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Schwinger Pair Production via Polons and the Origin of Stokes Phenomena

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Schwinger pair production of electrons and positrons in a strong electric field is a prediction of nonperturbative quantum field theory, in which the out-vacuum is superposed of multi-particle states of the in-vacuum. Solving the Dirac or Klein-Gordon equation in the background field, though a linear wave equation, and finding the pair-production rate is a difficult or nontrivial job. The phase-integral method has recently been introduced to compute the pair production in space-dependent electric fields, and a complex analysis method has been employed to calculate the pair production in time-dependent electric fields. In this paper, we apply the complex analysis method to a Sauter-type electric field and other hyperbolic-type electric fields that vanish in the past and future and show that the Stokes phenomena in pair production occur when the time-dependent frequency for a given momentum has finite simple poles (polons) with pure imaginary residues.

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I. INTRODUCTIONS FOR PUBLICATION

Quantum field theory in strong background fields has attracted much attention in theoretical and experimental physics and numerous papers have been published since Heisenberg and Euler [1] and Schwinger [2] obtained the one-loop effective action in a strong electromagnetic field in quantum electrodynamics (QED). One prominent feature of the one-loop QED action is the spontaneous pair production of electrons and positrons, known as the Schwinger effect, in a strong electric field in a proper Lorentz frame, which could not be explained by any perturbative method of summing a finite number of Feynman diagrams. The other aspect is the vacuum polarization due to the interaction of background photons with virtual electron-positron pairs. Contrary to the real effective action without simple poles in a pure magnetic field, the effective action in an electric field takes a complex value, whose real part is the vacuum polarization and whose imaginary part results from the pair production [2].

The gamma-function regularization has recently been introduced in the in-out formalism [3], which via the Cauchy residue theorem leads to a complex effective action in electric fields [4,5]. The vacuum persistence amplitude, twice the imaginary part of the effective action, is the sum of either all instantons for multi-pairs [6] or all residues of the vacuum polarization [4,5] and is thus determined by the mean number of produced pairs. The vacuum persistence amplitude implies the instability of the Dirac vacuum due to pair production. The complex effective action through the gamma-function regularization seems to provide a consistent quantum field theory for spontaneous pair production from the background [7].

Kim and Page have advanced the phase-integral method to explain the Schwinger effect in a spatial profile of electric field, for which the Dirac or Klein-Gordon equation becomes a tunneling problem with at least more than one pair of complex turning points [8]. However, the phase-integral method proved to extend to a temporal profile of electric field, for which the Dirac or Klein-Gordon equation becomes a scattering problem over a barrier, and the coefficient for the negative frequency solution is also determined by the relativistic instanton

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actions [8]. Dumlu and Dunne have further elaborated the phase-integral method in various profiles of time-dependent electric fields and found Stokes phenomena for specific profiles of electric fields with more than two pairs of turning points in the complex plane, which give rise to substructures of Schwinger pair production [9–11]. Some WKB approximations have been adopted to the Schwinger effect [12–16].

Recently Kim has found in the in-in formalism that the particle production originates from the geometric transition in the complex plane for time-dependent electric fields or expanding universes [17–19]. In the formalism the vacuum persistence amplitude is determined by the scattering matrix between the in-vacuum and the transported one along loops in the complex plane that starts from and returns to the base point for the in-vacuum *á la* the functional Schrödinger picture. The scattering matrix gives a null result when the in-vacuum evolves along the real time and then returns to the same base point. However, the time-dependent frequencies for the relativistic theory for a charge may have multiple simple poles including the one at the infinity and gain residues that lead to the nontrivial vacuum persistence amplitude and exhibit a rich structure such as the Stokes phenomena for the Schwinger effect. In short, the mean number for pair production is prescribed by [17–19]

$$N = \left| \sum_{L\geq 1, J} \exp\left[-i \oint_{C_J^{(L)}(t_0)} \omega(z) dz \right] \right|, \tag{1}$$

where the summation is over all possible, independent, loops J of winding number L with the base point t_0 for the in-vacuum. In fact, the structure of simple poles, the so-called polons, determines the mean number for the Schwinger effect.

To illustrate how the formula (1) works and to show the advantages of the complex analytical method, we apply Eq. (1) to a constant electric field with the vector potential

$$E_{\parallel} = E_0, \quad A_{\parallel}(t) = -E_0 t, \tag{2}$$

in which a charge q with mass m has the time-dependent frequency in the complex plane

$$\omega_{\mathbf{k}}(z) = \sqrt{\mu^2 + \Big(k_{\parallel} + qE_0 z\Big)^2}, \tag{3}$$

where $\mu^2 = m^2 + \mathbf{k}_{\perp}^2$ is an effective mass that the charge would experience along the longitudinal direction. Using the branch-cut in Ref. [8] and expanding the square root for large z in a power series

$$\begin{aligned} &\omega_{\mathbf{k}}(z) \\ &= qE_{\parallel} z \Bigg[1 - \sum_{l=1}^{\infty} \frac{(2l-2)!}{2^{2l-1} l!(l-1)!} \Bigg(\Big(\frac{z_0}{z}\Big)^l + \Big(\frac{z_0^*}{z}\Big)^l \Bigg) \\ &\quad + \sum_{k,l=1}^{\infty} \frac{(2l-2)!(2k-2)!}{2^{2l+2k-2} l!(l-1)!k!(k-1)!} \Big(\frac{z_0}{z}\Big)^l \Big(\frac{z_0^*}{z}\Big)^k \Bigg], \end{aligned} \tag{4}$$

where $z_0 = (k_{\parallel} + i\mu)/qE_0$, the simple pole at $z = 0$ occurs when $l = 2$ in the first sum and $k = l = 1$ in the second sum. Then, the sum of the residues is $-qE_0(z_0 - z_0^*)^2/8$ and hence the contour integral clockwise along a loop of winding number one recovers the Schwinger formula [2]

$$N_{\mathbf{k}} = e^{-i \oint_{C_J^{(1)}} dz \omega_{\mathbf{k}}(z)} = e^{-\frac{\pi \mu^2}{qE_0}}. \tag{5}$$

Note that all loops not enclosing the pole at the infinity with winding number $L \geq 2$ is homotipically equivalent to that of winding number one. Thus, this implies that the Schwinger formula (5) is exact. Similarly, the higher order corrections to the Wentzel-Kramers-Brillouin (WKB) approximation vanish in the phase-integral method, which explain why the relativistic action from the WKB approximation gives the exact result [8].

The organization of this paper is as follows. In Sec. II we revisit the Schwinger effect in a Sauter-type electric field, for which the exact solutions of the relativistic wave equation are known in terms of a special function, the in-vacuum and the out-vacuum are well-defined, and the Bogoliubov transformation is established. This field model has provided an arena for various approximation schemes to be tested. In Sec. III the formula for Schwinger pair production is applied to another hyperbolic electric field which vanishes at the past and future infinities, and whose direction reverses symmetrically in time. The solitonic nature of the Schwinger effect is explained via the Stokes phenomena. In Sec. IV the Schwinger effect is studied when the electric field effectively acts for a finite period of time but the electric field changes slower than the Sauter-type field.

II. SAUTER-TYPE ELECTRIC FIELD REVISITED

The Sauter-type electric field [20]

$$E_{\parallel}(t) = \frac{E_0}{\cosh^2(t/\tau)}, \quad A_{\parallel}(t) = -E_0\tau \tanh(t/\tau), \quad (6)$$

is one of nontrivial electric fields that has been most extensively studied in strong QED. In this work we focus on the scalar QED problem, though the study can be straightforwardly extended to spinor QED. Then the time-dependent frequencies of a charge q with mass m after the Fourier mode-decomposition of the Klein-Gordon equation take the form (in units of $\hbar = c = 1$)

$$\omega_{\mathbf{k}}(t) = \sqrt{\mu^2 + \Big(k_{\parallel} + qE_0\tau \tanh(t/\tau)\Big)^2}, \quad (7)$$

where $\mu^2 = m^2 + \mathbf{k}_{\perp}^2$. A conformal transformation $z = e^{t\tau}$ leads to a contour integral in the complex plane

$$\oint \omega_{\mathbf{k}}(t) = \tau \oint \frac{dz}{z(z^2+1)} \sqrt{\mu^2 (z^2+1)^2 + \Big(k_{\parallel}(z^2+1) - qE_0(z^2-1)\Big)^2}. \quad (8)$$

In order to apply the Cauchy residue theorem to the pair production formula (1), the frequency (8) should be made an analytical function in the complex plane, which can be done by properly choosing a branch-cut. In fact, the square root of the frequency (7) has the same form as in Ref. [19], in which the branch-cut and independent loops are discussed in detail. Using the branch-cut in Ref. [19], a simple pole at the infinity ($z = \infty$) leads to the relativistic action, after symmetrizing the momenta $k_{\parallel}$ and $-k_{\parallel}$ for equally produced pair with opposite directions along the longitudinal direction,

$$S_{(\infty)} := 2\pi \mathrm{Res}(\infty) = -\pi\tau \left[\sqrt{\mu^2 + (k_{\parallel} + qE_0\tau)^2} + \sqrt{\mu^2 + (k_{\parallel} - qE_0\tau)^2} \right]. \quad (9)$$

There are also a pair of finite simple poles at $z = \pm i$, and enclosing clockwise the pole at $z = i$ from the causality reason yields another action

$$S_{(i)} := 2\pi \mathrm{Res}(i) = 2\pi qE_0\tau^2. \quad (10)$$

Finally, adding the contributions from the independent loops of winding number one, the leading term for the mean number of pairs produced with the momentum $\mathbf{k}$ is given by

$$N_{\mathbf{k}} = e^{-\pi\tau\left(\omega_{\mathbf{k}}(t=-\infty) + \omega_{\mathbf{k}}(t=\infty)\right)} \left(1 + e^{2\pi\lambda_{\mathrm{sp}}}\right), \quad (11)$$

where the first term in the parenthesis comes from any loop not enclosing the pole at $z = i$, which is universal, while the second exponent comes from any loop enclosing the pole at $z = i$. Noting that $\lambda_{\mathrm{sp}} = qE_0\tau^2$ is a quantity for spinor QED, Eq. (11) is the leading term of the exact mean number from the Bogoliubov transformation between the in-vacuum and the out-vacuum [8].

A few comments are in order. The pair production and the total pairs produced per unit volume could be expressed in terms of the two dimensionless parameters introduced in Ref. [8]: the inverse of the integrated power multiplied by the Compton time

$$\frac{q}{2}\int_{-\infty}^{\infty} dt E_{\parallel}(t) = qE_0\tau, \quad \epsilon = \frac{m}{qE_0\tau}, \quad (12)$$

and the peak strength to the Schwinger critical strength

$$\delta = \frac{qE_0}{\pi m^2}. \quad (13)$$

These dimensionless parameters may play a role similar to the Keldysh parameter for the Schwinger effect [21], provided that an electric field profile is characterized by both the peak strength and the effective duration of the field, as will be shown in Secs. III and IV. Then the total number of pairs integrated over the momentum in Ref. [8] is indeed expressed in terms of dimensionless parameters (12) and (13). Note that $\pi\epsilon^2\delta = 1/(qE_0\tau^2) = /\lambda_{\mathrm{sp}}$.

It was shown that the leading WKB result for scalar QED is equivalent the sum of the leading and the next-to-leading order WBK results for spinor QED [8], which was numerically confirmed in Ref. [22]. For a practical purpose, when the field decreases slower than the Sauter-type (6) but is still exponentially decaying, one may use the power integrated over a finite period of a few multiple of τ for $p \geq 1$:

$$\frac{q}{2}\int_{-p\tau}^{p\tau} dt E_{\parallel}(t) = qE_0\tau \tanh p, \quad \epsilon_p = \frac{m}{qE_0\tau \tanh p}. \tag{14}$$

For $p \gg 1$, the parameter ϵ_p rapidly approaches ϵ.

III. DIRECTION-REVERSING TIME-DEPENDENT ELECTRIC FIELD

An interesting model is provided by an electric field, for which the Klein-Gordon equation resembles a reflectionless scattering problem in quantum mechanics. For certain parameters for the profile of the field, Schwinger pair production sustains effectively for a finite period of time and then vanishes in the future infinity, exhibiting a solitonic nature [23,24]. The model electric field and the vector potential are

$$E_{\parallel}(t) = \frac{E_0 \sinh(t/\tau)}{\cosh^2(t/\tau)}, \quad A_{\parallel}(t) = \frac{E_0\tau}{\cosh(t/\tau)}. \tag{15}$$

The electric field and the vector potential are anti-symmetric and symmetric, respectively, with respect to $t = 0$. Note that the parameter (14) vanishes while the parameter (13) still has the same form. Then the charge with a given momentum has the time-dependent frequency

$$\omega_{\mathbf{k}}(t) = \sqrt{\mu^2 + \Big(k_{\parallel} - \frac{qE_0\tau}{\cosh(t/\tau)}\Big)^2}. \tag{16}$$

The solution to Eq. (16) is known in terms of special functions [25].

Using the same conformal transformation as in Sec. II, the frequency in the complex plane is given by

$$\oint \omega_{\mathbf{k}}(t) = \tau \oint \frac{dz}{z(z^2+1)}\sqrt{\mu^2(z^2+1)^2 + \Big(k_{\parallel}(z^2+1) - 2qE_0\tau z\Big)^2}. \tag{17}$$

The large-z expansion leads to a simple pole at the infinity, whose residue gives an action

$$S_{(\infty)} = 2\pi \mathrm{Res}(\infty) = -2\pi\tau\omega_{\mathbf{k}}(t=\infty). \tag{18}$$

Together with the imaginary actions from a pair of poles at $z = \pm i$

$$S_{(\pm i)} = 2\pi \mathrm{Res}(\pm i) = -2i\pi qE_0\tau^2, \tag{19}$$

the mean number (1) is a sum of four loops of winding number one

$$\begin{aligned} N_{\mathbf{k}} &= e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)}\big|1 + 2e^{-i2\pi qE_0\tau^2} + e^{-i4\pi qE_0\tau^2}\big| \\ &= 4e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)}\cos^2\big(\pi qE_0\tau^2\big), \end{aligned} \tag{20}$$

where the first term comes from any loop not enclosing any pole at $z = \pm i$, the second term from two loops enclosing only one pole at $z = i$ or $z = -i$, and the last term from any loop enclosing both poles. The phase-integral in the complex time of t may lead to the interference effect due to infinite pairs of complex tunneling points, which shows the substructure of the Schwinger effect [9–11].

Note that when the parameters satisfy

$$qE_0\tau^2 = \Big(n + \frac{1}{2}\Big), \tag{21}$$

any charged pair cannot be detected in the future infinity. The pair production from the exact Bogoliubov transformation under the condition (21) confirms that the Schwinger effect indeed exhibits a solitonic nature, which corresponds to the reflectionless scattering of the wave equation [23,24]. Here, we have shown that the solitonic characteristic is a consequence of Stokes phenomena due to a destructive interference among possible loops enclosing simple poles with pure imaginary residues. The Stokes phenomenon also occurs for Gibbons-Hawking radiation in the global coordinates in any odd-dimensional spacetime [18]. A physical interpretation may be that each pair of electron and positron produced before $t = 0$ reverses the acceleration after $t = 0$ in an exactly symmetric way and annihilates each other.

IV. UNIDIRECTIONAL TIME-DEPENDENT ELECTRIC FIELD

As the last model, we consider an electric field similar to Eq. (15) but time-symmetrical

$$E_{\parallel}(t) = \frac{E_0 \cosh(t/\tau)}{\sinh^2(t/\tau)}, \quad A_{\parallel}(t) = \frac{E_0 \tau}{\sinh(t/\tau)}. \tag{22}$$

The electric field is symmetric with respect to $t = 0$ and does not change the direction. Further, the electric field blows up immediately before and after $t = 0$ and in a physical sense is not realizable. The model (22) is theoretically interesting since one may test whether pairs are produced even for a cusp profile with an effective duration of $2p\tau$. The parameter (14) now given by

$$\epsilon_p = \frac{m \sinh p}{q E_0 \tau} \tag{23}$$

grows exponentially for $p \gg 1$. The time-dependent frequency for each momentum becomes

$$\omega_{\mathbf{k}}(t) = \sqrt{\mu^2 + \Big(k_{\parallel} - \frac{qE_0\tau}{\sinh(t/\tau)}\Big)^2}. \tag{24}$$

The solution to Eq. (24) is not known in terms of special functions [25].

Using the same conformal transformation as in Sec. II, the frequency in the complex plane is given by

$$\oint \omega_{\mathbf{k}}(t) = \tau \oint \frac{dz}{z(z^2-1)} \sqrt{\mu^2 (z^2-1)^2 + \Big(k_{\parallel}(z^2-1) - 2qE_0\tau z\Big)^2}. \tag{25}$$

The large-z expansion gives a simple pole at the infinity, whose residue contributes the action

$$S_{(\infty)} = 2\pi \mathrm{Res}\big(\infty\big) = -2\pi\tau\omega_{\mathbf{k}}(t=\infty). \tag{26}$$

However, the pair of poles at $z = \pm 1$ along a counter-clockwise loop enclosing one pole now give the real action

$$S_{(\pm 1)} = 2\pi \mathrm{Res}\big(\pm 1\big) = -2\pi q E_0 \tau^2. \tag{27}$$

The mean number (1) after summing four loops of winding number one is

$$\begin{aligned} N_{\mathbf{k}} &= e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty)} \big|1 + 2e^{-2\pi qE_0\tau^2} + e^{-4\pi qE_0\tau^2}\big| \\ &= 4e^{-2\pi\tau\omega_{\mathbf{k}}(t=\infty) - 2\pi qE_0\tau^2} \cosh^2\big(\pi qE_0\tau^2\big). \end{aligned} \tag{28}$$

A few comments are in order. The existence of a pair of finite simple poles does not always guarantee the destructive interference among independent loops. As the two model fields in Secs. III and IV explicitly show the pair production, the presence of pure imaginary residues at pairs of finite poles is a necessary condition for the formula (1) to exhibit an oscillatory nature and thus the Stokes phenomena. The orientation of loops should be chosen in a physically meaningful sense, otherwise the residues change the sign and give exponentially blowing up of produced pairs. The choice of the loop orientation may be related to the causality.

V. CONCLUSION

Schwinger pair production, a nonperturbative quantum effect, of electrons and positrons in a strong electric field is a consequence that a positive frequency solution for the in-vacuum in the past infinity splits into another positive solution and a negative frequency solution in the future infinity when scattered by the time-dependent vector potential. The Bogoliubov transformation between the in-vacuum and the out-vacuum gives rise to the mean number of pairs produced by the electric field. The exact solutions are known only for a few profiles of electric fields and appropriate approximation schemes have to be employed to compute the pair production.

In this paper, we have applied a recently introduced complex analytical method for the Schwinger effect to a few time-dependent electric field profiles, which include the Sauter-type field in Sec. II, a direction-reversing field in Sec. III, and a unidirectional field in Sec. IV. The time-dependent frequency for each momentum exhibits an interesting and rich structure when it is conformally transformed to the complex plane. The complex analytical method rests on the Cauchy residue theorem in finding the vacuum persistence amplitude and thereby the mean number of pairs produced by the electric field. In other words, the pair production (1) is entirely determined by the pole structure, polons, in the complex plane and by all independent loops enclosing poles in various combinations.

We have found that the frequencies (8) and (17) in the Sauter-type electric field (6) and the direction-reversing electric field (15), respectively, have the same pole structure. However, all the residues take real values for the Sauter-type electric field while the residues at a pair of finite simple poles become pure imaginary for the direction-reversing field (15), which leads to an oscillatory factor for the pair production. On the other hand, the unidirectional electric field (22) with an infinite cusp does have a different pole structure and an exponential behavior for pair production. It can be argued that the oscillatory nature of the pair production is a consequence of the destructive interference among finite simple poles with pure imaginary residues, which is the origin of the Stokes phenomenon.

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