Proceedings of XIII Fall Workshop on
Geometry and Physics
Murcia, September 20–22, 2004
Publ. de la RSME, Vol. 9 (2005), 115–125

On warped and double warped space-times

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Abstract. Warped and double warped space-times are defined and classified according to the existence of preferred vector and tensor fields on the locally decomposable space-times they are associated to by means of special conformal transformations. An invariant geometric characterization is also presented, as well as a characterization based on the Newman-Penrose formalism.

Keywords: Exact solutions of Einstein's equations; NP formalism

2000 Mathematics Subject Classification: 83C20; 83C60

1. Introduction

In [1] warped product manifolds are obtained by homothetically warping the product metric on a semi-Riemannian product manifold. Warped space-times are then viewed as conformal to locally decomposable ones, the conformal factor depending on the variables defined only on one of the spaces used to build the decomposable space. They are classified and characterized in an invariant manner in [2] and [3], where their geometry is expressed in terms of the warping function and the geometry of the underlying locally decomposable space-times. It is also referred (see references listed in [2]) that there is a variety of exact solutions to Einstein's equations which can be viewed as warped space-times, for example Schwarzchild, Bertotti-Robinson, Robertson-Walker, Reissner-Nordstrom, de Sitter, therefore their study being quite relevant in general relativity. From the point of view of describing solutions to Einstein's equations, it is an interesting problem to investigate double warped space-times. These are also conformal to locally decomposable ones, however the conformal factor is now separable on the coordinates defined on each space used to build the decomposable space-time. It should also be referred that warped and double warped space-times are are special cases of twisted products [4]. In [5] an invariant characterization for double warped space-times is presented in terms of the Newman-Penrose formalism [6] and a classification scheme is proposed, as well as a detailed study of their conformal algebra.

Here warped and double warped space-times are defined and classified according to the existence of preferred vector and tensor fields on the underlying decomposable space-times. Therefore it is convenient to start with a brief summary of results on locally decomposable space-times [7].

Given two metric manifolds (M_1, h_1) and (M_2, h_2), one can build a new metric manifold (M, g) by setting

$$M = M_1 \times M_2 \tag{1}$$

and

$$g = {\pi_1}^{\star} h_1 \oplus {\pi_2}^{\star} h_2, \tag{2}$$

or simply $g = h_1 \otimes h_2$, π_1 and π_2 being the canonical projections onto M_1 and M_2, respectively.

Throughout this text we will be interested in space-time manifolds (M, g), therefore we will consider that $dimM = dimM_1 + dimM_2 = 4$ and that g is a Lorentzian metric. Consequently one of the manifolds (M_1, h_1), (M_2, h_2) is Lorentzian and the other is Riemannian.

Since the considerations to be presented are mainly local, we will assume that, for each point p in M, there exists a neighborhood U of p where a coordinate system adapted to the product structure exists, namely $(u = x^0, x^1, x^2, x^3)$. Two main cases then arise, as summarized below.

(A) (M, g) is a 1+3 locally decomposable space-time, if it admits a global, non-null, nowhere zero covariantly constant vector field. For this case the line element associated with g can be written in the following form

$$ds^2 = \epsilon du^2 + h_{\alpha\beta}(x^{\gamma}) dx^{\alpha} dx^{\beta}, \tag{3}$$

where greek indices take the values $1, 2, 3$ and $\epsilon = \pm 1$. Here $h_1 = \epsilon du \otimes du$ and $h_2 = h_{\alpha\beta}(x^{\gamma}) dx^{\alpha} \otimes dx^{\beta}$. Moreover, when the covariantly constant vector field $\partial/\partial u$ is timelike (respectively spacelike), then $\epsilon = -1$ (respectively $\epsilon = +1$) and the space-time is locally $1+3$ spacelike (respectively timelike) decomposable.

It should be noticed that, if the space-time admits another global covariantly constant non-null vector field, nowhere zero, then the space-time decomposes further and can be referred to as being $1+1+2$ spacelike or $1+1+2$ timelike in an obvious notation.

(B) (M, g) is a $2+2$ locally decomposable space-time, in which case no global, covariantly constant, nowhere zero vector field exists in M, but the space-time admits two global recurrent null vector fields, linearly independent. This is equivalent to saying that in (M, g) there are two global, linearly independent covariantly constant tensor fields of rank 2, namely P and Q, such that $g_{ab} = P_{ab} + Q_{ab}$ with $P_{ab;c} = Q_{ab;c} = 0$ ($a, b, c, \cdots = 0, 1, 2, 3$). The line element associated with g can have, for this case, the following form:

$$ds^2 = h_{1\ \alpha\beta}(x^\gamma)dx^\alpha dx^\beta + h_{2\ AB}(x^D)dx^A dx^B, \tag{4}$$

where h_1 and h_2 are two 2-dimensional metrics on M_1 and M_2 respectively such that $\pi_1^* h_1 = P$ and $\pi_2^* h_2 = Q$.

2. General Concepts

We now consider the two metric manifolds (M_1, h_1) and (M_2, h_2) introduced above and a smooth function $\theta : M_1 \to \mathbb{R}$, which will be called a *warping function*. We can now build a new manifold (M, g), namely a *warped product manifold* (see [1], [8]) by setting $M = M_1 \times M_2$ and

$$g = h_1 \otimes \mathrm{e}^{2\theta} \pi^* h_2, \tag{5}$$

or simply $g = h_1 \otimes \mathrm{e}^{2\theta} h_2$, where π is the canonical projection onto M_2 and will be omitted where there is no risk of confusion.

It should be noticed that (5) can also be written as

$$g = \mathrm{e}^{2\theta}(h_1' \otimes h_2) = \mathrm{e}^{2\theta} g', \tag{6}$$

where we have defined $h_1' \equiv \mathrm{e}^{-2\theta} h_1$ as a new metric on M_1. Therefore a warped space-time is conformal to a decomposable space-time (M, g'), the conformal factor depending only on the coordinates on either M_1 or M_2.

Similarly, if one considers two smooth functions $\theta_1 : M_1 \to \mathbb{R}$, $\theta_2 : M_2 \to \mathbb{R}$ (*warping functions*), one can build a new metric manifold (M, g) by setting $M = M_1 \times M_2$ and

$$g = \mathrm{e}^{2\theta_2} \pi_1^* h_1 \otimes \mathrm{e}^{2\theta_1} \pi_2^* h_2, \tag{7}$$

where π_1, π_2 are the canonical projections onto M_1 and M_2 respectively. As before, if no risk of confusion exists we will simply write $g = \mathrm{e}^{2\theta_2} h_1 \otimes \mathrm{e}^{2\theta_1} h_2$.

This metric manifold (M,g) will be called a *double warped product manifold* (see [5]). It is now clear that, if either θ_1 or θ_2 are constant, then (M,g) becomes a warped manifold. If, moreover, both warping functions are constant then (M,g) is a decomposable metric manifold.

It should be noticed that (7) can be written as follows:

$$g = \mathrm{e}^{2(\theta_1+\theta_2)}[\mathrm{e}^{-2\theta_1}h_1 \otimes \mathrm{e}^{-2\theta_2}h_2] = \mathrm{e}^{2(\theta_1+\theta_2)}[h_1' \otimes h_2'] = \mathrm{e}^{2(\theta_1+\theta_2)}g'. \tag{8}$$

Due to the definition of the warping functions, it is clear that h_1' and h_2' are metrics on M_1 and M_2, respectively. Therefore, a double warped space-time can be thought of as being conformal to a locally decomposable one, say (M,g'), the conformal factor being separable on the coordinates defined on those submanifolds used to construct M.

On what follows, we will write h_1 and h_2 instead of h_1' and h_2', whenever there is no risk of confusion.

3. Canonical Forms for the Metric

When studying warped and double warped space-times (M,g), it is our goal to express their geometry in terms of the warping functions and the geometries of the underlying spaces M_1 and M_2. It appears natural to consider two main classes of warped space-times: one class includes space-times conformally related to $1+3$ locally decomposable space-times, as shown in (3), the other class containing those space-times which are obtained from $2+2$ locally decomposable space-times, as in (4). Analogous considerations apply to double warped metrics.

3.1. Warped Space-Times

According to [2], warped space-times (M,g) are always included in one of the classes described below.

Class A: (M,g) is conformally related to a $1+3$ locally decomposable space-time (M,g'). Two different sub-classes are distinguished, namely class A_1 corresponding to a $1+3$ spacelike (M,g') and class A_2 corresponding to a $1+3$ timelike (M,g').

Class B: (M,g) is conformally related to a $2+2$ locally decomposable space-time (M,g').

It is now convenient to write canonical forms for the line element associated with the metric of **warped** space-times in each of the referred classes. If one uses (3), (5) and (6), the following canonical forms arise.

Class A_1:

$$ds^2 = \epsilon du^2 + \mathrm{e}^{2\theta(u)} h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta \tag{9}$$

Class A_2:

$$ds^2 = \epsilon \mathrm{e}^{2\theta(x^\gamma)} du^2 + h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta, \tag{10}$$

Class B:

$$ds^2 = h_{1\ \alpha\beta}(x^\gamma) dx^\alpha dx^\beta + \mathrm{e}^{2\theta(x^\gamma)} h_{2\ AB}(x^C) dx^A dx^B. \tag{11}$$

Here we have followed the notation used for writing (3) and (4).

Since $h_{\alpha\beta}(x^\gamma)$ is a three-dimensional metric it can always be written in a diagonal form so that the line elements (9) and (10) can be rewritten taking this property into account. Similarly (11) can also be modified if one takes into account that a two-dimensional space is always conformally flat.

3.2. Double Warped Space-Times

The similarity between the construction of warped and of double warped space-times induces a similar procedure for writing canonical forms for the line element of double warped space-times. In fact, according to [5], **double warped** space-times (M, g) are always included in one of the following classes:

Class A: (M, g) is conformally related to a $1+3$ locally decomposable space-time (M, g'). Two different sub-classes are distinguished, namely class A_1 corresponding to a locally $1+3$ decomposable spacelike (M, g') and class A_2 corresponding to a locally $1+3$ decomposable timelike (M, g').

Class B: (M, g) is conformally related to a $2+2$ locally decomposable space-time (M, g').

Canonical forms for the line element associated with the metric in each of these classes are presented, following a procedure similar to the one used in studing warped space-times.

Class A_1:

$$ds^2 = \mathrm{e}^{2(\theta_1(u)+\theta_2(x^D))} \left[-du^2 + h_{AB}(x^D) dx^A dx^B\right], \tag{12}$$

Class A_2:

$$ds^2 = \mathrm{e}^{2(\theta_1(x^\gamma)+\theta_2(u))}\left[h_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + du^2\right], \tag{13}$$

Class B:

$$ds^2 = \mathrm{e}^{2(\theta_1(x^\gamma)+\theta_2(x^D))}\left[h_{1\ \alpha\beta}(x^\gamma)dx^\alpha dx^\beta + h_{2\ AB}(x^D)dx^A dx^B\right]. \tag{14}$$

Again the notation used for writing (3) and (4) is applied in these canonical forms. The canonical forms written above can be cast in a different manner if one applies the considerations referred to in the previous section.

4. Geometric Characterization

We recall that a vector field $\vec{X}$ on M is a *conformal Killing vector* whenever

$$\mathcal{L}_{\vec{X}} g = 2\phi g.$$

Here ϕ (*conformal factor*) is a scalar field on U and, as usual, $\mathcal{L}_{\vec{X}}$ is the Lie derivative operator with respect to the vector field $\vec{X}$. The special cases ϕ = constant and $\phi = 0$ correspond respectively to $\vec{X}$ being a *homothetic vector field* (HV) and a *Killing vector field* (KV). Moreover, a CKV is said to be *proper* whenever it is non-homothetic (i.e. $\phi \neq$ const). Similarly, a HV which is not a KV (i.e. $\phi = \text{const} \neq 0$) will be designated as 'proper homothetic'. A proper CKV is said to be a *special CKV* (SCKV) whenever its associated conformal factor ϕ satisfies $\phi_{a;b} = 0$ in any coordinate chart.

In order to establish an invariant geometric characterization of warped and double warped space-times (M, g) one can look at preferred vector fields living on the underlying decomposable space-time (M, g') and investigate how those vector fields are transformed when the warping functions are introduced.

For example, it is easy to see from (3) that a global, non null, nowhere zero, covariantly constant vector field for a $1+3$ locally decomposable space-time (M, g') can be rescaled so as to become a KV in the associated warped (M, g).

For warped space-times of classes A_1 and A_2 the characterization can be summarized as follows. The proofs of the following theorem can be found in [2].

Theorem 1. *The necessary and sufficient condition for (M, g) to be warped class A is that a global, non-null, nowhere zero, hypersurface orthogonal unit vector field exists that is shearfree* *(a) and such that it is geodesic with expansion* Θ *satifying* $\Theta_{,c}h_a^c = 0$, *where* $h_{ab} \equiv g_{ab} - \epsilon u_a u_b$ *is the orthogonal projector to* u^a, *in which case* (M, g) *is warped class* A_1.

(b) and such that it is non-expanding, its acceleration being a gradient, in which case (M, g) *is warped class* A_2.

The definitions used for expansion and shear can be found in [9], if u is timelike, and in [10], if u is spacelike.

This theorem is proved in [2]. However results of (a) can also be found in [11].

The following theorem was proved in [5] and establishes an invariant characterization for class A double warped space-times.

Theorem 2. *The necessary and sufficient condition for* (M, g) *to be a double warped class A spacetime is that it admits a non-null, nowhere vanishing CKV* $\vec{X}$ *which is hypersurface orthogonal and such that the gradient of its associated conformal factor* ψ *is parallel to* $\vec{X}$.

The characterization of warped spacetimes can now be easily obtained, as stated in the following corollary.

Corollary 3. *If the CKV* $\vec{X}$ *in theorem 2 is a Killing Vector (KV) then the spacetime is warped of class* A_2. *If* $\vec{X}$ *is a proper (non-KV) gradient CKV (i.e. if the associated conformal bivector* $F_{ab} = X_{a;b} - X_{b;a}$ *vanishes) the spacetime is class* A_1 *warped.*

It is worthwhile noticing that theorem 2 provides also an invariant characterization of spacetimes conformal to 1+3 locally decomposable spacetimes:

Corollary 4. *The necessary and sufficient condition for* (M, g) *to be conformally related to a 1+3 decomposable spacetime* (M, g') *is that it admits a non-null, nowhere vanishing conformal Killing vector (CKV)* $\vec{X}$ *which is hypersurface orthogonal.*

On the other hand, if the underlying space-time (M, g') is locally $2+2$ decomposable the characterization of warped and double warped space-times in given in the following theorem. The proof of this theorem is provided in [2] and [5].

Theorem 5. *The necessary and sufficient condition for* (M, g) *to be conformally related to a 2+2 decomposable spacetime* (M, g') *with* $g = \mathrm{e}^{2\theta} g'$ *(*θ *being a real function), is that there exist null vectors* $\vec{l}$ *and* $\vec{k}$ *(*$l^a k_a = -1$*) satisfying*

$$l_{a;b} = A\mathrm{e}^{-\theta} l_a l_b - \theta_{,a} l_b + (\theta_{,c} l^c) g_{ab} \qquad (15)$$

$$k_{a;b} = -A\mathrm{e}^{-\theta} k_a l_b - \theta_{,a} k_b + (\theta_{,c} k^c) g_{ab} \tag{16}$$

for some function A. Further, (M, g) is class B double warped if and only if

$$H_a^c \left(h_b^d \theta_{,d} \right)_{;c} + 2 \left(h_b^d \theta_{,d} \right) (H_a^c \theta_{,c}) = 0, \tag{17}$$

where

$$h_{ab} \equiv -2k_{(a} l_{b)} \ \text{ and } \ H_{ab} \equiv g_{ab} + h_{ab}. \tag{18}$$

5. Newman-Penrose Characterization

The invariant characterization of double warped space-times using the Newman-Penrose (NP) formalism was obtained in [5]. Here we present this classification for class A_1, class A_2 and class B double warped space-times. However the corresponding classification for warped space-times of those classes can easily be obtained if one restricts the form of the conformal factor in an obvious manner.

With the notation used in Theorem 2 we have that for a class A_1 double warped spacetime a coordinate chart $\{u, x^K\}$ exists such that the line element takes the form (12). Then $\vec{X} = \partial_u$ is a timelike hypersurface orthogonal CKV with associated conformal factor $\psi(u) = \theta_{1,u}(u)$, and $\vec{u} = \mathbf{e}^{-U} \partial_u$ is a unit timelike vector field parallel to $\vec{X}$. For convenience we will write $U(u, x^K) = \theta_1(u) + \theta_2(x^K)$. The following theorems were proved in [5].

Theorem 6. *(M, g) is a class A_1 double warped spacetime if and only if there exist a function $U : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ ($k^a l_a = -m^a \bar{m}_a = -1$) in which:*

$$DU = \epsilon + \bar{\epsilon} \tag{19}$$

$$\triangle U = -(\gamma + \bar{\gamma}) \tag{20}$$

$$\delta U = \kappa + \bar{\pi} = -(\tau + \bar{\nu}) \tag{21}$$

$$\sigma + \bar{\lambda} = 0 \tag{22}$$

$$\alpha + \bar{\beta} = 0 \tag{23}$$

$$\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma} = \rho + \bar{\mu} \tag{24}$$

$$D(\rho + \bar{\mu}) = -\Phi \tag{25}$$

$$\triangle(\rho + \bar{\mu}) = \Phi \tag{26}$$

$$\delta(\rho + \bar{\mu}) = \bar{\delta}(\rho + \bar{\mu}) = 0 \tag{27}$$

where $\Phi = \Phi(u)$ is a real function of the timelike coordinate u.

A similar notation is used to establish the classification of A_2 double warped space-times within NP formalism. The result is summarized in the following theorem.

Theorem 7. *(M, g) is a class A_2 double warped spacetime if and only if there exist a function $U : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ ($k^a l_a = -m^a \bar{m}_a = -1$) in which one of the following sets of equations holds:*

(i)

$$DU = \epsilon + \bar{\epsilon} \qquad \triangle U = -(\gamma + \bar{\gamma}) \qquad \delta U = -\kappa + \bar{\pi} = \tau + \bar{\nu} \tag{28}$$

$$\sigma - \bar{\lambda} = \alpha + \bar{\beta} = 0 \tag{29}$$

$$\epsilon + \bar{\epsilon} - (\gamma + \bar{\gamma}) = \rho - \bar{\mu} \tag{30}$$

$$D(\rho - \bar{\mu}) = \triangle(\rho - \bar{\mu}) = \Phi \tag{31}$$

$$\delta(\rho - \bar{\mu}) = \bar{\delta}(\rho - \bar{\mu}) = 0 \tag{32}$$

(ii)

$$DU = \sigma + \bar{\rho} \qquad \triangle U = -(\bar{\lambda} + \mu) \qquad \delta U = \bar{\alpha} - \beta \tag{33}$$

$$\delta U + \bar{\delta} U = \pi + \bar{\pi} = -(\tau + \bar{\tau}) \tag{34}$$

$$\kappa + \bar{\kappa} = \nu + \bar{\nu} = 0 \tag{35}$$

$$\epsilon - \bar{\epsilon} = 0 \gamma - \bar{\gamma} = 0 \tag{36}$$

$$\delta(\pi + \bar{\pi}) = \bar{\delta}(\pi + \bar{\pi}) = \Phi' \tag{37}$$

$$\triangle(\pi + \bar{\pi}) = D(\pi + \bar{\pi}) = 0 \tag{38}$$

(iii)

$$DU = -\sigma + \bar{\rho} \qquad \triangle U = \bar{\lambda} - \mu \qquad \delta U = \bar{\alpha} - \beta \tag{39}$$

$$\delta U - \bar{\delta} U = -\pi + \bar{\pi} = -\tau + \bar{\tau} \tag{40}$$

$$\kappa - \bar{\kappa} = \nu - \bar{\nu} = 0 \tag{41}$$

$$\epsilon - \bar{\epsilon} = \gamma - \bar{\gamma} = 0 \tag{42}$$

$$-\delta(\pi - \bar{\pi}) = \bar{\delta}(\pi - \bar{\pi}) = -\Phi'' \tag{43}$$

$$\triangle(\pi - \bar{\pi}) = D(\pi - \bar{\pi}) = 0 \tag{44}$$

where Φ, Φ' and Φ'' are real functions of the spacelike coordinate u.

In order to establish a characterization of class B double warped space-times, using the NP formalism, a complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ is chosen such that $\vec{k}$ and $\vec{l}$ are the vectors in (15) and (16), i.e. $k^a l_a = -m^a \bar{m}_a = -1$ all other inner products vanishing. The following theorem then holds.

Theorem 8. *The necessary and sufficient condition for (M, g) to be conformally related to a 2+2 decomposable spacetime (M, g'), with $g = \mathrm{e}^{2\theta} g'$, is that there exist a function $\theta : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ as described above such that*

$$\kappa = \sigma = \lambda = \nu = \alpha + \bar{\beta} = \pi + \bar{\tau} = \rho + (\epsilon + \bar{\epsilon}) = 0$$
$$A\mathrm{e}^{-\theta} = \mu + (\gamma + \bar{\gamma})$$
$$\rho = -D\theta, \quad \mu = \triangle\theta, \quad \tau = -\delta\theta \tag{45}$$

where A is the real function appearing in (15) and (16). Furthermore, (M, g) is class B double warped if and only if

$$\delta\rho = -2\rho\tau, \quad \delta\mu = -2\mu\tau, \quad \rho\mu = 0 \tag{46}$$

The characterization of class A and class B double warped spacetimes given in the results stated here should prove useful in formulating an algorithm for classifying such metrics. This is the case since such characterization is coordinate independent although tetrad dependent. In what follows the tetrads described above will be designated as *double warped tetrads of class A and B*, as appropriate.

Thus, in order to determine whether a given metric g represents a double warped spacetime, one can use the theorems stated either with a coordinate or a tetrad approach through the following scheme:

1. Determine the Petrov type of the Weyl tensor associated with the metric g and choose a canonical tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ such that $g_{ab} = 2[-l_{(a}k_{b)} + m_{(a}\bar{m}_{b)}]$.

2. Determine the NP spin coefficients and their NP derivatives in the chosen tetrad (1).

3. If the scalars determined in step (2) satisfy the relations of theorem 6 or 7 (respectively 8) for some function U (respectively θ), then the spacetime is double warped of class A (respectively B) and the algorithm stops here, otherwise continue the algorithm.

4. If possible, find the Lorentz transformation of the invariance group that transforms tetrad (1) into a double warped tetrad, i.e. such that the corresponding NP spin coefficients and NP derivatives obey the conditions in theorem 6 or 7 (respectively 8). If such a transformation exists, the spacetime is double warped of class A (respectively B), otherwise it is not double warped.

The Lorentz transformations considered in step (4) must belong to the invariance group of the Petrov type of the metric g since in step (1) one chooses a canonical tetrad. Thus, for instance, if the given metric is of Petrov type D or N, then in step (4) one looks for spin and boost transformations or for null rotations respectively.

Acknowledgments

The author wishes to thank Dr. J. Carot and the Departament de Física, Universitat de les Illes Baleares, for their support and hospitality during the elaboration of this work.

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