\|_{\frac{r}{2}}$$

that $Te^{\bar{a}kz} \to 0$ in s and hence $e^{\bar{a}kz} \to 0$ (as T is an isomorphism) which is not true. The contradiction implies the result in view of the fact that two differential operators of order one with constant coefficients are equivalent on s [18]. □

Remark 1. When $n > 1$, it is still true that a differential operator of infinite order equivalent to D^n is of the form $p_n(D)e^{aD}$, where $p_n(D)$ a polynomial of degree n. We conjecture that $a = 0$.

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S-Expansions of Three-Dimensional Lie Algebras

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S-expansions of three-dimensional real Lie algebras are considered. It is shown that the expansion operation allows one to obtain a non-unimodular Lie algebra from a unimodular one. Nevertheless S-expansions define no ordering on the variety of Lie algebras of a fixed dimension.

1 Introduction

In 1961 Zaitsev [8] supposed that all solvable Lie algebras of a fixed dimension could be obtained via contractions from the semisimple algebras of the same dimension. He called such solvable Lie algebras ‘limiting solvable’ and proposed the contraction procedure for the ‘limiting classification’ of solvable Lie algebras. Later the same conjecture was also formulated by other scientists, e.g., Celeghini and Tarlini [2]. Complexity of the actual state of affairs was illustrated in [6], where contractions of real and complex low-dimensional Lie algebras were studied. The incorrectness of the above conjecture was illustrated by the fact that all semisimple (and reductive) Lie algebras are unimodular and any continuous contraction of a unimodular algebra necessarily results in a unimodular algebra. Moreover, semisimple Lie algebras exist not in all dimensions.

The aim of this paper is to revisit Zaitsev’s conjecture in terms of S-expansions. We will study S-expansions of the real three-dimensional Lie algebras. It was shown in [4] that generalized Inönü–Wigner contractions give a particular case of S-expansions. This is why we consider only pairs of such algebras that are not connected by a contraction.

The paper is arranged as follows. Section 2 contains preliminary information on S-expansions and relevant objects. In Section 3 we construct several key examples of three-dimensional S-expansions and discuss the possibility of application of S-expansions to the classification of solvable Lie algebras.

2 Basic properties of S-expansions

Roughly speaking, the *S-expansion* is the “product” of the semigroup and the Lie algebra with a Lie algebra structure defined in a special way. The notion was introduced in [4]. It generalizes the notions of expansion, deformation (under a proper choice of the corresponding semigroup), extension and generalized Inönü–Wigner contraction.

Let $S = \{\lambda_\alpha\}$ be an Abelian semigroup of order N and let $\mathfrak{g}$ be an n-dimensional Lie algebra with the structure constants C_{ij}^k in a fixed basis $\{e_i\}$ of the underlying vector space V. Here and in what follows the indices α, β and γ run from 1 to N and the indices i, j and k run from 1 to $n = \dim V$. The nN-dimensional *S-expanded Lie algebra* $\mathfrak{G} := S \times \mathfrak{g}$, the underlying vector space of which is spanned by the basis elements $e_{(i,\alpha)} = \lambda_\alpha e_i$, is defined by the structure constants

$$C_{(i,\alpha)(j,\beta)}^{(k,\gamma)} = \begin{cases} C_{ij}^k & \text{if} \quad \lambda_\alpha\lambda_\beta = \lambda_\gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Remark 1. It directly follows from (1) that the product of elements $e_{(i,\alpha)}$ and $e_{(j,\beta)}$ in the algebra $\mathfrak{G}$ is zero if $[e_i, e_j] = 0$. Therefore, an appropriate representative from the isomorphism class of the initial Lie algebra $\mathfrak{g}$ should be chosen if S-expansion is used for a certain purpose. In other words, sometimes the structure constants tensor has to be transformed by an element of the group $\mathrm{GL}(V)$ before the expansion procedure. Of course commutation relations of the expanded algebra $\mathfrak{G}$ can also be reduced to an appropriate form using a change of basis.

Let a Lie algebra $\mathfrak{g}$ admit the decomposition $\mathfrak{g} = \check{V} \oplus \hat{V}$ with $[\check{V}, \hat{V}] \subset \hat{V}$, namely

$$[\check{v}_{\check{i}}, \hat{v}_{\hat{j}}] = \hat{C}_{\check{i}\hat{j}}^{\hat{k}} \hat{v}_{\hat{k}}, \quad [\check{v}_{\check{i}}, \check{v}_{\check{j}}] = \check{C}_{\check{i}\check{j}}^{\check{k}} \check{v}_{\check{k}} + \hat{C}_{\check{i}\check{j}}^{\hat{k}} \hat{v}_{\hat{k}}, \quad [\hat{v}_{\hat{i}}, \hat{v}_{\hat{j}}] = \check{C}_{\hat{i}\hat{j}}^{\check{k}} \check{v}_{\check{k}} + \hat{C}_{\hat{i}\hat{j}}^{\hat{k}} \hat{v}_{\hat{k}}, \tag{2}$$

where $\{\check{v}_{\check{i}}\}$ is a basis of $\check{V}$ and $\{\hat{v}_{\hat{i}}\}$ is a basis of $\hat{V}$. Then the elements $\{\check{v}_{\check{i}}\}$ form a well-defined *reduced* Lie algebra $|\check{V}|$ with the Lie product defined by the structure constants $\check{C}_{\check{i}\check{j}}^{\check{k}}$.

2.1 Algebras of lower dimensions

In general case the dimension of the S-expanded Lie algebra $\mathfrak{G}$ is greater than n and it has a more complicated structure than $\mathfrak{g}$. Therefore for some purposes it is reasonable to consider certain subalgebras or reduced algebras of less dimensions instead of working with the whole expanded algebra.

The problem of the construction of algebras of less dimension that are related to an expanded Lie algebra is rather complicated. Two classes of such algebras were presented in [4] for S-expanded Lie algebras admitting an agreed decomposition of the associated semigroups and Lie algebras. A semigroup S and a Lie algebra $\mathfrak{g}$ are said to admit *a resonant subset decomposition* if they can be decomposed as $\mathfrak{g} = \bigoplus_{p\in I} V_p$ (where the direct sum is interpreted in the sense of vector spaces only) and $S = \bigcup_{p\in I} S_p$ with some index set I in such a way that $[V_p, V_q] \subset \bigoplus_{r\in i(p,q)} V_r$ and $S_p S_q \subset \bigcap_{r\in i(p,q)} S_r$, where for every $p, q \in I$ the index set $i(p,q)$ is a subset of I. In the resonance case, $\mathfrak{G}_R = \bigoplus_{p\in I}(S_p \times V_p)$ is a *resonance* *subalgebra* of the S-expanded Lie algebra $\mathfrak{G}$. For each resonant decomposition this gives a Lie algebra of dimension less than the dimension of $\mathfrak{G}$. Algebras of the other class are constructed by means of the reduction of Lie algebras. Suppose that for each $p \in I$ the set S_p is partitioned into the subsets $\check{S}_p$ and $\hat{S}_p$ such that $\check{S}_p\hat{S}_q \subset \bigcap_{r\in i(p,q)} \hat{S}_r$. This partition induces the decomposition $\mathfrak{G}_R = \hat{\mathfrak{G}} \oplus \check{\mathfrak{G}}$, where direct sums are interpreted in the sense of vector spaces, $\check{\mathfrak{G}} = \bigoplus_{p\in I}(\check{S}_p \times V_p)$ and $\hat{\mathfrak{G}} = \bigoplus_{p\in I}(\hat{S}_p \times V_p)$. As $[\check{\mathfrak{G}}, \hat{\mathfrak{G}}] \subset \hat{\mathfrak{G}}$, the projection of the Lie bracket of $\mathfrak{G}_R$ to $\check{\mathfrak{G}}$ gives a well-defined Lie bracket on $|\check{\mathfrak{G}}|$. In other words, the Lie algebra $\check{\mathfrak{G}}$ with this bracket is the reduced algebra for $\mathfrak{G}_R$.

Given an Abelian semigroup S with a zero $\lambda_N = 0_S \in S$, i.e., $\lambda_\alpha 0_S = 0_S$ for all $\alpha = 1, \dots, N$, any S-expanded Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$ can be decomposed as $\mathfrak{G} = \check{V} \oplus \hat{V}$ with $[\check{V}, \hat{V}] \subset \hat{V}$, where $\hat{V} = \langle 0_S e_1, 0_S e_2, \dots, 0_S e_n\rangle$. In this case the reduced Lie algebra $\mathfrak{G}_0 := |\check{V}|$ is called the 0_S*-reduced algebra* of $\mathfrak{G}$.

Remark 2. The condition (1) implies that the 0_S-reduced algebra $\mathfrak{G}_0$ has the following commutation relations in the basis $\{\lambda_\alpha e_i \mid \alpha = 1, \dots, N-1,\ i = 1, \dots, n\}$:

$$[\lambda_\alpha e_i, \lambda_\beta e_j] = \begin{cases} \sum\limits_{k=1}^{n} C_{ij}^k \lambda_\alpha\lambda_\beta e_k & \text{if} \quad \lambda_\alpha\lambda_\beta \neq 0_S, \\ 0 & \text{if} \quad \lambda_\alpha\lambda_\beta = 0_S. \end{cases} \tag{3}$$

2.2 *S*-expansions and contractions

As generalized Inönü–Wigner contractions are related to gradings of the contracted Lie algebras (i.e., to special decompositions of the underlying vector space), it is clear that all generalized IW-contractions can be obtained by means of resonant S-expansions. See [4] for details.

At the same time, this claim does not imply that S-expansions exhaust all possible contractions since there exist contractions that are not realized by generalized IW-contractions [1, 7].

Example 1. Consider the example of such a contraction constructed in [1] for the seven-dimensional Lie algebras $\mathfrak{g}_F$ and $\mathfrak{g}_E$. These algebras are defined by the commutation relations

$$\begin{aligned} \mathfrak{g}_F\colon & [e_1, e_i] = e_{i+1},\ 2 \leqslant i \leqslant 6, \\ & [e_2, e_3] = e_6, \quad [e_2, e_4] = e_7, \quad [e_2, e_5] = e_7, \quad [e_3, e_4] = -e_7, \\ \mathfrak{g}_E\colon & [e_1, e_i] = e_{i+1},\ 2 \leqslant i \leqslant 6, \quad [e_2, e_3] = e_6 + e_7, \quad [e_2, e_4] = e_7, \end{aligned}$$

and are characteristically nilpotent since their differentiation algebras

$$\begin{aligned} \mathrm{Der}(\mathfrak{g}_F) = \langle & 2E_{21} + E_{42} + E_{53} + 3E_{64} + 5E_{75} + 2E_{76},\ E_{31} + E_{52} - E_{75}, \\ & E_{32} + E_{43} + E_{54} + E_{65} + E_{76},\ E_{61},\ E_{41} - E_{51} + E_{74}, \\ & E_{51} + E_{62}\rangle, \end{aligned}$$

$$\begin{aligned}\mathrm{Der}(\mathfrak{g}_E) = \langle & E_{31} - E_{41} + E_{52},\ E_{41} + E_{62},\ -E_{31} + E_{41} + E_{63} + E_{74}, \\ & - E_{21} - E_{31} + E_{41} + 2E_{42} + 2E_{53} + E_{63} + E_{64},\ E_{71},\ E_{51}, \\ & E_{21} + E_{31} - E_{41} - E_{42} - E_{53} - E_{63} + E_{75},\ -E_{41} + E_{73}, \\ & E_{32} + E_{43} + E_{54} + E_{65} + E_{76},\ E_{72},\ E_{61}\rangle\end{aligned}$$

are nilpotent. Here E_{ij} denotes the 7×7 matrix with only nonzero (i, j)th entry that equals 1. (It is obvious that both the differentiation algebras consist of lower triangular matrices.) This means that the algebras $\mathfrak{g}_F$ and $\mathfrak{g}_E$ admit no group grading and hence none of them can be a result of a generalized IW-contraction. On the other hand, there exists the contraction from $\mathfrak{g}_F$ to $\mathfrak{g}_E$ provided by the following contraction matrix at $\varepsilon \to 0$:

$$U_\varepsilon = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\varepsilon^4 & 0 & \varepsilon^5 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\varepsilon^5 & 0 & \varepsilon^6 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\varepsilon^6 & 0 & \varepsilon^7 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\varepsilon^7 & 0 & \varepsilon^8 \end{pmatrix}.$$

Examples on non-universality of the generalized IW-contractions in dimension four can be found in [7]. This is why it is still unclear whether all contractions can be obtained via S-expansions.

On the other part, there exist S-expansions which are not equivalent to contractions. See, e.g., the example on a connection between the Lie algebras $\mathrm{sl}(2, \mathbb{R})$ and $A_{2.1} \oplus A_1$ that is presented in the next section.

Other objects closely related to S-expansions are given by the purely algebraic notion of *graded contractions* [3]. The graded contraction procedure is the following. Structure constants of a graded Lie algebra are multiplied by numbers which are chosen in such a way that the multiplied structure constants define a Lie algebra with the same grading. Graded contractions include discrete contractions as a subcase but do not cover all continuous ones.

Contractions of real and complex three-dimensional Lie algebras were completely studied in [6]. As all these contractions are realized by the generalized Inönü–Wigner contractions, they can be realized by S-expansions. Moreover, S-expansions lead to establishing new relations between such algebras.

3 Three-dimensional S-expansions

According to Mubarakzyanov's classification [5], nonisomorphic real three-dimensional Lie algebras are exhausted by the two simple algebras $\mathrm{sl}(2, \mathbb{R})$ and $\mathrm{so}(3)$; the two parameterized series of solvable algebras $A^a_{3.4}$, $0<|a|\leqslant 1$ and $A^b_{3.5}$, $b\geqslant 0$; the three single indecomposable solvable algebras $A_{3.1}$, $A_{3.2}$ and $A_{3.3}$; and the two decomposable solvable algebras $3A_1$ and $A_{2.1} \oplus A_1$.

In [6] it was proven that all unimodular three-dimensional algebras (namely, $\mathrm{sl}(2,\mathbb{R})$, $\mathrm{so}(3)$, $A_{3.4}^{-1}$, $A_{3.5}^{0}$, $A_{3.1}$ and $3A_1$) belong to the orbit closure of at least one of the simple algebras.

All contractions are shown on Fig. 1 by dashed lines (each arrow indicates the direction of the corresponding contraction). Repeated contractions (i.e., contractions of the kind: $\mathfrak{g}$ contracts to $\mathfrak{g}_0$ if $\mathfrak{g}$ contracts to $\mathfrak{g}_1$ and $\mathfrak{g}_1$ contracts to $\mathfrak{g}_0$) are implied but not indicated on this figure. All three-dimensional contractions were realized by generalized IW-contractions, therefore the corresponding S-expansions also exist.

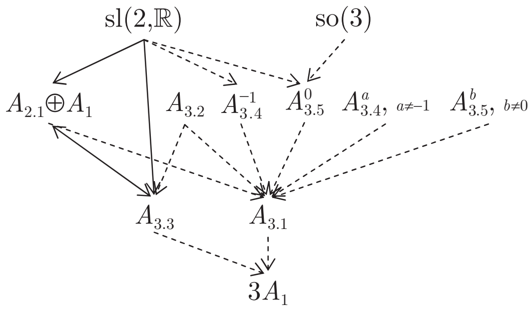

Figure 1. Contractions of real three-dimensional Lie algebras are marked by the dashed lines and S-expansions which are not equivalent to contractions are marked by solid lines.

There are a number of necessary conditions to be satisfied for a pair of Lie algebras connected by a contraction, see, e.g., [6]. The following examples show that S-expansions of Lie algebras do not obey the major part of these rules. These examples involve the algebras (for each algebra we present only nonzero commutation ralations)

$$\mathrm{sl}(2,\mathbb{R})\colon \quad [e_1,e_2]=e_1, \quad [e_2,e_3]=e_3, \quad [e_1,e_3]=2e_2; \tag{4}$$

$$A_{2.1}\oplus A_1\colon \quad [e_1,e_2]=e_1; \tag{5}$$

$$A_{3.3}\colon \quad [e_1,e_3]=e_1, \quad [e_2,e_3]=e_2. \tag{6}$$

3.1 Unimodularity of S-expansions

Given a unimodular Lie algebra $\mathfrak{g}\langle e_1,\dots,e_n\rangle$, we have

$$\mathrm{tr}(\mathrm{ad}_{e_i})=\sum_{j=1}^{n}C_{ij}^{j}=0 \qquad \forall\, i=1,\dots,n. \tag{7}$$

Using (7) for the basis elements $E_{(i-1)N+\alpha}:=\lambda_\alpha e_i$ of the S-expanded Lie algebra $\mathfrak{G}$, we get

$$\mathrm{tr}(\mathrm{ad}_{E_{(i-1)N+\alpha}})=\sum_{\beta=1}^{N}\sum_{j=1}^{n}K_{\alpha\beta}^{\beta}C_{ij}^{j}=\sum_{\beta=1}^{N}K_{\alpha\beta}^{\beta}\left(\sum_{j=1}^{n}C_{ij}^{j}\right)=0,$$

where

$$K^{\gamma}_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \lambda_\alpha \lambda_\beta = \lambda_\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Thereby the unimodularity property is necessarily preserved by any sole S-expansion. At the same time, a unimodular Lie algebra may contain a non-unimodular subalgebra or may be reduced to a non-unimodular Lie algebra.

3.2 Examples of non-unimodular S-expansions

Consider the expansion of the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ by means of the Abelian semigroup $S_2 := \{\lambda_1, \lambda_2\}$, where $\lambda_1\lambda_1 = \lambda_1\lambda_2 = \lambda_2\lambda_1 = \lambda_2\lambda_2 = \lambda_2$, i.e., λ_2 is the zero element 0_{S_2} of the semigroup. The elements $E_{2(i-1)+\alpha} := \lambda_\alpha e_i$, $\alpha = 1, 2$, $i = 1, 2, 3$, form a basis of six-dimensional S_2-expanded Lie algebra $\mathfrak{G}$. Then from (4) we obtain the nonzero commutation relations

$$\begin{aligned} &[E_1, E_3] = E_2, \quad [E_1, E_4] = E_2, \quad [E_1, E_5] = 2E_4, \quad [E_1, E_6] = 2E_4, \\ &[E_2, E_3] = E_2, \quad [E_2, E_4] = E_2, \quad [E_2, E_5] = 2E_4, \quad [E_2, E_6] = 2E_4, \\ &[E_3, E_5] = E_6, \quad [E_3, E_6] = E_6, \quad [E_4, E_5] = E_6, \quad [E_4, E_6] = E_6. \end{aligned}$$

Elements E_1, E_2 and E_3 span a three-dimensional subalgebra with the nonzero commutation relations $[E_1, E_3] = E_2$, $[E_2, E_3] = E_2$. The basis change $\tilde{E}_1 = E_2$, $\tilde{E}_2 = E_3$, $\tilde{E}_3 = E_1 - E_2$ leads to the unique nonzero relation $[\tilde{E}_1, \tilde{E}_2] = \tilde{E}_1$ and, therefore, we obtain the Lie algebra $A_{2.1} \oplus A_1$, cf. (5).

Another example concerns the Lie algebras $\mathrm{sl}(2,\mathbb{R})$ and $A_{3.3}$. Let a three-element Abelian semigroup with a zero $S_3 = \{0_{S_3} = \lambda_1, \lambda_2, \lambda_3\}$ satisfies the conditions $\lambda_2\lambda_3 = \lambda_3\lambda_2 = \lambda_2$ and $\lambda_2\lambda_2 = \lambda_3\lambda_3 = \lambda_1$. The above properties of S_3 are consistent with the semigroup structure and are sufficient for defining S_3-expansions. Thus, the S_3-expansion of $\mathrm{sl}(2,\mathbb{R})$ is the nine-dimensional Lie algebra $S_3 \times \mathrm{sl}(2,\mathbb{R})$ with the nonzero commutation relations

$$\begin{aligned} &[E_1, E_4] = E_1, \quad [E_1, E_5] = E_1, \quad [E_1, E_6] = E_1, \\ &[E_1, E_7] = 2E_4, \quad [E_1, E_8] = 2E_4, \quad [E_1, E_9] = 2E_4, \\ &[E_2, E_4] = E_1, \quad [E_2, E_5] = E_1, \quad [E_2, E_6] = E_2, \\ &[E_2, E_7] = 2E_4, \quad [E_2, E_8] = 2E_4, \quad [E_2, E_9] = 2E_5, \\ &[E_3, E_4] = E_1, \quad [E_3, E_5] = E_2, \quad [E_3, E_6] = E_1, \\ &[E_3, E_7] = 2E_4, \quad [E_3, E_8] = 2E_5, \quad [E_3, E_9] = 2E_4, \\ &[E_4, E_7] = E_7, \quad [E_4, E_8] = E_7, \quad [E_4, E_9] = E_7, \\ &[E_5, E_7] = E_7, \quad [E_5, E_8] = E_7, \quad [E_5, E_9] = E_8, \\ &[E_6, E_7] = E_7, \quad [E_6, E_8] = E_8, \quad [E_6, E_9] = E_7. \end{aligned}$$

From the algebra $S_3 \times \mathrm{sl}(2,\mathbb{R})$ we can extract the three-dimensional subalgebra $\langle E_1, E_2, E_6 \rangle$ isomorphic to $A_{3.3}$, cf. (6).

The two above constructions give us examples on connection between unimodular and non-unimodular three-dimensional Lie algebras by means of the expansion and subalgebra extraction.

Remark 3. Concerning the rest of non-unimodular Lie algebras, namely $A_{3.2}$, $A^a_{3.4}$ and $A^b_{3.5}$, it seems to be impossible to construct them from simple three-dimensional Lie algebras by means of S-expansion. This conjecture is motivated by the disagreement of the right-hand sides of the respective canonical commutators that can not be overcome by basis changes. Nevertheless, this conjecture needs a rigorous proof.

3.3 S-expansion from $A_{2.1} \oplus A_1$ to $A_{3.3}$ and vice versa

Consider S_3-expansion of the Lie algebra $A_{2.1} \oplus A_1$. To skip the tedious commutation relations of the nine-dimensional Lie algebra we consider only those which concern the basis elements $E_1 = \lambda_1 e_1$, $E_2 = \lambda_2 e_1$ and $E_6 = \lambda_3 e_2$. They are

$$[E_1, E_6] = E_1, \quad [E_2, E_6] = E_2, \quad [E_1, E_2] = 0.$$

This implies that the basis elements E_1, E_2 and E_6 form a subalgebra isomorphic to $A_{3.3}$, cf. (6).

The inverse connection can be obtained by means of S_2-expansion of the algebra $A_{3.3}$. The basis elements E_1, E_2 and E_6 span the subalgebra isomorphic to $A_{2.1} \oplus A_1$. Indeed, the nontrivial commutation relations between these elements are

$$[E_1, E_2] = 0, \quad [E_1, E_6] = E_1, \quad [E_2, E_6] = E_1.$$

After the basis change $\tilde{E}_1 = E_1 - E_2$, $\tilde{E}_2 = E_6$, $\tilde{E}_3 = E_2$, we obtain the unique nonzero commutation relation $[\tilde{E}_1, \tilde{E}_2] = \tilde{E}_1$.

In contrast to the contraction procedure, the last expansion creates a Lie algebra of more complicated structure. For example, the dimension of the center decreases from 1 to 0, the dimension of the Cartan subalgebra decreases from 2 to 1 and the dimension of the derivative enlarges from 1 to 2 after the S-expansion from the Lie algebra $A_{2.1} \oplus A_1$ to the algebra $A_{3.3}$.

At the same time, the S-expansion, even combined with algebra reduction and singling out a subalgebra, preserves certain properties of Lie algebras. Thus, we have $[S \times \mathfrak{g}, S \times \mathfrak{g}] = (SS) \times [\mathfrak{g}, \mathfrak{g}]$. Hence the algebra $S \times \mathfrak{g}$ is solvable (resp. nilpotent) if and only if the algebra $\mathfrak{g}$ is, and then the solvability (resp. nilpotency) degrees of these algebras coincide. The procedures of reducing the algebra $S \times \mathfrak{g}$ and singling out a subalgebra may not increase the solvability (resp. nilpotency) degree.

Note that all four of the discussed examples of expansions can be obtained from the non-Abelian two-dimensional Lie algebra, since in all cases the key role is played by the commutation relation $[e_1, e_2] = e_1$.

Remark 4. S-expansions do not set any ordering relationship on the variety of Lie algebras of a fixed dimension. This statement is illustrated by the example of Lie algebras $A_{2.1} \oplus A_1$ and $A_{3.3}$ that can be connected by S-expansion in both directions. Therefore, in spite of the fact that it is possible to construct non-unimodular Lie algebras from unimodular ones, there is still a question whether S-expansions fit to the classification of solvable Lie algebras of a fixed dimension by means of S-expansions of simple (semisimple) Lie algebras of the same dimension.

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Superintegrable and Supersymmetric Systems of Schrödinger Equations

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Superintegrable and supersymmetric models described by systems of two coupled Schrödinger equations are presented. All additive shape invariant potentials for such systems are classified and first order integrals of motion with matrix coefficients are specified. Physically, the discussed systems simulate a neutral fermions with non-trivial dipole moment, interacting with the external electromagnetic field.

1 Introduction

Exactly solvable problems in quantum mechanics are very attractive. They can be described fully and in a straightforward way free of various complications caused by the perturbation method. The very existence of exact solutions of these problems is usually connected with their non-trivial symmetries which are mostly of particular interest by themselves. In addition, exact solutions form convenient complete sets of vectors which can be used to expand solutions of other problems.

There are two properties of quantum mechanical systems which can make them exactly solvable: *supersymmetry* and *superintegrability*. Both of them are guide signs in searches for exactly solvable problems. Moreover, some of quantum mechanical systems, like the Hydrogen atom or isotropic harmonic oscillator, are both superintegrable and supersymmetric, and exactly such systems as a rule are very interesting and important. Notice that these qualities appear together naturally, and there exists a tight coupling between superintegrability and supersymmetry [7]. A more contemporary discussion of such coupling can be found in [3, 8–10, 19].

A quantum mechanical system with n degrees of freedom is called superintegrable if it admits more than $n-1$ integrals of motion. Moreover, at least $n-1$ integrals of motion should commute each other. The maximal possible number of constants of motion including Hamiltonian is equal $2n-1$.

The system is treated as supersymmetric in two cases: when some of its integrals of motion form a superalgebra, and when it has a specific symmetry called shape invariance.

In the present paper a classification of supersymmetric and superintegrable models of quantum mechanics is presented. Mathematically, the subject of this classification are systems of coupled Schrödinger equations of the following form

$$H\psi = E\psi, \tag{1}$$

where

$$H = -\frac{\nabla^2}{2m} + V(\boldsymbol{x}). \tag{2}$$

Here ψ is a *multicomponent* wave function and $V(\boldsymbol{x})$ is a *matrix potential.* Physically, just equations of generic form (1) are requested to construct models of neutral particles which have nontrivial dipole moments. A perfect example of such particle is neutron. Its electrical charge is zero, but the magnetic moment and, probably, the electric one, are nontrivial.

Symmetries of systems (1) including two and three equations have been investigated in [1] and [2]. However, it was done only for diagonal potentials V depending on time and one spatial variable.

Higher symmetries of 2d and 3d systems (1), which are responsible for their superintegrability have been studied in recent papers [20,21] and [4]. Moreover the class of the considered potentials was restricted to linear combinations of scalar and spin-orbit terms, and the related systems include only two equations.

In the present paper we adduce classification results of one dimensional systems with arbitrary number of equations, which have a special symmetry called shape invariance. In addition we present a classification of 2d integrable and superintegrable models of neutral fermions with a non-trivial dipole moments, and give examples of 2d and 3d systems which lead to shape invariant equations after separation of variables. The presented results are based on the recent papers [5,11–17].

2 An example: Pron'ko–Stroganov model

Let us start with an example of a system with matrix potential, which is both superintegrable and supersymmetric. This system (discovered by Pron'ko and Stroganov [18] and denoted in the following text as PS system) is based on the following version of the Schrödinger–Pauli Hamiltonian

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + \lambda\frac{\sigma_1 y - \sigma_2 x}{r^2} \tag{3}$$

and simulates a neutral spinor anomalously interacting with the magnetic field generated by a straight line current directed along the z coordinate axis. Here

$$p_x = -\mathrm{i}\frac{\partial}{\partial x}, \quad p_y = -\mathrm{i}\frac{\partial}{\partial y}, \quad r^2 = x^2 + y^2,$$

σ_1 and σ_2 are Pauli matrices, λ is the integrated coupling constant.

Thus the PS system has a clear and important physical content. In addition it is a nice "symmetry toy" which is both superintegrable and supersymmetric (shape invariant). Indeed, Hamiltonian (3) is invariant w.r.t. rotations in $x-y$ plane, since it commutes with the total angular momentum operator

$$J = xp_y - yp_x + \frac{1}{2}\sigma_3. \tag{4}$$

There are two more integrals of motion for the PS system

$$\begin{aligned} A_x &= \frac{1}{2}(Jp_x + p_xJ) + \frac{m}{r}\mu(\boldsymbol{n})y, \\ A_y &= \frac{1}{2}(Jp_y + p_yJ) - \frac{m}{r}\mu(\boldsymbol{n})x, \end{aligned} \tag{5}$$

where $\mu(\boldsymbol{n}) = \lambda(\sigma_1 y - \sigma_2 x))/r$. Operators J, A_x and A_y commute with $\mathcal{H}$ and satisfy the following commutation relations:

$$[J, A_x] = iA_y, \quad [J, A_y] = -iA_x, \quad [A_x, A_y] = -iJ\mathcal{H}.$$

In other words, we have 3 integrals of motion for a system with two degrees of freedom, but only two of them are functionally independent. Thus the PS system is superintegrable.

Introducing the polar coordinates and expanding solutions via eigenfunctions of J we reduce the related equation (1) to the following eigenvalue problem for radial functions:

$$\mathcal{H}_\kappa\psi_\kappa \equiv \left(-\frac{\partial^2}{\partial r^2} + \kappa(\kappa - \sigma_3)\frac{1}{r^2} + \sigma_1\frac{\lambda}{r}\right)\psi_\kappa = \tilde{\mathcal{E}}\psi_\kappa.$$

Here $\kappa = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are eigenvalues of J and parameter λ can be reduced to 1 by rescaling independent variable r. The effective potential

$$V_\kappa(r) = \kappa(\kappa - \sigma_3)\frac{1}{r^2} + \sigma_1\frac{\lambda}{r} \tag{6}$$

is *shape invariant*. To show this let us represent V_κ as

$$V_k(x) = -\frac{\partial W_k}{\partial r} + W_k^2 - \frac{1}{(2\kappa + 1)^2} \tag{7}$$

and solve this Riccati equation for W_κ. We obtain

$$W_\kappa = \frac{1}{2r}\sigma_3 - \frac{1}{2\kappa + 1}\sigma_1 - \frac{2\kappa + 1}{2r}. \tag{8}$$

Using (8) we can define the so called superpartner for V_κ

$$V_k^+ = \frac{\partial W_k}{\partial r} + W_k^2 = (\kappa + 1)(\kappa + 1 - \sigma_3)\frac{1}{r^2} + \sigma_1\frac{1}{r} + \frac{1}{(2\kappa + 1)^2}$$

which appears to be shape invariant, i.e.,

$$V_\kappa^+ = V_{\kappa+1} + C_\kappa, \qquad C_\kappa = \frac{1}{(2\kappa + 1)^2} - \frac{1}{(2\kappa + 3)}. \tag{9}$$

Thus the PS system admits supersymmetry with shape invariance and can be solved algebraically using the standard technique of supersymmetric quantum mechanics.

Alternatively, the energy values for coupled states (which by definition are negative) and the related eigenvectors can be found using the representation theory of algebra $o(3)$, whose generators are given by equations (4) and (5).

Equation (6) gives an example of *matrix shape invariant potential.* In contrast to the scalar shape invariant potentials, the matrix ones were practically unknown. In the following we classify shape invariant potentials which are $n \times n$ with arbitrary n.

3 Matrix superpotentials

We restrict ourselves to additive shape invariant potentials which are characterized by superpartners of type (9) with a shifted parameter. We start with superpotentials of the special form

$$W_k = kQ + \frac{1}{k}R + P, \tag{10}$$

where P, R and Q are $n \times n$ Hermitian matrices depending on x. Moreover, we suppose that $Q = Q(x)$ is proportional to the unit matrix. This supposition can be motivated by two reasons:

- our goal is to generalize the superpotential appearing in the Pron'ko–Stroganov problem which has exactly this form;
- restricting ourselves to such Q it is possible to make a complete classification of matrix superpotentials (10) satisfying shape invariance condition Moreover, it can be done for *matrix potentials of arbitrary dimension.*

By definition superpotentials should satisfy the shape invariance condition

$$W_k^2 + W_k' = W_{k+\alpha}^2 - W_{k+\alpha}' + C_k. \tag{11}$$

Substituting (10) into (11) and equating coefficients for the same powers of variable parameter κ we obtain the following system of determining equations

$$Q^2\alpha - Q' + \nu\alpha I = 0, \tag{12}$$

$$P' - \alpha QP + \mu I = 0, \tag{13}$$

$$\{R, P\} + \lambda I = 0, \tag{14}$$

$$R^2 = \omega^2 I, \tag{15}$$

where I is the unit matrix, the prime denotes derivation w.r.t. r and Greek letters denote arbitrary real parameters.

Solving equations (12)–(15) it is reasonable to restrict ourselves to irreducible matrices P and R. It is possible to show [11] that to obtain a non-trivial solutions it is necessary to set $\lambda = 0$. Then algebraic equations (14) and (15) can be decoupled to a direct sum of equations for 2×2 and 1×1 matrices [11].

Irreducible matrices satisfying (13)–(15) can have dimension 1×1 or 2×2. The general solution of the determining equations for 2×2 matrices includes five non-equivalent superpotentials:

$$W_{\kappa,\mu} = ((2\mu+1)\,\sigma_3 - 2\kappa - 1)\,\frac{1}{2x} + \frac{\omega}{2\kappa+1}\sigma_1, \quad \mu > -\frac{1}{2}, \tag{16}$$

$$W_{\kappa,\mu} = \lambda\left(-\kappa + \mu\exp(-\lambda x)\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right), \tag{17}$$

$$W_{\kappa,\mu} = \lambda\left(\kappa\tan\lambda x + \mu\sec\lambda x\sigma_3 + \frac{\omega}{\kappa}\sigma_1\right), \tag{18}$$

$$W_{\kappa,\mu} = \lambda\left(-\kappa\coth\lambda x + \mu\,\mathrm{cosech}\,\lambda x\,\sigma_3 - \frac{\omega}{\kappa}\sigma_1\right), \quad \mu < 0,\ \omega > 0, \tag{19}$$

$$W_{\kappa,\mu} = \lambda\left(-\kappa\tanh\lambda x + \mu\,\mathrm{sech}\,\lambda x\,\sigma_1 - \frac{\omega}{\kappa}\sigma_3\right) \tag{20}$$

while for 1×1 matrices we obtain ten non-equivalent solutions:

$$W = -\frac{\kappa}{x} + \frac{\omega}{\kappa} \qquad \text{(Coulomb)}, \tag{21}$$

$$W = \lambda\kappa\tan\lambda x + \frac{\omega}{\kappa} \qquad \text{(Rosen 1)}, \tag{22}$$

$$W = \lambda\kappa\tanh\lambda x + \frac{\omega}{\kappa} \qquad \text{(Rosen 2)}, \tag{23}$$

$$W = -\lambda\kappa\coth\lambda x + \frac{\omega}{\kappa} \qquad \text{(Eckart)}, \tag{24}$$

$$W = \mu x \qquad \text{(harmonic oscillator)}, \tag{25}$$

$$W = \mu x - \frac{\kappa}{x} \qquad \text{(3d oscillator)}, \tag{26}$$

$$W = \lambda\kappa\tan\lambda x + \mu\sec\lambda x \qquad \text{(Scarf 1)}, \tag{27}$$

$$W = \lambda\kappa\tanh\lambda x + \mu\,\mathrm{sech}\,\lambda x \qquad \text{(Scarf 2)}, \tag{28}$$

$$W = \lambda\kappa\coth\lambda x + \mu\,\mathrm{cosech}\,\lambda x \qquad \text{(Pöschl–Teller)}, \tag{29}$$

$$W = \kappa - \mu\exp(-x) \qquad \text{(Morse)}. \tag{30}$$

Thus all superpotential of generic form (10) are exhausted by direct sums of matrix potentials (16)–(20) and scalar potentials (21)–(30). The potentials corresponding to the found matrix superpotentials are easy calculated using definition (7)

$$\hat{V}_\kappa = \left(\mu(\mu+1) + \kappa^2 - \kappa(2\mu+1)\sigma_3\right)\frac{1}{x^2} - \frac{\omega}{x}\sigma_1, \tag{31}$$

$$\hat{V}_\kappa = \lambda^2\left(\mu^2\exp(-2\lambda x) - (2\kappa-1)\mu\exp(-\lambda x)\sigma_1 + 2\omega\sigma_3\right), \tag{32}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 &\left((\kappa(\kappa-1)+\mu^2)\sec^2\lambda x + 2\omega\tan\lambda x\sigma_1\right.\\ &\left.+\mu(2\kappa-1)\sec\lambda x\tan\lambda x\sigma_3\right),\end{aligned} \tag{33}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 &\left((\kappa(\kappa-1)+\mu^2)\,\mathrm{cosech}^2(\lambda x) + 2\omega\coth\lambda x\sigma_1\right.\\ &\left.+\mu(1-2\kappa)\coth\lambda x\,\mathrm{cosech}\,\lambda x\sigma_3\right),\end{aligned} \tag{34}$$

$$\begin{aligned}\hat{V}_\kappa = \lambda^2 &\left((\mu^2-\kappa(\kappa-1))\,\mathrm{sech}^2\lambda x + 2\omega\tanh\lambda x\sigma_3\right.\\ &\left.-\mu(2\kappa-1)\,\mathrm{sech}\,\lambda x\tanh\lambda x\sigma_1\right).\end{aligned} \tag{35}$$

Moreover potentials (31), (32), (33), (34) and (35) are generated by the superpotentials (16), (17), (18), (19) and (20), respectively.

Notice that our approach makes it possible to find all known scalar superpotentials (21)–(30) in a very easy way.

4 Dual shape invariance

Starting with (16)–(19) we find the related potentials (31)–(34) in a unique fashion. But let us state the inverse problem: to find possible superpotentials corresponding to given potentials. This problem is very interesting since this is a way to generate families of isospectral Hamiltonians. For matrix superpotentials everything is much more interesting since there exist additional superpotentials compatible with the shape invariance condition.

To find additional superpotentials we note that, in contrast to (32) and (35), potentials (31), (33) and (34) are invariant with respect to the simultaneous change

$$\mu \to \kappa - \frac{1}{2}, \quad \kappa \to \mu + \frac{1}{2}.$$

However, the corresponding superpotentials are not invariant w.r.t. this change and take the following forms

$$\widetilde{W}_{\mu,\kappa} = \frac{\kappa\sigma_3 - \mu - 1}{x} + \frac{\omega}{2(\mu+1)}\sigma_1, \quad c_\mu = \frac{\omega^2}{4(\mu+1)^2} \tag{36}$$

for $\hat{V}_k$ given by equation (31) and

$$\widetilde{W}_{\mu,\kappa} = \frac{\lambda}{2}\left((2\mu+1)\tan\lambda x + (2\kappa-1)\sec\lambda x\sigma_3 + \frac{4\omega}{2\mu+1}\sigma_1\right)$$

for potential (33). The superpotential for (34) has the form

$$\widetilde{W}_{\mu,\kappa} = \frac{\lambda}{2}\left(-(2\mu+1)\coth\lambda x + (2\kappa-1)\,\mathrm{cosech}\,\lambda x\sigma_3 - \frac{4\omega}{2\mu+1}\sigma_1\right).$$

Thus potentials (16), (18) and (19) admit a dual supersymmetry, i.e., they are shape invariant w.r.t. shifts of two parameters, namely, κ and μ. More exactly, superpartners for potentials (31), (33) and (34) can be obtained either by shifts of κ or by shifts of μ while simultaneous shifts are forbidden. We call this property of potentials *dual shape invariance.*

5 Multidimensional shape invariant systems

Any of the presented potentials corresponds to the exactly solvable model, which can be solved algebraically using tools of SUSY quantum mechanics. And it is very interesting to search for realistic 3d quantum mechanical problems which can be solved exactly using shape invariance of their effective potentials in separated variables.

In particular case $\mu = 0$ potential (31) coincides with (6). Thus we obtain a generalized potential of Pron'ko–Stroganov problem with arbitrary parameter μ. A natural question is whether there exist realistic quantum mechanical problems whose effective potentials are given by equation (31) with $\mu \neq 0$. In this section we present examples of such problems.

5.1 Neutral fermion with dipole moment in an external field

Consider Hamiltonian (2) with the potential

$$\hat{V} = \alpha \frac{\boldsymbol{\sigma} \cdot \boldsymbol{x}}{x^2}, \tag{37}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose components are Pauli matrices,

$$x^2 = x_1^2 + x_2^2 + x_3^2.$$

Equation (1) describes a neutral fermion with a non-trivial dipole moment, interacting with an external electric field. It is transparently invariant w.r.t. the rotation group since the corresponding Hamiltonian commutes with the orbital momentum vector $\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} + \frac{1}{2}\boldsymbol{\sigma}$. In addition, this Hamiltonian commutes with the generalized Runge–Lenz vector

$$\boldsymbol{R} = \frac{1}{2m}(\boldsymbol{p} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{p}) + \boldsymbol{x}V, \tag{38}$$

and so we have an example of 3d superintegrable system with Fock symmetry [16].

Expanding solutions of equation (1) with potential (37) via spherical spinors we come to the following equations for radial functions

$$\mathcal{H}_j \Phi_{j,\kappa} \equiv \left(-\frac{\partial^2}{\partial r^2} + V_j \right) \Phi_{j,\kappa} = \varepsilon \Phi_{j,\kappa}. \tag{39}$$

Here

$$\boldsymbol{r} = 2m\alpha\boldsymbol{x}, \quad \varepsilon = \frac{E}{2m\alpha^2}, \quad \Phi_{j,\kappa} = \begin{pmatrix} \psi_{j,-\frac{1}{2},\kappa}(r) \\ \psi_{j,\frac{1}{2},\kappa}(r) \end{pmatrix},$$

and V_j is the matrix potential

$$V_j = \left(j(j+1) + \frac{1}{4} - \sigma_3\left(j + \frac{1}{2}\right)\right)\frac{1}{r^2} - \sigma_1\frac{1}{r}, \tag{40}$$

where j are half integer numbers labeling eigenvectors of the squared total orbital momentum $\boldsymbol{J}$.

Effective potential (40) coincides with (31) if we set $\omega = 1$ and $\mu = j$. Thus the radial equation (39) is shape invariant and can be solved with using tools of SUSY quantum mechanics. The ground state $\Phi^0_{j,k} = \text{column}(\psi^0, \xi^0)$ can be found as a solution of the first order equation

$$a_j\Phi^0_{j,k} = 0, \tag{41}$$

where $a_j = -\partial_r + W_j$ and W_j is superpotential (36) with $\mu = j$ and $\omega = 1$. Solving (41) we obtain

$$\Phi^0_{j,k} = c_k \begin{pmatrix} r^{j+\frac{3}{2}}K_1\left(\frac{r}{2(j+1)}\right) \\ r^{j+\frac{3}{2}}K_0\left(\frac{r}{2(j+1)}\right) \end{pmatrix}, \tag{42}$$

where K_1 and K_0 are the modified Bessel functions, c_k are arbitrary constants. The corresponding eigenvalue ε_0 is equal to c_j, i.e., $\varepsilon_0 = -\frac{1}{4}(j+1)^{-2}$. The solution $\Phi^n_{j,k}$ corresponding to n^{th} excited state and eigenvalue ε_n for this state are:

$$\Phi^n_{j,k} = a^+_j a^+_{j+1} \cdots a^+_{j+n-1}\Phi^0_{j+n,k}, \quad \varepsilon_n = -\frac{1}{4(j+n+1)^2}, \tag{43}$$

where $a^+_j = \partial_r + W_j$. The related energy value in (1) is given by the equation

$$E = -\frac{m\alpha^2}{2N^2}, \tag{44}$$

where $N = n + j + 1$, $n = 0, 1, 2, \ldots,$

5.2 Shape invariant 2d systems with arbitrary spin

In this section we present a countable set of shape invariant systems, which describe particles with arbitrary spin s interacting with external fields. Their Hamiltonians have the following form [14]:

$$\mathcal{H}_s = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{\tilde{x}}\mu_s(\boldsymbol{n}), \tag{45}$$

where $\boldsymbol{n} = \dfrac{\tilde{\boldsymbol{x}}}{\tilde{x}}$, $\quad \tilde{\boldsymbol{x}} = (x_1, x_2)$, $\quad \tilde{x} = \sqrt{x_1^2 + x_2^2}$,

$$\mu_s(\boldsymbol{n}) = \alpha \sum_{\nu=-s}^{s} (-1)^\nu \Lambda_\nu, \tag{46}$$

and Λ_ν are projectors onto eigenspaces of matrix $\boldsymbol{S} \cdot \boldsymbol{n}$ corresponding to the eigenvalue ν $(\nu, \nu' = s, s-1, \ldots, -s)$

$$\Lambda_\nu = \prod_{\nu' \neq \nu} \frac{\boldsymbol{S} \cdot \boldsymbol{n} - \nu'}{\nu - \nu'}.$$

In particular, for $s = 1$ Hamiltonian (45) takes the form

$$\mathcal{H}_1 = \frac{p_1^2 + p_2^2}{2m} + \alpha \left(\frac{2(\boldsymbol{S} \cdot \tilde{\boldsymbol{x}})^2}{\tilde{x}^3} - \frac{1}{\tilde{x}} \right),$$

where $\boldsymbol{S} = (S_1, S_2)$ with S_1 and S_2 are the spin one matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For $s = \frac{3}{2}$ Hamiltonian $\mathcal{H}_\int$ looks as follows

$$\mathcal{H}_{\frac{3}{2}} = \frac{p_1^2 + p_2^2}{2m} + \omega \left(\frac{1}{\tilde{x}^2} \boldsymbol{S} \cdot \tilde{\boldsymbol{x}} - \frac{4}{7\tilde{x}^4} (\boldsymbol{S} \cdot \tilde{\boldsymbol{x}})^3 \right).$$

Here $\mathbf{S}$ is a vector of spin $\frac{3}{2}$ whose explicit form can be found, e.g., in [6].

Let us consider the eigenvalue problem (1) for Hamiltonian (45)–(46). We introduce polar coordinates and expand solutions via eigenvectors of operator $J = x_1 P_2 - x_2 P_1 + S_3$ where S_3 is a matrix of spin s. These eigenvectors can be represented as:

$$\psi_\kappa = \frac{1}{\sqrt{r}} \exp(\mathrm{i}(\kappa - S_z)\theta) \Phi_\kappa(r), \tag{47}$$

where

$$\Phi_\kappa(r) = \mathrm{column}(\phi_s, \phi_{s-1}, \ldots, \phi_{-s}). \tag{48}$$

Substituting (47), (48) into (1) we obtain the following equation for radial functions Φ_k:

$$\hat{\mathcal{H}}_\kappa \Phi_\kappa \equiv \left(-\frac{\partial^2}{\partial \tilde{x}^2} + V_\kappa \right) \Phi_\kappa = \tilde{\mathcal{E}} \Phi_\kappa, \tag{49}$$

where $\tilde{\mathcal{E}} = 2mE$ and

$$V_\kappa = \left((k - S_z)^2 - \frac{1}{4}\right)\frac{1}{r^2} + \tilde{\mu}_s\frac{1}{r}, \quad \tilde{\mu}_s = 2m\mu_s(\boldsymbol{n})|_{n_y=0}.$$

Hamiltonian $\hat{\mathcal{H}}_\kappa$ and matrix S_3^2 commute each other and so they have a mutual set of eigenfunctions. Matrix S_3^2 can be chosen diagonal and its eigenfunctions ψ_ν corresponding to eigenvalues $\nu^2 = s^2, (s-1)^2, (s-2)^2, \dots$ have two or one non-zero components for $\nu^2 > 0$ and $\nu = 0$ respectively. In the standard representation of spin matrix with diagonal S_3 and symmetric S_1[1], the matrix $\tilde{\mu}_s$ is symmetric and antidiagonal, and has the unit non-zero entries.

Thus the eigenvalue problem (49) can be decoupled to the following equations

$$\hat{\mathcal{H}}_{\kappa,\nu}\psi_{\kappa,\nu} \equiv \left(-\frac{\partial^2}{\partial\tilde{x}^2} + V_\kappa\right)\psi_{\kappa,\nu} = \tilde{\mathcal{E}}\psi_{\kappa,\nu}, \tag{50}$$

where $\psi_{\kappa,\nu} = \begin{pmatrix}\phi_\nu \\ \phi_{-\nu}\end{pmatrix}$ are two-component functions (non-zero entries of eigenvectors of matrix S_3^2), and

$$V_{\kappa,\nu} = \frac{(\kappa - \nu\sigma_3)^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r}\sigma_1, \quad \nu \neq 0. \tag{51}$$

For $\nu = 0$ the corresponding reduced potential is one-dimensional

$$V_{\kappa,0} = \frac{\kappa^2 - \frac{1}{4}}{r^2} + \frac{\tilde{\lambda}}{r}. \tag{52}$$

Potentials (51) coincide with the shape-invariant potential (31) if we denote $\nu = \mu + \frac{1}{2}$ and $\kappa = \frac{1}{2}$. Thus, in analogy with Section 5.1 we can find the corresponding exact solution algebraically. Doing this we should exploit the dual shape invariance of potential (51) and use the superpotentials (16) for $\kappa \geq \nu$ and superpotentials (36) for $0 \leq \kappa \leq \nu$. Another choice of the superpotentials leads to solutions which are not square integrable.

Finally we obtain the following components of the ground state vectors $\psi_\nu^0 = \text{column}(\phi_\nu^0, \phi_{-\nu}^0)$, $\nu = s, s-1, s-2, \dots, \nu > 0$:

$$\phi_\nu^0 = d_\nu r^{\kappa+1}K_{\nu+\frac{1}{2}}\left(\frac{\tilde{\lambda}r}{2\kappa+1}\right), \quad \phi_{-\nu}^0 = d_\nu(-1)^{\nu-\frac{1}{2}}r^{\kappa+1}K_{\nu-\frac{1}{2}}\left(\frac{\tilde{\lambda}r}{2\kappa+1}\right),$$

$\kappa \geq \nu$,

$$\phi_\nu^0 = d_\nu r^{\nu+1}K_{\kappa+\frac{1}{2}}\left(\frac{\tilde{\lambda}r}{2\nu+1}\right), \quad \phi_{-\nu}^0 = d_\nu(-1)^{\kappa-\frac{1}{2}}r^{\nu+1}K_{\kappa-\frac{1}{2}}\left(\frac{\tilde{\lambda}r}{2\nu+1}\right),$$

$0 \leq \kappa < \nu$,

[1] See, e.g., Eq. (4.65) in [6].

where d_ν are integration constants. Solution for $\nu = 0$, i.e., the component ϕ_0^0, is given by the following equation:

$$\phi_0^0 = du^{\kappa+\frac{1}{2}} \exp\left(-\tilde{\lambda} r\right), \quad \kappa = 0, 1, 2, \ldots \tag{53}$$

The vectors of exited states and the corresponding energy levels again are given by relations (43) and (44), respectively, where j should be replaced by κ.

5.3 2d system with matrix Morse potential

Finally, let us consider a 2d system with a periodic potential. This system appeared as a particular case of superintegrable systems classified in [12]. It is described by the following equation:

$$\begin{aligned} H\psi \equiv \big(p_1^2 + p_2^2 + \lambda(1-2\kappa)\exp(-x_2)(\sigma_1\cos x_1 - \sigma_2 \sin x_1) \\ + \lambda^2 \exp(-2x_2)\big)\psi = 2mE\psi. \end{aligned} \tag{54}$$

The Hamiltonian H in (54) admits integral of motion $Q_2 = P_1 - \frac{\sigma_3}{2}$. Thus it is possible to expand solutions of (54) via eigenvectors of Q_2 which look as follows:

$$\psi_p = \begin{pmatrix} \exp(i(p+\frac{1}{2})x_1)\varphi(x_2) \\ \exp(i(p-\frac{1}{2})x_1)\xi(x_2) \end{pmatrix} \tag{55}$$

and satisfy the condition $Q\psi_p = p\psi_p$. Substituting (55) into (54) we come to the equation

$$\left(-\frac{\partial^2}{\partial y^2} + V_\kappa\right)\Phi = \varepsilon\Phi,$$

where we denote $y = x_2$, $\varepsilon = 2mE - p^2 - \frac{1}{4}$, $\Phi = \begin{pmatrix}\varphi \\ \xi\end{pmatrix}$, and

$$V_\kappa = \lambda^2\exp(-2y) - \lambda(2\kappa-1)\exp(-y)\sigma_1 - p\sigma_3. \tag{56}$$

Potential (56) coincides with the shape invariant potential (32). Using the shape invariance it is possible to integrate equation (54) in a simple and straightforward way with using tools of SUSY quantum mechanics [11]. The eigenvalues ε and the corresponding state vectors are enumerated by natural numbers $n = 0, 1, \ldots$. The ground state vector $\Phi_0(\kappa, y) = \begin{pmatrix}\varphi_0 \\ \xi_0\end{pmatrix}$ should solve the equation

$$a_\kappa^- \Phi_0(\kappa, y) \equiv \left(\frac{\partial}{\partial x} + W_\kappa\right)\Phi_0(\kappa, y) = 0,$$

thus $\varphi_0 = y^{\kappa+1}K_{\nu+1}(z)$, $\quad \xi_0 = y^{\kappa+1}K_\nu(z)$, where $K_\nu(z)$ is modified Bessel function, $\nu = \frac{p}{\kappa} + \frac{1}{2}$ and $z = -py/(4\kappa+2) \geq 0$.

Solutions which correspond to n^{th} excited state can be calculated using equation (43), and the corresponding spectral parameter ε have the following form [11]

$$\varepsilon = -N^2 - \frac{p^2}{4N^2},$$

where $N = \kappa + n$ and n is a natural number.

6 Classification of 2d superintegrable systems

Consider a special class of Schrödinger–Pauli equations describing neutral fermions with non-trivial dipole momentum. The corresponding stationary Schrödinger–Pauli equation is given by formula (1), where

$$V = V(\tilde{\boldsymbol{x}}) = \frac{\lambda}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}. \tag{57}$$

Here $\boldsymbol{\sigma}$ is the matrix three vector whose components are Pauli matrices, $\mathbf{B}$ is a three component vector of external field, depending on two spatial variables. A particular example of potential (57) is discussed in Section 2.

To simplify the following formulae we rescale variables and reduce Hamiltonian (1) with potential (57) to the form

$$H = -\nabla^2 + \boldsymbol{\sigma}\cdot\boldsymbol{B}. \tag{58}$$

For this purpose it is sufficient to change in (1) and (57) $E \to \frac{1}{2m}E$ and $\boldsymbol{B} \to \frac{1}{\lambda}\boldsymbol{B}$.

Let us search for integrals of motion for Hamiltonian (58) of the generic form:

$$Q = \sigma^\mu \left(\mathrm{i}\{\Lambda^{\mu a}, \nabla_a\} + \Omega^\mu\right), \tag{59}$$

where summation is imposed over the repeated indices $\mu = 0, 1, 2, 3$ and $a = 1, 2$, $\Lambda^{\mu a}$ and Ω^a are functions of $\tilde{\boldsymbol{x}}$, $\{\Lambda^{\mu a}, \nabla_a\} = \Lambda^{\mu a}\nabla_a + \nabla_a\Lambda^{\mu a}$, $\nabla_a = \frac{\partial}{\partial x_a}$, σ_μ are Pauli matrices. In other words, we restrict ourselves to first-order differential operators with matrix coefficients.

Integrals of motion should commute Hamiltonian,

$$[H, Q] \equiv HQ - QH = 0. \tag{60}$$

Substituting (58) and (59) into (60) and equating coefficients for linearly independent matrices and differential operators, we obtain the following system of determining equations [17]:

$$\Lambda^{\mu a}_b + \Lambda^{\mu b}_a = 0, \tag{61}$$

$$\Omega^0_a = 0, \tag{62}$$

$$\Lambda^{ab} B^a_b = 0, \quad \Lambda^{0b} B^a_b = \varepsilon^{abc}\Omega^b B^c, \tag{63}$$

$$\Omega^a_b = 2\varepsilon^{acd}\Lambda^{cb} B^d, \tag{64}$$

where the subindices denote derivatives w.r.t. the corresponding independent variables. Solving them we can find both admissible external fields $\mathbf{B}$ and the corresponding integrals of motion.

The classification results are presented in the following tables.

Table 1. External fields with Lie symmetries.

no.	External field	Symmetry operators
1	$B^1 = \cos(k\theta) f_1(r) + \sin(k\theta) f_2(r)$, $B^2 = \cos(k\theta) f_2(r) - \sin(k\theta) f_1(r), \quad B^3 = f_3(r)$	$\tilde{Q}_1 = L + \frac{k}{2}\sigma_3$
2	$B^1 = \mu\cos(k\theta) r^{-k}, \quad B^2 = \mu\sin(k\theta) r^{-k}, \quad B^3 = f_3(r)$	$\tilde{Q}_1$
3	$B^1 = \mu\cos(\theta) f_1(r), \quad B^2 = \mu\sin(\theta) f_1(r), \quad B^3 = f_2(r)$	$Q_1 = L + \frac{1}{2}\sigma_3$
4	$B^1 = \cos(\delta x_1) f_1(x_2) + \sin(\delta x_1) f_2(x_2)$, $B^2 = \cos(\delta x_1) f_2(x_2) - \sin(\delta x_1) f_1(x_2), \quad B^3 = f_3(x_2)$	$\tilde{Q}_2 = P_1 - \frac{\delta}{2}\sigma_3$
5	$B^1 = \mu\exp(-x_2)\cos x_1$, $B^2 = -\mu\exp(-x_2)\sin x_1, \quad B^3 = f_3(x_2)$	$Q_2 = P_1 - \frac{1}{2}\sigma_3$
6	$B^1 = B^2 = 0, \quad B^3 = f(\boldsymbol{x})$	σ_3
7	$B^1 = B^2 = 0, \quad B^3 = f(x_1)$	$P_2, \ \sigma_3$
8	$B^1 = B^2 = 0, \quad B^3 = f(r)$	$L, \ \sigma_3$

Here $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan\frac{x_2}{x_1}$, $P_a = p_a = -\mathrm{i}\frac{\partial}{\partial x_a}, a = 1, 2$, $L = x_1 p_2 - x_2 p_1$, $f_1(\cdot)$, $f_2(\cdot)$, $f_3(\cdot)$ and $f(\cdot)$ are arbitrary functions, $\delta \in \{0, 1\}$. For B given in Cases 1 and 4 to be single valued and $\tilde{Q}_1$ to generate finite rotations for arbitrary values of θ, the parameter k must be an integer.

Table 2. External fields and higher symmetries.

no.	External field	Symmetry operators
1	$B^1 = \mu\cos x_1, \ B^2 = \mu\sin x_1$, $B^3 = \nu$	$Q_2 = P_1 - \frac{1}{2}\sigma_3, \ P_2$, $Q_3 = \sigma_3(P_1 - \nu) - \mu(\sigma_1\cos x_1 + \sigma_2\sin x_1)$
2	$B^1 = \dfrac{\mu k\sin(k\theta)}{r^2}$, $B^2 = -\dfrac{\mu k\cos(k\theta)}{r^2}, \ B^3 = \dfrac{k\nu}{r^2}$	$\tilde{Q}_1 = L + \frac{k}{2}\sigma_3$, $Q_4 = \sigma_3(\tilde{Q}_1 + \nu)$ $- \mu(\sigma_1\sin(k\theta) - \sigma_2\cos(k\theta))$
3	$B^1 = \dfrac{\mu^2 x_2}{2\sqrt{\nu^2 - \mu^2 r^2}}$, $B^2 = -\dfrac{\mu^2 x_1}{2\sqrt{\nu^2 - \mu^2 r^2}}, \ B^3 = \dfrac{\mu}{2}$	$Q_1 = L + \frac{1}{2}\sigma_3$, $Q_5 = \sigma_1 P_1 + \sigma_2 P_2 - \frac{\mu}{2}(\sigma_1 x_2$ $- \sigma_2 x_1) - \frac{1}{2}\sigma_3\sqrt{\nu^2 - \mu^2 r^2}$
4	$B^1 = \dfrac{x_2\varphi'}{r}$, $B^2 = -\dfrac{x_1\varphi'}{r}, \ B^3 = -\mu(r\varphi)'$	$Q_1, \ Q_6 = \sigma_1 P_1 + \sigma_2 P_2 + \mu(\sigma_3 Q_1$ $+ \sigma_1 x_2\varphi - \sigma_2 x_1\varphi) + \sigma_3(\varphi + \nu)$

Here the function $\varphi = \varphi(r)$ is a solution of the algebraic equation $\left(\mu^2 r^2 + 1\right)\varphi^2 + 2\nu\varphi = c$, where μ, ν, k and c are real parameters.

The results presented in Tables 1 and 2 extend the list of exactly solvable problems of quantum mechanics. Moreover, they exhaust all matrix potentials (57) which correspond to integrable Schrödinger equations admitting first order constants of motion.

7 Discussion

We present an extended list of new exactly solvable systems of coupled Schrödinger equations with matrix potentials. An important property of this list is that it is completed in the classes of equations which are discussed.

In this paper we restrict ourselves to shape invariant superpotentials of special form (10), where matrix Q is proportional to a unit matrix. This restriction makes it possible to solve completely the corresponding classification problem. In [12] superpotentials with an arbitrary matrix Q are described. In particular, in this case there are 17 non-equivalent matrix superpotentials of dimension 2×2 [12].

The other property of exactly solvable systems, i.e., superintegrability, is exploited in Section 6. Here a classification of matrix potentials (57) is presented which give rise to Schrödinger equations admitting the first order integrals of motion with matrix coefficients. Notice that integrals of motion presented in Tables 1 and 2 form interesting superalgebraic structures which are discussed in paper [17].

Some of the classified systems include external fields which do not solve Maxwell equations with physically realizable currents. However, as it was shown in papers [13] and [15], all these fields solve equations of axion electrodynamics.

There are interesting links between non-relativistic systems discussed in the present paper and relativistic systems described by Dirac equation. In paper [5] a relativistic counterpart of the PS system was proposed. It is shape invariant and exactly solvable, and its non-relativistic limit is nothing but the Pron'ko–Stroganov system. A relativistic version of the model discussed in Section 5.1 which keeps its shape invariance and superintegrability was proposed in [16]. There are now doubts that the list of relativistic superintegrable and supersymmetric systems can be added by other systems. The work on classification of such systems is in progress.

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Normalized Classes of Generalized Burgers Equations

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A hierarchy of normalized classes of generalized Burgers equations is studied. The equivalence groupoids of these classes are computed. The equivalence groupoids of classes of linearizable generalized Burgers equations are related to those of the associated linear counterparts using the Hopf–Cole transformation.

1 Introduction

We consider certain generalizations of the well known Burgers equation

$$u_t + uu_x + u_{xx} = 0, \tag{1}$$

which has been widely used as a one-dimensional turbulence model [1]. A review of its properties can be found in [24, Chapter 4]. Burgers equation can be generalized in various ways. The purpose of this paper is to study a hierarchy of classes of generalized Burgers equations. We show that the majority of naturally arising classes are normalized. This property simplifies considerably the problem of the group classification for such classes. Specifically, these problems reduce to subgroup analysis of the corresponding equivalence groups.

A class of differential equations is said to be *normalized* if its equivalence groupoid is generated by its equivalence group [12, 13, 15, 17]. The *equivalence groupoid* of a class of differential equations is the set of admissible transformations in this class with the natural groupoid structure, where the composition of mappings is the groupoid operation [14, p. 7]. An *admissible transformation* is a triple of an initial equation, a target equation and a mapping between them.

The notion of normalized classes is quite natural and useful for applications. For a normalized class of differential equations the following hold: 1) its complete group classification coincides with its preliminary group classification and 2) there are no additional equivalence transformations between cases of the classification list. This notion can be weakened. For example, weakly normalized classes maintain the first of the aforementioned features but lose the second, and for semi-normalized classes the situation is opposite (see [14, 17] for precise definitions).

Hierarchies of normalized subclasses arise in the course of solving group classification problems. Note that a single differential equation forms a normalized class. Any set of all possible equations with a prescribed number of independent variables and a fixed equation order is a normalized class likewise.

In order to prove the normalization property of a class of differential equations we compare its equivalence group with its equivalence groupoid. Practically, a class is normalized if there are no classifying conditions among the determining equations for admissible transformations. A classifying condition is, roughly speaking, a determining equation that simultaneously involves arbitrary elements of the class and parameters of admissible transformations and leads to a furcation while solving the determining equations.

Section 2 is devoted to a normalized superclass, which contains all other classes under consideration. In Section 3 we consider the relation between equivalence groupoids of classes of linear (1+1)-dimensional evolution equations and those of the associated classes of equations linearized by the Hopf–Cole transformation $u = 2v_x/v$. In Section 4 we consider classes of generalized Burgers equations with variable diffusion coefficients. One of these classes is not normalized but it can be partitioned into two normalized subclasses. Section 5 treats the classical Burgers equation as a normalized class.

2 Normalized superclass

It is known that the t-component of every point (or even contact) transformation between any two fixed (1+1)-dimensional evolution equations depends only on t [9, 10]. Moreover, as proved in [6, Lemma 2], any point transformation between two equations from the class

$$u_t = F(t,x,u)u_{xx} + G(t,x,u,u_x) \tag{2}$$

has the form $\tilde t = T(t)$, $\tilde x = X(t,x)$, and $\tilde u = U(t,x,u)$ with $T_tX_xU_u \neq 0$. The coefficients F and G are arbitrary smooth functions of their arguments with $F \neq 0$.

This class is normalized in the usual sense [6], and any contact transformation between equations of this class, is generated by a point transformation [19]. However, class (2) is too wide for a generalization of Burgers equation. For our purpose it is more convenient to consider its subclass,

$$u_t + F(t,x,u)u_{xx} + H^1(t,x,u)u_x + H^0(t,x,u) = 0, \tag{3}$$

where the coefficients F, H^1, and H^0 are arbitrary smooth functions in their arguments with $F \neq 0$. In the present paper we consider this class as the initial superclass. As it contains all subclasses to be studied in the subsequent analysis, any transformation between two fixed equations from each specified subclass obeys the restrictions marked for class (2).

In order to find the general form of admissible transformations for class (3), we write an equation of this class in tilded variables, $\tilde u_{\tilde t} + \tilde F\tilde u_{\tilde x\tilde x} + \tilde H^1\tilde u_{\tilde x} + \tilde H^0 = 0$, and replace $\tilde u_{\tilde t}$, $\tilde u_{\tilde x}$, and $\tilde u_{\tilde x\tilde x}$ with their expressions in terms of untilded variables. After restricting the result to the manifold defined by the initial equation using the substitution $u_t=-Fu_{xx}-H^1u_x-H^0$, we split it with respect to u_{xx} and u_x and obtain the determining equations for admissible transformations. They imply

$$\begin{aligned}&\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U(t,x,u)=U^1(t,x)u+U^0(t,x),\\&\tilde F=\frac{X_x^2}{T_t}F,\quad \tilde H^1=\frac{1}{T_t}\left(X_xH^1+X_{xx}F-2X_x\frac{U^1_x}{U^1}F+X_t\right),\\&\tilde H^0=U^1H^0+\frac{2U_xU^1_x}{T_tU^1}F-\frac{1}{T_t}\big(U_t+FU_{xx}+H^1U_x\big),\end{aligned}\tag{4}$$

where $T=T(t)$, $X=X(t,x)$, $U^1=U^1(t,x)$, and $U^0=U^0(t,x)$ are arbitrary smooth functions in their arguments with $T_tX_xU^1\neq 0$. Note that we obtain no additional equations (classifying conditions) on the arbitrary elements. This means that all admissible transformations in this class are generated by the transformations from the corresponding equivalence group, so class (3) is normalized.

To derive admissible transformations of any subclass of (3) it is sufficient to specify the arbitrary elements F, H^1, H^0, $\tilde F$, $\tilde H^1$, and $\tilde H^0$.

3 Linearizable generalized Burgers equations

We relate the equivalence groupoids of the class of second-order linear evolution equations

$$v_t+a(t,x)v_{xx}+b(t,x)v_x+c(t,x)v=0,\tag{5}$$

and the class of linearizable generalized Burgers equations

$$u_t+au_{xx}+(au+a_x+b)u_x+\frac12a_xu^2+b_xu+f=0.\tag{6}$$

Here a, b, c are smooth functions of (t,x) with $a\neq 0$, and $f=2c_x$. Class (6) is the widest class of differential equations that can be linearized to linear equations of form (5) by the Hopf–Cole transformation $u=2v_x/v$. This linearization was implicitly presented in [4, p. 102, Exercise 3]. Class (6) is a subclass of (3), where the arbitrary elements are specified as $F=a$, $H^1=au+a_x+b$, and $H^0=\frac12a_xu^2+b_xu+f$. Substituting these and the corresponding tilded expressions into (4) and splitting the result with respect to u, we derive the general form of admissible transformations between two equations from class (6),

$$\begin{aligned}&\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=\frac{1}{X_x}u+U^0(t,x),\\&\tilde a=\frac{X_x^2}{T_t}a,\quad \tilde b=\frac{1}{T_t}\big(X_xb+X_{xx}a-X_x^2U^0a+X_t\big),\\&\tilde f=\frac{f}{T_t}-\frac{\big(X_xU^0b\big)_x}{T_t}+\frac{\big(X_xU^0\big)^2-2\big(X_xU^0\big)_x}{2T_t}a_x\\&\qquad+\frac{X_xU^0\big(X_xU^0\big)_x-\big(X_xU^0\big)_{xx}}{T_t}a-\frac{\big(X_xU^0\big)_t}{T_t},\end{aligned}\tag{7}$$

where $T=T(t)$, $X=X(t,x)$, and $U^0=U^0(t,x)$ are arbitrary smooth functions in their arguments with $T_tX_x\neq 0$. There are no classifying conditions, so, transformations (7) form the (usual) equivalence group, and class (6) is normalized (in the usual sense).

Arbitrary elements of class (6) can be gauged to simple fixed values by equivalence transformations. At the first step we set $a=1$ using the transformation

$$\tilde t = t\,\mathrm{sign}\,a(t,x),\quad \tilde x=\int\frac{dx}{\sqrt{|a(t,x)|}},\quad \tilde u=u.$$

Thereby we obtain the class of equations of the general form

$$u_t+u_{xx}+(u+b)u_x+b_xu+f=0, \tag{8}$$

where $b=b(t,x)$ and $f=f(t,x)$ are arbitrary smooth functions. The linear counterpart of (8) is $v_t+v_{xx}+bv_x+\left(\frac12\int f dx\right)v=0$.

The equivalence group of class (8) can be calculated directly or by means of the substitution $a=\tilde a=1$ into (7). It consists of the transformations

$$\begin{aligned}
&\tilde t=T(t),\quad \tilde x=\varepsilon\left(\sqrt{T_t}x+X^0(t)\right),\quad \tilde u=\varepsilon\left(\frac{1}{\sqrt{T_t}}u+U^0(t,x)\right),\\
&\tilde b=\varepsilon\left(\frac{b}{\sqrt{T_t}}+\frac{T_{tt}}{T_t^{3/2}}x+\frac{X^0_t}{\sqrt{T_t}}-U^0\right),\\
&\tilde f=\varepsilon\left(\frac{f}{T_t^{3/2}}-\frac{\left(U^0b\right)_x}{T_t}+\frac{U^0U^0_x}{\sqrt{T_t}}-\frac{U^0_t}{T_t}-\frac{U^0_{xx}}{T_t}-\frac{T_{tt}U^0}{2T_t^2}\right),
\end{aligned}\tag{9}$$

where $T=T(t)$, $X^0=X^0(t)$, and $U^0=U^0(t,x)$ are arbitrary smooth functions with $T_t>0$, and $\varepsilon=\pm1$. Hence, class (8) is normalized.

As the next step we set the arbitrary element b to zero by means of the transformation

$$\tilde t=t,\quad \tilde x=x,\quad \tilde u=u+b,\quad \tilde f=f-b_t-bb_x-b_{xx},$$

which leads to the simplest reduced form for linearizable generalized Burgers equations containing the single arbitrary smooth function $f=f(t,x)$,

$$u_t+u_{xx}+uu_x+f=0. \tag{10}$$

Substituting $b=\tilde b=0$ into (9) we derive the general form of admissible transformations between two equations of form (10),

$$\begin{aligned}
&\tilde t=T(t),\quad \tilde x=\varepsilon\left(\sqrt{T_t}x+X^0(t)\right),\quad \tilde u=\varepsilon\left(\frac{1}{\sqrt{T_t}}u+\frac{T_{tt}}{2T_t^{3/2}}x+\frac{X^0_t}{T_t}\right),\\
&\tilde f=\varepsilon\left(\frac{1}{T_t^{3/2}}f+\frac{3T_{tt}^2-2T_tT_{ttt}}{4T_t^{7/2}}x+\frac{X^0_tT_{tt}-X^0_{tt}T_t}{T_t^3}\right),
\end{aligned}$$

where $T(t)$ is a monotonically increasing smooth function, $X^0(t)$ is an arbitrary smooth function, and $\varepsilon=\pm1$. Class (10) is normalized. Its linear counterpart consists of equations of the form $v_t+v_{xx}+\left(\frac12\int f dx\right)v=0$.

Every equation from class (6) (resp. (8) or (10)) is connected with its linear counterpart via the Hopf–Cole transformation, as well as the admissible transformations in any of these classes are connected with transformations in the corresponding linear classes.

Consider now the equivalence groupoid of the class of linear equations (5). It is determined by the transformations [18]

$$\begin{aligned}
&\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde v=V^1(t,x)v+V^0(t,x),\\
&\tilde a=\frac{X_x^2}{T_t}a,\quad \tilde b=\frac{1}{T_t}\left(X_xb+X_{xx}a-\frac{2X_xV_x^1}{V^1}a+X_t\right),\\
&\tilde c=\frac{1}{T_t}\left(c-\frac{V_x^1}{V^1}b+\frac{2\left(V_x^1\right)^2-V^1V_{xx}^1}{\left(V^1\right)^2}a-\frac{V_t^1}{V^1}\right),
\end{aligned}\tag{11}$$

where $T=T(t)$, $X=X(t,x)$, $V^1=V^1(t,x)$, and $V^0=V^0(t,x)$ are arbitrary smooth functions in their arguments satisfying $T_tX_xV^1\neq0$ and the classifying condition

$$\left(\frac{V^0}{V^1}\right)_t+a\left(\frac{V^0}{V^1}\right)_{xx}+b\left(\frac{V^0}{V^1}\right)_x+c\frac{V^0}{V^1}=0.$$

This means that V^0/V^1 is a solution of the initial equation (5). The equivalence group $G^\sim$ of class (5) consists of the transformations of form (11) with $V^0=0$. Class (5) is not normalized but semi-normalized because every transformation of form (11) is a composition of the Lie symmetry transformation $\overline{v}=v+V^0/V^1$ of the initial equation and an element of $G^\sim$, namely the transformation (11) with $V^0=0$.

A correspondence between the equivalence groupoids (resp. groups) of classes (5) and (6) can be established using the Hopf–Cole transformation. Indeed,

$$\tilde u=2\frac{\tilde v_{\tilde x}}{\tilde v}=\frac{2}{X_x}\frac{V^1v_x+V_x^1v+V_x^0}{V^1v+V^0}=\frac{1}{X_x}\frac{\left(V^1u+2V_x^1\right)v+2V_x^0}{V^1v+V^0},$$

which can be expressed in terms of (t,x,u) only if $V^0=0$. The transformation component for u in this case is

$$\tilde u=\frac{1}{X_x}u+\frac{2V_x^1}{X_xV^1},\quad\text{i.e.,}\quad U^0=\frac{2V_x^1}{X_xV^1}.$$

The constraint on V^0 is related to the general form of transformations from the equivalence group of class (5). The admissible transformations with $V^0\neq0$ in class (5) have no counterparts in the equivalence groupoid of class (6).

Roughly speaking, the semi-normalization of class (5) of linear equations induces the normalization of class (6) of linearizable equations.

4 Generalized Burgers equations with arbitrary diffusion coefficient

Now we set $F = f(t, x)$, $H^1 = u$, and $H^0 = 0$ in (3). This leads to the class of generalized Burgers equations with an arbitrary nonvanishing smooth coefficient $f = f(t, x)$ of u_{xx},

$$u_t + uu_x + f(t, x)u_{xx} = 0. \tag{12}$$

Class (12) was considered, e.g., in [8,11]. Note that [8] is the first paper where the exhaustive study of admissible transformations of a class of differential equations was carried out. The equivalence group of class (12) is finite dimensional and consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma},$$
$$\tilde{f} = \frac{\kappa^2}{\alpha\delta - \beta\gamma} f, \tag{13}$$

where the constant tuple $(\alpha, \beta, \gamma, \delta, \kappa, \mu_0, \mu_1)$ is defined up to a nonzero multiplier and satisfies the constraints $\alpha\delta - \beta\gamma \neq 0$ and $\kappa \neq 0$. The form of these transformations can be calculated directly or by means of the substitutions $F = f$, $\tilde{F} = \tilde{f}$, $H^1 = u$, $\tilde{H}^1 = \tilde{u}$, and $H^0 = \tilde{H}^0 = 0$ into (4). Since all transformations between any two fixed equations from (12) are exhausted by (13), class (12) is normalized.

The class of equations of the form

$$u_t + uu_x + \big(f(t, x)u_x\big)_x = 0 \tag{14}$$

with f running through the set of nonvanishing smooth functions of (t, x) admits the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = \varkappa\sqrt{|T_t|}x + X^0(t),$$
$$\tilde{u} = \varkappa\frac{\sqrt{|T_t|}}{T_t}u + \varkappa\frac{T_{tt}\sqrt{|T_t|}}{2T_t^2}x + \frac{X^0}{T_t}, \quad \tilde{f} = \varkappa^2 f, \tag{15}$$

where $\varkappa$ is an arbitrary nonzero constant and the smooth functions T and X^0 of t satisfy the equation

$$\varkappa\sqrt{|T_t|}\,T_{tt}f_x + 2T_tX_{tt} - 2T_{tt}X_t = 0. \tag{16}$$

Unlike the previous classes, class (14) is not normalized. At the same time, its subclass singled out by the inequality $f_{xxx} \neq 0$ is normalized. In this case equation (16) split with respect to f_x leads to the constraints $X_{tx} = 0$ and $T_{tt} = 0$. Hence the associated equivalence groupoid is determined by the transformations

$$\tilde{t} = c_1^2 t + c_0, \quad \tilde{x} = \varkappa c_1 x + c_2 t + c_3, \quad \tilde{u} = \frac{\varkappa c_1 u + c_2 t + c_3}{c_1^2}, \quad \tilde{f} = \varkappa^2 f,$$

where c_0, c_1, c_2, c_3, and $\varkappa$ are arbitrary constants with $\varkappa c_1 \neq 0$, which form the equivalence group of this subclass.

The complementary subclass, which is defined by the constraint $f_{xxx} = 0$, i.e., $f = f^2(t)x^2 + f^1(t)x + f^0(t)$, possesses a wider equivalence groupoid. Namely, all admissible transformations in this subclass are of form (15), where the parameter-functions $T = T(t)$ and $X^0 = X^0(t)$ additionally satisfy the system of ODEs

$$4T_tT_{tt}f^2 + 2T_tT_{ttt} - 3T_{tt}^{\,2} = 0, \quad \frac{\varkappa}{2}\sqrt{|T_t|}\,T_{tt}f^1 + T_tX^0_{tt} - T_{tt}X^0_t = 0,$$

and $\varkappa$ is an arbitrary nonzero constant. Although the general solution of this system is parameterized by the arbitrary elements f^1 and f^2 in a nonlocal way,

$$T = \pm\int\left(C_2\int e^{-2\int f^2\,\mathrm{d}t}\,\mathrm{d}t + C_1\right)^{-2}\mathrm{d}t + C_0,$$
$$X^0 = -\frac{\varkappa}{2}\int T_t\int\frac{\sqrt{|T_t|}T_{tt}}{T_t^{\,2}}f^1\,\mathrm{d}t\,\mathrm{d}t + C_3T + C_4,$$

the solution structure is the same for all values of the parameters. In other words, the subclass singled out from class (14) by the constraint $f_{xxx} = 0$ possesses a nontrivial generalized extended equivalence group, and it is normalized with respect to this group. See, e.g., [6, 16, 17, 20–22] for the related definitions and other examples of generalized extended equivalence groups.

Note that the class of equations $u_t + uu_x + f(t)u_{xx} = 0$, which differs from classes (12) and (14) only in arguments of f and is the intersection of these classes, is normalized with respect to the equivalence group (13) of the whole class (12). The group analysis of this class was performed in [3, 23].

5 Classical Burgers equation

Finally, we consider the class consisting of the single equation (1). It is well known [2, 5] that its linear counterpart is the heat equation $v_t + v_{xx} = 0$. The maximal Lie invariance algebra of the classical Burgers equation (1) is spanned by the vector fields [7]

$$\partial_t, \quad 2t\partial_t + x\partial_x - u\partial_u, \quad t^2\partial_t + tx\partial_x + (x - ut)\partial_u, \quad \partial_x, \quad t\partial_x + \partial_u.$$

The complete point symmetry group of equation (1) consists of the transformations

$$\tilde t = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde x = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde u = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma},$$

where $(\alpha, \beta, \gamma, \delta, \kappa, \mu_0, \mu_1)$ is an arbitrary set of constants defined up to a nonzero multiplier, and $\alpha\delta - \beta\gamma = \kappa^2 > 0$. Up to composition with continuous point symmetries, this group contains the single discrete symmetry $(t, x, u) \to (t, -x, -u)$.

6 Conclusion

This paper deals with a hierarchy of normalized classes of generalized Burgers equations. Due to the normalization property, the group classification for these classes can be carried out using the algebraic method. There are several examples of normalized classes the equivalence groups of which are finite dimensional, which is an unexpected result.

It is important to emphasize the following phenomenon in the relationship between the classes of linearizable generalized Burgers equations (6) and linear equations (5) as well as their subclasses via the Hopf–Cole transformation. In view of the superposition principle for solutions of linear equations, class (5) possesses the wider set of admissible transformations than class (6). Transformations associated with the linear superposition depend on arbitrary elements of the corresponding initial equations. This obstacle destroys the normalization property of class (5), though this class is still semi-normalized in the usual sense. At the same time, the linear superposition principle has no counterpart for the linearizable equations among local transformations. This is why class (6) is normalized.

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Generalized IW-Contractions of Low-Dimensional Lie Algebras

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Generalized Inönü–Wigner contractions (generalized IW-contractions) between Lie algebras are discussed. The contemporary principle results on such contractions are reviewed. An algorithm of finding generalized IW-contractions or proving their nonexistence for a certain pair of Lie algebras is presented.

1 Introduction

Usual or generalized Inönü–Wigner contractions (*IW-contractions*) are widely used for realizing contractions of Lie algebras. In fact, the concept of contractions of Lie algebras became well known only after the invention of IW-contractions by Inönü and Wigner in [11, 12]. The contraction from the Poincaré algebra to the Galilean one can be realized as a simple IW-contraction. The other significant example, the contractions from the Heisenberg algebras to the Abelian ones of the same dimensions, which form a symmetry background of limit processes from relativistic and quantum mechanics to classical mechanics, is a trivial contraction. Any Lie algebra is contracted to the Abelian algebra of the same dimension via the IW-contraction corresponding to the zero subalgebra.

The name "generalized Inönü–Wigner contraction" was first used in [9] for so-called p-contractions by Doebner and Melsheimer [7]. Generalizing IW-contractions, Doebner and Melsheimer studied contractions whose matrices become diagonal after choosing appropriate bases of initial and contracted algebras, and diagonal elements being powers of the contraction parameter with real exponents. In the algebraic literature, similar contractions with integer exponents are called *one-parametric subgroup degenerations* [1, 2, 4, 8]. The notion of degenerations of Lie algebras extends the notion of contractions to the case of an arbitrary algebraically closed field and is defined in terms of orbit closures with respect to the Zariski topology. Note that in fact a one-parametric subgroup degeneration is induced by a one-parametric matrix group only under an agreed choice of bases in the corresponding initial and contracted algebras.

All continuous contractions appearing in the physical literature are realized as generalized IW-contractions. The question whether every contraction is equivalent to a generalized IW-contraction was posed in [20]. As shown in [15], the conjecture that the answer is positive was proved in [21] incorrectly. Counterexamples to this conjecture are obviously given by contractions to characteristically nilpotent Lie algebras, which were studied in [1, 2]. Indeed, each proper generalized IW-contraction induces a proper grading on the contracted algebra. There exists a bijection between proper group gradings of a Lie algebra and its nonzero diagonalizable derivations. Each characteristically nilpotent Lie algebra possesses only nilpotent derivations and hence has no nonzero diagonalizable derivations and no proper gradings. Therefore, no contraction to a characteristically nilpotent Lie algebra can be realized by a generalized IW-contraction. Since the minimal dimension for which characteristically nilpotent Lie algebras exist is equal to seven, this fact cannot be used for lower dimensions.

The proof of nonexistence of generalized IW-contractions to a Lie algebra that is not characteristically nilpotent is much more delicate since then the contracted algebra admits proper gradings and the nonexistence of such contraction is related to an inconsistency between filtrations of the initial algebra and gradings of the contracted algebra. Examples on non-universality of generalized IW-contractions of the above kind were first presented in [17] using four-dimensional Lie algebras. Therefore, it may be impossible to realize a well-defined contraction by a generalized IW-contraction even though the contracted algebra admits a wide range of proper gradings. This establishes more precise bounds for applicability of generalized IW-contractions. It was proven in [17] that between four-dimensional Lie algebras over the field of real (resp. complex) numbers, there exist exactly two (resp. one) well-defined contractions that are inequivalent to generalized IW-contractions. The other contractions of four-dimensional Lie algebras were realized in [5, 14, 15] by generalized IW-contractions involving nonnegative integer parameter exponents not greater than three, and the upper bound proved to be exact [17]. Merging the above results leads to the complete description of generalized IW-contractions in dimension four. The similar problem for dimensions five and six is still not studied.

Considering classes of Lie algebras closed with respect to contractions or setting restrictions on the structure of contraction matrices, one can pose the problem on partial universality of generalized IW-contractions for specific kinds of contractions. In particular, generalized IW-contractions of low-dimensional nilpotent Lie algebras were studied in [3]. Analogously, the problem on generalized IW-contractions within the class of almost abelian Lie algebras can be posed since this class is also closed with respect to contractions. The notion of diagonal contraction is an extension of the notion of generalized IW-contraction. Namely, the nonconstant part of a diagonal contraction is still diagonal, but allowed to depend on the contraction parameter in an arbitrary way, rather than to consist of powers of the contraction parameter. At the same time, it was shown in [18] that every diagonal contraction is equivalent to a generalized IW-contraction.

This paper is an overview of progress on the question about universality of generalized IW-contractions. We also present a new method that allows one to construct a generalized IW-contraction if it exists or to proof that a contraction cannot be realized as a generalized IW-contraction. This method forms a basis for a step-by-step algorithm which can directly be implemented in symbolic calculation packages. We apply the algorithm described for optimizing the proof from [17] on that there exists a unique contraction among complex four-dimensional Lie algebras that is not equivalent to a generalized IW-contraction. This immediately implies that among real four-dimensional Lie algebras there exist precisely two contractions that cannot be realized as generalized IW-contractions.

2 General contractions of Lie algebras

Let $\mathfrak{g} = (V, [\cdot,\cdot])$ be an n-dimensional Lie algebra with an underlying vector space $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$, and a Lie bracket $[\cdot,\cdot]$. A common way for defining this algebra is to write down the commutation relations in a fixed basis $\{e_1,\dots,e_n\}$ of V. It suffices to present the nonzero commutators $[e_i,e_j] = c^k_{ij}e_k$, $i<j$, where c^k_{ij} are components of the structure constant tensor of $\mathfrak{g}$. In what follows the indices i, j, k, i', j' and k' run from 1 to n. We use the Einstein notation, implying the summation over the repeated indices unless otherwise explicitly stated. For a matrix A, a^i_j is the entry of A located at the intersection of the ith row and the jth column.

Given a continuous mapping $U\colon (0,1] \to \mathrm{GL}(V)$, we construct a parameterized family of the Lie algebras $\mathfrak{g}_\varepsilon = (V, [\cdot,\cdot]_\varepsilon)$, $\varepsilon \in (0,1]$, which are isomorphic to $\mathfrak{g}$, by defining, for each ε, the new Lie bracket $[\cdot,\cdot]_\varepsilon$ on V according to the formula $[x,y]_\varepsilon = {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$ for all $x,y\in V$. If for any $x,y\in V$ there exists the limit

$$\lim_{\varepsilon\to+0}[x,y]_\varepsilon = \lim_{\varepsilon\to+0} {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y] =: [x,y]_0$$

then $[\cdot,\cdot]_0$ is a well-defined Lie bracket.

Definition 1. The Lie algebra $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$ is called a *one-parametric continuous contraction* (or just a *contraction*) of the Lie algebra $\mathfrak{g}$. The procedure $\mathfrak{g}\to\mathfrak{g}_0$ that provides the algebra $\mathfrak{g}_0$ from the algebra $\mathfrak{g}$ is also called a *contraction*. The parameter ε is called a *contraction parameter*. The contraction $\mathfrak{g}\to\mathfrak{g}_0$ is *trivial* if $\mathfrak{g}_0$ is abelian and *improper* if $\mathfrak{g}_0$ is isomorphic to $\mathfrak{g}$.

Given a basis $\{e_1,\dots,e_n\}$ of V, the operator $U_\varepsilon\in\mathrm{GL}(V)$ is defined by the associated matrix and the above definition can be reformulated in terms of the structure constants c^k_{ij} of the algebra $\mathfrak{g}$ in this basis. Namely, the definition of the Lie bracket $[\cdot,\cdot]_0$ is reduced to the existence of the limit

$$\lim_{\varepsilon\to+0}(U_\varepsilon)^i_{i'}(U_\varepsilon)^j_{j'}({U_\varepsilon}^{-1})^{k'}_k c^k_{ij} =: c^{k'}_{0,i'j'}$$

for all values of i', j' and k', where $c^{k'}_{0,i'j'}$ are components of the well-defined structure constant tensor of the Lie algebra $\mathfrak{g}_0$ in the basis $\{e_1,\dots,e_n\}$. The matrix-valued function U_ε of the contraction parameter ε is called a *contraction matrix*.

Definition 2. Contractions $\mathfrak{g} \to \mathfrak{g}_0$ and $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_0$ are *(weakly) equivalent* if the algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_0$ are isomorphic to $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively.

Under using weak equivalence, only existence and results of contractions are essential and differences in the contractions ways are ignored. Different notions of stronger equivalence can be introduced for taking into account contraction ways. Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of automorphisms of the Lie algebra $\mathfrak{g}$. We identify automorphisms with the associated matrices in the basis fixed.

Definition 3. One-parametric contractions in the same pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}_0)$ with the contraction matrices $U(\varepsilon)$ and $\tilde{U}(\varepsilon)$ are called *strongly equivalent* if there exist such $\delta \in (0, 1]$, mappings $\hat{U}\colon (0, \delta] \to \mathrm{Aut}(\mathfrak{g})$ and $\check{U}\colon (0, \delta] \to \mathrm{Aut}(\mathfrak{g}_0)$ and a continuous monotonic function $\varphi\colon (0, \delta] \to (0, 1]$, $\lim\limits_{\varepsilon\to+0} \varphi(\varepsilon) = 0$ that

$$\tilde{U}_\varepsilon = \hat{U}_\varepsilon U_{\varphi(\varepsilon)} \check{U}_\varepsilon, \quad \varepsilon \in (0, \delta].$$

The notion of contraction is generalized to an arbitrary algebraically closed field in terms of orbit closures within the variety of Lie algebras [1, 2, 4, 8]. Consider an n-dimensional vector space V over an algebraically closed field $\mathbb{F}$. The set $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ of all possible Lie brackets on V is an algebraic subset of the variety $V^* \otimes V^* \otimes V$ of bilinear maps from $V \times V$ to V. Indeed, fixing a basis $\{e_1, \ldots, e_n\}$ of V leads to a bijection among $\mathcal{L}_n$ and

$$\mathcal{C}_n = \big\{(c_{ij}^k) \in \mathbb{F}^{n^3} \mid c_{ij}^k + c_{ji}^k = 0,\ c_{ij}^{i'} c_{i'k}^{k'} + c_{ki}^{i'} c_{i'j}^{k'} + c_{jk}^{i'} c_{i'i}^{k'} = 0\big\}$$

that is defined by $\mu(e_i, e_j) = c_{ij}^k e_k$ for any Lie bracket $\mu \in \mathcal{L}_n$ and any structure constant tensor $(c_{ij}^k) \in \mathcal{C}_n$. Under identifying $\mu \in \mathcal{L}_n$ with the corresponding Lie algebra $\mathfrak{g} = (V, \mu)$, the variety $\mathcal{L}_n$ of possible Lie brackets on V can be called the *variety of n-dimensional Lie algebras* (*over the field* $\mathbb{F}$). The action of group $\mathrm{GL}(V)$ on $\mathcal{L}_n$ is left,

$$(U \cdot \mu)(x, y) = U\big(\mu(U^{-1}x, U^{-1}y)\big) \quad \forall\, U \in \mathrm{GL}(V),\ \forall\, \mu \in \mathcal{L}_n,\ \forall\, x, y \in V,$$

in contrast to the right action, which is traditionally used for contractions in physics and defined by the formula $(U \cdot \mu)(x, y) = U^{-1}\big(\mu(Ux, Uy)\big)$. Of course, this difference is not essential. We use the right action throughout the rest of the paper.

Denote by $\mathcal{O}(\mu)$ the orbit of $\mu \in \mathcal{L}_n$ under the action of $\mathrm{GL}(V)$ and by $\overline{\mathcal{O}(\mu)}$ the closure of $\mathcal{O}(\mu)$ with respect to the Zariski topology on $\mathcal{L}_n$.

Definition 4. A Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *contraction* (or *degeneration*) of a Lie algebra $\mathfrak{g} = (V, \mu)$ if $\mu_0 \in \overline{\mathcal{O}(\mu)}$. The contraction is *proper* if $\mu_0 \in \overline{\mathcal{O}(\mu)} \backslash \mathcal{O}(\mu)$. The contraction is *nontrivial* if $\mu_0 \not\equiv 0$.

In the case of the field of complex numbers, $\mathbb{F} = \mathbb{C}$, the closure of an orbit with respect to the Zariski topology in $\mathbb{C}$ coincides with the closure of this orbit with respect to the Euclidean topology in $\mathbb{C}$. Hence, Definition 4 is reduced to the usual definition of contractions.

3 Generalized IW-contractions

An IW-contraction can be viewed as a result of a parameterized rescaling of the basis elements in a specially chosen bases of the initial and contracted algebras. This rescaling should be singular in a course of the limit process with respect to the associated parameter.

Definition 5. The contraction $\mathcal{C}$ from $\mathfrak{g}$ to $\mathfrak{g}_0$ (over $\mathbb{C}$ or $\mathbb{R}$) is called *a generalized Inönü–Wigner contraction* (or briefly, *generalized IW-contraction)* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where the matrices A and P are nonsingular and constant (i.e., they do not depend on ε) and $W_\varepsilon = \operatorname{diag}(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n})$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

The n-tuple of exponents $(\alpha_1, \dots, \alpha_n)$ is called the *signature* of the generalized IW-contraction $\mathcal{C}$. As the reparameterization $\varepsilon = \tilde{\varepsilon}^\beta$, where $\beta > 0$, results in a generalized IW-contraction strongly equivalent to $\mathcal{C}$, the signature of $\mathcal{C}$ is defined up to a positive multiplier. There should be nonzero components in the signature, since otherwise the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ are isomorphic, i.e., the contraction $\mathcal{C}$ is improper. Moreover, it is enough to consider only signatures with integer components as *any generalized IW-contraction is* (*weakly*) *equivalent to a generalized IW-contraction with an integer signature* (*and with the same associated constant matrices*). Although this claim was believed to hold for a long time, it was rigorously proved much later in [18].

The set of signatures of generalized IW-contractions with nonnegative integer parameter exponents among two fixed algebras can naturally be ordered up to component permutation. Namely, suppose that $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \dots, \beta_n)$, where $\alpha_i, \beta_i \in \mathbb{Z}$, $\alpha_1 \geqslant \dots \geqslant \alpha_n \geqslant 0$ and $\beta_1 \geqslant \dots \geqslant \beta_n \geqslant 0$, are signatures of generalized IW-contractions from $\mathfrak{g}$ to $\mathfrak{g}_0$. Up to equivalence of signatures components of $\bar{\alpha}$ (resp. $\bar{\beta}$) can be additionally assumed coprime. We say that $\bar{\alpha} < \bar{\beta}$ if $\alpha_1 = \beta_1, \dots, \alpha_{j-1} = \beta_{j-1}$ and $\alpha_j < \beta_j$ for some j. A signature of generalized IW-contractions from $\mathfrak{g}$ to $\mathfrak{g}_0$ is called *minimal* if it is minimal with respect to the above ordering. Finding minimal signatures is a necessary step for optimizing the choice of contraction matrices.

A particular case of generalized IW-contractions is given by usual IW-contractions, whose signatures equivalent to tuples of zeros and units. Most contractions of low dimensional Lie algebras are equivalent to such contractions. They present limit processes between Lie algebras with contraction matrices of the simplest possible type. The description of IW-contractions for three- and four-dimensional Lie algebras [6, 10] easily follows from the classifications of subalgebras of such algebras obtained in [16].

Similarly to generalized IW-contractions, we can define the class of diagonal contractions by weakening the related restrictions on the structure of contraction matrices. The contraction $\mathfrak{g} \to \mathfrak{g}_0$ (over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) is called *diagonal* if its matrix U_ε can be represented in the form $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \operatorname{diag}(f_1(\varepsilon), \dots, f_n(\varepsilon))$ for some continuous functions $f_i\colon (0,1] \to \mathbb{F}\backslash\{0\}$. The class of diagonal contractions is wider than the class of generalized IW-contractions and, at the same time, *any contraction from the first class is equivalent to a generalized IW-contraction involving only integer parameter powers* [18].

For some theoretical studies, it is convenient to set A and P equal to the identity matrix by changing bases in the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ or, in other words, replacing these algebras by isomorphic ones. However, this does not properly work for specific Lie algebras. If $U_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1}, \dots, \varepsilon^{\alpha_n})$ then the structure constants of the resulting algebra $\mathfrak{g}_0$ are given by the formula

$$c^k_{0,ij} = \lim_{\varepsilon\to+0} c^k_{ij}\,\varepsilon^{\alpha_i+\alpha_j-\alpha_k} = \begin{cases} c^k_{ij} & \text{if} \quad \alpha_i+\alpha_j=\alpha_k, \\ c^k_{0,ij}=0, & \text{otherwise} \end{cases}$$

with no sums over the repeated indices. Hence matrix U_ε defines a generalized IW-contraction if and only if $\alpha_i+\alpha_j \geqslant \alpha_k$ for any (i,j,k) when $c^k_{ij} \neq 0$. The conditions for existence of generalized IW-contractions and on the structure of contracted algebras are reformulated in the basis-independent fashion in terms of gradings of contracted algebras associated with filtrations on initial algebras [8,13]. In particular, the contracted algebra $\mathfrak{g}_0$ has to possess a derivation whose matrix is diagonalizable to $\mathrm{diag}(\alpha_1, \dots, \alpha_n)$.[1]

It is obvious that the generalized IW-contractions defined by the matrices $U_\varepsilon = AW_\varepsilon P$ and $\tilde U_\varepsilon = \tilde A \tilde W_\varepsilon \tilde P$, where

$$\tilde A = MAN, \quad \tilde P = N^{-1}PM_0, \quad \tilde W_\varepsilon = \mathrm{diag}(\varepsilon^{\beta\alpha_1}, \dots, \varepsilon^{\beta\alpha_n}) \text{ for some } \beta>0,$$

are strongly equivalent. Here M and M_0 are automorphisms matrices for algebras $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, and N is a matrix commuting with the diagonal parts W_ε and $\tilde W_\varepsilon$. In other words, the matrix N corresponds to an arbitrary change of basis within components of the grading of $\mathfrak{g}_0$ associated with W_ε and $\tilde W_\varepsilon$. Note that the diagonal matrices W_ε and $\tilde W_\varepsilon$ induce the same grading of $\mathfrak{g}_0$. Summing up the above consideration, we can say that a certain amount of freedom in choosing the matrices A and P is preserved even after fixing the canonical commutation relations.

Let the canonical basis of $\mathfrak{g}_0$ be associated with a grading which is isomorphic to the one induced by the matrix W_ε. Then the matrix P can be represented as a product $P_{\rm grad}P_{\rm aut}$, where $P_{\rm grad}$ and $P_{\rm aut}$ are matrices of a change of basis within the graded components and of an automorphism of $\mathfrak{g}_0$, respectively. Therefore, in such a case we can get rid of the matrix P by setting it equal to the unit matrix up to the above equivalence. If $U_\varepsilon = AW_\varepsilon$, the structure constants of $\mathfrak{g}_0$ read

$$c^k_{0,ij} = \lim_{\varepsilon\to+0} a^{i'}_i a^{j'}_j b^k_{k'} c^{k'}_{i'j'}\,\varepsilon^{\alpha_i+\alpha_j-\alpha_k}, \tag{1}$$

[1] An operator D in $\mathrm{GL}(V)$ is a derivation of a Lie algebra $\mathfrak{g}$ if $D[x,y]=[Dx,y]+[x,Dy]$ for any $x,y\in V$. The derivations of $\mathfrak{g}$ constitute a Lie algebra called the derivation algebra $\mathrm{Der}(\mathfrak{g})$ of $\mathfrak{g}$. After the basis $\{e_1,\dots,e_n\}$ of V is fixed, entries of the matrix $\Gamma=(\gamma^i_j)$ of a derivation D satisfy the system: $c^{k'}_{ij}\gamma^k_{k'} = c^k_{i'j}\gamma^{i'}_i + c^k_{ij'}\gamma^{j'}_j$.

where $A=(a^i_j)$, $A^{-1}=(b^i_j)$, and there is no sum over i, j and k. We introduce the notation

$$x^k_{ij}=a^{i'}_i a^{j'}_j b^k_{k'} c^{k'}_{i'j'}. \tag{2}$$

It is clear that $x^k_{ij}=-x^k_{ji}$. The condition (1) implies that there are three cases for values of x^k_{ij} depending on the sign of $\alpha_i+\alpha_j-\alpha_k$. The first two cases,

$$\begin{aligned} x^k_{ij}&=0 && \text{if}\quad \alpha_i+\alpha_j-\alpha_k<0,\\ x^k_{ij}&=c^k_{0,ij} && \text{if}\quad \alpha_i+\alpha_j-\alpha_k=0, \end{aligned} \tag{3}$$

can be formally united in the single case $x^k_{ij}=c^k_{0,ij}$, $\alpha_i+\alpha_j-\alpha_k\leqslant 0$ since $c^k_{0,ij}$ is also zero if $\alpha_i+\alpha_j-\alpha_k<0$. The entries x^k_{ij} with $\alpha_i+\alpha_j-\alpha_k>0$ are not constrained due to the limit process. They can be computed jointly with a^i_j from the system of algebraic equations formed by equations (2) defining x^k_{ij}, where the entries x^k_{ij} with $\alpha_i+\alpha_j-\alpha_k\leqslant 0$ should be substituted from (3). We re-arrange this system. Namely, we multiply the tensors on the left and right hand sides of (2) by $a^{k''}_k$ and contract with respect to the index k. We additionally arrange the derived equations $a^{i'}_i a^{j'}_j c^{k''}_{i'j'}=a^{k''}_k x^k_{ij}$ by re-denoting indices $k\to k'$ and $k''\to k$ and obtain the system of quadratic equations

$$a^{i'}_i a^{j'}_j c^k_{i'j'}=a^k_{k'} x^{k'}_{ij} \tag{4}$$

with respect to the entries of A and x^k_{ij} with $\alpha_i+\alpha_j-\alpha_k>0$, where $i<j$ and $x^k_{ij}=c^k_{0,ij}$ if $\alpha_i+\alpha_j-\alpha_k\leqslant 0$. For low-dimensional Lie algebras, system (4) can easily be solved using a symbolic computation system, e.g., `Maple`.

In view of the above consideration, we can suggest the following algorithm for realizing a contraction between Lie algebras by a generalized IW-contraction or proving that such a realization is not possible. We fix two Lie algebras, $\mathfrak{g}$ and $\mathfrak{g}_0$. The starting point of the algorithm is the assumption that there exists the contraction $\mathfrak{g}\to\mathfrak{g}_0$.

1. It is convenient to begin the study of generalized IW-contractions with an exhaustive study of simple IW-contractions of $\mathfrak{g}$. This is equivalent to the study of sub-algebraic structure of $\mathfrak{g}$. If $\mathfrak{g}_0$ is among the listed contracted algebras, then the algorithm is completed. Otherwise, we continue the study, excluding simple IW-contractions from it.

2. We find the algebra of derivations of the contracted algebra $\mathfrak{g}_0$ and select diagonalizable differentiations. There exist a one-to-one correspondence between such differentiations and gradings of $\mathfrak{g}_0$. In other words, we study gradings of contracted Lie algebra in terms of diagonalizable differentiations. The consideration of the gradings aims at resolving a twofold challenge—to obtain possible values of parameter exponents of contraction matrices and to understand the structure of constant components of these matrices.

3. The signature of any generalized IW-contraction from $\mathfrak{g}$ to $\mathfrak{g}_0$ coincides with the diagonal of a diagonalized differentiation of $\mathfrak{g}_0$. Further restrictions on parameter exponents follow from the absence of simple IW-contractions from $\mathfrak{g}$ to $\mathfrak{g}_0$. We should use the fact that up to positive multiplier, any signature associated with a simple IW-contraction consists of zeros and units.
4. We fix a proper signature. The matrix P in the representation $U_\varepsilon = AW_\varepsilon P$ of the contraction matrix U_ε is determined up to changes of basis within graded components and up to automorphisms of the contracted algebra. Often the matrix P provides an isomorphism between gradings of g_0. Then we can set P equal to the identity matrix. If for a fixed signature there exist a few non-isomorphic gradings, they correspond to inequivalent values of the parameter-matrix P. We separately set each of these values to the identity matrix by carrying out the corresponding change the canonical basis of g_0 and continuing the consideration with the new structure constants. In the above two cases, which are typical for low-dimensional algebras, only the entries of the matrix A remain unknown. These entries satisfy a system of algebraic equations implied by the condition (1). If a signature possesses a parameterized family of inequivalent gradings, the algorithm requires further development.
5. A significant part of cases for parameter exponents can be ignored as the associated systems of equations for entries of the matrix A are extensions of their counterparts for other cases and hence the consistency of the former systems implies that of the latter ones.
6. We consider each tuple of parameter exponents for which the corresponding system of algebraic equations for entries of the matrix A is minimal. This nonlinear system is represented in a specific form that allows us to apply methods of solving systems of multi-variable quadratic equations.

4 Non-existence of generalized IW-contractions between four-dimensional Lie algebras

Almost all contractions of four-dimensional Lie algebras were realized in [5,15] by generalized IW-contractions. The exceptions are exhausted by the contractions

$$2A_{2.1} \to A_1 \oplus A_{3.2}, \quad A_{4.10} \to A_1 \oplus A_{3.2} \quad \text{and} \quad 2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$$

for the real and complex cases, respectively, since the complexification of the algebra $A_{4.10}$ is isomorphic to the complexification $2\mathfrak{g}_{2.1}$ of the algebra $2A_{2.1}$. The above real Lie algebras are defined by the following nonzero commutation relations in their canonical bases:

$$\begin{aligned}
2A_{2.1}\colon &\quad [e_1, e_2] = e_1,\ [e_3, e_4] = e_3;\\
A_1 \oplus A_{3.2}\colon &\quad [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3;\\
A_{4.10}\colon &\quad [e_1, e_3] = e_1,\ [e_2, e_3] = e_2,\ [e_1, e_4] = -e_2,\ [e_2, e_4] = e_1.
\end{aligned}$$

For the complex case, $\mathfrak{g}_{\dots}$ denotes the complexification of the corresponding algebra $A_{\dots}$.

In fact, the contraction $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ cannot be realized as a generalized IW-contraction. First this assertion was proved in [17]. Here we provide an optimized proof of it, which illustrates algorithm presented in the previous section. As all other contractions relating complex four-dimensional Lie algebras are equivalent to generalized IW-contraction, the following theorem is true [17].

Theorem 1. *There exists a unique contraction among complex four-dimensional Lie algebras* (*namely,* $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$) *that cannot be realized by a generalized Inönü–Wigner contraction.*

Proof. We prove this theorem by contradiction. Suppose that the contraction $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ is equivalent to a generalized IW-contraction. We begin with finding the gradings of the algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ that can be associated with this contraction.

The derivation algebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ consists of linear mappings whose matrices in the canonical basis have the form [19]

$$\Gamma = \begin{pmatrix} \gamma^1_1 & 0 & 0 & \gamma^1_4 \\ 0 & \gamma^2_2 & \gamma^2_3 & \gamma^2_4 \\ 0 & 0 & \gamma^2_2 & \gamma^3_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the superscript and subscript of a matrix entry denote the corresponding row and column numbers, respectively, and all entries expect identically zero ones are arbitrary. Therefore, the matrix of any diagonalizable derivation of $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ is reduced, by changing the basis, to the form $\operatorname{diag}(\beta, \alpha, \alpha, 0)$, which also gives the only possible form $(\beta, \alpha, \alpha, 0)$ of signatures for generalized IW-realizations of $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$. In other words, each grading of the contraction $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ admits at most three nonzero components, and one of these components correspond to zero exponent. The signature $(\beta, \alpha, \alpha, 0)$ includes at least two nonzero values since otherwise the contraction $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ would be equivalent to a usual IW-contraction, which is not true [10]. Hence the contraction $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ may induce only gradings with three nonzero components, L_β, L_α and L_0, where $0 \neq \alpha \neq \beta \neq 0$, $\dim L_\beta = \dim L_0 = 1$ and $\dim L_\alpha = 2$. We prove that any such grading $\tilde{\mathcal{G}}$ is equivalent, up to automorphisms of $\mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$, to the grading $\mathcal{G}$ with $L_\beta = \langle e_1 \rangle$, $L_\alpha = \langle e_2, e_3 \rangle$ and $L_0 = \langle e_4 \rangle$.

Indeed, let Γ be the matrix (in the canonical basis $\{e_i\}$) of a derivation associated with a grading $\tilde{\mathcal{G}} = (\tilde{L}_\beta, \tilde{L}_\alpha, \tilde{L}_0)$. Then $\gamma^2_3 = 0$ since the matrix Γ is diagonalizable. We choose a new basis $\tilde{e}_i = e_j s^j_i$, where $\det(s^i_j) \neq 0$, so that $\tilde{L}_\beta = \langle \tilde{e}_1 \rangle$, $\tilde{L}_\alpha = \langle \tilde{e}_2, \tilde{e}_3 \rangle$ and $L_0 = \langle \tilde{e}_4 \rangle$. Upon this choice the matrix Γ has to be transformed into a diagonal form. Hence $s^2_1 = s^3_1 = s^4_1 = 0$ and $s^1_2 = s^4_2 = s^1_3 = s^4_3 = 0$. Then the change of basis in question can be represented as a composition of the change of basis within the graded components

$$\hat e_1 = e_1 s_1^1, \quad \hat e_2 = e_2 s_2^2 + e_3 s_2^3, \quad \hat e_3 = e_2 s_3^2 + e_3 s_3^3, \quad \hat e_4 = e_4 s_4^4$$

with $s_1^1 s_4^4 (s_2^2 s_3^3 - s_2^3 s_3^2) \neq 0$, which does not affect Γ in any substantial way, and of the automorphism

$$\tilde e_1 = \hat e_1, \quad \tilde e_2 = \hat e_2, \quad \tilde e_3 = \hat e_3, \quad \tilde e_4 = \hat e_4 + \hat e_1 \hat s_4^1 + \hat e_2 \hat s_4^2 + \hat e_3 \hat s_4^3$$

setting $\gamma_4^1 = \gamma_4^2 = \gamma_4^3 = 0$. (Here the coefficients $\hat s_4^1$, $\hat s_4^2$ and $\hat s_4^3$ are expressed via s_j^i.) This means that up to the automorphism we can assume $\tilde L_\beta = L_\beta$, $\tilde L_\alpha = L_\alpha$ and $\tilde L_0 = L_0$.

The general form of the matrices for the generalized IW-contractions from $2\mathfrak g_{2.1}$ to $\mathfrak g_1 \oplus \mathfrak g_{3.2}$ is $U_\varepsilon = AW_\varepsilon P$, where A and P are constant nonsingular matrices and $W_\varepsilon = \operatorname{diag}(\varepsilon^\beta, \varepsilon^\alpha, \varepsilon^\alpha, 1)$. Since P is a matrix of transition between two graded bases with the same signature $(\beta, \alpha, \alpha, 0)$, it can be represented as $P = P_{\rm grad} P_{\rm aut}$, where $P_{\rm grad}$ is a matrix of basis change within the graded components and $P_{\rm aut}$ is an automorphism matrix of $\mathfrak g_1 \oplus \mathfrak g_{3.2}$. The matrix $P_{\rm grad}$ commutes with W_ε and is absorbed into the matrix A by passing from A to $\hat A = AP_{\rm grad}$. The matrix $P_{\rm aut}$ can be ignored as it does not affect the commutation relations of the contracted algebra $\mathfrak g_1 \oplus \mathfrak g_{3.2}$. Therefore, it suffices to study only contraction matrices of the form $U_\varepsilon = AW_\varepsilon$ assuming that P is the unit matrix.

Each of the transformed structure constants $(U_\varepsilon)_{i'}^i (U_\varepsilon)_{j'}^j ({U_\varepsilon}^{-1})_k^{k'} c_{ij}^k$ includes a single power of the contraction parameter ε. The set of possible values for exponents of such powers is

$$\mathcal E = \{0,\ \alpha,\ \beta,\ \alpha+\beta,\ \alpha-\beta,\ \beta-\alpha,\ 2\alpha,\ 2\alpha-\beta\}.$$

We treat the two possible cases $\alpha > \beta$ and $\beta > \alpha$ separately.

In the first case, $\alpha > \beta$, we may have two definitely nonpositive exponents, 0 and $\beta - \alpha$. Then we set $x_{ij}^k = c_{0,ij}^k$ for the corresponding values i, j and k and substitute such x_{ij}^k into (4). If additionally α and β are positive, we have no more nonpositive elements in $\mathcal E$ and, therefore, no other constraints for x_{ij}^k. In other words, the system of equations for x_{ij}^k and a_j^i associated with the condition $\alpha > \beta > 0$ is minimal among (i.e., is a subsystem of) all such systems arising in the case $\alpha > \beta$. Moreover, any such system does not involve the parameters α and β. For this reason specific values of them are not essential, and we can set, e.g., $\alpha = 2$ and $\beta = 1$ for convenience of symbolic computation. The resulting system

$$\begin{aligned}
&a_4^1 x_{41}^4 = a_4^1 a_1^2 - a_4^2 a_1^1, && a_1^1 x_{42}^1 - a_2^1 + a_4^1 x_{42}^4 = a_4^1 a_2^2 - a_4^2 a_2^1,\\
&a_4^2 x_{41}^4 = 0, && a_1^2 x_{42}^1 - a_2^2 + a_4^2 x_{42}^4 = 0,\\
&a_4^3 x_{41}^4 = a_4^3 a_1^4 - a_4^4 a_1^3, && a_1^3 x_{42}^1 - a_2^3 + a_4^3 x_{42}^4 = a_4^3 a_2^4 - a_4^4 a_2^3,\\
&a_4^4 x_{41}^4 = 0, && a_1^4 x_{42}^1 - a_2^4 + a_4^4 x_{42}^4 = 0,
\end{aligned}$$

$$
\begin{aligned}
&a^1_1x^1_{43} - a^1_2 - a^1_3 + a^1_4x^4_{43} = a^1_4a^2_3 - a^2_4a^1_3,\\
&a^2_1x^1_{43} - a^2_2 - a^2_3 + a^2_4x^4_{43} = 0,\\
&a^3_1x^1_{43} - a^3_2 - a^3_3 + a^3_4x^4_{43} = a^3_4a^4_3 - a^4_4a^3_3,\\
&a^4_1x^1_{43} - a^4_2 - a^4_3 + a^4_4x^4_{43} = 0
\end{aligned}
$$

has no solution with $\det(a^i_j) \neq 0$, which is easily checked by `Maple`.

The second case, $\beta > \alpha$, is considered in a similar way. If additionally $\beta > \alpha > 0$ and $2\alpha > \beta$, the set $\mathcal{E}$ contains only two nonpositive elements, 0 and $\alpha - \beta$, and both these elements are always nonpositive. Hence the system of equations for x^k_{ij} and a^i_j associated with the inequalities $\beta > \alpha > 0$ and $2\alpha > \beta$ is minimal among (i.e., is a subsystem of) all such systems arising in the case $\beta > \alpha$. Specific values of the parameters α and β are again inessential as any such system does not involve them, and we can set, e.g., $\alpha = 2$ and $\beta = 3$ for convenience of symbolic computation. It can again be checked by `Maple` that the resulting system of this case also has no solution with $\det(a^i_j) \neq 0$.

$$
\begin{aligned}
&a^1_2x^2_{41} + a^1_3x^3_{41} + a^1_4x^4_{41} = a^1_4a^2_1 - a^2_4a^1_1, &&-a^1_2 + a^1_4x^4_{42} = a^1_4a^2_2 - a^2_4a^1_2,\\
&a^2_2x^2_{41} + a^2_3x^3_{41} + a^2_4x^4_{41} = 0, &&-a^2_2 + a^2_4x^4_{42} = 0,\\
&a^3_2x^2_{41} + a^3_3x^3_{41} + a^3_4x^4_{41} = a^3_4a^4_1 - a^4_4a^3_1, &&-a^3_2 + a^3_4x^4_{42} = a^3_4a^4_2 - a^4_4a^3_2,\\
&a^4_2x^2_{41} + a^4_3x^3_{41} + a^4_4x^4_{41} = 0, &&-a^4_2 + a^4_4x^4_{42} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-a^1_2 - a^1_3 + a^1_4x^4_{43} = a^1_4a^2_3 - a^2_4a^1_3,\\
&-a^2_2 - a^2_3 + a^2_4x^4_{43} = 0,\\
&-a^3_2 - a^3_3 + a^3_4x^4_{43} = a^3_4a^4_3 - a^4_4a^3_3,\\
&-a^4_2 - a^4_3 + a^4_4x^4_{43} = 0.
\end{aligned}
$$

□

Corollary 1. *There exist precisely two contractions among real four-dimensional Lie algebras* (*namely,* $2A_{2.1} \to A_1 \oplus A_{3.2}$ *and* $A_{4.10} \to A_1 \oplus A_{3.2}$) *which cannot be realized as generalized Inönü–Wigner contractions.*

Combining the results of [5, 15, 17] also yields the following assertion.

Theorem 2. *Any generalized Inönü–Wigner contraction between complex* (*resp. real*) *four-dimensional Lie algebras is equivalent to a generalized Inönü–Wigner contraction whose signature consists of nonnegative integers that are not greater than three. This upper bound is exact. The only generalized Inönü–Wigner contractions necessarily involving exponents which do not belong to* $\{0, 1, 2\}$ *are* $2A_{2.1} \to A_{4.1}$, $A_{4.10} \to A_{4.1}$ *and* $\mathrm{so}(3) \oplus A_1 \to A_{4.1}$ *in the real case and* $2\mathfrak{g}_{2.1} \to \mathfrak{g}_{4.1}$ *in the complex case. The minimal tuple of exponents for each of these contractions is* $(3, 2, 1, 1)$.

5 Conclusion

Although generalized IW-contractions of four-dimensional Lie algebras over the real and complex fields were exhaustively studied in [5, 15, 17], there are still a number of open problems about generalized IW-contractions in higher dimensions including even dimensions five and six. In particular, no examples on non-universality of generalized IW-contractions contractions for five-dimensional (resp. six-dimensional) Lie algebras are known. Note that all contractions of five-dimensional nilpotent Lie algebras and elementary contractions of six-dimensional nilpotent Lie algebras proved to be equivalent to generalized IW-contractions [3, 8]. (A contraction is called elementary if it is not equivalent to a repeated contraction.) The problem posed can be additionally specified to the following question: Is there an elementary contraction between five- or six-dimensional Lie algebras that cannot be realized by generalized IW-contractions? The examples on non-universality of generalized IW-contractions for four-dimensional Lie algebras over the real and complex fields are in fact given by repeated contractions. Both the corresponding contractions in the real case $2A_{2.1} \to A_1 \oplus A_{3.2}$ and $A_{4.10} \to A_1 \oplus A_{3.2}$ and their complex counterpart $2\mathfrak{g}_{2.1} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$ can be represented as compositions of generalized IW-contractions, $2A_{2.1} \to A^0_{4.8} \to A_1 \oplus A_{3.2}$, $A_{4.10} \to A^0_{4.8} \to A_1 \oplus A_{3.2}$, $2\mathfrak{g}_{2.1} \to \mathfrak{g}^0_{4.8} \to \mathfrak{g}_1 \oplus \mathfrak{g}_{3.2}$, respectively. Here the real Lie algebra $A^0_{4.8}$ is defined by the following nonzero commutation relations in its canonical basis:

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2,$$

and $\mathfrak{g}^0_{4.8}$ denotes the complexification of the algebra $A^0_{4.8}$. At the same time, the contraction between the seven-dimensional characteristically nilpotent Lie algebras $\mathfrak{g}_F$ and $\mathfrak{g}_E$, which was constructed by Burde [2], is elementary since the dimensions of the orbits of these algebras are $\dim \mathcal{O}(\mathfrak{g}_F) = 39$ and $\dim \mathcal{O}(\mathfrak{g}_E) = 38$, i.e., their difference equals 1.

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On Differential Equations with Infinite Conservation Laws

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We consider partial differential equations admitting infinite-dimensional symmetry algebras parametrized by arbitrary functions of dependent variables and their derivatives. These symmetries were shown to lead to infinite sets of (essential) conservation laws unlike infinite-dimensional symmetries with arbitrary functions of independent variables. We discuss the problem of finding all Lagrangian PDE's of the second order that possess an infinite set of conservation laws with an arbitrary function of the dependent variable and its first and second derivatives. We show that the problem leads to two classes of PDE's, and among them are equations of Liouville type.

1 Introduction

Infinite-dimensional (or infinite) symmetry algebras parametrized by arbitrary functions and their relations to conservation laws have been studied considerably less extensively than finite-dimensional Lie symmetry groups and corresponding conservation laws. According to the classic Noether result (the Second Noether Theorem [9], see also [10]) infinite variational symmetries with arbitrary functions of all independent variables do not lead to conservation laws but to a certain relation between equations of the original differential system (which means that the original system is underdetermined). Infinite variational symmetries with arbitrary functions of not all independent variables were studied in [12] and shown to lead to a finite number of essential (integral) local conservation laws. Each essential conservation law is determined by a specific form of boundary conditions, see e.g. [12, 13, 15, 16]. It was noted in [14] that the situation with infinite symmetry algebras parametrized by arbitrary functions of dependent variables is radically different leading to an infinite set of essential conservation laws (local conserved densities). Two known examples of this situation are equations of Liouville type, (see [22–24]) that can be integrated by the Darboux method, see e.g. [1, 2, 7, 21], and hydrodynamic-type equations [4, 20]. A general case of one scalar Lagrangian equation of the second order admitting infinite variational symmetries with arbitrary functions of a dependent variable u ($u = u(x,t)$) and its first derivatives u_x, u_t was analyzed in [17] where classes of partial differential equations possessing infinite symmetries with arbitrary functions of a dependent variable and its first derivatives were found along with corresponding infinite sets of essential conserved quantities. Extension of this work to the case of systems of two equations for two dependent variables was demonstrated in [18]. Thus, equations with infinite-dimensional variational symmetry algebras parameterized by arbitrary functions of dependent variables have a remarkable property: they possess infinite sets of essential conservation laws.

In the present paper we discuss a problem of finding all Lagrangian scalar PDE's of the second order that possess an infinite set of conservation laws with an arbitrary function of the dependent variable and its first and second derivatives; in general form this problem has not been posed or solved in the literature. We show that this problem leads to two classes of PDE's and analyze equations of each class.

2 Infinite symmetries and essential conservation laws

Let us briefly outline the approach we follow, for details see [12, 17]. By a conservation law for a differential system

$$\omega^a(x, u, u_{(1)}, u_{(2)}, \dots) = 0, \qquad a = 1, \dots, n,$$

is meant a divergence expression

$$\mathrm{D}_i K_i(x, u, u_{(1)}, u_{(2)}, \dots) \doteq 0,$$

vanishing for all solutions of the original system ($\doteq$). Here and in what follows $x = (x^1, x^2, \dots, x^{m+1})$ and $u = (u^1, u^2, \dots, u^n)$ are the tuples of independent and dependent variables, respectively, and $x^{m+1} = t$; $u_{(r)}$ is the tuple of rth-order derivatives of u, $r = 1, 2, \dots$; ω^a and K_i are differential functions, i.e., smooth functions of x, u and a finite number of derivatives of u; see [10] for a precise definition of differential functions. The index a runs from 1 to n and the indices i and j run from 1 to $m+1$, $a = 1, \dots, n$ and $i, j = 1, \dots, m+1$. Summation over repeated indices is assumed.

Two conservation laws K and $\tilde{K}$ are equivalent if they differ by a trivial conservation law [10]. A conservation law $\mathrm{D}_i P_i \doteq 0$ is trivial if a linear combination of two kinds of triviality is taking place: 1. $(m{+}1)$-tuple P vanishes on the solutions of the original system: $P_i \doteq 0$. 2. The divergence identity is satisfied in the whole space: $\mathrm{D}_i P_i = 0$.

By an **essential** conservation law [12], we mean such non-trivial conservation law $\mathrm{D}_i K_i \doteq 0$, which gives rise to a non-vanishing conserved quantity

$$\mathrm{D}_t \int_D K_t \, dx^1 dx^2 \cdots dx^m \doteq 0, \qquad x \in D \subset \mathbb{R}^{m+1}, \qquad K_t \mathrel{\dot{\neq}} 0. \tag{1}$$

We consider functions $u^a = u^a(x)$ defined on a region D of $(m{+}1)$-dimensional space-time. Let

$$S = \int_D L(x, u, u_{(1)}, \dots) \, d^{m+1}x$$

be the action functional, where L is the Lagrangian density. Then the equations of motion are

$$E^a(L) \equiv \omega^a(x, u, u_{(1)}, u_{(2)}, \dots) = 0, \tag{2}$$

where

$$E^a = \frac{\partial}{\partial u^a} - \sum_i \mathrm{D}_i \frac{\partial}{\partial u^a_i} + \sum_{i \leqslant j} \mathrm{D}_i \mathrm{D}_j \frac{\partial}{\partial u^a_{ij}} + \cdots \tag{3}$$

is the Euler–Lagrange operator. Consider an infinitesimal (one-parameter) transformation with the canonical infinitesimal operator

$$X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + \sum_i (\mathrm{D}_i \alpha^a) \frac{\partial}{\partial u^a_i} + \sum_{i \leqslant j} (\mathrm{D}_i \mathrm{D}_j \alpha^a) \frac{\partial}{\partial u^a_{ij}} + \cdots, \tag{4}$$
$$\alpha^a = \alpha^a(x, u, u_{(1)}, \dots).$$

Variation of the functional S under the transformation with operator X_α is

$$\delta S = \int_D X_\alpha L \, d^{m+1} x \,. \tag{5}$$

X_α is a variational (Noether) symmetry if

$$X_\alpha L = \mathrm{D}_i M_i, \tag{6}$$

where $M_i = M_i(x, u, u_{(1)}, \dots)$ are smooth functions of their arguments. The Noether identity [11] (see also [5], or [19] for the version used here) relates the operator X_α to E^a,

$$X_\alpha = \alpha^a E^a + \mathrm{D}_i R_{\alpha i}, \tag{7}$$

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u^a_i} + \left\{ \sum_{k \geqslant i} (\mathrm{D}_k \alpha^a) - \alpha^a \sum_{k \leqslant i} \mathrm{D}_k \right\} \frac{\partial}{\partial u^a_{ik}} + \cdots . \tag{8}$$

Applying the identity (7) with (8) to L and using (6), we obtain

$$\mathrm{D}_i (M_i - R_{\alpha i} L) = \alpha^a \omega^a, \tag{9}$$

which on the solution manifold ($\omega = 0$, $\mathrm{D}_i \omega = 0$, …) is

$$\mathrm{D}_i (M_i - R_{\alpha i} L) \doteq 0, \tag{10}$$

leads to the statement of the First Noether Theorem: any variational one-parameter symmetry transformation with infinitesimal operator X_α (4) gives rise to a conservation law (10). Consider now infinite variational symmetries.

2.1 Arbitrary functions of independent variables

Consider an infinite variational symmetry with a characteristic α of the form

$$\alpha^a = \alpha^{a0} p(x) + \alpha^{ai} \mathrm{D}_i p(x) + \sum_{i \leqslant j} \alpha^{aij} \mathrm{D}_i \mathrm{D}_j p(x) + \cdots , \tag{11}$$

where $p(x)$ runs through a set of smooth functions of x and the coefficients α^{a0}, α^{ai}, α^{aij}, ... are some differential functions.

The Second Noether Theorem [9] deals with the case when $p(x)$ is an arbitrary function of all base variables of the space. The situation when $p(x)$ is an arbitrary function of only some of base variables was analyzed in [12]. For a Noether symmetry generator X_α (4) in this case we have

$$\delta S = \int_D \delta L \, d^{m+1}x = \int_D X_\alpha L \, d^{m+1}x = \int_D \mathrm{D}_i M_i \, d^{m+1}x = 0. \tag{12}$$

Therefore, the following conditions for M_i (Noether boundary conditions) should be satisfied [12]

$$M_i(x, u, \dots)\Big|_{x \xrightarrow{i} \partial D} = 0, \qquad i = 1, \dots, m+1, \tag{13}$$

where $\xrightarrow{i}$ denotes the limit along the ith axis. Equations (13) are usually satisfied for a "regular" asymptotic behavior: $u^a \to 0$ and $u^a_i \to 0$ at infinity, or for periodic solutions. Integrating equation (10) over the space $\Omega\,(x^1, x^2, \dots, x^m)$ we get

$$\int_\Omega dx^1 \dots dx^m \mathrm{D}_t (M_t - R_{\alpha t} L) \doteq \int_\Omega dx^1 \dots dx^m \sum_{i=1}^{m} \mathrm{D}_i (R_{\alpha i} L - M_i). \tag{14}$$

Applying the Noether boundary condition (13) and requiring the LHS of (14) to vanish on the solution manifold leads to the "strict" boundary conditions [12]

$$R_{\alpha 1} L\Big|_{x \xrightarrow{1} \partial\Omega} = R_{\alpha 2} L\Big|_{x \xrightarrow{2} \partial\Omega} = \cdots = R_{\alpha m} L\Big|_{x \xrightarrow{m} \partial\Omega} = 0. \tag{15}$$

In the case $L = L(x, u, u_{(1)})$, the strict boundary conditions (15) take the simple form

$$\alpha^a \frac{\partial L}{\partial u^a_l}\Big|_{x \xrightarrow{l} \partial\Omega} = 0, \qquad l = 1, \dots, m. \tag{16}$$

It was shown in [12] that in the case of an arbitrary function of (not all) independent variables conditions (13) and (15) necessary for the existence of Noether conservation laws allow only a finite number of essential conservation laws corresponding to infinite symmetry of the system.

2.2 Arbitrary functions of dependent variables

Consider now infinite symmetries whose characteristics contain an arbitrary smooth functions f of dependent variables and their derivatives [12], i.e.

$$\alpha^a = \alpha^{a0} f + \alpha^{as}\partial_s f + \cdots ,$$

where index s numerates arguments of the function $f = f(u, u_{(1)}, \dots)$ and the coefficients α^{a0}, α^{as}, ... are some differential functions. The conservation law (10) then has the form

$$\mathrm{D}_l(M_l - R_{\alpha l}L) + \mathrm{D}_t(M_t - R_{\alpha t}L) \doteq 0, \qquad l = 1, \dots, m, \tag{17}$$

where

$$\begin{aligned} M_i &= M_i^0 f + M_i^s \partial_s f + \cdots , \\ R_{\alpha i}L &= P_i^0 f + P_i^s \partial_s f + \cdots \end{aligned} \tag{18}$$

for some differential functions M_i^0, M_i^s, ..., and P_i^0, P_i^s, ... In order for the system to possess (Noether) local conserved quantities, both Noether (13) and strict boundary conditions (15) have to be satisfied. Let

$$f(u, u_{(1)}, \dots) = g(\xi(u, u_{(1)}, \dots)), \tag{19}$$

where g runs through the set of smooth function of a single argument. Assuming regular boundary conditions $u^a \to 0$, $u_i^a \to 0$, ... at infinity, the Noether boundary conditions (13) take the form

$$\begin{aligned} &M_i\left(g(\xi(0,\dots,0), g'(\xi(0,\dots,0),\dots\right) \\ &\qquad - M_i\left(g(\xi(0,\dots,0), g'(\xi(0,\dots,0),\dots\right) = 0. \end{aligned} \tag{20}$$

Conditions (20) are satisfied for any smooth function g and

$$\left|\xi(0,\dots,0)\right| < \infty. \tag{21}$$

It is easy to see that the strict boundary conditions (15) are also satisfied if restrictions (21) are met.

Thus, in the case of infinite symmetries with arbitrary functions of dependent variables $\alpha^a = f^a(u, u_{(1)}, \dots)$ unlike the case with independent variables, there are no serious restrictions for functions f^a to lead to local conservation laws, Therefore, in this case the continuity equation (17) provides an infinite number of essential conservation laws [17]. The corresponding Noether conserved quantities can be found in the form

$$\mathrm{D}_t \int_\Omega dx^1 dx^2 \dots dx^m \left(M_t - R_{\alpha t}L\right) \doteq 0. \tag{22}$$

3 Equations possessing infinite variational symmetries with an arbitrary function $f(u, u_{(1)}, u_{(2)})$

Consider the case $m = n = 1$ and denote $x^1 = x$ and $x^2 = t$. We look for equations with first-order Lagrangians, $L = L(u, u_x, u_t)$, possessing X_α as a variational symmetry for each characteristic α of the form

$$\alpha = Pf(\xi) + Qf'(\xi) + Rf''(\xi), \tag{23}$$

where f is an arbitrary smooth functions of a single argument, ξ is a smooth function of u and its derivatives up to order two, $\xi = \xi(u, u_{(1)}, u_{(2)})$, and P, Q and R are some differential functions. Then we have

$$X_\alpha L = \mathrm{D}_x M_x + \mathrm{D}_t M_t \tag{24}$$

with

$$\begin{aligned} M_x &= Af(\xi) + Bf'(\xi) + Cf''(\xi),\\ M_t &= Ef(\xi) + Ff'(\xi) + Gf''(\xi), \end{aligned} \tag{25}$$

where A, B, C, E, F and G are some differential functions.

Calculating the LHS of (24) ($X_\alpha L$) we obtain terms with $f'''(\xi)$, $f''(\xi)$, $f'(\xi)$, and $f(\xi)$. Using the fact that the function $f(\xi)$ is arbitrary we can equate coefficients of derivatives of f to corresponding coefficients in the RHS. We obtain the following four constraints, respectively:

$$\begin{aligned} &\xi_x\left(RL_z - C\right) + \xi_t\left(RL_p - G\right) = 0,\\ &\xi_x\left(QL_z - B\right) + \xi_t\left(QL_p - F\right)\\ &\qquad\qquad = \mathrm{D}_x C - (\mathrm{D}_x R)L_z + \mathrm{D}_t G - (\mathrm{D}_t R)L_p - RL_u,\\ &\xi_x\left(PL_z - A\right) + \xi_t\left(PL_p - E\right)\\ &\qquad\qquad = \mathrm{D}_x B - (\mathrm{D}_x Q)L_z + \mathrm{D}_t F - (\mathrm{D}_t Q)L_p - QL_u,\\ &PL_u = \mathrm{D}_x A + \mathrm{D}_t E, \end{aligned} \tag{26}$$

where the following notations were used: $u_x \equiv z$, $u_t \equiv w$; $\xi_x \equiv \mathrm{D}_x\xi$, $\xi_t \equiv \mathrm{D}_t\xi$. Solving the first equation of this system for G and using

$$\omega = L_u - \mathrm{D}_x(L_z) - \mathrm{D}_t(L_p) \tag{27}$$

we have

$$\begin{aligned} &G = RL_p + \frac{\xi_x}{\xi_t}\left(RL_z - C\right),\\ &\xi_x\left(QL_z - B\right) + \xi_t\left(QL_p - F\right)\\ &\qquad\qquad = -\mathrm{D}_x\left(RL_z - C\right) + \mathrm{D}_t\left[\frac{\xi_x}{\xi_t}(RL_z - C)\right] - R\omega,\\ &\xi_x\left(PL_z - A\right) + \xi_t\left(PL_p - E\right)\\ &\qquad\qquad = -\mathrm{D}_x\left(QL_z - B\right) - \mathrm{D}_t\left(QL_p - F\right) - Q\omega,\\ &PL_u = \mathrm{D}_x A + \mathrm{D}_t E. \end{aligned} \tag{28}$$

Introducing $\bar{B}, \bar{C}, \bar{F}$

$$\bar{B} = QL_z - B, \qquad \bar{C} = RL_z - C, \qquad \bar{F} = QL_p - F, \tag{29}$$

and expressing E from the third equation of system (28), we obtain

$$\begin{aligned} G &= RL_p + \frac{\xi_x}{\xi_t}\bar{C}, \\ E &= PL_p + \frac{\xi_x}{\xi_t}(PL_z - A) + \frac{1}{\xi_t}\left[\mathrm{D}_x\bar{B} + \mathrm{D}_t\bar{F} + Q\omega\right], \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\xi_x\bar{B} + \xi_t\bar{F} = -\mathrm{D}_x\bar{C} + \mathrm{D}_t\left[\frac{\xi_x}{\xi_t}\bar{C}\right] - R\omega, \\ &\beta\mathrm{D}_tP - P\left[L_u - \mathrm{D}_t\left(L_p + \frac{\xi_x}{\xi_t}L_z\right)\right] = -\mathrm{D}_xA + \mathrm{D}_t\left[\frac{\xi_x}{\xi_t}A - H\right], \end{aligned} \tag{31}$$

where

$$\beta \;=\; L_p + \frac{\xi_x}{\xi_t}L_z, \qquad H = \frac{\mathrm{D}_x\bar{B} + \mathrm{D}_t\bar{F} + Q\omega}{\xi_t}. \tag{32}$$

Since we are looking for local conservation laws we require all coefficients to be local functions. The second equation in (31) will have local solutions for the function P if the integrating factor e^α is local

$$e^\alpha = e^{-\int \frac{L_u - \mathrm{D}_t\beta}{\beta}\,dt} = \beta e^{-\int \frac{L_u}{\beta}\,dt}.$$

The integrating factor is local if the function $e^{-\int \frac{L_u}{L_p + \xi_x L_z/\xi_t}\,dt}$ is local. Therefore we require that the integral

$$\int \frac{L_u}{L_p + \xi_x L_z/\xi_t}\,dt$$

be a local function. The last condition means the existence of function $\psi \in C^1$ such that

$$\frac{L_u}{L_p + \xi_x L_z/\xi_t} = \psi_t(u, u_{(1)}, u_{(2)}). \tag{33}$$

Since $L = L(u, u_{(1)})$ and $\xi = \xi(u, u_{(1)}, u_{(2)})$, the LHS of (33) is nonlinear with respect to the third derivatives of function u, u_{ijk} while the RHS is linear with u_{ijk}. The equation (33) therefore, can be satisfied only if $L_z = 0$ (not interesting) or $L_u = 0$.

Thus, the first class of equations possessing an infinite symmetry algebra (25) is given by $L_u = 0$, meaning that

$$L = L(u_x, u_t), \tag{34}$$

see also [17]. Any equations with Lagrangians of the form (34) have an infinite number of essential conservation laws parametrized by an arbitrary function of the dependent variable and its first and second derivatives given by expression (17) ((22)). An interesting example of the class (34) is two-dimensional Born–Infeld equation

$$(1-u_t^2)u_{xx}+2u_tu_xu_{xt}-(1+u_x^2)u_{tt}=0, \tag{35}$$

with the Lagrangian density

$$L=\left(1+u_x^2-u_t^2\right)^{1/2}. \tag{36}$$

Born–Infeld Lagrangian (36) is related to classical relativistic string Lagrangian in a four-dimensional space, see e.g. [3]. Infinite set of conservation laws for Born–Infeld equation (35) and the relation to its infinite group of contact transformations was studied in [8]. Another example of the class (34) is hyperbolic Fermi–Pasta–Ulam equation [6]

$$u_{tt}-ku_x^2u_{xx}=0, \tag{37}$$

with the Lagrangian density

$$L=\frac{u_t^2}{2}-\frac{ku_x^3}{3}. \tag{38}$$

Note that the condition $L_u=0$ came as a requirement that solutions of the second equation of the system (31) for the function P (and its integrating factor) be local. There is no need for this condition in the case when

$$\mathrm{D}_tP=0,\qquad P=P(x). \tag{39}$$

In this case the second equation of (31) takes the form

$$P\left[L_u-\mathrm{D}_t\left(L_p-\frac{\xi_x}{\xi_t}L_z\right)\right]=\mathrm{D}_xA-\mathrm{D}_t\left[\frac{\xi_x}{\xi_t}A-H\right]. \tag{40}$$

We can rewrite the equation (40) in the form

$$\mathrm{D}_tH=P\left[L_u-\mathrm{D}_t\left(L_p+\frac{\xi_x}{\xi_t}L_z\right)\right]+\mathrm{D}_t\left[\frac{\xi_x}{\xi_t}A\right]-\mathrm{D}_xA, \tag{41}$$

and solve it for H

$$H=P\int L_u dt-P\left(L_p+\frac{\xi_x}{\xi_t}L_z\right)+\frac{\xi_x}{\xi_t}A-\int(\mathrm{D}_xA)dt+\bar{g}(x), \tag{42}$$

where $\bar{g}(x)$ is arbitrary. Since $L_u\neq 0$ from the requirement that function H be local we obtain $P=0$ and

$$H=\frac{\xi_x}{\xi_t}A-\int(\mathrm{D}_xA)dt+\bar{g}(x). \tag{43}$$

The function A is local if

$$\mathrm{D}_x A = \psi_t, \qquad \psi = \psi(u, u_{(1)}, u_{(2)}), \tag{44}$$

meaning that

$$A = u_{xt}\varphi(u, u_{(1)}) + u_{tt}\beta(u, u_{(1)}) + \mu(u, u_{(1)})$$
$$\psi = u_{xt}\beta(u, u_{(1)}) + u_{xx}\varphi(u, u_{(1)}) + \nu(u, u_{(1)}), \tag{45}$$

with $\varphi, \beta, \mu, \nu \in C^1$. As a result we have two constraints (31) for the coefficients $P, Q, R, A, B, C, E, F, G$ determining our infinite symmetry (23) and infinite essential conservation laws (17) ((22)) along with the expressions (29) for finding $\bar{B}, \bar{C}, \bar{F}$ and (30) for finding G and E. Using the definition of H (32) we can write these two constraints in the form

$$\xi_x \bar{B} = -\xi_t \bar{F} - \mathrm{D}_x \bar{C} + \mathrm{D}_t \left[\frac{\xi_x}{\xi_t}\bar{C}\right] - R\omega, \tag{46}$$
$$\xi_t H = \mathrm{D}_x \bar{B} + \mathrm{D}_t \bar{F} + Q\omega.$$

Solving the first equation of the system for $\bar{B}$ and substituting it into the second equation we obtain

$$\xi_t H = -\mathrm{D}_x \frac{\xi_t \bar{F}}{\xi_x} - \mathrm{D}_x \frac{\bar{C}_x}{\xi_x} + \mathrm{D}_x \frac{\mathrm{D}_t(\xi_x \bar{C}/\xi_t)}{\xi_x} - \mathrm{D}_x \frac{R\omega}{\xi_x} + \mathrm{D}_t \bar{F} + Q\omega. \tag{47}$$

In a special case $\bar{F} = \bar{C} = 0$ we obtain

$$\xi_t H = \mathrm{D}_x \frac{R\omega}{\xi_x} + Q\omega, \tag{48}$$

where ω is the LHS of the original equation, see (30). Thus, we obtain two classes of solutions

$$1. \quad (\mathrm{D}_t \xi) H \doteq 0, \qquad \mathrm{D}_x \xi \not\doteq 0; \tag{49}$$

and

$$2. \quad \mathrm{D}_x \xi \doteq 0. \tag{50}$$

Case 1 naturally splits into two sub-cases

$$1a. \quad H \doteq 0, \tag{51}$$

and

$$1b. \quad \mathrm{D}_t \xi \doteq 0. \tag{52}$$

It can be shown that solution of the system (31) in general case also gives rise to two classes (49) and (50), and that case $1a$ (51) leads to trivial conservation laws (the details will be given elsewhere). The case (52) is known to lead to equations of Liouville type, see e.g. [1, 2, 7, 21–24]. The most common equation of this class is the Liouville equation

$$u_{xt} = e^u \qquad \text{with} \qquad L = \frac{u_x u_t}{2} + e^u, \tag{53}$$

for which we have [23]

$$\alpha = u_x f'(\xi) + \mathrm{D}_x f'(\xi) = u_x f'(\xi) + \xi_x f''(\xi), \quad \text{where} \quad \xi = u_{xx} - \frac{u_x^2}{2}.$$

Indeed,

$$\mathrm{D}_t\xi = u_{xxt} - u_x u_{xt} = \mathrm{D}_x(e^u - \omega) - u_x(e^u - \omega) = -\mathrm{D}_x\omega + u_x\omega \doteq 0,$$

where $\omega = e^u - u_{xt}$. The coefficients P, Q, R, A, B, C, E, F and G in (23), (24), (25) take the form

$$P = 0, \quad Q = u_x, \quad R = \xi_x, \quad A = 0, \quad B = \frac{u_x u_t}{2} - u_{xt} + e^u,$$
$$C = \frac{u_t \xi_x}{2}, \quad E = -1, \quad F = \frac{{u_x}^2}{2}, \quad G = \frac{u_x \xi_x}{2}.$$

Clearly, the case (50) also leads to the equations of Liouville type ($x \leftrightarrow t$).

4 Conclusion

We have demonstrated that the problem of finding all Lagrangian PDE's of the second order possessing an infinite set of conservation laws with an arbitrary function of the dependent variable and its first and second derivatives leads to equations of two classes: equations of the form (34) with Lagrangians depending only on first derivatives of the function, and equations of Liouville type characterized by relation (52).

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Miura-Reciprocal Transformations for Two Integrable Hierarchies in 1+1 Dimensions

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We present two different hierarchies of PDEs in 1+1 dimensions whose first and second members are the shallow water wave Camassa–Holm and Qiao equations, correspondingly. These two hierarchies can be transformed by reciprocal methods into the Calogero–Bogoyavlenskii–Schiff equation (CBS) and its modified version (mCBS), respectively. Considering that there exists a Miura transformation between the CBS and mCBS, we obtain a relation between the initial hierarchies by means of a composition of a Miura and a reciprocal transforms.

1 Introduction

Reciprocal methods are based on transformations in which *the role of the independent and dependent variables is interchanged*. When the variables are switched, the space of independent variables is called the *reciprocal space*, or in the case of two dimensions, the *reciprocal plane*. As a physical interpretation, whereas the independent variables play the usual role of positions, in the reciprocal space, this number is increased by turning certain fields (usually velocities or parameterized heights of waves, in the case of fluid mechanics) into independent variables and vice-versa [4]. Recently, a lot of attention has been paid upon reciprocal transformations, as they appear to be a very useful instrument for the identification of ordinary or partial differential equations and high-order hierarchies which, a priori, do not have the Painlevé property (PP) [6, 8, 9].

According to the Painlevé criterion of integrability, we say that a non-linear equation is integrable if its solutions are single-valued in the neighborhood of the movable singularity manifold. The PP can be checked using an algorithmic procedure developed by Weiss [16, 17], which gives us a set of solutions relying on the truncation of an infinite Laurent expansion. In this situation, one can prove that the manifold of movable singularities satisfies a set of equations known as *the singular manifold equations* (SME). Also, the SME are a subject of our rising interest due to the possibility of classification of O/PDEs in terms of their SME used as a canonical form. If this were the case, two apparently unrelated equations

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sharing the same SME must be tantamount versions of a unique equation. In this way, we are reducing the ostensible bundle of non-linear integrable equations into series of equivalent ones.

The PP is non-invariant under changes of independent/dependent variables. For this reason, we can transform an initial equation which does not pass the Painlevé test into another which successfully accomplishes the Painlevé requirements. In most of the cases, the transformed equation is well-known, as well as its properties, integrability and so on. In this way, reciprocal transformations can be very useful, for instance, in the derivation of Lax pairs (LP) in such a way that given the LP for the transformed equation, we shall perform the inverse reciprocal transformation to obtain the LP for the initial one.

Nevertheless, finding a suitable reciprocal transform is usually a complicated task and of very limited use. Notwithstanding, in cases of fluid mechanics, a change of this type is usually reliable. In [5, 10], similar transformations were introduced to turn peakon equations into other integrable ones.

Sometimes, the composition of a Miura and a reciprocal transform, hence the name *Miura-reciprocal transform*, helps us to relate two different hierarchies, which is the purpose of the present paper. We shall illustrate this matter through a particular example: the Camassa–Holm (CH1+1) and Qiao (Qiao1+1) hierarchies in 1+1 dimensions. We shall prove that a combination of a Miura and reciprocal transforms can relate the parameterized height of the wave in Camassa–Holm's $U = U(X, T)$ with Qiao's $u = u(x, t)$ hierarchies. A summary of the process is included within the following diagram:

		Reciprocal T.		
	CH1+1	$\Longleftrightarrow$	**CBS**	
Miura-reciprocal T.	$\Updownarrow$		$\Updownarrow$	*Miura T.*
	Qiao1+1	$\Longleftrightarrow$	**mCBS**	
		Reciprocal T.		

In view of this, we shall present the plan of the paper as follows. In Section 2, we establish the reciprocal transformations that link CH1+1 and Qiao1+1 to CBS and mCBS, respectively. Section 3 is devoted to the introduction of the Miura transformation between CBS and mCBS, as well as the inverse reciprocal transform from CBS and mCBS to the initial CH1+1 and Qiao1+1. In the conclusion, we underline several important relations between the fields and the independent variables of the two initial hierarchies.

2 Reciprocal links

In this section we review previous results from references [7,9], in which the reciprocal links for the Camassa–Holm and Qiao hierarchies in 2+1 dimensions were introduced, and particularize these to the case of 1+1 dimensions that concerns us in this paper. Notice that technical details will be omitted but can be checked in the aforemention references. It is important to mention that henceforth we shall use capital letters for the independent variables and fields appearing in the Camassa–Holm's hierarchy and lower case letters for those variables and fields concerning Qiao's hierarchy.

2.1 Reciprocal link for CH1+1

The n-component Camassa–Holm hierarchy in 1+1 dimensions can be written in a compact form in terms of a recursion operator as follows:

$$U_T = R^{-n} U_X,$$

where $R = KJ^{-1}$ with $K = \partial_{XXX} - \partial_X$ and $J = -\frac{1}{2}(\partial_X U + U\partial_X)$. The factor $-1/2$ is conveniently added in J for future calculations. If we include auxiliary fields $\Omega^{(i)}$ with $i = 1, \ldots, n$ when the inverse of an operator appears, the hierarchy can be written as:

$$\begin{aligned} &U_T = J\Omega^{(1)}, \\ &K\Omega^{(i)} = J\Omega^{(i+1)}, \quad i = 1, \ldots, n-1, \\ &U_X = K\Omega^{(n)}. \end{aligned} \tag{1}$$

It is also useful to introduce the change $U = P^2$ such that the final equations are of the form

$$P_T = -\frac{1}{2}\left(P\Omega^{(1)}\right)_X, \tag{2}$$

$$\Omega^{(i)}_{XXX} - \Omega^{(i)}_X = -P\left(P\Omega^{(i+1)}\right)_X, \quad i = 1, \ldots, n-1, \tag{3}$$

$$P^2 = \Omega^{(n)}_{XX} - \Omega^{(n)}. \tag{4}$$

If $i = 1$ we recover the celebrated Camassa–Holm equation in 1+1 dimensions [3].

Given the conservative form of equation (2), the following transformation arises naturally:

$$dT_0 = P dX - \frac{1}{2} P\Omega^{(1)} dT, \quad dT_1 = dT, \tag{5}$$

such that $d^2T_0 = 0$, recovering (2). We shall now propose a reciprocal transformation [5] by considering the former independent variable X as a dependent field of the new pair of independent variables $X = X(T_0, T_1)$, and therefore, $dX = X_0\, dT_0 + X_1\, dT_1$, where the subscripts zero and one refer to partial derivative of the field X with respect to T_0 and T_1, correspondingly. The inverse transformation takes the form

$$dX = \frac{dT_0}{P} + \frac{1}{2}\Omega^{(1)} dT_1, \quad dT = dT_1, \tag{6}$$

using which, by direct comparison with the total derivative of the field X, we obtain

$$X_0 = \frac{1}{P}, \quad X_1 = \frac{\Omega^{(1)}}{2}.$$

We can now extend the transformation [9] by introducing $n-1$ independent variables $T_2, \ldots, T_n$ which account for the transformation of the auxiliary fields $\Omega^{(i)}$ in such a way that $\Omega^{(i)} = 2X_i$, with $i = 2, \ldots, n$ and $X_i = \frac{\partial X}{\partial T_i}$. Then, X is a function $X = X(T_0, T_1, T_2, \ldots, T_n)$ of $n+1$ variables. It requires some computation to transform the hierarchy (2)–(4) into the equations that $X = X(T_0, T_1, T_2, \ldots, T_n)$ should obey. For this matter, we use the symbolic calculus package Maple. Equation (2) is identically satisfied by the transformation and (3), (4) lead to the following set of PDEs

$$-\left(\frac{X_{i+1}}{X_0}\right)_0 = \left\{\left(\frac{X_{00}}{X_0} + X_0\right)_0 - \frac{1}{2}\left(\frac{X_{00}}{X_0} + X_0\right)^2\right\}_i, \quad i = 1, \ldots, n-1, \quad (7)$$

which constitutes $n-1$ copies of the same system, each of which is written in three variables T_0, T_i, T_{i+1}. Considering the conservative form of (7), we introduce the change

$$M_i = -\frac{1}{4}\left(\frac{X_{i+1}}{X_0}\right),$$
$$M_0 = \frac{1}{4}\left\{\left(\frac{X_{00}}{X_0} + X_0\right)_0 - \frac{1}{2}\left(\frac{X_{00}}{X_0} + X_0\right)^2\right\} \quad (8)$$

with $M = M(T_0, T_i, T_{i+1})$ and $i = 1, \ldots, n-1$. The compatibility condition of X_{000} and X_{i+1} in this system gives rise to a set of equations written entirely in terms of M:

$$M_{0,i+1} + M_{000i} + 4M_iM_{00} + 8M_0M_{0i} = 0, \quad i = 1, \ldots, n-1, \quad (9)$$

that are $n-1$ CBS equations [1, 2] each of which depends on three different variables $M = M(T_0, T_i, T_{i+1})$. These equations have the PP [11, 12] and the singular manifold method (SMM) can be applied to derive its LP. Making use of CBS's LP and with the aid of the inverse reciprocal transform, we could derive a LP for the initial CH1+1. The detailed process can be consulted in [8, 9].

2.2 Reciprocal link for Qiao1+1

The n-component Qiao hierarchy in 1+1 dimensions can be written in a compact form in terms of a recursion operator:

$$u_t = r^{-n}u_x$$

such that $r = kj^{-1}$ with $k = \partial_{xxx} - \partial_x$ and $j = -\partial_x u(\partial_x)^{-1} u\partial_x$. If we introduce n additional fields $v^{(i)}$ when we encounter the inverse of an operator, the expanded equations take the form

$$
\begin{aligned}
&u_t = jv^{(1)}, \\
&kv^{(j)} = jv^{(i+1)}, \quad i = 1, \dots, n-1, \\
&u_x = kv^{(n)}.
\end{aligned} \tag{10}
$$

This hierarchy was firstly introduced in [7] as a generalization of the Qiao hierarchy to 2+1 dimensions, depicted in [13] and whose second member was studied in [14]. If we now introduce the value of the operators k and j, we obtain the following equations:

$$u_t = -\left(u\omega^{(1)}\right)_x, \tag{11}$$

$$v^{(i)}_{xxx} - v^{(i)}_x = -\left(u\omega^{(i+1)}\right)_x, \quad i = 1, \dots, n-1, \tag{12}$$

$$u = v^{(n)}_{xx} - v^{(n)}, \tag{13}$$

in which the change $\omega^{(i)}_x = uv^{(i)}_x$ with other n auxiliary fields $\omega^{(i)}$ has been necessarily included to operate with the inverse term present in j.

Given the conservative form of (11), the following change reciprocal transformation [5] arises naturally

$$dT_0 = u\,dx - u\omega^{(1)}dt, \quad dT_1 = dt \tag{14}$$

such that $d^2T_0 = 0$ recovers (11). We now propose a reciprocal transformation [9] by considering the initial independent variable x as a dependent field of the new independent variables such that $x = x(T_0, T_1)$, and therefore, $dx = x_0\,dT_0 + x_1\,dT_1$. The inverse transformation adopts the form:

$$dx = \frac{dT_0}{u} + \omega^{(1)}dT_1, \quad dt = dT_1. \tag{15}$$

By direct comparison of the inverse transform with the total derivative of x, we obtain that:

$$x_0 = \frac{1}{u}, \quad x_1 = \omega^{(1)}.$$

We shall prolong this transformation [15] in such a way that we introduce new variables $T_2, \dots, T_n$ such that $x = x(T_0, T_1, \dots, T_n)$ according to the following rule $\omega^{(i)} = x_i$, $x_i = \frac{\partial x}{\partial T_i}$ for $i = 2, \dots, n$. In this way, (11) is identically satisfied by the transformation and (12), (13) are transformed into $n-1$ copies of the following equation, which is written in terms of the three variables T_0, T_i, T_{i+1}:

$$\left(\frac{x_{i+1}}{x_0} + \frac{x_{i00}}{x_0}\right)_0 = \left(\frac{x_0^2}{2}\right)_i, \quad i = 1, \dots, n-1.$$

The conservative form of these equations allows us to write them in the form of a system as:

$$m_0 = \frac{x_0^2}{2}, \tag{16}$$
$$m_i = \frac{x_{i+1}}{x_0} + \frac{x_{i00}}{x_0}, \quad i = 1, \ldots, n-1, \tag{17}$$

which can be considered as modified CBS equation with

$$m = m\left(T_0, \ldots, T_i, T_{i+1}, \ldots, T_n\right).$$

The modified CBS equation has been extensively studied from the point of view of the Painlevé analysis in [8], its LP was derived and hence, a version of a LP for Qiao is available in [7].

3 Miura-reciprocal transformations

In the previous section, we have seen that both hierarchies CH1+1 and Qiao1+1 are related to the CBS and mCBS, respectively, through reciprocal transformations. These final equations possess the PP, whereas the initial did not accomplish it. In this way, we are able to obtain their LPs, solutions and many other properties. The inverse reciprocal transform allows us to obtain LPs for the initial. From the literature [8], we know that CBS and mCBS can be transformed one into another. In this manner, the fields present in CBS and mCBS are related through the following formula

$$4M = x_0 - m. \tag{18}$$

This is the point at which the question of whether Qiao1+1 could possibly be a modified version of CH1+1 arises. Nevertheless, the relation between these two hierarchies cannot be a simple Miura transform, since each of them is written in different triples of variables (X, Y, T) and (x, y, t). However, both triples lead to the same final triple (T_0, T_1, T_n). Then, by combining (5) and (14) we have

$$PdX - \frac{1}{2}P\Omega^{(1)}dT = udx - u\omega^{(1)}dt, \quad dt = dT, \tag{19}$$

that yields a relationship between the variables in CH1+1 and Qiao1+1. Using formula (18), we obtain the relations between fields X and x

$$4M_0 = x_{00} - m_0 \Rightarrow \frac{X_{00}}{X_0} + X_0 = x_0, \tag{20}$$
$$4M_i = x_{0i} - m_i \Rightarrow -\frac{X_{i+1}}{X_0} = x_{0i} - \frac{x_{00i}}{x_0} - \frac{x_{i+1}}{x_0}, \quad i = 1, \ldots, n-1, \tag{21}$$

where we have also employed (8), (16) and (17). Now, by using the inverse reciprocal transformations proposed in (6) and (15) in equations (20) and (21), we obtain (see Appendix):

$$\frac{1}{u} = \left(\frac{1}{P}\right)_X + \frac{1}{P}, \tag{22}$$

$$P\Omega^{(i+1)} = 2(v^{(i)} - v_x^{(i)}) \quad \Rightarrow \quad \omega^{(i+1)} = \frac{\Omega_X^{(i+1)} + \Omega^{(i+1)}}{2}, \tag{23}$$

where $i = 1, \ldots, n-1$. Furthermore, if we isolate dx in (19) and use expression (22), we have

$$dx = \left(1 - \frac{P_X}{P}\right) dX + \left(\omega^{(1)} - \frac{\Omega^{(1)}}{2}\left(1 - \frac{P_X}{P}\right)\right) dT. \tag{24}$$

The condition $d^2x = 0$ implies that the cross derivative satisfies

$$\left(1 - \frac{P_X}{P}\right)_T = \left(\omega^{(1)} - \frac{\Omega^{(1)}}{2}\left(1 - \frac{P_X}{P}\right)\right)_X \quad \Rightarrow \quad \omega^{(1)} = \frac{\Omega_X^{(1)} + \Omega^{(1)}}{2}. \tag{25}$$

Finally, with the aid of (25), we write (24) in the form

$$dx = \left(1 - \frac{P_X}{P}\right) dX - \frac{P_T}{P} dT$$

which can be integrated to give

$$x = X - \ln P.$$

This equation gives us an important relation between the initial independent variables X and x in CH1+1 and Qiao1+1, respectively. We point out that the rest of independent variables, y, t and Y, T do not appear in these expressions, since they were left untouched in the transformations proposed in (5) and (14).

Summarizing, hierarchies (1) and (10) are related through the Miura-reciprocal transformation

$$x = X - \frac{\ln U}{2},$$

$$\frac{1}{u} = \left(\frac{1}{\sqrt{U}}\right)_X + \frac{1}{\sqrt{U}},$$

which means that the Qiao hierarchy can be considered as the modified version of the celebrated Camassa–Holm hierarchy.

4 Conclusions

Our aim was to highlight the utility of the reciprocal transformations for the identification of integrable equations, for the study of their properties as well as for classification purposes in the case of disguised versions of a given equation. To illustrate this we have presented the example of the Camassa–Holm and Qiao hierarchies in $1+1$ dimensions. For these two hierarchies we have derived a transformation between their fields and independent variables. This has been achieved by a combination of a Miura and a reciprocal transform, which we have called *Miura-reciprocal transformation.*

Appendix

- A method for obtaining equation (22).

  Equation (20) provides $x_0 = X_0 + \partial_0(\ln X_0)$. Since $X_0 = \frac{1}{P}$ and $x_0 = \frac{1}{u}$, the latter equation becomes $\frac{1}{u} = \frac{1}{P} - \partial_0(\ln P)$, and by using (5) we have $\frac{1}{u} = \frac{1}{P} - \frac{1}{P}(\ln P)_X$, which yields (22).

- A method for obtaining equation (23).

  If we use (6) and (15), then (21) becomes

  $$-\frac{P\Omega^{(i+1)}}{2} = \partial_0(\omega^{(i)}) - u\partial_{00}(\omega^{(i)}) - u\omega^{(i+1)}.$$

  Here and in all equations below $i = 1, \ldots, n-1$. And now (14) gives us

  $$-\frac{P\Omega^{(i+1)}}{2} = \frac{\omega_x^{(i)}}{u} - \left(\frac{\omega_x^{(i)}}{u}\right)_x - u\omega^{(i+1)}.$$

  Using the expressions $\omega_x^{(i)} = uv_x^{(i)}$, $u\omega^{(i+1)} = v^{(i)} - v_{xx}^{(i)}$, arising from (13), we arrive at

  $$-\frac{P\Omega^{(i+1)}}{2} = v_x^{(i)} - v^{(i)},$$

  as is required in (23), and $u\omega^{(i+1)} = v^{(i)} - v_{xx}^{(i)}$ can be written as

  $$u\omega^{(i+1)} = \left(v^{(i)} - v_x^{(i)}\right) + \left(v_x^{(i)} - v_{xx}^{(i)}\right) = \left(\frac{P\Omega^{(i+1)}}{2}\right) + \left(\frac{P\Omega^{(i+1)}}{2}\right)_x.$$

  From (19), we have $\partial_x = \frac{u}{P}\partial_X$, therefore,

  $$u\omega^{(i+1)} = \left(\frac{P\Omega^{(i+1)}}{2}\right) + \frac{u}{P}\left(\frac{P\Omega^{(i+1)}}{2}\right)_X,$$

$$\omega^{(i+1)} = \left(\frac{P\Omega^{(i+1)}}{2u}\right) + \frac{1}{2P}\left(P_X\Omega^{(i+1)} + P\Omega_X^{(i+1)}\right).$$

We can eliminate u using (22). This results in

$$\omega^{(i+1)} = \frac{\Omega_X^{(i+1)} + \Omega^{(i+1)}}{2}, \quad i = 1, \ldots, n-1.$$

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Preliminary Classification of Realizations of Two-Dimensional Lie Algebras of Vector Fields on a Circle

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Finite-dimensional subalgebras of a Lie algebra of smooth vector fields on a circle, as well as piecewise-smooth global transformations of a circle on itself, are considered. A canonical form of realizations of two-dimensional noncommutative algebra is obtained. It is shown that all other realizations of smooth vector fields are reduced to this form using global transformations.

1 Introduction

The description of Lie algebra representations by vector fields on a line and a plane was first considered by S. Lie [3, S. 1–121]. However, this problem is still of great interest and widely applicable. In spite to its importance for applications, only recently a complete description of realizations begun to be investigated systematically. Furthermore, only since the late eighties of the last century papers on that subject were published regularly. In particular, different problems of realizations were studied such as realizations of first order differential operators of a special form in [2], realizations of physical algebras (Galilei, Poincaré and Euclid ones) in [9, 11]. In [4] it was constructed a complete set of inequivalent realizations of real Lie algebras of dimension no greater than four in vector fields on a space of an arbitrary (finite) number of variables. In that paper one can obtain a more complete review on the subject and a list of references.

Almost in all works on the subject realizations are considered up to local equivalence transformations. Attempts to classify realizations of Lie algebras in vector fields on some manifold with respect to global equivalence transformations (on the whole manifold) have been made only in a few papers (see, e.g., [6,8,10]). In these papers it is proved that (up to isomorphism) there are only three algebras, namely, one-dimensional, noncommutative two-dimensional and three-dimensional isomorphic to $\mathrm{sl}(2, \mathbb{R})$, that can be realized by analytic vector fields on the circle.

The purpose of this paper is to construct all inequivalent realization of the two-dimensional algebras on the circle. For this reason, we are not limited by the requirement of analyticity, as it is considered in [6, 8, 10].

On the circle S^1 we introduce the parameter $\theta \in \mathbb{R}$, $0 \leqslant \theta < 2\pi$. Then the vector fields on S^1 can be represented as a vector field $v(\theta)\frac{d}{d\theta}$, where $v(\theta)$ is a smooth real function on the circle [5]. One more possibility is to assume that $\theta \in \mathbb{R}$, and $v(\theta)$ is a smooth 2π-periodic function on a line. We consider the vector fields of the class C^1 (with continuously-differentiable functions $v(\theta)$), which is natural to require when calculate the commutators of two vector fields.

We also introduce a class of transformations $f\colon S^1 \to S^1$, which is defined by the following properties:

- it is one-to-one mapping of the circle onto itself;
- $f(\theta)$ is continuous at any point $\theta \in S^1$;
- it is continuously differentiable at all points except a finite number of them;
- the derivative $f'(\theta)$ tends to $-\infty$ or $+\infty$ at all points of discontinuity;
- under a change of the coordinate $\tilde{\theta} = f(\theta)$ a vector field of the class C^1 transforms to a vector field of the same class.

Such transformations are defined as *equivalence transformations* of vector fields. We call two realizations of an algebra of vector fields inequivalent, if it is impossible to transform realizations to each other by compositions of equivalence transformations.

We assume that $f(0) = 0$, without loss of generality. Indeed, any equivalence transformation is obvious a composition of some equivalence transformations with fixed zero and a rotation of the circle. So, we can perform preliminary classification of inequivalent realizations up to such transformations (with fixed zero) and then complete classification taking into account the rotations of the circle. Then it is easy to see that $f(\theta)$ is monotone on the whole interval $0 \leqslant \theta < 2\pi$. Degree of the map equals ± 1, $\deg f = \pm 1$ (see [1]). Depending on the sign of the degree the function is monotonically decreasing or increasing.

Besides the transformations of f, we introduce the following class of *homotopy* of a circle into itself. Let us take two points $\theta_1, \theta_2 \in S^1$, $\theta_1 < \theta_2$. We define a family of transformations $F_{\theta_1,\theta_2}(t,\theta)\colon [0,1] \times S^1 \to S^1$ of the circle the as follows.

In the case $\theta_1 \neq 0$

$$F_{\theta_1,\theta_2}(t,\theta) = \begin{cases} \theta + t\dfrac{\theta}{\theta_1}(\theta_2 - \theta_1) & \text{if } \ 0 \leq \theta < \theta_1, \\ \theta + t(\theta_2 - \theta) & \text{if } \ \theta_1 \leq \theta < \theta_2, \\ \theta & \text{if } \ \theta_2 \leq \theta < 2\pi. \end{cases}$$

In the case $\theta_1 = 0$

$$F_{\theta_1,\theta_2}(t,\theta) = \begin{cases} \theta(1-t) & \text{if } \ 0 \leq \theta < \theta_2, \\ \theta - t\dfrac{2\pi - \theta}{2\pi - \theta_2}\theta_2 & \text{if } \ \theta_2 \leq \theta < 2\pi. \end{cases}$$

Obviously, if some subset of singular points of a vector field (i.e., zeros of its coefficient [1]) forms an open set (θ_1, θ_2) on S^1 then this interval can be constricted to a point by an appropriate homotopy. Therefore, we can consider vector fields singularities which do not form intervals, although the number of singularities may be infinite. We denote the class of such vector fields by $\mathcal{C}$. They form the set of realizations of the one-dimensional algebra. In general case such vector fields cannot be simplified by equivalence transformations, because of infinity the number of singular points.

We are interested in all inequivalent realizations of finite-dimensional Lie algebras by vector fields from the class $\mathcal{C}$.

2 Two-dimensional commutative algebra

Further we denote vector fields $v(\theta)\frac{d}{d\theta}$ and $w(\theta)\frac{d}{d\theta}$ by V and W, correspondingly. Let they commute, and $V \in \mathcal{C}$. A singular point θ_0 of the field V is said to be *degenerate* if $v'(\theta_0) = 0$. It is easily to show that if $0 \leqslant \theta_0 < \theta_1 < 2\pi$ are two degenerate points, such that the interval (θ_0, θ_1) does not contain additional degenerate points, then $w(\theta) = \lambda v(\theta)$ on (θ_0, θ_1), where $\lambda \neq 0$ is an arbitrary constant. This follows from the fact that $W \in \mathcal{C}$ and the continuity of the derivative of the function $w(\theta)$. In particular, if there is no degenerate point, or it is unique, then $w(\theta) = \lambda v(\theta)$ on S^1. In this case the vector fields V and W are linearly dependent, and there is no realization of two-dimensional commutative Lie algebra (see [6, 8, 10]).

We assume that we have more than one degenerate point. Without loss of generality, we may assume that point 0 (and 2π) is degenerate. Then the function $w(\theta)$ can be described as follows. We take an arbitrary point $\theta \in S^1$. If it is non-degenerate for function $v(\theta)$, then, obviously, there is a maximum interval (θ_0, θ_1) with two degenerate endpoints on it, such that $0 \leqslant \theta_0 < \theta < \theta_1 < 2\pi$. Then $w(\theta) = \lambda v(\theta)$ on this interval. Further, considering point $\theta' \notin [\theta_0, \theta_1]$, we repeat the procedure, if it is not degenerate. Again we have relation $w(\theta) = \lambda' v(\theta)$ on some interval. Here the values λ and λ' can be not equal.

If the point θ' is degenerate, then, by virtue of the fact that $V \in \mathcal{C}$, we can find arbitrarily close to it a non-degenerate point θ''. So we repeat the above procedure for θ''. Thus if we know the function $v(\theta)$, we can construct values of the function $w(\theta)$ in all non-degenerate points. For degenerate points we can construct values of $w(\theta)$ in arbitrary close to them points. If the number of degenerate points of the function $v(\theta)$ is infinite, then the number of non-equivalent realizations of the vector field W is infinite. So the number of realizations of two-dimensional commutative algebra is infinite.

3 Two-dimensional noncommutative algebra. Auxiliary lemmas

Let vector fields V and W generate the noncommutative algebra. It can be assumed up to their linear combining that they satisfy the commutation relation

$[V, W] = W$, which implies the relation between the functions $v(\theta)$ and $w(\theta)$:

$$v(\theta)w'(\theta) - v'(\theta)w(\theta) = w(\theta). \tag{1}$$

Lemma 1. *There is a singular point for the field W.*

Proof. Assume that the field W has no singular points, i.e., $w(\theta) > 0$ (or $w(\theta) < 0$) for all values $0 \leqslant \theta < 2\pi$. Then from (1) we can obtain the solution for the function $v(\theta)$ on the whole interval $[0, 2\pi)$:

$$v(\theta) = \left(-\int_0^\theta \frac{d\vartheta}{w(\vartheta)} + \lambda \right) w(\theta), \tag{2}$$

where λ is some constant. The function $w(\theta)$ is 2π-periodic on the line. Since the integrand in (2) is positive, we have $v(0) \neq v(2\pi)$. It contradicts the periodicity of the function $v(\theta)$. □

Lemma 2. *Singular points of W are singular points of V.*

Proof. Assume that $w(\theta_0) = 0$. Suppose that $v(\theta_0) \neq 0$. Then there exists some neighborhood U_{θ_0} of this point, where $v(\theta) \neq 0$. In this neighborhood, the equation (1) can be rewritten as:

$$w'(\theta) = \frac{1 + v'(\theta)}{v(\theta)} w(\theta). \tag{3}$$

Since $w(\theta_0) = 0$ and the right-hand side of equation (3) satisfies the Lipschitz condition with respect to w uniformly on θ then, by virtue of Picard's theorem [7], the differential equation (3) has a unique solution in the neighborhood of θ_0. Obviously, such a solution is $w \equiv 0$, what contradicts the fact that $W \in \mathcal{C}$. □

Lemma 3. *The number of singular points of the field W is finite.*

Proof. Assume that the number of singular points is infinite. Since S^1 is a compact set then there is a monotonically increasing (or decreasing) a sequence $\{\theta_n\}$ converging to some point θ_0 such that $w(\theta_n) = 0$. It is easy to show that for any n there is a non-singular point $\hat{\theta}_n \in (\theta_n, \theta_{n+1})$ satisfying the condition $w'(\hat{\theta}_n) = 0$. Then, from equation (1) we see that $v'(\hat{\theta}_n) = -1$. Since $\lim\limits_{n\to\infty} \hat{\theta}_n = \theta_0$, then, by virtue of continuous differentiability of function $v(\theta)$, we have $\lim\limits_{n\to\infty} v'(\hat{\theta}_n) = v'(\theta_0) = -1$. On the other hand, from Lemma 2 we get that $v(\theta_n) = v(\theta_{n+1}) = 0$. Hence there is a point $\tilde{\theta}_n \in (\theta_n, \theta_{n+1})$ such that $v'(\tilde{\theta}_n) = 0$. And since $\lim\limits_{n\to\infty} \tilde{\theta}_n = \theta_0$, then $\lim\limits_{n\to\infty} v'(\tilde{\theta}_n) = v'(\theta_0) = 0$. Therefore, we have a contradiction. □

Lemma 4. *If θ_0 is a singular point of W, then it is degenerate for this field (i.e., $w'(\theta_0) = 0$).*

Proof. Since $v(\theta_0) = 0$ (see Lemma 2), then

$$v(\theta) = v'(\theta_0)(\theta - \theta_0) + h(\theta), \;\; w(\theta) = w'(\theta_0)(\theta - \theta_0) + g(\theta),$$

where h, g are continuously differentiable functions and

$$h(\theta_0) = g(\theta_0) = h'(\theta_0) = g'(\theta_0) = 0. \tag{4}$$

Furthermore, taking to account equation (1) we have

$$\begin{aligned}\left[v(\theta)\frac{d}{d\theta}, w(\theta)\frac{d}{d\theta}\right] &= \big[(\theta - \theta_0)(v'(\theta_0)g'(\theta) - w'(\theta_0)h'(\theta)) \\ &+h(\theta)(w'(\theta_0) + g'(\theta)) - g(\theta)(v'(\theta_0) + h'(\theta))\big]\,\frac{d}{d\theta} \\ &= \big[w'(\theta_0)(\theta - \theta_0) + g(\theta)\big]\,\frac{d}{d\theta}.\end{aligned}$$

Let us divide both sides of this equality by $\theta - \theta_0$ and take the limit $\theta \to \theta_0$. Then from relations (4) and L'Hopital theorem one can easily see that $w'(\theta_0) = 0$. □

Lemma 5. *Under the equivalence transformation $\tilde{\theta} = f(\theta)$ singular (resp. regular) points of the vector field W are mapped to singular (resp. regular) points of the vector field $\tilde{W} = \tilde{w}(\tilde{\theta})\frac{d}{d\tilde{\theta}}$.*

Proof. a) Let $w(\theta_0) \neq 0$. Then there is a finite derivative $f'(\theta_0)$. If it is not, then there is a neighborhood of the singular point θ_0, where it is continuous and $\lim\limits_{\theta\to\theta_0} f'(\theta) = \pm\infty$ (the sign depends on $\deg f$). For the transformed vector field $\tilde{W}$ we have the relation $\tilde{w}(\tilde{\theta})\frac{d}{d\tilde{\theta}} = w(\theta)f'(\theta)\frac{d}{d\theta}$. Hence, $\tilde{w}(f(\theta)) = w(\theta)f'(\theta)$. Since $w(\theta) \neq 0$ in the above neighborhood, then

$$f'(\theta) = \frac{\tilde{w}(f(\theta))}{w(\theta)}, \tag{5}$$

and from the continuity of the functions f and $\tilde{w}$ it is following that the limit $\lim\limits_{\theta\to\theta_0} f'(\theta)$ is finite. Recall that a continuity of the function $\tilde{w}$ follows from the properties that any equivalence transformation f maps C^1-vector fields to C^1-vector fields.

Now suppose that $\tilde{w}(f(\theta_0)) = 0$ (i.e., $f(\theta_0)$ is singular). Since the right side of differential equation (5) satisfies Lipschitz condition for the argument $f(\theta)$ uniformly on θ (function $\tilde{w}$ is continuously differentiable), then in a neighborhood of the point θ_0 a unique solution $f(\theta)$ exists. The constant solution in this neighborhood $f(\theta) = f(\theta_0) = \text{const}$ satisfies equation (5). But the definition of the equivalence transformation contradicts to the property of one-to-one mapping. That is $\tilde{\theta}_0 = f(\theta_0)$ is a regular point.

b) Let θ_0 be a singular point and suppose that $\tilde{w}(f(\theta_0)) \neq 0$ (i.e., $f(\theta_0)$ is regular). By Lemma 3 there is a neighborhood of θ_0, in which all points are regular (except θ_0). From the previous part of the proof we get (see (5)) for the regular points relations $f'(\theta) > 0$ ($f'(\theta) < 0$) if $\deg f > 0$ ($\deg f < 0$). Therefore, it is easy to show that the inverse map $\theta = f^{-1}(\tilde{\theta})$ belongs to the class of equivalence transformations. Applying the reasoning of the previous part, we see that the regular point $\tilde{\theta}_0$ goes to the regular point θ_0 under the mapping f^{-1}. So we have a contradiction. □

Corollary 1. *The number of singular points is an invariant of any equivalence transformation.*

4 Realizations of a two-dimensional noncommutative algebra

Taking into account Lemmas 1 and 3, we suppose that there is a vector field W with $n \geqslant 1$ singular points θ_k. It is easy to show that applying the composition of equivalence transformations and rotation of the circle we achieve that $\theta_k = \dfrac{2\pi k}{n}$, $k = 0, 1, \ldots, n-1$. Consider the interval $\triangle_k = (\theta_k, \theta_{k+1})$ and denote

$$\bar{\theta}_k = \frac{\theta_k + \theta_{k+1}}{2} = \frac{\pi(2k+1)}{n}.$$

We construct the following continuously differentiable transformation for f on $\triangle_k$ satisfying the conditions

$$f(\theta_k) = \theta_k, \quad f(\theta_{k+1}) = \theta_{k+1}, \quad f(\bar{\theta}_k) = \bar{\theta}_k. \tag{6}$$

Suppose that $w(\theta) > 0$, $\theta \in \triangle_k$. Consider the Cauchy problem for this interval:

$$w(\theta) f'(\theta) = 1 - \cos(n f(\theta)), \quad f(\bar{\theta}_k) = \bar{\theta}_k. \tag{7}$$

Its solution is

$$f(\theta) = \frac{2}{n} \arctan\left(-n I(\theta)\right) + \theta_k, \quad \text{where} \quad I(\theta) = \int_{\bar{\theta}_k}^{\theta} \frac{d\theta}{w(\theta)}. \tag{8}$$

The integral $I(\theta)$ converges for any point of the interval $\triangle_k$. By virtue of Lemma 4 the integral diverges at the ends of this interval:

$$\lim_{\theta \to \theta_k + 0} I(\theta) = -\infty, \quad \lim_{\theta \to \theta_{k+1} - 0} I(\theta) = +\infty.$$

It is easy to show that the transformation (8) satisfies conditions (6) and maps the vector field $w(\theta)\frac{d}{d\theta}$ to the vector field $(1 - \cos(n\tilde{\theta}))\frac{d}{d\tilde{\theta}}$.

If $w(\theta) < 0$ for $\theta \in \triangle_k$ then we can analogously obtained the equivalence transformation that maps the vector field $w(\theta)\frac{d}{d\theta}$ to $(\cos(n\tilde{\theta}) - 1)\frac{d}{d\tilde{\theta}}$.

Now, if we consider the vector field at the intervals $\triangle_k$, $k = 0, 1, \ldots, n-1$, then substituting the function $w(\theta) = \pm(\cos(n\theta) - 1)$ (omitting the tilde) in equation (1), it is easy to obtain the solution for the function $v(\theta)$ on these specified intervals:

$$v(\theta) = \frac{1}{n}\sin(n\theta) + \lambda_k(1-\cos(n\theta)), \quad \lambda_k \in \mathbb{R}. \tag{9}$$

As a result, we have the following assertion.

Theorem 1. *Any realization of the two-dimensional noncommutative algebra of vector fields on a circle is equivalent to the form*

$$\left\langle \left(\frac{1}{n}\sin(n\theta) + \lambda_k(\theta)(1-\cos(n\theta))\right)\frac{d}{d\theta},\ \sigma_k(\theta)(1-\cos(n\theta))\frac{d}{d\theta} \right\rangle, \tag{10}$$

where $\lambda_k(\theta)$ and $\sigma_k(\theta) = \pm 1$ are constants on the intervals $\left(\frac{2\pi k}{n}, \frac{2\pi(k+1)}{n}\right)$.

The questions about the reducibility of realizations (10) to simpler ones, their inequivalence and the number of inequivalent realizations will be discussed in a forthcoming work.

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On Some Exact Solutions of Convection Equations with Buoyancy Force

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Some exact solutions of the equations of convective motion are constructed. Examples of the non-stationary, stationary and self-similar flows are considered.

1 Introduction

The natural convection is a type of macroscopic flows which are intensively studied in modern fundamental sciences. The development of the experimental and theoretical researches has led to the isolation of the convection in a separate field of fluid mechanics. The natural convection mechanisms define different processes which have wide applications and educational values. The results of investigations in this field are applied in the power engineering, metallurgy, meteorology, chemistry and crystal physics [3].

The equations of the natural convection are complicated because the gradients of the temperature, the concentration and the density are taken into account. Also the equation of the state must be considered. Symmetry methods are successfully used to study mathematical models of convective flows. In particular, in [8] the group classification problem is solved for all constant physical parameters in the case that the buoyancy force depends on the temperature and concentration linearly. Group classification for transport coefficients depending on the temperature is carried out in [4] for just thermal convection model. A number of exact solutions for the description of the convective flows is presented, e.g., in the papers of Russian researchers V.V. Pukhnachev, V.K. Andreev, R.V. Birikh and members of their research groups.

To study the basic laws of convection, the equations of motion are usually chosen in according to the simplified model of the process. Let us consider a binary mixture with the equation of state

$$\rho = \rho_0 F(T, C),$$

where ρ_0 is the mixture density at the mean values of temperature T_0 and concentration C_0, T and C are the deviations from their mean values, F is arbitrary positive function defining the buoyancy force. The equations of motion under the

Soret effect influence have the form

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\rho_0^{-1}\nabla p + \nu\nabla^2\boldsymbol{u} + F(T,C)\,\mathbf{g}, \tag{1}$$

$$T_t + \boldsymbol{u}\cdot\nabla T = \chi\nabla^2 T, \tag{2}$$

$$C_t + \boldsymbol{u}\cdot\nabla C = D\nabla^2 C + D_T\nabla^2 T, \tag{3}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4}$$

where $\boldsymbol{x} = (x^1, x^2, x^3)$ is the coordinate vector, $\boldsymbol{u} = (u^1, u^2, u^3)$ is the velocity vector, p is the pressure, $\mathbf{g} = (0, 0, -\mathrm{g})$ is the acceleration of external force vector, ν, χ, D are the kinematic viscosity, thermal diffusivity and diffusion coefficients respectively, D_T is the thermal diffusion coefficient. The case $D_T < 0$ corresponds to positive Soret effect, when the lighter component is driven towards the higher temperature region. In the case of negative Soret effect we have $D_T > 0$, and the opposite situation is observed.

2 Symmetry properties of the governing equations

We assume that the constants ν, χ, D, and D_T do not vanish and $D \neq \chi$. Using the notations

$$T = \frac{\chi - D}{D_T}u^5, \quad C = u^5 + u^6, \quad p = \rho_0 u^4,$$

system (1)–(4) takes the form

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla u^4 + \nu\nabla^2\boldsymbol{u} + F(u^5, u^6)\,\mathbf{g}, \tag{5}$$

$$u^5_t + \boldsymbol{u}\cdot\nabla u^5 = \chi\nabla^2 u^5, \tag{6}$$

$$u^6_t + \boldsymbol{u}\cdot\nabla u^6 = D\nabla^2 u^6, \tag{7}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{8}$$

We note that equations (6) and (7) have the same differential form. The detailed solution of the group classification problem for system (5)–(8) with respect to the function F is carried out in [1]. There are 43 forms of the classified function F and the admitted algebras of the generators are presented in this paper. The function F has power, logarithmic and exponential dependencies on the temperature, the concentration and their combinations. The kernel L_0 of the admitted Lie algebras is spanned by the generators

$$X_0 = \partial_t, \quad X_{12} = x^1\partial_{x^2} - x^2\partial_{x^1} + u^1\partial_{u^2} - u^2\partial_{u^1},$$

$$H^0(f_0) = f_0\partial_{u^4}, \quad H^i(f_i) = f_i\partial_{x^i} + f_{it}\partial_{u^i} - f_{itt}x^i\partial_{u^4},$$

where $f_i = f_i(t)$, $i = 0, 1, 2, 3$, are arbitrary smooth functions of t. It should be noted that the generators L_0 are inherited by equations (5)–(8) from the set of generators from the Lie symmetry algebra for the Navier–Stokes equations [2].

The extension of the algebra L_0 depends on the forms of function F and consists of the linear combinations of the generators

$$Z = 2t\partial_t + \sum_{i=1}^{3}\left(x^i\partial_{x^i} - u^i\partial_{u^i}\right) - 2u^4\partial_{u^4}, \quad Y = x^3\partial_{u^4},$$
$$T^1 = u^5\partial_{u^5}, \quad T^2 = \partial_{u^5}, \quad C^1 = u^6\partial_{u^6}, \quad C^2 = \partial_{u^6}.$$

3 Exact solutions

It is well known that symmetries can be used to construct exact solutions. In practice it is very important to know the exact solutions of partial differential equations for describing simpler mathematical models or for testing numerical methods. Usually the goal is to reduce the governing equations to ordinary differential equations or to partial differential equations with less independent variables, which can be solved. Of course this reduction gives only a class of solutions which have certain properties. But often these solutions help to define the specificity of some physical phenomena. The simplest classes of exact solutions are stationary and self-similar solutions. In terms of the symmetry approach the stationary solutions are invariant under translations and the self-similar solutions are invariant under dilations. By means of these simpler classes of solutions we can study certain properties of new phenomena and use them as initial point to construct solutions of more complex problems with approximate methods. That is why the study of such types of solutions is very important.

I. Firstly, we present the example of the stationary solutions. These solutions are characterized by the independence of the unknown functions on time t. We consider the subalgebra

$$\langle X_0,\ H^1(1) + \lambda H^0(1),\ H^2(1)\,\rangle,$$

where λ is an arbitrary constant. Then the solution of the governing equations can be found in the form

$$u^1 = u(x^3), \quad u^2 = v(x^3), \quad u^3 = w(x^3),$$
$$u^4 = P(x^3) + \lambda x^1, \quad u^5 = \widehat{T}(x^3), \quad u^6 = \widehat{C}(x^3).$$

The factor system can be written as

$$wu_{x^3} = -\lambda + \nu u_{x^3x^3}, \quad wv_{x^3} = \nu v_{x^3x^3}, \quad w\widehat{T}_{x^3} = \chi\widehat{T}_{x^3x^3},$$
$$ww_{x^3} = \nu w_{x^3x^3} - P_{x^3} - F(\widehat{T},\widehat{C})\mathrm{g}, \quad w_{x^3} = 0, \quad w\widehat{C}_{x^3} = D\widehat{C}_{x^3x^3}.$$

In the case $w = w_0 = 0$ the solution has the simple form

$$
\begin{aligned}
&u = \frac{\lambda}{2\nu}(x^3)^2 + k_1 x^3 + k_2, \quad v = k_3 x^3 + k_4, \quad w = 0, \quad \widehat{T} = k_5 x^3 + k_6,\\
&\widehat{C} = k_7 x^3 + k_8, \quad u^4 = -\mathrm{g}\int F(k_5 x^3 + k_6, k_7 x^3 + k_8)dx^3 + \lambda x^1 + k_9.
\end{aligned} \tag{9}
$$

If $w = w_0 \neq 0$ the solution can be written as

$$
\begin{aligned}
&u = \frac{\lambda}{w_0}x^3 + k_1 e^{x^3 w_0/\nu} + k_2, \quad v = k_3 e^{x^3 w_0/\nu} + k_4, \quad w = w_0,\\
&\widehat{T} = k_5 e^{x^3 w_0/\chi} + k_6, \quad \widehat{C} = k_7 e^{x^3 w_0/D} + k_8,\\
&u^4 = -\mathrm{g}\int F(k_5 e^{x^3 w_0/\chi} + k_6, k_7 e^{x^3 w_0/D} + k_8)dx^3 + \lambda x^1 + k_9.
\end{aligned} \tag{10}
$$

In formulas (9) and (10) k_i, $i = 1, \ldots, 9$, are constants. Solution (9) is the generalization of solution describing the Poiseuille flow which arises between two rigid walls under the horizontal pressure gradient action. There is a description of the classical Poiseuille flow in [5].

II. Here we present an example of a non-stationary solution. We consider the subalgebra of generators

$$\langle H^2(1),\ H^1(t)\rangle.$$

The solution has the form

$$
\begin{aligned}
&u^1 = \frac{x^1}{t} + U(t, x^3), \quad u^2 = v(t, x^3), \quad u^3 = w(t, x^3),\\
&u^4 = P(t, x^3), \quad u^5 = \widehat{T}(t, x^3), \quad u^6 = \widehat{C}(t, x^3).
\end{aligned}
$$

From the fourth equation of system (5)–(8) we obtain

$$w = -\frac{x^3}{t} + w_0(t).$$

Then the factor system can be written as

$$
\begin{aligned}
&U_t = \nu U_{x^3x^3} - \left(w_0 - \frac{x^3}{t}\right)U_{x^3} - \frac{1}{t}U, \quad v_t = \nu v_{x^3x^3} - \left(w_0 - \frac{x^3}{t}\right)v_{x^3},\\
&P_{x^3} = -F(\widehat{T}, \widehat{C})\mathrm{g} - \frac{2x^3}{t^2} - \frac{1}{t}w_0 - w_{0t},\\
&\widehat{T}_t = \chi\widehat{T}_{x^3x^3} - \left(w_0 - \frac{x^3}{t}\right)\widehat{T}_{x^3}, \quad \widehat{C}_t = D\widehat{C}_{x^3x^3} - \left(w_0 - \frac{x^3}{t}\right)\widehat{C}_{x^3}.
\end{aligned}
$$

Let us consider the change of variables [6]

$$\tau = \frac{1}{3}B^2 t^3 + A, \quad \xi = Btx^3 - B\int w_0\ dt + C,$$

$$U = -tQ_1(\tau, \xi), \quad v = Q_2(\tau, \xi), \quad \widehat{T} = Q_3(\tau, \xi), \quad \widehat{C} = Q_4(\tau, \xi),$$

where A, B, C are arbitrary constants. Using these notations we obtain the system of parabolic equations for the functions Q_i, $i = 1, \ldots, 4$,

$$Q_{1\tau} = \nu Q_{1\xi\xi}, \quad Q_{2\tau} = \nu Q_{2\xi\xi}, \quad Q_{3\tau} = \chi Q_{3\xi\xi}, \quad Q_{4\tau} = DQ_{4\xi\xi}.$$

After integration of these equations, we find the solution of the governing equations (1)–(4). Such form of solution allows to use well-known data about boundary-initial problems for parabolic equations.

III. It is well-known that the class of self-similar solutions is a wide-used class of solutions describing the motion in continuum mechanics. We present an example of a such solution for convection equations. One of the forms of the buoyancy force function presented in [1] is $F = (u^5)^\gamma f(u^6)$, where $\gamma \neq 0$ is arbitrary constant, f is arbitrary smooth function of u^6. Considering the subalgebra

$$\langle H^1(1),\ H^2(1),\ 3T^1 - \gamma Z \rangle$$

we find the solution in the form

$$u^1 = \frac{U(\xi)}{\sqrt{t}}, \quad u^2 = \frac{V(\xi)}{\sqrt{t}}, \quad u^3 = \frac{W(\xi)}{\sqrt{t}},$$
$$u^4 = \frac{P(\xi)}{t}, \quad u^5 = \frac{\widehat{T}(\xi)}{\sqrt[2\gamma]{t^3}}, \quad u^6 = \widehat{C}(\xi),$$

where $\xi = x^3/\sqrt{t}$ is the new independent variable.

From the fourth equation of the system (5)–(8) we find that $W = w_0 =$ constant and the factor system can be written as

$$U'\left(w_0 - \frac{1}{2}\xi\right) - \frac{1}{2}U - \nu U'' = 0, \quad V'\left(w_0 - \frac{1}{2}\xi\right) - \frac{1}{2}V - \nu V'' = 0,$$

$$P' - \frac{1}{2}w_0 + \mathrm{g}\widehat{T}^\gamma f(\widehat{C}) = 0, \quad \widehat{T}'\left(w_0 - \frac{1}{2}\xi\right) - \frac{3}{2\gamma}\widehat{T} - \chi\widehat{T}'' = 0,$$

$$\widehat{C}'\left(w_0 - \frac{1}{2}\xi\right) - D\widehat{C}'' = 0,$$

where the prime denotes the derivative with respect to ξ.

After the integration of this system of ODEs with reference to the handbook [7], we obtain the solution

$$U = k_1\Phi\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4\nu}\zeta^2\right) + k_2\frac{\zeta}{2\sqrt{\nu}}\Phi\left(1, \frac{3}{2}, \frac{1}{4\nu}\zeta^2\right),$$

$$V = k_3\Phi\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4\nu}\zeta^2\right) + k_4\frac{\zeta}{2\sqrt{\nu}}\Phi\left(1, \frac{3}{2}, \frac{1}{4\nu}\zeta^2\right), \tag{11}$$

$$\widehat{T} = k_5\Phi\left(\frac{3\chi}{2\gamma}, \frac{1}{2}, \frac{1}{4\chi}\zeta^2\right) + k_6\frac{\zeta}{2\sqrt{\chi}}\Phi\left(\frac{3\chi}{2\gamma} + \frac{1}{2}, \frac{3}{2}, \frac{1}{4\chi}\zeta^2\right),$$

$$\widehat{C} = k_7 + k_8 \operatorname{erf}\left(\frac{\zeta}{2i\sqrt{D}}\right), \qquad P = \frac{w_0}{2}\xi - \mathrm{g}\int \widehat{T}^\gamma f(\widehat{C})\, d\xi + k_9.$$

In the formulas (11) $\zeta = \xi - 2w_0$, k_j, $j = 1, \ldots, 9$, are constants, the functions $\Phi(a, b, \psi)$ are the Kummer functions and $\operatorname{erf}(\psi)$ is the error function, $i = \sqrt{-1}$. Real physical velocities, pressure, temperature and concentration of the fluid can be found from the corresponding expressions. It is interesting to note that the class of self-similar solutions can be used as intermediate asymptotics for numerical or practical experiment and understanding nature of flows.

All constructed solutions describe the flows of a binary mixture under the thermodiffusion effect and buoyancy force action. Different boundary value problems can be studied for these solutions. For example behavior of the velocity, temperature and concentration fields can be carried out for layers of liquids between rigid walls, with the free surface or with the interface.

4 Conclusion

Equations describing convection in a binary mixture with the thermodiffusion effect under buoyancy force action are considered. The results of group classification with respect to the buoyancy force function are used to construct certain exact solutions of the governing equations. These solutions can be useful to study simpler models of convection and for testing numerical methods.

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Group Classification of the Fisher Equation with Time-Dependent Coefficients

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The class of generalized Fisher equations with time-dependent coefficients is studied from the point of view of Lie symmetry. We find the associated equivalence groupoid and perform the exhaustive group classification of this class.

1 Introduction

The classical Fisher equation

$$u_t = bu_{xx} + au(1-u), \quad ab \neq 0, \tag{1}$$

first appeared seventy-five years ago in the seminal paper [5]. This equation was originally derived to model the propagation of a gene in a population. More precisely, the dependent variable, u, stands for the frequency of the mutant gene in a population distributed in a linear habitat, such as a shore line, with uniform density. Theorems on existence and uniqueness of bounded solutions for equations of the more general form $u_t = u_{xx} + F(t,x,u)$ were proved in [10]. Traveling-wave solutions of (1) were constructed in [1] (see also [3, 4, 11, 14]).

In recent papers [6, 15] it was proposed to consider generalized Fisher equations with a time-dependent diffusion coefficient, b, and a time-dependent favorability coefficient, a, namely the equations of the general form

$$u_t = b(t)u_{xx} + a(t)u(1-u), \tag{2}$$

where $a(t)$ and $b(t)$ are smooth nonvanishing functions. In practice these coefficients could represent long term changes in climate or short term seasonality [6]. Solutions for equations from this class were constructed in [6, 15].

The aim of this paper is to study symmetry properties of equations from class (2). In the next section we first look for equivalence and all admissible transformations (called also form-preserving or allowed ones) of this class. We describe the equivalence groupoid of class (2) by proving that this class is normalized in the generalized extended sense. Two simplest gauges of the arbitrary elements a and b are discussed in Section 3, and the gauge $a = 1$ is shown to be optimal. The exhaustive group classification of class (2) is performed in Section 4. The results obtained and further development of them are discussed in the Conclusion.

2 Admissible transformations

To solve the group classification problem for a class of differential equations it is important to describe exhaustively point transformations that preserve the general form of equations from the class and transform only its arbitrary elements. Such transformations are called *equivalence transformations* and form a group called the *equivalence group* of this class [17]. According to Ovsiannikov the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and the arbitrary elements of the class, where transformation components for independent and dependent variables do not depend upon arbitrary elements, i.e., they are projectible onto the space of independent and dependent variables. Neglect of the projectibility restriction leads to the notion of *generalized* equivalence group [13]. If new arbitrary elements are allowed to depend upon old ones in some nonpoint (possibly, nonlocal) way, then the corresponding equivalence group is called *extended*. The first examples of a generalized equivalence group and an extended equivalence group are presented in [13] and [7], respectively. The set of point transformations admissible in pairs of fixed equations from the class naturally possesses the groupoid structure with respect to the composition of transformations and hence it is called the *equivalence groupoid* of the class [18]. If the equivalence groupoid is generated by the equivalence group of a certain kind (usual, generalized, extended, etc.), the class of differential equations is called *normalized* in the same sense [18, 20].

Proposition 1. *The usual equivalence group $G^{\sim}$ of class* (2) *comprises the transformations*

$$\tilde t = T(t), \quad \tilde x = \delta_1 x + \delta_2, \quad \tilde u = \varepsilon u + \frac{1-\varepsilon}{2}, \quad \tilde a = \frac{a}{T_t\varepsilon}, \quad \tilde b = \frac{\delta_1^2}{T_t} b,$$

where $T(t)$ is an arbitrary smooth function with $T_t \neq 0$, δ_1 and δ_2 are arbitrary constants with $\delta_1 \neq 0$ and $\varepsilon = \pm 1$.

Remark 1. Up to composing to each other and to continuous equivalence transformations, the equivalence group $G^{\sim}$ contains three independent discrete transformations

$$\mathcal{T}_1\colon\ (t,x,u,a,b) \mapsto (-t,x,u,-a,-b),$$
$$\mathcal{T}_2\colon\ (t,x,u,a,b) \mapsto (t,-x,u,a,b),$$
$$\mathcal{T}_3\colon\ (t,x,u,a,b) \mapsto (t,x,1-u,-a,b).$$

Proposition 2. *Class* (2) *is normalized with respect to its generalized extended equivalence group* $\hat{G}^{\sim}$ *formed by the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \omega(t)u + \theta(t),$$
$$\tilde{a} = \frac{a}{T_t\omega}, \quad \tilde{b} = \frac{{\delta_1}^2}{T_t}b,$$

where $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$*,* δ_1 *and* δ_2 *are arbitrary constants with* $\delta_1 \neq 0$*,*

$$\omega = \frac{(\alpha e^{\int a\,dt} + \beta)(\gamma e^{\int a\,dt} + \delta)}{(\alpha\delta - \beta\gamma)e^{\int a\,dt}}, \quad \theta = -\gamma\frac{\alpha e^{\int a\,dt} + \beta}{\alpha\delta - \beta\gamma},$$

the constant pairs (α, β) *and* (γ, δ) *are defined up to nonvanishing multipliers and* $\alpha\delta - \beta\gamma \neq 0$*. In other words, the equivalence groupoid of class* (2) *is generated by the generalized extended equivalence group* $\hat{G}^{\sim}$ *of this class.*

Proof. Suppose that an equation from class (2) is connected with another equation

$$\tilde{u}_{\tilde{t}} = \tilde{b}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{a}(\tilde{t})\tilde{u}(1 - \tilde{u}) \tag{3}$$

from the same class by a point transformation of the general form

$$\tilde{t} = T(t, x, u), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u),$$

where $|\partial(T, X, U)/\partial(t, x, u)| \neq 0$. Admissible point transformations between quasilinear evolution equations are known to satisfy the constraints $T_x = T_u = X_u = 0$ [19]. (The first two constraints are valid for all evolution equations [9].) Therefore from the very beginning we can assume that

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U(t, x, u),$$

where $T_t X_x U_u \neq 0$. The prolongation of this transformation to the derivatives $\tilde{u}_{\tilde{t}}$ and $\tilde{u}_{\tilde{x}\tilde{x}}$ is

$$\tilde{u}_{\tilde{t}} = \frac{1}{T_t X_x}\left(X_x(U_t + U_u u_t) - X_t(U_x + U_u u_x)\right),$$
$$\tilde{u}_{\tilde{x}\tilde{x}} = \frac{1}{X_x^3}\left(X_x(U_{xx} + 2U_{xu}u_x + U_u u_{xx} + U_{uu}u_x^2) - X_{xx}(U_x + U_u u_x)\right).$$

After we substitute the expressions for the tilded variables into (3), we obtain an equation in the variables without tildes. It should be an identity on the manifold $\mathcal{L}$ determined by (2) in the second-order jet space J^2 with the independent variables (t, x) and the dependent variable u. To involve the constraint between variables of J^2 on the manifold $\mathcal{L}$ we substitute the expression of u_t implied by equation (2). The splitting of the obtained condition with respect to the derivatives u_{xx} and u_x results in the determining equations for the functions T, X and U, which are parameterized by a, b, $\tilde a$ and $\tilde b$,

$$\begin{aligned}
&U_{uu}=0,\\
&bX_x^2-\tilde bT_t=0,\\
&X_tX_x^2U_u+2\tilde bT_tX_xU_{xu}-\tilde bT_tX_{xx}U_u=0,\\
&\tilde aT_tU(1-U)-aU_uu(1-u)-U_t+\frac{X_t}{X_x}U_x-\tilde bT_t\frac{X_{xx}}{X_x^3}U_x+\tilde b\frac{T_t}{X_x^2}U_{xx}=0.
\end{aligned}$$

The first two determining equations respectively imply that U is a function linear in u and X is a function linear in x, i.e.,

$$U=\omega(t,x)u+\theta(t,x)\quad\text{and}\quad X=\varphi(t)x+\psi(t).$$

Additionally the second equation gives the transformation component for b,

$$\tilde b=\frac{\varphi^2}{T_t}b.$$

When we substitute the expressions for U, X and $\tilde b$ into the other determining equations and then split the last one with respect to u, we get

$$2b\varphi\omega_x+(\varphi_tx+\psi_t)\omega=0, \tag{4}$$
$$\tilde aT_t\omega-a=0, \tag{5}$$
$$\tilde aT_t\omega\varphi(1-2\theta)-\varphi\omega_t-a\varphi\omega+\omega_x(\varphi_tx+\psi_t)+b\varphi\omega_{xx}=0, \tag{6}$$
$$\tilde aT_t\varphi\,\theta(1-\theta)+b\varphi\theta_{xx}-\varphi\theta_t+\theta_x(\varphi_tx+\psi_t)=0. \tag{7}$$

We obtain from (5) that ω is a function of t only, $\omega=\omega(t)$, and

$$\tilde a=\frac{a}{T_t\omega}.$$

Then it follows from (4) that $\varphi_t=\psi_t=0$ and Eq. (6) reduces to

$$\omega_t=a(1-\omega-2\theta). \tag{8}$$

It is easy to see from Eq. (8) that the function θ depends only upon t, $\theta=\theta(t)$. Therefore Eq. (7) can be rewritten as

$$\omega\theta_t=a\theta(1-\theta). \tag{9}$$

Looking for the usual equivalence group of class (2), we should split Eqs. (8) and (9) with respect to the arbitrary element a (b is not involved in these equations). Thus we get $\omega_t=\theta_t=0$, $\theta(\theta-1)=0$ and $\omega=1-2\theta$. The solution of this system results in Proposition 1.

In order to find the generalized equivalence group of class (2) and describe the whole equivalence groupoid of this class we have to solve the system of first-order nonlinear ODEs (8) and (9) with respect to the functions ω and θ. This system reduces to the system of the same form with $a = 1$ by the transformation $\bar{t} = \int a\,dt$. The simplest way to integrate this system is to consider three cases for values of θ. If $\theta = 0$ or $\theta = 1$, then $\omega = C_1 e^{-\bar{t}} + 1$ or $\omega = C_1 e^{-\bar{t}} - 1$, respectively. If $\theta \neq 0, 1$, then from Eq. (9) we have $\omega = \theta(1-\theta)/\theta_{\bar{t}}$ and hence Eq. (8) is equivalent to the equation $\theta_{\bar{t}\bar{t}} = \theta_{\bar{t}}$. As a result in this case we get $\theta = C_1 e^{\bar{t}} + C_0$ and $\omega = (C_1 e^{\bar{t}} + C_0)(1 - C_1 e^{\bar{t}} - C_0) C_1^{-1} e^{-\bar{t}}$. It is easy to show that the union of the above three parameterized families of solutions coincides with the family of (ω, θ) presented in Proposition 2.

Since the transformation component for u nonlocally depends upon the arbitrary element a, the corresponding transformations form the generalized extended equivalence group $\hat{G}^{\sim}$ of class (2). We have also shown that every admissible point transformation in class (2) is generated by an element of $\hat{G}^{\sim}$, i.e., the equivalence groupoid of class (2) is generated by $\hat{G}^{\sim}$. Therefore, this class is normalized in the generalized extended sense. □

It is obvious that there are equations in class (2) that are $\hat{G}^{\sim}$-equivalent but not $G^{\sim}$-equivalent. Therefore the usage of the group $\hat{G}^{\sim}$ strongly simplifies the group analysis of class (2).

Corollary 1. *Equation* (2) *reduces to the classical Fisher equation*

$$u_t = u_{xx} + u(1-u) \tag{10}$$

by a point transformation if and only if for some positive constant λ the coefficients a and b satisfy the condition

$$\lambda b^2 - 2\frac{b_{tt}}{b} + 3\frac{b_t^2}{b^2} = a^2 - 2\frac{a_{tt}}{a} + 3\frac{a_t^2}{a^2}. \tag{11}$$

Remark 2. After the exclusion of the constant λ by differentiation with respect to t, the condition (11) reduces to the condition

$$\frac{b_{ttt}}{b} + 6\frac{b_t^3}{b^3} - 6\frac{b_{tt}b_t}{b^2} = \frac{a_{ttt}}{a} - 4\frac{a_{tt}a_t}{a^2} + 3\frac{a_t^3}{a^3} - aa_t - 2\frac{a_{tt}}{a}\frac{b_t}{b} + 3\frac{a_t^2}{a^2}\frac{b_t}{b} + a^2\frac{b_t}{b},$$

which is more convenient for checking using a computer algebra package.

Remark 3. The condition (11) is satisfied if and only if the function b is expressed in terms of a as

$$b = \frac{\lambda(\alpha\delta - \beta\gamma) a e^{\int a\,dt}}{(\alpha e^{\int a\,dt} + \beta)(\gamma e^{\int a\,dt} + \delta)},$$

where λ is a positive constant, the constant pairs (α, β) and (γ, δ) are defined up to nonvanishing multipliers and $\alpha\delta - \beta\gamma \neq 0$.

3 Gauging of arbitrary elements

Equivalence transformations allow us to simplify the group classification problem by gauging arbitrary elements, often reducing their number. For example, there is one arbitrary parameter-function $T(t)$ in the equivalence groups $G^\sim$ and $\hat G^\sim$ of class (2). It means that we can gauge an arbitrary element, either a or b, to a simple constant value, e.g., to 1. Thus the equivalence transformation

$$\tilde t=\int b\,dt,\quad \tilde x=x,\quad \tilde u=u$$

belonging to the group $G^\sim$ maps class (2) onto its subclass singled out by the constraint $b=1$. The arbitrary element $\tilde a$ of the mapped class equals a/b. The gauge $a=1$ is realized by the similar point transformation from $G^\sim$

$$\tilde t=\int a\,dt,\quad \tilde x=x,\quad \tilde u=u.$$

In the corresponding mapped class we have $\tilde a=1$ and $\tilde b=b/a$.

Since class (2) is normalized in the generalized sense, it is easy to find the equivalence group of its subclass with $b=1$ (resp. $a=1$) by setting $\tilde b=b=1$ (resp. $\tilde a=a=1$) in the transformations from $\hat G^\sim$. We obtain the following corollaries of Proposition 2.

Corollary 2. *The class of equations of the general form*

$$u_t=u_{xx}+a(t)u(1-u),\tag{12}$$

where a runs through the set of nonvanishing smooth functions of t, is normalized in the generalized extended sense. The generalized extended equivalence group $\hat G^\sim_a$ of this class consists of the transformations

$$\tilde t=\delta_1^2t+\delta_0,\quad \tilde x=\delta_1x+\delta_2,\quad \tilde u=\omega(t)u+\theta(t),\quad \tilde a=\frac{a}{\delta_1^2\omega},$$

where δ_i, $i=0,1,2$, are arbitrary constants with $\delta_1\neq0$,

$$\omega=\frac{(\alpha e^{\int a\,dt}+\beta)(\gamma e^{\int a\,dt}+\delta)}{(\alpha\delta-\beta\gamma)e^{\int a\,dt}},\quad \theta=-\gamma\frac{\alpha e^{\int a\,dt}+\beta}{\alpha\delta-\beta\gamma},$$

the constant pairs (α,β) and (γ,δ) are defined up to nonvanishing multipliers and $\alpha\delta-\beta\gamma\neq0$.

Corollary 3. *The class of equations of the general form*

$$u_t=b(t)u_{xx}+u(1-u),\tag{13}$$

where b runs through the set of nonvanishing smooth functions of t, is normalized in the usual sense. The usual equivalence group $G^\sim_b$ of this class consists of the transformations

$$\tilde t=\ln\frac{\alpha e^t+\beta}{\gamma e^t+\delta},\quad \tilde x=\delta_1x+\delta_2,$$
$$\tilde u=\frac{(\alpha e^t+\beta)(\gamma e^t+\delta)}{(\alpha\delta-\beta\gamma)e^t}u-\gamma\frac{\alpha e^t+\beta}{\alpha\delta-\beta\gamma},\quad \tilde b=\frac{\delta_1^2(\alpha e^t+\beta)(\gamma e^t+\delta)}{(\alpha\delta-\beta\gamma)e^t}b,$$

where δ_j, $j=1,2$, are arbitrary constants with $\delta_1\neq0$, the constant quadruple $(\alpha,\beta,\gamma,\delta)$ is defined up to a nonzero multiplier and $\alpha\delta-\beta\gamma\neq0$.

Remark 4. The group $G^\sim_b$ contains two discrete equivalence transformations

$$\mathcal{T}'\colon (t,x,u,b)\mapsto(t,-x,u,b),\quad \mathcal{T}''\colon (t,x,u,b)\mapsto(-t,x,1-u,-b).$$

An interesting question is which of the above two gauges is preferable for further consideration. Class (12) is still normalized only in the generalized extended sense. At the same time class (13) is normalized with respect to its usual equivalence group. This is why we can expect that it is easier to perform the group classification in class (13) rather than in class (12).

Corollary 4. *Eq.* (13) *reduces to the classical Fisher equation* (10) *by a point transformation if and only if the coefficient b has the form*

$$b(t)=\frac{\lambda(\alpha\delta-\beta\gamma)e^t}{(\alpha e^t+\beta)(\gamma e^t+\delta)},$$

where λ is a positive constant, the constant quadruple $(\alpha,\beta,\gamma,\delta)$ is defined up to a nonzero multiplier and $\alpha\delta-\beta\gamma\neq0$.

4 Lie symmetries

We study the Lie symmetries of equations from class (13) using the classical approach [17]. We fix an equation, $\mathcal{L}$, from class (13) and search for vector fields of the form

$$Q=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u$$

that generate one-parameter point symmetry groups of $\mathcal{L}$. These vector fields form the maximal Lie invariance algebra, $A^{\max}=A^{\max}(\mathcal{L})$, of the equation $\mathcal{L}$. Any such vector field Q satisfies the infinitesimal invariance criterion, i.e., the action of the second prolongation, $Q^{(2)}$ of Q, on the equation $\mathcal{L}$ results in the condition identically satisfied for all solutions of this equation. Namely we require that

$$Q^{(2)}\big(u_t-b(t)u_{xx}-u(1-u)\big)\Big|_{\mathcal{L}}=0. \qquad (14)$$

After the elimination of u_t by means of (13), equation (14) becomes an identity in six variables, t, x, u, u_x, u_{xx} and u_{tx}. In fact, equation (14) is a polynomial in the variables u_x, u_{xx} and u_{tx}. The coefficients of different powers of these variables are zero, which gives the determining equations for the coefficients τ, ξ and η. When we solve these equations, we immediately find that $\tau=\tau(t)$ and $\xi=\xi(t,x)$. This completely agrees with the general results on point transformations between evolution equations [9]. Then the remaining determining equations take the form

$$\eta_{uu}=0,\quad 2b\,\xi_x=(b\tau)_t,\quad 2b\,\eta_{xu}=b\xi_{xx}-\xi_t,$$
$$\eta-\eta_t+b\eta_{xx}+(\tau_t-2\eta-\eta_u)u+(\eta_u-\tau_t)u^2=0.$$

The integration of the first and the second equations of this system results in

$$\xi=\frac{(b\tau)_t}{2b}x+\zeta(t)\quad\text{and}\quad \eta=\eta^1(t,x)u+\eta^0(t,x),$$

where ζ, η^1 and η^0 are arbitrary functions of their arguments. Then the third equation becomes

$$\left(\frac{1}{2b}(b\tau)_t\right)_t x+\zeta_t+2b\eta^1_x=0. \tag{15}$$

After substituting the above expression for η into the fourth equation and splitting this equation with respect to u, we get

$$\eta^1=-\tau_t,\quad \eta^0_t-b\eta^0_{xx}-\eta^0=0,\quad 2\eta^0=\tau_t+b\eta^1_{xx}-\eta^1_t.$$

The last system implies that η^1 and η^0 do not depend upon x and are expressed via derivatives of the function τ as follows

$$\eta^1=-\tau_t,\quad \eta^0=\frac{1}{2}(\tau_t+\tau_{tt}).$$

The function τ satisfies the equation $\tau_{ttt}-\tau_t=0$, i.e.,

$$\tau=c_1e^t+c_2e^{-t}+c_3$$

for some constants c_1, c_2 and c_3. We take into account all the constraints derived and split Eq. (15) with respect to x. As a result we immediately obtain $\zeta=c_0=\mathrm{const}$ and the classifying equation $((b\tau)_t/b)_t=0$ which essentially includes both the residuary uncertainties in the coefficients of the vector field Q and the arbitrary element b. We integrate the classifying equation once to obtain

$$(c_1e^t+c_2e^{-t}+c_3)b_t=(-c_1e^t+c_2e^{-t}+c_4)b \tag{16}$$

with one more constant, c_4. To find the common part of the maximal Lie invariance algebras (the kernel algebra) of equations from class (13), we split equation (16) with respect to b and b_t, which gives $c_i=0$, $i=1,2,3,4$. The only nonzero constant c_0 corresponds to the operator ∂_x (Case 0 of Table 1).

The set V_b of coefficient tuples of equations of the form (16) satisfied by a fixed value of the parameter-function b is a linear space. The dimension, k_b, of this space coincides with the dimension of Lie symmetry extension for the same value of b. It is easy to prove that $k_b < 3$ and, if $k_b = 2$, any element of V_b satisfies the equation $c_4^2 = c_3^2 - 4c_1c_2$.

An at least one-dimensional extension of Lie symmetry exists only for values of b satisfying (16) with $(c_1, c_2, c_3) \neq (0,0,0)$, i.e., if

$$b(t) = c_5 \exp\left(\int \frac{-c_1e^t + c_2e^{-t} + c_4}{c_1e^t + c_2e^{-t} + c_3}\mathrm{dt}\right) \tag{17}$$

for some nonzero constant c_5. Then an extension operator is of the form

$$X_2 = (c_1e^t + c_2e^{-t} + c_3)\partial_t + \frac{c_4}{2}x\partial_x + \left[(c_2e^{-t} - c_1e^t)u + c_1e^t\right]\partial_u.$$

The integration in (17) gives the following values of b:

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\left|\frac{2c_1e^t + c_3 - \nu}{2c_1e^t + c_3 + \nu}\right|^{\frac{c_4}{\nu}} \quad \text{if} \quad D > 0,\ c_1 \neq 0,$$

$$b = \frac{c_5 e^{\frac{c_4}{c_3}t}}{(c_2e^{-t} + c_3)^{1-\frac{c_4}{c_3}}} \quad \text{if} \quad D > 0,\ c_1 = 0,$$

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\exp\left(-\frac{2c_4}{2c_1e^t + c_3}\right) \quad \text{if} \quad D = 0,\ c_1 \neq 0,$$

$$b = c_5 \exp\left(t + \frac{c_4}{c_2}e^t\right) \quad \text{if} \quad D = 0,\ c_1 = 0,$$

$$b = \frac{c_5}{c_1e^t + c_2e^{-t} + c_3}\exp\left(\frac{2c_4}{\nu}\arctan\frac{2c_1e^t + c_3}{\nu}\right) \quad \text{if} \quad D < 0.$$

Here $D = c_3^2 - 4c_1c_2$ and $\nu = \sqrt{|D|}$. When one uses the scaling transformation with respect to x, the constant c_5 can be set to $\operatorname{sign} c_5$, i.e., $c_5 = \pm 1 \bmod G_b^{\sim}$.

In fact the above expressions for b can be simplified more by transformations from the group $G_b^{\sim}$. Up to $G_b^{\sim}$-equivalence the parameter quadruple (c_1, c_2, c_3, c_4) can be assumed to belong to the set

$$\{(0,0,1,\sigma),\ (0,1,0,\kappa),\ (1,1,0,\rho) \mid \sigma \geqslant 0,\ \kappa = \pm 1,\ \rho \in \mathbb{R}\}.$$

Indeed, combined with multiplication by a nonzero constant, each transformation from the equivalence group $G_b^{\sim}$ is extended to the coefficient quadruple of equation (16) in the following way:

$$\tilde{c}_1 = c_1\delta^2 - c_3\gamma\delta + c_2\gamma^2, \qquad \tilde{c}_2 = c_1\beta^2 - c_3\alpha\beta + c_2\alpha^2,$$
$$\tilde{c}_3 = -2c_1\beta\delta + c_3(\alpha\delta + \beta\gamma) - 2c_2\alpha\gamma, \qquad \tilde{c}_4 = (\alpha\delta - \beta\gamma)c_4.$$

There are three $G^\sim_b$-inequivalent reduced forms of the triple (c_1, c_2, c_3) depending upon the sign of D,

$$(0,0,1) \quad \text{if} \quad D>0, \qquad (0,1,0) \quad \text{if} \quad D=0, \qquad (1,1,0) \quad \text{if} \quad D<0.$$

So, up to $G^\sim_b$-equivalence, which coincides with the general point equivalence, there are three types of equations from class (2) the maximal Lie symmetry algebras of which are two-dimensional. They are represented by Cases 1–3 of Table 1. In view of the constraint $c_4^2 = c_3^2 - 4c_1c_2$ any case of extension of the kernel algebra by two linearly independent operators reduces by equivalence transformations to Case 4 of Table 1, where $b = \pm e^t$. Since class (13) is normalized, there are no additional equivalence transformations between the cases listed in Table 1.

As a result we have proven the following theorem.

Theorem 1. *The kernel algebra of class* (13) *is* $A^\cap = \langle \partial_x \rangle$. $G^\sim_b$*-inequivalent Lie symmetry extensions for class* (13) *are exhausted by those presented in Table 1.*

Table 1. The group classification list for class (13).

no.	b	Basis elements of $A^{\max}$
0	$\forall$	∂_x
1	$\varepsilon e^{\sigma t}$	$\partial_x, \quad \partial_t + \dfrac{\sigma}{2} x\partial_x$
2	$\varepsilon \exp(t + \kappa e^t)$	$\partial_x, \quad e^{-t}\partial_t + \dfrac{\kappa}{2} x\partial_x + e^{-t}u\partial_u$
3	$\dfrac{\varepsilon}{\cosh t} \exp(\rho \arctan e^t)$	$\partial_x, \quad 2\cosh t\, \partial_t + \dfrac{\rho}{2} x\partial_x + (e^t - 2\sinh t\, u)\, \partial_u$
4	εe^t	$\partial_x, \quad \partial_t + \dfrac{1}{2} x\partial_x, \quad e^{-t}(\partial_t + u\partial_u)$

Here ε, σ, κ, ρ are constants, $\varepsilon = \pm 1 \bmod G^\sim_b$, $\sigma \geqslant 0 \bmod G^\sim_b$, $\sigma \neq 1$ and $\kappa = \pm 1 \bmod G^\sim_b$.

The classification list adduced in Table 1 represents by itself the result of group classification problem for class (2) up to $G^\sim$-equivalence.

5 Conclusion

On the face of it, the class (2) of variable-coefficient Fisher equations is a quite simple object for symmetry analysis. It consists of semilinear (1+1)-dimensional second-order evolution equations and is parameterized by only two functions of the single variable t. Group classification problems have already been solved for much more general classes of evolution equations, see, e.g., [2, 12]. At the same time the group classification for class (2) cannot be directly extracted from existing group classifications of wider classes with reasonable effort. Moreover it turns out that class (2) has a number of interesting properties. In particular it possesses a nontrivial generalized extended equivalence group and it is normalized with respect to this group. It is also mapped by proper gauging of arbitrary elements to its subclass (13) that is normalized in the usual sense and the equivalence algebra of which is finite dimensional. Therefore it is convenient to solve the group classification problem in class (13) by the algebraic method although in this paper we have applied the standard method, which is based upon the direct solution of the determining equations for Lie symmetries up to point equivalence.

When one analyzes results of group classification problems for various classes of variable-coefficient PDEs (see, e.g., [8, 19–22]), one can observe that constant coefficient equations usually admit the widest Lie symmetry groups within such classes. In other words they represent the most symmetric cases. We unexpectedly discovered that the Lie symmetry group of the classical Fisher equation (1) is not widest within class (2). This fact can easily be interpreted in terms of mappings between classes of differential equations [20–22]. Namely class (2) is mapped by a family of point transformations to a class of similar but simpler structure. The mapped class consists of equations $u_t = f(t)u_{xx} + g(t)u^2$, where the arbitrary elements f and g run though the set of smooth nonvanishing functions of t, and this class is more convenient for group classification than the initial class (2). Under the above mapping the classical Fisher equation is transformed to a variable-coefficient equation and the equation presenting the Lie symmetry extension of highest dimension in class (2) (Case 4 of Table 1) is transformed to a constant-coefficient equation. An extended consideration of the aforementioned fact will be the subject of a forthcoming paper.

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Hidden and Conditional Symmetry Reductions of Second-Order PDEs and Classification with respect to Reductions

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We investigate hidden and conditional symmetries of second-order PDEs with linear and nonlinear additional conditions, and classes of equations having symmetries of such types. We also discuss some connections of hidden symmetry and reductions of differential equations and approaches to classification of equations with respect to reductions.

1 Introduction

In this paper we study reductions of PDEs and non-Lie symmetries of PDEs resulting in reductions. We will discuss some general ideas and look at some examples, e.g., of nonlinear wave equations with linear and nonlinear additional conditions.

Here we mean by reduction of a PDE a transformation of this PDE into another PDE with smaller number of independent variables. Solutions of this new PDE form a subset of solutions of the initial PDE. Such reduction may be performed either just to simplify the process of finding of exact (or approximate) solutions, or to separate some special set of solutions.

Within the standard Lie framework [17, 19], the process of search for exact solutions starts with search for Lie symmetries. Then we classify these Lie symmetries to get inequivalent reductions: then use the inequivalent subalgebras of the Lie invariance algebra to find “inequivalent” reductions. However, with this algorithm we get first far too many reductions and solutions; and lose far too many reductions and solutions.

Though within the Lie framework it is possible to find inequivalent reductions. In the 1983 paper by Fushchych and Serov [8] Lie reductions were found by means of the same principle as was later used in finding of Q-conditional symmetries (see [9]): ansatzes reducing equations under consideration were found as solutions of the condition (as in the definition of a Q-conditional operator)

$$Q[u] = 0,$$

but the operator Q was the general form of the Lie invariance operator. The benefit of this approach as compared to finding of inequivalent subalgebras and their invariants [20], and then building of Lie invariant ansatzes from them is that we get no redundant (equivalent or identical) reductions and solutions.

We should also note that application of the direct method for finding of reductions [3, 18] also gives inequivalent reductions. Actually what was done in [8] was similar to the direct method, but with limitation of symmetry operators only with Lie operators, and with the resulting limitation of the reductions. Though the direct method does not give any new results as compared to non-classical reductions [16] (and, in some cases, even as compared to classical Lie reductions), it gives a reasonably algorithmic method to obtain inequivalent reductions.

However, it is more or less feasible to use the direct method (and thus to obtain inequivalent reductions straightforwardly) only for equations with small number of dimensions. For multidimensional equations, Lie reductions can be algorithmically obtained using inequivalent subalgebras and their invariants (as to such reductions of the multidimensional wave equations see e.g. [4–7, 13, 20]).

So it is interesting to find some ways to study non-Lie symmetries and to classify reductions for wider classes of equations than it would be feasible by means of the direct method of reduction.

Why it is important to study reductions of PDEs

It is already well known that reductions of a PDE usually show much more than its Lie symmetries, not only with respect to exact solutions, but also with respect to the properties of the equation [3, 18].

Another reason to study and classify reductions even for the equations were only Lie reductions exist: it is interesting to have classification of the exact solutions of a PDE that were already found.

This research has also some negative motivation: there exist lots of papers that present solutions that are in reality equivalent to already known ones, and many papers listing exact solutions of a PDE present lots of redundant solutions.

When studying interesting PDE and their symmetries people are actually interested in their solutions. Very often an equation is regarded to be good and interesting if it has just one good and interesting solution.

A very popular subject for studies may be higher dimensional equations that combine or inherit good properties and solutions of lower dimensional good equations.

Solutions obtained by the direct reduction are related to the equation's symmetry properties — Q-conditional symmetry of this equation (such symmetries are also called non-classical or non-Lie symmetries) as was rigourously proved in [26].

Discussing relations between reductions and the conditional symmetry, we should note that that symmetry of two-dimensional reduced equations is often wider than symmetry of the initial equation. Then, reduction to two-dimensional equations allows finding new non-Lie solutions and hidden symmetries of the initial equation (see, e.g., papers by Abraham-Schrauner [1, 2]). Such study of new symmetries of reduced equations was probably done first in the papers by Kapitanskii [14, 15] without using of a special term for such symmetry.

The classification problem for systems of evolution equations with respect to generalised conditional (Lie–Bäcklund) symmetries was solved in [21, 22].

New (as compared to Lie) reductions may be also obtained by adding of another equation to the initial equation. One prominent example of such new reduction with an additional condition is the d'Alembert–Hamilton system [12].

2 Reductions and hidden symmetry

The concept of "hidden symmetry" has quite a few different meanings in various contexts, and it is usually a symmetry not obtainable by some standard and straightforward procedure applicable to the models in this context.

We will consider hidden symmetry of PDE similarly to Type II hidden symmetry of ODE within the context of papers by Abraham-Schrauner [1, 2].

With respect to ODE, such symmetry arises as symmetry of equations with reduced order that is not a symmetry of the original equations. For a PDE, it is symmetry of the reduced equation (with reduced number of independent variables) not present in the original equation. We will consider all possible reductions to find hidden symmetries, not only symmetry reductions.

We use a definition of Type II hidden symmetry in terms of the conditional symmetry (see [23, 24]).

3 Examples of classes of PDEs reducible by means of certain ansatzes

One problem related to investigation of reduction of PDE is finding all equations in some class that can be reducible by a certain ansatz.

3.1 Equations reducible using translation operators

First example is description of all equations in the general class of the second-order PDEs with three independent variables that are invariant and conditionally invariant with respect to translation operators.

The example is very simple, but it is instrumental to illustrate difference in classification with respect to Lie symmetries and classification with respect to reductions.

We consider a general class

$$F = F(t, x, y, u, \underset{1}{u}, \underset{2}{u}) = 0. \qquad (1)$$

We describe all such equations having Lie symmetry with respect to the operator ∂_x and hidden symmetry with respect to the operator ∂_y after reduction by means of the operator ∂_x. The condition of such Lie and hidden symmetry is invariance of the equation (1) under the operator ∂_y on condition that $u_x = 0$:

$$\partial_x F\big|_{F=0} = 0, \qquad \partial_y F\big|_{F=0,\ u_x=0} = 0. \tag{2}$$

The general solution of the conditions (2) will be a function of all invariants of the operators ∂_x and ∂_y, that is of t, u, u_t, u_x, u_y, and of the conditional invariants

$$q^1 = u_x R^1(t, y, u, \underset{1}{u}, \underset{2}{u}), \quad q^2 = u_{xt} R^2(t, y, u, \underset{1}{u}, \underset{2}{u}),$$
$$q^3 = u_{xx} R^3(t, y, u, \underset{1}{u}, \underset{2}{u}), \quad q^4 = u_{xy} R^4(t, y, u, \underset{1}{u}, \underset{2}{u})$$

(being absolute invariants of ∂_x), where R^k are arbitrary functions that is reasonably determined on the relevant manifolds $u_x = 0$, $u_{xt} = 0$, $u_{xx} = 0$, $u_{xy} = 0$:

$$F(q^1, q^2, q^3, q^4, t, u, \underset{1}{u}, \underset{2}{u}) = 0.$$

3.2 Equations reducible using radial variables

We describe all equations in the general class (1) that can be reducible by means of radial variables

$$r = x^2 + y^2, \tag{3}$$
$$\rho = t^2 - x^2 - y^2. \tag{4}$$

We may use reduction of space variables using the Euclidean radial variable r and leave the "old" time variable t, or use the radial variable ρ in the Minkovsky space.

Reduction of equation (1) by means of a new variable (3) (here we have two new independent variables r and t) is equivalent to its conditional invariance under the rotation operator

$$J = x\partial_y - y\partial_x. \tag{5}$$

Conditional differential invariants with the condition

$$xu_y - yu_x = 0 \tag{6}$$

may be chosen as follows:

$$t, \quad r = x^2 + y^2, \quad u, \quad u_t, \quad xu_x + yu_y, \quad u_x^2 + u_y^2, \tag{7}$$

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}, \quad xu_x u_{xx} + (xu_y + yu_x)u_{xy} + yu_y u_{yy},$$

$$u_{tt}, \quad u_{xx} + u_{yy}, \quad u_{xt}^2 + u_{yt}^2, \quad xu_{xt} + yu_{yt},$$

$$\frac{u_k}{x_k}, \quad \frac{u_{kt}}{x_k}, \quad \frac{u_{kl}}{x_k x_l} - \epsilon_{kl}\frac{u_k}{x_k^3}. \tag{8}$$

We used notations $x_1 = x$, $x_2 = y$, $u_1 = u_x$, $u_2 = u_y$ etc.; $\epsilon_{kl} = 1$ if $k = l$ or $\epsilon_{kl} = 0$ if $k \neq l$. Note that expressions in (7) do not imply summation over k and l.

Obviously we can use as conditional differential invariants any of these invariants with adequately differentiable functional multipliers.

Invariants (7) represent a functional basis of absolute differential invariants for the operator (5) (see, e.g., [11]), and invariants (8) are proper conditional differential invariants under condition (6). It is easy to check directly that they are really differential invariants under such condition. The listed proper conditional differential invariants do not actually represent a functional basis, e.g., from (6) $\dfrac{u_x}{x} = \dfrac{u_y}{y}$, but we adduced all such invariants just to show their general structure.

The general form of the equation (1) reducible with the ansatz

$$u = \phi(t, r), \tag{9}$$

will be

$$F\left(I_A, \frac{u_k}{x_k}, \frac{u_{kt}}{x_k}, \frac{u_{kl}}{x_k x_l} - \epsilon_{kl}\frac{u_k}{x_k^3}\right) = 0, \tag{10}$$

where I_A is the functional basis of absolute differential invariants (7), and the remaining variables are represented by proper conditional differential invariants.

It is easy to check that equation (10) can be reduced by means of ansatz (9) to the form

$$f(t, r, \phi, \phi_t, \phi_r, \phi_{tt}, \phi_{tr}, \phi_{rr}) = 0, \tag{11}$$

and the class (11) may be studied to find equations having new symmetries. Equations with hidden symmetries then will be described by conditional differential invariants under (6) and these new symmetries.

Reduction of equation (1) by means of the new variable (4) is equivalent to its conditional invariance under the operators of Lorentz algebra

$$J_{01} = t\partial_x + x\partial_t, \qquad J_{02} = t\partial_y + y\partial_t, \qquad J = x\partial_y - y\partial_x. \tag{12}$$

Conditional differential invariants with the conditions

$$tu_x + xu_t = 0, \qquad tu_y + yu_t = 0, \qquad xu_y - yu_x = 0 \tag{13}$$

may be chosen as follows:

$$u, \quad x_\mu x_\mu, \quad x_\mu u_\mu, \quad u_\mu u_\mu, \quad \Box u, \quad u_\mu u_{\mu\nu} u_\nu, \quad u_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha, \tag{14}$$
$$u_{\mu\nu} u_{\nu\alpha} u_{\mu\alpha}, \quad x_\mu u_{\mu\nu} u_\nu, \quad x_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha,$$

and in (14) we imply summation over μ, ν, α;

$$\frac{u_\mu}{x_\mu}, \quad \frac{u_{\mu\nu}}{x_\mu x_\nu} - g_{\mu\nu}\frac{u_\mu}{x_\mu^3}, \tag{15}$$

and in (15) we do not imply summation over μ, ν, α. Here μ, ν, α take values from 0 to 2, and we used notations $x_0 = t$, $x_1 = x$, $x_2 = y$, $u_0 = u_t$, $u_1 = u_x$, $u_2 = u_y$ etc.; $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1)$.

Invariants (14) represent a functional basis of absolute differential invariants for the operator (12), and invariants (15) are proper conditional differential invariants under condition (13). It is easy to check directly that they are really differential invariants under such condition. The listed proper conditional differential invariants do not actually represent a functional basis, e.g., from (13) $\dfrac{u_x}{x} = \dfrac{u_y}{y}$, but we adduced such invariants just to show their general structure.

The general form of the equation (1) reducible with the ansatz

$$u = \phi(\rho), \tag{16}$$

will be

$$F\left(I_A, \frac{u_\mu}{x_\mu}, \frac{u_{\mu\nu}}{x_\mu x_\nu} - g_{\mu\nu}\frac{u_\mu}{x_\mu^3}\right) = 0, \tag{17}$$

where I_A is the functional basis of absolute differential invariants (14), and the remaining variables are represented by proper conditional differential invariants.

It is easy to check that equation (17) can be reduced by means of ansatz (16) to the form

$$f(\rho, \phi, \phi', \phi'') = 0, \tag{18}$$

and the class (18) may be studied to find equations having new symmetries. Equations with hidden symmetries then will be described by conditional differential invariants under (13) and these new symmetries.

The presented results can be naturally extended to arbitrary number of space dimensions.

An example of a nonlinear wave equation having conditional symmetry with respect to the Lorentz group with n space dimensions was given in 1985 paper by Fushchych and Tsyfra [10]:

$$\Box u = \frac{\lambda_0 u_0^2}{x_0^2} + \frac{\lambda_1 u_1^2}{x_1^2} + \cdots + \frac{\lambda_n u_n^2}{x_n^2}.$$

It is easy to see that this equation is actually constructed with first-order conditional differential invariants of the type $\dfrac{u_\mu}{x_\mu}$.

4 Classification of PDEs with respect to reductions

Usual classification of the classes of PDE with respect to symmetries allows utilization of algebraic methods.

Operators of conditional symmetry in general do not form Lie algebras, so we cannot use the standard algorithms of the Lie group classification for classification of equations with respect to their reductions.

The easiest example of inequivalent equations within well-known classes is equations that have special symmetries in lower dimensions, and thus have hidden symmetries for a higher dimension under consideration.

The equation

$$\Box u = \lambda u^k$$

is conformally invariant when the number of spatial variables $n = 3$, $k = 3$, but also has hidden conformal invariance when $k = 5$ (this is a "descendant" conformal invariance from $n = 2$).

Scheme of classification of PDEs with respect to reductions

Equivalence condition in our classification is equivalence of the system consisting of the equation (or class of equations) and the reduction condition.

1. Study possible reductions for a class of equations.
2. Study equivalence of the resulting class of the reduced equations and get its classification.
3. Then we find "original" equations from our reduced equations, and thus have some classification of the original class of equations.
4. If we have no conditional or hidden symmetries then our problem reduces to the usual Lie group classification.

Example of classification of a PDE with respect to reduction is

$$u_{tt} - u_{xx} - u_{yy} = F(u). \qquad (19)$$

We consider classification with respect to "absolute" reductions by means of the ansatzes $u = \varphi(v)$, $u = \varphi(v, w)$.

Following the results of [25] on compatibility of the reduction conditions, for the special case of two spatial variables, we can find possible types of the reduced equations:

$$\varphi'' = F(\varphi), \qquad \varphi'' + \frac{1}{v}\varphi' = F(\varphi), \qquad \varphi'' + \frac{2}{v}\varphi' = F(\varphi),$$

$$\varphi_{vv} - \varphi_{ww} = F(\varphi), \qquad \varphi_{vv} - \frac{1}{w}\varphi_w - \varphi_{ww} = F(\varphi).$$

Classification of these reduced equations will produce lists of special nonlinearities. To obtain such full classification, we used the direct reduction procedure. Even though we were unable to complete the procedure, we succeeded to describe all possible reductions and possible reduced equations, and thus enable full classification of reductions for this equation.

Such classification with respect to reduction also finally gives full description of all inequivalent Type II hidden symmetries for the initial equation (19) being Lie and conditional symmetries of the reduced equations.

5 Conclusion

We briefly presented the main ideas and outlines of algorithms related to classification problems with respect to possible reductions of PDEs, together with some simple examples. Here we do not consider rigorous proofs needed for description of equations conditionally invariant under arbitrary sets of operators.

Further research

1. Study of Lie and non-Lie symmetries of the systems of the reduction conditions for the class of equations under study.
2. Investigation of Lie and non-Lie symmetries of the classes of reduced equations. Finding exact solutions of the reduced equations. Note that many interesting classes of PDEs may be considered as classes of reduced equations for some classes of PDEs or single PDEs.
3. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.

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