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Novel Outlook on the Eigenvalue Problem for the Orbital Angular Momentum Operator

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Abstract: Based on the novel prescription for the power function, $(x + iy)^m$, the new expression for $\Psi(x, y|m)$, the eigenfunction of the operator of the third component of the angular momentum, \hat{M}_z , is presented. These functions are normalizable, single valued and, distinct to the traditional presentation, $(x + iy)^m = \rho^m e^{im\phi}$, are invariant under the rotations at 2π for any, not necessarily integer, m —the eigenvalue of \hat{M}_z . For any real m the functions $\Psi(x, y|m)$ form an orthonormal set, therefore they may serve as a quantum mechanical eigenfunction of \hat{M}_z . The eigenfunctions and eigenvalues of the angular momentum operator squared, \hat{M}^2 , derived for the two different prescriptions for the square root, $(m^2)^{1/2}$, $(m^2)^{1/2} = |m|$ and $(m^2)^{1/2} = \pm m$, are reported. The normalizable eigenfunctions of \hat{M}^2 are presented in terms of hypergeometric functions, admitting integer as well as non-integer eigenvalues. It is shown that the purely integer spectrum is not the most general solution but is just the artifact of a particular choice of the Legendre functions as the pair of linearly independent solutions of the eigenvalue problem for the \hat{M}^2 .

Keywords: nonrelativistic quantum mechanics; angular momentum; power of a complex number



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1. Introduction

It is commonly accepted that, from theoretical quantum mechanics, it follows that the spectrum of the eigenvalues of the angular momentum operator is discrete and is comprised of the integer values only; see, e.g., [1–4].

Non-integer values of angular momentum do not contradict the principles of quantum theory and were considered few times from different viewpoints. Back in 1932, Majorana noted that in the framework of relativistic quantum mechanics, the general formulation of a one-particle equation admits a solution with an arbitrary angular momentum, a predecessor of the theory of infinite dimensional representations of the Lorentz group [5]. Working on the analytical properties of the scattering amplitude, Regge Regge considered angular momentum as a continuous complex variable, and derived the singularities in the plane of the complex angular momentum that became universally known as Regge poles [6]. Götte et al., exploiting the freedom in fixing the orientation of phase discontinuity, introduced states with non-integer angular momentum and applied formalism of the propagation of light modes with the fractional angular momentum in the paraxial and non-paraxial regime [7]. Exploring polar solutions for the harmonic oscillator, Land [8] discovered that the Fock space equivalent to the Hilbert space wave functions, found by solving the Schrödinger equation in spherical coordinates is realized by acting with the creation and annihilation operators, allowing states with both integer and non-integer angular momentum.

In [9,10], we argued that if only physical conditions are imposed, what can be derived from the principles of quantum mechanics is that the spectrum is discrete with the only

condition that the difference $L - |m|$ is integer while L and m could be integer as well as non-integer. Throughout, $L(L + 1)$ is the eigenvalue of the angular momentum operator squared and m is the eigenvalue of the operator of the third component of the angular momentum.

In this paper, a solution of the eigenvalue problem for the quantum-mechanical orbital angular momentum (hereafter referred to as angular momentum) operator is reported obtained when only the physical requirement is imposed on the eigenfunction and show that in the framework of theoretical quantum mechanics, the eigenfunctions with both integer and non integer eigenvalues are allowed.

The paper is organized as follows. In Section 2, the multivaluedness and periodicity of the eigenfunctions of the operator of the third component of the angular momentum, \hat{M}_z , are discussed. A new prescription for the power of a complex variable, differing from the Euler-de Moivre prescription, $(x + iy)^m = \rho^m e^{im\phi}$, used in quantum mechanics, is presented. Based on this prescription, the eigenfunction of \hat{M}_z in terms of Gaussian hypergeometric series is given. This wave function is normalizable and distinct from the traditional eigenfunction being proportional to $e^{im\phi}$, is single valued and invariant under the rotations at 2π for any, not necessarily integer m . In other words, the requirement of single-valuedness of the wave function does not necessarily lead to the solution with only integer m . This eigenfunction satisfies the physical requirement of orthonormality, and, therefore, it can be considered as the wave function describing the physical state with the eigenvalue m , being not necessarily integer. In Section 3, it is discussed how the different prescriptions for the power function alter the eigenfunctions and the spectrum of the angular momentum operator squared \hat{M}^2 . The eigenfunction of \hat{M}^2 is found which is normalizable and satisfies physical requirements for an integer as well as non-integer L ; to a fixed value of L and a discrete spectrum of m , defined by the relation $|m| = L - k$, $k = \{0, 1, \dots, [L]\}$, where $[L]$ is an integer part of L . It is shown that the statement that the spectrum of eigenvalues consists of only integer L (see, e.g., [1–4]) is just an artifact of choosing the Legendre function, P_L^m , as an eigenfunction of \hat{M}^2 . Results are discussed in Section 4.

2. Eigenfunctions of \hat{M}_z That Are Single-Valued and Periodic for Integer as Well as for Non-Integer Eigenvalues

$\Psi(x, y|m)$, the eigenfunction of the operator of the third component of the angular momentum, \hat{M}_z , is defined as the solution of the following eigenvalue equation:

$$\hat{M}_z \Psi(x, y|m) = i \left(y \frac{d}{dx} - x \frac{d}{dy} \right) \Psi(x, y|m) = m \Psi(x, y|m), \quad (1)$$

where m is the eigenvalue and throughout the reduced Planck constant $\hbar = 1$. Solving Equation (1) for the complex $\Psi(x, y|m) = \Psi_R(x, y|m) + i\Psi_I(x, y|m)$ is the equivalent to solve the following system of the two coupled equations for the real and imaginary parts:

$$\left(y \frac{d}{dx} - x \frac{d}{dy} \right) \Psi_R(x, y|m) = -m \Psi_I(x, y|m), \quad \left(y \frac{d}{dx} - x \frac{d}{dy} \right) \Psi_I(x, y|m) = m \Psi_R(x, y|m). \quad (2)$$

Acting on Equation (2) with the operator $(y d/dx - x d/dy)$ results into the one and the same equation for both Ψ_R and Ψ_I :

$$\left(x \frac{d}{dy} - y \frac{d}{dx} \right)^2 \Psi_{R, I}(x, y|m) = -m^2 \Psi_{R, I}(x, y|m). \quad (3)$$

The two linearly independent solutions of the homogeneous differential Equation (3) can be presented as $\Psi_1(x, y|m) = C_1(x; y|m)F_1(x; y|m)$ and $\Psi_2 = C_2(x; y|m)F_2(x; y|m)$, where $F_{1,2}$ are linearly independent particular solutions of Equation (3) and $C_{1,2}$ satisfy the condition, $(yd/dx - xd/dy)C_{1,2} = 0$. If one chooses the $F_{1,2}$ that satisfy Equation (2), then $C_1 = C_2$ and the general solution of the eigenvalue Equation (1) is $\Psi(x, y|m) = C(x; y|m)[F_1(x; y|m) + iF_2(x; y|m)]$, where $C(x; y|m)$ is a complex function with an abso-

lute value, fixed by the physical requirement of normalizability, and the phase remains undetermined.

After (x, y) is transformed to another set of independent variables, $(f(x^2 + y^2), \zeta(x, y))$, where f is any differentiable function of $x^2 + y^2$, Equation (1) turns into an equation with the only variable, ζ . This technique of separating variables is used below but first let us quote and discuss the function that is cited in textbooks as a solution of Equation (1) [1–4]:

$$F(x, y|m) \sim (x + iy)^m. \quad (4)$$

If m is non-integer, $F(x, y|m)$ is undetermined, since for non-integer exponents, the power function is multivalued. In order for this function to be a solution of Equation (1) it must be defined as a differentiable function of x and y . This may be achieved using, e.g., the Euler–de Moivre prescription for the power of a complex number [11]:

$$(x + iy)^m = (\rho e^{i\phi})^m = \rho^m e^{im\phi} = \rho^m (\cos \phi + i \sin \phi)^m = \rho^m (\cos m\phi + i \sin m\phi), \quad (5)$$

where

$$\rho = |(x^2 + y^2)^{1/2}|, \sin \phi = \frac{y}{|(x^2 + y^2)^{1/2}|}, \cos \phi = \frac{x}{|(x^2 + y^2)^{1/2}|}, \quad (6)$$

and $|z|$ stands for the absolute value of z .

Note that in the chain of Equation (5) rotational symmetry of the original expression $(x + iy)^m$ is violated when m is non-integer. Indeed, due to the invariance of the Cartesian coordinates, x, y , under the 2π -rotation, $(x + iy)^m$ is formally rotationally invariant for any m . Expressions $(\rho e^{i\phi})^m$ and $\rho^m (\cos \phi + i \sin \phi)^m$ are invariant with respect to $\phi \rightarrow \phi + 2\pi$ for any m , while $\rho^m e^{im\phi}$ and $\rho^m (\cos m\phi + i \sin m\phi)$ violate rotational symmetry for a non-integer m .

If one requires all expressions in Equation (5) to be invariant with respect to $\phi \rightarrow \phi + 2\pi$, then, m must be an integer. Then, $(x + iy)^m$ is a single valued function of x and y . This connection between the rotational invariance and the single valuedness of $(x + iy)^m$ is caused the following assertion: if one requires the invariance of the wave function $(x + iy)^m \rightarrow \rho^m e^{im\phi}$ with respect to $\phi \rightarrow \phi + 2\pi$, this is equivalent to the single valuedness of this function [1–4]. Both these conditions are satisfied when m is integer and that is a reason why, based on the requirements of single valuedness or/and periodicity, it was declared that m can only be integer and these requirements were formalized in theoretical quantum mechanics as follows [1–4]:

$$\Psi(\rho, \phi|m) = \Psi(\rho, \phi + 2\pi k|m), \quad (7)$$

where the polar coordinates ρ and ϕ are given by Equation (6) and k is integer.

Imposing these physical conditions of single valuedness and rotational invariance on the wave function that is not observable has been criticized by Pauli [12] (see also in [13]). We agree with Pauli's criticism and emphasize that the purely integer spectrum of the eigenvalues is obtained only when the conditions of single valuedness and/or periodicity are realized in the framework of the Euler–de Moivre prescription (5). In fact, there are other possible prescriptions for determining $(x + iy)^m$ and it turns out that for one of these prescriptions, the eigenfunction $\Psi(x, y|m)$ will be single valued, differentiable with respect to its variables, invariant with respect to $\phi \rightarrow \phi + 2\pi k$ for any m , integer as well as non-integer. In other words, even if one imposes the requirement of single valuedness/rotational invariance on wave function, this still does not necessarily lead to a purely integer spectrum.

To demonstrate this, in Equation (3), the technique of separating variables is used transforming from (x, y) to (ρ, ζ) , where $\rho = |(x^2 + y^2)^{1/2}|$. Now the equation for $F(x, y|m)$ depends only on ζ and ζ is chosen so that Equation (3) takes a form of an equation, solutions

of which are well documented. Defining $\zeta = [1/2 - x/(2|(x^2 + y^2)^{1/2}|)] = [1/2 - x/(2\rho)]$ turns Equation (3) into the Gaussian hypergeometric equation:

$$\left[\zeta(1 - \zeta) \frac{d^2}{d\zeta^2} + \left(\frac{1}{2} - \zeta \right) \frac{d}{d\zeta} + m^2 \right] F(\zeta|m) = 0. \quad (8)$$

As known, any pair from the Kummer's 24 solutions can be chosen as a set of linearly independent solutions to the Gauss equation [11]; here the pair,

$$\begin{aligned} F_1(\zeta|m) &= {}_2F_1\left(m, -m; \frac{1}{2}; \zeta\right) = (1 - \zeta)^{\frac{1}{2}} {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \frac{1}{2}; \zeta\right), \\ F_2(\zeta|m) &= \zeta^{\frac{1}{2}} {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \frac{3}{2}; \zeta\right) = \zeta^{\frac{1}{2}}(1 - \zeta)^{\frac{1}{2}} {}_2F_1\left(1 + m, 1 - m; \frac{3}{2}; \zeta\right), \end{aligned} \quad (9)$$

is chosen, where ${}_2F_1(a, b; c; \zeta)$ is the Gaussian hypergeometric function [11,14,15]. Finally, $\Psi(x, y|m) = C(F_1 + iF_2)$, the eigenfunction of the operator of the third component of the angular momentum, is given by

$$\begin{aligned} \Psi(x, y|m) &= C(\rho|m) \left[{}_2F_1\left(m, -m; \frac{1}{2}; \frac{1}{2} - \frac{x}{2\rho}\right) \right. \\ &\quad \left. - im \frac{y}{\rho} \left(\frac{1}{2} + \frac{x}{2\rho} \right)^{-1/2} {}_2F_1\left(\frac{1}{2} + m, \frac{1}{2} - m; \frac{3}{2}; \frac{1}{2} - \frac{x}{2\rho}\right) \right], \end{aligned} \quad (10)$$

where the square root is determined via prescription $(f^2(x))^{1/2} = |f(x)|$.

It is straightforward to verify that the eigenfunction (10), as a function of x, y , is single-valued, invariant under the rotation at $2\pi k$, is continuous and has continuous derivatives of all orders up to infinity for any real, not necessarily integer m . Though $\Psi(x, y|m)$ contains a square root, it is infinitely differentiable. This is guaranteed as soon as $d(x/|x|)/dx = 0$ and $d(y/|y|)/dy = 0$ are satisfied which is readily demonstrated using Equation (6). Indeed, from $\cos \phi(x, y) = x/|(x^2 + y^2)^{1/2}|$, i.e., $\cos \phi(x, y)|_{y=0} = x/|x|$ it follows that

$$\frac{d \cos \phi(x, y)}{dx} \Big|_{y=0} = \frac{y^2}{|(x^2 + y^2)^{3/2}|} \Big|_{y=0} = 0 \quad \rightarrow \quad \frac{d(x/|x|)}{dx} = 0. \quad (11)$$

Similar to that of Equation (11), from $d \sin \phi(x, y)/dy|_{x=0} = d(y/|y|)/dy = 0$.

Let us consider particular values of m . Let us start with the integer $m = \pm N$, $N = 1, 2, 3 \dots$. The corresponding hypergeometric functions ${}_2F_1(N, -N; 1/2; z)$ and ${}_2F_1(1/2 + N, 1/2 - N; 3/2; z)$ are tabulated (see, e.g., [14]). Then, Equation (10) reads:

$$\Psi(x, y| \pm N) = \frac{C(\rho| \pm N)}{\rho^N} (x \mp iy)^N. \quad (12)$$

So, for an integer m , $\Psi(x, y|N)$ reproduces, up to the factor $C(\rho| \pm N)/\rho^N$, solution (4), $(x \mp iy)^N$.

Let us consider next the half-integer values of m . Explicit expressions are lengthy and involved; results for $m = \pm 1/2$ and $m = \pm 3/2$ are:

$$\begin{aligned} \Psi\left(x, y| \pm \frac{1}{2}\right) &= C\left(\rho| \pm \frac{1}{2}\right) \left[\left(\frac{1}{2} + \frac{x}{2\rho} \right)^{\frac{1}{2}} \mp i \frac{y}{2\rho} \left(\frac{1}{2} + \frac{x}{2\rho} \right)^{-\frac{1}{2}} \right], \\ \Psi\left(x, y| \pm \frac{3}{2}\right) &= C\left(\rho| \pm \frac{3}{2}\right) \left[\left(\frac{1}{2} + \frac{x}{2\rho} \right)^{\frac{1}{2}} \left(\frac{2x}{\rho} - 1 \right) \mp i \frac{y}{2\rho} \left(\frac{1}{2} + \frac{x}{2\rho} \right)^{-\frac{1}{2}} \left(1 + \frac{2x}{\rho} \right) \right]. \end{aligned} \quad (13)$$

A relation between the wave functions for the integer and half-integer m is found being verified for $m = 1/2, 3/2, 5/2$:

$$\left[\frac{\Psi(x, y| \pm \frac{N}{2})}{C(\rho| \pm \frac{N}{2})} \right]^2 = \frac{\Psi(x, y| \pm N)}{C(\rho| \pm N)}, \quad (14)$$

i.e., wave function for the half-integer m , $\Psi(x, y| \pm N/2)$, satisfies $\Psi^2(x, y| \pm N/2) \sim \Psi(x, y| \pm N)$, a relation similar to that which holds for Equation (4), $((x \pm iy)^{N/2})^2 = (x \pm iy)^N$. This result, along with Equation (12), indicates that the eigenfunction $\Psi(x, y|m)$, given by Equation (10), presents one possible prescription for the power function $(x + iy)^m$.

Let us demonstrate with the example of the half-integer m the importance of choosing prescription for the square root as $(f^2(x))^{1/2} = |f(x)|$. To this end, one moves from Cartesian to polar coordinates, $(x, y) \rightarrow (\rho, \phi)$, see Equation (6). In polar coordinates, the argument, $(1/2 - x/2\rho)$, of the hypergeometric functions reads: $(1 - \cos \phi)/2 = \sin^2(\phi/2)$. First, the prescription, $(f^2(x))^{1/2} = f(x)$ is used. Using in Equation (13) the known relation ${}_2F_1(a, b; b; z) = (1 - z)^{-a}$ [14,15], one obtains:

$$\begin{aligned} \Psi\left(x, y| \pm \frac{1}{2}\right) \rightarrow \Psi\left(\rho, \phi| \pm \frac{1}{2}\right) &= C\left(\rho| \pm \frac{1}{2}\right) \left[\left(\cos^2 \frac{\phi}{2}\right)^{\frac{1}{2}} \mp \frac{i}{2} \sin \phi \left(\cos^2 \frac{\phi}{2}\right)^{-\frac{1}{2}} \right] = \\ &C\left(\rho| \pm \frac{1}{2}\right) \left(\cos \frac{\phi}{2} \mp i \sin \frac{\phi}{2} \right) = C\left(\rho| \pm \frac{1}{2}\right) e^{\mp i \frac{\phi}{2}}. \end{aligned} \quad (15)$$

Equation (15), apart from the normalization factor, is the known $e^{im\phi}$, $m = \pm 1/2$, originated by the Euler–de Moivre prescription for the $(x + iy)^{1/2}$ and presented as a standard expression for the eigenfunction of \hat{M}_z [1–4]. Obviously, because of the rotational invariance of Cartesian coordinates, $x(\phi) = x(\phi + 2k\pi)$, $y(\phi) = y(\phi + 2k\pi)$, the left hand side of Equation (15), $\Psi(x, y| \pm 1/2)$, given by Equation (13), is invariant under the rotation $\phi \rightarrow \phi + 2\pi k$. On the other hand, the right-hand side (r.h.s.) of Equation (15) is invariant under the translations $\phi \rightarrow \phi + 4\pi k$, but not under $\phi \rightarrow \phi + 2\pi k$. This inconsistency stems from the prescription, $(f^2(x))^{1/2} = f(x)$ while deriving Equation (15). For example, $\cos(\phi/2)$ appeared in the real part of Equation (15) because $\cos(z)$ is used for $(\cos^2(z))^{1/2}$:

$${}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \sin^2\left(\frac{\phi}{2}\right)\right) = \left[1 - \sin^2\left(\frac{\phi}{2}\right)\right]^{1/2} = \left[\cos^2\left(\frac{\phi}{2}\right)\right]^{1/2} = \cos\left(\frac{\phi}{2}\right). \quad (16)$$

If $\phi = 2\pi k$, the left-hand side (l.h.s.) of this relation is unity, ${}_2F_1(-1/2, 1/2; 1/2; 0) = +1$, while, for r.h.s., one gets $\cos \pi k$, which, depending on k , can be either $+1$ or -1 . If $\phi = \pi(2k + 1)$, both l.h.s. and r.h.s. of Equation (16) vanish. This means that Equation (16) is valid only for ϕ with $\cos(\phi/2) \geq 0$; this condition, lacking from [14,15], is also noted in Ref. [16]. Meantime, both l.h.s. and r.h.s. of Equation (16) exist and are well defined for all values of ϕ and that calls for the question of how relation (16) has to be interpreted when $\cos(\phi/2) < 0$. Note that the inconsistency does not arise if, as an alternative, instead of Equation (16), the following relation,

$${}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \sin^2\left(\frac{\phi}{2}\right)\right) = \left|\cos\left(\frac{\phi}{2}\right)\right|, \quad (17)$$

is used, i.e., if one applies the prescription $(\cos^2(\phi/2))^{1/2} = |\cos(\phi/2)|$, but not $(\cos^2(\phi/2))^{1/2} = \cos(\phi/2)$.

If, instead of the prescription, $(f^2(x))^{1/2} = f(x)$, $(f^2(x))^{1/2} = |f(x)|$ is applied, one obtains:

$$\Psi\left(x, y| \pm \frac{1}{2}\right) \rightarrow \Psi\left(\rho, \phi| \pm \frac{1}{2}\right) = C\left(\rho| \pm \frac{1}{2}\right) \left(\left|\cos \frac{\phi}{2}\right| \mp \frac{i}{2} \sin \phi \left|\cos \frac{\phi}{2}\right|^{-1} \right). \quad (18)$$

Expression (18) is well defined for all values of ϕ and, most importantly, it is invariant under translations, $\phi \rightarrow \phi + 2\pi k$; the above-mentioned inconsistency disappears. Similarly, using, for the case of $m = 3/2$, the prescription, $(f^2(x))^{1/2} = |f(x)|$, one obtains the same result, $\Psi(\rho, \phi | \pm 3/2) = \Psi(\rho, \phi + 2k\pi | \pm 3/2)$ but for the prescription, $(f^2(x))^{1/2} = f(x)$, the resulting wave function is no longer invariant under $\phi \rightarrow \phi + 2\pi$.

From Equation (1) and its conjugation, using the properties of the Gaussian hypergeometric functions (see, e.g., [11]), one finds that $|\Psi(x, y | m)|^2$ only depends on $\rho = |(x^2 + y^2)^{1/2}|$ and the real and imaginary parts of $\Psi(x, y | m)$ satisfy the relation resembling the trigonometric identity, $\cos^2 x + \sin^2 x = 1$:

$$\frac{\Psi_R^2(x, y | m) + \Psi_I^2(x, y | m)}{|C(\rho | m)|^2} = 1. \quad (19)$$

Particular examples of the general result (19) are cases of integer $m = N$, when Equation (19) reduces to $\cos^2 N\phi + \sin^2 N\phi = 1$, and of half-integer m , quoted here for $m = 1/2$: $(\Psi_R^2(x, y | 1/2) + \Psi_I^2(x, y | 1/2)) / |C(\rho | 1/2)|^2 = |\cos(\phi/2)|^2 + \sin^2(\phi) / (4|\cos(\phi/2)|^2) = 1$. Relation (19) is another indication that the functions (4) and (10) belong to the same class since for any m , both Equations (4) and (10) satisfy relation $|\Psi / C|^2 = 1$.

The physical requirement the solution should satisfy is that $\Psi(x, y | m)$ must be orthonormal. To verify normalizability, let us use relation (19). Normalizability condition,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy |\Psi(x, y | m)|^2 = \pi \int_0^{\infty} d\rho |C(\rho | m)|^2 < \infty \quad (20)$$

can be readily realized by the appropriate choice of C . It suffices to choose $|C|^2 \sim \rho^{\gamma}$ with $\gamma < -1$ for $\rho \rightarrow \infty$ and C finite for $\rho \rightarrow 0$.

Orthogonality follows from the relation which is obtained from Equation (1):

$$\begin{aligned} & i(m' - m) \int \int_{-\infty}^{\infty} dx dy \Psi^*(x, y | m') \Psi(x, y | m) \\ &= \int_{-\infty}^{\infty} dy y \Psi^*(x, y | m') [\Psi(x, y | m)|_{x=\infty} - \Psi(x, y | m)|_{x=-\infty}] \\ & - \int_{-\infty}^{\infty} dx x \Psi^*(x, y | m') [\Psi(x, y | m)|_{y=\infty} - \Psi(x, y | m)|_{y=-\infty}]. \end{aligned} \quad (21)$$

Using the above-mentioned constraints on $C(\rho | m)$ one obtains $\Psi(x, y | m)|_{x,y=\infty} - \Psi(x, y | m)|_{x,y=-\infty} = 0$, from which follows that, if the integral in l.h.s. of Equation (21) exists, then, for $(m' - m) \neq 0$, the integral is zero being the condition of the orthogonality. Therefore, $\Psi(x, y | m)$ from Equation (10) fulfills every physical requirement which the eigenfunction of the third component of the quantum mechanical angular momentum operator should satisfy.

Using a certain prescription for the power function may lead to an expression of the wave function that is not invariant under the translations at $2k\pi$ for a non-integer eigenvalues. This case is realized by the Euler-de Moivre prescription, $(\rho e^{i\phi})^m = \rho^m e^{im\phi}$. On the other hand, if another prescription is applied, this may result in a wave function that is invariant under the translations at $2k\pi$ for integer as well as for non-integer eigenvalues. This case is realized by the eigenfunction (10), where the prescription, $(f^2(x))^{1/2} = |f(x)|$, is used. Prescription for the power function affects not only the features of eigenfunctions of \hat{M}_z but also the features of the eigenvalues of the operator of the angular momentum squared, as described in Section 3 below.

3. Eigenfunctions and Eigenvalues of \hat{M}^2

The eigenvalue equation for $\hat{M}^2 = \hat{M}_x^2 + \hat{M}_y^2 + \hat{M}_z^2$, the operator of the angular momentum squared,

$$\hat{M}^2 \Psi_M(L, m|\theta) = \left(\sin^2 \theta \frac{d^2}{d \cos^2 \theta} - 2 \cos \theta \frac{d}{d \cos \theta} - \frac{m^2}{\sin^2 \theta} \right) \Psi_M(L, m|\theta) = L(L+1) \Psi_M(L, m|\theta), \quad (22)$$

where θ is the polar angle and $L > 0$ is the eigenvalue, reduces to the equation for the Gaussian hypergeometric series, solutions of which can be represented by various linearly independent pairs of functions. A possible pair is $\Psi_{M(1)}$, $\Psi_{M(2)}$ [9]:

$$\begin{aligned} \Psi_{M(1)}(L, m|\theta) &= f_1(m|\theta) \Phi_1(L, m|\theta), \quad f_1(m|\theta) = (\sin^2 \theta)^{\frac{(m^2)^{1/2}}{2}}, \\ \Phi_1(L, m|\theta) &= {}_2F_1\left(\frac{1}{2} + \frac{(m^2)^{1/2}}{2} + \frac{L}{2}, \frac{(m^2)^{1/2}}{2} - \frac{L}{2}; \frac{1}{2}; \cos^2 \theta\right); \\ \Psi_{M(2)}(L, m|\theta) &= f_2(m|\theta) \Phi_2(L, m|\theta), \quad f_2(m|\theta) = \cos \theta (\sin^2 \theta)^{\frac{(m^2)^{1/2}}{2}}, \\ \Phi_2(L, m|\theta) &= {}_2F_1\left(1 + \frac{(m^2)^{1/2}}{2} + \frac{L}{2}, \frac{1}{2} + \frac{(m^2)^{1/2}}{2} - \frac{L}{2}; \frac{3}{2}; \cos^2 \theta\right). \end{aligned} \quad (23)$$

Any linear superposition of $\Psi_{M(1)}(L, m|\theta)$ and $\Psi_{M(2)}(L, m|\theta)$ is also a solution of Equation (22).

The only physical requirement for the functions $\Psi_{M(1,2)}(L, m|\theta)$ is the normalizability. The necessary condition for the normalizability is that $\Psi_M(L, m|\theta)$, presented as a product, $\Psi = f\Phi$, must be a regular function. Unfortunately it is not known how to realize such a condition for the product and the only option left is that normalizability can be achieved when the factors $f_j(m|\theta)$ and $\Phi_j(L, m|\theta)$ are regular for all values of their arguments which leads to a regular product $\Psi = f\Phi$.

Functions, $f_j(m|\theta) \sim (\sin^2 \theta)^{(m^2)^{1/2}/2}$, are regular for any θ when $(m^2)^{1/2} \geq 0$. Hypergeometric series, ${}_2F_1(a, b; c; \cos^2 \theta)$, converge for $\cos^2 \theta < 1$; for $\cos^2 \theta = 1$, the series converge only if $a + b - c < 0$ [11]. This condition for ${}_2F_1$ from Equation (23) reads: $a + b - c = (m^2)^{1/2} < 0$, which is opposite to the condition of the regularity of $f_j(m|\theta)$, $(m^2)^{1/2} \geq 0$. Therefore, $f_j(m|\theta)$ and infinite series ${}_2F_1$ cannot simultaneously be regular, and, in order $\Psi_{M(j)}(L, m|\theta)$ to be regular, the infinite hypergeometric series ${}_2F_1$ should terminate resulting in polynomials; hereafter, this truncation of infinite hypergeometric series is referred to as “polynomialization.” As known, the infinite hypergeometric series ${}_2F_1(a; b; c; z)$ are polynomials if either a or b is a non-positive integer [11]. Since the parameters of ${}_2F_1(a; b; c; \cos^2 \theta)$ depend on L and m (see Equation (23)), setting a or b to a non-positive integer results into constraints on L and m . It is obvious that a different prescription for $(m^2)^{1/2}$ generates different restrictions on eigenvalues of the angular momentum.

Let us report results for the eigenvalue problem of the operator of the angular momentum squared; for explicit but somewhat lengthy calculations, see [9]. It is essential to specify which prescription is used for the $(m^2)^{1/2}$ appearing in $\Psi_{M(j)}(L, m|\theta)$, given by Equation (23), since, as shown below, after the normalizability is required, different prescriptions lead to the different results for the spectrum.

There are two possible prescriptions, namely, $(m^2)^{1/2} = |m|$ and $(m^2)^{1/2} = \pm m$. First, the case when the prescription is $(m^2)^{1/2} = |m|$ is considered. Equating parameters a and b of the two hypergeometric functions from Equation (23) to non-positive integers, $-k$, results in four conditions: two conditions for $\Psi_{M(1)}$ and two conditions for $\Psi_{M(2)}$. One condition out of the two for $\Psi_{M(1)}$ generates singular functions and, thus, to be dropped; the same is true for $\Psi_{M(2)}$ and, finally, one remains with only two conditions of polynomialization, generating regular eigenfunctions [9].

The spectrum of eigenvalues corresponding to the two left regular eigenfunctions is obtained from the following polynomialization conditions:

$$\left| \frac{m_{(1)}}{2} \right| - \frac{L}{2} = -k_1; \quad L - 2 \left[\frac{L}{2} \right] \leq |m_{(1)}| \leq L, \quad (24)$$

$$\left| \frac{m_{(2)}}{2} \right| - \frac{L-1}{2} = -k_2; \quad (L-1) - 2 \left[\frac{L-1}{2} \right] \leq |m_{(2)}| \leq (L-1). \quad (25)$$

Here, $[X]$ stands for the integer part of X satisfying $X - [X] \geq 0$, $k_1 = 0, 1, 2, \dots [L/2]$ and $k_2 = 0, 1, 2, \dots [(L-1)/2]$. The sets (24) and (25) are comprised of the numeric sequence of positive and negative elements $m_{(j),k}$, $j = 1, 2$ with the step size 2, e.g., $|m_{(1)}| = L - 2k_1$. From Equations (24) and (25) it follows that the spectrum is discrete with the only condition that $L - |m|$ is necessarily integer, while there are no constraints on L and m separately; the solution (23) is regular for integer as well as for non-integer L, m .

The sets of the eigenvalues, $\{m_{(1)}\}$ and $\{m_{(2)}\}$ (and their corresponding eigenfunctions), can be formally combined into one set, comprised of the positive and negative elements, m_k , with the step size 1: $m_k - m_{k-1} = \pm 1$. Using numerical ordering from the smallest to the largest value, the combined set of all possible eigenvalues reads as follows:

$$\{m\}|_{(m^2)^{1/2} = |m|} = \{-L, -L+1, -L+2, \dots, -m_0; m_0, \dots, L-2, L-1, L\}, \quad (26)$$

where, depending on a numeric value of L , m_0 is either $(L - 2[L/2])$, the minimal positive value from the set (24), or $(L - 1 - 2[(L-1)/2])$, the minimal positive value from the set (25) [9].

Starting from the subset of Equation (26) with m positive, applying $\hat{M}_- = \hat{M}_x - i\hat{M}_y$ leads to a subset with the negative m and vice versa, $\hat{M}_+ \Psi(m < 0) = (\hat{M}_x + i\hat{M}_y) \Psi(m < 0) \rightarrow \Psi(m > 0)$ only when L is either integer or half-integer, where the arrow is a symbol for "goes to" or "tends to". Acting by \hat{M}_- on the regular functions with m positive leads to the regular functions with m negative, $\hat{M}_- \Psi_{\text{reg}}(m > 0) \rightarrow \Psi_{\text{reg}}(m < 0)$ only when L is integer. When L is half-integer, acting by \hat{M}_- on the regular functions with m positive leads to the singular functions with m negative, $\hat{M}_- \Psi_{\text{reg}}(m > 0) \rightarrow \Psi_{\text{sing}}(m < 0)$. A symmetric result is valid when applying the rising operator: $\hat{M}_+ \Psi_{\text{reg}}(m < 0) \rightarrow \Psi_{\text{reg}}(m > 0)$ only when L is integer and when L is half-integer, $\hat{M}_+ \Psi_{\text{reg}}(m < 0) \rightarrow \Psi_{\text{sing}}(m > 0)$ [9].

Evidently, if it is required that, when moving with the step size 1, starting from the wave function with $m = (-L)$, one should arrive at the wave function with $m = +L$ and vice versa, this will be possible only when m is either integer or half-integer. In this case, no analysis of the eigenvalue problem is necessary since the spectrum is already predefined to consist of only integer or half-integer m . The requirement that, starting from the state with $m = \mp L$ one arrives, moving with step size 1, to the state with $m = \pm L$, is postulated in the method of commutator algebra of the angular momentum operators [1–4].

This requirement, customarily taken for granted to be a physical postulate, is actually a mathematical condition, imposed by hand which filters out possible non-integer and non-half-integer m from the spectrum, similarly to how imposing the non physical condition of periodicity on $e^{im\phi}$, filters out non integer m from the spectrum of \hat{M}_z . When L is non-integer, the requirement that starting from the state with $m = \mp L$ one arrives at the state with $m = \pm L$, cannot be satisfied. Indeed, e.g., for $L = 1.7$, acting with the lowering operator $\hat{M}_- = \hat{M}_x - i\hat{M}_y$ on $\Psi(1.7, 1.7|\theta)$ would never result in $\Psi(1.7, -1.7|\theta)$ and then terminate; instead, one gets: $\Psi(1.7, 1.7|\theta) \rightarrow \Psi(1.7, 0.7|\theta) \rightarrow \Psi(1.7, -0.3|\theta) \rightarrow \Psi(1.7, -1.3|\theta) \rightarrow \Psi(1.7, -2.3|\theta) \rightarrow \dots$. Let us recall that as an alternative to an unphysical requirement of single valuedness of the wave function, Pauli suggested that acting by the rising and lowering operators $\hat{M}_x \pm i\hat{M}_y$ on regular wave functions one should find: $\Psi_M(L, -L|\theta) \leftrightarrow \Psi_M(L, -L+1|\theta) \leftrightarrow \dots \leftrightarrow \Psi_M(L, -1+L|\theta) \leftrightarrow \Psi_M(L, L|\theta)$. Pauli justified this by postulating that as a result of acting on regular wave functions by $\hat{M}_x \pm i\hat{M}_y$,

no singular functions appear [12,13]. In the case of the prescription, $(m^2)^{1/2} = |m|$, moving up and down in spectrum with step size 2, indeed no singular functions are generated for any, integer or non-integer, m , as follows from the conditions of polynomialization [9].

However, singular functions appear if, instead of $(m^2)^{1/2} = |m|$, the prescription, $(m^2)^{1/2} = \pm m$ is used. In this case, the operators, $\hat{M}_\pm = \hat{M}_x \pm i\hat{M}_y$, connecting wave functions and $m \rightarrow m \pm 1$, can be defined and, acting by \hat{M}_\pm , results in a set of eigenfunctions with the eigenvalues [9],

$$\begin{aligned} \{m_1\} \downarrow_{(m^2)^{1/2}=\pm m} &= (L - 2k_1) = \{L; L - 2; \dots; L - 2[L/2]; L - 2[L/2] - 2; \dots; -\infty\}, \\ \{m_1\} \uparrow_{(m^2)^{1/2}=\pm m} &= (-L + 2k_2) = \{-L; -L + 2; \dots; -L + 2[L/2]; -L + 2[L/2] + 2; \dots; \infty\}, \\ \{m_2\} \downarrow_{(m^2)^{1/2}=\pm m} &= (L - 1 - 2k_3) = \{L - 1; L - 3; \dots; \\ &\quad L - 1 - 2[(L - 1)/2]; L - 1 - 2[(L - 1)/2] - 2; \dots; -\infty\}, \\ \{m_2\} \uparrow_{(m^2)^{1/2}=\pm m} &= (-L + 1 + 2k_4) = \{-L + 1; -L + 3; \dots; \\ &\quad -L + 1 + 2[(L - 1)/2]; -L + 1 + 2[(L - 1)/2] + 2; \dots; \infty\}, \end{aligned} \quad (27)$$

where k_1, k_2, k_3, k_4 are positive integers and the elements of the sets $\{m_j\}$ can be any real number, not necessarily integer or half-integer. For ms from Equation (27), the corresponding hypergeometric functions, ${}_2F_1$, are regular, some $f(m|\theta) \sim (\sin^2 \theta)^{(m^2)^{1/2}/2} = (\sin \theta)^{\pm m}$ are singular and, therefore, some $\Psi_{M(1,2)}(L, m|\theta)$ also are singular; for explicit calculations and technicalities, see [9]. For the case of $(m^2)^{1/2} = \pm m$, one obtains, similar to the spectrum, resulting from Equations (24) and (25), that the spectrum is discrete with the only condition that $L - |m|$ is necessarily integer, while L and m can be integer as well as non-integer each. If L is integer, the sequences (27) do not extend to $\pm\infty$ but truncate at $\pm L$ and reproduce the set of eigenvalues (26), obtained using prescription, $(m^2)^{1/2} = |m|$.

Let us note that in the group theoretical framework, the eigenfunctions, corresponding to the eigenvalue spectrum (27), form an irreducible representation of $SO(3)$, the three dimensional rotation group; see, e.g., [17]. As mentioned just above, when L and m are integer, infinite sequences (27) truncate into a finite set of eigenvalues, $-L \leq m \leq L$, and the corresponding eigenfunctions are regular and form a finite set. In terms of the representation theory, these are the finite dimensional irreducible representations of the $SO(3)$ group. When L and m are non integer (half-integers included), the sequences (27) do not truncate and remain infinite. Then, the corresponding infinite set of eigenfunctions is formed of both singular and regular functions. In terms of the representation theory, these are the infinite dimensional irreducible representations of the $SO(3)$ group. Using prescription $(m^2)^{1/2} = \pm m$ results in an infinite set of eigenfunctions (containing both singular and regular functions), corresponding to an infinite dimensional representation of the rotation group. Using $(m^2)^{1/2} = |m|$, the finite set of eigenvalues $-L \leq m \leq L$, symmetric with respect to $m \rightarrow -m$, is filtered out from the infinite set (27) because of $|m| \geq 0$. The set of corresponding eigenfunctions is comprised of regular functions only. In other words, the set of regular eigenfunctions is being filtered out from a general infinite set of eigenfunctions exactly the same way, similar to the case of integer L and m .

So, depending on a prescription for $(m^2)^{1/2}$, eigenfunctions of the operator of angular momentum square could be regular or singular and the eigenvalues could be given either by Equations (26) or (27). When the prescription, $(m^2)^{1/2} = |m|$, is used, all eigenfunctions are regular and the eigenvalue spectrum is given by Equation (26). When the prescription is $(m^2)^{1/2} = \pm m$, some eigenfunctions are regular, while some are singular and the eigenvalue spectrum is given by Equation (27).

Finally, let us discuss what could cause the statement that the eigenvalue problem for the \hat{M}^2 admits normalizable solutions only when L is integer [1–4]. The eigenfunctions and the spectrum, e.g., the set of regular functions and eigenvalues (26), are obtained by requiring normalizability of a solution that is presented in terms of a specific pair of linearly independent functions, $\Psi_{M(1)}$ and $\Psi_{M(2)}$. Quite a different picture arises when the normalization condition is applied to another pair of linearly independent functions,

e.g., to the Legendre functions, $P_L^m(\theta)$ and $Q_L^m(\theta)$, which were, from the early days of quantum mechanics, considered as eigenfunctions of the operator of the angular momentum squared [1–4].

Certainly, both pairs, $\Psi_{M(1)}(L, m|\theta)$, $\Psi_{M(2)}(L, m|\theta)$ and $P_L^m(\theta)$, $Q_L^m(\theta)$, are solutions of the eigenvalue equation (22). Legendre functions can be written as linear combinations of hypergeometric functions, $\Psi_{M(1,2)}(L, m|\theta)$ [11]:

$$\begin{aligned} P_L^m(\theta) &= C_{11}\Psi_{M(1)}(L, m|\theta) + C_{12}\Psi_{M(2)}(L, m|\theta); \\ Q_L^m(\theta) &= C_{21}\Psi_{M(1)}(L, m|\theta) + C_{22}\Psi_{M(2)}(L, m|\theta). \end{aligned} \quad (28)$$

Using just a polynomialization, it is not sufficient to normalize both $P_L^m(\theta)$ and $Q_L^m(\theta)$ simultaneously. The reason is that the sets $\{m_{(1)}\}$ and $\{m_{(2)}\}$, generated by the two conditions of polynomialization (24) and (25), have no common element, and, thus, it is impossible to satisfy these conditions simultaneously. As soon as the polynomialization conditions are applied, either $\Psi_{M(1)}(L; m|\theta)$ or $\Psi_{M(2)}(L; m|\theta)$ is singular [9]. Therefore, in order to carry out the normalizability of $P_L^m(\theta)$ and $Q_L^m(\theta)$, only polynomialization would not suffice and an additional requirement to filter out singular parts of $P_L^m(\theta)$ and $Q_L^m(\theta)$ is necessary. This can be achieved by choosing the coefficients, C_{ij} in Equation (28). Namely, as soon as the polynomialization condition (24) is satisfied, what leads to regular $\Psi_{M(1)}(L; m|\theta)$ and singular $\Psi_{M(2)}(L; m|\theta)$, the coefficients, C_{12} and C_{22} , of $\Psi_{M(2)}(L; m|\theta)$ must vanish. Similarly, as soon as the polynomialization condition (25) is satisfied what leads to regular $\Psi_{M(2)}(L; m|\theta)$ and singular $\Psi_{M(1)}(L; m|\theta)$, the coefficients, C_{11} and C_{21} , of $\Psi_{M(1)}(L; m|\theta)$ must vanish.

Coefficients C_{ij} are calculated in [9]:

$$\begin{aligned} C_{11}^{-1} &\sim \Gamma\left(\frac{1}{2} - \frac{L}{2} + \frac{|m|}{2}\right) \Gamma\left(1 + \frac{L}{2} + \frac{|m|}{2}\right), \quad C_{12}^{-1} \sim \Gamma\left(\frac{1}{2} + \frac{L}{2} + \frac{|m|}{2}\right) \Gamma\left(-\frac{L}{2} + \frac{|m|}{2}\right), \\ C_{21} &\sim \frac{\Gamma\left(\frac{1}{2} + \frac{L}{2} - \frac{|m|}{2}\right)}{\Gamma\left(1 + \frac{L}{2} + \frac{|m|}{2}\right)}, \quad C_{22} \sim \frac{\Gamma\left(1 + \frac{L}{2} - \frac{|m|}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{L}{2} + \frac{|m|}{2}\right)}. \end{aligned} \quad (29)$$

Here, Γ is the Euler Gamma function.

It is straightforward to show that, after applying the polynomialization conditions (24) and (25) and using the property of the Γ function such that $1/\Gamma(\text{nonpositive integer}) = 0$ [11,14], one obtains that $C_{11} = 0$ and $C_{12} = 0$ can be realized only for integer L and m and that $C_{21} = 0$ and $C_{22} = 0$ can never be satisfied [9]. Therefore, $Q_L^m(\theta)$ has to be excluded, and only $P_L^m(\theta)$ remains as the quantum mechanical eigenfunction. Consequently, for the pair $P_L^m(\theta)$ and $Q_L^m(\theta)$, normalizability is achieved only when L and m are integer. This is not a general result; one explicit example of the eigenfunction, normalizable for integer as well as for non-integer eigenvalues, is presented by the pair $\Psi_{M(1)}(L, m|\theta)$ and $\Psi_{M(2)}(L, m|\theta)$, given in Equation (23) and leading to spectra (26) or (27), depending on the prescription for the power function. Hence, the statement that from theoretical quantum mechanics it follows that the eigenvalue spectrum of \hat{M}^2 is comprised of only integers is not necessarily correct in sense that it corresponds to a special case when $P_L^m(\theta)$ and $Q_L^m(\theta)$ are chosen as a pair of linearly independent solutions of the eigenvalue Equation (22).

4. Conclusions

In this paper, the eigenvalue problem for the operator of the angular momentum is studied in the framework of nonrelativistic quantum mechanics. The general result for the spectrum is that it is discrete, namely, $|m| = L - k$ with k been integer, $k = \{0, 1, \dots, [L]\}$, where $[L]$ is the integer part of L . L and m can be integer as well as non-integer and this does not contradict any physical principle.

The above is in stark contrast with the known statement that from theoretical quantum mechanics it follows that m and L can only be integer or half-integer [1–4]. An explanation of the contradiction is that the result, obtained here, does not impose a non-physical

requirements of either periodicity of the wave function or postulating that, moving with the step size 1 and starting from a state with $m = -L$, one should arrive at the state with $m = +L$ and vice versa. The discreteness condition, $|m| = L - k$, does not require that moving with the step size 1 from $\Psi_M(L, -L|\theta)$ one should end with $\Psi_M(L, +L|\theta)$.

Using the Legendre functions, $P_L^m(\theta)$ and $Q_L^m(\theta)$, as a pair of linearly independent solutions for the eigenvalue equation, $\hat{M}^2\Psi = L(L+1)\Psi$, is a specific choice that does not encompasses the most general case. When $P_L^m(\theta)$ and $Q_L^m(\theta)$ are used, the normalizability requirement filters out non-integer L and m , but the eigenvalue equation solution that is normalizable may exist for any real eigenvalues, integer and non-integer. Another solution, presented here, Equation (23), satisfies the necessary physical requirement of the normalizability for integer and non-integer L and m .

Imposing the condition of the single valuedness on the eigenfunction of the third component of the angular momentum $(x + iy)^m$ does not necessarily lead to the violation of the rotational invariance. Indeed, it is found such a representation of the power function, Equation (10), which for any m is a single valued function of (x, y) and is invariant under the $2\pi k$ -rotation. Two results indicate that $\Psi(x, y|m)$ from Equation (10) and $(x + iy)^m$ belong to the same class of functions. First, according to Equation (12), solution $\Psi(x, y|m)$, coincide, up to the normalization factor, with $(x + iy)^m$ when m is integer. Second, for the half-integer values of $m = N/2$, one obtaines $(\Psi(x, y|N/2))^2 \sim \Psi(x, y|N)$ (Equation (14)), relation mirroring the one valid for the function (4), $((x + iy)^{N/2})^2 = (x + iy)^N$. For the rational $m = p/n$, it is not shown that $(\Psi(x, y|p/n))^n \sim \Psi(x, y|p)$ but particular cases of integer and half-integer m are indications that $(\Psi(x, y|\alpha))^\beta \sim \Psi(x, y|\alpha\beta)$ may well enough be true for any α and β . Another important property, indicating that $(x + iy)^m$ and $\Psi(x, y|m)$ belong to the same class, is that for any m , both the functions satisfy the relation, $|\Psi/C|^2 = 1$; see Equation (19). To summarize, the wave function (10), solution of the eigenvalue equation $\hat{M}_z\Psi(x, y|m) = m\Psi(x, y|m)$, represents a possible prescription for the power function, $(x + iy)^m$.

In the general case, when the only condition imposed on a wave function is the physical requirement of the normalizability, i.e., when the periodicity requirement for a wave function is lifted or when the different pair of linearly independent functions is chosen, there is no constraint on L and m to be integer only. From the physics point of view, the only self-consistent approach is to drop all non-physical conditions and consider the problem in the presence of only the physical requirements. This is what is done in this paper and, as a result, a new quantum-mechanical solution of the eigenvalue problem for the angular momentum operator is obtained.

To conclude, the main result of this paper is that from the framework of theoretical quantum mechanics it does not follow that the eigenvalues of the angular momentum operator should only be integer.

Surely, the spectrum of the angular momentum cannot be defined from theoretical quantum mechanics alone but has to be established by comparing theoretical calculations with experiments. However, this is not a goal of the current paper, which seeks to analyze the eigenvalue problem for the angular momentum operator from the purely theoretical viewpoint.

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