MATHEMATICAL ASPECTS OF FIELD THEORY: NAHM'S EQUATIONS AND JACOBI FORMS

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Abstract

This thesis consists of two projects wherein we explore some mathematical aspects of field theory.

In the first project, we address Nahm's equations, which is an integrable system with a Lax pair. We consider boundary conditions on Nahm's equations that correspond to the Dirac multimonopole in Yang-Mills theory. The algebro-geometric integration method is to construct solutions via a linear flow in the Jacobian of the spectral curve associated to the Lax pair. We construct a frame of sections of this linear flow, which allows us to obtain exact solutions to Nahm's equations for arbitrary rank n.

Nahm's equations with our boundary conditions correspond to the Dirac multimonopole via the ADHMN construction. The ADHMN construction requires us to find normalizable zero modes of Dirac operators. We again use the frame of sections of the linear flow on the Jacobian of the spectral curve to construct these normalizable zero modes.

In the second project, we consider weak Jacobi forms of weight 0. The polar coefficients of such weak Jacobi form are known to uniquely determine the weak Jacobi form, and we improve on the number of polar coefficients that determine the weight 0 form.

Weak Jacobi forms of weight 0 may be exponentially lifted to Siegel modular forms, which appear in the string-theory of black holes. In connection to this, the growth of a certain sum about a term $q^a y^b$ in the Fourier-Jacobi expansion of the underlying weak Jacobi form is of interest to us.

We discover that the weak Jacobi forms which are quotients of theta functions give us a large class of forms that are slow growing about their most polar term. Additionally, the characteristics of growth behavior for a weak Jacobi form about a term y^b are known, here we investigate growth behavior about an arbitrary $q^a y^b$ term and find several analogues.

Chapter 1

Introduction

Gauge theory is a powerful mathematical framework for studying a wide variety of physical phenomena. Its roots developed as early as the second half of the 19^{th} century with Maxwell's equations describing electromagnetism, and entered its golden period in the 1970s and 1980s, culminating with the establishment of the standard model of particle physics. However, the standard model does not adequately explain some fundamental physical phenomena, such as gravity. Attempts to go beyond the standard model have proved quite challenging, given our available methods and computing power. Instead, we can look at simpler models in an attempt to better understand the mathematical structures at hand. It is in this spirit that this thesis is undertaken.

In this thesis, we describe two different projects, unified by the common goal of understanding some mathematical aspect of simple field theories. In the first, we consider Nahm's equations, which arise from Yang-Mills gauge theory. We construct exact solutions to those equations, subject to a simple boundary condition. Chapters 2, 3, and 4 are dedicated to this project. In the second, we consider weak Jacobi forms and the growth behavior of a certain sum of their Fourier-Jacobi coefficients. This growth behavior is of interest to us as it has connections to the holographic behavior of a 2D conformal field theory. Chapters 6, 7, and 8 are dedicated to this project.

In the following sections of this chapter, we elaborate on the motivations behind each project and summarize our findings.

1.1 Overview of Nahm's Equations

Nahm's equations for antihermitian matrix-valued functions $T_1(s)$, $T_2(s)$, $T_3(s)$ with s in some interval in $\mathbb{R}$ are

$$\begin{aligned}\frac{dT_1}{ds} &= T_2T_3 - T_3T_2,\\ \frac{dT_2}{ds} &= T_3T_1 - T_1T_3,\\ \frac{dT_3}{ds} &= T_1T_2 - T_2T_1.\end{aligned} \tag{1.1}$$

Nahm's equations are an integrable system, a system of nonlinear ordinary differential equations with sufficiently many conserved quantities to be solved by means of algebraic geometry. The starting point for this algebro-geometric integration method is to discover a Lax representation with spectral parameter ζ for matrix-valued functions L and M such that the system is equivalent to

$$\frac{d}{ds}L(s,\zeta) = [L(s,\zeta), M(s,\zeta)]. \tag{1.2}$$

This Lax equation implies the spectrum of $L(s,\zeta)$ is independent of the variable s, indeed $\mathrm{tr}L^k$ are the conserved quantities of the system. For Nahm's equations, $\zeta \in \mathbb{P}^1$ and we obtain an algebraic curve S in $T\mathbb{P}^1$

$$\det(\eta\mathbb{1} - L(s,\zeta)) = 0, \tag{1.3}$$

for $\eta\frac{\partial}{\partial\zeta} \in T\mathbb{P}^1$. One then defines for generic $L(s,\zeta)$ a line bundle F^s over S with fiber at (ζ,η) the η-eigenspace of $L(s,\zeta)$ and one obtains from the flow of $L(s,\zeta)$ a linear flow F^s in the Jacobian $\mathrm{Jac}(S)$ [1]. The method concludes with writing a solution for the system in terms of an appropriate basis of the linear flow. For some overviews on integrable systems from this point of view, see [2] and [3].

One goal of this paper is to carry out the algebro-geometric method of integration explicitly and obtain exact solutions to Nahm's equations in boundary conditions we specify. In general, exact solutions have proved difficult to construct but one classical solution is provided by the substitutions $T_i(s) = f_i(s)\rho_i$ for ρ_i a constant matrix and $f_i(s)$ a scalar function, in which case the Nahm equations implies ρ_i must form a representation of $\mathfrak{su}(2)$ and the functions $f_i(s)$ must satisfy the Euler top system

$$\begin{aligned} \dot{f}_1 &= f_2 f_3, \\ \dot{f}_2 &= f_3 f_1, \\ \dot{f}_3 &= f_1 f_2, \end{aligned} \tag{1.4}$$

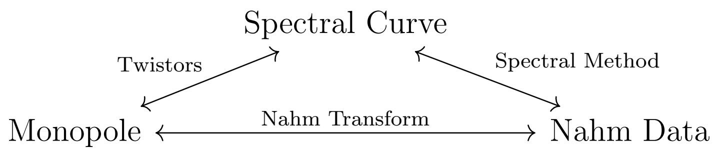
and exact solutions can then be given in terms of Jacobi elliptic functions [4].

Nahm's equations arose in the context of four-dimensional gauge theory where Nahm in [5], [6], [7], [8], and [9] showed that solutions of the Nahm equations are in 1-to-1 correspondence with monopoles, in the sense that there is a sort of non-abelian Fourier transform, called the Nahm transform or the ADHMN transform, that takes solutions of one system to the other. We will later place this transform in its wider context when we discuss the anti-self dual Yang-Mills equation.

We now introduce the monopole. The magnetic monopole was first proposed by Dirac in [10], a pointlike magnetic charge with its charge at the origin of $\mathbb{R}^3$ solving the Maxwell equations in electromagnetism, with a singularity at the origin. The monopole is a topological soliton and a major part of its attraction for physicists is that its existence would complete the electromagnetic duality of Maxwell's equations, putting electricity and magnetism on equal footing, and provide an explanation for the observed quantization of electric charge.

The monopole attracted considerably more attention after 't Hooft [11] and Polyakov [12] demonstrated that non-abelian gauge theories admitted magnetic monopoles as regular solutions on the whole $\mathbb{R}^3$ space to the Yang-Mills equations of motion. For a comprehensive survey on monopoles, see [13]. Exact solutions for the gauge group $G = U(1)$ were given by Dirac in [10]. For $G = SU(2)$ Prasad and Sommerfield [14] discovered exact solutions for the spherically symmetric monopole, and Prasad and Rossi [15] [16] discovered exact solutions for the axially symmetric monopole. Exact solutions in terms of elliptic functions are also known for monopoles having a Platonic solid symmetry [17] [18] [19]. Later, Braden and Enolski constructed exact solutions for any charge 2 $SU(2)$ monopole [20].

In general, there is rich array of tools to construct monopole and Nahm data such as rational maps [21] [22], Backlund transformations [23], and the twistorial approach [24]. Exact solutions are in general difficult to obtain, see the book by Weinberg [25] for a more complete history of such solutions. In [26], Hitchin proved that there is a spectral curve associated to each $SU(2)$ monopole solution and that this spectral curve is the same as the one appearing in its Nahm counterpart. In [27], Hurtubise classified the spectral curves for charge 2 $SU(2)$ monopoles and in [28], he and Murray extended the spectral curve correspondence to monopoles for all compact Lie groups G. Here we will follow the program established by Nahm [7] and Hitchin [26], illustrated in the diagram below.



To fully appreciate this story, we must give the context of Yang-Mills theory in four-dimensions and situate the Nahm transform in this background. In four-dimensional gauge theory, the anti-self dual Yang-Mills equation (ASD) in a complex oriented Riemannian four-manifold is a completely integrable system that arises naturally from Yang-Mills theory. Given a principal G-bundle over the manifold and A a connection 1-form with $F_A = dA + A \wedge A$ its curvature 2-form, the anti-self dual Yang-Mills equation is

$$*F_A = -F_A, \tag{1.5}$$

where $*$ is the Hodge star operation on differential forms. The anti-self dual equation arises in Yang-Mills theory as the minimizer of the action S_{YM}, with

$$S_{\text{YM}}[A] = \int_M \text{Tr}(F \wedge *F). \tag{1.6}$$

The four-dimensional space of interest to us is $\mathbb{R}^4$. Over this space, Ward [29] applied the twistorial methods of Penrose to provide a paradigm for integrability of ASD, and shortly thereafter its solution was given by the now-famous ADHM construction [30] of Atiyah, Drinfeld, Hitchin, and Manin.

The space $\mathbb{R}^4$ is in fact a hyperkahler manifold and in [31], the ASD equation is shown to be the hyperkahler moment map for the action of the gauge group and the moduli space of solutions to ASD is then a formal hyperkahler space. A hyperkahler manifold is a Riemannian manifold of dimension $4n$ with holonomy in $Sp(n)$ so that its tangent space is quaternionic. It has a S^2 family of complex structures and may be thought of as the quaternionic counterpart of complex Kahler manifolds. Penrose showed that an arbitrary hyperkahler manifold X admits a twistor space $\mathrm{Tw}(X)$ over the space X

$$\begin{array}{c} \mathrm{Tw}(X) \\ \big\downarrow \\ X \end{array}$$

with S^2 fibers given by the complex structures.

Ward used this in [29] to establish a 1-1 correspondence

$$\begin{array}{ccc} \mathcal{E} & \xleftrightarrow{\text{Ward}} & ASD \\ \big\downarrow & & \big\downarrow \\ \mathrm{Tw}(X) & \longrightarrow & X \end{array}$$

between ASD connections and holomorphic bundles $\mathcal{E}$ satisfying some additional conditions over the twistor space $\mathrm{Tw}(X)$. Thus, the ASD equation is equivalent to a problem in complex analysis and then finally to one of algebraic geometry.

The remarkable connection of ASD to Penrose's twistor paradigm above begins by expressing the ASD equation as saying that two operators commute for every choice of complex structure on $\mathbb{R}^4$. Let us use the coordinates (x^1, x^2, x^3, x^4) on $\mathbb{R}^4$, with the Euclidean metric and volume form $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. In a chosen trivialization, the connection one-form A may be written $A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3 + A_4 dx^4$. With this, ASD may be written in an equivalent system of two equations:
The Complex Equation,

$$[D_1 - iD_2, D_3 + iD_4] = 0,$$

and the Real Equation,

$$[D_1 + iD_2, (D_1 + iD_2)^\dagger] = [D_3 - iD_4, (D_3 - iD_4)^\dagger].$$

Here, $D_\mu = D_{\frac{\partial}{\partial x^\mu}}$ is the covariant derivative, where $D_\mu \Phi = \partial_{\frac{\partial}{\partial x^\mu}} \Phi + [A_\mu, \Phi]$.

One ought to think of this from the complexified point of view. Let $z = x_1 - ix_2$ and $w = x_3 + ix_4$, then this particular choice of complex structure establishes an isomorphism between $\mathbb{R}^4$ and $\mathbb{C}^2$. For $D_z = \frac{1}{2}(D_1 + iD_2)$, $D_w = \frac{1}{2}(D_3 - iD_4)$ the Complex Equation states $[D_{\bar{z}}, D_{\bar{w}}] = 0$.

But there is a $S^2 \cong \mathbb{C}P^1$ family of complex structures on $\mathbb{R}^4$ and there is nothing special about our given choice of complex structure. Let $\zeta \in \mathbb{C}P^1$ be the north coordinate, then the Complex Equation for the choice of complex structure parameterized by ζ gives

$$[D_{\bar{z}} - D_w\zeta, D_{\bar{w}} + D_z\zeta] = 0. \tag{1.7}$$

Requiring the above commutation to hold for every choice ζ of complex structure is equivalent to the ASD equation! For a specific choice of ζ with corresponding complex coordinates z_ζ, w_ζ, solving the Complex Equation is not sufficient to solve the ASD equation. We will use these operators extensively.

The ASD equation admits symmetry reductions by requiring the bundle and its connection A to be invariant under some subgroup of the conformal group on $\mathbb{R}^4$, and one has a corresponding quotient manifold and reduced twistor space. Many well-known integrable systems are obtained through such symmetry reductions, including the KdV and the nonlinear Schroedinger equations (NLS)! The latter two systems are obtained by starting with gauge group $SL(2,\mathbb{R})$ and the $(2,2)$ metric signature on $\mathbb{R}^4$. The KdV and NLS then appear as the two possible cases from the symmetry reduction of requiring the bundle and connection to be invariant under an orthogonal timelike translation and a null translation. See the book [32] for more details on these and other symmetry reductions.

The reduced form of (1.7) holds and the 1-1 correspondence between solutions to the reduced ASD equation and holomorphic bundles over the reduced twistor space survives but with additional conditions on the bundle coming from boundary conditions, reality structures, etc. The symmetry cases we consider will be the abelian subgroup of the conformal group on $\mathbb{R}^4$ given by translations. Let Γ be a closed subgroup of $\mathbb{R}^4$ (the group of translations) and $\Gamma^\vee = \{f \in \mathbb{R}^{4^*} \mid f(\Gamma) \in \mathbb{Z}\}$ be its dual group.

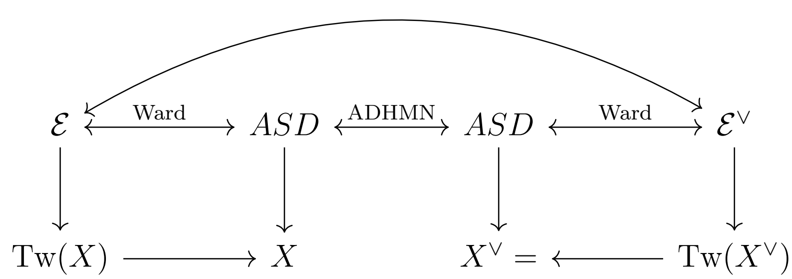
There is a stunning correspondence called the Nahm transform between ASD solutions over the quotient space $X = \mathbb{R}^4/\Gamma$ and ASD solutions over the quotient space $X^\vee = \mathbb{R}^{4^*}/\Gamma^*$ [33]. We loosely describe it here by drawing analogues to the Fourier transform, but an explicit description of the Nahm transform is given in Chapter 4.

Recall that in elementary Fourier analysis, one has a position space $\mathbb{R}^n$ and a momentum space $\mathbb{R}^n$. The Fourier transform begins with a function

$f(x)$ on the position space, and obtains a function on $\mathbb{R}^n \times \mathbb{R}^n$ by twisting $f(x)$ with the characters $e^{-2\pi i x \cdot y}$, then by integrating the twisted function over the position space, $f(x)$ is transformed to a function $\hat{f}(y)$ on the momentum space defined by $\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot y} dx$.

The Nahm transform proceeds in an analogous manner. We take a bundle with ASD connection over the manifold X, twist the ASD connection with trivial connections parametrized by the dual manifold X^*, and after finding a certain object (the kernel of the Dirac operator coupled to the twisted connection) on $X \times X^*$, we push the object down to X^* by integrating over X and obtain the transformed bundle and ASD connection over the dual manifold X^*.

We emphasize here that the appropriate additional conditions on the bundles $\mathcal{E}$ and $\mathcal{E}^\vee$ are crucial and must be examined before establishing a Nahm transform. They are highly nontrivial and much of the literature on Nahm transforms is dedicated to this task, see the survey by Jardim [34] for systems corresponding to different choices of Γ, such as calorons, Hitchin's equations, periodic monopoles, doubly periodic instantons, amongst others. We then have a spectacularly rich interplay of analytical and geometrical structures, illustrated in the diagram below.



The ASD reductions we consider are $\Gamma = \mathbb{R}^3$, invariance in three directions of $\mathbb{R}^4$, which reduces ASD to Nahm's equations on $\mathbb{R}^4/\mathbb{R}^3 = \mathbb{R}$, and its dual $\Gamma^* = \mathbb{R}$ which reduces ASD to the Bogomolny equation on $\mathbb{R}^4/\mathbb{R} = \mathbb{R}^3$. These are the original spaces for which Nahm proposed his transform [5] [6].

The dimensional reduction of ASD on $\mathbb{R}^4$ to $\mathbb{R}$ is Nahm's equations. The convention is to set invariance in the x_1, x_2, x_3 directions, relabel $A_4 = T_0$, $(A_1, A_2, A_3) = (T_1, T_2, T_3)$ and relabel x_4 as simply s. Then ASD (1.5) becomes Nahm's equation,

$$\begin{aligned} \frac{dT_1}{ds} + [T_0, T_1] = [T_2, T_3], \\ \frac{dT_2}{ds} + [T_0, T_2] = [T_3, T_1], \\ \frac{dT_3}{ds} + [T_0, T_3] = [T_1, T_2]. \end{aligned} \tag{1.8}$$

Traditionally, we also choose a specific gauge so that $T_0 = 0$.

We will consider the case where the Nahm data $(T_1(s), T_2(s), T_3(s))$ are $n \times n$ matrix-valued functions over the interval $(0, \infty)$ of $\mathbb{R}$ and the boundary conditions we consider are that the limits $\lim_{s\to\infty} T_j(s)$, $\lim_{s\to 0} sT_j(s)$ exist and furthermore,

$$\lim_{s\to\infty}(T_1(s), T_2(s), T_3(s)) \in \mathrm{ad}_{U(n)}(i\tau_1, i\tau_2, i\tau_3), \qquad \lim_{s\to 0} sT_j(s) = \frac{i\sigma_j}{2}, \tag{1.9}$$

for a chosen irreducible representation σ_j of dimension n of $\mathfrak{su}(2)$, i.e. $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, and a chosen triplet (τ_1, τ_2, τ_3) that is regular, in the sense that the stabilizer of τ_1, τ_2, and τ_3 form a maximal torus in $U(n)$.

In general, Nahm's equations might be over intervals $(-\infty, \lambda_1) \cup (\lambda_1, \lambda_2) \cup (\lambda_2, \lambda_3) \cup \cdots \cup (\lambda_k, \infty)$ with varying ranks for (T_1, T_2, T_3) on each interval and conditions on the jump data between any two intervals, describing how the Nahm data changes from one interval to the next. A very nice approach to visualizing these interval and jump conditions uses the D-brane approach, first proposed in [35]. We recommend [36] for an overview from this perspective.

The boundary conditions on Nahm's equations can vary, from choosing an arbitrary representation (not necessarily irreducible) of $\mathfrak{su}(2)$ for the poles to choosing irregular triplets (τ_1, τ_2, τ_3). The study of the moduli space of solutions to Nahm's equations was first undertaken by Donaldson [21] for the interval $(0, 2)$ with irreducible $\mathfrak{su}(2)$ representation at the poles, which corresponds to the $SU(2)$ monopole. More general boundary conditions were considered by [37] and [38], where it was clarified that the moduli space of solutions to Nahm's equations arises from intersections of the adjoint orbit of $\tau_1 + i\tau_2$ with Slodowy slices in the Lie algebra. See [39] for an overview from this perspective.

The dimensional reduction of ASD on $\mathbb{R}^4$ to $\mathbb{R}^3$ is Bogomolny's equation. The convention is to set invariance in the x_4 direction and relabel $A_4 = -\Phi$.

Then ASD becomes Bogomolny's equation,

$$D\Phi = *F_A. \tag{1.10}$$

Here, we write the reduced connection as $A = A_1dx^1 + A_2dx^2 + A_3dx^3$ and F_A is the curvature of this connection with $D\Phi$ the differential. We consider the Dirac monopole, which is the counterpart under the Nahm Transform of Nahm's equations with our boundary conditions (1.9). The bundle is the pullback to $\mathbb{R}^3\setminus\{0\}$ of the Hopf bundle over S^2 with gauge group $U(1)$. Since $U(1)$ is abelian, we may take superpositions of Dirac monopoles positioned at different points in $\mathbb{R}^3$ to form a Dirac multimonopole. The Nahm transform from $U(1)$ monopole (A, Φ) data to obtain Nahm data requires finding L^2 solutions ψ to the Dirac equation:

$$\mathcal{D}_s^\dagger \psi = 0, \tag{1.11}$$

where $\mathcal{D}_s^\dagger$ is a certain twisted Dirac operator coupled to the (A, Φ) data. Linear superpositions in (A, Φ) do not lead to linear superpositions in the corresponding $\mathcal{D}_s^\dagger$ so (1.11) is difficult to solve for Dirac multimonopoles. We reduce the problem to a linear system in Proposition 4.4 and present in Chapter 3 several methods for construction of these solutions.

1.2 Atiyah's Conjecture

We introduce Atiyah's conjecture for 'stellar' polynomials. A major motivation of the thesis is to examine the relationship between the conjecture and Nahm's equations, in the hope that this will lead to a proof of Atiyah's conjecture. In [40], [41] Atiyah constructed a set of n polynomials in $\mathbb{C}[\zeta]$ built from any collection of n distinct points in $\mathbb{R}^3$. These polynomials turn out to play a prominent role in the solutions to Nahm's equations. We describe informally his construction, which will account for our labeling of these polynomials as 'stellar'.

Think of each point as a star, with its celestial sphere which we identify with the Riemann sphere. Stand at a star, call it the sun, and look at all the other points. From our vantage view, the other stars will appear as dots on our celestial sphere – these are the directions! Associate to our sun the unique monic polynomial with these dots as its roots. Now, do this for each star so that each of the n stars has its own degree $n-1$ polynomial. Atiyah then conjectured that these n polynomials are always linearly independent.

Formally, the polynomials are defined in the following way. Label n distinct points for our point configuration in $\mathbb{R}^3$. Given $x_i \in \mathbb{R}^3$, for every point x_j with $j \neq i$ we have a direction $a_{ij} \in \mathbb{C} \cup \{\infty\}$ from x_i to x_j.

Definition 1.1. Define the i^{th} *Atiyah polynomial*, arising from the point x_i, to be the degree $n-1$ polynomial

$$A_i(\zeta) = \prod_{\substack{j=1 \\ j \neq i}}^{n} (\zeta - a_{ij}), \tag{1.12}$$

where $\zeta - \infty$ does not contribute a factor, that is, we consider it to be 1.

Atiyah's Conjecture. *For any point configuration of n distinct points in $\mathbb{R}^3$, the Atiyah polynomials $A_1(\zeta), A_2(\zeta), \ldots, A_n(\zeta)$ are linearly independent.*

Consider a simple example.

Example 1.1. Suppose the n points are located along the z-axis and we label them in descending order. At any point, the direction to all points north of it is 0 and the direction to all points south of it is ∞. Thus the Atiyah polynomial for the point i is $A_i(\zeta) = \zeta^i$. The Atiyah polynomials $A_1(\zeta) = 1, A_2(\zeta) = \zeta, A_3(\zeta) = \zeta^2, \ldots, A_n(\zeta) = \zeta^{n-1}$ are, of course, linearly independent.

The conjecture for $n \geq 4$ is hard. In its current state, it has been proved for up to $n = 4$ and also for specific point configurations. Numerical evidence was gathered for $n \leq 20$ in [42]. Eastwood and Norbury proved the conjecture for $n = 4$ [43] using a combination of extensive Maple calculations and geometric arguments, with Khuzam and Johnson providing a lower bound on the determinant of the matrix of coefficients for the Atiyah polynomials in [44]. Dokovic proved the conjecture for the point configuration where l of the points lie on the vertices of a l-gon in the plane and the remaining $n-l$ points lie on a line perpendicular to the plane passing through the centroid of the polygon [45]. Mazur and Petrenko [46] proved the conjecture for vertices of regular n-gons, convex quadrilaterals, and inscribed quadrilaterals.

The conjecture (1.2) arose from Atiyah's answer [40] to a problem posed by Berry and Robbins [47] in their study of the spin statistics theorem from the point of view of quantum theory. Here, Berry and Robbins considered n

distinct particles idealized as points. In the spirit of Roger Penrose's twistorial ideas, the Atiyah polynomials $A_1(\zeta), \ldots, A_n(\zeta)$ may then be thought as 'quantum states' associated to the classical point states $\vec{a}_1, \ldots, \vec{a}_n$. Let us describe the mathematical setting.

Consider two spaces, the configuration space $C_n(\mathbb{R}^3)$ of n distinct ordered points in $\mathbb{R}^3$ and the well-known flag manifold $U(n)/T^n$, where T^n is the subset of $U(n)$ consisting of the diagonal matrices. The configuration space is an open subset in $\mathbb{R}^{3n}$, obtained by removing the linear subspaces of codimension 3 where any two points coincide. The flag manifold $U(n)/T^n$ represents collections of n orthonormal vectors, each ambiguous up to a phase, in $\mathbb{C}^n$. The permutation group S_n acts freely on both spaces. On the configuration space, S_n permutes the points. On $U(n)/T^n$, S_n permutes the orthonormal vectors.

Berry-Robbins Problem: *Is there a continuous map*

$$C_n(\mathbb{R}^3) \longrightarrow U(n)/T^n \tag{1.13}$$

compatible with the action of the symmetric group S_n?

Atiyah confirmed this in [40] using an elementary construction but also gave a more elegant answer relying on the linear independence of the Atiyah polynomials. We describe how the Atiyah polynomials, if they are linearly independent, provide a solution to the Berry Robbins problem (1.13).

A set of n linearly independent vectors in $\mathbb{C}^n$ can be orthogonalized in a way compatible with S_n, i.e. not to depend on an ordering of them, by taking $U = MP^{-1}$ from $M = UP$ of the polar decomposition of the matrix M representing the vectors, with $P = (M^*M)^{1/2}$ [40]. Thus, equation (1.13) is equivalent to defining n points in $\mathbb{C}P^{n-1}$ that do not lie in a proper linear subspace. We may think of $\mathbb{C}P^{n-1}$ as the space of polynomials of degree less than or equal to $n-1$ in the projective variable $\zeta \in \mathbb{C}P^1 = S^2$ by assigning to the element $[a_0 : a_1 : \cdots : a_{n-1}] \in \mathbb{C}P^{n-1}$ the polynomial $a_0 + a_1\zeta + \cdots + a_{n-1}\zeta^{n-1}$, so that the map

$$\begin{gathered} C_n(\mathbb{R}^3) \to \mathbb{C}P^{n-1} \\ \{\vec{a}_1, \ldots, \vec{a}_n\} \mapsto \{A_1(\zeta), \ldots, A_n(\zeta)\} \end{gathered} \tag{1.14}$$

solves the Berry Robbins problem (1.13) so long as the Atiyah polynomials are linearly independent.

There are generalizations to the Berry-Robbins problem (1.13) in two directions. One is to generalize the space $\mathbb{R}^3$ so that the points become time-like lines in the Minkowski space [41] and another is to generalize $U(n)$ to an arbitrary compact Lie group [48].

In the Minkowski space generalization, denote by $C_n(M^{3+1})$ the configuration space of n non-intersecting straight world-lines $\xi_1, \ldots, \xi_n$ of n moving stars (or particles). The Berry-Robbins problem (1.13) generalizes to

Generalization 1 of Berry-Robbins Problem: *Is there a continuous map*

$$C_n(M^{3+1}) \longrightarrow U(n)/T^n \tag{1.15}$$

compatible with the action of the symmetric group S_n?

A generalization of the stellar polynomials provides a solution, again on the supposition that they are linearly independent. On each worldline ξ_i at the time t_i, the observer standing there sees on his celestial sphere the light of the $n-1$ other stars. These positions on the sphere describe the light rays from the other stars, emitted at some time in their past, which arrive at star i at time t_i. These marked points on the celestial sphere again correspond to $a_{ij} \in \mathbb{P}^1$, which varies as t_i varies, and we obtain our polynomials $A_1(\zeta), \ldots, A_n(\zeta)$ along with the generalization of Atiyah's conjecture.

Generalized Atiyah's Conjecture. *For any configuration of n straight non-intersecting worldlines in M^{3+1} and times $t_1, \ldots, t_n$, the Atiyah polynomials $A_1(\zeta), A_2(\zeta), \ldots, A_n(\zeta)$ are linearly independent.*

The generalization reduces to the Euclidean space $\mathbb{R}^3$ we previously considered when all the stars are static. Another case of interest reducing to the hyperbolic space $\mathbb{H}^3$ is when the stars all begin at a common origin, i.e. a 'big bang', and move at uniform velocities (straight lines in Minkowski space). A proof that the Atiyah polynomials are linearly independent in the Hyperbolic case for $n = 3$ is found in [40].

In the Lie-theoretic generalization posed and solved in [48], $U(n)$ is replaced with a compact Lie group G and S_n with its Weyl group W. Let T be a maximal torus with Lie algebra $\mathfrak{h}$. The Weyl group acts on G/T and on $\mathfrak{h}$. Define $\mathfrak{h}^3 := \mathfrak{h} \otimes \mathbb{R}^3$ and denote by Δ the singular subset of $\mathfrak{h}^3$ under the Weyl action. This set Δ is the union of the codimension 3 subspaces that are the kernels of root homomorphisms $\alpha \otimes 1 : \mathfrak{h}^3 \to \mathbb{R}^3$. Then W acts freely on $\mathfrak{h}^3 - \Delta$ and this space is the appropriate generalization of the configuration space $C_n(\mathbb{R}^3)$.

Generalization 2 of Berry-Robbins Problem: *Does there exist a map f*

$$f : (\mathfrak{h}^3 - \Delta) \longrightarrow G/T \tag{1.16}$$

compatible with the action of the Weyl group W?

The authors of [48] then solve problem (1.16) in the affirmative using a construction derived from Nahm's equations. We describe the case of $G = U(n)$. The point configuration corresponds to a regular triple $\tau = (\tau_1, \tau_2, \tau_3)$ in $\mathfrak{h}^3 - \Delta$. The map f is accomplished in the following way. Fix a choice of maximal irreducible representation σ of $\mathfrak{su}2$. Given a regular triple (τ_1, τ_2, τ_3), let us consider Nahm's equations over the interval $(0, \infty)$.

As shown in Chapter 2, we obtain an unique solution to Nahm's equations if we prescribe the following boundary conditions: $T_1(s), T_2(s), T_3(s)$ have a pole at $s = 0$ given by the irreducible representation σ and as $s \to \infty$, $T_1(s), T_2(s), T_3(s)$ decay to some regular triple lying in the adjoint orbit of $\tau = (\tau_1, \tau_2, \tau_3)$. Then $\lim_{s\to\infty}(T_1(s), T_2(s), T_3(s)) = g(\tau_1, \tau_2, \tau_3)g^{-1}$ for some $g \in U(n)$. Note, g is not unique since we may multiply it on the right with any element of the maximal torus T stabilizing τ, however the coset gT is unique. We have then obtained a map from $\mathfrak{h} - \Delta$ to $U(n)/T$, and one may check this map is compatible with the action of S_n and solves the Berry-Robbins problem (1.13).

1.3 Objectives for Nahm's Equations

Our objective is to build high rank solutions to Nahm's equations with the boundary conditions

$$\lim_{s\to\infty}(T_1(s), T_2(s), T_3(s)) \in \mathrm{ad}_{U(n)}(i\tau_1, i\tau_2, i\tau_3), \qquad \lim_{s\to 0} sT_j(s) = \frac{i\sigma_j}{2},$$

for a chosen irreducible representations σ_j of $\mathfrak{su}(2)$ and a chosen regular triplet (τ_1, τ_2, τ_3) and to use the resulting picture of the ADHMN Transform on the Nahm and monopole sides to investigate Atiyah's Conjecture (1.2) on stellar polynomials (1.12).

We are now able to

1. give an algorithm for finding Nahm solutions (T_0, T_1, T_2, T_3) for matrices of arbitrary rank n,

2. find L^2 zero modes of the multimonopole Dirac operator, and
3. carry out the Up and Down Transforms of ADHMN for arbitrary charge n multimonopoles of the Dirac $U(1)$ monopole.

This is new in the literature. Currently there are no algorithms for finding exact solutions to Nahm's equations or for carrying out the Nahm Transforms for general gauge group G. This resolves many questions about the Nahm Transform for the $G = U(1)$ monopoles (with a drawback that we do not know the necessary gauge to set $T_0 = 0$).

Specifically, we achieve the following results. Let L^s denote the linear flow in the Jacobian of S of the eigenline bundles over the spectral curve S, from the algebro-geometric approach to solving integrable systyems.

1. The bundles $L^s \otimes \mathcal{O}_S(n-2)$ in the Jacobian of S have no nonzero global sections for $s \in (0, \infty)$.
2. We give algorithms relying only on linear algebra to find an orthonormal basis of sections of $L^s \otimes \mathcal{O}_S(n-1)$ such that the corresponding solution to Nahm's equations satisfies the boundary conditions at $s = \infty$. However, T_0 is not zero and the boundary behavior as $s \to 0$ has a phase ambiguity.
3. We give a method to write a perturbation expansion of an orthonormal basis of sections to the eigenline bundle $L^s \otimes \mathcal{O}_S(n-1)$.
4. Lamy-Poirier showed that a collection of polynomials satisfying a certain set of algebraic equations can be used to construct L^2 zero modes of the monopole Dirac operator [49]. We linearized these equations and showed they are sections of $L^s \otimes \mathcal{O}_S(n-1)$, thereby proving the existence of n linearly independent such polynomials and their construction is given using the algorithm in item 3. above.

In Chapter 2, we discuss Nahm's equations with our boundary conditions. We prove uniqueness of the solution by considering Nahm's Real and Complex equations. The solution to the Complex equation is well known, it corresponds to the intersection of the Slodowy slice given by the boundary conditions at $s = 0$ with the adjoint orbit given by boundary conditions at $s = \infty$, see [48] for our case or e.g. [36] for more general boundary conditions. For the Real equation, we adapt analytical results from [21] and [37] to show that for every family of solutions to the Complex equation, there exists a unique solution that also solves the Real equation.

We continue Chapter 2 with the proof that the line bundle $L_S^s(n-2)$ over the spectral curve S has no nonzero global sections by considering the behavior of sections as $s \to \infty$. Hitchin proved this result for spectral curves of the nonsingular monopoles of the gauge group $G = SU(2)$ [26], but our result is new for monopoles of the gauge group $U(1)$, which are necessarily singular.

In Chapter 3 we present the algorithm to build high rank solutions to Nahm's equations using sections of $L_S^s(n-1)$. We adapt Bielawski's idea for constructing an orthonormal basis of sections [50]. There, the basis is constructed via theta functions (which are polynomial functions for our spectral curve) and finding zeros of such theta functions. In contrast, we present a linear system for our basis. In particular, this method does not require finding roots of polynomials of degree $n-1$ (we do not however discover the special gauge necessary to set $T_0 = 0$).

We continue in Section 3.4 to present a perturbation expansion for sections of $L_S^s(n-1)$ in terms of the exponentials of the transition function for $L_S^s(n-1)$. We give a demonstration for the case $n = 2$ and $n = 3$.

In Chapter 4, we introduce the Dirac monopole as well as the Down Transform of the ADHMN construction. We discuss Lamy-Poirier's ansatz for the L^2 zero modes of the monopole Dirac operator [49]. We linearize the system of algebraic conditions presented there and reinterpret his ansatz in terms of a basis of $H^0(L_S^s(n-1))$. We affirm in the positive Lamy-Poirier's conjecture that this method produces n zero modes by showing that this statement is equivalent to the fact $h^0(L_S^s(n-1)) = n$, proved in Corollary 3.4.

1.4 Motivation: Siegel Modular Forms and Black Holes

In this section, we discuss the motivation for the second part of the thesis, and we adopt an informal style. The precise definitions are given in later sections.

A *weak Jacobi form* is a holomorphic function $\varphi_{k,t}(\tau, z) = \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, with $\mathbb{H}$ the upper half plane, satisfying the transformation rules

$$\varphi_{k,t}(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}) = (c\tau+d)^k e^{i2\pi t\frac{cz^2}{c\tau+d}}\varphi_{k,t}(\tau,z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$
$$\varphi_{k,t}(\tau, z+\lambda\tau+\mu) = e^{-i2\pi t(\lambda^2\tau+2\lambda z)}\varphi_{k,t}(\tau,z), \qquad (\lambda,\mu)\in\mathbb{Z}^2 \tag{1.17}$$

and having a Fourier-Jacobi expansion of the form

$$\sum_{\substack{n,l\in\mathbb{Z} \\ 4tn-l^2\geq -t^2}} c(n,l)q^n y^l, \tag{1.18}$$

with $q := e^{2\pi i\tau}$ and $y := e^{2\pi i z}$.

The coefficient $c(n,l)$ has a *discriminant* equal to $4tn - l^2$. The terms $c(n,l)q^n y^l$ with $4tn - l^2 < 0$ are called *polar terms*.

In this thesis, we study weak Jacobi forms of weight $k = 0$ and our chief interests are the polar terms as well as the growth behavior of the sums

$$f_{a,b}(n,l) = \sum_{r\in\mathbb{Z}} c(nr + ar^2, l - br) \tag{1.19}$$

of Fourier-Jacobi coefficients.

The growth behavior of the Fourier coefficients of an automorphic form is a common theme in mathematics, and the work that our project undertakes is naturally situated here. The most classical example of such mathematical investigations is, perhaps, the asymptotics of the partition function $p(n)$. Indeed, the generating function $P(q) = \sum_{n=0}^{\infty}$ for the partition function is given by $P(q) = \frac{q^{1/24}}{\eta(\tau)}$, where $\eta(\tau)$ is the Dedekind eta function, a modular form of weight $1/2$. Thus, the question of the asymptotic behavior of $p(n)$ is precisely the question of the growth behavior of the Fourier coefficients of $\frac{1}{\eta(\tau)}$. Of course, Hardy and Ramanujan famously proved in 1918 that $p(n) \sim \frac{1}{4n\sqrt{3}}\exp(\pi\sqrt{\frac{2n}{3}})$.

Our interest in the growth behavior of $f_{a,b}(n,l)$ for a weak Jacobi form $\varphi_{0,m}$ comes from the fact that it determines the asymptotic growth of the Fourier coefficients of a Siegel modular form lifted from $\varphi_{0,m}$. We will expand on this later in the section.

To consider the possible behaviors of $f_{a,b}(n,l)$, we note that the asymptotic growth of the Fourier-Jacobi coefficient $c(n,l)$ for large discriminant is

$$c(n,l) \sim \exp \pi \sqrt{\frac{|\Delta_{\min}|}{t^2}(4tn - l^2)}, \tag{1.20}$$

where $\Delta_{\min}$ is the maximal polarity of the weak Jacobi form [51, Equation B.6]. Then roughly speaking, if there are not substantial cancellations inside the sum of $f_{a,b}(n,l)$, then $f_{a,b}(n,l)$ will be dominated by the most polar term in its sum and have exponential growth. However, in nongeneric cases, there are significant cancellations between the coefficients in the sum of $f_{a,b}(n,l)$ leading to subexponential growth in $f_{a,b}(n,l)$.

Now, let us return to our interest in the sums $f_{a,b}(n,l)$ of (1.19). As mentioned, they determine the growth behavior of the Fourier coefficients for a class of Siegel modular forms, which we now explain.

Let $\Phi_k(\Omega) : \mathbb{H}_2 \to \mathbb{C}$ with $\Omega = \begin{pmatrix} \tau & z \\ z & \rho \end{pmatrix}$ be a *Siegel modular form* of weight k and degree 2, i.e. it is a holomorphic function satisfying the following transformation law under the action of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_4(\mathbb{Z})$:

$$\Phi_k\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^k \Phi_k(\Omega). \tag{1.21}$$

Φ_k is, then, periodic in each variable τ, z, and ρ. Writing $p = e^{2\pi i \rho}$ and taking its Fourier expansion

$$\Phi_k(\Omega) = \sum_{t=0}^{\infty} \varphi_{k,t}(\tau, z) p^t, \tag{1.22}$$

The weak Jacobi form $\varphi_{0,t}(\tau, z)$ admits a *lift* [52] to a Siegel modular form $\Phi_\varphi : \mathbb{H}_2 \to \mathbb{C}$,

$$\Phi_\varphi(\Omega) = \text{Exp-Lift}(\varphi)(\Omega), \tag{1.23}$$

which transforms as (1.21) under the action of a subgroup of $Sp_4(\mathbb{Z})$, rather than the full group. We summarize the details of this lift in Chapter 6.

The sums $f_{a,b}(n,l)$ of (1.19) indicate the growth behavior of the Fourier coefficients $d(m,n,l)$ with negative *discriminant* $4mn - l^2 < 0$ of the meromorphic Siegel modular form

$$\frac{1}{\text{Exp-Lift}(\varphi)(\Omega)} = \sum_{m,n,l} d(m,n,l) p^m q^n y^l, \tag{1.24}$$

where here we expand the Fourier coefficients in the region $Im(\rho) \gg Im(\tau) \gg Im(z) > 0$.

For terms with very large negative discriminant, the possible behaviors [53] [54] are

(i) $\log |d(m,n,l)|$ grows linearly, in which case $d(m,n,l)$ is *fast growing*,

(ii) $\log |d(m,n,l)|$ grows as a square root, in which case $d(m,n,l)$ is *slow growing*.

The growth of the coefficients $d(m,n,l)$ with negative discriminant depends on the underlying weak Jacobi form $\varphi_{0,t}$, indeed they depend on the sums $f_{a,b}(n,l)$. Roughly speaking, slow growing $f_{a,b}(n,l)$ leads to slow growing $d(m,n,l)$ and fast growing $f_{a,b}(n,l)$ leads to fast growing $d(m,n,l)$. We summarize more precisely this relationship in Chapter 6.

Now, we remark on the physical motivation behind our investigation into these mathematical objects. However, we make a note that the findings of our project are purely mathematical, and the following physical discussion will not appear as part of our results.

The exponentially-lifted Siegel modular forms arise in the string-theory of black holes, where their Fourier-coefficients $d(m,n,l)$ count the dimension of certain eigenspaces.

A classical example is the Igusa cusp form [55]

$$\Phi_{10}(\Omega) = \text{Exp-Lift}(2\phi_{0,1}), \tag{1.25}$$

where the Fourier coefficients $\psi_m(\tau,z)$ of

$$\frac{1}{\Phi_{10}(\Omega)} = \sum_{m=-1}^{\infty} \psi_m(\tau,z)p^m \tag{1.26}$$

turn out to be the partition functions of degeneracy 1/4-BPS black holes in four-dimensional $\mathcal{N}=4$ supergravity on a type II compactification on the product of a K3 surface and an elliptic curve.

In general, a weak Jacobi form $\varphi_{0,t}(\tau,z)$ appears as the elliptic genus of a worldvolume theory of a propagating string in a Calabi-Yau manifold. If $\varphi_{0,t}$ is the elliptic genus χ of a conformal field theory on a manifold M, then $\Phi_\varphi = \text{Exp-Lift}(\varphi_{0,t})$ is the generating function for the elliptic genera of the symmetric orbifolds of that theory,

$$\Phi_\varphi = \sum_{r=0}^{\infty} p^{tr} \chi(\tau, z; Sym^r(M)), \tag{1.27}$$

and the Fourier coefficients of the reciprocal $\frac{1}{\Phi_\varphi}$ correspond to black hole states. For a review of weak Jacobi forms and Siegel modular forms in string theory, see [56] and [57].

The physical significance of the growth behavior of the coefficients $d(m, n, l)$ for $\frac{1}{\Phi_\varphi}$ is that it indicates whether the symmetric product orbifolds are candidates for marginal deformation to a supergravity CFT: $\varphi_{0,t}$ lifts to a viable candidate whenever $d(m, n, l)$ is slow-growth [58].

1.5 Objectives for Jacobi Forms

Our objectives are to further explore the relationship between polar coefficients and the weak Jacobi form, and to investigate the space of weak Jacobi forms with slow growing $f_{a,b}(n, l)$. We are guided by the following questions.

Weight 0 weak Jacobi forms are uniquely determined by their polar terms. However, this is an overdetermined system as for a fixed index m, the number of polar terms exceed the dimension of weak Jacobi forms. This leads to the question:

Question 1. *Which polar terms determine a weak Jacobi form of weight 0 and index m?*

In addition to this, we are also motivated by the following conjecture from [58].

Conjecture 1.1. *For every index m, there exists a weak Jacobi form $\varphi_{0,m}$ that has slow growing $f_{0,b}(n, l)$ about its most polar term y^b.*

This leads us to consider the question:

Question 2. *Given $a, b \in \mathbb{Z}$, what is the space of weak Jacobi forms that have slow growing $f_{a,b}(n, l)$?*

In pursuing these questions, we obtain the following results in this thesis.

1. In Proposition 6.2, we gain an analytic bound on the number of polar terms $p(m)$: for any $\epsilon > 0$ there exists a computable C_ϵ such that the total number of polar terms $p(m)$ for index m satisfies the bound

$$|p(m) - \frac{m^2}{12} - \frac{5m}{8}| \leq C_\epsilon m^{1/2+\epsilon}. \tag{1.28}$$

2. In Proposition 6.3, we prove that the polar coefficients of polarity $\leq -m/6$ uniquely determine the weak Jacobi form.

We define a largest value $P(m)$ for index m such that the polar terms of polarity $\leq -P(m)$ uniquely determine the weak Jacobi form $\varphi_{0,m}$.

3. We compute $P(m)$ for $1 \leq m \leq 61$, displayed in Figure 6.1.

4. We propose Conjecture 6.6, which if confirmed, implies

$$|P(m) - \frac{m}{2}| \leq Cm^{1/2} \tag{1.29}$$

   for some constant C.

In our investigations of slow growing $f_{0,b}(n, l)$, we discovered a large class of slow growing forms from quotients of theta functions and we obtained the following results.

5. We implement a fast algorithm for computing the table of polar coefficients for a basis of $J_{0,m}$.

6. We produce an enlarged table in Table 7.1 for index $1 \leq m \leq 61$, listing the dimension of the space of weak Jacobi forms with most polar term y^b that have slow growing $f_{0,b}(n, l)$. This table expands Table 2 of [59, pp.19], which contains data for index $1 \leq m \leq 18$.

7. We give a simple criterion for slow growth of a theta quotient.

8. In Proposition 7.7, we classify all single quotient theta functions that are slow growing about its most polar term y^b.

9. In Lemma 7.8, we prove the conjecture in [60] that the class of weak Jacobi forms from the $M = 2$ Kazama-Suzuiki models are slow growth.

10. We produce Table 7.2, containing the dimensions of slow growing theta quotients for quotients of size ≤ 7.

Unlike for sums $f_{0,b}(n,l)$, the behavior of the sums $f_{a,b}(n,l)$ with $a > 0$ do not appear in the literature. A big obstacle in their investigation is, unlike $f_{0,b}(n,l)$, we are required to compute the Fourier-Jacobi expansion of $\varphi_{0,t}$ to very high order to be able to compute $f_{a,b}(n,l)$. However, we were able to make substantial computational progress and create new numerical findings, in addition to some limited analytical results. We summarize our progress below.

11. We are able to give a fast implementation for computing the Fourier-Jacobi coefficients to very high order, partially based on a new formula in Lemma A.1 for a generating function.

12. Based on our numerical findings, we propose Conjecture 8.1 that the behavior of $f_{a,b}(n,l)$ is the same as that for $f_{0,b}(n,l)$. That is, $f_{a,b}(n,l)$ grows either exponentially fast in n, l or they attain only finitely many distinct values.

Part II of this thesis is organized as follows.

In Chapter 6, we explore the interaction between the weak Jacobi form and its polar part and we give the results of items 1-4. We compute $P(m)$ for $1 \leq m \leq 61$, displayed in Figure 6.1, as the difference between $P(m)$ and the dimension of $J_{0,m}$, displayed in Figure 6.2. The latter scatterplot shows that for many m, the polar part of polarity $\leq -P(m)$ does not overdetermine the space of weak Jacobi forms so that every polar part of polarity $\leq -P(m)$ indeed has a corresponding weak Jacobi form.

We discuss an upper bound $P^+(m)$ for $P(m)$, where $P^+(m)$ is such that the number of polar terms of polarity $\leq \delta_{j(m)}$ equals the dimension $j(m)$ of the space of weak Jacobi forms of weight 0 and index m. Based on the computed data of $P^+(m)$ for $1 \leq m \leq 1000$, we propose Conjecture 6.6 which bounds $P^+(m)$ from above by $\frac{m}{2} + Cm^{1/2}$.

In Chapter 7, we further develop the exploration of weak Jacobi forms with slow growing $f_{0,b}(n,l)$. We expand Table 2 of [59, pp.19] to index $1 \leq m \leq 61$, listing the dimension of the space of weak Jacobi forms with most polar term y^b that have slow growing $f_{0,b}(n,l)$.

We discuss a class of weak Jacobi forms given by quotients of theta functions. This class of functions is found to contain a large amount of slow growing weak Jacobi forms. In Proposition 7.7, we classify all single quotient theta functions that are slow growing about its most polar term y^b.

We find a simple criterion (7.40) for a quotient of theta functions to be slow growing about its y^b term and we use this to produce a table of the dimensions of slow growing theta quotients for quotients of size ≤ 7. This criterion is also used to prove the conjecture in [60] that the class of weak Jacobi forms from the $M = 2$ Kazama-Suzuiki models are slow growth, and as another application, we give a simplified proof of [60] that the weak Jacobi forms from the minimal models of type A, D, E are slow growing.

In Chapter 8, we discuss weak Jacobi forms with slow growing $f_{a,b}(n, l)$ for $a > 0$. The exploration of $f_{a,b}(n, l)$ for $a > 0$ was suggested in [61] but was previously inaccessible as computation of $f_{a,b}(n, l)$ requires Fourier-Jacobi expansions of the weak Jacobi forms to order 1000 or more. For low index $1 \leq m \leq 12$, we compute some values of $f_{a,b}(n, l)$ for $q^a y^b$ of relatively low polarity and present a table of the dimensions of slow growing weak Jacobi forms.

We find that the growth characteristics of $f_{a,b}(n, l)$ are precisely the same as $f_{0,b}(n, l)$, that is, $f_{a,b}(n, l)$ either grows exponentially fast or it attains only finitely many values.

For $f_{0,b}(n, l)$, this is explained by the fact found in [61] that the generating functions for $f_{0,b}(n, l)$ are modular forms of weight 0 and that these modular forms are constant functions whenever $f_{0,b}(n, l)$ are slow growing. In this case, $f_{0,b}(n, l)$ attains only finitely many distinct values as $f_{0,b}(n, l)$ is nonzero only when it is the constant coefficient of these generating functions.

For $f_{a,b}(n, l)$ with $a > 0$, generating functions are not known but we discover some special cases where the generating functions may be given in terms of specializations of an Atkin-Lehner involution of the underlying weak Jacobi form $\varphi_{0,m}$.

In the Appendix, we present the essential Mathematica code we used for obtaining our numerical results. We chose to use Gritsenko's generating functions $\phi_{0,1}, \phi_{0,2}, \phi_{0,3}$ for the space of weak Jacobi forms of weight 0. We present the code for fast computation of the matrix of polar coefficients of a basis of the space of weak Jacobi forms of weight 0 and index k. This extends the indexes m that are reasonable (less than two weeks) to compute from $m \leq 20$ to $m \leq 71$.

We present a novel formula in Lemma A.1 that allows for fast Fourier-Jacobi expansion of the generating function $\phi_{0,3}$. We then present the Mathematica code for fast Fourier-Jacobi expansion of all the generating functions for a basis of the space of weak Jacobi forms that allows for the computation of $f_{a,b}(n, l)$ with $a > 0$.

Part I

Chapter 2

Nahm's Equations

2.1 Introduction

We want to solve Nahm's equations on the real interval $(0, \infty)$ with specified boundary conditions that correspond to the Dirac $U(1)$ monopoles of charge n [5]. The problem is to find antihermitian $n \times n$ matrix-valued functions $T_1(s)$, $T_2(s)$, and $T_3(s)$ over the interval $(0, \infty)$ solving

$$\begin{aligned} \frac{dT_1}{ds} &= [T_2, T_3], \\ \frac{dT_2}{ds} &= [T_3, T_1], \\ \frac{dT_3}{ds} &= [T_1, T_2], \end{aligned} \tag{2.1}$$

with the following boundary conditions at infinity and zero for $j = 1, 2, 3$:

$$\lim_{s\to\infty} (T_1(s), T_2(s), T_3(s)) \in \mathrm{ad}_{U(n)}(i\tau_1, i\tau_2, i\tau_3), \quad \lim_{s\to 0} sT_j(s) = \frac{i\sigma_j}{2}. \tag{2.2}$$

The triplet $i\sigma_j$ is a chosen irreducible unitary rank n representation of $\mathfrak{su}(2)$: $[i\sigma_1, i\sigma_2] = -2i\sigma_3$ with $\sigma_3 = \mathrm{diag}(n{-}1, n{-}3, \ldots, -n{+}1)$ and $\sigma_+ = \frac{1}{2}(\sigma_1{+}i\sigma_2)$ with $(\sigma_+)_{j,i} = \sqrt{j(n-j)}\delta_{(j+1),i}$. The triplet $i\tau_j = \mathrm{diag}(ip_j^1, ip_j^2, \ldots, ip_j^n)$ is regular, i.e. the set of matrices that commute with all three is a maximal torus T of U, here T is the set of all diagonal matrices.

We note that $(\frac{i\sigma_1}{2s}, \frac{i\sigma_2}{2s}, \frac{i\sigma_3}{2s})$ as well as $(i\tau_1, i\tau_2, i\tau_3)$ are model solutions to Nahm's equations. Analytically, we require the Nahm data to satisfy the conditions

$$T_j(s) = \frac{i\sigma_j}{2s} + ig_0\tau_j g_0^{-1} + b_j(s), \tag{2.3}$$

for some $g_0 \in U(n)$ and $b_j(s) \in L^2[0,\infty)$. Infact, given these conditions, $b_j(s) \in L^\infty[0,\infty)$ thanks to the two lemmas below from the literature.

Lemma 2.1 ([62]). *Suppose the Nahm data $(T_1(s), T_2(s), T_3(s))$ satisfies the following condition.*

$$sT_j - \frac{i\sigma_j}{2s} \in L^2[0,\epsilon), \tag{2.4}$$

for some $\epsilon > 0$. Then

$$sT_j - \frac{i\sigma_j}{2s} \in L^\infty([0,\epsilon]). \tag{2.5}$$

The previous lemma deals with the behavior of the Nahm data near the pole. Asymptotically as $s \to \infty$, the solutions (T_1, T_2, T_3) approach their limit exponentially fast with the precise statement given by the following lemma.

Lemma 2.2. *Let (T_1, T_2, T_3) satisfy Nahm's equations with the boundary condition (2.2) where τ_j is a regular triple and $\lim_{s\to\infty} T_j(s) = g_0 i\tau_j g_0^{-1}$ for some $g_0 \in U(n)$. Then away from $s = 0$, e.g. for $s > 2$, there exists a constant $\eta > 0$ depending only on (τ_1, τ_2, τ_3) such that $|T_j - Ad(g_0)i\tau_j| \leq const \times e^{-\eta s}$.*

Proof. This result is [37, Lemma 3.4]. The reason is that Nahm's equations are the gradient-flow equations for the function

$$\psi(T_1, T_2, T_3) = \langle T_1, [T_2, T_3]\rangle = \mathrm{tr} T_1[T_2, T_3],$$

where $\langle\ ,\ \rangle$ is an Ad-invariant inner product. The critical set C of this flow consists of triples (T_1, T_2, T_3) which commute and the condition of regularity makes C a smooth manifold in the neighborhood of $\mathrm{Ad}(g_0)(i\tau_1, i\tau_2, i\tau_3)$, then exponential decay holds in general for any gradient system in the neighborhood of such a 'hyperbolic' critical set with non-degenerate Hessian. Any η smaller than the smallest positive eigenvalue of the Hessian of ψ will do. However, if we do not have a regular triplet then this behavior will not hold in general. Simple counterexamples are in [37, pp.207]. □

It is useful to extend the equations by introducing a fourth $L^2[0,\infty)$ antihermitian matrix $T_0(s)$ and writing

$$\begin{aligned} \frac{dT_1}{ds} + [T_0, T_1] &= [T_2, T_3] \\ \frac{dT_2}{ds} + [T_0, T_2] &= [T_3, T_1] \\ \frac{dT_3}{ds} + [T_0, T_3] &= [T_1, T_2]. \end{aligned} \tag{2.6}$$

These new equations and corresponding boundary conditions are invariant under an action of the group $\mathcal{G}$ of $g : [0,\infty) \to U(n)$ with $g(0) = 1$ and $\lim_{s\to\infty} g(s) \to$ diagonal. The action of $\mathcal{G}$ on (T_0, T_1, T_2, T_3) is

$$\begin{aligned} &T_0 \to gT_0g^{-1} - \dot{g}g^{-1}, \\ &T_i \to gT_ig^{-1}, \quad i = 1, 2, 3. \end{aligned} \tag{2.7}$$

Analytically, we take the gauge group to be

$$\mathcal{G} = \{g : [0,\infty) \to U(n) \mid g(0) = 1, \lim_{s\to\infty} g = \text{diagonal}, g^{-1}(s)\dot{g}(s) \in L^2[0,\infty)\}. \tag{2.8}$$

2.2 Boundary Conditions

This section is devoted to the proof of the following theorem:

Theorem 2.3. *There exists a unique solution (T_1, T_2, T_3) to Nahm's equations with the boundary conditions*

$$\lim_{s\to\infty} (T_1(s), T_2(s), T_3(s)) \in ad_{U(n)}(i\tau_1, i\tau_2, i\tau_3), \qquad \lim_{s\to 0} sT_j(s) = \frac{i\sigma_j}{2}$$

with regular triplets $i\tau_j$ and chosen irreducible representations σ_j of $\mathfrak{su}(2)$ as in (2.2).

The theorem was proved in [48, Section 6]. We provide an explicit proof by piecing together results found in the literature. In [21], Donaldson considers the interval $(0, 2)$ with boundary conditions on both sides given by poles with residues that are irreducible representations of $\mathfrak{su}(2)$. In [37], Kronheimer considers the interval $[0,\infty)$ with boundary condition given by a regular triple at infinity but regular behavior at $s = 0$. Our case is the interval $(0, \infty)$ with an irreducible pole at $s = 0$ and a commuting triplet at infinity, so we will adapt both to our situation. The standard method of proof is to choose a complex structure, thereby losing the cyclic symmetry between T_1, T_2, T_3 in Nahm's equations, and looking at a system equivalent to Nahm's equations of two equations called the Complex and Real equations. One then shows that every solution to the Complex equation has a unique complexified gauge transform that transforms the solution into one that satisfies both the Complex and Real equations.

2.2.1 The Complex Equation

Set $\mathcal{X} = -iT_1 + T_2$ and $\mathcal{A} = T_0 - iT_3$, which takes values in the complexified Lie algebra $M_n(\mathbb{C})$ of $\mathfrak{u}(n)$. Two of Nahm's equations combine to a single equation called the Nahm's Complex equation,

$$\dot{\mathcal{X}} = [\mathcal{X}, \mathcal{A}]. \tag{2.9}$$

The action of the gauge group extends to the gauge group $\mathcal{G}^c$ taking values in the complexification $GL(n, \mathbb{C})$ of $U(n)$ which preserves the Complex equation (2.9). The complexified gauge group acts on $(\mathcal{X}, \mathcal{A})$ by $g \cdot (\mathcal{X}, \mathcal{A}) = (g\mathcal{X}g^{-1}, g\mathcal{A}g^{-1} - \dot{g}g^{-1})$.

The Complex equation is easily solved since it is locally trivial, indeed the Lax equation (2.9) for $(\mathcal{X}, \mathcal{A})$ implies that the spectrum of $\mathcal{X}$ does not evolve in time and the solution can be given locally as $\mathcal{X}(s) = g(s)Xg^{-1}(s)$, $\mathcal{A}(s) = -\dot{g}(s)g^{-1}(s)$ for some choice of constant matrix X and $g(s)$ any $GL(n, \mathbb{C})$-valued function. We will construct a solution $(\mathcal{X}, \mathcal{A})$ of the complex equation with the properties:

$$(\mathcal{X}, \mathcal{A})(s) = \begin{cases} (\frac{\sigma_+}{s} + O(s^0), \frac{\sigma_3}{2s} + O(s^0)) \text{ as } s \to 0 \\ (\tau_1 + i\tau_2, \tau_3) \text{ for } 2 < s < \infty. \end{cases} \tag{2.10}$$

The existence of such a solution follows from the local triviality of the complex equation. We use the exposition as in Gaiotto-Witten [36].

Near $s = 0$, we have from the boundary conditions of Theorem 2.3

$$\lim_{s \to 0} s\mathcal{X}(s) = \sigma_+, \qquad \lim_{s \to 0} s\mathcal{A}(s) = \frac{\sigma_3}{2}. \tag{2.11}$$

We may use a complex gauge to transform $\mathcal{A}(s)$ to only its pole part, i.e. $\mathcal{A}(s) = \frac{\sigma_3}{2s}$. Indeed, this is equivalent to solving the linear ordinary differential equation

$$\dot{g} = g\left(\mathcal{A} - \frac{\sigma_3}{2s}\right) + g\frac{\sigma_3}{2s} - \frac{\sigma_3}{2s}g,$$

with initial condition $g(0) = 1$. In particular, $\mathcal{A}(s) - \frac{\sigma_3}{2s} \in L^\infty([0,\epsilon)$ so the terms having a singular point at $s = 0$ are $g\frac{\sigma_3}{2s}$ and $\frac{\sigma_3}{2s}g$. The limit $\lim_{s\to 0} s\frac{\sigma_3}{2s}$ is clearly finite so $s = 0$ is a regular singular point, and the solution g exists [63, Section 2].

The weights of $\frac{\sigma_3}{2}$ under the action $[\cdot, \frac{\sigma_3}{2}]$ on the set of $n \times n$ matrices are $-n+1, \ldots, -1, 0, 1, \ldots, n-1$. The matrix $E_{pq} = \left(\delta_{ip}\delta_{jq}\right)$ is a weight vector of weight $p - q$ under this action. To aid in visualization, we write a $n \times n$ matrix below and assign to its pq entry the weight of E_{pq}.

$$\begin{pmatrix} 0 & -1 & -2 & \ldots & -n+1 \\ 1 & 0 & -1 & \ldots & -n+2 \\ 2 & 1 & 0 & & \vdots \\ & & & \ddots & \\ n-1 & n-2 & \ldots & & 0 \end{pmatrix}. \tag{2.12}$$

Take as basis V_α the matrices of definite weight

$$[V_\alpha, \frac{\sigma_3}{2}] = \nu_\alpha V_\alpha, \tag{2.13}$$

where $\nu_\alpha \in \{-n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-1\}$. Now with $\mathcal{A}(s) = \frac{\sigma_3}{2s}$, let us solve the Complex Nahm equation on the interval $(0, 1)$. We may write

$$\mathcal{X}(s) = \sum \epsilon_\alpha V_\alpha f_\alpha(s), \tag{2.14}$$

with arbitrary powers μ_α and complex coefficients ϵ_α. The Complex Nahm equation then states

$$\begin{aligned} \dot{\mathcal{X}}(s) &= [\mathcal{X}(s), \frac{\sigma_3}{2s}] \\ \sum \epsilon_\alpha V_\alpha \dot{f}_\alpha(s) &= \sum \nu_\alpha \epsilon_\alpha V_\alpha \frac{f_\alpha(s)}{s}, \end{aligned} \tag{2.15}$$

thus $f_\alpha(s) = c_\alpha s^{\nu_\alpha}$.

The Complex Nahm equation on the interval $(0,1)$ then has the general solution

$$\mathcal{X} = \sum_\alpha \epsilon_\alpha V_\alpha s^{\nu_\alpha}, \quad \mathcal{A} = \frac{\sigma_3}{2s}. \tag{2.16}$$

Since $\mathcal{X}$ must have prescribed pole, we exclude the ineligible negative weights and we have

$$\mathcal{X} = \frac{\sigma_+}{s} + \sum_{\nu_\alpha \geq 0} \epsilon_\alpha V_\alpha s^{\nu_\alpha}, \quad \mathcal{A} = \frac{\sigma_3}{2s}. \tag{2.17}$$

The form $\mathcal{A} = \frac{\sigma_3}{2s}$ is still preserved by remaining gauge transformations generated by the infinitesimal gauge transformations ϕ

$$\phi = \sum_{v_\alpha > 0} f_\alpha V_\alpha s^{\nu_\alpha}, \tag{2.18}$$

with arbitrary coefficients f_α. To see this, the gauge transformations preserving $\frac{\sigma_3}{2s}$ satisfy the equation

$$\begin{aligned} g\frac{\sigma_3}{2s}g^{-1} - \dot{g}g^{-1} &= \frac{\sigma_3}{2s} \\ \dot{g} &= [g, \frac{\sigma_3}{2s}], \end{aligned} \tag{2.19}$$

and the same argument as in (2.15) applies.

ϕ shifts $\mathcal{X}$ by $[\mathcal{X}, \phi]$. The matrix σ_+ lowers E_{pq} of weight $p-q$ to the weight space $p-q-1$. Since the weight space of the weight k has dimension $n-|k|$, we may use such gauge transformations to remove everything from $\mathcal{X}$ except the pole and subdiagonals with constant coefficients. Every solution to the Complex Equation (2.9) on the interval $(0,1)$ with the prescribed pole at $s=0$ given by (2.11) is then gauge equivalent to the solutions

$$\mathcal{X}(s) = \begin{pmatrix} a & s^{-1} & 0 & \dots & 0 \\ bs & a & s^{-1} & & 0 \\ cs^2 & bs & a & & \vdots \\ & & & \ddots & s^{-1} \\ ds^{n-1} & es^{n-2} & \dots & & a \end{pmatrix}, \quad \mathcal{A} = \frac{\sigma_3}{2s}. \tag{2.20}$$

Next, we incorporate the prescribed boundary condition at infinity.

Lemma 2.4. *There is one and only one solution* $(\mathcal{X}, \mathcal{A})$ *to the Complex Equation satisfying the conditions* (2.10)*, i.e.*

$$(\mathcal{X}, \mathcal{A})(s) = \begin{cases} (\frac{\sigma_+}{2s} + O(s), \frac{\sigma_3}{2s} + O(s)) \ \textit{as } s \to 0 \\ (\tau_1 + i\tau_2, \tau_3) \ \textit{for } 2 < s < \infty, \end{cases}$$

modulo smooth complex gauge transformations $g : [0, \infty) \to GL(n, \mathbb{C})$ *with support in* $[0, 2]$.

Proof. Let $\mathcal{X}(s)$ be a solution to the Complex equation with the conditions (2.10). We showed $\mathcal{X}(s)$ can be gauged using complex gauge transformations to the form (2.20) on an interval $(0, 1 + \epsilon)$ with $0 < \epsilon < 1$. We extended the right endpoint of the interval to be slightly larger than 1. Since $\mathcal{X}(s)$ satisfies the Lax equation (2.9), its spectrum does not evolve in s.

The boundary condition at infinity gives the spectrum of $\mathcal{X}$ so the coefficients of its characteristic polynomial are determined. A formula for the x^{n-k} term of the characteristic equation $\det(x1 - A) = 0$ for a matrix A is

$$\frac{1}{k!} \det \begin{vmatrix} \mathrm{tr}A & k-1 & 0 & \dots & \\ \mathrm{tr}A^2 & \mathrm{tr}A & k-2 & \dots & \\ \vdots & \vdots & & \ddots & \vdots \\ \mathrm{tr}A^{k-1} & \mathrm{tr}A^{k-2} & & \dots & 1 \\ \mathrm{tr}A^k & \mathrm{tr}A^{k-1} & & \dots & \mathrm{tr}A \end{vmatrix}.$$

Appying this formula to the matrix $\mathcal{X}(1)$ of (2.20),

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & 1 & \dots & \vdots \\ & \ddots & & \ddots & & 0 \\ a_{n-1} & & \dots & & a_1 & 1 \\ a_n & a_{n-1} & & \dots & a_2 & a_1 \end{pmatrix}, \tag{2.21}$$

the coefficient of x^{n-k} in its characteristic polynomial has a_k as a linear term along with terms a_j for $j < k$. For example, the coefficient of x^{n-1} is $-na_1$. This implies that for a given spectrum of $\mathcal{X}(s)$, the entries $\{a_k \mid 1 \le k \le n\}$ of $\mathcal{X}(1)$ are uniquely determined.

Approaching $s = 1$ from infinity, $\mathcal{X}(1) = g(\tau_1 + i\tau_2)g^{-1}$ for some constant complex gauge g. One may then take any smooth path $g(s)$ in $GL(n, \mathbb{C})$ with support in $[1, 2]$ from this fixed g to the identity matrix 1, and any other choice of smooth path is gauge equivalent. This path allows the Complex solution over $[2, \infty)$ to flow to the Complex solution over $(0, 1]$, the resulting solution then satisfies the Complex equation over the entire interval $(0, \infty)$. We have shown both existence and uniqueness of the solution to the Complex equation, up to complex gauge transformations. □

Let us describe in broader terms the features in the proof of the above lemma. Matrices of the form (2.21) are known as the Slodowy slice S_{σ_+}. It is transverse to the nilpotent orbit of σ_+; we give the formal definition of the Slodowy slice as in [64], which will be slightly different than the matrices of (2.21).

Definition 2.1. Let $\rho : \mathfrak{su}(2) \to \mathfrak{g}$ be a Lie algebra homomorphism with complexification $\rho^{\mathbb{C}} : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$ and let the resulting triple be (e, h, f) where

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2.22}$$

The *Slodowy slice* corresponding to ρ is

$$S(\rho) = f + C(e), \tag{2.23}$$

where $C(e)$ denotes the centralizer of e in $\mathfrak{g}^{\mathbb{C}}$.

The set of matrices in (2.21) is precisely the Slodowy slice $S(\rho_+)$ for the regular nilpotent matrix

$$\rho_+ = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \vdots \\ & & & \ddots & 1 \\ 0 & 0 & \dots & & 0 \end{pmatrix}.$$

This Slodowy slice is, of course, gauge equivalent under a complex gauge transformation to the Slodowy slice $S(\sigma_+)$ for the nilpotent matrix σ_+ of the irreducible unitary $\mathfrak{su}(2)$ representation at the pole of our Nahm data.

The Slodowy slice transverse to a regular nilpotent matrix has the property that it meets adjoint orbits at a single point, which we demonstrated in the proof of the lemma. The boundary conditions at infinity fix the spectrum of $\mathcal{X}(s)$ so that $\mathcal{X}(s)$ resides in the adjoint orbit $\mathcal{O}_{\tau_1+i\tau_2}$. The boundary conditions at $s=0$ state that, after applying complex gauge transformations, $\mathcal{X}(1)$ belongs to the Slodowy slice S_{σ_+}. Since Slodowy slices intersect adjoint orbits at a single point, $\mathcal{X}(1)$ is uniquely determined. Indeed, functions $\mathcal{X}(s)$ of the form (2.20) are in one-to-one correspondence with points in the Slodowy slice $\mathcal{S}_{\sigma_+}$ via $\mathcal{X}(1) \in \mathcal{S}_{\sigma_+}$.

2.2.2 The Real Equation

Now we discuss the Real equation. As mentioned, the Complex equation restates two of three of Nahm's equation. The third of Nahm's equations is equivalent to the Real equation

$$\hat{F}(\mathcal{X},\mathcal{A}) := \frac{d}{ds}(\mathcal{A}^* + \mathcal{A}) + [\mathcal{A},\mathcal{A}^*] + [\mathcal{X},\mathcal{X}^*] = 0. \tag{2.24}$$

Given a solution $(\mathcal{A}_0,\mathcal{X}_0)$ of the Complex equation (2.9), we seek a complex gauge transform g such that $g(\mathcal{A}_0,\mathcal{X}_0) = (g\mathcal{A}_0 g^{-1} - \dot{g}g^{-1}, g\mathcal{X}g^{-1})$ solves the Real equation (2.24). As in Donaldson [21, Section 2], we will consider $\hat{F}(g(\mathcal{X},\mathcal{A}))$ as a functional of $GL(n,\mathbb{C})$-valued maps $g(s)$. $\hat{F}(g(\mathcal{X},\mathcal{A}))$ is zero when $g(\mathcal{X},\mathcal{A})$ satisfies the Real equation. The Real equation is invariant under the group of real gauge transformations, i.e. $U(n)$-valued maps $g(s)$, thus the functional $\hat{F}(g(\mathcal{X},\mathcal{A}))$ depends only on the projection $h = g^*g$ of g as a path in the complete Riemannian manifold $\mathcal{H} = G^c/G$ the set of positive hermitian matrices.

Proposition 2.5. *For the solution $(\mathcal{X}',\mathcal{A}')$ to the Complex equation as in* (2.10)*, there is an unique bounded complexified gauge transformation g with $g(0)=1$ such that $g(\mathcal{X}',\mathcal{A}')$ satisfies the Real equation and $g(\mathcal{A}') = g\mathcal{A}'g^{-1} - \dot{g}g^{-1}$ is hermitian.*

Proof. First, we work on the interval $[1/N, N]$ and write $(\mathcal{X},\mathcal{A}) = g(\mathcal{X}',\mathcal{A}')$. As in Proposition 2.8 of [21], the equation $\hat{F}(g(\mathcal{X}',\mathcal{A}')) = 0$ on the interval $[1/N, N]$ is the Euler-Lagrange equation for the functional $\hat{F}$ of g given by

$$\mathcal{L}_N(g) = \frac{1}{2}\int_{1/N}^{N} |g(\mathcal{A}') + g(\mathcal{A}')^*|^2 + 2|g(\mathcal{X}')|^2. \tag{2.25}$$

The Lagrangian integrand, when written in terms of $h = g^*g$, is the standard GL-invariant Riemannian metric $|\cdot|^2_{\mathcal{H}}$ with a smooth nonnegative potential $V(h) = \mathrm{tr}(\mathcal{X}'h^{-1}\mathcal{X}'^*h)$ on the complete Riemannian manifold $\mathcal{H}$. By the usual calculus of variations, there exists an extremizing path $h_N(s)$ on $[1/N, N]$ taking the value 1 at the end points that minimizes the Lagrangian. Taking $g_N = (h_N)^{1/2}$, the transformed $g_N(\mathcal{X}', \mathcal{A}')$ then satisfies the Real equation.

For $h \in \mathcal{H}$ a positive $n \times n$ hermitian matrix with eigenvalues $\{\lambda_i\}$, define $\Phi(s) = \log \max(\lambda_i)_{1\le i\le n} \in \mathbb{R}$. Using the following differential inequality found in Lemma 2.10 of [21] that depends only on $h = g^*g$ for a given fixed $(\mathcal{X}', \mathcal{A}')$:

$$\begin{aligned} \frac{d^2}{ds^2}\Phi(h) &\ge -2(|\hat{F}\left(g(\mathcal{X}), g(\mathcal{A})\right)| + |\hat{F}(\mathcal{X}', \mathcal{A}')|), \\ \frac{d^2}{ds^2}\Phi(h^{-1}) &\ge -2(|\hat{F}\left(g(\mathcal{X}), g(\mathcal{A})\right)| + |\hat{F}(\mathcal{X}', \mathcal{A}')|), \end{aligned} \tag{2.26}$$

one obtains uniqueness of h_N and an uniform C^0 bound on g_N so that $h_N = g_N^* g_N$ has a C^∞ limit h_∞ as $N \to \infty$ by the following convexity argument. Let g'_N be another gauge of the transformed solution also satisfying the real condition with $g'_N(1/N) = 1, g'_N(N) = 1$. Set $h'_N = g'^*_N g'_N$. As in Donaldson, we can assume $g'_N(s) = \mathrm{diag}(e^{t_1(s)}, e^{t_2(s)}, \ldots, e^{t_n(s)})$ with $t_1(s) > t_2(s) > \cdots > t_n(s)$. We then have $t_1 = \Phi(h)$ and $t_n = -\Phi(h^{-1})$. By (2.26), we have

$$\begin{aligned} \frac{d^2 t_1}{ds^2} &\ge 0, \\ \frac{d^2 t_n}{ds^2} &\le 0. \end{aligned} \tag{2.27}$$

Note, $t_1 = 0$ and $t_n = 0$ at both endpoints $s = 1/N, N$ so $t_1(s) \le 0$ on $[1/N, N]$ and $t_n(s) \ge 0$ on $[1/N, N]$. By definition, $t_1 = t_2 = \cdots = t_k$ so that h'_N is the identity. The uniqueness reflects the more general fact that in any simply connected manifold with a complete Riemannian metric of negative curvature and a positive convex potential function, there is a unique stationary path for the corresponding Lagrangian between any two points.

The gauge transformation $g_\infty = (h_\infty)^{1/2}$ then yields a solution to the Real equation over the whole interval $(0, \infty)$. While h_∞ is unique, there may be other gauges g with $h_\infty = g^*g$. Every such gauge g differs by an unitary gauge transformation. The requirement that $g(\mathcal{A}')$ is hermitian means $T_0 = 0$ and this fixes the unique g. □

Remark 2.1. The above proposition is a proof of existence and uniqueness for $h(s)$, proved by Donaldson. We want to note that Theorem 5.1 of [65] has improved on this existence result of Donaldson, by giving an explicit construction for the function $h(s)$ for any spectral curve.

Lemma 2.6. *The transformed solution* $(\mathcal{X}, \mathcal{A}) = g \cdot (\mathcal{X}', \mathcal{A}')$ *of the above proposition 2.5 provides* (T_1, T_2, T_3) *solving the Nahm equations* (2.1) *and satisfying the correct boundary conditions* (2.2) *at* $s = 0$ *and at infinity.*

Proof. Taking the anithermitian and hermitian parts of $(\mathcal{X}, \mathcal{A})$ gives the Nahm solution (T_0, T_1, T_2, T_3). Since $\mathcal{A}$ is hermitian, $T_0 = 0$. The gauge satisfies $g(0) = 1$ so the boundary conditions of T_1, T_2, T_3 at $s = 0$ is satisfied. The gauge g is bounded so away from $s = 0$, we know that this (T_1, T_2, T_3) is bounded. As observed in Lemma 2.2, (T_1, T_2, T_3) satisfy the gradient-flow equations for the function $\psi(T_1, T_2, T_3) = \mathrm{tr} T_1 [T_2, T_3]$ so any bounded solution must have a limit point as $s \to \infty$ and this must be a critical point of ψ, i.e. a commuting triple. Thus, $\lim_{s\to\infty} T_j(s) = i\tau_j'$ for some τ_j' a commuting triple. The triple $(i\tau_1', i\tau_2', i\tau_3')$ must lie in the same adjoint orbit as $(i\tau_1, i\tau_2, i\tau_3)$, so we have $(i\tau_1', i\tau_2', i\tau_3') = g_0(i\tau_1, i\tau_2, i\tau_3)g_0^{-1}$ for some $g_0 \in G = U(n)$ and our (T_1, T_2, T_3) have the correct limits at infinity as well. □

We are at last ready to prove Theorem 2.3 stated at the start of the section. Suppose (T_1, T_2, T_3) and (T_1', T_2', T_3') are two solutions satisfying Nahm's equations with the boundary conditions (2.2). Let $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}', \mathcal{A}')$ be the corresponding solutions to the Complex equation. We may use complex gauge transformations so that the transformed pairs satisfy conditions (2.10). By Lemma 2.4, $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{X}', \mathcal{A}')$ are then equivalent to each other by a complex gauge transformation. Proposition 2.5 shows that there is only one (T_1, T_2, T_3) that can arise from this complex solution, thus $(T_1, T_2, T_3) = (T_1', T_2', T_3')$.

Chapter 3

Spectral Curve

There is an unique spectral curve corresponding to a Nahm solution modulo gauge transformations. The tangent bundle $T\mathbb{P}^1$ of the complex projective space $\mathbb{P}^1$ provides the setting for our spectral curve. The space $T\mathbb{P}^1$ is referred to as the minitwistor space, as Hitchin in [66] showed that the twistor space $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of $\mathbb{R}^4$ reduces to the minitwistor space $T\mathbb{P}^1$ of $\mathbb{R}^3$ under the dimensional reduction of $\mathbb{R}^4$ to $\mathbb{R}^3$.

We parametrize $\mathbb{P}^1$ as the Riemann sphere, denoting $\zeta \in \mathbb{C}$ as our North coordinate and $\tilde{\zeta} \in \mathbb{C}$ as our South coordinate with transition function $\tilde{\zeta} = \zeta^{-1}$. Points on the Riemann sphere correspond to directions in $\mathbb{R}^3$, which may be thought of as unit vectors, under stereographic projection.

The tangent bundle $T\mathbb{P}^1$ over the base space $\mathbb{P}^1$ has coordinates (ζ, η) for $\eta\frac{d}{d\zeta} \in T\mathbb{P}^1$ and $(\tilde{\zeta}, \tilde{\eta})$ for $\tilde{\eta}\frac{d}{d\tilde{\zeta}}$, where η and $\tilde{\eta}$ are the coordinates of the tangent fiber. Since $\tilde{\eta}\frac{d}{d\tilde{\zeta}} = \eta\frac{d}{d\zeta}$, the transition function is given by $\tilde{\zeta} = \zeta^{-1}$, $\tilde{\eta} = \eta/\zeta^2$. Observe that these transition functions identify $T\mathbb{P}^1$ as the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(2)$: $T\mathbb{P}^1 \cong \mathcal{O}_{\mathbb{P}^1}(2)$.

Hitchin in [26] showed the spectral curve for the $SU(2)$ monopole is an algebraic curve $S \subset T\mathbb{P}_1$ of the form

$$\eta^n + a_1(\zeta)\eta^{n-1} + \cdots + a_{n-r}(\zeta)\eta^r + \cdots + a_{n-1}(\zeta)\eta + a_n(\zeta) = 0, \qquad (3.1)$$

with each $a_r(\zeta)$ a polynomial in ζ of maximum degree $2r$ satisfying the conditions

A1. The spectral curve is invariant under the involution $(\eta, \zeta) \mapsto (-\frac{1}{\bar{\zeta}}, -\frac{\bar{\eta}}{\bar{\zeta}^2})$, i.e.

$$a_r(\zeta) = (-\zeta^2)^r \overline{a_r(-\frac{1}{\bar{\zeta}})}.$$

A2. For L^s the holomorphic line bundle on $T\mathbb{P}^1$ defined by transition function $\exp(-s\eta/\zeta)$, the restriction of L^2 to S is trivial on S.

A3. $L(n-1)$ is real, in the sense that the Hermitian inner product defined on sections of $L(n-1)$ as in Proposition 3.5 is real-valued and positive.

A4. $H^0(S, L^s(n-2)) = 0$ for $s \in (0,2)$, i.e. $L^s(n-2)\mid_S$ has no nontrivial holomorphic sections.

The paper then established that the above spectral curve of the $SU(2)$ monopole is the same as the spectral curve of Nahm data $(T_1(s), T_2(s), T_3(s))$ over the interval $(0,2)$ satisfying

B1. $T_i(s) = -\bar{T}_i(2-s)$,

B2. T_i has a simple pole at $s = 0$ and $s = 2$, and for each pole, the residue defines an irreducible representation of $\mathfrak{su}(2)$.

The spectral curve correspondence was then generalized for monopoles of arbitrary compact gauge groups G and various Nahm data in [28]. In general, the spectral curve is not known other than for a few special cases, such as charge 1 and 2 $SU(2)$ monopoles [27] and, of course, the $U(1)$ Dirac monopole. For a list of all currently known spectral curves, the reader may consult Table 1 of [67].

Both spectral curve data and monopole data for the Dirac multimonopole are known. In this chapter, we explain the spectral correspondence and then we use this to carry out the construction of Nahm solutions.

3.1 MiniTwistor Space

A line l in $\mathbb{R}^3$ can be specified by its direction $\vec{\zeta}$ and the normal vector $\vec{\eta} \in \mathbb{R}^3$ to the line, giving the displacement of that line from the origin.

In this way, the minitwistor space $T\mathbb{P}^1$ is identified with the space of oriented lines in $\mathbb{R}^3$,

$$\left\{ (\vec{\zeta}, \vec{\eta}) \in \mathbb{R}^3 \times \mathbb{R}^3 : |\vec{\zeta}| = 1, \quad \vec{\zeta} \cdot \vec{\eta} = 0 \right\}.$$

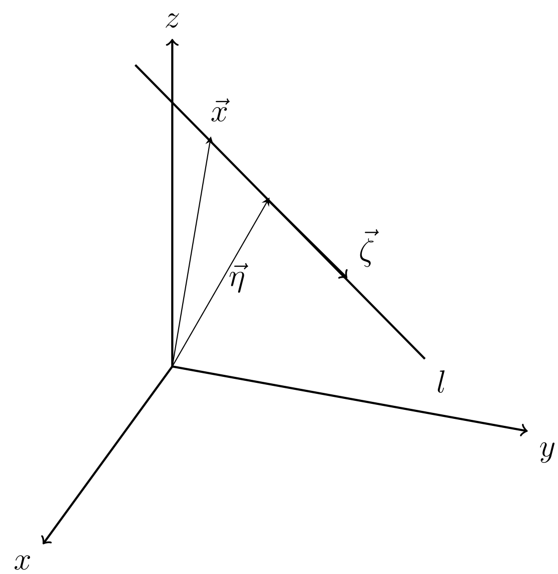
The complex structure J of $T\mathbb{P}^1$ inherited from $\mathbb{P}^1$ as a complex manifold is given by $J(\eta) = i\eta$, it corresponds to the complex structure $J(\vec{\eta}) = \vec{\zeta} \times \vec{\eta}$ on the space of oriented lines.

$T\mathbb{P}^1$ also carries a real structure τ, i.e. an antiholomorphic involution,

$$\zeta \mapsto -1/\bar{\zeta}, \quad \eta \mapsto -\bar{\eta}/\bar{\zeta}^2. \tag{3.2}$$

induced by reversing the orientation of each line.

The line l can also be specified by giving a point $x \in \mathbb{R}^3$ that the line passes through along with the direction $\vec{\zeta}$.



The minitwistor correspondence between points in $\mathbb{R}^3$ and sections of the minitwistor space is accomplished in the following manner. A point $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ corresponds to a section $p(\zeta) : \mathbb{P}^1 \to T\mathbb{P}^1$ where we take all lines that pass through $\vec{x}$ and assign to every direction $\vec{\zeta}$ the point η in the tangent space corresponding to the displacement vector $\vec{\eta}$ of that line.

Let us write $p(\zeta)$ explicitly. The line passing through x with direction $\vec{\zeta}$ has displacement vector $\vec{\eta} = x - (x \cdot \vec{\zeta})\vec{\zeta}$. The value of $\eta \in \mathbb{C}$ is given by $\eta = \frac{1}{2} S_*^{-1}(\vec{\eta})$ where S_*^{-1} is the Jacobian of the inverse stereographic projection $S^{-1} : \mathbb{C} \to \mathbb{P}^1$. Through this, we obtain that the section $(\zeta, p(\zeta))$ in terms of the North pole coordinates has description given by

$$p(\zeta) = (x^1 + ix^2) - 2x^3\zeta - (x^1 - ix^2)\zeta^2. \tag{3.3}$$

The boundary conditions (2.2) of Nahm's equations for $n \times n$ matrices as $s \to \infty$ states

$$\lim_{s\to\infty}(T_1(s), T_2(s), T_3(s)) \in \mathrm{ad}_{U(n)}\left(\begin{pmatrix} ip_1^1 & & & \\ & ip_1^2 & & \\ & & \ddots & \\ & & & ip_1^n \end{pmatrix}, \begin{pmatrix} ip_2^1 & & & \\ & ip_2^2 & & \\ & & \ddots & \\ & & & ip_2^n \end{pmatrix}, \begin{pmatrix} ip_3^1 & & & \\ & ip_3^2 & & \\ & & \ddots & \\ & & & ip_3^n \end{pmatrix}\right)$$

and so this condition marks out n distinct points $x_i = (p_1^i, p_2^i, p_3^i),\ i = 1, \dots, n$ in $\mathbb{R}^3$. The twistor section $p_i(\zeta)$ corresponding to point x_i appears as the factor in the spectral curve (3.9). Under stereographic projection $S : \mathbb{P}^1 \mapsto \mathbb{C}$, one may verify that the two roots a_{ij} and a_{ji} of $p_i(\zeta) - p_j(\zeta) = 0$ are the two directions $\vec{a}_{ij}$ from x_i to x_j and $\vec{a}_{ji}$ from x_j and x_i.

3.2 Spectral Curve of Nahm data

Nahm's equations form an integrable system, as previously mentioned in the introduction, with a Lax Pair (L, M). The Lax pair arises from the antiself-dual commutator equation on $\mathbb{R}^4$. Indeed, if we modify (1.7) to

$$[D_{\bar{z}} - D_w\zeta - (D_{\bar{w}} + D_z\zeta)\zeta, D_{\bar{w}} + D_z\zeta] = 0 \tag{3.4}$$

and take the ASD reduction to $\mathbb{R}$ as in (1.8), i.e. relabeling $A_z = \frac{1}{2}(T_1 + iT_2)$, $A_w = \frac{1}{2}(T_3 - iT_0)$ and requiring that these functions depend only on $s := x_4$, we obtain the Lax pair (L, M)

$$\frac{d}{ds}L = [L, M] \tag{3.5}$$

equivalent to Nahm's equations. In North and South coordinates, (L, M) are

$$\begin{aligned} L^N &= -i(T_1 + iT_2) + 2iT_3\zeta + i(T_1 - iT_2)\zeta^2,\ M^N = T_0 - i(T_3 + (T_1 - iT_2)\zeta), \\ L^S &= i(T_1 - iT_2) + 2iT_3\frac{1}{\zeta} - i(T_1 + iT_2)\frac{1}{\zeta^2},\ \ M^S = T_0 + i(T_3 - (T_1 + iT_2)\frac{1}{\zeta}). \end{aligned} \tag{3.6}$$

In particular, the spectrum of $L(s, \zeta)$ does not evolve in s and the spectral curve S for Nahm solution is given as the spectrum,

$$\det(\eta - L(s, \zeta)) = 0. \tag{3.7}$$

The boundary conditions of our Nahm solution state that

$$\lim_{s\to\infty} L^N(s,\zeta) \to ad_{U(n)} \begin{pmatrix} p_1(\zeta) & 0 & \dots & 0 \\ 0 & p_2(\zeta) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & p_n(\zeta) \end{pmatrix}, \tag{3.8}$$

where $p_j(\zeta) = x_j^1 + ix_j^2 - 2x_j^3\zeta - (x_j^1 - ix_j^2)\zeta^2$ is the twistor section corresponding to the point $(x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3$ from the boundary condition as $s \to \infty$.

Therefore, we find that the spectral curve of Nahm solution is:

Lemma 3.1. *The spectral curve $S \subset T\mathbb{P}^1$ for Nahm's equations with our boundary conditions* (2.2) *is*

$$S = \left\{ (\zeta,\eta) \in T\mathbb{P}^1 : \prod_{j=1}^{n} (\eta - p_j(\zeta)) = 0 \right\}, \tag{3.9}$$

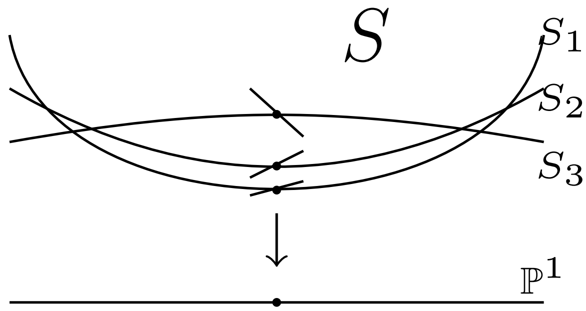
for $p_j(\zeta) = x_j^1 + ix_j^2 - 2x_j^3\zeta - (x_j^1 - ix_j^2)\zeta^2$ the twistor section of the point $x_j = (x_j^1, x_j^2, x_j^3)$ from the boundary conditions as $s \to \infty$.

This S is a degenerate n-branched covering of $\mathbb{P}^1$. For two distinct points x_i and x_j with distance r_{ij}, each branch $(\zeta, p_i(\zeta))$ intersects another $(\zeta, p_j(\zeta))$ at the double points $\zeta = a_{ij}$ and $\zeta = a_{ji}$ where

$$a_{ij} = \frac{x_i^3 - x_j^3 + r_{ij}}{x_j^1 - x_i^1 + i(x_i^2 - x_j^2)}. \tag{3.10}$$

Note, if x_i and x_j are vertically separated (i.e. $x_i^1 = x_j^1$ and $x_i^2 = x_j^2$) and, say $x_i^3 > x_j^3$, then $a_{ij} = \infty \in \mathbb{P}^1$ and $a_{ji} = 0 \in \mathbb{P}^1$.

In the algebro-geometric integration method, the η-eigenspace of $L(s,\zeta)$ gives us from the flow of $L(s,\zeta)$ a linear flow F^s in the Jacobian $\mathrm{Jac}(S)$, from which we carry out the integration of the Lax pair. This was proved by Griffiths in [1] for regular curves. Here we shall prove it for our spectral curves, which are singular. A sketch of our spectral curve S with eigenline bundle is



To obtain the linear flow F^s in $\mathrm{Jac}(S)$, consider solutions $U_j = (U_j^N, U_j^S)$ to the Lax linear problem associated to (L, M)

$$\begin{cases} (\frac{d}{ds} + M^N)U_j^N = 0, \\ (L^N - p_j(\zeta))U_j^N = 0, \end{cases}$$
$$\begin{cases} (\frac{d}{ds} + M^S)U_j^S = 0, \\ (L^S - \frac{p_j(\zeta)}{\zeta^2})U_j^S = 0. \end{cases} \tag{3.11}$$

That is, each U_j is an eigenvector of L for the eigenvalue $p_j(\zeta)$ and evolves in s as $\frac{dU}{ds} = -MU$. Set $U = (U_1, \ldots, U_n)$, we now look for the transition function $F(s, \zeta)$ of $U^N = U^S F(s, \zeta)$. To begin, we have

$$L^S = \frac{1}{\zeta^2} L^N \text{ and } M^S = M^N + \frac{1}{\zeta} L^N. \tag{3.12}$$

We have $L^N = \zeta^2 L^S$ so L^N and L^S have the same eigenspace and $U^N = U^S F(s, \zeta)$ for a diagonal matrix $F(s, \zeta)$. Now,

$$\begin{aligned} 0 &= (\frac{d}{ds} + M^N)U^N \\ &= (\frac{d}{ds} + M^S - \frac{1}{\zeta}L^N)(U^S F) \\ &= (\frac{d}{ds}U^S)F^{-1} + U^S \frac{dF^{-1}}{ds} + M^S U^S F^{-1} - \frac{1}{\zeta} L^N U^S F \\ &= [(\frac{d}{ds} + M^S)U^S]F + U^S \frac{dF}{ds} - U^S \frac{\eta^N}{\zeta} F \\ &= U^S (\frac{d}{ds} - \frac{\eta}{\zeta})F. \end{aligned} \tag{3.13}$$

Solving $\frac{d}{ds}F = \frac{\eta}{\zeta}F$ is easy, it gives $F(\zeta, s) = g(\zeta)e^{s\frac{\eta}{\zeta}}$ for some s-independent diagonal matrix $g(\zeta)$. However $g(\zeta)$ is still undetermined. We can set $g(\zeta) = \zeta^{n-1}$, the justification for this is provided later in Corollary 3.4. This gives us

$$F(s,\zeta) = \zeta^{n-1}\mathrm{diag}(e^{s\frac{p_j}{\zeta}}). \tag{3.14}$$

The spectral curve S is degenerate with double points at a_{ij} of (3.10). To see what happens at the eigenspace of $L(s,\zeta)$ at $\zeta = a_{ij}$, we consider the boundary conditions of Nahm's equations (2.2). Note that

$$e^{\zeta\sigma_-}\sigma_3 e^{-\zeta\sigma_-} = \sigma_3 + 2\zeta\sigma_-, \quad e^{\zeta\sigma_-}\sigma_+ e^{-\zeta\sigma_-} = \sigma - \zeta\sigma_3 - \zeta^2\sigma_-. \tag{3.15}$$

Using (3.15), the boundary conditions implies that near $s = 0$,

$$L^N = e^{\zeta\sigma_-}\big(\frac{\sigma_+}{s} + O(s^0)\big)e^{-\zeta\sigma_-}, \qquad \frac{d}{ds} + M^N = e^{\zeta\sigma_-}\big(\frac{d}{ds} + \frac{\sigma_3}{2s} + O(s^0)\big)e^{-\zeta\sigma_-}. \tag{3.16}$$

As in Section 2.2.1 and analogous to (2.20), we may use a complex gauge transformation that is identity at $s = 0$ to gauge M^N and L^N to

$$L^N = \begin{pmatrix} a(\zeta) & s^{-1} & 0 & \dots & 0 \\ b(\zeta)s & a(\zeta) & s^{-1} & & 0 \\ c(\zeta)s^2 & b(\zeta)s & a & & \vdots \\ & & & \ddots & s^{-1} \\ d(\zeta)s^{n-1} & e(\zeta)s^{n-2} & \dots & & a(\zeta) \end{pmatrix}, \quad M^N = \frac{\sigma_3}{2s}, \tag{3.17}$$

where the coefficients of L^N above are determined by the spectral curve. The rational canonical form $\tilde{L}^N$ of L^N is then a single companion matrix,

$$\tilde{L}^N = \begin{pmatrix} 0 & \frac{1}{s} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{s} & \dots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{s} \\ -a_0(\zeta)s^{n-1} & -a_1(\zeta)s^{n-2} & -a_2(\zeta)s^{n-3} & \dots & -a_{n-1}(\zeta) \end{pmatrix}, \tilde{M}^N = \frac{\sigma_3}{2s}. \tag{3.18}$$

The solution to the Lax linear problem associated to the Lax pair $(\tilde{L}, \tilde{M})$ is

$$\tilde{U}^N = s^{-\sigma_3/2}\begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1(\zeta) & p_2(\zeta) & \dots & p_n(\zeta) \\ \vdots & & & \vdots \\ p_1(\zeta)^{n-1} & p_2(\zeta)^{n-1} & \dots & p_n(\zeta)^{n-1} \end{pmatrix} Diag. \tag{3.19}$$

At $\zeta = a_{ij}$, the eigenspaces of L remain one-dimensional and we may set the $Diag$ matrix to the identity matrix so that

$$U_i(a_{ij}) = U_j(a_{ij}). \tag{3.20}$$

We conclude that the rows $(u_{l1}, u_{l2}, \ldots, u_{ln})$ for $1 \leq l \leq n$ of U have the transition functions from North to South patch and at $\zeta = a_{ij}$ given by

$$\begin{aligned} (u_{l1}, u_{l2}, \ldots, u_{ln})_0 &= (u_{l1}, u_{l2}, \ldots, u_{ln})_\infty \zeta^{n-1} \mathrm{diag}(e^{s\frac{p_k}{\zeta}}), \\ u_{li}(s, a_{ij}) &= u_{lj}(s, a_{ij}). \end{aligned} \tag{3.21}$$

The transition functions above determine the eigenline bundle of the Nahm spectral curve.

3.3 Eigenline Bundle over Spectral Curve

The spectral curve $S \subset T\mathbb{P}^1$ is an n-branched covering of $\mathbb{P}^1$ via $\pi : S \to \mathbb{P}^1$ given as $\pi(\zeta, \eta) = \zeta$. Each branch $S_i = \{(\zeta, p_i(\zeta)) \in T\mathbb{P}^1\}$, $i = 1, \ldots, n$ is isomorphic to $\mathbb{P}^1$. The spectral curve S has genus $g = (n-1)^2$ by the adjunction formula [26, p.159].

Consider a line bundle $F_S \to S$ defined in the atlas using our branches S_i as a cover by specifying transition functions $\lambda_{ij} \in \mathbb{C} \setminus \{0\}$ and $\lambda_{0\infty}(\zeta, \eta) : U_{North} \cap U_{South} \to \mathbb{C} \setminus \{0\}$ such that for a section $u \in \Gamma(S, F_S)$, $u_i(a_{ij}) = \lambda_{ij} u_j(a_{ij})$ on the points of intersection $a_{ij} \in S_i \cap S_j$ and $u_0(\zeta) = \lambda_{0\infty} u_\infty(\zeta)$ on the overlap of the North and South patches.

Denote by $\mathcal{O}_S(k) \to S$ the pullback of $\mathcal{O}_{\mathbb{P}^1}(k) \to \mathbb{P}^1$ to S and by $F_S(k)$ the line bundle $F_S \otimes \mathcal{O}_S(k)$.

The spectral curve S is invariant under the real structure τ of Equation (3.2) on $T\mathbb{P}^1$, so $\tau : S \to S$ induces an antiholomorphic involution σ on the set of line bundles $\mathrm{Pic}(S)$ given by $\sigma(F_S) = \overline{\tau_S^*(F)}$. There is a corresponding map on sections:

$$\sigma(u(\zeta, \eta)) = \overline{u(-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)}. \tag{3.22}$$

For line bundles with transition functions $\lambda_{ij} = 1$ so that $u_i(a_{ij}) = u_j(a_{ij})$, we may consider F_S as the pullback under the inclusion $i : S \hookrightarrow T\mathbb{P}^1$ of the line bundle $F_{T\mathbb{P}^1}$ with same transition function $\lambda_{0\infty}(\zeta, \eta)$. Sections of F_S in

North patch may be written as $[u_0] \in \mathbb{C}[\zeta,\eta]/(\prod_{i=1}^{n}(\eta - p_i(\zeta)))$ with unique representative

$$u_0(\zeta,\eta) = c_0(\zeta)\eta^{n-1} + c_1(\zeta)\eta^{n-2} + \cdots + c_{n-1}(\zeta). \tag{3.23}$$

Alternatively, a section u may be written in North patch as

$$u_0(\zeta) = (u_1(\zeta), u_2(\zeta), \ldots, u_n(\zeta)), \tag{3.24}$$

where the i^{th} entry is the value of u on the branch S_i. The two versions of writing u_0 are related by, in North patch, evaluating η on S_i and obtaining $u_i(\zeta) = c_0(\zeta)p_i(\zeta)^{n-1} + c_1(\zeta)p_i(\zeta)^{n-2} + \cdots + c_{n-1}(\zeta)$. Conversely, given $u_0(\zeta) = (u_1(\zeta), \ldots, u_n(\zeta))$, its expression in (ζ,η) is given by the Lagrangian interpolation $u_0(\zeta,\eta) = \sum_{i=1}^{n} \frac{u_i(\zeta)\prod_{j\neq i}(\eta - p_j(\zeta))}{\prod_{j\neq i}(p_i(\zeta)-p_j(\zeta))}$ and $u_j(\zeta) = u_0(\zeta, \eta_j(\zeta))$.

As in (3.21), we define the eigenline bundle $L^s_S(n-1)$ of the Nahm spectral curve to be the following line bundle.

Definition 3.1. Define the *eigenline bundle* $L^s_S(n-1)$ for $s \in (0,\infty)$ over S by the transition functions $\lambda_{ij} = 1$, $i,j = 1,2,\ldots,n$ and

$$u_0(\zeta) = \zeta^{n-1}u_\infty(\zeta)\mathrm{diag}(e^{sp_1(\zeta)/\zeta}, \ldots, e^{sp_n(\zeta)/\zeta}), \tag{3.25}$$

for u a section of $L^s_S(n-1)$. Denote by $H^0(S, L^s_S(n-1))$ the space of sections of $L^s_S(n-1)$.

The following lemma gives a wonderful way to think about a section of $L^s_S(n-1)$ as a row of polynomials satisfying matching conditions:

Lemma 3.2. *There is a bijection between sections $u \in H^0(S, L^s_S(n-1))$ and degree $n-1$ (or less) rows of polynomials $(Q_1(\zeta), \ldots, Q_n(\zeta))$ satisfying the matching condition $Q_i(a_{ij}) = e^{-sr_{ij}}Q_j(a_{ij})$, where r_{ij}, as usual, denotes the distance between the two points x_i, x_j.*

Proof. Observe that $\frac{p_j(\zeta)}{\zeta} = \frac{p^1_j + ip^2_j}{\zeta} - 2p^3_j - (p^1_j - ip^2_j)\zeta$ in the transition functions of (3.25) is invariant under the involution τ. Splitting it into the ζ and $1/\zeta$ part, $\frac{p_j(\zeta)}{\zeta} = \frac{p^1_j + ip^2_j}{\zeta} - p^3_j - p^3_j - (p^1_j - ip^2_j)\zeta = -h^+_j(\zeta) - h^-_j(\frac{1}{\zeta})$ with

$$h^+_j(\zeta) = p^3_j + (p^1_j - ip^2_j)\zeta, \quad h^-_j(\frac{1}{\zeta}) = -\frac{p^1_j + ip^2_j}{\zeta} + p^3_j, \tag{3.26}$$

define

$$q_0(\zeta) = u_0(\zeta)\mathrm{diag}(e^{sh_j^+(\zeta)}), \quad q_\infty(\zeta) = u_\infty(\zeta)\mathrm{diag}(e^{-sh_j^-(1/\zeta)}) \tag{3.27}$$

then on the overlap of North and South patches we have

$$q_0(\zeta) = \zeta^n q_\infty(\zeta). \tag{3.28}$$

This means $q_0(\zeta) = (Q_1(\zeta), \dots, Q_n(\zeta))$ is a row of polynomials of degree at most $n-1$. The matching conditions on q follow from $u_i(a_{ij}) = u_j(a_{ij})$. Conversely, given such q it is easy to obtain u via (3.27). □

The following proposition will be crucial,

Proposition 3.3. *For* $s \in (0, \infty)$, *the dimension* $h^0(L_S^s(n-2))$ *of* $H^0(L_S^s(n-1))$ *is* 0.

Proof. Sections of $L_S^s(n-2)$ correspond to degree $n-2$ polynomial rows satisfying matching conditions $P_i(a_{ij}) = e^{-sr_{ij}}P_j(a_{ij})$ as in Lemma 3.2. Taking the limit of $s \to \infty$ of the matching condition $Q_i(a_{ij}) = e^{-sr_{ij}}Q_j(a_{ij})$ shows $Q_i(a_{ij}) = 0$. A polynomial that satisfies $Q_i(a_{ij}) = 0$ is either a multiple of the Atiyah polynomial $A_i(\zeta) = \prod_{j\neq i}(\zeta - a_{ij})$ or it is identically zero. Since $A_i(\zeta)$ has degree exceeding $n-2$, $Q_i \equiv 0$ as $s \to \infty$.

The matching conditions form a linear system on the unknown polynomial coefficients. The s dependence enters the linear system via functions $e^{-sr_{ij}}$. We may write a perturbation expansion of the polynomials as

$$P_i(\zeta) = e^{-s\Delta_i}g_i(\zeta) + e^{-s\Delta_i'}g_i'(\zeta) + e^{-s\Delta_i''}g_i''(\zeta) + \dots \tag{3.29}$$

with $0 < \Delta_i < \Delta_i' < \Delta_i'' < \dots$ and s-independent polynomials $g_i(\zeta)$,$g_i'(\zeta)$, $g_i''(\zeta)$, $\dots$ of degree $n-2$.

The expansion of the matching conditions $P_i(a_{ij}) = e^{-sr_{ij}}P_j(a_{ij})$ at zero-order is the system

$$g_i(a_{ij}) = e^{-s(r_{ij}+\Delta_j-\Delta_i)}g_j(a_{ij}), \qquad \text{for } i \neq j. \tag{3.30}$$

If $g_i(a_{ij}) \neq 0$, then the s-independence of the perturbation polynomials implies $r_{ij} + \Delta_j - \Delta_i = 0$. However, we make the observation that for a fixed pair of i, j we cannot have both

$$\Delta_j + r_{ij} - \Delta_i = 0 \text{ and } \Delta_i + r_{ij} - \Delta_j = 0 \tag{3.31}$$

as that implies $r_{ij} = 0$. Therefore, if $g_i(a_{ij}) \neq 0$, we must have $g_i(a_{ji}) = 0$ to avoid this case. This gives $n-1$ many points that $g_i(\zeta)$ must be zero at. However $g_i(\zeta)$ is degree at most $n-2$ so the solution to zero-order is $g_i(\zeta) \equiv 0$.

By induction, subsequent orders are identical to the zero-order case. We conclude from the perturbation expansion of our section that

$$(P_1(\zeta), \ldots, P_n(\zeta)) \equiv (0, \ldots, 0).$$

□

Corollary 3.4 ([26])**.** *For $s \in (0, \infty)$, $h^0(L_S^s(n-1)) = n$.*

Proof. This is an elementary application of the Riemann-Roch formula [68, p.472]. The argument that follows may be found in [26, pp.165-166]. We state it for completeness. The degree of $L_S^s(n-2)$ is $(n-2)n$ and $L_S^s(n-1)$ is $(n-1)n$. Recall the genus of S is $g = (n-1)^2$.

By Serre duality we get $\dim H^1(S, L_S^s(n-2)) = \dim H^0(S, K \otimes (L_S^s)^\vee(n-2))$. From the Riemann-Roch,

$$\begin{aligned} \dim H^0(L_S^s(n-2)) - \dim H^1(L_S^s(n-2)) &= \deg L_S^s(n-2) - g + 1 \\ &= 0, \end{aligned} \tag{3.32}$$

hence $h^0(L_S^s(n-2)) = 0$ implies $H^1(L_S^s(n-2)) = 0$. Fix $\zeta_0 \in \mathbb{P}^1$ and let D_{ζ_0} be the zero divisor of $(\zeta - \zeta_0)$ on S, i.e. $D_{\zeta_0} \subset S$ is the fiber above $\zeta \in \mathbb{P}^1$. Since S is an n-fold covering of $\mathbb{P}^1$, D_{ζ_0} is a set of n points in S (counting multiplicity). We have exact sequence of sheaves

$$0 \to \mathcal{O}_S(n-2) \xrightarrow{\zeta - \zeta_0} \mathcal{O}_S(n-1) \xrightarrow{ev_{\zeta_0}} \mathcal{O}_{D_{\zeta_0}}(n-1) \to 0, \tag{3.33}$$

with the first map given by multiplication by $(\zeta - \zeta_0)$ and the second map given by evaluation of the section at ζ_0. Tensoring with L_S^s results in the long exact sequence

$$0 \to \cancelto{0}{H^0(L_S^s(n-2))} \to H^0(L_S^s(n-1)) \to H^0(L_{D_{\zeta_0}}^s(n-1)) \to \cancelto{0}{H^1(L_S^s(n-2))} \to \ldots \tag{3.34}$$

so $H^0(S, L_S^s(n-1)) \cong H^0(D_{\zeta_0}, L_{D_{\zeta_0}}^s(n-1))$. Since D_{ζ_0} is simply n distinct points of the fiber above ζ_0, $H^0(D_\zeta, L_{D_{\zeta_0}}^s(n-1))$ is isomorphic to $\mathbb{C}^n$. □

The space of sections $H^0(L_S^s(n-1))$ is an inner product space.

Proposition 3.5. *For the spectral curve S, given sections u, v of $L_S^s(n-1)$, $u\sigma(v)$ is a section of $\mathcal{O}_S(2n-2)$ and can be written uniquely as*

$$u\sigma(v)_0 = c_0\eta^{n-1} + c_1(\zeta)\eta^{n-2} + \cdots + c_{n-1}(\zeta), \tag{3.35}$$

for $c_i(\zeta)$ degree $2i$. Then

$$\langle u, v\rangle := c_0 \tag{3.36}$$

defines an Hermitian inner product [26], [50] on $H^0(S, L_S^s(n-1))$.

Hitchin in [26, pp.179-181] first discovered this inner product for the eigenline bundle of the $SU(2)$ and also contains the proof of definiteness. Bielawski then demonstrated in [50, Prop 4.2] that the same is true for a class of degenerate spectral curves, including the spectral curve of our Nahm data and also has the proof of positivity.

In our case, the inner product $\langle u, v\rangle$ may be written in terms of the corresponding row of polynomials (3.2). For $p = (P_1(\zeta), \ldots, P_n(\zeta))$ and $r = (R_1(\zeta), \ldots, R_n(\zeta))$,

$$\langle u, v\rangle = \langle p, r\rangle = (-\zeta)^{n-1}\sum_{i=1}^{n} \frac{P_i(\zeta)\overline{R_i(-1/\bar{\zeta})}}{\prod\limits_{i\neq j}(p_i(\zeta) - p_j(\zeta))}. \tag{3.37}$$

3.4 Spectral Method for Nahm's Equations

3.4.1 Solving Nahm's Equations

For an orthonormal basis $\{U_1(s,\zeta), \ldots, U_n(s,\zeta)\}$ of sections[1] of the eigenline bundle $L_S^s(n-1)$, where $s \in (0, \infty)$, with respect to the inner product (3.5), let

$$U(s,\zeta) = \begin{pmatrix} U_1(s,\zeta) \\ U_2(s,\zeta) \\ \vdots \\ U_n(s,\zeta) \end{pmatrix} \tag{3.38}$$

[1]Recall, $U_j(s,\zeta)$ is a section of $L_S^s(n-1)$, which in our case of the curve $\prod\limits_{j=1}^{n}(\eta - p_j(\zeta)) = 0$ is given by the row of its values on each of the n sheets.

be a $n \times n$ matrix where each row is a section. This matrix exists for all $s \in (0, \infty)$ as Corollary 3.4 states $h^0(L_S^s(n-1)) = n$ over this interval.

Define a Lax pair (L, M) of matrix-valued functions such that in North patch

$$\begin{aligned} L^N(s,\zeta) &:= U^N(s,\zeta) \begin{pmatrix} p_1(\zeta) & & & \\ & p_2(\zeta) & & \\ & & \ddots & \\ & & & p_n(\zeta) \end{pmatrix} U^N(s,\zeta)^{-1}, \\ M^N(s,\zeta) &:= -\frac{dU^N(s,\zeta)}{ds} U^N(s,\zeta)^{-1}, \end{aligned} \tag{3.39}$$

and in South patch

$$\begin{aligned} L^S(s,\frac{1}{\zeta}) &:= U^S(s,\frac{1}{\zeta}) \begin{pmatrix} \frac{p_1(\zeta)}{\zeta^2} & & & \\ & \frac{p_2(\zeta)}{\zeta^2} & & \\ & & \ddots & \\ & & & \frac{p_n(\zeta)}{\zeta^2} \end{pmatrix} U^S(s,\frac{1}{\zeta})^{-1}, \\ M^S(s,\frac{1}{\zeta}) &:= -\frac{dU^S(s,\frac{1}{\zeta})}{ds} U^S(s,\frac{1}{\zeta})^{-1}, \end{aligned}$$

with each $p_j(\zeta) = x_j^1 + ix_j^2 - 2x_j^3\zeta - (x_j^1 - ix_j^2)\zeta^2$ determined by the point $x_j = (x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3$ from the desired boundary conditions as $s \to \infty$.

Here L is a linear operator acting on the n-dimensional space of sections $H^0(L_S^s(n-1))$ for each value of s. And $\frac{d}{ds} + M$ is a connection for a frame U of $H^0(L_S^s(n-1))$. After a choice of frame U, L and M written in coordinates are $n \times n$ matrices. We prove that (L, M) is the Lax pair of the Nahm's equations when choosing U orthonormal.

Under the real structure (3.2), L and M satisfy the reality relationship

$$L^N(-1/\bar{\zeta})^\dagger = -L^S(\zeta), \qquad M^N(-1/\bar{\zeta})^\dagger = -M^S(\zeta). \tag{3.40}$$

Lemma 3.6. *The transition functions for L and M are*

$$L^S = \frac{1}{\zeta^2} L^N, \quad M^S = M^N + \frac{1}{\zeta} L^N. \tag{3.41}$$

Proof. By Definition 3.1, U has the transition function

$$U^N(s,\zeta) = \zeta^{n-1} U^S(s, 1/\zeta) e^{s\eta/\zeta}$$

as $\eta = p_j(\zeta)$ on the j-th branch of the spectral curve S.

$$\begin{aligned}L^N(s,\zeta) &= U^N(s,\zeta)\mathrm{diag}(p_1(\zeta),\dots,p_n(\zeta))U^N(s,\zeta)^{-1}\\ &= \zeta^2 U^S(s,1/\zeta)\zeta^{n-1}e^{s\eta/\zeta}\mathrm{diag}(\frac{p_1}{\zeta^2},\dots,\frac{p_n}{\zeta^2})e^{-s\eta/\zeta}\frac{1}{\zeta^{n-1}}U^S(s,\zeta)^{-1}\\ &= \zeta^2 U^S(s,\zeta)\mathrm{diag}(\frac{p_1}{\zeta^2},\dots,\frac{p_n}{\zeta^2})U^S(s,\zeta)^{-1}\\ &= \zeta^2 L^S(s,\zeta).\end{aligned} \tag{3.42}$$

For M, differentiate the transition relation $\frac{d}{ds}(U^N) = \frac{d}{ds}(\zeta^{n-1}U^S e^{s\eta/\zeta})$ to get

$$\dot{U}^N = \zeta^{n-1}\dot{U}^S e^{s\eta/\zeta} + \zeta^{n-1}U^S\mathrm{diag}(\frac{p_1}{\zeta},\dots,\frac{p_n}{\zeta})e^{s\eta/\zeta}.$$

Substituting into $M^N = -\dot{U}^N(U^N)^{-1}$, we obtain

$$\begin{aligned}M^N &= -(\zeta^{n-1}\dot{U}^S e^{s\eta/\zeta} + \zeta^{n-1}U^S\mathrm{diag}(\frac{p_1}{\zeta},\dots,\frac{p_n}{\zeta})e^{s\eta/\zeta})e^{-s\eta/\zeta}(U^S)^{-1}\frac{1}{\zeta^{n-1}}\\ &= -\dot{U^S}(U^S)^{-1} - U^S\mathrm{diag}(\frac{p_1}{\zeta},\dots,\frac{p_n}{\zeta})(U^S)^{-1}\\ &= M^S - \frac{1}{\zeta}L^N.\end{aligned}$$

□

Our next goal is identifying an orthonormal basis of $H^0(S, L_S^s(n-1))$ in terms of the degree $n-1$ polynomials rows $(Q_1(\zeta),\dots,Q_n(\zeta))$ satisfying the matching conditions $Q_i(a_{ij}) = e^{-sr_{ij}}Q_j(a_{ij})$.

Denote the $n \times n$ polynomial matrix $Q(s,\zeta)$

$$Q(s,\zeta) = U^N(s,\zeta)\mathrm{diag}(e^{sh_j^+(\zeta)}). \tag{3.43}$$

The columns of U fail to be linearly independent at the branch points $\zeta = a_{ij}$ because of the matching conditions (3.20) that U satisfies, so the inverse of U is meromorphic in ζ, rather than holomorphic. Explicitly, U is orthonormal with respect to the inner product on $H^0(L_S^s(n-1))$, so $Q(s,\zeta)^{-1}$ satisfies

$$Q(s,\zeta)^{-1} = (-\zeta)^{n-1}\begin{pmatrix}\frac{1}{\prod_{j=2}^{n}(p_1(\zeta)-p_j(\zeta))} & & \\ & \ddots & \\ & & \frac{1}{\prod_{j=1}^{n-1}(p_n(\zeta)-p_j(\zeta))}\end{pmatrix}Q(s,-1/\bar{\zeta})^\dagger, \tag{3.44}$$

where † denotes the complex conjugate transpose. Although U is meromorphic in ζ, we will prove that L and M are not.

Define the Vandermonde matrix

$$\mathrm{Vand}(p_1,\ldots,p_n) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ p_1(\zeta) & p_2(\zeta) & \ldots & p_n(\zeta) \\ & \vdots & & \\ p_1(\zeta)^{n-1} & p_2(\zeta)^{n-1} & \ldots & p_n(\zeta)^{n-1} \end{pmatrix}. \tag{3.45}$$

We shall need the formula for the inverse of the Vandermonde matrix. For variables $\{x_1,\ldots,x_k\}$, recall that the m-th elementary symmetric function is

$$e_m\left(\{x_1,\ldots,x_k\}\right) = \sum_{1\le j_1<j_2<\ldots j_m\le k} x_{j_1}x_{j_2}\cdots x_{j_m},$$

for $m=0,1,\ldots,k$. Then

$$\left[\mathrm{Vand}(p_1,\ldots,p_n)^{-1}\right]_{ji} = \frac{(-1)^{n-i}e_{n-i}(\{p_1(\zeta),\ldots,p_n(\zeta)\}\setminus\{p_j(\zeta)\})}{\prod\limits_{m\neq j}^{n}\left(p_j(\zeta)-p_m(\zeta)\right)}. \tag{3.46}$$

Lemma 3.7. *$L^N(s,\zeta)$ defined in Equation* (3.39) *is a quadratic polynomial in ζ.*

Proof. The uv-th entry of L^N is given by

$$L^N_{uv} = \sum_{s=1}^{n} p_s(\zeta)\frac{(-\zeta)^{n-1}Q_{us}(\zeta)\overline{Q_{vs}(-1/\bar\zeta)}}{\prod\limits_{l\neq s}^{n}(p_s(\zeta)-p_l(\zeta))}. \tag{3.47}$$

The possible poles of L^N_{uv} are at $\zeta = a_{ij}$ arising from the two terms

$$p_i(\zeta)\frac{(-\zeta)^{n-1}Q_{ui}(\zeta)\overline{Q_{vi}(-1/\bar\zeta)}}{\prod\limits_{l\neq i}(p_i-p_l)} + p_j(\zeta)\frac{(-\zeta)^{n-1}Q_{uj}(\zeta)\overline{Q_{vj}(-1/\bar\zeta)}}{\prod\limits_{l\neq j}(p_j-p_l)}. \tag{3.48}$$

We have $p_i(a_{ij}) = p_j(a_{ij})$, and as $-1/\overline{a_{ij}} = a_{ji}$, the matching conditions make the numerators equal when $\zeta = a_{ij}$, so we may factor (3.48) at $\zeta = a_{ij}$ as

$$p_i(a_{ij})(a_{ij})^{n-1}Q_{ui}(a_{ij})\overline{Q_{vi}(a_{ji})}\left(\frac{1}{\prod\limits_{l\neq i}(p_i-p_l)} + \frac{1}{\prod\limits_{l\neq j}(p_j-p_l)}\right)\Bigg|_{\zeta=a_{ij}}. \tag{3.49}$$

The latter factor appears as the only possible poles of $\zeta = a_{ij}$ in the sum representing the first row of $\mathrm{Vand}(p_1, \ldots, p_n)$ multiplied with the last column of $\mathrm{Vand}(p_1, \ldots, p_n)^{-1}$:

$$\sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}(p_i(\zeta) - p_j(\zeta))} = 0. \tag{3.50}$$

Since this sum is equal to zero, there cannot be a pole at $\zeta = a_{ij}$ and L^N_{uv} is holomorphic in ζ. Similarly, L^S_{uv} is holomorphic in $1/\zeta$. The transition relation (3.41) of L implies L^N_{uv} is a quadratic polynomial in ζ. □

Lemma 3.8. *$M^N(s, \zeta)$ is a linear function in ζ.*

Proof. The uv-th entry of M^N is given by

$$M^N_{uv} = \sum_{s=1}^{n} \frac{\Big(- \dot{Q}_{us}(\zeta) + Q_{us}(\zeta) h^+_s(\zeta) \Big) (-\zeta)^{n-1} \overline{Q_{vs}(-1/\overline{\zeta})}}{\prod_{l\neq s}^{n} (p_s(\zeta) - p_l(\zeta))}. \tag{3.51}$$

The possible poles of M^N_{uv} are at $\zeta = a_{ij}$, arising from the two terms

$$\frac{\Big(-\dot{Q}_{ui}(\zeta) + Q_{ui}(\zeta) h^+_i(\zeta)\Big) (-\zeta)^{n-1} \overline{Q_{vi}(-1/\overline{\zeta})}}{\prod_{l\neq i}(p_i - p_l)} + \frac{\big(- \dot{Q}_{uj}(\zeta) + Q_{uj}(\zeta) h^+_j(\zeta)\big)(-\zeta)^{n-1} \overline{Q_{vj}(-1/\overline{\zeta})}}{\prod_{l\neq j}(p_j - p_l)}. \tag{3.52}$$

We have $r_{ij} = h^+_j(a_{ij}) - h^+_i(a_{ij})$ as is easy to check, so the matching conditions make the numerators equal when $\zeta = a_{ij}$, and we may factor (3.52) at $\zeta = a_{ij}$ as

$$\Big(- \dot{Q}_{ui}(a_{ij}) + Q_{ui}(a_{ij}) h^+_i(a_{ij}) \Big) (-a_{ij})^{n-1} \times \overline{Q_{vi}(a_{ji})} \left(\frac{1}{\prod_{l\neq i}(p_i - p_l)} + \frac{1}{\prod_{l\neq j}(p_j - p_l)} \right) \Bigg|_{\zeta = a_{ij}}. \tag{3.53}$$

Again, the latter factor appears as the only possible poles of $\zeta = a_{ij}$ in the sum representing the first row of $\mathrm{Vand}(p_1, \ldots, p_n)$ multiplied with the last column of $\mathrm{Vand}(p_1, \ldots, p_n)^{-1}$. This sum equals zero identically, so the factor cannot contain any poles at $\zeta = a_{ij}$.

We conclude M^N_{uv} is holomorphic in ζ, and similarly M^S_{uv} is holomorphic in $1/\zeta$. The transition function (3.41) of M implies M^N_{uv} is a linear function of ζ. □

Let us discuss the behavior of $L(s,\zeta)$ as s approaches 0. At $s = 0$, the matching conditions become

$$Q_i(a_{ij}) = Q_j(a_{ij}), \tag{3.54}$$

i.e. $q = (Q_1(\zeta), Q_2(\zeta), \ldots, Q_n(\zeta))$ is a section of $\mathcal{O}_S(n-1)$, the pullback of $\mathcal{O}_{\mathbb{P}^1}(n-1)$ to the spectral curve S. That is, each section is $r(\zeta)\begin{pmatrix}1 & 1 & \ldots & 1\end{pmatrix}$ for some polynomial $r(\zeta)$ of degree $\leq n-1$. Note that such rows are multiples in $\mathbb{C}[\zeta]$ of the same row $\begin{pmatrix}1 & 1 & \ldots & 1\end{pmatrix}$ so that any $n \times n$ matrix with these rows is not invertible.

Thus, at $s = 0$, the $n \times n$ polynomial matrix $Q(s,\zeta)$ of (3.43) fails to be invertible and so $L(s,\zeta)$ and $M(s,\zeta)$ have a pole at $s = 0$. Nahm's equations then show this pole must give a representation of $\mathfrak{su}(2)$. Hitchin in [26, Eq.(5.17)] concludes that this representation must be the rank n maximal representation of $\mathfrak{su}(2)$.

Proposition 3.9. *The set of orthonormal bases $\{U_1(s,\zeta), \ldots, U_n(s,\zeta)\}$ of $L^s_S(n-1)$ is in 1-1 correspondence with the set of solutions (T_0, T_1, T_2, T_3) to Nahm's equations satisfying the boundary conditions $\lim_{s\to\infty}(T_0, T_1, T_2, T_3) \in ad_{U(n)}(0, i\tau_1, i\tau_2, i\tau_3)$ and $\lim_{s\to 0}(T_0, T_1, T_2, T_3) = (0, \frac{i\rho_1}{2s}, \frac{i\rho_2}{2s}, \frac{i\rho_3}{2s})$, with $\vec{\rho}$ some n dimensional irreducible representation of $\mathfrak{su}(2)$.*

Proof. Given an orthonormal basis $\{U_1(s,\zeta), \ldots, U_n(s,\zeta)\}$, the corresponding Lax pair (L, M) of (3.39) may be written, thanks to the reality (3.40) of (L, M), Lemmas 3.7, 3.8, as

$$L^N = -i(T_1 + iT_2) + 2iT_3\zeta + i(T_1 - iT_2)\zeta^2, \;\; M^N = T_0 - i(T_3 + (T_1 - iT_2)\zeta), \tag{3.55}$$

with (T_0, T_1, T_2, T_3) solving the Nahm's equations on the interval $s \in (0, \infty)$. Near $s = 0$, the residue of L at $s = 0$ gives us an n dimensional irreducible representation of $\mathfrak{su}(2)$. By taking the decomposition of L into $\vec{T}$ using (3.55), the residues of (T_0, T_1, T_2, T_3) then form an irreducible representation of $\mathfrak{su}(2)$ by [26, Prop 5.24].

Conversely, given (T_0, T_1, T_2, T_3) define the Lax pair (L, M) as in (3.55) and $U = (U^N, U^S)$ is obtained as the solution of the Lax linear problem associated to (L, M) as in (3.11). By (3.21), the rows of U are sections of $L^s_S(n-1)$. We show now that these sections are indeed orthonormal with respect to the inner product on $L^s_S(n-1)$.

From the reality (3.40) of (L, M), $U^N(s,\zeta)$ and $(U^S(s,-1/\bar{\zeta})^\dagger)^{-1}$ solve the same linear system so that $D^N(\zeta) := U^S(s,-1/\bar{\zeta})^\dagger U^N(s,\zeta)$ is diagonal and s-independent. Similarly for $D^S := (-1)^{n-1}U^N(-1/\bar{\zeta})^\dagger U^S(\zeta)$. D^N and D^S are well defined in the Northern and Southern patches, respectively. Since $U^N(\zeta) = U^S(\zeta)\zeta^{n-1}e^{s\eta/\zeta}$, we have

$$\begin{aligned} D^S &= (-1)^{n-1}\left(U^S(-\frac{1}{\bar{\zeta}})(-\frac{1}{\bar{\zeta}})^{n-1}e^{s\frac{\eta(-1/\bar{\zeta})}{-1/\bar{\zeta}}}\right)^\dagger U^N(\zeta)\frac{1}{\zeta^{n-1}}e^{-s\eta/\zeta} \\ &= e^{s\eta/\zeta}(\frac{1}{\zeta})^{n-1}U^S(-\frac{1}{\bar{\zeta}})^\dagger U^N(\zeta)\frac{1}{\zeta^{n-1}}e^{-s\eta/\zeta} = \frac{1}{\zeta^{2n-2}}D^N. \end{aligned} \quad (3.56)$$

Thus, (D^N, D^S) is a section of $\mathcal{O}_S(2n-2)$. From the matching condition $U_i(s,\alpha_{ij}) = U_j(s,\alpha_{ij})$ the i-th component on the diagonal of D^N vanishes at α_{ij} and α_{ji} for all $j \neq i$, so that D^N is proportional to $\prod_{\substack{j=1 \\ j\neq i}}(p_i(\zeta) - p_j(\zeta))$. We may gauge this proportionality factor to 1, giving us

$$D^N = U^S(s,-1/\bar{\zeta})^\dagger U^N(s,\zeta) = \begin{pmatrix} \prod_{j=2}^{n}(p_1(\zeta)-p_j(\zeta)) & & \\ & \ddots & \\ & & \prod_{j=1}^{n-1}(p_n(\zeta)-p_j(\zeta)) \end{pmatrix} \quad (3.57)$$

or in terms of U^N,

$$(-\zeta)^{n-1}e^{-s\eta/\zeta}U^N(s,-1/\bar{\zeta})^\dagger U^N(s,\zeta) = \begin{pmatrix} \prod_{j=2}^{n}(p_1(\zeta)-p_j(\zeta)) & & \\ & \ddots & \\ & & \prod_{j=1}^{n-1}(p_n(\zeta)-p_j(\zeta)) \end{pmatrix} \quad (3.58)$$

The equation (3.58) above shows that U is orthonormal with respect to the inner product on $H^0(S, L^s_S(n-1))$ given in Proposition 3.5. □

The gauge group $\mathcal{G}$ of maps $[0,\infty) \to U(n)$ for Nahm's equations relates different choices of bases for $H^0(L_S^s(n-1))$.

3.4.2 Basis Constructions

We discuss methods to construct bases of polynomial rows satisfying the matching conditions

$$Q_i(a_{ij}) = e^{-sr_{ij}} Q_j(a_{ij}). \tag{3.59}$$

It is here that the Atiyah polynomials of Definition 1.12 make their appearance. As in [49], fix values $Q_j(a_{kj})$ for $j \neq k$, then Lagrangian interpolation for the degree $n-1$ polynomial $Q_k(\zeta)$ using $Q_k(a_{kj}) = e^{-sr_{kj}} Q_j(a_{kj})$ for $j \neq k$ gives the degree $n-1$ Langrangian interpolation polynomial

$$Q_k(\zeta) = C_k A_k(\zeta) + \sum_{j\neq k} e^{-sr_{kj}} Q_j(a_{kj}) \prod_{l\neq k,j} \frac{\zeta - a_{kl}}{a_{kj} - a_{kl}}, \ k = 1, \ldots, n. \tag{3.60}$$

The polynomial $A_k(\zeta)$ is precisely the k-th Atiyah polynomial. The value C_k is some constant independent of ζ, although we do not need to make it independent of s.

We used the data $Q_k(a_{kj}) = e^{-sr_{kj}} Q_j(a_{kj})$ to interpolate $Q_k(\zeta)$, but $Q_k(\zeta)$ must also satisfy the other half of the matching conditions $Q_k(a_{jk}) = e^{sr_{kj}} Q_j(a_{jk})$. We discuss the linear system obtained from this. We will write the linear system in terms of only $Q_j(a_{kj})$ for $k < j$, which is half of the total $n(n-1)$ many $Q_j(a_{kj})$ values.

Lemma 3.10. *If $\{Q_j(a_{kj}) : k < j\}$ is known, then the whole set of values $\{Q_j(a_{kj}) : k \neq j, j = 1, \ldots, n, k = 1, \ldots, n\}$ is known.*

Proof. $Q_1(\zeta)$ in (3.60) is written in terms of $Q_j(a_{1j})$ for $1 < j$, which are known by the assumption of the lemma, so $Q_1(\zeta)$ is known. We use induction for $k = 2, \ldots, n$. For $Q_k(\zeta)$, the induction hypothesis is that $Q_j(\zeta)$ for $j < k$ is known. Then for $Q_k(\zeta)$ of (3.60),

$$\begin{aligned} Q_k(\zeta) = C_k A_k(\zeta) &+ \sum_{j<k} e^{-sr_{kj}} \underbrace{Q_j(a_{kj})}_{\text{known by induction hypothesis}} \prod_{l\neq k,j} \frac{(\zeta - a_{kl})}{(a_{kj} - a_{kl})} \\ &+ \sum_{k>j} e^{-sr_{kj}} \underbrace{Q_j(a_{kj})}_{\text{known by assumption}} \prod_{l\neq k,j} \frac{(\zeta - a_{kl})}{(a_{kj} - a_{kl})}. \end{aligned} \tag{3.61}$$

Thus $Q_k(\zeta)$ is known completely. □

We describe the method to obtain the linear system of $\frac{n(n-1)}{2}$ equations for the unknowns $Q_u(a_{vu})$, $u = 2, \ldots, n$ with $v < u$. The Lagrangian interpolation polynomials (3.60) altogether utilize all $n(n-1)$ many $Q_k(a_{jk})$, $k = 1, \ldots, n$ values with both $j < k$ and $j > k$. We may evaluate the interpolation polynomial for $Q_k(\zeta)$ at a_{jk} for $j > k$ and use recursion to write each $Q_k(a_{jk})$ for $j > k$ in terms of $Q_u(a_{vu}), u = 2, \ldots, n$ with $v < u$. This is the explicit form of Lemma 3.10. Now, the interpolation polynomials $\{Q_k(\zeta) : k = 1, \ldots, n\}$ are written in terms of only $Q_u(a_{vu})$, $u = 2, \ldots, n$ with $v < u$. Evaluate $Q_k(\zeta)$ at a_{jk}, $j < k$ to get the linear system in $n(n-1)/2$ many unknowns $Q_u(a_{vu})$ with $v < u$. Denote this linear system

$$Ax = b. \tag{3.62}$$

Let us illustrate this by an example.

Example 3.1. The $n = 3$ linear system $Ax = b$ obtained by the procedure above for x representing the unknowns $\begin{pmatrix} Q_2(a_{12}) \\ Q_3(a_{13}) \\ Q_3(a_{23}) \end{pmatrix}$ is

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

with

$$\begin{aligned}
A_{11} &= -1 + e^{-2sr_{12}} \frac{a_{21} - a_{13}}{a_{12} - a_{13}} \frac{a_{12} - a_{23}}{a_{21} - a_{23}}, \\
A_{12} &= e^{-s(r_{12}+r_{13})} \frac{a_{21} - a_{12}}{a_{13} - a_{12}} \frac{a_{12} - a_{23}}{a_{21} - a_{23}}, \\
A_{13} &= e^{-sr_{23}} \frac{a_{12} - a_{21}}{a_{23} - a_{21}}, \\
A_{21} &= e^{-s(r_{13}+r_{12})} \frac{a_{31} - a_{13}}{a_{12} - a_{13}} \frac{a_{13} - a_{32}}{a_{31} - a_{32}} + e^{-s(r_{23}+2r_{12})} \frac{a_{32} - a_{23}}{a_{21} - a_{23}} \frac{a_{21} - a_{13}}{a_{12} - a_{13}} \frac{a_{13} - a_{31}}{a_{32} - a_{31}}, \\
A_{22} &= -1 + e^{-2sr_{13}} \frac{a_{31} - a_{12}}{a_{13} - a_{12}} \frac{a_{13} - a_{32}}{a_{31} - a_{32}} + e^{-s(r_{23}+r_{12}+r_{13})} \frac{a_{32} - a_{23}}{a_{21} - a_{23}} \frac{a_{21} - a_{12}}{a_{13} - a_{12}} \frac{a_{13} - a_{31}}{a_{32} - a_{31}}, \\
A_{23} &= e^{-2sr_{23}} \frac{a_{32} - a_{21}}{a_{23} - a_{21}} \frac{a_{13} - a_{31}}{a_{32} - a_{31}}, \\
A_{31} &= e^{-s(r_{13}+r_{12})} \frac{a_{23} - a_{32}}{a_{31} - a_{32}} \frac{a_{31} - a_{13}}{a_{12} - a_{13}} + e^{-s(r_{23}+2r_{12})} \frac{a_{21} - a_{13}}{a_{12} - a_{13}} \frac{a_{23} - a_{31}}{a_{32} - a_{31}} \frac{a_{32} - a_{23}}{a_{21} - a_{23}}, \\
A_{32} &= e^{-2sr_{13}} \frac{a_{23} - a_{32}}{a_{31} - a_{32}} \frac{a_{31} - a_{12}}{a_{13} - a_{12}} + e^{-s(r_{23}+r_{12}+r_{13})} \frac{a_{21} - a_{12}}{a_{13} - a_{12}} \frac{a_{32} - a_{23}}{a_{21} - a_{23}} \frac{a_{23} - a_{31}}{a_{32} - a_{31}}, \\
A_{33} &= -1 + e^{2sr_{23}} \frac{a_{32} - a_{21}}{a_{23} - a_{21}} \frac{a_{23} - a_{31}}{a_{32} - a_{31}}.
\end{aligned} \tag{3.63}$$

and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_2 \end{pmatrix},$$

with

$b_1 = C_2 A_2(a_{12}) + e^{-sr_{12}} C_1 A_1(a_{21}) \frac{a_{12}-a_{23}}{a_{21}-a_{23}}$,

$$b_2 = C_3 A_3(a_{13}) + e^{-sr_{13}} C_1 A_1(a_{31}) \frac{a_{13}-a_{32}}{a_{31}-a_{32}} + e^{-sr_{23}} C_2 A_2(a_{32}) \frac{a_{13}-a_{31}}{a_{32}-a_{31}} + e^{-s(r_{23}+r_{12})} C_1 A_1(a_{21}) \frac{a_{32}-a_{23}}{a_{21}-a_{23}} \frac{a_{13}-a_{31}}{a_{32}-a_{31}},$$

$$b_3 = C_3 A_3(a_{23}) + e^{-sr_{13}} C_1 A_1(a_{31}) \frac{a_{23}-a_{32}}{a_{31}-a_{32}} + e^{-sr_{23}} C_2 A_2(a_{32}) \frac{a_{23}-a_{31}}{a_{32}-a_{31}} + e^{-s(r_{23}+r_{12})} C_1 A_1(a_{21}) \frac{a_{23}-a_{31}}{a_{32}-a_{31}} \frac{a_{32}-a_{23}}{a_{21}-a_{23}}. \quad (3.64)$$

Proof. We follow the procedure. The Lagrangian interpolation polynomials of (3.60) are

$$Q_1(\zeta) = C_1 A_1(\zeta) + e^{-sr_{12}} Q_2(a_{12}) \frac{\zeta - a_{13}}{a_{12}-a_{13}} + e^{-sr_{13}} Q_3(a_{13}) \frac{\zeta - a_{12}}{a_{13}-a_{12}},$$
$$Q_2(\zeta) = C_2 A_2(\zeta) + e^{-sr_{12}} Q_1(a_{21}) \frac{\zeta - a_{23}}{a_{21}-a_{23}} + e^{-sr_{23}} Q_3(a_{23}) \frac{\zeta - a_{21}}{a_{23}-a_{21}},$$
$$Q_3(\zeta) = C_3 A_3(\zeta) + e^{-sr_{13}} Q_1(a_{31}) \frac{\zeta - a_{32}}{a_{31}-a_{32}} + e^{-sr_{23}} Q_2(a_{32}) \frac{\zeta - a_{31}}{a_{32}-a_{31}}.$$

One sees that all six values of $Q_u(a_{vu})$ for $v < u$ and $v > u$ are present in this system of polynomials. The three values $Q_u(a_{vu})$ for $v > u$ may be written in terms of the other three $Q_u(a_{vu})$ for $v < u$ by evaluating the interpolation polynomial for $Q_k(\zeta)$ at a_{jk} for $j > k$ and using recursion.

$$Q_1(a_{21}) = C_1 A_1(a_{21}) + e^{-sr_{12}} Q_2(a_{12}) \frac{a_{21} - a_{13}}{a_{12}-a_{13}} + e^{-sr_{13}} Q_3(a_{13}) \frac{a_{21} - a_{12}}{a_{13}-a_{12}},$$
$$Q_1(a_{31}) = C_1 A_1(a_{31}) + e^{-sr_{12}} Q_2(a_{12}) \frac{a_{31} - a_{13}}{a_{12}-a_{13}} + e^{-sr_{13}} Q_3(a_{13}) \frac{a_{31} - a_{12}}{a_{13}-a_{12}},$$
$$Q_2(a_{32}) = C_2 A_2(a_{32}) + e^{-sr_{12}} Q_1(a_{21}) \frac{a_{32} - a_{23}}{a_{21}-a_{23}} + e^{-sr_{23}} Q_3(a_{23}) \frac{a_{32} - a_{21}}{a_{23}-a_{21}}.$$

Afterwards, the interpolation polynomials for $Q_k(\zeta)$, $k = 2, \dots, n$, are in terms of the three unknowns $Q_2(a_{12}), Q_3(a_{13}), Q_3(a_{23})$ we want to solve for.

$$Q_2(\zeta) = C_2A_2(\zeta) + e^{-sr_{12}}\frac{\zeta - a_{23}}{a_{21} - a_{23}}\Bigg(C_1A_1(a_{21}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{21} - a_{13}}{a_{12} - a_{13}} + e^{-sr_{13}}Q_3(a_{13})\frac{a_{21} - a_{12}}{a_{13} - a_{12}}\Bigg) + e^{-sr_{23}}Q_3(a_{23})\frac{\zeta - a_{21}}{a_{23} - a_{21}}.$$

$$\begin{aligned} Q_3(\zeta) &= C_3A_3(\zeta) \\ &+ e^{-sr_{13}}\frac{\zeta - a_{32}}{a_{31} - a_{32}}\left(C_1A_1(a_{31}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{31} - a_{13}}{a_{12} - a_{13}} + e^{-sr_{13}}Q_3(a_{13})\frac{a_{31} - a_{12}}{a_{13} - a_{12}}\right) \\ &+ e^{-sr_{23}}\frac{\zeta - a_{31}}{a_{32} - a_{31}}\Bigg(C_2A_2(a_{32}) + e^{-sr_{12}}\frac{a_{32} - a_{23}}{a_{21} - a_{23}}\Bigg(C_1A_1(a_{21}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{21} - a_{13}}{a_{12} - a_{13}} \\ &\qquad + e^{-sr_{13}}Q_3(a_{13})\frac{a_{21} - a_{12}}{a_{13} - a_{12}}\Bigg) + e^{-sr_{23}}Q_3(a_{23})\frac{a_{32} - a_{21}}{a_{23} - a_{21}}\Bigg). \end{aligned}$$

Now evaluate $Q_2(\zeta)$ at $\zeta = a_{12}$ and $Q_3(\zeta)$ at $\zeta = a_{13}, a_{23}$ to get the actual linear system in the three unknowns $Q_2(a_{12})$, $Q_3(a_{13})$, $Q_3(a_{23})$.

$$\begin{aligned} Q_2(a_{12}) &= C_2A_2(a_{12}) \\ &+ e^{-sr_{12}}\frac{a_{12} - a_{23}}{a_{21} - a_{23}}\left(C_1A_1(a_{21}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{21} - a_{13}}{a_{12} - a_{13}} + e^{-sr_{13}}Q_3(a_{13})\frac{a_{21} - a_{12}}{a_{13} - a_{12}}\right) \\ &\qquad + e^{-sr_{23}}Q_3(a_{23})\frac{a_{12} - a_{21}}{a_{23} - a_{21}}. \end{aligned}$$

$$\begin{aligned} Q_3(a_{13}) &= C_3A_3(a_{13}) \\ &+ e^{-sr_{13}}\frac{a_{13} - a_{32}}{a_{31} - a_{32}}(C_1A_1(a_{31}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{31} - a_{13}}{a_{12} - a_{13}} + e^{-sr_{13}}Q_3(a_{13})\frac{a_{31} - a_{12}}{a_{13} - a_{12}}) \\ &+ e^{-sr_{23}}\Bigg(C_2A_2(a_{32}) + e^{-sr_{12}}\frac{a_{32} - a_{23}}{a_{21} - a_{23}}(C_1A_1(a_{21}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{21} - a_{13}}{a_{12} - a_{13}} \\ &\qquad + e^{-sr_{13}}Q_3(a_{13})\frac{a_{21} - a_{12}}{a_{13} - a_{12}}) + e^{-sr_{23}}Q_3(a_{23})\frac{a_{32} - a_{21}}{a_{23} - a_{21}}\Bigg)\frac{a_{13} - a_{31}}{a_{32} - a_{31}}. \end{aligned}$$

$$\begin{aligned}Q_3(a_{23}) = C_3A_3(a_{23}) + e^{-sr_{13}}\frac{a_{23}-a_{32}}{a_{31}-a_{32}}\bigg(C_1A_1(a_{31}) + e^{-sr_{12}}Q_2(a_{12})\frac{a_{31}-a_{13}}{a_{12}-a_{13}} \\ + e^{-sr_{13}}Q_3(a_{13})\frac{a_{31}-a_{12}}{a_{13}-a_{12}}\bigg) \\ + e^{-sr_{23}}\frac{a_{23}-a_{31}}{a_{32}-a_{31}}\bigg(C_2A_2(a_{32}) + e^{-sr_{12}}\frac{a_{32}-a_{23}}{a_{21}-a_{23}}\bigg(C_1A_1(a_{21}) \\ + e^{-sr_{12}}Q_2(a_{12})\frac{a_{21}-a_{13}}{a_{12}-a_{13}} + e^{-sr_{13}}Q_3(a_{13})\frac{a_{21}-a_{12}}{a_{13}-a_{12}}\bigg) + e^{-sr_{23}}Q_3(a_{23})\frac{a_{32}-a_{21}}{a_{23}-a_{21}}\bigg).\end{aligned}$$

□

Lemma 3.11. *The $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix A of (3.62) representing the linear system in the unknowns $Q_k(a_{jk})$, $k = 1, \dots, n$ with $j < k$ is invertible when $s \in (0, \infty)$.*

Proof. Set all $C_k = 0$ so that the Atiyah polynomials in the Lagrangian interpolations are turned off and $Q_k(\zeta)$ are only degree $n-2$ polynomials satisfying the matching conditions $Q_i(a_{ij}) = e^{-sr_{ij}}Q_j(a_{ij})$. This is the system $Ax = 0$. A nontrivial solution to $Ax = 0$ then corresponds to a section belonging to $H^0(S, L^s_S(n-2))$. But by Proposition 3.3, $h^0(S, L^s_S(n-2)) = 0$, so that $Ax = 0$ admits no nontrivial solutions and A is invertible. □

Proposition 3.12. *There exists a basis of $H^0(S, L^s_S(n-1))$ where the j-th section corresponds to the row of polynomials $q_j = (Q_1(\zeta), Q_2(\zeta), \dots, Q_n(\zeta))$ with $Q_j(\zeta)$ of degree exactly $n-1$ and all other polynomials $Q_{i\neq j}(\zeta)$ are of degree smaller than $n-1$.*

Proof. The j-th row of polynomials corresponds to the choice

$$(C_1 = 0, \dots, C_j \neq 0, \dots, C_n = 0)$$

in (3.60). The matrix A is invertible by Lemma 3.11 so we need to show b is nonzero. The vector b is of the form

$$b = \begin{pmatrix} C_2A_2(a_{12}) + \sum\limits_{\substack{k>i\\ i<2}} d^1_{ki}C_iA_i(a_{ki}) \\ C_3A_3(a_{13}) + \sum\limits_{\substack{k>i\\ i<3}} d^2_{ki}C_iA_i(a_{ki}) \\ C_3A_3(a_{23}) + \sum\limits_{\substack{k>i\\ i<3}} d^3_{ki}C_iA_i(a_{ki}) \\ \vdots \\ C_jA_j(a_{lj}) + \sum\limits_{\substack{k>i\\ i<j}} d_{ki}C_iA_i(a_{ki}) \\ \vdots \\ C_nA_n(a_{n-1}) + \sum\limits_{\substack{k>i\\ i<n}} d^{n(n-1)/2}_{ki}C_iA_i(a_{ki}) \end{pmatrix}, \tag{3.65}$$

where the collection of d^l_{ki} are coefficients we do not specify. For the j-th section with $j > 1$, C_j is nonzero and every other $C_i = 0$, $i \neq j$. Looking at (3.65), the entry of b with $C_jA_j(a_{kj}) + \sum\limits_{\substack{k>i\\ i<j}} d_{ki}C_iA_i(a_{ki})$ reduces to $C_jA_j(a_{kj})$, which is nonzero. Thus b is nonzero for this section. The only case remaining is the first section with $(C_1 \neq 0, C_2 = 0, \ldots, C_n = 0)$. Here we explicitly write the first entry of b, which is $C_2A_2(a_{12}) + e^{-sr_{12}}C_1A_1(a_{21})\prod\limits_{k=2}^{n}\frac{a_{12}-a_{2k}}{a_{21}-a_{2k}}$. When $C_2 = 0$ and $C_1 \neq 0$, this entry of b is nonzero.

The polynomial rows are linearly independent since the j-th polynomial row corresponding to $(C_1 = 0, \ldots, C_j \neq 0, \ldots, C_n = 0)$ has in the j-th position a degree $n-1$ polynomial but the other polynomial rows have a degree $n-2$ polynomial in this position. □

To obtain an orthonormal basis of sections, one may use the Gram-Schmidt process. But as in [50, Proposition 2.2], there is a prescription to create an orthonormal basis based on the following observation. Fix two distinct sheets S_j, S_k of the spectral curve with $1 \leq j \neq k \leq n$ and a pair of antipodal points $\zeta_0, -1/\bar{\zeta}_0$ that are not double points on the curve S. If the row of polynomials $p = (P_1(\zeta), \ldots, P_n(\zeta))$ and $r = (R_1(\zeta), \ldots, R_n(\zeta))$ has the following vanishing conditions at the antipodal points

$$\begin{aligned} P_i(\zeta_0) = 0 \text{ for } i < j, \quad P_i(-1/\bar{\zeta}_0) = 0 \text{ for } i > j, \\ R_i(\zeta_0) = 0 \text{ for } i < k, \quad R_i(-1/\bar{\zeta}_0) = 0 \text{ for } i > k, \end{aligned} \tag{3.66}$$

then using (3.37), the polynomial rows are easily seen to be orthogonal to one another, that is, $\langle p, r \rangle = 0$.

The vanishing conditions on a row of polynomials ψ states that the corresponding ψ vanishes at the points on sheets above a fixed point ζ_0 for all sheets before some chosen S_j sheet, then after S_j all sheets have ψ vanishing above the antipodal point of ζ_0, i.e.

$$\psi\big(\eta_i(\zeta_0), \zeta_0\big) = 0 \text{ for } i < j, \quad \psi\left(\eta_i(-\frac{1}{\bar{\zeta}_0}), -\frac{1}{\bar{\zeta}_0}\right) = 0 \text{ for } i > j. \tag{3.67}$$

As in [50, Section 2], we shall take $\zeta_0 = 0$ and $-1/\bar{\zeta}_0 = \infty$.

Proposition 3.13. *The orthogonal basis $\{\psi_1, \ldots, \psi_n\}$ of $L^s_S(n-1)$ for $s \in (0, \infty)$, with $\psi_j = (P_1(\zeta), P_2(\zeta), \ldots, P_n(\zeta))$ for $P_j(\zeta)$ a monic polynomial of maximal degree $n-1$ and satisfying the vanishing conditions $P_i(\infty) = 0$ for $i < j$ and $P_i(0) = 0$ for $i > j$, is constructible from the linear system* (3.62) *of the Lagrangian interpolation polynomials.*

Proof. We refer to the Lagrangian interpolation polynomial (3.60). The polynomial row has $C_1, \ldots, C_n$ as free parameters. The condition that $P_j(\zeta)$ is a monic polynomial of maximal degree $n-1$ is $C_j = 1$. Rotating the spectral curve if necessary so that the spectral curve does not have double points above $\zeta = 0$ and $\zeta = \infty$, the condition $P_i(0) = 0$ imposes $C_i = 0$ and the condition $P_i(\infty) = 0$ also fixes C_i.

Each row of polynomials is the solution to the linear system (3.62) with b determined by the above conditions on the polynomial row. We may write the overall system on the n rows of the polynomials as

$$AX = B, \tag{3.68}$$

with n columns of X for the n many polynomial rows, with the corresponding n columns of B determined by the vanishing conditions. The orthogonal basis of sections of $L^s_S(n-1)$ is given by $X = A^{-1}B$. □

There is a more pedestrian approach to the linear system of polynomials satisfying the matching conditions, building on the notes of Braden and Cherkis [69]. For a row of polynomials $(P_1, \dots, P_n)$ we can simply take as unknowns all the coefficients

$$\{p_{ij} : \ i = 0, \dots, n-1, j = 1, \dots, n\}$$

of

$$\begin{aligned} P_1(\zeta) &= p_{01} + p_{11}\zeta + \dots + p_{(n-1)1}\zeta^{n-1}, \\ &\vdots \\ P_n(\zeta) &= p_{0n} + p_{1n}\zeta + \dots + p_{(n-1)n}\zeta^{n-1}, \end{aligned}$$

and subject $\{p_{ij}\}$ to the matching conditions $P_l(a_{lk}) = e^{-sr_{lk}} P_k(a_{lk})$.

This forms the linear system

$$\Xi p = 0, \tag{3.69}$$

where $p = (p_{01}, p_{02}, \dots, p_{0n}, \dots, p_{(n-1)1}, p_{(n-1)n}, \dots, p_{(n-1)n})^T$ and the ijth row of Ξ corresponds to the matching condition $P_i(a_{ij}) = e^{-sr_{ij}} P_j(a_{ij})$, that is

$$\Xi_{ij} = \begin{pmatrix} 1 & \alpha_{ij} & \dots & \alpha_{ij}^{n-1} \end{pmatrix} \otimes (e^{sr_{ij}/2}\hat{e}_i - e^{-sr_{ij}/2}\hat{e}_j). \tag{3.70}$$

Proposition 3.14. *The orthogonal basis $\{p_1, \dots, p_n\}$ of $L^s_S(n-1)$ for $s \in (0, \infty)$, with $p_j = (P_1(\zeta), P_2(\zeta), \dots, P_n(\zeta))$ for $P_j(\zeta)$ a monic polynomial of maximal degree $n-1$ and satisfying the vanishing conditions $P_i(\infty) = 0$ for $i < j$ and $P_i(0) = 0$ for $i > j$, is constructible from the pedestrian linear system* (3.69).

Proof. Let $\mathcal{P}_{n^2 \times n}$ be the matrix of unknowns where all of the coefficients in $p_j = (P^j_1(\zeta), \dots, P^j_n(\zeta))$ are placed in column j. The vanishing condition $P^j_i(\infty) = 0$, i.e. $P^j_i(\zeta)$ has degree $n-2$, for $i < j$ and monic condition makes the bottom n rows of $\mathcal{P}$ a lower triangular matrix with 1's along the diagonal. The vanishing condition $P^j_i(0) = 0$ for $i > j$ makes the top n rows of $\mathcal{P}$ an upper triangular matrix. The linear system $\Xi\mathcal{P} = 0$ of (3.69) becomes

$$\begin{pmatrix} a & B & c \end{pmatrix} \begin{pmatrix} U \\ \tilde{P} \\ \hat{L} \end{pmatrix} = 0. \tag{3.71}$$

It can be solved the following way,

$$\begin{aligned} \begin{pmatrix} a & B \end{pmatrix} \begin{pmatrix} U \\ \tilde{P} \end{pmatrix} &= -c\hat{L}, \\ \begin{pmatrix} a & B \end{pmatrix} \begin{pmatrix} U \\ \tilde{P} \end{pmatrix} \hat{L}^{-1} &= -c, \\ \begin{pmatrix} U\hat{L}^{-1} \\ \tilde{P}\hat{L}^{-1} \end{pmatrix} &= -\begin{pmatrix} a & B \end{pmatrix}^{-1} c. \end{aligned} \tag{3.72}$$

Take UL decomposition [70] of the top n rows of $-\begin{pmatrix} a & B \end{pmatrix}^{-1} c$ to solve for the unknowns U and $\hat{L}^{-1}$. Apply $\hat{L}$ to the remaining rows of $-\begin{pmatrix} a & B \end{pmatrix}^{-1} c$ to solve for $\tilde{P}$.

The matrix $\begin{pmatrix} a & B \end{pmatrix}$ is invertible as this block of Ξ encodes the matching conditions for $L_S^s(n-2)$ and so $\det \begin{pmatrix} a & B \end{pmatrix} \neq 0$ is equivalent to $h^0(S, L_S^s(n-2)) = 0$, which is proved in Proposition 3.3. The UL decomposition of $-\begin{pmatrix} a & B \end{pmatrix}^{-1} c$ exists since the existence of such a basis is proven in Proposition 3.13. □

Lemma 3.15. *The Nahm data (T_0, T_1, T_2, T_3) constructed from normalizing the basis of Propositions 3.13 and 3.14, and using the Lax pair of* (3.39)*, is such that*

$$\lim_{s\to\infty} T_j(s) = \begin{pmatrix} p_1^j & & & \\ & p_2^j & & \\ & & \ddots & \\ & & & p_n^j \end{pmatrix}. \tag{3.73}$$

Proof. We have

$$\begin{aligned} L^N(s,\zeta) &= U^N(s,\zeta) \begin{pmatrix} p_1(\zeta) & & & \\ & p_2(\zeta) & & \\ & & \ddots & \\ & & & p_n(\zeta) \end{pmatrix} U^N(s,\zeta)^{-1}, \\ M^N(s,\zeta) &= -\frac{dU^N(s,\zeta)}{ds} U^N(s,\zeta)^{-1}. \end{aligned}$$

We must then show that $\lim_{s\to\infty} Q^N(s,\zeta)$ is diagonal so that $L^N(s,\zeta)$ has a diagonal matrix as its limit as $s \to \infty$. As in the proof of [50, Proposition 4.2], we take the limit of $s \to \infty$ in the matching conditions for the row of polynomials

$$Q_i(a_{ij}) = e^{-sr_{ij}} Q_j(a_{ij}).$$

This shows that the limit of $s \to \infty$ for the row of polynomials is generated by $s^l = (P_1(\zeta), \ldots, P_n(\zeta))$, $1 \le l \le n$, with $P_l(\zeta) = A_l(\zeta)$ the l-th Atiyah polynomial and $P_j(\zeta) \equiv 0$ for $j \ne l$. The vanishing conditions that $\psi_l = (P_1(\zeta), \ldots, P_n(\zeta))$ of our basis satisfies implies its limit at infinity must be exactly

$$\lim_{s\to\infty} \psi_l = (0, \ldots, 0, A_l(\zeta), 0, \ldots, 0).$$

Thus our choice of basis gives a diagonal matrix Q^N at infinity. □

3.4.3 Perturbation Expansion

We seek an orthogonal basis in the space of n-tuplets

$$(Q_1(\zeta), Q_2(\zeta), \ldots, Q_n(\zeta))$$

of degree $\le n-1$ polynomials satisfying

$$Q_i(a_{ij}) = e^{-sr_{ij}} Q_j(a_{ij}) \tag{3.74}$$

with respect to the inner product (3.37)

$$\langle P, R\rangle = (-\zeta)^{n-1} \sum_{i=1}^{n} \frac{P_i(\zeta)\overline{R_i(-1/\bar{\zeta})}}{\prod_{i\ne j}(p_i(\zeta) - p_j(\zeta))}. \tag{3.75}$$

Even though ζ appears in the right hand side of the inner product, this inner product is independent of ζ, by Proposition 3.5.

One such choice of basis is (3.62), which requires that the jth element of the basis $q_j = (Q_1(\zeta), Q_2(\zeta), \ldots, Q_n(\zeta))$ has

$$Q_i(\infty) = 0, \text{i.e.} \deg Q_i(\zeta) < n-1 \text{ for } i < j \quad \text{and} \quad Q_i(0) = 0, \text{ for } i > j, \tag{3.76}$$

with $Q_j(\zeta)$ a monic polynomial of maximal degree $n-1$.

We obtained exact solutions to this problem in terms of a linear system in Propositions 3.13 and 3.14. We also know from Lemma 3.15 that

$$\lim_{s\to\infty} q_j = (0, \ldots, 0, A_j(\zeta), 0, \ldots, 0). \tag{3.77}$$

However, we would still like to understand the behavior for large s better.

In this section, we construct an approximate solution to the choice of basis above for large s in terms of a formal series in the small parameters $e^{-sr_{ij}}$. The higher-order terms in the series become successively smaller and we give a method for constructing the approximate solution to arbitrary order.

Let us now consider the first element of the basis (the story is analogous for other elements). We describe again how to obtain the zeroth order of our expansion given in (3.77). The limit as $s \to \infty$ of the matching conditions $Q_j(a_{ji}) = e^{-sr_{ij}} Q_i(a_{ji})$ is

$$Q_j^{(0)}(a_{ji}) = 0.$$

The vanishing conditions on the first basis element, in addition to the equation above, states that $Q_{j\neq 1}^{(0)}(\zeta)$ vanishes at 0 and $n-1$ many other points a_{jk}. Since $Q_{j\neq 1}^{(0)}(\zeta)$ is a polynomial of degree at most $n-1$, we must have $Q_{j\neq 1}^{(0)}(\zeta) \equiv 0$. $Q_1^{(0)}(\zeta)$ vanishes at a_{1k} and is a monic polynomial, thus $Q_1^{(0)}(\zeta) = A_1(\zeta) = \prod_{k=2}^{n} (\zeta - a_{1k})$.

We are seeking an expansion, which we name the perturbation expansion for large s, with

$$Q_1(\zeta) = A_1(\zeta) + e^{-s\Delta_1} q_1(\zeta) + e^{-s\Delta_1'} q_1'(\zeta) + \dots \tag{3.78}$$

$$Q_{j\neq 1}(\zeta) = e^{-s\Delta_j} \zeta q_j(\zeta) + e^{-s\Delta_j'} \zeta q_j'(\zeta) + \dots \tag{3.79}$$

with $0 < \Delta_k < \Delta_k' < \Delta_k'' < \dots$. The polynomials $q_k(\zeta)$ are degree less or equal to $n-2$ in ζ and independent of s.

We give a method to find the perturbation expansion to arbitrary order. The zeroth order is $Q_1 = A_1(\zeta)$, $Q_{j\neq 1}(\zeta) = 0$. By induction, given the expansion at n-th order, the $(n+1)$-st order of Q_1 is the Lagrangian interpolation polynomial for the points $(a_{1i}, e^{-sr_{1i}} e^{-s\Delta_i^{(n)}} \tilde{q}_i^{(n)}(a_{1i}))$ for $1 < i \leq n$, where $e^{-s\Delta_i^{(n)}} \tilde{q}_i^{(n)}(\zeta)$ is the n-th order of $Q_i(\zeta)$. That is, the $n+1$-st order of Q_1 is given by

$$\sum_{i=2}^{n} e^{-s(\Delta_i^{(n)} + r_{i1})} \tilde{q}_i^{(n)}(a_{1i}) \prod_{k\neq 1,i} \frac{\zeta - a_{1k}}{a_{1i} - a_{1k}}. \tag{3.80}$$

The $n+1$-st order of $Q_{j\neq 1}$ is similar, except each term has an extra factor to account for the vanishing conditions on the basis. Explicitly, the $n+1$-st order is

$$\sum_{i\neq j}^{n} e^{-s(\Delta_i^{(n)}+r_{ij})}\tilde{q}_i^{(n)}(a_{ji})\frac{\zeta}{a_{ji}}\prod_{k\neq j,i}\frac{\zeta-a_{jk}}{a_{ji}-a_{jk}}. \tag{3.81}$$

The points for the Lagrangian interpolation for the $n+1$ order of Q_j comes from the matching conditions

$$Q_j(a_{ji}) = e^{-sr_{ij}}Q_i(a_{ji}). \tag{3.82}$$

In general, for the l-th basis element, the factor $\frac{\zeta}{a_{ji}}$ is either kept or eliminated according to the vanishing conditions.

We present the $n=2$ case as a simple example.

Example 3.2. The first section is $(Q_1(\zeta), Q_2(\zeta)) = (Q_1^1(\zeta), Q_2^1(\zeta))$ with

Order	Zeroth	First	Second	...
$Q_1^1(\zeta) =$	$\zeta - a_{12}$	0	$e^{-s2r_{12}}(a_{21}-a_{12})\frac{a_{12}}{a_{21}}$	...
$Q_2^1(\zeta) =$	0	$e^{-sr_{12}}(a_{21}-a_{12})\frac{\zeta}{a_{21}}$	0	...

$$\begin{aligned} Q_1^1(\zeta) &= \zeta - a_{12} + e^{-s2r_{12}}(a_{21}-a_{12})\frac{a_{12}}{a_{21}} + e^{-s4r_{12}}(a_{21}-a_{12})(\frac{a_{12}}{a_{21}})^2 + \dots \\ &= \zeta - a_{12} + \frac{a_{12}(a_{21}-a_{12})}{a_{21}e^{2sr_{12}} - a_{12}}, \end{aligned} \tag{3.83}$$

$$\begin{aligned} Q_2^1(\zeta) &= e^{-sr_{12}}(a_{21}-a_{12})\frac{\zeta}{a_{21}} + e^{-3sr_{12}}(a_{21}-a_{12})\frac{a_{12}\zeta}{a_{21}^2} + e^{-5sr_{12}}(a_{21}-a_{12})\frac{a_{12}^2\zeta}{a_{21}^3} + \dots \\ &= \frac{\zeta(a_{21}-a_{12})}{a_{21}e^{sr_{12}} - e^{-sr_{12}}a_{12}}. \end{aligned} \tag{3.84}$$

The second section is $(Q_1(\zeta), Q_2(\zeta)) = (Q_1^2(\zeta), Q_2^2(\zeta))$ with

Order	Zeroth	First	Second	...
$Q_1^2(\zeta) =$	0	$e^{-sr_{12}}(a_{12}-a_{21})$	0	...
$Q_2^2(\zeta) =$	$\zeta - a_{21}$	0	$e^{-s2r_{12}}(a_{12}-a_{21})$	...

$$\begin{aligned} Q_1^2(\zeta) &= e^{-sr_{12}}(a_{12} - a_{21}) + e^{-s3r_{12}}(a_{12} - a_{21}) + \dots \\ &= \frac{a_{12} - a_{21}}{e^{sr_{12}} - e^{-sr_{12}}}, \end{aligned} \tag{3.85}$$

$$\begin{aligned} Q_2^2(\zeta) &= \zeta - a_{21} + e^{-s2r_{12}}(a_{12} - a_{21}) + e^{-s4r_{12}}(a_{12} - a_{21}) + \dots \\ &= \zeta - a_{21} + \frac{a_{12} - a_{21}}{e^{2sr_{12}} - 1}. \end{aligned} \tag{3.86}$$

We present the $n = 3$ case up to second order for the first section to illustrate the method. For the remaining sections, we limit our formulas to first order.

Example 3.3. The first section $(Q_1(\zeta), Q_2(\zeta), Q_3(\zeta)) = (Q_1^1(\zeta), Q_2^1(\zeta), Q_3^1(\zeta))$ is

Order	Zeroth	First	Second
$Q_1^1 =$	$A_1(\zeta)$	0	$e^{-2sr_{12}}A_1(a_{21})\frac{a_{12}}{a_{21}}\frac{a_{12}-a_{23}}{a_{21}-a_{23}}\frac{\zeta-a_{13}}{a_{12}-a_{13}} +$ $e^{-2sr_{13}}A_1(a_{31})\frac{a_{13}}{a_{31}}\frac{a_{13}-a_{32}}{a_{31}-a_{32}}\frac{\zeta-a_{12}}{a_{13}-a_{12}}$
$Q_2^1 =$	0	$e^{-sr_{12}}A_1(a_{21})\frac{\zeta}{a_{21}}\frac{\zeta-a_{23}}{a_{21}-a_{23}}$	$e^{-s(r_{12}+r_{13})}A_1(a_{31})\frac{a_{23}}{a_{31}}\frac{a_{23}-a_{32}}{a_{31}-a_{32}}\frac{\zeta}{a_{23}}\frac{\zeta-a_{21}}{a_{23}-a_{21}}$
$Q_3^1 =$	0	$e^{-sr_{13}}A_1(a_{31})\frac{\zeta}{a_{31}}\frac{\zeta-a_{32}}{a_{31}-a_{32}}$	$e^{-s(r_{12}+r_{13})}A_1(a_{21})\frac{a_{32}}{a_{21}}\frac{a_{32}-a_{23}}{a_{21}-a_{23}}\frac{\zeta}{a_{32}}\frac{\zeta-a_{31}}{a_{32}-a_{31}}$

The second section $(Q_1(\zeta), Q_2(\zeta), Q_3(\zeta)) = (Q_1^2(\zeta), Q_2^2(\zeta), Q_3^2(\zeta))$ to first order is

$$\begin{aligned} Q_1^2(\zeta) &= e^{-sr_{12}}A_2(a_{12})\frac{\zeta - a_{13}}{a_{12} - a_{13}}, \\ Q_2^2(\zeta) &= A_2(\zeta), \\ Q_3^2(\zeta) &= e^{-sr_{23}}A_2(a_{32})\frac{\zeta}{a_{32}}\frac{\zeta - a_{31}}{a_{32} - a_{31}}. \end{aligned} \tag{3.87}$$

The third section $(Q_1(\zeta), Q_2(\zeta), Q_3(\zeta)) = (Q_1^3(\zeta), Q_2^3(\zeta), Q_3^3(\zeta))$ to first order is

$$\begin{aligned} Q_1^3(\zeta) &= e^{-sr_{13}}A_3(a_{13})\frac{\zeta - a_{12}}{a_{13} - a_{12}}, \\ Q_2^3(\zeta) &= e^{-sr_{23}}A_3(a_{23})\frac{\zeta - a_{21}}{a_{23} - a_{21}}, \\ Q_3^3(\zeta) &= A_3(\zeta). \end{aligned} \tag{3.88}$$

3.5 Nahm Solutions

In this section, we illustrate the procedure of Section 3.4 for obtaining rank n Nahm solutions by giving the examples of an exact Nahm solution for $n = 2$ and a perturbative Nahm solution for $n = 3$.

Recall that by Proposition 3.9, an orthonormal basis $\{U_1(s,\zeta),\ldots,U_n(s,\zeta)\}$ of sections of the eigenline bundle $L^s_S(n-1)$, where $s \in (0,\infty)$, over the spectral curve S may be used to obtain Nahm solutions. To summarize, we set

$$U(s,\zeta) = \begin{pmatrix} U_1(s,\zeta) \\ U_2(s,\zeta) \\ \vdots \\ U_n(s,\zeta) \end{pmatrix} \tag{3.89}$$

and our Lax pair (L, M) in North patch is

$$\begin{aligned} L^N(s,\zeta) &= U^N(s,\zeta) \begin{pmatrix} p_1(\zeta) & & & \\ & p_2(\zeta) & & \\ & & \ddots & \\ & & & p_n(\zeta) \end{pmatrix} U^N(s,\zeta)^{-1}, \\ M^N(s,\zeta) &= -\frac{dU^N(s,\zeta)}{ds} U^N(s,\zeta)^{-1}, \end{aligned} \tag{3.90}$$

with each $p_j(\zeta) = x^1_j + ix^2_j - 2x^3_j\zeta - (x^1_j - ix^2_j)\zeta^2$ determined by the point $\vec{x}_j = (x^1_j, x^2_j, x^3_j) \in \mathbb{R}^3$ from the desired boundary conditions as $s \to \infty$.

From the proof of Proposition 3.9, we see we may evaluate L^N and M^N at $\zeta = 0$ and obtain

$$\begin{aligned} T_1(s) &= \frac{i}{2}(L^N(s,0) + L^N(s,0)^\dagger), \\ T_2(s) &= \frac{1}{2}(L^N(s,0) - L^N(s,0)^\dagger), \\ T_3(s) &= \frac{i}{2}(M^N(s,0) + M^N(s,0)^\dagger), \\ T_0(s) &= \frac{1}{2}(M^N(s,0) - M^N(s,0)^\dagger). \end{aligned} \tag{3.91}$$

We obtained bases of the eigenline bundle in terms of Q, consisting of rows of polynomials of degree $\leq n-1$, for $n = 2$ in Example 3.2 and for $n = 3$ in Example 3.3. For each example, we now find the corresponding Nahm solution, using 3.91.

3.5.1 n=2 Exact Solution

The exact solution to Nahm's equations for $n = 2$ is known to be given by $T_i = f_i(s)\sigma_i$, for $f_i(s)$ the hyperbolic functions satisfying the Euler top system of (1.4).

For example, taking the boundary conditions in (1.9) to be

$$\lim_{s\to\infty}(T_1(s), T_2(s), T_3(s)) \in \mathrm{ad}_{U(n)}\left(0, 0, i\begin{pmatrix} c/2 & 0 \\ 0 & -c/2 \end{pmatrix}\right), \qquad \lim_{s\to 0} sT_j(s) = \frac{i\sigma_j}{2}, \tag{3.92}$$

the unique solution to Nahm's equations is

$$\begin{aligned} T_1 &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{c}{2\sinh(cs)}, \\ T_2 &= i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{c}{2\sinh(cs)}, \\ T_3 &= i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{c}{2\tanh(cs)}. \end{aligned} \tag{3.93}$$

However, as an illustration, we shall rederive the Nahm solution in terms of the basis in Example 3.2.

The matrix Q of Example 3.2 is

$$Q = \begin{pmatrix} \zeta - a_{12} + \frac{a_{12}(a_{21}-a_{12})}{a_{21}e^{2sr_{12}}-a_{12}} & \frac{\zeta(a_{21}-a_{12})}{a_{21}e^{sr_{12}}-a_{12}e^{-sr_{12}}} \\ \frac{a_{12}-a_{21}}{e^{sr_{12}}-e^{-sr_{12}}} & \zeta - a_{21} + \frac{a_{12}-a_{21}}{e^{2sr_{12}}-1} \end{pmatrix}. \tag{3.94}$$

The rows of Q must be normalized with respect to the norm of (3.37). Let $z_j := x_j^1 + ix_j^2$ and $z_{jk} := z_j - z_k$. The norm of the row $\psi_j(s,\zeta) = (Q_{j1}, Q_{j2})$ for $1 \le j \le 2$ is

$$\begin{aligned} ||\psi_1||^2 &= \frac{-1 + e^{-2sr_{12}}}{-\bar{z}_{12}(a_{21} - a_{12}e^{-2sr_{12}})}, \\ ||\psi_2||^2 &= \frac{a_{21} - a_{12}e^{-2sr_{12}}}{-\bar{z}_{12}a_{12}a_{21}(1 - e^{-2sr_{12}})}. \end{aligned} \tag{3.95}$$

The resulting Nahm solution from (3.91) is

$$
\begin{aligned}
T_1 &= i\begin{pmatrix} x_1^1 & -\frac{r_{12}}{2\sinh(sr_{12})} \\ -\frac{r_{12}}{2\sinh(sr_{12})} & x_2^1 \end{pmatrix},\\
T_2 &= i\begin{pmatrix} x_1^2 & \frac{ir_{12}}{2\sinh(sr_{12})} \\ \frac{-ir_{12}}{2\sinh(sr_{12})} & x_2^2 \end{pmatrix},\\
T_3 &= i\begin{pmatrix} x_1^3 & 0 \\ 0 & x_2^3 \end{pmatrix} + \\
&\begin{pmatrix} \frac{x_{12}^3 r_{12}\sinh(sr_{12})\cosh(sr_{12}) - (x_{12}^3)^2\sinh^2(sr_{12}) - r_{12}^2}{\sinh(sr_{12})\left(x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})\right)} & \frac{-r_{12}z_{12}}{x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})} \\ \frac{-r_{12}\overline{z_{12}}}{x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})} & \frac{x_{12}^3\sinh^2(sr_{12}) - x_{12}^3 r_{12}\sinh(sr_{12})\cosh(sr_{12}) + r_{12}^2}{\sinh(sr_{12})\left(x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})\right)} \end{pmatrix},\\
T_0 &= \begin{pmatrix} 0 & \frac{e^{-sr_{12}}r_{12}^2\left(x_{12}^3\cosh(sr_{12}) - r_{12}\sinh(sr_{12})\right)}{2\bar{z_{12}}\left(x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})\right)} \\ \frac{-e^{-sr_{12}}r_{12}^2\left(x_{12}^3\cosh(sr_{12}) - r_{12}\sinh(sr_{12})\right)}{2z_{12}\left(x_{12}^3\sinh(sr_{12}) - r_{12}\cosh(sr_{12})\right)} & 0 \end{pmatrix}.
\end{aligned}
\tag{3.96}
$$

When taking the same points $(0,0,c/2)$ and $(0,0,-c/2)$ of $\mathbb{R}^3$ as the boundary conditions (1.9) for $s \to \infty$, the above solution is

$$
\begin{aligned}
T_1 &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{-c}{2\sinh(cs)},\\
T_2 &= i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{-c}{2\sinh(cs)},\\
T_3 &= i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{c}{2\tanh(cs)}.
\end{aligned}
\tag{3.97}
$$

Note this fails to satisfy the boundary conditions (3.92) at $s = 0$, indeed the procedure described in the thesis is only guaranteed to construct a solution that is some gauge transform of the unique solution to the prescribed boundary conditions. Here, the gauge taking our solution to the one in (3.93) is simply $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

3.5.2 Perturbative Expansion of the n=3 Solution

We give the perturbative solution for large s to Nahm's equations for $n = 3$, up to the first order. In Example 3.3, we found the perturbative basis of

$L^s_S(n-1)$, up to the first order, in terms of rows of polynomials of degree ≤ 2.

The perturbation expansion for large s of a Nahm solution is an expansion with the form

$$T_j(s) = T_j^{(0)} + e^{-s\Delta_j} T_j^{(1)} + e^{-s\Delta'_j} T_j^{(2)} + \dots \tag{3.98}$$

with $0 < \Delta_j < \Delta'_j < \Delta''_j < \dots$. The matrices $T_j^{(k)}$ are independent of s.

We will use the following notation. For the point $\vec{x}_j = (x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3$, we write

$$z_j := x_j^1 + i x_j^2, \quad x_j := x_j^3, \quad z_{jk} := z_j - z_k,$$

and

$$r_{jk} := ||\vec{x}_j - \vec{x}_k||, \quad a_{jk} := \frac{x_j^3 - x_k^3 + r_{jk}}{-\bar{z}_{jk}}.$$

The perturbation expansion for large s of Q to the first order is

$$Q(s,\zeta) = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix},$$

with

$$\begin{aligned}
Q_{11} &= (\zeta - a_{12})(\zeta - a_{13}), \\
Q_{12} &= e^{-sr_{12}}(a_{21} - a_{12})(a_{21} - a_{13}) \frac{\zeta}{a_{21}} \frac{\zeta - a_{23}}{a_{21} - a_{23}}, \\
Q_{13} &= e^{-sr_{13}}(a_{31} - a_{12})(a_{31} - a_{13}) \frac{\zeta}{a_{31}} \frac{\zeta - a_{32}}{a_{31} - a_{32}}, \\
Q_{21} &= e^{-sr_{12}}(a_{12} - a_{21})(a_{12} - a_{23}) \frac{\zeta - a_{13}}{a_{12} - a_{13}}, \\
Q_{22} &= (\zeta - a_{21})(\zeta - a_{23}), \\
Q_{23} &= e^{-sr_{23}}(a_{32} - a_{21})(a_{32} - a_{23}) \frac{\zeta}{a_{32}} \frac{\zeta - a_{31}}{a_{32} - a_{31}}, \\
Q_{31} &= e^{-sr_{13}}(a_{13} - a_{31})(a_{13} - a_{32}) \frac{\zeta - a_{12}}{a_{13} - a_{12}}, \\
Q_{32} &= e^{-sr_{23}}(a_{23} - a_{31})(a_{23} - a_{32}) \frac{\zeta - a_{21}}{a_{23} - a_{21}}, \\
Q_{33} &= (\zeta - a_{31})(\zeta - a_{32}).
\end{aligned} \tag{3.99}$$

We must normalize Q so that each row has norm 1 with respect to the norm (3.37). The norm of the row $\psi_j(s,\zeta) = (Q_{j1}, Q_{j2}, Q_{j3})$ for $1 \le j \le 3$, to the first order, is

$$\begin{aligned} ||\psi_1|| &= \frac{1}{\sqrt{\bar{z}_{12}a_{21}\bar{z}_{13}a_{31}}}, \\ ||\psi_2|| &= \frac{1}{\sqrt{-\bar{z}_{12}a_{12}\bar{z}_{23}a_{32}}}, \\ ||\psi_3|| &= \frac{1}{\sqrt{\bar{z}_{13}a_{13}\bar{z}_{23}a_{23}}}. \end{aligned} \tag{3.100}$$

The Lax pair (L, M) obtained from the normalized Q via (3.90), evaluated at $\zeta = 0$, is the following:

$$L^N(s,0) = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix},$$

with

$$\begin{aligned} L_{11} &= z_1, \\ L_{21} &= e^{-sr_{12}}\sqrt{-\bar{z}_{12}a_{21}\bar{z}_{13}a_{31}\bar{z}_{12}a_{12}\bar{z}_{23}a_{32}} \\ &\qquad \left(\frac{z_1 z_{23}a_{13}a_{31}(a_{21}-a_{12})(a_{12}-a_{23})}{z_{12}z_{13}z_{23}a_{31}(a_{12}-a_{13})} - \frac{z_2 z_{13}a_{23}a_{32}(a_{21}-a_{12})(a_{12}-a_{31})}{z_{12}z_{13}z_{23}a_{31}(a_{12}-a_{32})}\right), \\ L_{22} &= z_2, \end{aligned}$$

$$\begin{aligned} L_{31} &= e^{-sr_{13}}\sqrt{\bar{z}_{12}a_{21}\bar{z}_{13}a_{13}\bar{z}_{13}a_{31}\bar{z}_{23}a_{23}}\left(\frac{z_1 a_{12}(a_{13}-a_{31})(a_{13}-a_{32})}{z_{12}z_{13}(a_{12}-a_{13})}\right. \\ &\qquad\qquad \left. - \frac{z_3 a_{23}a_{32}(a_{13}-a_{31})(a_{13}-a_{21})}{z_{13}z_{23}a_{21}(a_{13}-a_{23})}\right), \\ L_{32} &= e^{-sr_{23}}\sqrt{-\bar{z}_{12}a_{12}\bar{z}_{13}a_{13}\bar{z}_{23}a_{23}\bar{z}_{23}a_{32}}\,\frac{a_{23}-a_{32}}{z_{12}z_{13}z_{23}a_{12}(a_{13}-a_{23})(a_{23}-a_{21})}, \\ L_{33} &= z_3. \end{aligned}$$

$$M(s,0) = \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{33} \end{pmatrix},$$

with

$$\begin{aligned} M_{11} &= x_1, \\ M_{21} &= e^{-sr_{12}}\sqrt{-\bar{z}_{12}a_{12}\bar{z}_{12}a_{21}\bar{z}_{13}a_{31}\bar{z}_{23}a_{32}}(a_{12}-a_{21}) \\ &\left(\frac{x_2 z_{13}a_{32}a_{23}(a_{12}-a_{31})(a_{12}-a_{13}) - (r_{12}+x_1)z_{23}a_{13}a_{31}(a_{12}-a_{23})(a_{12}-a_{32})}{z_{12}z_{13}z_{23}a_{31}(a_{12}-a_{13})(a_{12}-a_{32})}\right), \end{aligned}$$

$$M_{22} = x_2,$$

$$M_{31} = e^{-sr_{13}}\sqrt{\bar{z}_{12}a_{21}\bar{z}_{13}a_{13}\bar{z}_{13}a_{31}\bar{z}_{23}a_{23}}\left(\frac{(x_1+r_{13})z_{23}a_{12}a_{21}(a_{13}-a_{32})}{z_{12}z_{13}z_{23}a_{21}(a_{12}-a_{13})}\right.$$
$$\left.-\frac{x_3z_{12}a_{23}a_{32}(a_{13}-a_{21})}{z_{12}z_{13}z_{23}a_{21}(a_{13}-a_{23})}\right),$$

$$M_{32} = e^{-sr_{23}}\sqrt{-\bar{z}_{12}a_{12}\bar{z}_{13}a_{13}\bar{z}_{23}a_{23}\bar{z}_{23}a_{32}}(a_{23}-a_{32})$$
$$\left(\frac{x_3z_{12}a_{13}a_{31}(a_{12}-a_{23})(a_{21}-a_{23})+(r_{23}+x_2)z_{13}a_{12}a_{21}(a_{13}-a_{23})(a_{23}-a_{31})}{z_{12}z_{13}z_{23}a_{12}(a_{21}-a_{23})(a_{23}-a_{13})}\right),$$

$$M_{33} = x_3. \tag{3.101}$$

From $L^N(s,0) = T_2 - iT_1$ and $M^N(s,0) = T_0 - iT_3$, we obtain the following Nahm solutions:

$$\begin{aligned}
T_1(s) &= \frac{i}{2}\begin{pmatrix} z_1+\bar{z}_1 & \bar{L}_{21} & \bar{L}_{31} \\ L_{21} & z_2+\bar{z}_2 & \bar{L}_{32} \\ L_{31} & L_{32} & z_3+\bar{z}_3 \end{pmatrix} + O\left(e^{-\alpha s}\right), \\
T_2(s) &= \frac{1}{2}\begin{pmatrix} z_1-\bar{z}_1 & -\bar{L}_{21} & -\bar{L}_{31} \\ L_{21} & z_2-\bar{z}_2 & -\bar{L}_{32} \\ L_{31} & L_{32} & z_3-\bar{z}_3 \end{pmatrix} + O\left(e^{-\alpha s}\right), \\
T_3(s) &= \frac{i}{2}\begin{pmatrix} 2x_1 & \bar{M}_{21} & \bar{M}_{31} \\ M_{21} & 2x_2 & \bar{M}_{32} \\ M_{31} & M_{32} & 2x_3 \end{pmatrix} + O\left(e^{-\alpha s}\right), \\
T_0(s) &= \frac{1}{2}\begin{pmatrix} 0 & -\bar{M}_{21} & -\bar{M}_{31} \\ M_{21} & 0 & -\bar{M}_{32} \\ M_{31} & M_{32} & 0 \end{pmatrix} + O\left(e^{-\alpha s}\right),
\end{aligned} \tag{3.102}$$

where α is the minimum of the values $2r_{12}$, $2r_{13}$, and $2r_{23}$.

Chapter 4

Dirac Monopole and ADHMN

4.1 Introduction

We address the Nahm transform for the Dirac multimonopole in this chapter. The Nahm transform may be thought of as a nonabelian generalization of the Fourier transform sending a (reduced) self-dual pair (E, A), consisting of a Hermitian bundle $E \to X$ over a manifold $X = \mathbb{R}^4/\Lambda$ with a connection A on it, to a (reduced) self-dual pair $(\hat{E}, \hat{A})$ over a dual space $X^* = \mathbb{R}^4/\Lambda^*$ [21] [71]. The four-dimensional space for the self-duality equations that concerns us is $\mathbb{R}^4$, and the reductions to SD are from imposing invariance under some abelian subgroup $\Lambda \subset \mathbb{R}^4$. We then have $X = \mathbb{R}^4/\Lambda$ and $X^* = \mathbb{R}^4/\Lambda^*$. For a more general setting of the transform, see [34].

The first major result in this area was the ADHM construction of instantons on $\mathbb{R}^4$ [30]. In that setting, the ADHM construction may be thought of as the case $\Lambda = \{0\}$, in which case $\Lambda^* = \mathbb{R}^4$. Monopoles and Nahm's equations arise from the choices $\Lambda = \mathbb{R}$ and $\Lambda^* = \mathbb{R}^3$, giving the following picture.

$$\text{Monopole} \xleftrightarrow{\text{Nahm Transform}} \text{Nahm Solution.}$$

The Nahm transform starts with a self-dual connection A on a Hermitian bundle $E \to X$. Let S be the spin bundle over X, we have Dirac operators $\mathcal{D}^\dagger_{x^*}$ for a family of connections of A twisted by $x^* \in X^*$. The resulting family of spaces of L^2 zero modes of $\mathcal{D}^\dagger_{x^*}$ over the parameters of X^* form a vector bundle $\hat{E} \to X^*$ forming a subbundle in $\underline{L^2(X, S \otimes E)}$, the trivial infinite rank bundle with fiber the vector space of L^2 sections of $S \otimes E$. Projecting the trivial connection of $\underline{L^2(X, S \otimes E)}$ on $\hat{E}$ gives the self-dual connection $\hat{A}$ [72, Section 4].

The crucial ingredient of the transform is then index theory, namely the index of $\mathcal{D}^\dagger_{x^*}$. In our case, there are no normalizable zero modes of $\mathcal{D}_{x^*}$ and so the number of zero modes of $\mathcal{D}^\dagger_{x^*}$ equals the Dirac index. To perform the Nahm transform, it is required to compute the zero modes.

Only a few explicit results for zero modes are known for single monopoles [73] and double monopoles [74]. For generic n point configurations of the Dirac multimonopole, Lamy-Poirier in [49] presented a general formula for the zero modes in terms of a finite set of algebraic equations. We prove that this system is equivalent to a set of matching conditions on n many polynomials of degree $n-1$ and we present its solution. In doing so, we complete the programme of finding exact solutions for the zero modes of the generic Dirac multimonopole.

4.2 Multimonopole Configuration

For a compact Lie group G and its Lie algebra $\mathfrak{g} = Lie(G)$, a monopole is a principal G-bundle E over a 3-dimensional Riemannian manifold M and a pair (A, Φ) of a $\mathfrak{g}$-valued connection A and a $\mathfrak{g}$-valued section Φ of the associated Lie algebra bundle $ad(E)$ satisfying Bogomolny's equation

$$\boldsymbol{D}\Phi = *F_A, \tag{4.1}$$

where $F_A = dA + A \wedge A$ is the curvature form of the connection form A, $\boldsymbol{D}\Phi = d\Phi + [A, \Phi]$ is the differential, and $*$ is the Hodge star.

For the Dirac monople with a singularity at the origin of charge m, the principal G-bundle is $E = P^* H^m$ the pullback of the Hopf bundle H^m of degree m over the sphere S^2 under the projection $\mathbb{R}^3 \setminus \{0\} \cong S^2 \times \mathbb{R}_+ \xrightarrow{P} S^2$. Then the Dirac multimonopole over $\mathbb{R}^3$ with singularities at $\vec{a}_k$ and respective charges m_k for $k = 1, \ldots, n$ has base manifold $\mathbb{R}^3 \setminus \{\vec{a}_k \mid k = 1, \ldots, n\}$ and the principal G-bundle is $E = \bigotimes_{k=1}^{n} (i \circ P)^*_{\vec{a}_k} H^{m_k}$ with $(i \circ P)^*_{\vec{a}_k} H^{m_k}$ the pullback of the Hopf bundle of degree m_k under the map

$$\mathbb{R}^3 \setminus \{\vec{a}_k \mid k = 1, \ldots, n\} \xhookrightarrow{i} \mathbb{R}^3 \setminus \vec{a}_k \cong S^2 \times \mathbb{R}_+ \xrightarrow{P} S^2.$$

The gauge group $G = U(1)$ is abelian, so the Bogomolny equation is linear, and a multimonopole (A, Φ) is a superposition of single monopoles. We use the coordinates $z := x_1 + ix_2$ and $x := x^3$ for $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ with the volume form $dx^1 \wedge dx^2 \wedge dx^3 = \frac{i}{2} dz \wedge d\bar{z} \wedge dx$. For a multimonopole configuration of n point monopoles of unit charge and distinct locations $\vec{a}_k,\ k = 1, \ldots, n$ in $\mathbb{R}^3$ the pair (A, Φ) can be written as [73]

$$\Phi(x) = \sum_{1}^{n} \frac{i}{2r_k}, \quad A(x) = \sum_{k=1}^{n} \frac{z_k d\bar{z}_k - \bar{z}_k dz_k}{4r_k(r_k + x_k)}, \tag{4.2}$$

where for the vector $\vec{x}_k = \vec{x} - \vec{a}_k$ we set $r_k := |\vec{x}_k|,\ z_k := x_k^1 + ix_k^2$, and $x_k := x_k^3$.

To avoid pathologies in the gauge, we rotate the monopole configuration in $\mathbb{R}^3$, if necessary, so that no two monopoles are separated by a translation in the x^3 direction.

4.3 Dirac Operators

4.3.1 Dirac Operators Coupled to a Monopole

Let S be the spin bundle over $\mathbb{R}^3$ with chiral decomposition $S = S^+ \oplus S^-$. With respect to this decomposition, the monopole Dirac operator $\not{\mathcal{D}} : \Gamma(S \otimes E) \to \Gamma(S \otimes E)$ coupled to the gauge connection A and Higgs field Φ for the bundle E has the form

$$\not{\mathcal{D}} = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix},$$

with

$$\mathcal{D}^\dagger = -\sum_{j=1}^{3} \sigma_j \otimes D_j - \mathbb{1} \otimes i\Phi, \quad \mathcal{D} = \sum_{j=1}^{3} \sigma_j \otimes D_j - \mathbb{1} \otimes i\Phi, \tag{4.3}$$

where σ_j are the Pauli sigma matrices[1] and $D_j = \frac{\partial}{\partial x^j} + A_j$ are the components of the covariant derivative $\boldsymbol{D}$.

We consider the L^2 kernel of $\mathcal{D}^\dagger$ in the L^2 subspace of functions from $S^- \otimes E \to S^+ \otimes E$.

[1] The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and satisfies $[\sigma_1, \sigma_2] = 2i\sigma_3$, $[\sigma_2, \sigma_3] = 2i\sigma_1$, and $[\sigma_3, \sigma_1] = 2i\sigma_2$.

Definition 4.1. Define the *normalizable zero mode* $\binom{\psi}{\phi}$ of the monopole Dirac operators to consist of spinors $\psi \in L^2(S^- \otimes E)$ and $\phi \in L^2(S^+ \otimes E)$ satisfying

$$\mathcal{D}^\dagger \psi = 0, \quad \mathcal{D}\phi = 0. \tag{4.4}$$

The operator $\mathcal{D}^\dagger \mathcal{D}$ is

$$\mathcal{D}^\dagger \mathcal{D} = \mathbb{1} \otimes \left(-\Phi^2 - \sum_{j=1}^{3} D_j^2 \right) + i\sigma_1 \otimes ([D_3, D_2] + [D_1, \Phi]) + \\ i\sigma_2 \otimes ([D_1, D_3] + [D_2, \Phi]) + i\sigma_3 \otimes ([D_2, D_1] + [D_3, \Phi]) \,. \tag{4.5}$$

Vanishing of the pure quaternion part of $\mathcal{D}^\dagger \mathcal{D}$ is equivalent to (A, Φ) satisfying the Bogomolny equation (4.1). For a monopole (A, Φ), the operator $\mathcal{D}^\dagger \mathcal{D} = \mathbb{1} \otimes (-\Phi^2 - \sum_{j=1}^{3} D_j^2)$ is a positive operator since

$$\langle \mathcal{D}^\dagger \mathcal{D} \chi, \chi \rangle = ||\Phi \chi||^2 + \sum_{j=1}^{3} ||D_j \chi||, \tag{4.6}$$

so there are no normalizable zero modes of $\mathcal{D}$.

4.3.2 Dirac Operator Coupled to a Nahm Solution

The Nahm Dirac operator $\not{\mathfrak{D}} : \Gamma(S \otimes \hat{E}) \to \Gamma(S \otimes \hat{E})$ coupled to the Nahm data (T_0, T_1, T_2, T_3) for the hermitian bundle $\hat{E}$ of rank n over the interval $(0, \infty)$ has the form

$$\not{\mathfrak{D}} = \begin{pmatrix} 0 & \mathfrak{D} \\ \mathfrak{D}^\dagger & 0 \end{pmatrix},$$

with $\mathfrak{D}^\dagger : L^2(S^- \otimes \hat{E}) \to H^{-1}(S^+ \otimes \hat{E})$ and $\mathfrak{D} : H^1(S^+ \otimes \hat{E}) \to L^2(S^- \otimes \hat{E})$. Here,

$$\mathfrak{D}^\dagger = i\frac{d}{ds} + iT_0 - \sum_{j=1}^{3} \sigma_j \otimes T_j, \quad \mathfrak{D} = i\frac{d}{ds} + iT_0 + \sum_{j=1}^{3} \sigma_j \otimes T_j. \tag{4.7}$$

Definition 4.2. The *normalizable zero mode* of the Nahm Dirac operator is $\binom{\psi}{\phi}$ with spinors $\psi \in L^2(S^- \otimes \hat{E})$ and $\phi \in L^2(S^+ \otimes \hat{E})$ satisfying

$$\mathfrak{D}^\dagger \psi = 0, \quad \mathfrak{D}\phi = 0. \tag{4.8}$$

4.3.3 Zero Modes of Dirac Operators

We use an observation of Nahm [8] that the commuting pair of operators in (1.7) may be used to discover zero modes of the Dirac operators. Here, we follow the presentation of Braden and Enolski [65].

The Bogomolny pair $(\mathcal{L}, \mathcal{M})$ of (1.7) in our self-dual conventions for monopoles becomes

$$\begin{aligned} \mathcal{L}^N &= D_1 + iD_2 - (D_3 + i\Phi)\zeta, & \mathcal{M}^N &= D_3 - i\Phi + (D_1 - iD_2)\zeta, \\ \mathcal{L}^S &= -(D_1 - iD_2) + (-D_3 + i\Phi)\frac{1}{\zeta}, & \mathcal{M}^S &= -D_3 - i\Phi + (D_1 + iD_2)\frac{1}{\zeta}, \end{aligned} \tag{4.9}$$

and Bogomolny's equation is equivalent to $[\mathcal{L}, \mathcal{M}] = 0$ for all ζ, as in Equation (1.7).

Definition 4.3. The spinor χ is parallel for the pair $(\mathcal{L}, \mathcal{M})$ if it satisfies the associated equation

$$\mathcal{L}\chi = 0, \quad \mathcal{M}\chi = 0. \tag{4.10}$$

Note that $\mathcal{L}^S = -\frac{\mathcal{M}^N}{\zeta}$ and $\mathcal{M}^S = \frac{\mathcal{L}^N}{\zeta}$, so any spinor parallel for $(\mathcal{L}^N, \mathcal{M}^N)$ is also parallel for $(\mathcal{L}^S, \mathcal{M}^S)$ and vice versa.

Example 4.1. The parallel spinor $\chi = (\chi^N, \chi^S)$ for the trivial monopole $(A, \Phi) = (0, is)$ is

$$\chi^N = e^{-s(x+\zeta\bar{z})}, \quad \chi^S = e^{s(x-\frac{z}{\zeta})}. \tag{4.11}$$

Proof. In North patch, χ solves

$$\begin{aligned} 0 =& \mathcal{L}^N\chi^N = \left(2D_{\bar{z}} - (D_3 + i\Phi)\zeta\right)\chi^N, \\ 0 =& \mathcal{M}^N\chi^N = \left(D_3 - i\Phi + 2D_z\zeta\right)\chi^N. \end{aligned}$$

If we assume χ^N is independent of z then χ^N is holomorphic in ζ. Indeed in this case,

$$\begin{aligned} (\mathcal{L}^N + \zeta M^N)\chi^N &= 0, \\ \frac{\partial_{\bar{z}}\chi^N}{\chi^N} &= -s\zeta, \end{aligned}$$

so that $\chi^N = C(x_3, \zeta)e^{-s\zeta\bar{z}}$. Then

$$\mathcal{M}^N \chi^N = 0,$$
$$\frac{\partial_3 \chi^N}{\chi^N} = -s,$$

leading to $\chi^N = C(\zeta)e^{-s(x_3+\zeta\bar{z})}$. Assuming χ^S is independent of $\bar{z}$ leads to χ^S being holomorphic in $\frac{1}{\zeta}$ and a similar computation applies. □

The monopole Dirac operator $\mathcal{D}$ is related to $(\mathcal{L}, \mathcal{M})$ via the following formula:

$$\begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \mathcal{D} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathcal{M}^N \\ \mathcal{L}^N \end{pmatrix}, \quad \begin{pmatrix} -1 & 1/\zeta \\ 0 & 1 \end{pmatrix} \mathcal{D} \begin{pmatrix} 1/\zeta \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{L}^S \\ \mathcal{M}^S \end{pmatrix}. \tag{4.12}$$

One might think this relationship is not so useful since it is stated for the Dirac operator $\mathcal{D}$ with no normalizable zero modes, rather than for the desired $\mathcal{D}^\dagger$. However, Nahm observed the following fact.

Lemma 4.1 (Nahm [8])**.** *Let χ be a parallel section of the Bogomolny pair $(\mathcal{L}, \mathcal{M})$ of commuting holomorphic operators, then the spinors*

$$\begin{pmatrix} 1 \\ \zeta \end{pmatrix} \chi(x, s, \zeta), \ \begin{pmatrix} \frac{1}{\zeta} \\ 1 \end{pmatrix} \chi(x, s, \zeta) \tag{4.13}$$

are (non-normalizable) zero modes of $\mathcal{D}$ and the spinors

$$\mathcal{D} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \chi(x, s, \zeta), \quad \mathcal{D} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \chi(x, s, \zeta), \tag{4.14}$$

are (perhaps, non-normalizable) zero modes of $\mathcal{D}^\dagger$. Here, we may use χ^N and χ^S interchangeably for χ.

Proof. We have $\begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix} \mathcal{D} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \otimes \chi^N = \begin{pmatrix} \mathcal{M}^N \chi^N \\ \mathcal{L}^N \chi^N \end{pmatrix} = 0$. The matrix $\begin{pmatrix} 1 & 0 \\ -\zeta & 1 \end{pmatrix}$ is invertible so $\mathcal{D} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \chi^N(x, s, \zeta) = 0$.

This implies

$$\mathcal{D}^\dagger \mathcal{D} \begin{pmatrix} \chi^N(x, s, \zeta) \\ \zeta \chi^N(x, s, \zeta) \end{pmatrix} = 0.$$

However, (A, Φ) solves the Bogomolny equation so that $\mathcal{D}^\dagger\mathcal{D}$ is a diagonal operator, i.e. $\mathcal{D}^\dagger\mathcal{D} = \mathbb{1}\otimes(-\Phi^2 - \sum_{j=1}^{3} D_j^2)$. We conclude that $\nabla^*\nabla\chi^N(x,s,\zeta) = 0$ so $\chi^N(x,s,\zeta)$ is a harmonic function, giving that $\mathcal{D}^\dagger\mathcal{D}\begin{pmatrix}1\\0\end{pmatrix}\chi^N(x,s,\zeta)$ is zero. A similar argument applies to the remaining spinors. □

The story for the zero modes of the Nahm Dirac operators is similar to the monopole Dirac operators, with some differences. In fact, the basis of sections of $L_S^s(n-1)$ may be used to give zero modes to both $\mathfrak{D}$ and $\mathfrak{D}^\dagger$ in a straight-forward manner.

The Nahm Dirac operator $\mathfrak{D}$ is related to the Nahm Lax pair (L, M) of (3.55) via the same formula:

$$\begin{pmatrix}1 & 0\\ -\zeta & 1\end{pmatrix}\mathfrak{D}\begin{pmatrix}1\\ \zeta\end{pmatrix} = \begin{pmatrix}M^N\\ L^N\end{pmatrix}, \quad \begin{pmatrix}-1 & 1/\zeta\\ 0 & 1\end{pmatrix}\mathfrak{D}\begin{pmatrix}1/\zeta\\ 1\end{pmatrix} = \begin{pmatrix}L^S\\ M^S\end{pmatrix}. \tag{4.15}$$

We also have $\mathfrak{D}^\dagger\mathfrak{D} = -(\frac{d}{ds}+T_0)^2 - \sum_{j=1}^{3} T_j^2$ so $\mathfrak{D}^\dagger\mathfrak{D}$ is a positive operator with no normalizable solutions to $\mathfrak{D}\chi = 0$.

Lemma 4.2 (Nahm [8])**.** *Let $U_j(s,\zeta)$ be a solution to the Lax linear problem associated to (L, M) for the eigenvalue $p_j(\zeta)$ of L^N. Let a_{j0} and a_{0j} be the two roots of $p_j(\zeta)$. Then the spinors*

$$\begin{pmatrix}1\\ a_{j0}\end{pmatrix}\otimes U_j^N(s, a_{j0}), \quad \begin{pmatrix}1\\ a_{0j}\end{pmatrix}\otimes U_j^N(s, a_{0j})$$

are (non-normalizable) zero modes of $\mathfrak{D}$.

Proof. We have

$$\begin{pmatrix}1 & 0\\ -\zeta & 1\end{pmatrix}\mathfrak{D}\begin{pmatrix}1\\ \zeta\end{pmatrix}\otimes U_j^N = \begin{pmatrix}M^N U_j^N\\ L^N U_j^N\end{pmatrix} = \begin{pmatrix}0\\ p_j(\zeta)U_j^N\end{pmatrix}.$$

Evaluating this at $\zeta = a_{j0}$ and $\zeta = a_{0j}$ gives zero. The matrix $\begin{pmatrix}1 & 0\\ -\zeta & 1\end{pmatrix}$ is invertible for all values of ζ so this proves the lemma. □

Lemma 4.3 (Nahm [8])**.** *Given a basis U of $H^0(S, L_S^s(n-1))$. The $2n\times 2n$ fundamental matrices W and V of solutions to*

$$\mathfrak{D}W = 0, \quad \mathfrak{D}^\dagger V = 0,$$

are given by W *the collection of spinors* $\begin{pmatrix} U_j^N(s,a_{j0}) \\ a_{j0}U_j^N(s,a_{j0}) \end{pmatrix}$, $\begin{pmatrix} U_j^N(s,a_{0j}) \\ a_{0j}U_j^N(s,a_{0j}) \end{pmatrix}$ *for* $1 \leq j \leq n$, *and by* $V = (W^\dagger)^{-1}$.

Proof. Lemma 4.2 gives the stated form of W. V is easy to see, but we state the proof for completeness. $\mathfrak{D}W = 0$ states

$$i\frac{dW}{ds} + iT_0W + \sum_{j=1}^{3} \sigma_j \otimes T_jW = 0.$$

Take the transpose of the above equation to obtain

$$-i\frac{dW^\dagger}{ds} + iW^\dagger T_0 - W^\dagger \sum_{j=1}^{3} \sigma_j \otimes T_j = 0.$$

Conjugating both sides of the equation by $W^{\dagger^{-1}}$ gives

$$-iW^{\dagger^{-1}}\frac{dW^\dagger}{ds}W^{\dagger^{-1}} + iT_0W^{\dagger^{-1}} - \sum_{j=1}^{3} \sigma_j \otimes T_jW^{\dagger^{-1}} = 0.$$

Since $\frac{d(W^{\dagger^{-1}})}{ds} = -W^{\dagger^{-1}}\frac{dW^\dagger}{ds}W^{\dagger^{-1}}$, the above equation states $\mathfrak{D}^\dagger W^{\dagger^{-1}} = 0$ and the proof is complete. □

4.3.4 Down Transform

The Down Transform of the ADHMN is the map

$$\text{Monopole} \xrightarrow{\text{Down Transform}} \text{Nahm Solution}.$$

The Bogomolny equation for $\mathbb{R}^3 = \mathbb{R}^4/\mathbb{R}$ is the dimensional reduction of ASD by imposing invariance under the subgroup $\mathbb{R}$ of shifts of $\mathbb{R}^4$. The parameter s of $\mathbb{R}$ parametrize the trivial solutions $(A, \Phi) = (0, is)$ to the Bogomolny equation (4.1) for the trivial line bundle L over $\mathbb{R}^3$. The Down Transform of the ADHMN construction is carried out by twisting the monopole Dirac operators (4.3) with this parameter s to obtain $\mathcal{D}_s^\dagger : H^{-1}(S^-\otimes E\otimes L) \to L^2(S^+ \otimes E \otimes L)$ and $\mathcal{D}_s : L^2(S^+ \otimes E \otimes L) \to H^1(S^- \otimes E \otimes L)$ as follows:

$$\mathcal{D}_s^\dagger = -\sum_{j=1}^{3} \sigma_j \otimes D_j - i\Phi + s, \quad \mathcal{D}_s = \sum_{j=1}^{3} \sigma_j \otimes D_j - i\Phi + s. \tag{4.16}$$

After choosing an orthonormal basis $\Psi(x,s)$ of L^2 zero modes to $\mathcal{D}_s^\dagger$, the matrix-valued functions

$$T_i(s) = -i\int_{\mathbb{R}^3} d^3x \Psi^\dagger x^i \Psi,\ i=1,2,3\ ,\ T_0(s) = \int_{\mathbb{R}^3} dx^3 \Psi^\dagger \frac{\partial}{\partial s}\Psi \qquad (4.17)$$

form a solution to Nahm's equations (2.6) [7, pp.7].

Recall from Section 4.3 that parallel spinors for the Bogomolny pair $(\mathcal{L},\mathcal{M})$ may be used to give zero modes of $\mathcal{D}$ and $\mathcal{D}^\dagger$. In the case of the Dirac monopole, the gauge group $U(1)$ is abelian and because of this, superpositions of solutions (A,Φ) lead to superpositions in $(\mathcal{L},\mathcal{M})$. To be precise, the Bogomolny pair $(\mathcal{L},\mathcal{M})$ in North patch for a superposition $(A^1+A^2,\Phi^1+\Phi^2)$ is

$$\mathcal{L}^N = \frac{\partial}{\partial x_1} + A_1^1 + A_1^2 + i\frac{\partial}{\partial x_2} + iA_2^1 + iA_2^2 + \left(-\frac{\partial}{\partial x_3} - A_3^1 - A_3^2 - i\Phi_1 - i\Phi_2\right)\zeta,$$
$$\mathcal{M}^N = \frac{\partial}{\partial x_3} + A_3^1 + A_3^2 - i\Phi_1 - i\Phi_2 + \left(\frac{\partial}{\partial x_1} + A_1^1 + A_1^2 - i\frac{\partial}{\partial x_2} - iA_2^1 - iA_2^2\right)\zeta.$$

Thus, parallel χ of a superposition of $\{(\mathcal{L}_i,\mathcal{M}_i) \mid i=1,\dots,n\}$ is simply the product of parallel χ_i of its constituents,

$$\chi = \prod_{i=1}^{n}\chi_i. \qquad (4.18)$$

We need to twist the operators for the ADHMN construction. The twisted pair $(\mathcal{L}_s,\mathcal{M}_s)$ of $(\mathcal{L},\mathcal{M})$ corresponding to the twisted Bogomolny equation is the superposition $(A,\Phi-is)$ of (A,Φ) with the trivial solution $(0,-is)$,

$$\mathcal{L}_s = \mathcal{L},\quad \mathcal{M}_s = \mathcal{M} + s. \qquad (4.19)$$

The parallel spinor for the trivial solution is $(e^{s(x+\zeta\bar{z})}, e^{-s(x-\frac{z}{\zeta})})$ so for parallel (χ^N,χ^S) of $(\mathcal{L},\mathcal{M})$, the parallel spinor for the twisted operators $(\mathcal{L}_s,\mathcal{M}_s)$ is then the product

$$\chi_s^N = e^{s(x+\zeta\bar{z})}\chi^N,\quad \chi_s^S = e^{-s(x-\frac{z}{\zeta})}\chi^S. \qquad (4.20)$$

These, however, are not normalizable. We will discuss in Section 4.5 the construction of the *normalizable* zero modes of $\mathcal{D}_s^\dagger$ necessary for the Down Transform.

4.3.5 Up Transform

The Up Transform of the ADHMN is the map

$$\text{Monopole} \xleftarrow{\text{Up Transform}} \text{Nahm Solution.}$$

Nahm's equations over $\mathbb{R} = \mathbb{R}^4/\mathbb{R}^3$ is the dimensional reduction of ASD by imposing invariance under the subgroup $\mathbb{R}^3$ of $\mathbb{R}^4$. The parameter (x_1, x_2, x_3) of $\mathbb{R}^3$ parametrize the trivial solutions $(T_1, T_2, T_3) = (ix_1, ix_2, ix_3)$ to Nahm's equation (2.1) with the trivial bundle I over the interval $(0, \infty)$.

The Up Transform of the ADHMN construction is carried out by twisting the Nahm Dirac operators (4.7) with this parameter (x_1, x_2, x_3) to obtain $\mathfrak{D}^\dagger_{\vec{x}} : L^2(S^- \otimes \hat{E} \otimes I) \to H^{-1}(S^+ \otimes \hat{E} \otimes I)$ and $\mathfrak{D}_{\vec{x}} : H^1(S^+ \otimes \hat{E} \otimes I) \to L^2(S^- \otimes \hat{E} \otimes I)$ where

$$\mathfrak{D}^\dagger_{\vec{x}} = i\frac{d}{ds} + iT_0 - \sum_{j=1}^{3} \sigma_j \otimes (T_j - ix_j), \quad \mathfrak{D}_{\vec{x}} = i\frac{d}{ds} + iT_0 + \sum_{j=1}^{3} \sigma_j \otimes (T_j - ix_j) \tag{4.21}$$

After choosing an orthonormal basis $\mathfrak{v}$ of L^2 zero modes to $\mathfrak{D}^\dagger_{\vec{x}}$, the functions

$$\Phi(x) = i\int ds\ s\mathfrak{v}^\dagger\mathfrak{v}, \qquad A_i(x) = \int ds \mathfrak{v}^\dagger \frac{\partial}{\partial x_i}\mathfrak{v}, \tag{4.22}$$

form a monopole solution [6].

In Section 3.4.2, we constructed a basis of sections of $H^0(L^s_S(n-1))$. Now, the zero modes of the twisted Dirac operators can be obtained from a basis $U(s,\zeta)$ of $H^(L^s_S(n-1))$ in the following way. The corresponding twisted Lax pair $(L_{\vec{x}}, M_{\vec{x}})$ is $L_{\vec{x}} = L - p_x(\zeta)$ and $M_{\vec{x}} = M - h_x(\zeta)$, for $p_x(\zeta)$ the twistor line section corresponding to $\vec{x}$ and $h_x(\zeta) = x + \zeta\bar{z}$. Note the eigenvalues of $L_{\vec{x}}$ are now $p_j(\zeta) - p_x(\zeta)$, with roots $\zeta = a_{jx}$ and $\zeta = a_{xj}$.

The basis of solutions to the twisted Lax linear problem in the North patch is then $e^{sh_x(\zeta)}U^N(s,\zeta)$. From Lemma 4.3, the fundamental matrix W of zero modes to $\mathfrak{D}_{\vec{x}}$ is the collection of spinors $\begin{pmatrix} e^{sh_x(a_{jx})}U^N_j(s,a_{jx}) \\ a_{jx}e^{sh_x(a_{jx})}U^N_j(s,a_{jx}) \end{pmatrix}$, $\begin{pmatrix} e^{sh_x(a_{xj})}U^N_j(s,a_{xj}) \\ a_{xj}e^{sh_x(a_{xj})}U^N_j(s,a_{xj}) \end{pmatrix}$ for $1 \le j \le n$, and $V = (W^\dagger)^{-1}$ is the fundamental matrix of zero modes to $\mathfrak{D}^\dagger_{\vec{x}}$.

4.4 Spectral Curve of Monopole Data

In this section, we discuss the spectral curve of the multimonopole data for the gauge group $U(1)$. We will follow the $U(1)$ analogue of the description of the singular $U(2)$ monopole in [75]. For any straight line

$$\gamma = \{\vec{x} \mid \vec{x} = \vec{\zeta} t + \vec{\eta}, \vec{\zeta}\cdot\vec{\zeta} = 1, \vec{\zeta}\cdot\vec{\eta} = 0\},$$

define

$$\begin{aligned} \gamma_+ &= \{\vec{x} \mid \vec{x} = \vec{\zeta} t + \vec{\eta}, t > R\}, \\ \gamma_- &= \{\vec{x} \mid \vec{x} = \vec{\zeta} t + \vec{\eta}, t < R\}, \end{aligned} \tag{4.23}$$

where R is a positive number far greater than the absolute value of the location of any singularity $|\vec{a}|$. Define two complex line bundles L^+ and L^- over $T\mathbb{P}^1$:

$$\begin{aligned} L^+ &= \{s \in \Gamma(\gamma_+, E) \mid D_\gamma s - i\Phi s = 0\}, \\ L^- &= \{s \in \Gamma(\gamma_-, E) \mid D_\gamma s - i\Phi s = 0\}. \end{aligned} \tag{4.24}$$

Bogomolny's equations imply these bundles are holomorphic, as in [66].

Let us find the solutions s of the scattering equations in (4.24) for the Dirac multimonopole. Let $\vec{\zeta}$ be the unit direction of the line γ so that $D_\gamma = \vec{\zeta}\cdot\vec{D}$. In terms of the North patch for $\mathbb{P}^1$, $\vec{\zeta} = \frac{1}{1+|\zeta|^2}(\zeta+\bar{\zeta}, i(\bar{\zeta}-\zeta), 1-|\zeta|^2)$ so that

$$D_{\vec{\zeta}} = \vec{\zeta}\cdot\vec{D} = \frac{\zeta+\bar{\zeta}}{1+|\zeta|^2}D_1 + \frac{i(\bar{\zeta}-\zeta)}{1+|\zeta|^2}D_2 + \frac{1-|\zeta|^2}{1+|\zeta|^2}D_3$$

is the covariant derivative along the line γ in the direction $\vec{\zeta}$. Observe that

$$\frac{1}{1+|\zeta|^2}\begin{pmatrix}1 & \bar{\zeta}\end{pmatrix}\mathcal{D}\begin{pmatrix}1\\ \zeta\end{pmatrix} = D_{\vec{\zeta}} - i\Phi. \tag{4.25}$$

From Lemma 4.1, we see that the parallel spinor $\chi = (\chi^N, \chi^S)$ gives us the solution to the scattering equations, where χ^N belongs to γ_+ and χ^S belongs to γ_-. For the unit charge Dirac monopole

$$\Phi(x) = \frac{i}{2r_k}, \quad A(x) = \frac{z_k d\bar{z}_k - \bar{z}_k dz_k}{4r_k(r_k + x_k)}, \tag{4.26}$$

one finds that

$$\chi^N(x,\zeta) = \frac{z_k - (r_k + x_k)\zeta}{(r_k + x_k)^{1/2}}, \quad \chi^S(x,\zeta) = \frac{(r_k + x_k)^{1/2}}{r_k + x_k + \bar{z}_k\zeta}. \tag{4.27}$$

By (4.18), the parallel spinor χ for the Dirac multimonopole is the product of the above spinors of its constituent monopoles.

Denote the total space of the twistor section corresponding to a point $\vec{x} \in \mathbb{R}^3$ by $P_{\vec{x}} := \{(\zeta, p_x(\zeta)) \mid \zeta \in \mathbb{P}^1\}$. Let S be the union in $T\mathbb{P}^1$ of all the twistor sections P_a corresponding to the singularities. This is, of course, the spectral curve of the monopole data in the specific case of $U(1)$, but this is not how the authors of [75] defined the spectral curve for the singular $U(2)$ monopole and we will find it useful to continue the analogy.

If γ does not pass through any singularity $\vec{a}$, then any solution s can be continued from γ_+ to γ_- and this defines an isomorphism

$$h : L^+|_{T\mathbb{P}^1\setminus S} \xrightarrow{\sim} L^-|_{T\mathbb{P}^1\setminus S}. \tag{4.28}$$

The Ward correspondence for the $U(1)$ monopole is then the analogue of [75, pp.5]: there is a bijection between $U(1)$ multimonopoles modulo gauge transformations and triplets (L^+, L^-, h) of holomorphic bundles over $T\mathbb{P}^1$ satisfying the following conditions.

(a) For any $\vec{x} \neq \vec{a}$, there is a splitting $P_x = P_x^+ \cup P_x^-$ such that E_x is trivial.

(b) In the vicinity of each point of S, there exist trivializations of L^+ and L^- such that h takes the form

$$h = \prod_{k=1}^{n} (\eta - p_k(\zeta)). \tag{4.29}$$

(c) The real structure τ on $T\mathbb{P}^1$ lifts to an antilinear antiholomorphic map $\sigma : (L^+) \to (L^-)^*$.

As we have the solutions to the scattering equations given by (4.27), we can verify item (b). For the Dirac monopole with singularity at $\vec{a}_k$, we can rewrite the parallel spinors as

$$\chi^N(x,s,\zeta) = -(r+x)^{1/2}(\zeta - a_{0x}), \quad \chi^S(x,s,\zeta) = \frac{(r+x)^{1/2}}{\bar{z}} \frac{1}{(\zeta - a_{x0})}, \tag{4.30}$$

where a_{x0} is the direction from $\vec{x}$ to 0 and a_{0x} is the direction from 0 to $\vec{x}$, with explicit formula in (3.10). From this, we have

$$\chi^N = (p_x(\zeta) - p_k(\zeta))\chi^S. \tag{4.31}$$

For the Dirac multimonopole, we then have

$$\chi^N = \prod_{k=1}^{n} (p_x(\zeta) - p_k(\zeta))\, \chi^S, \tag{4.32}$$

so that $h = \prod_{k=1}^{n} (\eta - p_k(\zeta))$. In [75, pp.6], the spectral curve S of the $U(2)$ singular monopole was defined as the zero level of the map from L^+ to L^-. In our case of the $U(1)$ monopole, this map is given by h and we obtain the following spectral curve of the Dirac multimonopole.

Definition 4.4. The *spectral curve* $S \subset T\mathbb{P}^1$ for the Dirac multimonopole with n singularities at the points $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^3$ is

$$S = \left\{ (\zeta, \eta) \in T\mathbb{P}^1 : \prod_{j=1}^{n} (\eta - p_j(\zeta)) = 0 \right\}, \tag{4.33}$$

for $p_j(\zeta) = a_j^1 + ia_j^2 - 2a_j^3\zeta - (a_j^1 - ia_j^2)\zeta^2$ the twistor section of the singular point $a_j = (a_j^1, a_j^2, a_j^3) \in \mathbb{R}^3$.

In [75, pp.6], the behaviors as $t \to \pm\infty$ of the solutions χ to the scattering equations along the line γ were examined. Let us do the same here. Again, recall that χ for the Dirac multimonopole with n singularities at $\vec{a}_1, \ldots, \vec{a}_n$ is the product of χ_k for the Dirac monopole at $\vec{a}_k$ so we need only consider the case of the single monopole located at the origin of $\mathbb{R}^3$. The scattering equation for a line γ parametrized by $\vec{\zeta} t$, belonging to the spectral curve, is simply

$$D_\gamma - i\Phi = \partial_t + \frac{1}{2|t|}, \tag{4.34}$$

with solutions

$$\chi^N(t) = \sqrt{-t}, \quad \chi^S(t) = \frac{1}{\sqrt{t}}. \tag{4.35}$$

In general for the Dirac multimonopole, the solutions χ belonging to the line bundle L^+ is such that $\chi(t) \to \infty$ when $t \to -\infty$ and the solutions χ belonging to L^- is such that $\chi(t) \to 0$ as $t \to \infty$.

4.5 Dirac Zero Modes from Rows of Polynomials

Recall from Section 4.3 that zero modes of the Nahm operator $\mathfrak{D}^\dagger_{\vec{x}}$ may be constructed from an orthonormal basis of rows of polynomials. One may then ask if the zero modes of the monopole Dirac operator $\mathcal{D}^\dagger_s$ may be written in terms of these rows, i.e. in terms of a basis of $H^0(L^s_S(n-1))$. It turns out that the answer is yes. In this section, we present the construction of normalizable zero modes of $\mathcal{D}^\dagger_s$.

The relationship between the zero modes of the monopole operators $\mathcal{D}^\dagger$ and $\mathcal{D}$ is more ambiguous than the zero modes of the Nahm operators $\mathfrak{D}^\dagger$ and $\mathfrak{D}$. Recall from Lemma 4.3 that the $2n \times 2n$ fundamental matrix V of zero modes to $\mathfrak{D}^\dagger$ and W to $\mathfrak{D}$, we have $V = (W^\dagger)^{-1}$. However, on the monopole side the relation between the zero modes ψ and ϕ takes the form of the differential equation $\nabla \cdot (\psi \boldsymbol{\sigma} \phi) = 0$ as in equation (46) of [8]. We will not try to solve for the normalizable zero modes using that approach, but rather, we will follow an ansatz of Lamy-Poirier in [49].

We begin with the zero modes of the Dirac operator $\mathcal{D}^\dagger_s$ for the single unit charge monopole as an example. We then give the general formulation for the zero modes in terms of the rows of polynomials from Lemma 3.2 satisfying the matching conditions $Q_i(a_{ij}) = e^{-sr_{ij}} Q_j(a_{ij})$.

Recall the unit charge Dirac monopole located at $\vec{a}_k \in \mathbb{R}^3$ is the pullback $E = P^*H$ of the Hopf bundle $H \to S^2$ under the map $P : \mathbb{R}^3 \setminus \{\vec{a}_k\} \to S^2$ with (A, Φ) given in North chart by

$$\Phi(x) = \frac{i}{2r_k}, \quad A(x) = \frac{z_k d\bar{z}_k - \bar{z}_k dz_k}{4r_k(r_k + x_k)}. \tag{4.36}$$

For the parallel section of the associated operators $(\mathcal{L}, \mathcal{M})$, we shall use

$$\chi^S(x, \zeta) = \frac{(r_k + x_k)^{1/2}}{r_k + x_k + \bar{z}_k \zeta} F(\zeta), \tag{4.37}$$

with any function $F(\zeta)$ of ζ.

Recall the twistor line corresponding to $\vec{y}$ in $\mathbb{R}^3$ is $p_y(\zeta) = (y^1 + iy^2) - 2y^3\zeta - (y^1 - iy^2)\zeta^2$ and the root of the polynomial $p_{xk}(\zeta) = p_x(\zeta) - p_k(\zeta)$ corresponding to the direction from the vector $\vec{x}$ to the monopole located at $\vec{a}_k$ in $\mathbb{R}^3$ is $a_{xk} = \frac{x^3 - a^3_k + r_{xk}}{a^1_k - x^1 + i(x^2 - a^2_k)}$.

By Lemma 4.1, $\mathcal{D}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix} F(\zeta) e^{-s(x_k - \frac{z_k}{\zeta})} \frac{(r_k+x_k)^{1/2}}{r_k+x_k+\bar{z}_k\zeta}$ is a zero mode of $\mathcal{D}_s^\dagger$. However, we want to get a normalizable zero mode. The unique normalized zero mode of $\mathcal{D}_s^\dagger$ for the single monopole case is known [73] and given by

$$\psi(x,s) = \mathcal{D}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{-e^{-sr_k}}{(r_k+x_k)^{1/2}}. \tag{4.38}$$

The factor $\frac{-e^{-sr_k}}{(r_k+x_k)^{1/2}}$ of the normalizable zero mode above is not a parallel section of $(\mathcal{L}_s, \mathcal{M}_s)$, so a natural question to ask is, can we still recover this from the family of non-normalizable zero modes of $\mathcal{D}_s^\dagger$ in Lemma 4.1, which are given by application of the operator $\mathcal{D}_s$ to parallel sections of $(\mathcal{L}_s, \mathcal{M}_s)$?

Lamy-Porirer observed in [49] that the normalized zero mode (4.38) is obtained by taking the residue around $\zeta = a_{xk}$ of the parallel section (4.37) with choice $F(\zeta) = \frac{1}{\zeta}$. With this choice of $F(\zeta)$, we may rewrite the parallel section as

$$\chi^S(x,\zeta) = \frac{1}{\zeta} e^{-sh^-_{xk}(\zeta)} \sqrt{\frac{-a_{xk}}{\bar{z}_k}} \frac{1}{\zeta - a_{xk}}, \tag{4.39}$$

where $h^-_{xk}(\zeta) = x_k - z_k/\zeta$ is the component of the splitting $\frac{p_{xk}(\zeta)}{\zeta} = -h^+_{xk}(\zeta) - h^-_{xk}(\zeta)$. The residue of χ^S at $\zeta = a_{xk}$ is then

$$\frac{1}{2\pi i} \oint_{a_{xk}} \chi^S(x,\zeta) = \frac{1}{2\pi i} \oint_{a_{xk}} \frac{1}{\zeta} e^{-sh^-_{xk}(\zeta)} \sqrt{\frac{-a_{xk}}{\bar{z}_k}} \frac{1}{\zeta - a_{xk}} = -\frac{e^{-sr_k}}{(r_k+x_k)^{1/2}}. \tag{4.40}$$

Lamy-Poirier discovered that this residue formula generalizes to arbitrary monopole configurations [49]. In the single monopole case, the residue formula must be applied to a specific choice of function $F(\zeta)$ for the parallel section χ^S in (4.39) to recover the appropriate harmonic function. The n-multimonopole configuration is a superposition of monopoles so its parallel sections are a product of n many parallel sections from the constituent monopoles. In the general case, we then have a choice of some sum of these parallel sections $\chi_1, \ldots, \chi_j$ along with choices for $F_1(\zeta), \ldots, F_j(\zeta)$ to apply the residue formula to.

In the same paper [49], Lamy-Poirier wrote an ansatz to produce n normalizable zero modes by choosing a superposition of n parallel sections with choices $F_1(\zeta) = \frac{Q_1(\zeta)}{\zeta}, \ldots, F_n(\zeta) = \frac{Q_n(\zeta)}{\zeta}$ of polynomials $Q_k(\zeta)$ of degree (at most) $n-1$ satisfying a set of algebraic equations. The questions of whether this set of algebraic conditions produces n many linearly independent rows $(Q_1(\zeta),\ldots,Q_n(\zeta))$ as well as the method of constructing such $(Q_1(\zeta),\ldots,Q_n(\zeta))$ were left open in that paper.

We describe the ansatz and answer the remaining open questions. For a polynomial row $(Q_1(\zeta),\ldots,Q_n(\zeta))$, we take the parallel section to be

$$\chi^S(s,\zeta)=\sum_{i=1}^{n}\frac{Q_i(\zeta)}{\zeta}e^{-s(x_i-\frac{z_i}{\zeta})}\prod_{k=1}^{n}\sqrt{\frac{-a_{xk}}{\bar{z}_k}}\frac{1}{\zeta-a_{xk}}, \tag{4.41}$$

where we have a sum of parallel sections for the general multimonopole with choices $F_i(\zeta)=\frac{Q_i(\zeta)}{\zeta}$. To see that each term is a parallel section of the multimonopole, observe that the factor following $\frac{Q_i(\zeta)}{\zeta}$ in (4.41) is the product of parallel sections of the constituent monopoles of the configuration.

Given a polynomial row $(Q_1(\zeta),\ldots,Q_n(\zeta))$, define the residue operation for the parallel section (4.41) to be

$$Res[(Q_1(\zeta),\ldots,Q_n(\zeta))]:=\sum_{i=1}^{n}\frac{1}{2\pi i}\oint_{a_{xi}}\frac{Q_i(\zeta)}{\zeta}e^{-s(x_i-\frac{z_i}{\zeta})}\prod_{k=1}^{n}\sqrt{\frac{-a_{xk}}{\bar{z}_k}}\frac{1}{\zeta-a_{xk}}. \tag{4.42}$$

Proposition 4.4. *Let Q be a matrix of polynomials whose rows form a basis of the rows of polynomials $(Q_1(\zeta),\ldots,Q_n(\zeta))$ satisfying the matching conditions $Q_i(a_{ij})=e^{sr_{ij}}Q_j(a_{ij})$ for all double points $a_{ij}, i\neq j$ of the spectral curve S. Then the n normalizable zero modes of the Dirac operator $\mathcal{D}_s^\dagger$ are given by taking for each row $(Q_1(\zeta),\ldots,Q_n(\zeta))$ of Q the spinor*

$$\psi[(Q_1,\ldots,Q_n)]=\mathcal{D}_s\begin{pmatrix}1\\0\end{pmatrix}Res[(Q_1(\zeta),\ldots,Q_n(\zeta))]. \tag{4.43}$$

The components of ψ are given by

$$\psi_1[(Q_1,\ldots,Q_n)] = \prod_1^n (\frac{-a_{xk}}{\bar{z}_k})^{1/2} \quad \times$$
$$\sum_{i=1}^n \left[\frac{e^{-sr_i} a_{xi}^{-1} Q_i(a_{xi})}{\prod_{k\neq i}(a_{xi}-a_{xk})} \left(\frac{\partial_3(Q_i(a_{xi}))}{Q_i(a_{xi})} - \sum_{j\neq i}^n \frac{\partial_3(a_{xi}-a_{xj})}{a_{xi}-a_{xj}} + \sum_{j\neq i}^n a_{xj}^{-1}\partial_3 a_{xj} - s(\frac{x_i^3}{r_i}+1) \right) \right],$$
$$\psi_2[(Q_1,\ldots,Q_n)] = 2\prod_1^n (\frac{-a_{xk}}{\bar{z}_k})^{1/2} \quad \times$$
$$\sum_{i=1}^n \left[\frac{e^{-sr_i} a_{xi}^{-1} Q_i(a_{xi})}{\prod_{k\neq i}(a_{xi}-a_{xk})} \left(\frac{\bar{\partial}(Q_i(a_{xi}))}{Q_i(a_{xi})} - \sum_{j\neq i}^n \frac{\bar{\partial}(a_{xi}-a_{xj})}{a_{xi}-a_{xj}} + \sum_{j\neq i}^n a_{xj}^{-1}\bar{\partial}(a_{xj}) - s\frac{z_i}{2r_i} \right) \right]. \tag{4.44}$$

In particular, constructions for Q are given in Propositions 3.13 and 3.14 of Chapter 3 of this thesis.

Proof. If $(Q_1(\zeta),\ldots,Q_n(\zeta))$ satisfies the matching conditions, then by uniqueness of Lagrangian interpolation each $Q_k(\zeta)$ admits the form (3.60) and thus satisfies Lamy-Poirier's system of algebraic equations. The proof that such a solution gives rise to a normalizable zero mode is found in [49, Appendix A] and [49, A.5]. By Proposition 3.3, we get $h^0(S, L^s(n-1)) = n$ so that we obtain n zero modes by this method. □

Let us consider how the zero modes may behave for a specific choice of basis that singles out the Atiyah polynomials. Consider the basis where the diagonal entries are degree $n-1$ polynomials and the off-diagonal entries are degree $n-2$ polynomials. Recall that we gave a method to construct a perturbation expansion (3.78) for each basis element of an orthogonal basis of polynomial rows $(Q_1(\zeta),\ldots,Q_n(\zeta))$. In this basis, the zero order of the perturbation expansion for the ith row is $Q_i = A_i(\zeta)$ the Atiyah polynomial at $\vec{a}_i$ and $Q_{j\neq 1}(\zeta) = 0$. The zero order of the harmonic function $Res[(Q_1,\ldots,Q_n)]$ for this basis element is then given by the term

$$(\prod_{k=1}^n \frac{-a_{xk}}{\bar{z}_k})^{1/2} \frac{e^{-sr_i} a_{xi}^{-1} A_i(a_{xi})}{\prod\limits_{\substack{k\neq i\\k\neq i}}^n (a_{xi}-a_{xk})}. \tag{4.45}$$

If we approach the monopole located at $\vec{a}_i$, i.e. $r_i \to 0$, then the zero order of the harmonic function approaches

$$\left(\prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{-a_{ik}}{\bar{z}_{ik}} \right)^{1/2} (\frac{-a_{xi}}{\bar{z}_i})^{1/2} a_{xi}^{-1}, \tag{4.46}$$

which is precisely the harmonic function found in (4.38) for the single monopole configuration at location $\vec{a}_i$. The zero mode coming from the ith element of this basis then approaches near $\vec{a}_i$ the zero mode for the single monopole with charge centered at $\vec{a}_i$.

Chapter 5

Conclusions for Part I

We construct solutions to Nahm's equations with our prescribed boundary conditions via the algebro-geometric integration method for Lax pairs. In our case, the explicit formulation for our spectral curve is well-known, obtained from the boundary conditions. For other boundary conditions, the spectral curve is generally not explicitly known. As mentioned in Chapter 3, the reader may consult Table 1 of [67] for a list of all currently known spectral curves.

We then give two different linear systems for explicitly constructing an orthonormal basis of sections to the eigenline bundle over our spectral curve, which come from the associated linear problem to the Lax pair. We also give an algorithm for constructing a perturbation expansion of the sections for large s, to any order desired.

We solve Nahm's equations in terms of the orthonormal basis of eigenline sections. The perturbation expansion of the sections are also used to give a perturbative solution to Nahm's equations for large s, to any order. We illustrate this with the example of rank 3 Nahm matrices. Our Nahm solutions approach a diagonal limit at infinity and generally do not have $T_0 = 0$.

We fill in the gap of Lamy-Poirier's ansatz [49] for the explicit construction of the L^2 zero modes of the monopole-side Dirac operators. We show that the polynomials coming from a section of the eigenline bundle over the spectral curve satisfy his criteria [49][Equation (5.8)] for obtaining a L^2 zero mode. These polynomials are constructed explicitly from either one of our two linear systems.

Our results and algorithms do not depend on the truth or falsity of Atiyah's conjecture on the linear independence of stellar polynomials. Atiyah's polynomials appear in one of the linear systems for the construction of an orthonormal basis of eigenline sections. They also appear in the perturbation expansion of these sections. Note, for a Nahm solution of rank n, the dimension of eigenline sections is n, the dimension of L^2 zero modes is n, and there are exactly n stellar polynomials. We do not think these are coincidences, however, our present results do not immediately imply a proof of Atiyah's conjecture. It would be interesting to further examine whether Atiyah's stellar conjecture may be proven from the perspective of Nahm's equations.

Part II

Chapter 6

Polar Terms of Weak Jacobi Forms

6.1 Polar Part of Weak Jacobi Forms

A Jacobi form is an automorphic form for the Jacobi group, so we begin with the definition of the Jacobi group.

Definition 6.1. The *Jacobi group* Γ^J is $SL_2(\mathbb{Z})^J = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, with action defined as

$$(M, X) \cdot (M', X') = (MM', XM' + X'), \tag{6.1}$$

for $M \in SL_2(\mathbb{Z})$ and $X \in \mathbb{Z}^2$.

Now, we give the definition of a classical Jacobi form.

Definition 6.2. A *Jacobi form* of *weight* k and *index* t is a holomorphic function $\varphi_{k,t}(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, with $\mathbb{H}$ the upper-half plane, transforming the action of $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda & \mu \end{smallmatrix}\right)\right) \in SL_2(\mathbb{Z})^J$ as

$$\begin{aligned} \varphi_{k,t}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) &= (c\tau + d)^k e^{i2\pi t \frac{cz^2}{c\tau + d}} \varphi_{k,t}(\tau, z), \\ \varphi_{k,t}(\tau, z + \lambda\tau + \mu) &= e^{-i2\pi t(\lambda^2\tau + 2\lambda z)} \varphi_{k,t}(\tau, z), \end{aligned} \tag{6.2}$$

and has the Fourier-Jacobi expansion

$$\sum_{\substack{n,l \in \mathbb{Z} \\ 4tn - l^2 \geq 0}} c(n, l) e^{2\pi i n\tau} e^{2\pi i l z}. \tag{6.3}$$

For the Fourier-Jacobi expansion, we adopt the conventional notations $q := e^{2\pi i\tau}$ and $y := e^{2\pi iz}$.

However, we will need to work with a larger class of functions, the weak Jacobi forms.

Definition 6.3. A *weak Jacobi form* of *weight* k and *index* m is a holomorphic function $\varphi_{k,m}(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformation laws (6.2), and its Fourier-Jacobi expansion satisfies the weaker conditions

$$\tilde{\varphi}_{k,m}(\tau, z) = \sum_{n\geq 0} \sum_{l\in\mathbb{Z}} c(n, l) q^n y^l. \tag{6.4}$$

The transformation laws infact bounds the sum over l so that $4mn - l^2 \geq -m^2$. We denote the space of weak Jacobi forms by $J_{k,m}$.

Definition 6.4. Define $j(m)$ to be the dimension of the space of weak Jacobi forms of weight 0 and index m,

$$j(m) := \dim J_{0,m}. \tag{6.5}$$

We have the following formula for $j(m)$ from [76, Section 9],

$$j(m) = \begin{cases} \frac{m^2}{12} + \frac{m}{2} + 1 & m = 0 \bmod 6 \\ \frac{m^2}{12} + \frac{m}{2} + 5/12 & m = 1, 5 \bmod 6 \\ \frac{m^2}{12} + \frac{m}{2} + 2/3 & m = 2, 4 \bmod 6 \\ \frac{m^2}{12} + \frac{m}{2} + 3/4 & m = 3 \bmod 6. \end{cases} \tag{6.6}$$

Gritsenko in [77] gives us a basis for the space of all weak Jacobi forms of weight 0, freely generated by the weak Jacobi forms $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ as defined below. A basis of $J_{0,m}$ is, then, given by the set $\{\phi_{0,1}^a \phi_{0,2}^b \phi_{0,3}^c \mid a+2b+3c = m\}$. The generating functions can be written in terms of the following Jacobi theta functions. For

$$\begin{aligned} \theta_{00}(q, y) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n, \\ \theta_{01}(q, y) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n, \\ \theta_{10}(q, y) &= q^{1/8} y^{1/2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} y^n, \end{aligned} \tag{6.7}$$

define $\zeta_{00} := \frac{\theta_{00}(\tau,z)}{\theta_{00}(\tau,0)}$, $\zeta_{10} := \frac{\theta_{10}(\tau,z)}{\theta_{10}(\tau,0)}$, and $\zeta_{01} := \frac{\theta_{01}(\tau,z)}{\theta_{01}(\tau,0)}$.

Then [77, Equation (2.7)] (with the coefficient of $\phi_{0,3}$ adjusted to 4, rather than 16) states

$$\begin{aligned} \phi_{0,1}(\tau, z) &= 4(\zeta_{00}^2 + \zeta_{10}^2 + \zeta_{01}^2), \\ \phi_{0,2}(\tau, z) &= 2\big((\zeta_{00}\zeta_{10})^2 + (\zeta_{00}\zeta_{01})^2 + (\zeta_{10}\zeta_{01})^2\big), \\ \phi_{0,3}(\tau, z) &= 4\zeta_{00}^2\zeta_{10}^2\zeta_{01}^2. \end{aligned} \tag{6.8}$$

In the Appendix, we discuss an implementation of $\phi_{0,1}, \phi_{0,2}, \phi_{0,3}$ in Mathematica that allows for a fast computation of their Fourier-Jacobi expansions, up to order 10,000 in q. We used this code for our numerical computations in Chapter 8, which require Fourier-Jacobi expansions to order approximately 1000 (if not more) in q to properly investigate.

Definition 6.5. The *polar terms* of a weak Jacobi form of index m are the terms $c(n, l)q^n y^l$ in its Fourier-Jacobi expansion such that $4mn - l^2 < 0$, with $n \geq 0$ and $0 \leq l \leq m$[1]. We denote by $p(m)$ the total number of polar terms $q^n y^l$ with $n \geq 0, 0 \leq l \leq m$ for index m. Let $p_{\mathcal{P}}(m)$ be the number of pairs (n, l) with $n \geq 0, 0 \leq l \leq m$ such that $4mn - l^2 \leq -\mathcal{P}$.

It is known that the polar parts of a weight zero index m weak Jacobi form uniquely determine the form when $m > 0$. To describe this requires introducing the theta decomposition of a weak Jacobi form, which involves modular forms of fractional weights for congruence subgroups with multiplier systems and so we give their definitions here.

Definition 6.6. The *principal congruence subgroup of level* N for an integer $N > 1$ is the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{6.9}$$

We take $\Gamma(1)$ to be $SL(2, \mathbb{Z})$, but shall abbreviate it to simply Γ.

When extending modular forms from integral weights to fractional weights k, the factor $J(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := (c\tau + d)^k$ is no longer an automorphy factor. To correct this, we need multiplier systems.

[1]This definition may differ from part of the literature. In general, the restriction $n \geq 0$ and $0 \leq l \leq m$ may not be insisted upon. In this case, our definition will be the 'fundamental domain' of all polar terms, i.e. there exists an element of Γ^J taking a general polar term to one with $n \geq 0$, $0 \leq l \leq m$ so there is no loss of generality.

Definition 6.7. A *multiplier system* v is a map $v : \Gamma(N) \to \mathbb{C}^*$ satisfying

$$v(M_1M_2)J(M_1M_2, \tau) = v(M_1)v(M_2)J(M_1, M_2\tau)J(M_2, \tau), \tag{6.10}$$

for $J(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := (c\tau + d)^k$.

Note, multiplier systems are not necessarily characters of $\Gamma(N)$.

Now, we are able to define modular forms for a congruence subgroup $\Gamma(N)$ with fractional weights k.

Definition 6.8. A function $\phi_k : \mathbb{H} \to \mathbb{C}$ is called a *modular form* of *weight* k with *multiplier system* v for $\Gamma(N)$ if ϕ_k is holomorphic on $\mathbb{H}$ and satisfies the transformation property

$$\phi_k(M\tau) = \phi_k(\frac{a\tau + b}{c\tau + d}) = v(M)(c\tau + d)^k\phi_k(\tau), \quad M \in \Gamma(N), \tag{6.11}$$

and ϕ_k is holomorphic at all cusps $r \in \mathbb{Q} \cup \{\infty\}$. When k is fractional, we will take the principal branch cut of the root when defining $(c\tau + d)^k$. If ϕ_k is, instead, meromorphic at the cusps $r \in \mathbb{Q} \cup \{\infty\}$, then ϕ_k is called a *meromorphic modular form.*

We illustrate this with the important example of the Dedekind eta function, a meromorphic modular form of weight 1/2 with a nontrivial multiplier system. This function will play a role in some of our later proofs.

Example 6.1. Let $(\frac{c}{d})$ be the Kronecker symbol, which generalizes the Legendre symbol to all integers d. Define $(\frac{c}{d})_*$ to be the Kronecker symbol, except at $(\frac{0}{-1})_* := -1$. Define $(\frac{c}{d})^* := (\frac{c}{|d|})$. The Dedekind eta function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n), \tag{6.12}$$

is a meromorphic modular form of weight 1/2 for $SL(2, \mathbb{Z})$ with the multiplier system ν_η defined as

$$v_\eta(M) = \begin{cases} (\frac{d}{c})^* e^{\frac{2\pi i}{24}\left((a+d)c - bd(c^2-1) - 3c\right)} & c \text{ odd} \\ (\frac{c}{d})_* e^{\frac{2\pi i}{24}\left((a+d)c - bd(c^2-1) + 3d - 3 - 3cd\right)} & c \text{ even.} \end{cases} \tag{6.13}$$

The above is a nontrivial fact, for its proof we refer the reader to [78, Theorem 1.7].

With all the necessary definitions in place, we may now introduce the following lemma.

Lemma 6.1. *[79, Section 3] For index $m > 0$, the polar part of a weight zero weak Jacobi form uniquely determines the weak Jacobi form.*

Proof. Consider the theta decomposition [76, Equation (5.5)] of a weak Jacobi form.

$$\varphi_{0,m}(\tau, z) = \sum_{\mu \bmod 2m \in \mathbb{Z}/2m\mathbb{Z}} h_\mu(\tau)\theta_{m,\mu}(\tau, z), \tag{6.14}$$

with the vector-valued modular forms $h_\mu(\tau) = \sum_{N=-m^2}^{\infty} c(N, \mu)q^{N/4m}$.

The proof is accomplished by showing that if $\varphi_{0,m}$ has no polar part, it must be identically zero. Now, all the polar terms of $\varphi_{0,m}$ appear in the negative q-power part of the Fourier expansions of $\{h_\mu(\tau) : \mu \in \mathbb{Z}/2m\mathbb{Z}\}$.

The forms $h_\mu(\tau)$ are scalar modular forms of weight $-1/2$ for $\Gamma(4m)$ with multiplier system $(\frac{c}{d})$. Since $\varphi_{0,m}$ has no polar part, each $h_\mu(\tau)$ is a holomorphic modular form. However, there are no non-zero holomorphic modular forms of weight $-1/2$ for the subgroup $\Gamma(4m)$ with multiplier system $(\frac{c}{d})$, so that $h_\mu(\tau) \equiv 0$. □

While the polar terms determine the weak Jacobi form when the index is greater than 0, we also have that for $m \geq 5$, the number of polar terms $p(m)$ exceeds the dimension $j(m)$ of weak Jacobi forms of weight zero and index m. The polar terms then form an overdetermined system for $J_{0,m}$, in the sense that given an arbitrary list of polar coefficients, there may not be a corresponding weak Jacobi form with polar part having these coefficients.

The number of polar terms $p(m)$ for index m is, by [80, Equation (2.31)],

$$p(m) = \frac{m^2}{12} + \frac{5m}{8} + a(m), \tag{6.15}$$

for $a(m)$ defined as

$$a(m) = \frac{1}{4}\sum_{d|4m} h'(-d) - \frac{1}{2}\lfloor\frac{b}{2}\rfloor - \frac{1}{2}((\frac{m}{4})) + \frac{1}{24}, \tag{6.16}$$

where $h'(-3) = 1/3$, $h'(-4) = 1/2$, and otherwise $h'(-d)$ is the class number of the positive definite binary quadratic form of discriminant d, b is the largest integer such that $b^2 \mid m$, and $((\frac{m}{4}))$ is the sawtooth function[2].

A heuristic argument appears in [80, Section 2.2] for the claim that, asymptotically, $a(m) \sim m^{1/2}$. We prove here an explicit analytical bound for $a(m)$.

Proposition 6.2. *For any $\epsilon > 0$, there exists a constant C_ϵ such that*

$$|a(m)| < C_\epsilon m^{1/2+\epsilon}. \tag{6.17}$$

Proof. We put bounds on each term defining the quantity $a(m)$ in (6.16). The term $((\frac{m}{4}))$ is bounded between $-1/2$ and $1/2$. While we have $\frac{1}{2}\lfloor\frac{1}{2}(\frac{m}{p_1\cdots p_n})^{1/2}\rfloor \leq \frac{1}{2}\lfloor\frac{b}{2}\rfloor \leq \frac{1}{4}m^{1/2}$, where $p_1, \ldots, p_n$ are the prime divisors of m, we will trivially underestimate $\frac{1}{2}\lfloor\frac{b}{2}\rfloor$ by setting it to zero. We now overestimate the remaining term, $\frac{1}{4}\sum_{d|4m} h'(-d)$.

We have from [81, p.290] that

$$h'(-d) < \frac{1}{\pi}\sqrt{d}\log d, \tag{6.18}$$

and the total number of divisors of $4m$, denoted $\sigma_0(4m)$, satisfies [82, p.229]

$$\sigma_0(4m) \leq (4m)^{1.5379\log 2/\log\log 4m}. \tag{6.19}$$

We overestimate the sum of class numbers by overestimating the largest term $h'(-4m)$ with $\frac{1}{\pi}\sqrt{4m}\log 4m$ using (6.18), and then replacing each term in our sum with this largest term. Overestimating the total number of terms in the sum with (6.19), we have

$$\frac{1}{4}\sum_{d|4m} h'(-d) < \frac{1}{4}(4m)^{1.5379\log 2/\log\log 4m}\frac{1}{\pi}\sqrt{4m}\log 4m. \tag{6.20}$$

□

[2]The sawtooth function is defined as

$$((x)) := x - \frac{1}{2}(\lceil x\rceil + \lfloor x\rfloor) = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha - \frac{1}{2} & x = n + a, 0 < a < 1. \end{cases}$$

That is, the sawtooth returns a zero for an integral argument, and otherwise returns the decimal part subtracted by 1/2.

In [80, Figure 2], numerical analysis of the growth of $a(m)$ for indexes m to order several thousands shows that the difference $(p(m) - j(m) - \frac{m}{8})$ grows approximately as $m^{1/2}$.

We would like to eliminate the extraneous polar terms, and we achieve the following analytical result: we improve on Lemma 6.1 by proving that the polar terms of polarity less than or equal to $-m/6$ determine the corresponding weak Jacobi form.

Proposition 6.3. *For $m > 0$, the polar terms $c(n, l)$ of polarity $4mn - l^2 \leq -m/6$ uniquely determine the weak Jacobi form $\varphi_{0,m}$.*

Proof. As in Lemma 6.1, we consider the theta decomposition (6.14) of the weak Jacobi form. The polar terms of $\varphi_{0,m}$ appear in the negative q-power part of the Fourier expansions of $\{h_\mu(\tau) : \mu \in \mathbb{Z}/2m\mathbb{Z}\}$.

We will show that the product $\eta(\tau)h_\mu(\tau)$ with the Dedekind eta function is a scalar modular form of weight zero for the group $\Gamma\big(\mathrm{lcm}(24, 4m)\big)$. This implies that the product must be a constant.

By Example 6.1, the Dedekind eta function $\eta(\tau) = q^{1/24} \prod_{m>0} (1 - q^m)$ is a scalar modular form of weight $1/2$ for $\Gamma(24)$ with multiplier system $(\frac{c}{d})$. The forms $h_\mu(\tau)$ are scalar modular forms of weight $-1/2$ for $\Gamma(4m)$ with the same multiplier system $(\frac{c}{d})$.

Since $(\frac{c}{d})$ squares to the identity, the product $\eta(\tau)h_\mu(\tau)$ is a scalar modular form for $\Gamma\big(\mathrm{lcm}(24, 4m)\big)$ with trivial multiplier system.

Given a weak Jacobi form with no polar terms of polarity less than or equal to $-m/6$, we show this form must be identically zero. Let N be the most polar term of this weak Jacobi form, this term shall also be the most polar term of $h_\mu(\tau)$ for its theta decomposition. The Fourier expansion of $\eta(\tau)h_\mu(\tau)$ then begins at $c(N, \mu)q^{N/4m+\frac{1}{24}}$. We have $N > -\frac{m}{6}$ by assumption, so

$$N/4m + \frac{1}{24} > 0, \tag{6.21}$$

which implies that $\eta(\tau)h_\mu(\tau) \in M_0\left(\Gamma\big(\mathrm{lcm}(24, 4m)\big)\right)$. However, the only modular forms of $M_0\left(\Gamma\big(\mathrm{lcm}(24, 4m)\big)\right)$ are constants, whose Fourier expansion consists of only the q^0 term. This implies that $\eta(\tau)h_\mu(\tau)$ is zero. □

Continuing the same spirit of eliminating extraneous polar terms that over-determine the system of weak Jacobi forms, consider as in [80] the polarity value $P(m)$, where $P(m)$ is the largest number such that the polar terms $c(n,l)$ of polarity $4mn - l^2 \leq -P(m)$ uniquely determine the weak Jacobi form $\varphi_{0,m}$.

The formal definition of $P(m)$ is as follows.

Definition 6.9. Let $J_{0,m}^P := \{\varphi_{0,m} \in J_{0,m} \mid c(n,l) = 0 \text{ for } 4mn - l^2 < -P\}$. Define the positive integer $P(m)$ to be such that $J_{0,m}^{P(m)} = 0$ and $J_{0,m}^{P(m)+1} \neq 0$.

We have computed the values $P(m)$ for small index m, using Gritsenko's basis (6.8) for $J_{0,m}$. We plot the results in Figure 6.1 below.

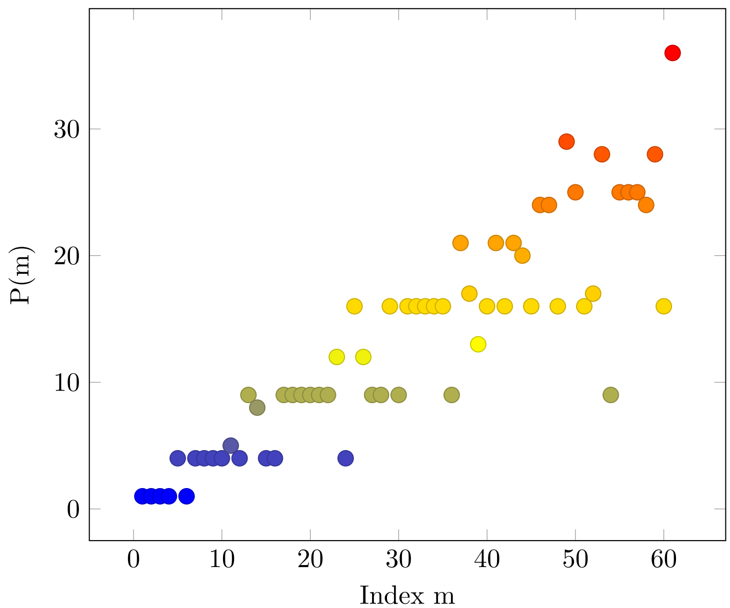


Figure 6.1: Scatterplot of $P(m)$, the largest polar value such that the polar terms of polarity less than or equal to $-P(m)$ uniquely determine the weak Jacobi form.

The polar terms of polarity less than or equal to $-P(m)$, then, form a linear system for the space of weak Jacobi forms that is not as overdetermined as the linear system coming from the entire collection of polar terms.

Let $p_{P(m)}$ be the total number of polar terms with polarity less than or equal to $-P(m)$. For small index m, we plot the difference $p_{P(m)} - j(m)$ in Figure 6.2, and this quantity represents how many extraneous polar terms we continue to have.

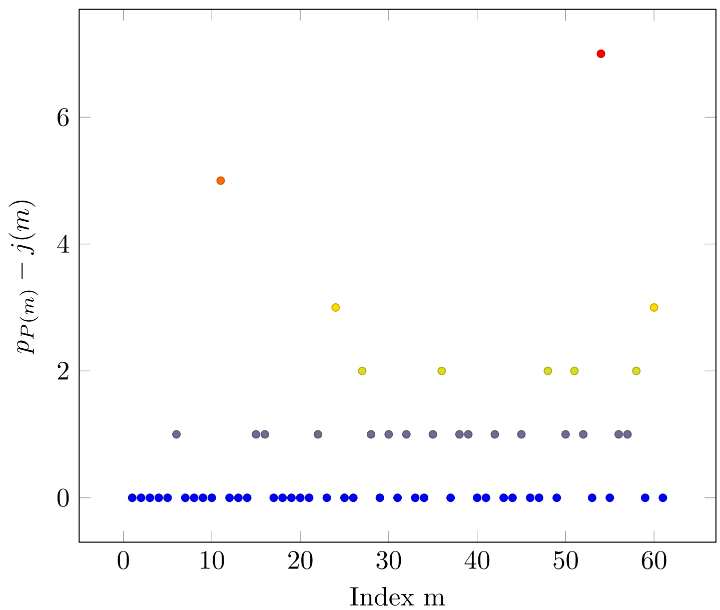


Figure 6.2: Scatterplot of the difference between the dimension of $J_{0,m}$ and the number of polar terms of polarity $\leq -P(m)$. This difference measures the extent to which these polar terms form an overdetermined system for $J_{0,m}$.

We see from the scatter plot of Figure 6.2 that frequently, the polar terms with polarity less than or equal to $-P(m)$ are sufficient to uniquely determine the weak Jacobi form.

Computing $P(m)$ for large m quickly becomes computer-intensive because of the requirement to extract Fourier-Jacobi coefficients from the polar part of each basis element, in order to form the linear system. We have managed to compute $P(m)$ up to $m = 61$ in Mathematica. We present the code for computing the polar coefficients of a basis of $J_{0,m}$ in the Appendix. Instead of continuing this computer-intensive task for higher m, we now discuss lower and upper bounds for $P(m)$.

In [80], a heuristic was given for approximating $P(m)$ but our goal is to obtain some analytic bounds on $P(m)$. To begin with, we already have a lower bound offered by Lemma 6.1.

6.1.1 A Lower Bound for $P(m)$

Corollary 6.4. *A lower bound for $P(m)$ is $P_-(m) := \lceil \frac{m}{6} \rceil$. $P(m)$ attains this lower bound at m if and only if there are constants C_μ such that*

$$\varphi_{0,m} = \sum_{\mu \ mod \ 2m \in \mathbb{Z}/2m\mathbb{Z}} \frac{C_\mu}{\eta(\tau)} \theta_{m,\mu}(\tau, z) \tag{6.22}$$

is a weak Jacobi form that is not identically zero.

Proof. The statement of Lemma 6.1 implies $\lceil \frac{m}{6} \rceil$ is a lower bound for $P(m)$ and its proof shows that whenever $P(m) = \lceil \frac{m}{6} \rceil$, there exists a weak Jacobi form in $J_{0,m}$ having the above theta decomposition. Such weak Jacobi forms do exist, e.g. at $m = 6$ we have $P(m) = 1$ with the Jacobi form $-4\phi_{0,3}^2 - \phi_{0,2}^3 + \phi_{0,1}\phi_{0,2}\phi_{0,3}$ having a theta decomposition of this type and lowest polar discriminant -1. □

6.1.2 An Upper Bound for $P(m)$

Counting the number of polar terms of polarity less than or equal to a fixed $-\mathcal{P}$ for $\mathcal{P} \in \mathbb{Z}_+$ will be important to us, as we will use this counting number to obtain an upper bound on $P(m)$ as in the following lemma.

Lemma 6.5. *For $p_\mathcal{P}(m) = \sum_{l=\lceil \sqrt{\mathcal{P}} \rceil}^{m} \lceil \frac{l^2 - \mathcal{P}}{4m} \rceil$ counting the number of polar terms with polarity less than or equal to $-\mathcal{P}$, we have*

$$P(m) \leq \mathcal{P}, \tag{6.23}$$

for any $\mathcal{P}$ satisfying the inequality $p_\mathcal{P}(m) \leq j(m) < p_{\mathcal{P}+1}(m)$. Denote by $P^+(m)$ the smallest such $\mathcal{P}$.

Proof. The polar terms for a given index m form a linear system for the space of weak Jacobi forms of index m. We order the polar terms according to their polarity. We may use the $j(m)$ basis elements to set $(j(m) - 1)$ of the most polar terms to zero, so that $P(m)$ is bounded above by the value of $\mathcal{P}$ such that $p_\mathcal{P}(m) \leq j(m) < p_{\mathcal{P}+1}(m)$. □

$P^+(m)$, the upper bound for $P(m)$, is easy to compute, we present a scatter plot of its value for $1 \leq m \leq 1000$ in Figure 6.3. Comparing this with the scatter plot for $P(m)$ for $1 \leq m \leq 61$ in Figure 6.1, we find that $P(m) = P^+(m)$ except at $m = 39, 51, 54$, and 58. A particularly wide gap is found at $m = 54$, where $P(m) = 9$ but $P^+(m) = 25$.

We expect that for generic m, $P(m)$ will be very close to $P^+(m)$. Equality between $P(m)$ and $P^+(m)$ holds whenever the linear system of polar coefficients with polarity less than or equal to $-P^+(m)$ has maximal rank. We expect the matrix of these polar coefficients to behave like a random matrix, and such matrices generically have maximal rank. In contrast, we expect $P^-(m)$ to be a weak lower bound.

Numerically, we find that

$$|P^+(m) - \frac{m}{2}| \leq 2.1016 m^{1/2}. \tag{6.24}$$

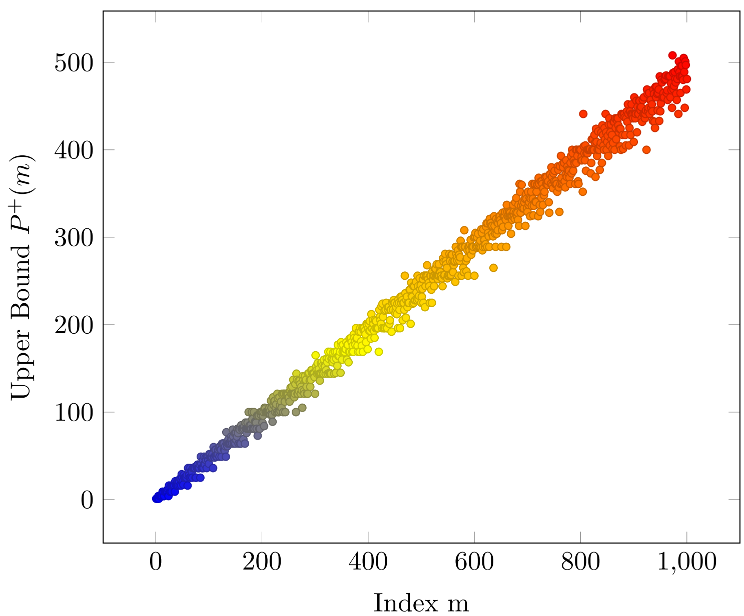


Figure 6.3: Scatterplot of the upper bound $P^+(m)$ for $P(m)$, where $P^+(m)$ is the polarity such that the number of polar terms of polarity $\leq -P^+(m)$ equals $j(m)$.

We discuss a heuristic approximation of the upper bound $P^+(m)$ for $P(m)$, building on [80]. Our approximation has one analytical gap, which is the assumption that $\sum_{l=1}^{m}((\frac{l^2-\mathcal{P}}{4m}))$ for any $\mathcal{P}$ has the 'same' behavior as a function in m as $\sum_{l=1}^{m}((\frac{l^2}{4m}))$. If this assumption holds, then it implies the following conjecture.

Conjecture 6.6. *$P^+(m)$ has the following upper bound*

$$P^+(m) \leq \frac{m}{2} + Cm^{1/2}. \tag{6.25}$$

Numerically for $m \leq 1000$, we have obtained $C \sim 1.35695$. We now describe this rough approximation of $P^+(m)$.

Argument. We fix m and try to solve the following equation

$$p_{P^+(m)} = j(m) \tag{6.26}$$

for $P^+(m)$ in terms of m. That is, we want to find the polarity $P^+(m)$ such that the number of polar terms of polarity $\leq -P^+(m)$ equals $j(m)$. From numerical data, we see $P^+(m)$ grows like αm for α approximately equal to $1/2$, and we shall later use this ansatz.

Equation (6.26) is

$$\begin{aligned} j(m) &= \sum_{l=l_0}^{m} \left\lceil \frac{l^2 - P^+(m)}{4m} \right\rceil \\ &= \sum_{l=l_0}^{m} \frac{l^2 - P^+(m)}{4m} - \sum_{l=l_0}^{m} \left(\left(\frac{l^2 - P^+(m)}{4m} \right) \right) \\ &\qquad + \frac{1}{2} \sum_{l=l_0}^{m} \left(\left\lceil \frac{l^2 - P^+(m)}{4m} \right\rceil - \left\lfloor \frac{l^2 - P^+(m)}{4m} \right\rfloor \right), \end{aligned} \tag{6.27}$$

for $l_0 = \lceil \sqrt{P^+(m)} \rceil$.

For any $\mathcal{P} < 4m$, define $\mu(m, \mathcal{P})$ to be the number of integers l with $0 \leq l \leq m$ solving $l^2 = \mathcal{P} \bmod 4m$. Asymptotically, $\mu(m, \mathcal{P})$ is dominated by $m^{1/2}$. To see this, take the prime factorization $4m = 2^k p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$. There are at most $4 \cdot 2^r$ solutions to $x^2 = \mathcal{P} \bmod 4m$ over the ring $\mathbb{Z}/4m\mathbb{Z}$, as may be seen in any standard book on classical number theory, e.g. [83, Section

5.1]. By [84, Theorem 12], the number of distinct prime factors of an integer $m > 2$ is bounded above by $\frac{\ln m}{\ln\ln m} + O(\frac{\ln m}{(\ln\ln m)^2})$ so we have

$$\mu(m, \mathcal{P}) \leq 4 \cdot 2^{\frac{\ln m}{\ln\ln m} + O(\frac{\ln m}{(\ln\ln m)^2})}. \tag{6.28}$$

The latter is rapidly dominated by $m^{1/2}$ as $m \to \infty$.

Now, we have

$$\frac{1}{2}\sum_{l=l_0}^{m}\left(\left\lceil \frac{l^2 - P^+(m)}{4m} \right\rceil - \left\lfloor \frac{l^2 - P^+(m)}{4m} \right\rfloor \right) = m + 1 - l_0 - \mu(m, \mathcal{P}^+(m)). \tag{6.29}$$

Let us use the anstaz $P^+(m) = \alpha m$ for some $\alpha < 1$. Then (6.26) becomes

$$\frac{m}{8} - \frac{\alpha}{4}m + \frac{2\alpha^{3/2} - 6\alpha^{1/2}}{12}m^{1/2} - \sum_{l=l_0}^{m}((\frac{l^2 - P^+(m)}{4m})) - \mu(m, P^+(m)) = \chi(m), \tag{6.30}$$

for $0 \leq \chi(m) \leq 2$. Explicitly,

$$\chi(m) = \begin{cases} \frac{\alpha}{8} + \frac{23}{24} + \frac{\alpha^{1/2}}{24m^{1/2}} & m = 0 \bmod 6 \\ \frac{\alpha}{8} + \frac{9}{24} + \frac{\alpha^{1/2}}{24m^{1/2}} & m = 1, 5 \bmod 6 \\ \frac{\alpha}{8} + \frac{15}{24} + \frac{\alpha^{1/2}}{24m^{1/2}} & m = 2, 4 \bmod 6 \\ \frac{\alpha}{8} + \frac{17}{24} + \frac{\alpha^{1/2}}{24m^{1/2}} & m = 3 \bmod 6. \end{cases}$$

Let us assume that the behavior of $\sum_{l=l_0}^{m}((\frac{l^2-P^+(m)}{4m}))$ is similar to $\sum_{l=0}^{m}((\frac{l^2}{4m}))$, by which we mean that for any $\epsilon > 0$ there exists a C_ϵ such that

$$|\sum_{l=l_0}^{m}((\frac{l^2 - P^+(m)}{4m}))| \leq C_\epsilon m^{1/2+\epsilon}.$$

Then we see (6.30) is of the form

$$\frac{m}{8} - \frac{\alpha}{4}m + O(m^{1/2+\epsilon}) = 0, \tag{6.31}$$

which forces

$$\alpha = \frac{1}{2}. \tag{6.32}$$

For any $\epsilon > 0$, we then have $P^+(m) \leq \frac{1}{2}m + C_\epsilon m^{1/2+\epsilon}$. □

Again, the above argument is based on the assumption that $\sum_{l=l_0}^{m}((\frac{l^2-P^+(m)}{4m}))$ behaves similarly to $\sum_{l=0}^{m}((\frac{l^2}{4m}))$, i.e. for any $\epsilon > 0$, there is a C_ϵ such that

$$\sum_{l=l_0}^{m}((\frac{l^2-P^+(m)}{4m})) \leq C_\epsilon m^{1/2+\epsilon}. \tag{6.33}$$

We now discuss the difficulty of giving an analytical proof of the assumption that the inequality (6.33) holds.

We begin by reviewing how this bound for $\sum_{l=0}^{m}((\frac{l^2}{4m}))$ is proved. One writes

$$\sum_{l=0}^{m}((\frac{l^2}{4m})) = \sum_{n \bmod 4m}((\frac{n}{4m})) \sum_{\substack{l\in\{0,\ldots,m\} \\ l^2=n \bmod 4m}} 1, \tag{6.34}$$

and we see the sum, then, is about counting the number of solutions to $l^2 = n \bmod 4m$. This is, of course, connected to quadratic reciprocity and the class number $h(-d)$ of the positive definite binary quadratic form of discriminant d. Indeed, this term equals $\frac{1}{4}\sum_{d|4m} h'(-d)$ by [76, pp.124], for which we can use the upper bound (6.20).

Let us attempt the same thing for $\sum_{l=l_0}^{m}\lceil\frac{l^2-\mathcal{P}}{4m}\rceil$. We have

$$\sum_{l=l_0}^{m}((\frac{l^2-\mathcal{P}}{4m})) = \sum_{l=0}^{m}((\frac{l^2-\mathcal{P}}{4m})) - \sum_{l=0}^{l_0-1}((\frac{l^2-\mathcal{P}}{4m})). \tag{6.35}$$

For $0 \leq l \leq l_0-1$, $\frac{l^2-\mathcal{P}}{4m}$ is a negative number between -1 and 0 so that for such l, $((\frac{l^2-\mathcal{P}}{4m})) = \frac{l^2-\mathcal{P}}{4m}+\frac{1}{2}$. Then

$$\sum_{l=l_0}^{m}((\frac{l^2-\mathcal{P}}{4m})) = \sum_{l=0}^{m}((\frac{l^2-\mathcal{P}}{4m})) - \left(\sum_{l=0}^{l_0-1}\frac{l^2-\mathcal{P}}{4m}+\frac{1}{2}\right). \tag{6.36}$$

The sawtooth function is periodic with period 1 so $((\frac{l^2-\mathcal{P}}{4m}))$ has a period of $2m$. This gives us

$$\sum_{l=0}^{m}((\frac{l^2-\mathcal{P}}{4m})) = \frac{1}{2}((\frac{-\mathcal{P}}{4m})) + \frac{1}{2}((\frac{m^2-\mathcal{P}}{4m})) + \frac{1}{2}\sum_{l \bmod 2m}((\frac{l^2-\mathcal{P}}{4m})). \tag{6.37}$$

Again, from periodicity, we clearly have

$$\sum_{l \bmod 2m} ((\frac{l^2-\mathcal{P}}{4m})) = \frac{1}{2} \sum_{l \bmod 4m} ((\frac{l^2-\mathcal{P}}{4m})). \tag{6.38}$$

Now,

$$\begin{aligned} \sum_{l \bmod 4m} ((\frac{l^2-\mathcal{P}}{4m})) &= \sum_{n \bmod 4m} ((\frac{n}{4m})) \sum_{\substack{l \bmod 4m \\ l^2=n+\mathcal{P} \bmod 4m}} 1 \\ &= \sum_{n \bmod 4m} ((\frac{n-\mathcal{P}}{4m})) \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 \end{aligned} \tag{6.39}$$

We can then write the latter expression as

$$\begin{aligned} \sum_{n \bmod 4m} ((\frac{n-\mathcal{P}}{4m})) \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 &= \sum_{n \bmod 4m} \frac{n-\mathcal{P}}{4m} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 \\ &\quad + \sum_{0\le n<\mathcal{P}} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 - \sum_{\mathcal{P}<n<4m} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 \\ &= -\mathcal{P} + \sum_{n \bmod 4m} \frac{n}{4m} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 \\ &\quad + \sum_{0\le n<\mathcal{P}} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 - \sum_{\mathcal{P}<n<4m} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1. \end{aligned} \tag{6.40}$$

The second term is exactly $\frac{1}{4} \sum_{d|4m} h'(-d)$ so we get

$$\begin{aligned} \sum_{n \bmod 4m} ((\frac{n-\mathcal{P}}{4m})) \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 &= -\mathcal{P} + \frac{1}{4} \sum_{d|4m} h'(-d) \\ &\quad + \sum_{0\le n<\mathcal{P}} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1 - \sum_{\mathcal{P}<n<4m} \frac{1}{2} \sum_{\substack{l \bmod 4m \\ l^2=n \bmod 4m}} 1. \end{aligned} \tag{6.41}$$

The issue is, we do not know how to analytically impose an upper bound on the remaining two terms on the right side of the equation above. A heuristic argument is that the quadratic residues are randomly distributed so that the latter two sums behaves like a random walk and are bounded by $m^{1/2}$. If we are able to analytically impose a bound of $m^{1/2+\epsilon}$, it would imply the inequality (6.33) and thus prove Conjecture 6.6.

Chapter 7

Slow Growth around y^b

In this chapter, we consider the weak Jacobi forms with some y^b as their most polar term in their Fourier-Jacobi expansion (6.4) (as opposed to the general case, where the most polar term is an arbitrary $q^a y^b$ term, addressed in Chapter 8). Before we begin, we want to fulfill our promise of filling in the details behind the motivation outlined in the introduction on why we are interested in the growth behaviors of the sums $f_{a,b}(n,l)$ in (1.19).

As mentioned in the introduction, a weak Jacobi form of weight 0 admits an exponential lift to a Siegel modular form, and the growth of $f_{a,b}(n,l)$ about its most polar term $q^a y^b$ indicates the growth of the Fourier coefficients of the lifted Siegel modular form. For completeness, we describe this lift and the emergence of the sums $f_{a,b}(n,l)$, but we will not need it for the remainder of the thesis.

The exponential lift is described in [52, Theorem 2.1], which we summarize here. Given a weak Jacobi form $\varphi_{0,t}$ of weight 0 with Fourier-Jacobi expansion

$$\varphi_{k,m}(\tau,z) = \sum_{n\geq 0}\sum_{l\in\mathbb{Z}} c(n,l) q^n y^l.$$

Define

$$A = \frac{1}{24}\sum_{l\in\mathbb{Z}} c(0,l), \quad B = \frac{1}{2}\sum_{l>0} l c(0,l), \quad C = \frac{1}{4}\sum_{l\in\mathbb{Z}} l^2 c(0,l). \tag{7.1}$$

The function $\varphi_{0,t}(\tau, z)$ admits a lift to a Siegel modular form $\Phi_\varphi : \mathbb{H}_2 \to \mathbb{C}$,

$$\Phi_\varphi(\Omega) = \text{Exp-Lift}(\varphi)(\Omega) = q^A y^B p^C \prod_{n>0} (1 - q^n y^l)^{c(0,l)} \times \prod_{l<0} (1 - y^l)^{c(0,l)} \times \prod_{r>0} (1 - q^n y^l p^{tr})^{c(nr,l)} \quad (7.2)$$

for the paramodular group

$$\Gamma_t := \begin{pmatrix} \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t^{-1}\mathbb{Z} \\ \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ t\mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(4, \mathbb{Q}). \quad (7.3)$$

We are interested in growth behavior of the Fourier coefficients $d(m, n, l)$ with negative *discriminant* $4mn - l^2 < 0$ of the meromorphic Siegel modular form

$$\frac{1}{\text{Exp-Lift}(\varphi)(\Omega)} = \sum_{m,n,l} d(m, n, l) p^m q^n y^l, \quad (7.4)$$

where here we expand the Fourier coefficients in the region $Im(\rho) \gg Im(\tau) \gg Im(z) > 0$.

Sen in [85] shows we may compute $d(m, n, l)$ by making a contour integration so that

$$d(m, n, l) = \sum_{\boldsymbol{p}_i} \frac{1}{2\pi i} \text{Res}(\frac{q^{-n} p^{-m} y^{-l}}{\Phi}, \boldsymbol{p}_i), \quad (7.5)$$

where $\boldsymbol{p}_i$ are the poles inside the contour.

To know which residues to take for a contour, we need to know the poles of Exp-Lift($\varphi_{0,t}$). Following [52, Section 1.3], the divisors of Exp-Lift($\varphi_{0,t}$) are the *Humbert surfaces* for $D := 4ta - b^2$ with $D < 0$ defined by

$$H_D(b) = \Gamma_t \cdot \{\Omega \in \mathbb{H}_2 \mid a\tau + bz + t\rho = 0\}, \quad (7.6)$$

i.e. the orbit of the level set of

$$a\tau + bz + t\rho = 0 \quad (7.7)$$

under the action of the paramodular group Γ_t.

The Humbert surfaces depend on D and b mod $2t$, and each polar term $q^a y^b$ of negative discriminant $D = 4ta - b^2$ in the Fourier-Jacobi coefficients of $\varphi_{0,t}$ has an associated Humbert surface $H_{D,b}$. The lifts of weak Jacobi forms of the same index t then have the same divisors $H_D(b)$, but with differing multiplicity.

The multiplicity of $m_{D,b}$ of $H_D(b)$ is given by

$$m_{D,b} = \sum_{n>0} c(n^2 a, nb). \tag{7.8}$$

Since $q^a y^b$ is polar, $c(n^2 a, nb)$ are polar coefficients as well and the multiplicity of the Humbert surface $H_{D,b}$ is determined by the polar terms of $\varphi_{0,t}$.

We may evaluate the residue at the representative (7.7) of the Humbert surface, i.e. at

$$\boldsymbol{p}_i : p^t = q^a y^b. \tag{7.9}$$

For simple poles $m_{D,b} = 1$, by [58, (3.21)] one obtains

$$\begin{aligned} \operatorname{Res}\left(\frac{q^{-n}p^{-m}y^{-p}}{\Phi_\varphi}, \boldsymbol{p}_i\right) = (-1)^{2B} q^{-A+\frac{a}{t}C} y^{B+\frac{b}{t}C} \prod_{l>0} (1-y^l)^{-c(0,l)} \\ \times \prod_{\substack{n,l\in\mathbb{Z} \\ (n,l)\neq(0,0)}} (1-q^n y^l)^{-f_R(n,l)}, \end{aligned} \tag{7.10}$$

where

$$f_R(n,l) = \sum_{r=0}^{\infty} c(nr + ar^2, l - br). \tag{7.11}$$

Here is how $f_{a,b}(n,l)$ arises. From (7.10), the growth behavior of the residue, and thus of $d(m,n,l)$, is determined by the sums $f_R(n,l)$. Because of the bound $4tn - l^2 < -t^2 \implies c(n,l) = 0$, $f_R(n,l)$ differs from $f_{a,b}(n,l)$ by only finitely many terms. The latter sum is preferred to work with as [58] discovered generating functions for $f_{a,b}(n,l)$ in the case of $a = 0$, discussed in Proposition 7.3 of the next section.

Let us interpret the two possible behaviors of $d(m,n,l)$ in terms of $f_{a,b}(n,l)$ in this simplified regime. The asymptotic growth of $c(n,l)$ for large discriminant is

$$c(n,l) \sim \exp \pi \sqrt{\frac{|\Delta_{\min}|}{t^2}(4tn - l^2)}, \tag{7.12}$$

where $\Delta_{\min}$ is the maximal polarity of the weak Jacobi form. If there are not substantial cancellations inside the sum of $f_{a,b}(n,l)$, then $f_{a,b}(n,l)$ will be dominated by the most polar term in its sum and have exponential growth. This leads to fast growth for $d(m,n,l)$. However, in nongeneric cases, there are significant cancellations between the coefficients in the sum of $f_{a,b}(n,l)$, leading to subexponential growth in $f_{a,b}(n,l)$. This gives us slow growth for $d(m,n,l)$.

Now that we've established the importance of the sums $f_{a,b}(n,l)$ for the lifted Siegel modular form, we illustrate the dramatic difference between the two possible growth behaviors of $f_{a,b}(n,l)$ with a simple example.

Example 7.1. The weak Jacobi form $\phi_{0,1}$ has y^1 as its most polar term and its sums $f_{0,1}(n,l)$ are slow growing. The table below presents a selection of their values.

(n,l)	$f_{0,1}(n,l)$
(0,0)	12
(7,10)	0
(14,20)	0

In contrast, the weak Jacobi form $\phi_{0,1}^2$ has y^2 as its most polar term and its sums $f_{0,2}(n,l)$ are fast growing. The table below of a few selected values clearly demonstrates this.

(n,l)	$f_{0,2}(n,l)$
(0,0)	104
(3,2)	2390434947
(7,10)	8074095060829281900923310709

The reader may wonder why there is such a dramatic difference in the growth behaviors. As shown in [58], there are generating functions for $f_{0,b}(n,l)$ in terms of modular forms. We will give a summary of these generating functions in the next section, but for now, in the case of $\phi_{0,1}$, there is a single generating function for $f_{0,1}(n,l)$ and it is a holomorphic modular form of weight 0, i.e. it is a constant. So, in fact, $f_{0,1}(n,l)$ may only attain the values 12 or 0. In the case of $\phi_{0,1}^2$, the generating functions for $f_{0,2}(n,l)$ are nonholomorphic modular forms of weight 0, and therefore $f_{0,2}(n,l)$ grows exponentially in n, l.

The remainder of this chapter is devoted to our findings on weak Jacobi forms with slow growing $f_{0,b}(n,l)$, where y^b is their most polar term.

7.1 Slow Growth Forms

For a weak Jacobi form $\phi_{0,m}$ of weight 0 and index m, define a sum of its Fourier-Jacobi coefficients,

$$f_{0,b}(n,l) = \sum_{r\in\mathbb{Z}} c(rn, l-br), \tag{7.13}$$

where $c(rn, l-br)$ is the Fourier-Jacobi coefficient of $q^{rn}y^{l-br}$. This sum is a finite sum as $c(n,l) = 0$ whenever $4mn - l^2 < -m^2$.

In [61], the behavior of $f_{0,b}(n,l)$ was classified, summarized in the theorem below.

Theorem 7.1. *[61] The functions $f_{0,b}(n,l)$ have two types of asymptotic behavior, as a function of n and l. In the slow growth case, $f_{0,b}(n,l)$ takes on only finitely many distinct values as n and l range over $\mathbb{Z}$. In the fast growth case, $f_{0,b}(n,l)$ is unbounded and grows exponentially with n and l. In this case, its growth is roughly of the form*

$$f_{0,b}(n,l) \sim \exp 2\pi\sqrt{4\gamma(tn^2/b^2 + nl/b)}, \tag{7.14}$$

for some $\gamma \leq 1$.

From Theorem 7.1, we establish the following definition.

Definition 7.1. A weak Jacobi form $\varphi_{0,m}$ has slow growth at y^b if $f_{0,b}(n,l)$ exhibits subexponential growth.

Theorem 7.1, of course, gives us a stronger conclusion about the possible growth cases of $f_{0,b}(n,l)$ beyond simply subexponential or exponential growth. It states that if $f_{0,b}(n,l)$ has subexponential growth, then $f_{0,b}(n,l)$

has only finitely many distinct values, which is surprising. To prove this, the authors of [61] found generating functions for the coefficients $f_{0,b}(n,l)$ for $\varphi_{0,m}$ in terms of a sum of specializations $q^{mr^2/b^2}\varphi_{0,m}(\tau,(r\tau+s)/b)$, $r,s=0,\dots,b-1$ of the underlying weak Jacobi form. They concluded that the weak Jacobi form $\varphi_{0,m}$ is slow growth if and only if these specializations are holomorphic modular forms, in which case, the specializations are constant functions and there are only finitely many nonzero $f_{0,b}(n,l)$. Before giving our findings, we review their results regarding these generating functions.

The specialization $\chi_{r,s}(\tau)=q^{mr^2/b^2}\varphi_{0,m}(\tau,(r\tau+s)/b)$ is indeed a modular form, per the following theorem of Eichler and Zagier.

Theorem 7.2. *[76, Theorem 1.3] Let $\phi_{k,m}(\tau,z)$ be a Jacobi form on Γ of weight k and index m. Let α and β be rational numbers. The specialization $f(\tau)=e^{2\pi im(\alpha^2\tau)}\phi_{k,m}(\tau,\alpha\tau+\beta)$ is a modular form of weight k on some subgroup Γ' of finite index depending only on Γ,α,β.*

The generating functions of Belin et al. are given by the following proposition:

Proposition 7.3. *[58, (4.16)] The generating functions for $f_{0,b}(n,l)$ are given by*

$$F_{n_b,k}(\tau)=\frac{1}{b}\sum_{j=0}^{b-1}\chi_{n_b,j}(\tau)e^{-2\pi ikj/b}, \tag{7.15}$$

for $n_b=0,\dots,b-1$ and $k=0,\dots,b-1$. Here, $\chi_{n_b,j}(\tau)=q^{tn_b^2/b^2}\varphi(\tau,(n_b\tau+j)/b)$ are specializations of the weak Jacobi form.

Proof. This proof is a rephrasing of the same argument in [58, Section 4.3]. We do so because we will later show that it is not possible to make a similar argument for $f_{a,b}(n,l)$ with $a\neq 0$. The goal of this proof is to be able to use Theorem 7.2. That is, the specialization

$$e^{2\pi it\alpha^2\tau}\phi(\tau,\alpha\tau+\beta)=q^{t\alpha^2}\sum_{n\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}c(n,l)q^nq^{l\alpha}e^{2\pi il\beta} \tag{7.16}$$

is a modular form of weight 0.

To be able to use this, we must change the form of the sum

$$f_{0,b}(n,l)=\sum_{r\in\mathbb{Z}}c(rn,l-br)$$

to match the form of the first sum of (7.16), i.e.

$$f_{0,b}(n,l) = \sum_{\hat{m}\in\mathbb{Z}} c(\_\_, \hat{m}).$$

This will allow $f_{0,b}(n,l)$ to arise from specializations of weak Jacobi forms.

The Fourier-Jacobi coefficient $c(n,l)$ depends only on its discriminant $4tn - l^2$ and on the value l mod $2t$, which we encode as

$$c(n,l) = c(n + l\lambda + t\lambda^2, l + 2t\lambda), \ \lambda \in \mathbb{Z}. \tag{7.17}$$

We apply this to $c(rn, l - br)$. Set $k := l + 2t\lambda$ and we obtain

$$\begin{aligned} f_{0,b}(n,l) &= \sum_{r\in\mathbb{Z}} c(rn + (l-br)\lambda + t\lambda^2, l - br + 2t\lambda) \\ &= \sum_{\hat{m}\in b\mathbb{Z}+k} c(2tn\lambda/b + nl/b + \hat{m}(-n/b + \lambda) - t\lambda^2, \hat{m}) \\ &= \sum_{\hat{m}\in\mathbb{Z}} c(2tn\lambda/b + nl/b + \hat{m}(-n/b + \lambda) - t\lambda^2, \hat{m})\delta^{(b)}_{\hat{m},k} \\ &= \frac{1}{b}\sum_{j=0}^{b-1}\sum_{\hat{m}\in\mathbb{Z}} c(2tn\lambda/b + nl/b + \hat{m}(-n/b + \lambda) - t\lambda^2, \hat{m}) e^{2\pi i \frac{j}{b}\hat{m}} e^{-2\pi i \frac{k}{b} j}, \end{aligned} \tag{7.18}$$

where we used the b-periodic Kronecker delta function

$$\delta^{(b)}_{\hat{m},k} = \frac{1}{b}\sum_{j=0}^{b-1} e^{2\pi i(\hat{m}-k)j/b} = \begin{cases} 1 & \text{if } \hat{m} \in b\mathbb{Z} + k \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

The factor $e^{2\pi i \frac{j}{b}\hat{m}}$ in 7.18 imposes the constraint

$$\beta = j/b.$$

We must make one comment. Observe that

$$\tilde{M} := 2tn\lambda/b + nl/b + \hat{m}(-n/b + \lambda) - t\lambda^2$$

is integral when $\hat{m} \in b\mathbb{Z}+k$. When the index $\hat{m}$ is changed to run over $\mathbb{Z}$, the coefficient $c(2tn\lambda/b + nl/b + \hat{m}(-n/b + \lambda) - t\lambda^2, \hat{m})$ is no longer well-defined for $\hat{m} \notin b\mathbb{Z} + k$. Instead, we understand we may place any number here without changing the value of the sum (7.18) because of the presence of the Kronecker delta function (7.19).

To free up subscripts, let us write $f(n, l) := f_{0,b}(n, l)$. Looking at the form (7.16), a generating function $F(\tau) = \sum_M f(n, l)q^M$ suggests the following (nonsensical) equation

$$\begin{aligned} \sum_M f(n,l)q^M &= \sum_{\tilde{M}\in\mathbb{Z}} f(n,l)q^{\tilde{M}+\hat{m}\alpha+t\alpha^2} \\ &= \frac{1}{b}\sum_{j=0}^{b-1}\left(\sum_{\tilde{M}\in\mathbb{Z}}\sum_{\hat{m}\in\mathbb{Z}} c(\tilde{M},\hat{m})q^{\tilde{M}+\hat{m}\alpha+t\alpha^2}e^{2\pi i\frac{j}{b}\hat{m}}\right)e^{-2\pi i\frac{k}{b}j}. \end{aligned} \tag{7.20}$$

Note (7.20) is currently nonsensical since there is no relationship between n, l and $\tilde{M}$. The index set for M is also unspecified. However, the requirement to have a single index M already imposes a constraint on α. From the first equality of (7.20), $\tilde{M} + \hat{m}\alpha + t\alpha^2$ must reduce to a single variable. Since $\hat{m}$ appears in $\tilde{M}$ as the term $(-n/b + \lambda)\hat{m}$, we see that we must have

$$\alpha = n/b - \lambda.$$

Now, let us accomplish a sensible form of (7.20). This means k must be fixed. Also, in order to get finitely many specializations, $\alpha = n/b - \lambda$ must reduce to a finite set of values. So a single generating function (7.20) cannot have every $f_{0,b}(n, l)$ as coefficients, but only a subset thereof. In fact, the desired equation clearly tells us that we must reduce the variables n, l to a single variable $\hat{M}$.

Write $n_b = n \bmod b \in \{0, 1, \ldots, b - 1\}$. Then $n = bs + n_b$ for some s. In terms of a fixed n_b, $\alpha = n_b/b + s - \lambda$. Clearly, we must take $\lambda = s$, i.e.

$$\lambda = (n - n_b)/b.$$

Thus, for fixed n_b and k, we may write such $f(n, l)$ as $f_{n_b,k}(s)$ with one single variable $s \in \mathbb{Z}$. Then $M = \tilde{M} + \hat{m}\alpha + t\alpha^2 = -ts^2 + sk + kn_b/b + tn_b^2/b$ and we may use instead the variables $f_{n_b,k}(M)$.

We now write a generating function $F_{n_b,k}(\tau)$ for $f_{n_b,k}(M)$ with $M \in \mathbb{Z} + kn_b/b + tn_b^2/b$.

$$F_{n_b,k}(\tau) = \sum_{M\in\mathbb{Z}+n_bk/b+n_b^2t/b^2} f_{n_b,k}(M)q^M = \frac{1}{b}\sum_{j=0}^{b-1} q^{n_b^2t/b^2}\varphi_{0,t}(\tau,\frac{n_b\tau+j}{b})e^{-2\pi ikj/b}$$
$$= \frac{1}{b}\sum_{j=0}^{b-1}\chi_{n_b,j}(\tau)e^{-2\pi ikj/b}, \tag{7.21}$$

where we define the specializations

$$\chi_{r,s}(\tau) := q^{tr^2/b^2}\varphi(\tau,(r\tau+s)/b), \qquad r,s=0,\dots,b-1. \tag{7.22}$$

□

The remaining results of [58, Section 4.4] follows as a corollary of Proposition 7.3, which we summarize below.

Corollary 7.4. *$f_{0,b}(n,l)$ is slow growing if and only if the specializations $\chi_{r,s}(\tau)$ are holomorphic functions, in which case $\chi_{r,s}$ are constant functions.*

For $\varphi_{0,t}(\tau,z)$ with slow growing $f_{0,b}(n,l)$, the values of $f_{0,b}(n,l)$ are given by [58, Equation (4.24)]

$$f_{0,b}(n,l) = \begin{cases} \frac{1}{b}\sum_{j=0}^{b-1}\chi_{n_b,j}e^{-2\pi ikj/b} & : tn+bl=0 \text{ or } n=0 \\ 0 & : \text{ else,} \end{cases} \tag{7.23}$$

where $n_b = n \bmod b$ with $n_b \in \{0,1,\dots,b-1\}$ and $k = 2(n-n_b)t/b+l$. Here, $\chi_{n_b,j}$ are constants.

In terms of the Fourier-Jacobi coefficients of $\varphi_{0,t}(\tau,z)$,

$$f_{0,b}(n,l) = \begin{cases} \sum_{\hat{m}\in b\mathbb{Z}-l-n_bt/b} c(-n_b\hat{m}/b - n_b^2t/b^2,\hat{m}) & : tn+bl=0 \text{ or } n=0 \\ 0 & : \text{ else.} \end{cases} \tag{7.24}$$

Given a weak Jacobi form, we can check slow growth of $f_{0,b}(n,l)$ using Corollary 7.4. We follow [58][Section 5.1] here. The specialization $\chi_{r,s}(\tau)$ is holomorphic if it has no q^β term with $\beta < 0$. The term $q^n y^{-l}$ in φ leads to q^β in $\chi_{r,s}$ with

$$\beta = tr^2/b^2 + n - lr/b,$$

so by taking α to be the max of $-\beta$ over all the specializations, i.e.

$$\alpha := \max_{r=0,\ldots,b-1} [-\beta], \tag{7.25}$$

we see φ has slow growth if and only if none of the terms $q^n y^l$ with $\alpha > 0$ appear in the Fourier-Jacobi expansion of φ. Note, a nonopolar term $q^n y^l$ automatically has $\alpha \leq 0$ so we only need to consider the polar terms of φ.

We have computed the following dimensions of the space of weak Jacobi forms that are slow growth about its maximal polar term y^b, for $m \leq 61$. The code for fast extraction of polar coefficients for a basis of $J_{0,m}$ is given in the Appendix. The table expands Table 2 of [59], which ends at $m = 18$. We also computed $m = 71$, which we include here. The table gives further experimental evidence for Conjecture 1.1 that there exists a weak Jacobi form of slow growth for each index m.

However, we find at $m = 61$ an exception to the observation in [59] that there exists a weak Jacobi form with slow growth for every $m, b = \lfloor\sqrt{m}\rfloor$. At $m = 61$, $b = \lfloor\sqrt{61}\rfloor$ there are no slow growing weak Jacobi forms.

Note, the table has finitely many b for each m. Terms y^b with $b > \lfloor\sqrt{m}\rfloor$ have $\alpha > 0$ so we need not consider weak Jacobi forms with these terms in their Fourier-Jacobi expansions.

m	b	dim	m	b	dim	m	b	dim	m	b	dim
1	1	1	8	1	0	12	1	0	16	1	0
2	1	1	8	2	2	12	2	2	16	2	1
3	1	1	9	1	0	12	3	3	16	3	2
4	1	1	9	2	1	13	1	0	16	4	4
4	2	2	9	3	3	13	2	0	17	1	0
5	1	0	10	1	0	13	3	1	17	2	0
5	2	1	10	2	1	14	1	0	17	3	0
6	1	1	10	3	2	14	2	0	17	4	2
6	2	2	11	1	0	14	3	1	18	1	0
7	1	0	11	2	0	15	1	0	18	2	0
7	2	1	11	3	1	15	2	1	18	3	3
						15	3	2	18	4	3
									19	2	0
									19	3	1
									19	4	1

m	b	dim	m	b	dim	m	b	dim
20	2	0	28	3	1	37	3	0
20	3	1	28	4	3	37	4	0
20	4	4	28	5	2	37	5	0
21	2	0	29	3	0	37	6	3
21	3	1	29	4	1	38	3	0
21	4	2	29	5	1	38	4	0
22	2	0	30	3	1	38	5	1
22	3	1	30	4	2	38	6	3
22	4	2	30	5	4	39	3	0
23	2	0	31	3	0	39	4	0
23	3	0	31	4	0	39	5	1
23	4	1	31	5	1	39	6	4
24	2	1	32	3	0	40	3	0
24	3	2	32	4	3	40	4	2
24	4	4	32	5	2	40	5	3
25	3	0	33	3	0	40	6	4
25	4	1	33	4	2	41	3	0
25	5	4	33	5	1	41	4	0
26	3	0	34	3	0	41	5	0
26	4	2	34	4	1	41	6	1
26	5	2	34	5	1	42	3	0
27	3	1	35	3	0	42	4	1
27	4	1	35	4	0	42	5	2
27	5	2	35	5	3	42	6	6
			36	3	1	43	3	0
			36	4	3	43	4	0
			36	5	2	43	5	0
			36	6	7	43	6	3

m	b	dim	m	b	dim	m	b	dim
44	3	0	50	6	2	56	6	2
44	4	0	50	7	3	56	7	5
44	5	2	51	3	0	57	4	0
44	6	3	51	4	0	57	5	0
45	3	0	51	5	1	57	6	2
45	4	1	51	6	3	57	7	2
45	5	2	51	7	3	58	4	0
45	6	4	52	3	0	58	5	0
46	3	0	52	4	0	58	6	3
46	4	0	52	5	1	58	7	2
46	5	1	52	6	3	59	4	0
46	6	3	52	7	2	59	5	0
47	3	0	53	3	0	59	6	0
47	4	0	53	4	0	59	7	1
47	5	0	53	5	0	60	4	1
47	6	1	53	6	0	60	5	3
48	3	0	53	7	2	60	6	5
48	4	2	54	3	1	60	7	3
48	5	2	54	4	0	61	4	0
48	6	6	54	5	1	61	5	3
49	3	0	54	6	6	61	6	1
49	4	0	54	7	2	61	7	0
49	5	0	55	4	0	71	4	0
49	6	2	55	5	1	71	5	0
49	7	5	55	6	2	71	6	0
50	3	0	55	7	1	71	7	1
50	4	0	56	4	0			
50	5	4	56	5	0			

Table 7.1: Dimension of the Space of Weak Jacobi Forms of Weight 0 and Index m that are Slow Growing About Their Most Polar y^b Term

We introduce a lower bound on the dimension of the space of weak Jacobi forms that have slow growth about their most polar term y^b, derived from linear algebra.

Proposition 7.5. *For index m and integer b, let $\rho(m,b)$ be the number of polar terms $q^n y^l$ with $4mn - l^2 \geq -b^2$ and $\omega(m,b)$ be the number of such polar* *terms with $\alpha > 0$. The dimension of weak Jacobi forms with slow growth about their most polar term y^b is bounded below by $j(m) - \rho(m, b) - \omega(m, b)$.*

Proof. The requirement that a weak Jacobi form has Fourier expansion beginning at y^b can be encoded as the solution to a linear system, with respect to a basis of $J_{0,m}$ and the polar terms for index m of polarity $> -b^2$, along with the polar term y^b itself. Indeed, given a basis of $J_{0,m}$, let A be the matrix where the j-th row is the polar coefficients $c(n, l)$ for $-b^2 < 4mn - l^2 < 0$, along with $c(0, b)$, of the j-th basis element. The linear system is

$$Ax = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{7.26}$$

Of course, if the linear system has any solution x', then all the solutions are given by $x = x' + \ker A$. The dimension of the space of weak Jacobi forms with most polar term y^{-b} is then either 0 (no solution) or $1 + \mathrm{null}(A)$ (when there exists a solution).

The matrix A is onto if rank $A = 1 + \rho(m, b)$, in which case the linear system clearly has a solution. The nullspace of A is bounded below by $j(m) - (1 + \rho(m, b))$. When there exists a solution, the dimension of the space of such weak Jacobi forms is bounded below by $j(m) - \rho(m, b)$. We expect that generically, this lower bound is optimal for b such that $1 + \rho(m, b) \leq j(m)$.

Now, let us address slow growth. Recall that for fixed b, each polar term $q^n y^l$ has a value α (7.25) such that $q^n y^l$ leads to fast growth in φ if $\alpha > 0$. We can encode as a linear system the condition for a weak Jacobi form to have none of these terms, in which case the weak Jacobi form has slow growth.

Given a basis of weak Jacobi forms whose most polar term is given by y^b, the space of weak Jacobi forms that have slow growth about y^b is the space of solutions to the linear system

$$By = 0, \tag{7.27}$$

where each row of B is given by the coefficients $c(n, l)$, with $\alpha > 0$, of a basis element.

The nullspace of B is bounded below by $1 + (A) - \omega(m, b)$. We expect equality to happen for generic m, b when $1 + (A) > \omega(m, b)$.

We conclude that whenever there exists a weak Jacobi form with y^b as its most polar term, the dimension of slow growth weak Jacobi forms is bounded below by $j(m) - \rho(m, b) - \omega(m, b)$. This lower bound is optimal whenever A and B have maximal rank, which occurs for generic matrices. □

Comparing to the actual dimensions in Table 7.1, this lower bound is nearly always optimal. The indices m and values of b for which the lower bound fails to be equal to the actual dimension are $(m = 41, b = 6)$, $(m = 47, b = 6)$, $(m = 55, b = 7)$, $(m = 59, b = 7)$, $(m = 61, b = 6)$, $(m = 71, b = 7)$, and $(m = 71, b = 8)$. For all of these, the lower bound is zero but the actual dimension is one. These preliminary results suggest this lower bound is optimal for generic m, b, however we suspect this is deceptive. The lower bound is easy to compute, so we present several scatter plots of its value for large m.

We check whether this lower bound gives us a slow growing weak Jacobi form for every $m, b = \lfloor\sqrt{(m)}\rfloor$. Under the assumption that there always exists a weak Jacobi form with most polar term $y^{\lfloor\sqrt{(m)}\rfloor}$, we have the following scatter plot of the lower bound for index m, $b = \lfloor\sqrt{(m)}\rfloor$. We find the lower bound is frequently zero for large indices m.

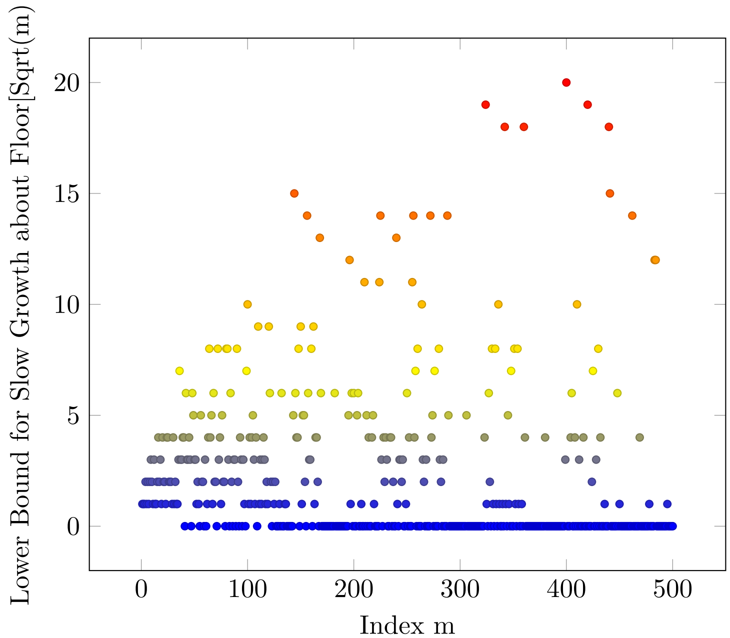


Figure 7.1: Scatterplot of the lower bound for the dimension of weak Jacobi forms that are slow growth about its most polar term y^b with $b = \lfloor\sqrt{m}\rfloor$.

More generally, let us address Conjecture 1.1. Under the assumption that there exists a weak Jacobi form with most polar term y^b for $\lceil \frac{m}{6} \rceil \leq b \leq \lfloor\sqrt{m}\rfloor$, we have the following scatter plot of the lower bound for the dimension of slow growing weak Jacobi forms of index m.

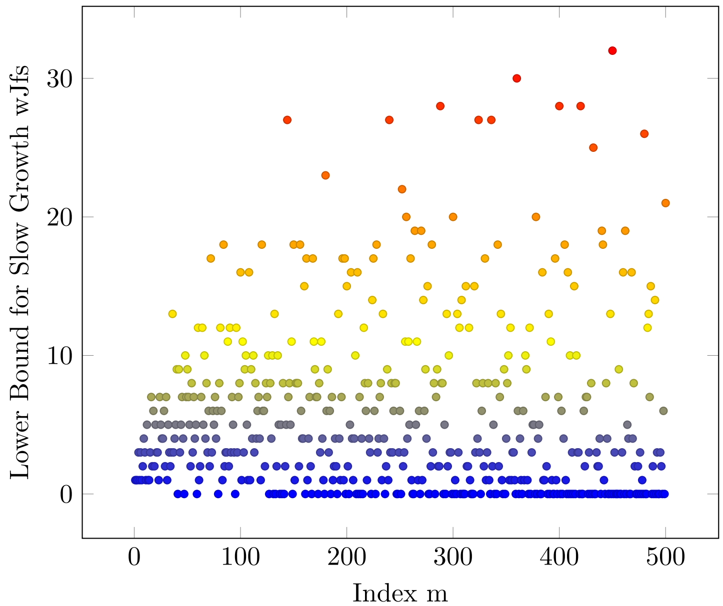


Figure 7.2: Scatterplot of the lower bound for the dimension of weak Jacobi forms that are slow growth about its most polar term y^b, for any b.

These scatter plots show that for large m, the lower bound for the dimension is frequently zero, i.e. there are no slow growing weak Jacobi forms for those indices. This suggests that Conjecture 1.1 may be false, but we will next discuss the discovery of a large class of slow growing weak Jacobi forms, given by theta quotients. Because of this discovery, we suspect the lower bound fails to be optimal for most m as m becomes large.

7.2 Theta Quotients

We introduce the theta function

$$\theta_1(\tau, z) = -q^{1/8}y^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}y)(1-q^ny^{-1}). \tag{7.28}$$

We will find that quotients of such functions give rise to a large amount of slow growth Jacobi forms.

The theta function $\theta_1(\tau, z)$ of (7.28) is not a classical integral weight and index weak Jacobi form as it has weight 1/2 and index 1/2, so we must introduce a more general definition of weak Jacobi forms. In order for the transformation laws (6.2) to give a proper action when we have half integral weights and indices, we must make a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ and we obtain the integral Heisenberg group $H(\mathbb{Z})$,

$$0 \to \mathbb{Z} \to H(\mathbb{Z}) \to \mathbb{Z} \times \mathbb{Z} \to 0. \tag{7.29}$$

Definition 7.2. The *Heisenberg group* $H(\mathbb{Z})$ is the group

$$H(\mathbb{Z}) = \{[\lambda, \mu : \kappa] \mid \lambda, \mu, \kappa \in \mathbb{Z}\}$$

with action $[\lambda, \mu : \kappa] \cdot [\lambda', \mu' : \kappa'] = [\lambda + \lambda', \mu + \mu' : \kappa + \kappa' + \lambda\mu' - \lambda'\mu]$. The subgroup $C_{\mathbb{Z}} = \{[0, 0 : \kappa], \kappa \in \mathbb{Z}\}$ is the center of $H(\mathbb{Z})$ and $H(\mathbb{Z})/C_{\mathbb{Z}} \cong \mathbb{Z}^2$.

Definition 7.3. We define the extended Jacobi group $\Gamma^J(\mathbb{Z})$ to be the semidirect product $SL(2, \mathbb{Z}) \ltimes H(\mathbb{Z})$, with $SL(2, \mathbb{Z})$ acting on $H(\mathbb{Z})$ via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [x, y : \kappa] = [dx - cy, -bx + ay : \kappa].$$

Earlier on, we introduced multiplier systems for half integral modular forms. We must now do the same for Jacobi forms.

Definition 7.4. A *multiplier system* of $\Gamma^J(\mathbb{Z})$ is a character $\nu : \Gamma^J(\mathbb{Z}) \to \mathbb{C}^*$. Characters ν of $\Gamma^J(\mathbb{Z})$ of finite order are of the form $\nu((M, [x, y : \kappa])) = \nu_\eta^D(M) \times \nu_H^t([x, y : \kappa])$ [86], where ν_η is the multiplier system of the Dedekind eta function, D is some integral power $0 \le D < 24$, $v_H([x, y : \kappa]) := (-1)^{x+y+xy+\kappa}$ is the unique binary character of $H(\mathbb{Z})$, and $t = 0$ or 1.

Note that while multiplier systems of the form $\nu_\eta^D(M) \times \nu_H^t([x, y : \kappa])$ are characters of the extended Jacobi group $\Gamma^J(\mathbb{Z})$, they are not guaranteed to descend to characters of the base Jacobi group Γ^J.

Definition 7.5. A holomorphic function $\varphi_{k,m}(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ for $k \in \frac{1}{2}\mathbb{Z}$, $m \in \frac{1}{2}\mathbb{Z}$ is a *weakly holomorphic Jacobi form* of weight k and index m with a multiplier system $\nu : \Gamma^J(\mathbb{Z}) \to \mathbb{C}^*$ of finite order if $\varphi_{k,m}$ satisfies

$$\begin{aligned} \varphi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= \nu(M)(c\tau + d)^k e^{i2\pi m \frac{cz^2}{c\tau+d}} \varphi_{k,m}(\tau, z), \\ \varphi_{k,m}(\tau, z + \lambda\tau + \mu) &= \nu([\lambda, \mu : \kappa]) e^{-i2\pi m(\lambda^2\tau + 2\lambda z)} \varphi_{k,m}(\tau, z), \end{aligned} \tag{7.30}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $[\lambda, \mu : \kappa] \in H(\mathbb{Z})$, and has a Fourier-Jacobi expansion of the form

$$\varphi_{k,m}(\tau, z) = \sum_{\substack{n \geq n_0, n \equiv \frac{D}{24} \bmod \mathbb{Z} \\ l \in \frac{1}{2}\mathbb{Z}}} f(n,l) q^n y^l. \tag{7.31}$$

Here, $n_0 \in \mathbb{Z}$ is a constant and $\nu(M) = \nu_\eta^D(M)$ with $0 \leq D < 24$. Denote by $\tilde{J}_{0,m}$ the space of weakly holomorphic Jacobi forms of weight 0 and index m with trivial multiplier system.

If the Fourier-Jacobi expansion of $\varphi_{k,m}$ begins at $n_0 \geq 0$, then $\varphi_{k,m}$ is a *weak Jacobi form* with multiplier system ν. When k and m are integral and ν is trivial, this definition agrees with the original definition of weak Jacobi forms in Definition 6.3.

Our theta function $\theta_1(\tau, z)$ is a weak Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with multiplier system $\nu_\eta^3 \times \nu_H$. The quotient of weak Jacobi forms $\frac{\varphi_{k,m}(\tau,\alpha z)}{\varphi_{k',m'}(\tau,\beta z)}$ is a meromorphic Jacobi form of weight $k - k'$ and index $\alpha^2 m - \beta^2 m'$. To get a theta quotient of weight 0, we then require the same number of theta functions in the numerator as in the denominator, and this cancels out the multiplier system of the theta function.

Lemma 7.6. *Given $N > 0$, there are $P(N)$ types of holomorphic weight 0 theta quotients*

$$\prod_{j=1}^{N} \frac{\theta_1(\tau, n_1 z)\theta_1(\tau, n_2 z) \cdots \theta_1(\tau, n_N z)}{\theta_1(\tau, m_1 z)\theta_1(\tau, m_2 z) \cdots \theta_1(\tau, m_N z)},$$

where $P(N)$ is the partition function. The partition $N = r_1 + \cdots + r_s$ corresponds to a holomorphic quotient with r_1 many coprime scalers in the denominator that divide n_1, r_2 many coprime scalers in the denominator that divide n_2, and so on.

Proof. The divisors of $\theta_1(\tau, z)$ are the zeros $z = \lambda + n\mu, (\lambda, \mu) \in \mathbb{Z}^2$. This implies for positive integers k, k' that a quotient $\frac{\theta_1(\tau,kz)}{\theta_1(\tau,k'z)}$ is holomorphic so long as $\{z = \frac{1}{k'}(\lambda\tau + \mu) \mid \lambda, \mu \in \mathbb{Z}\} \subset \{z = \frac{1}{k}(\lambda\tau + \mu) \mid \lambda, \mu \in \mathbb{Z}\}$ i.e. $k' \mid k$. This argument easily generalizes to the case of multiple theta quotients. □

To illustrate the necessity of coprimality among the partitioned $m_1, \dots, m_r$ dividing n, consider the nonholomorphic theta quotient

$$\frac{\theta_1(\tau, 8z)\theta_1(\tau, z)}{\theta_1(\tau, 4z)\theta_1(\tau, 2z)}. \tag{7.32}$$

Here $m_1 = 4$ and $m_2 = 2$ divide $n = 8$ but are not coprime. We have a simple zero at $\{z = \frac{1}{2}(\lambda\tau + \mu) \mid \mu \text{ odd}, \lambda \in \mathbb{Z}\}$ from the numerator $\theta_1(\tau, 8z)$ but a double pole at the same points $\{z = \frac{1}{2}(\lambda\tau + \mu) \mid \mu \text{ odd}, \lambda \in \mathbb{Z}\}$ from the denominators $\theta_1(\tau, 4z)$ and $\theta_1(\tau, 2z)$.

We give some examples that should clarify the above prescription:

$$\begin{array}{ccc} & \text{holomorphic ;} & \text{not holomorphic} \\ \dfrac{\theta_1(\tau, 6z)\theta_1(\tau, 5z)}{\theta_1(\tau, 2z)\theta_1(\tau, 3z)}, & \dfrac{\theta_1(\tau, 16z)\theta_1(\tau, 8z)}{\theta_1(\tau, 4z)\theta_1(\tau, 2z)}; & \dfrac{\theta_1(\tau, 4z)\theta_1(\tau, 5z)}{\theta_1(\tau, 2z)\theta_1(\tau, 2z)}. \end{array}$$

The denominator scaling factors must divide some numerator scaler, which occurs in all the examples above. In the first example, the theta quotient is of partition type $2 = 2+0$, where $2, 3$ are coprime and divide 6. In the second example, the theta quotient is of partition type $2 = 1 + 1$ where 4 divides 16 and 2 divides 8 (we could also say 4 divides 8 and 2 divides 16). However, in the third example, $2, 2$ divide 4 but are not coprime so the quotient fails to be holomorphic.

We are interested in weight 0 weak Jacobi forms with trivial multiplier systems that are theta quotients of scaled versions $\theta_1(\tau, \alpha z)$ of (7.28), with α a positive integer. We want to know which theta quotients have slow growth.

7.3 Slow growth of single theta quotient

We examine what conditions on the scaling factors α and β will result in slow growth about the most polar term $y^b = y^{\frac{1}{2}(\alpha-\beta)}$ in the single theta quotient

$$\varphi_{0,\frac{1}{2}(\alpha^2-\beta^2)}(\tau, z) = \frac{\theta_1(\tau, \alpha z)}{\theta_1(\tau, \beta z)}. \tag{7.33}$$

We prove that the pairs of scaling factors (α, β) that produce a holomorphic slow growing theta quotient of weight zero and integral index are given by the set $\{\big((k + 1)\beta, \beta\big) \mid k \text{ even or } \beta \text{ even}\}$.

From [58], we know slow growth about y^b is equivalent to regularity at $\tau \to i\infty$ of the collection of modular forms $\chi_{r,s}(\tau)$, $0 \le r, s \le b-1$ coming from specializations of the weak Jacobi form $\varphi_{0,\frac{1}{2}(\alpha^2-\beta^2)}$.

The specializations $\chi_{r,s}(\tau)$ of (7.22) for $0 \le r, s \le b-1$ of the quotient (7.33) are

$$\begin{aligned}\chi_{r,s}(\tau) &= \frac{q^{\frac{\alpha^2}{2}\frac{r^2}{b^2}}\theta_1(\tau, \alpha(\frac{r}{b}\tau + \frac{s}{b}))}{q^{\frac{\beta^2}{2}\frac{r^2}{b^2}}\theta_1(\tau, \beta(\frac{r}{b}\tau + \frac{s}{b}))} \\ &= \frac{-q^{\frac{\alpha^2}{2}\frac{r^2}{b^2}}q^{1/8}q^{-\frac{\alpha}{2}\frac{r}{b}}e^{-2\pi i\frac{\alpha}{2}\frac{s}{b}}}{-q^{\frac{\beta^2}{2}\frac{r^2}{b^2}}q^{1/8}q^{-\frac{\beta}{2}\frac{r}{b}}e^{-2\pi i\frac{\beta}{2}\frac{s}{b}}} \\ &\qquad \times \frac{\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}q^{\alpha\frac{r}{b}}e^{2\pi i\alpha\frac{s}{b}})(1-q^n q^{-\alpha\frac{r}{b}}e^{-2\pi i\alpha\frac{s}{b}})}{\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}q^{\beta\frac{r}{b}}e^{2\pi i\beta\frac{s}{b}})(1-q^n q^{-\beta\frac{r}{b}}e^{-2\pi i\beta\frac{s}{b}})}.\end{aligned} \tag{7.34}$$

Regularity of the specialized theta quotient (7.34) as $\tau \to i\infty$ is equivalent to it having only nonnegative powers of q in its Fourier expansion. Thus, we only need compare the lowest powers of q in the numerator and denominator: the lowest power of q in the numerator of $\chi_{r,s}(\tau)$ is greater than or equal to the power of q in the denominator if and only if $\chi_{r,s}(\tau)$ is regular at $i\infty$. Note for $\chi_{0,s}(\tau)$, this approach does not work as naive computation leads to an undefined $0/0$ quotient. However, this is easily fixed as the Fourier-Jacobi expansion of the form $\varphi_{0,\frac{1}{2}(\alpha^2-\beta^2)}(\tau, z)$ is nonnegative in q and for $\chi_{0,s}(\tau)$, its variable y is specialized to $e^{2\pi i\frac{s}{b}}$ which does not modify the powers of q. So $\chi_{0,s}(\tau, z)$ is always regular at $i\infty$.

The term with the lowest power of q in $q^{\frac{\kappa^2}{2}\frac{r^2}{b^2}}\theta_1(\tau, \kappa(\frac{r}{b}\tau + \frac{s}{b}))$ is given by multiplying out all $q^{n-\kappa\frac{r}{b}}$ with negative $n - \kappa\frac{r}{b}$ in the third factor of the product formula (7.28). The lowest power of q in the specialization (7.34) is then

$$\begin{aligned}&\frac{q^{\frac{\alpha^2}{2}\frac{r^2}{b^2}+\frac{1}{8}-\frac{\alpha}{2}\frac{r}{b}+\sum_{n=1}^{\lfloor\alpha\frac{r}{b}\rfloor}n-\alpha\frac{r}{b}}}{q^{\frac{\beta^2}{2}\frac{r^2}{b^2}+\frac{1}{8}-\frac{\beta}{2}\frac{r}{b}+\sum_{n=1}^{\lfloor\beta\frac{r}{b}\rfloor}n-\beta\frac{r}{b}}} \\ &\qquad = q^{\frac{\alpha^2-\beta^2}{2}\frac{r^2}{b^2}-\frac{\alpha-\beta}{2}\frac{r}{b}+\frac{\lfloor\alpha\frac{r}{b}\rfloor(\lfloor\alpha\frac{r}{b}\rfloor+1)}{2}-\frac{\lfloor\beta\frac{r}{b}\rfloor(\lfloor\beta\frac{r}{b}\rfloor+1)}{2}-\alpha\frac{r}{b}\lfloor\alpha\frac{r}{b}\rfloor+\beta\frac{r}{b}\lfloor\beta\frac{r}{b}\rfloor}.\end{aligned} \tag{7.35}$$

The condition for the theta quotient (7.33) to have slow growing $f_{0,b}(n,l)$, for $b = \frac{1}{2}(\alpha - \beta)$, is then

$$\begin{aligned} \frac{\alpha^2-\beta^2}{2}\frac{r^2}{b^2} - \frac{\alpha-\beta}{2}\frac{r}{b} + \frac{\lfloor \alpha\frac{r}{b}\rfloor(\lfloor \alpha\frac{r}{b}\rfloor+1)}{2} - \frac{\lfloor \beta\frac{r}{b}\rfloor(\lfloor \beta\frac{r}{b}\rfloor+1)}{2} \\ -\alpha\frac{r}{b}\lfloor \alpha\frac{r}{b}\rfloor + \beta\frac{r}{b}\lfloor \beta\frac{r}{b}\rfloor \geq 0, \quad 1 \leq r \leq b-1. \end{aligned} \tag{7.36}$$

Proposition 7.7. *The single theta quotients that have slow growing $f_{0,b}(n,l)$ about their most polar term y^b are given by quotients of the form*

$$\frac{\theta_1(\tau,(k+1)\beta z)}{\theta_1(\tau,\beta z)}, \tag{7.37}$$

for k even or β even. For such a quotient, the most polar term is $y^{k\beta/2}$ and the index is $\frac{\beta^2 k(k+2)}{2}$.

Proof. For the quotient (7.33) to be holomorphic on $\mathbb{H}\times\mathbb{C}$, we must have $\beta \mid \alpha$ so we may write $\alpha = (k+1)\beta$. To obtain an integral weight $t = \frac{\beta^2(k^2+2k)}{2}$, we must have that β is even or k is even.

Slow growth is equivalent to the condition (7.36). We have $b = \frac{k\beta}{2}$ and the left side of (7.36) may be written as

$$\begin{aligned} 2\frac{k+2}{k}r^2 - r + \frac{\lfloor (k+1)\frac{2r}{k}\rfloor(\lfloor (k+1)\frac{2r}{k}\rfloor+1)}{2} - \frac{\lfloor \frac{2r}{k}\rfloor(\lfloor \frac{2r}{k}\rfloor+1)}{2} \\ -(k+1)\frac{2r}{k}\lfloor (k+1)\frac{2r}{k}\rfloor + \frac{2r}{k}\lfloor \frac{2r}{k}\rfloor, \quad 0 < r < \frac{k\beta}{2}. \end{aligned} \tag{7.38}$$

Using $\lfloor \frac{k+1}{k}2r\rfloor = \lfloor 2r + \frac{2r}{k}\rfloor = 2r + \lfloor \frac{2r}{k}\rfloor$, (7.38) reduces to 0 for each value of r. Thus, we have slow growth. □

7.4 Slow growth of multiple theta quotients

Having classified the slow growing single theta quotients in the previous section, we now consider the general case with multiple theta quotients.

The same regularity argument preceding (7.35) shows that the condition for the theta quotient

$$\prod_{j=1}^{N}\frac{\theta_1(\tau,n_1 z)\theta_1(\tau,n_2 z)\cdots\theta_1(\tau,n_N z)}{\theta_1(\tau,m_1 z)\theta_1(\tau,m_2 z)\cdots\theta_1(\tau,m_N z)} \tag{7.39}$$

to be slow growth about $y^b = y^{\frac{1}{2}(\sum_{j=1}^{N} n_j - m_j)}$ is

$$\sum_{j=1}^{N} \frac{n_j^2 - m_j^2}{2}\frac{r^2}{b^2} - \frac{n_j - m_j}{2}\frac{r}{b} + \frac{\lfloor n_j \frac{r}{b}\rfloor(\lfloor n_j \frac{r}{b}\rfloor + 1)}{2} - \frac{\lfloor m_j \frac{r}{b}\rfloor(\lfloor m_j \frac{r}{b}\rfloor + 1)}{2}$$
$$-n_j\frac{r}{b}\lfloor n_j\frac{r}{b}\rfloor + m_j\frac{r}{b}\lfloor m_j\frac{r}{b}\rfloor \geq 0, \quad 1 \leq r \leq b-1. \tag{7.40}$$

Note that unlike the case of single theta quotients, the most polar term is not guaranteed to be y^b, indeed some $q^a y^b$ for $a > 00$ may be the most polar term.

We computed all theta quotients up to $N = 7$ quotients for index $1 \leq m \leq 61$ and checked them for slow growth about y^b using the condition (7.40). For each index m and $b = \frac{1}{2}\left(\sum_{j=1}^{N} n_j - m_j\right)$, we found the dimension of the space of theta quotients that have slow growth at y^b. We present our results in the following table, and we include the corresponding dimension of weak Jacobi forms that are slow growing about their most polar term y^b. Note that the two dimensions presented are not directly comparable, since a theta quotient may not have y^b as its most polar term.

m	b	slow dim	θ dim	m	b	slow dim	θ dim
3	1	1	1	22	3	1	1
4	1	1	1	22	4	2	2
6	1	1	1	24	2	1	2
6	2	2	1	24	3	2	3
7	2	1	1	24	4	4	4
8	2	2	1	25	4	1	1
9	2	1	1	25	5	4	2
9	3	3	1	26	4	2	1
10	2	1	1	26	5	2	1
10	3	2	1	27	3	1	2
11	3	1	1	27	4	1	1
12	2	2	2	27	5	2	2
12	3	3	2	28	3	1	2
13	3	1	1	28	4	3	3
14	3	1	1	30	3	1	2
15	2	1	1	30	4	2	3
15	3	2	2	30	5	4	4
16	2	1	1	31	5	1	1
16	3	2	2	32	4	3	2
16	4	4	2	32	5	2	1
17	4	2	1	33	4	2	2
18	2	0	1	33	5	1	1
18	3	3	3	34	4	1	2
18	4	3	2	34	5	1	1
19	3	1	1	35	5	3	1
19	4	1	1	36	3	1	3
20	3	1	1	36	4	3	4
20	4	4	2	37	6	3	3
21	3	1	1	38	6	3	3
21	4	2	2	39	3	0	1

Table 7.2: The dimension of slow growing theta quotients compared to the dimension of slow growing weak Jacobi forms, about a fixed y^b and index m.

In [60, Section 3.2], the class of weak Jacobi forms from the A, D and E minimal models were proved to be slow growing about their most polar term. We now give a simplified proof, as another application of (7.40).

Example 7.2. The following weak Jacobi forms from the A, D, and E minimal models have slow growth about y^b.

$$\varphi^{A_{k+1}}(\tau,z)=\frac{\theta_1(\tau,(k+1)z)}{\theta_1(\tau,z)},\qquad b=\frac{k}{2},\ t=\frac{k(k+2)}{2}:\qquad \text{A-series}, k \text{ even},$$

$$\varphi^{A_{k+1}}(\tau,z)=\frac{\theta_1(\tau,2(k+1)z)}{\theta_1(\tau,2z)},\qquad b=k,\ t=k(k+2):\qquad \text{A-series}, k \text{ odd},$$

$$\varphi^{D_{k/2+2}}(\tau,z)=\frac{\theta_1(\tau,\frac{k}{2}z)\theta_1(\tau,\frac{k+4}{4}z)}{\theta_1(\tau,\frac{k}{4}z)\theta_1(\tau,z)}, b=\frac{k}{4},\ t=\frac{k(k+2)}{8}: \text{D-series}, k\equiv 0 \bmod 4,$$

$$\varphi^{D_{k/2+2}}(\tau,z)=\frac{\theta_1(\tau,kz)\theta_1(\tau,\frac{k+4}{2}z)}{\theta_1(\tau,\frac{k}{2}z)\theta_1(\tau,2z)}, b=\frac{k}{2},\ t=\frac{k(k+2)}{2}: \text{D-series}, k\equiv 2 \bmod 4,$$

$$\varphi^{E_6}(\tau,z)=\frac{\theta_1(\tau,8z)\theta_1(\tau,9z)}{\theta_1(\tau,4z)\theta_1(\tau,3z)},\qquad b=5,\ t=60: \qquad E_6,$$

$$\varphi^{E_7}(\tau,z)=\frac{\theta_1(\tau,6z)\theta_1(\tau,7z)}{\theta_1(\tau,5z)\theta_1(\tau,3z)},\qquad b=4,\ t=36: \qquad E_7,$$

$$\varphi^{E_8}(\tau,z)=\frac{\theta_1(\tau,12z)\theta_1(\tau,10z)}{\theta_1(\tau,5z)\theta_1(\tau,3z)},\quad b=7,\ t=105: \qquad E_8.$$

Proof. The A-type weak Jacobi forms are slow growth by Proposition 7.7 and the E-type weak Jacobi forms can easily be computationally checked using the criterion (7.40) for slow growth. For the D-type weak Jacobi forms, the formula (7.40) is equal to zero for all $1 \le r \le b-1$, after inputting the respective scaling factors and simplifying. □

It was conjectured in [60] that the weak Jacobi forms for the A_{k+1} Kazama-Suzuiki model with $M=2$ have slow growth about their most polar y^b term. This was tested numerically up to $k=10$. We now prove this conjecture.

Lemma 7.8. *The weak Jacobi form corresponding to the $M=2$ A_{k+1} Kazama-Suzuiki model defined as*

$$\varphi^{2,k}(\tau,z)=\frac{\theta_1(\tau,(k+1)z)}{\theta_1(\tau,z)}\frac{\theta_1(\tau,(k+2)z)}{\theta_1(\tau,2z)},\quad k\in\mathbb{Z}_{>0} \tag{7.41}$$

has slow growth about the polar term y^k.

Proof. We use our criterion (7.40). $\varphi^{2,k}$ has slow growth about y^k if and only if for all $1 \leq r \leq k-1$, the following expression is greater than or equal to 0:

$$\frac{(k+1)^2-1}{2}\frac{r^2}{k^2} - \frac{(k+1)-1}{2}\frac{r}{k} + \frac{(k+2)^2-2^2}{2}\frac{r^2}{k^2} - \frac{k+2-2}{2}\frac{r}{k}$$
$$+ \frac{\lfloor (k+1)\frac{r}{k}\rfloor(\lfloor (k+1)\frac{r}{k}\rfloor+1)}{2} - \frac{\lfloor \frac{r}{k}\rfloor(\lfloor \frac{r}{k}\rfloor+1)}{2} - (k+1)\frac{r}{k}\lfloor (k+1)\frac{r}{k}\rfloor + \frac{r}{k}\lfloor \frac{r}{k}\rfloor$$
$$+\frac{\lfloor (k+2)\frac{r}{k}\rfloor(\lfloor (k+2)\frac{r}{k}\rfloor+1)}{2} - \frac{\lfloor 2\frac{r}{k}\rfloor(\lfloor 2\frac{r}{k}\rfloor+1)}{2} - (k+2)\frac{r}{k}\lfloor (k+2)\frac{r}{k}\rfloor + 2\frac{r}{k}\lfloor 2\frac{r}{k}\rfloor. \tag{7.42}$$

To simplify the expression, we have $0 < \frac{r}{k} < 1$ for each r so that

$$\lfloor (k+1)\frac{r}{k}\rfloor = r, \qquad \lfloor (k+2)\frac{r}{k}\rfloor = r + \lfloor 2\frac{r}{k}\rfloor. \tag{7.43}$$

This reduces (7.42) to

$$\frac{k^2+2k}{2}\frac{r^2}{k^2} - \frac{r}{2} + \frac{k^2+4k}{2}\frac{r^2}{k^2} - \frac{r}{2} + \frac{r(r+1)}{2} - (k+1)\frac{r^2}{k}$$
$$+\frac{(r+\lfloor 2\frac{r}{k}\rfloor)(r+\lfloor 2\frac{r}{k}\rfloor+1)}{2} - \frac{\lfloor 2\frac{r}{k}\rfloor(\lfloor 2\frac{r}{k}\rfloor+1)}{2} - (k+2)\frac{r}{k}(r+\lfloor 2\frac{r}{k}\rfloor) + 2\frac{r}{k}\lfloor 2\frac{r}{k}\rfloor. \tag{7.44}$$

Now, a simple matter of cancellations gives us that the above expression equals 0 for each value of r. □

Chapter 8

Slow Growth around $q^a y^b$

We consider the general case of growth behavior for a weight 0 weak Jacobi form with maximal polarity at the term $q^a y^b$ in this chapter. From now on, it is understood that $a > 0$, as we discussed the case $a = 0$ in the previous chapter.

8.1 Overview and Results

For a weight 0 weak Jacobi form $\varphi_{0,m}(\tau, z)$, define the sum of its Fourier-Jacobi coefficients

$$\begin{aligned} f_{a,b}(n,l) &= \sum_{r\in\mathbb{Z}} c(rn + ar^2, l - br) \\ &= \sum_{r=\lceil r_- \rceil}^{\lfloor r_+ \rfloor} c(nr + ar^2, l - br), \end{aligned} \tag{8.1}$$

where the finite sum comes from the polarity constraint of $q^a y^{-b}$. Set $-\Delta_0 = 4ma - b^2$ so Δ_0 is positive. The constraint is

$$4m(rn + ar^2) - (l - br)^2 \leq -\Delta_0. \tag{8.2}$$

Here, $r_\pm = (2mn + bl \pm \sqrt{(2mn + bl)^2 + \Delta_0(\Delta_0 - l^2)})/\Delta_0$.

The authors of [59] classified the asymptotic behavior of the sums $f_{0,b}(n,l)$ as functions of n, l but left open the question of the possible behavior of $f_{a,b}(n,l)$ as it presents some analytical and numerical challenges.

In this chapter, we present some analytical and numerical results of $f_{a,b}(n,l)$ for low index m and with varying values of a, b.

As in [51, Appendix B], the asymptotic growth of $c(n,l)$ for large discriminant is

$$c(n,l) \sim \exp \pi \sqrt{\frac{|\Delta_{\min}|}{t^2}(4tn - l^2)}, \tag{8.3}$$

where $\Delta_{\min}$ is the maximal polarity of the weak Jacobi form. Proceeding as in [58, Section 4.2], if there is not sufficient cancellation in the sum of $f_{a,b}(n,l)$ then $f_{a,b}(n,l)$ behaves the same as the largest term in $f_{a,b}(n,l)$ is $c(nr + ar^2, l - br)$ of maximal discriminant, occurring when $r_{\max} = \frac{2tn+lb}{b^2-4ta}$. We then expect generic $f_{a,b}(n,l)$ to grow as

$$\begin{aligned} f_{a,b}(n,l) &\sim c(nr_{\max} + ar_{\max}^2, l - br_{\max}) \\ &\sim \exp 2\pi \sqrt{\frac{|\Delta_{\min}|}{m(b^2-4ma)}(mn^2 + al^2 + bnl)}. \end{aligned} \tag{8.4}$$

Just as for $f_{0,b}(n,l)$, we discover that cancellation may occur. Let us continue to adopt the same definition for slow growth of $f_{a,b}(n,l)$.

Definition 8.1. A weak Jacobi form $\varphi_{0,m}$ has *slow growth about* $q^a y^{-b}$ if $f_{a,b}(n,l)$ has subexponential growth.

The conditions for slow growing $f_{0,b}(n,l)$ are well-understood [58, Section 5.1]. For $f_{a,b}(n,l)$, the general case is not currently known. We discover some surprising results in this direction. When $f_{a,b}(n,l)$ is slow growth, we numerically find for indexes $5 \leq m \leq 9$ that they exhibit the same behavior as in the case of slow growth $f_{0,b}(n,l)$. That is,

(1) Numerically, we find $f_{a,b}(n,l)$ assumes only finitely many distinct values when it is slow growth.

(2) We numerically find a nonvanishing constraint on $f_{a,b}(n,l)$ analogous to (7.24) for $f_{0,b}(n,l)$. Specifically, for $f_{a,b}(n,l)$ that are slow growing, we have numerically found integers $e, f, g, h \in \mathbb{Z}$ such that

$$f_{a,b}(n,l) = \begin{cases} \text{nonzero} & : en + fl = 0 \text{ or } gn + hl = 0 \\ 0 & : \text{ else.} \end{cases}$$

Index	(term,polarity)	dim wJf	dim slow growth
5	(q^1y^5,5)	2	1
6	(q^1y^5,1)	1	1
7	(q^1y^6,8)	2	1
8	(q^1y^6,4)	2	2
9	(q^2y^9,9)	3	2
10	(q^1y^7,9)	3	2
11	(q^1y^7,5)	1	1
11	(q^2y^{10},12)	3	2
12	(q^2y^{10},4)	2	2
12	(q^1y^8,16)	5	2

Table 8.1: Weak Jacobi Forms Slow Growth About Its Most Polar q^ay^b Term

Conjecture 8.1. *We conjecture that the behavior discovered in* (2) *holds true for all weak Jacobi forms that are slow growing at* $f_{a,b}(n,l)$.

Despite our 'fast' algorithm, the runtimes for computing the values $f_{a,b}(n,l)$ is still lengthy when n, l are large or when the underlying weak Jacobi form has large index m. Nevertheless, because of the features (1) and (2) that $f_{a,b}(n,l)$ are found to exhibit, it is easy to detect its growth behavior numerically even from data with only small n and l.

For a few select indexes m and polar terms q^ay^b term with polarity $4ma - b^2$, we numerically found the dimension of the space of weak Jacobi forms that were slow growth about its most polar term q^ay^b. We put our findings in the table below. In the table, we also record in the third column the dimension of the space of weak Jacobi forms with most polar term q^ay^b, regardless of its growth behavior.

We may compare the growth behaviors of a weak Jacobi form about terms q^ay^b and y^b of the same polarity by referring to Table 8.1 above and to Table 7.1. We have found weak Jacobi forms that exhibit slow growth about one term yet fast growth about another term of the same polarity. This can occur in both directions. We found index 9 has a weak Jacobi form with slow growth at y^3 but fast growth at q^2y^9. In the other direction, index 12 has a weak Jacobi form with slow growth at q^1y^8 but fast growth at y^4.

We summarize how the non-vanishing constraints for slow growing $f_{0,b}(n,l)$ of $\varphi_{0,m}(\tau, z)$ arise. The generating functions for $f_{0,b}(n,l)$ are given as sums of weight 0 modular forms $\chi_{r,s}(\tau)$ as in (7.23). $f_{0,b}(n,l)$ is slow growth if and only if all $\chi_{r,s}(\tau)$ are holomorphic, in which case each $\chi_{r,s}(\tau)$ is constant so that only the q^0 terms of the generating functions $F_{n_b,k}(\tau)$ are nonvanishing. From (7.21), one may verify that $f_{0,b}(n,l)$ appears as the constant term of some $F_{n_b,k}$ if and only if $n = 0$ or $mn + bl = 0$, giving the nonvanishing condition

$$f_{0,b}(n,l) \neq 0 \text{ only if } n = 0 \text{ or } mn + bl = 0. \tag{8.5}$$

We take these findings as evidence for Conjecture 8.1. Furthermore, our results suggest that an analogue for $f_{a,b}(n,l)$ of (7.23) for $f_{0,b}(n,l)$ may hold. That is, there exist generating functions for $f_{a,b}(n,l)$ in terms of modular functions and $f_{a,b}(n,l)$ is slow growing if and only if these modular functions are holomorphic.

Currently, no generating functions for general $f_{a,b}(n,l)$ are known in the case $a \neq 0$. The generating function for $f_{a,b}(n,l)$ cannot be given purely in terms of specializations $e^{2\pi im(\alpha^2\tau)}\varphi_{0,m}(\tau, \alpha\tau+\beta)$ of the underlying weak Jacobi form. As seen in (7.16), the arguments of the Fourier-Jacobi coefficients in these specializations are linear over its summation indices, i.e. we have the appearance of $\sum\limits_{n\in\mathbb{Z}}\sum\limits_{l\in\mathbb{Z}} c(n,l)$ in its formula. For

$$f_{0,b} = \sum_{r\in\mathbb{Z}} c(nr, l - br),$$

the arguments of the Fourier-Jacobi coefficients are linear in its summation index r so there existed a λ that allowed us to rewrite

$$c(nr, l - br) = c(n + l\lambda + m\lambda^2, l + 2m\lambda)$$

to the form we desired.

However, for

$$f_{a,b}(n,l) = \sum_{r\in\mathbb{Z}} c(rn + ar^2, l - br),$$

the arguments of the Fourier-Jacobi coefficients are quadratic in its summation index r and there is no choice of λ that linearizes the arguments, as required in the specialization $e^{2\pi im(\alpha^2\tau)}\varphi_{0,m}(\tau, \alpha\tau+\beta)$.

That being said, surprisingly, for the indexes 6 and 8, we have analytically discovered generating functions for $f_{a,b}(n,l)$ of $\varphi_{0,m}(\tau, z)$ in terms of specializations of $W_{m'}(\varphi_{0,m})$, where $W_{m'}$ is an Atkin-Lehner involution.

8.2 Analytical Results

8.2.1 Atkin-Lehner Involution

We present the definition of the Atkin-Lehner involutions [56, (4.39)].

Definition 8.2. For every m_1 such that $m = m_1m_2$ with m_1, m_2 coprime, the *Atkin-Lehner involution* $W_{m_1} : \tilde{J}_{k,m} \to \tilde{J}_{k,m}$ is defined in terms of the theta decomposition of Jacobi forms as the map

$$W_{m_1} : \sum_{l \bmod 2m} h_l(\tau)\theta_{m,l}(\tau, z) \mapsto \sum_{l \bmod 2m} h_{l^*}(\tau)\theta_{m,l}(\tau, z), \tag{8.6}$$

(or equivalently in terms of their Fourier-Jacobi coefficients, by $c(\Delta, l) \mapsto c(\Delta, l^*)$) where the involution $l \mapsto l^*$ on $\mathbb{Z}/2m\mathbb{Z}$ is defined by

$$l^* \equiv -l(\bmod 2m_1), \quad l^* \equiv +l(\bmod 2m_2). \tag{8.7}$$

In general, the operator W_{m_1} may not preserve the space of weak Jacobi forms as it may send a weak Jacobi form to only a weakly holomorphic Jacobi form.

We will also need a generalization of the Atkin-Lehner involution. An Atkin-Lehner involution is essentially a permutation $\theta_l \mapsto \theta_{l^*}$ of theta functions in the theta decomposition of $\phi_{k,m}$. We may generalize the involution by allowing ourselves to consider any permutation of the theta decomposition.

8.2.2 Index 6

For index 6, the q^ay^b term with smallest polarity is q^1y^5 of polarity 1. This has the same polarity as the term y^1. The next least polar q^ay^b term is q^1y^6 with large polarity 12. Because of the large polarity of q^1y^6, we restrict ourselves to considering growth behavior about the term q^1y^5.

For a weak Jacobi form $\varphi_{0,6}$, we are able to relate $f_{1,5}(n, l)$ to $f_{0,1}(n', l')$ through an Atkin-Lerner involution W_3 and to write a generating function for $f_{1,5}(n, l)$ using specializations of $W_3(\varphi_{0,6})$. In doing so, we show that the growth behavior of $\varphi_{0,6}$ about y^1 is equivalent to its growth behavior about q^1y^5.

Proposition 8.2. *For a weak Jacobi form $\varphi_{0,6}$, the growth behavior at y^1 is equivalent to the growth behavior at q^1y^5.*

Proof. Recall

$$f_{1,5}(n,l) = \sum_{r\in\mathbb{Z}} c(nr + r^2, l - 5r). \tag{8.8}$$

The key observation is that the coefficient $c(nr + r^2, l - 5r)$ has the same polarity as $c\left((2n + l)(r + 3n + 2l), -9n - 5l - (r + 3n + 2l)\right)$. Moreover,

$$\sum_{r\in\mathbb{Z}} c\left((2n + l)(r + 3n + 2l), -9n - 5l - (r + 3n + 2l)\right)$$
$$= \sum_{\hat{r}\in\mathbb{Z}} c((2n + l)\hat{r}, -9n - 5l - \hat{r}) \tag{8.9}$$

and the latter sum appears as

$$f_{0,1}(2n + l, -9n - 5l) = \sum_{\hat{r}\in\mathbb{Z}} c((2n + l)\hat{r}, -9n - 5l - \hat{r}). \tag{8.10}$$

However, we cannot say $c(nr + r^2, l - 5r)$ is equal to

$$c\left((2n + l)(r + 3n + 2l), -9n - 5l - (r + 3n + 2l)\right)$$

since the second arguments in these Fourier-Jacobi coefficients do not have the same modulo 12 value. But perhaps surprisingly, the second argument modulo 12 of the former is five times the latter, i.e. $5(l - 5r) \bmod 12 \equiv -12n - 7n - r \bmod 12$.

Consider the theta decomposition

$$\varphi_{0,m}(\tau, z) = \sum_{\mu \bmod 2m\in\mathbb{Z}/2m\mathbb{Z}} h_\mu(\tau)\theta_{m,\mu}(\tau, z). \tag{8.11}$$

The second arguments mod 12 tells us that for

$$c\left((2n + l)(r + 3n + 2l), -9n - 5l - (r + 3n + 2l)\right)$$

appearing in $h_\mu\theta_{6,\mu}$, $c(nr + r^2, l - 5r)$ instead appears in $h_{5\mu}\theta_{6,\mu}$.

We now establish the relationship between $f_{1,5}(n, l)$ and $f_{0,1}(n', l')$. We see that $f_{1,5}(n, l)$ is related to $f_{0,1}(2n + l, -9n - 5l)$ as follows. The generating function for $f_{0,1}(n', l')$ is given by the specialization $\varphi_{0,6}(\tau, 0)$ as in (7.23). From the discussion above, we see that for $f_{0,1}(2n+l, -9n-5l)$ appearing as the coefficient of some q^k of $\varphi_{0,6}(\tau,0)$, $f_{1,5}(n,l)$ appears as the coefficient of the same q^k of $(W_3\varphi_{0,6})(\tau,0)$ for the Atkin-Lehner involution W_3. This also means that the generating function for $f_{1,5}(n,l)$ is given by $(W_3\varphi_{0,6})(\tau,0)$.

We recall that the Atkin-Lehner involution does not preserve the space of weak Jacobi forms as it may send a weak Jacobi form to only a weakly holomorphic Jacobi form. To see $W_3(\varphi_{0,6})$ remains a weak Jacobi form, we must verify that the numerators in the powers of the terms $q^{(12k+\mu)^2/24}$, $q^{(12k+\mu^*)^2/24}$ of the respective theta functions $\theta_{6,\mu}$, θ_{6,μ^*} span the same modulo 24 set. This ensures that $h_{\mu^*}\theta_{6,\mu}$ does not have any fractional powers of q, so that $W_3(\varphi_{0,6})$ remains a weak Jacobi form.

Here, W_3 simply sends μ to 5μ and we have $(12k+\mu)^2 \equiv 1 \mod 24$ as well as $(12k+5\mu)^2 \equiv 1 \mod 24$. Thus W_3 takes weak Jacobi forms to weak Jacobi forms.

From this, we may conclude that $(W_3\varphi_{0,6})(\tau,0)$ is holomorphic if and only if $\varphi_{0,6}(\tau,0)$ is holomorphic. Since these specializations are the generating functions for $f_{1,5}(n,l)$ and $f_{0,1}(n',l')$, respectively, and $f_{1,5}(n,l)$ and $f_{0,1}(n',l')$ are slow growing if and only if their generating functions are holomorphic, this proves the proposition. □

The proof accomplishes more than the statement of the proposition. We have also given the generating function of $f_{1,5}(n,l)$, which we repeat in the following corollary.

Corollary 8.3. *The generating function for $f_{1,5}(n,l)$ is $(W_3\varphi_{0,6})(\tau,0)$. Morever, the space of weak Jacobi forms with maximal polarity 1 that are slow growth at q^1y^5 is one-dimensional, spanned by $\frac{\theta_1(\tau,4z)}{\theta_1(\tau,2z)}$ with its $f_{1,5}(n,l)$ given by*

$$f_{1,5}(n,l) = \begin{cases} -2 & 2n+l=0 \text{ or } 3n+l=0 \\ 0 & \text{else} \; . \end{cases} \tag{8.12}$$

Proof. The proof of Proposition 8.2 establishes that the generating function for $f_{1,5}(n,l)$ is $(W_3\varphi_{0,6})(\tau,0)$. Moreover, the proof also shows that $f_{1,5}(n,l)$ is the coefficient of q^0 in its generating function if and only if $f_{0,1}(2n+l,-9n-5l)$ is the coefficient of q^0 in its generating function. We have that $f_{0,1}(n',l')$ is the coefficient of q^0 whenever $n'=0$ or $6n'+l'=0$ by (7.24), under the shift $n'=2n+l$, $l'=-9n-5l$ this becomes $2n+l=0$ or $3n+l=0$.

Referring to Table 7.1, the space of weak Jacobi forms with maximal polarity -1 is one-dimensional. After computing a basis of $J_{0,6}$, one may see this space is spanned by $\frac{\theta_1(\tau,4z)}{\theta_1(\tau,2z)}$. A quick calculation of the q^0 term of $(W_3\frac{\theta_1(\tau,4z)}{\theta_1(\tau,2z)})(\tau, 0)$ gives us the values of $f_{1,5}(n, l)$. □

8.2.3 Index 8

For index 8, the q^ay^b term with the smallest polarity is q^1y^6 of polarity 4. This has the same polarity as the term y^2. The space of weak Jacobi forms with least polar term given by q^1y^6 is two-dimensional, spanned by

$$-4\phi_{0,2}\phi_{0,3}^2 - \phi_{0,2}^4 + \phi_{0,1}\phi_{0,2}^2\phi_{0,3},$$
$$60\phi_{0,2}\phi_{0,3}^2 + 19\phi_{0,2}^4 - 23\phi_{0,1}\phi_{0,2}^2\phi_{0,3} + 4\phi_{0,1}^2\phi_{0,3}^2.$$

Both are slow growing at y^2 and numerically, we found they were also slow growing at q^1y^6. We also discovered analytically a generating function for $f_{1,6}(n, 2l)$, which is half of the total $f_{1,6}(n, l)$ values.

Here, we need the generalized version of the Atkin-Lehner involution to obtain the generating function. Indeed, for index $8 = 2^3$ there are no Atkin-Lehner involutions. Recall that an Atkin-Lehner involution is a permutation $\theta_l \mapsto \theta_{l^*}$ of theta functions in the theta decomposition of $\phi_{k,m}$. We allow ourselves to consider any permutation, in particular the permutation $\sigma = (2\ 6)(10\ 14)$.

Definition 8.3. For the permutation $\sigma = (2\ 6)(10\ 14)$, define

$$\hat{\phi}_{0,8}(\tau, z) = \left(\sum_{\mu\neq 2,6,10,14} h_\mu(\tau)\right)\theta_{8,\mu} + h_2(\tau)\theta_{8,6} + h_6(\tau)\theta_{8,2} + h_{10}(\tau)\theta_{8,14} + h_{14}(\tau)\theta_{8,10}. \tag{8.13}$$

In other words, we modify $\phi_{0,8}$ by permuting θ_2 with θ_6 and θ_{10} with θ_{14}, keeping the other theta functions fixed.

The generating functions for $f_{1,6}(n, 2l)$ are given in terms of specializations of the permutation $\sum_{\mu \bmod 16} h_{\sigma(\mu)}(\tau)\theta_{8,\mu}(\tau, z)$ of the theta decomposition of its underlying index 8 weak Jacobi form $\varphi_{0,8}$. These generating functions are surprisingly similar to the ones for $f_{0,2}(n, l)$ and we now describe them.

Lemma 8.4. *For a weak Jacobi form $\phi_{0,8}(\tau, z)$, we have the following generating functions for $f(n, 2l)$:*

$$\frac{1}{2}(\chi_{0,0}(\tau) + \chi_{0,1}(\tau)) = \frac{1}{2}(\phi_{0,8}(\tau, 0) + \phi_{0,8}(\tau, 1/2))$$

generates $f_{0,2}(2n, 2l)$ *and*

$$\frac{1}{2}(\chi_{1,0}(\tau) + \chi_{1,1}(\tau)) = \frac{1}{2}(q^2\phi(\tau, \tau/2) + q^2\phi(\tau, (\tau+1)/2))$$

generates $f_{0,2}(2n+1, 2l)$.

$$\frac{1}{2}(\chi_{0,0}(\tau) + \chi_{0,1}(\tau)) = \frac{1}{2}(\hat{\phi}(\tau, 0) + \hat{\phi}(\tau, 1/2))$$

generates $f_{1,6}(2n, 2l)$ *and*

$$\frac{1}{2}(\chi_{1,0}(\tau) + \chi_{1,1}(\tau)) = \frac{1}{2}(q^2\hat{\phi}(\tau, \tau/2) + q^2\hat{\phi}(\tau, (\tau+1)/2))$$

generates $f_{1,6}(2n+1, 2l)$.

Proof. The generating functions for $f_{0,2}(n, l)$ are given by (7.23). For $f_{1,6}(n, l)$, the key observation is that the coefficient $c(nr+r^2, l-6r)$ has the same polarity as $c\big((2n+l)(r+n+\frac{3}{4}l), -6n - \frac{7}{2}l - 2(r+n+\frac{3}{4}l)\big)$ and the rest follows as in the proof of Proposition 8.2. □

For the weak Jacobi forms that have slow growing $f_{1,6}(n, l)$, we may analytically compute $f_{1,6}(n, 2l)$ from Lemma 8.4. For the remaining values of $f_{1,6}(n, l)$, we computed $f_{1,6}(n, l)$ for these forms at following values of (n, l):

$(n = 0, -10 \leq l \leq 10)$,

$(n = 1, -14 \leq l \leq -4 \text{ and } -2 \leq l \leq 8)$,

$(n = 2, -17 \leq l \leq -8 \text{ and } -4 \leq l \leq 5)$,

$(n = 3, -20 \leq -12 \text{ and } -6 \leq -2)$,

$(n = 4, -23 \leq l \leq -16 \text{ and } -8 \leq l \leq -1)$,

$(n = 5, -16 \leq l \leq -20 \text{ and } -10 \leq l - 4)$,

$(n = 6, -29 \leq l \leq -24 \text{ and } -12 \leq l \leq -7)$, and

$(n = 7, -33 \leq l \leq -28 \text{ and } -14 \leq l - 9)$.

For these data points, we have a nonvanishing condition as in the case of $f_{0,b}(n,l)$. Indeed, for slow growing $f_{1,6}(n,l)$ to be nonzero, we found we must have $2n+l=0$ or $4n+l=0$. Again, for $f_{1,6}(n,2l)$, these values follow analytically from the generating functions.

For $-4\phi_{0,2}\phi_{0,3}^2 - \phi_{0,2}^4 + \phi_{0,1}\phi_{0,2}^2\phi_{0,3}$, we found

$$f_{1,6}(n,l) = \begin{cases} 192 & 2n+l=0 \text{ or } 4n+l=0 \\ 0 & \text{otherwise .} \end{cases} \tag{8.14}$$

For $60\phi_{0,2}\phi_{0,3}^2 + 19\phi_{0,2}^4 - 23\phi_{0,1}\phi_{0,2}^2\phi_{0,3} + 4\phi_{0,1}^2\phi_{0,3}^2$, we found

$$f_{1,6}(n,l) = 0. \tag{8.15}$$

8.3 Numerical Results

Up until now, our results have been analytical. In this section, we present our numerical results for small index m. As mentioned previously, it is easy in practice to check whether a weak Jacobi form is slow growth or fast growth, even from small values of n, l.

For index 5, the q^ay^b term with smallest polarity is q^1y^5. The space of weak Jacobi forms with q^1y^5 as the most polar term is spanned by

$$\phi_{0,1}\phi_{0,2}^2 - \phi_{0,1}^2\phi_{0,3},$$
$$\phi_{0,1}\phi_{0,2}^2 - \phi_{0,1}^2\phi_{0,3} + 4\phi_{0,2}\phi_{0,3}.$$

Only the latter form is slow growth.

For the slow growing $\phi_{0,1}\phi_{0,2}^2 - \phi_{0,1}^2\phi_{0,3} + 4\phi_{0,2}\phi_{0,3}$, we computed $f_{1,5}(n,l)$ for the following values of (n,l):

$(n=0, -16 \le l \le 0)$,

$(n=1, -19 \le l \le 14)$,

$(n=2, -21 \le l \le -7)$,

$(n=3, -24 \le l \le -11)$,

$(n=4, -27 \le l \le -15)$,

$(n=5, -7 \le l \le 5)$,

$(n = 6, -8 \le l \le -2)$,

$(n = 7, -35 \le l \le -26)$.

In every instance, we found for the slow growing Jacobi form that

$$f_{1,5}(n,l) = 0.$$

Index 7 has an interesting slow growth Jacobi form. The $q^a y^b$ term with smallest polarity is $q^1 y^6$ of polarity 8, which exceeds the index 7. In [61, (5.5)], it was proven that all weak Jacobi forms are fast growth about terms y^b with polarity exceeding the index. We might then expect there to always be fast growth about any term $q^a y^b$ of polarity exceeding the index, but we find this not to be true.

The space of weak Jacobi forms with most polar term $q^1 y^6$ is two-dimensional, spanned by

$$4\phi_{0,2}^2\phi_{0,3} + \phi_{0,1}\phi_{0,2}^3 - \phi_{0,1}^2\phi_{0,2}\phi_{0,3},$$
$$4\phi_{0,1}\phi_{0,3}^2 + \phi_{0,1}\phi_{0,2}^3 - \phi_{0,1}^2\phi_{0,2}\phi_{0,3}.$$

Only $4\phi_{0,1}\phi_{0,3}^2 + \phi_{0,1}\phi_{0,2}^3 - \phi_{0,1}^2\phi_{0,2}\phi_{0,3}$ is found to be slow growth at $q^1 y^6$.

We computed the values of $f_{1,6}(n,l)$ for the following (n,l):

$(n = -2, 8 \le l \le 28)$,

$(n = -1, 4 \le l \le 26)$,

$(n = 0, 0 \le l \le 22)$,

$(n = 1, 0 \le l \le 20)$,

$(n = 2, -4 \le l \le 16)$,

$(n = 3, -5 \le l \le 13)$.

At each data point, we always found for the slow growing Jacobi form that

$$f_{1,6}(n,l) = 0. \tag{8.16}$$

We are able to compute $f_{a,b}(n,l)$ only for small values of (n,l) at the indexes $9, 10, 11, 12$. We describe the results we found.

For index 9, the term $q^a y^b$ of least polarity is q^2y^9 of polarity 9 with the same polarity as y^3. Numerically, we found that for slow growing $f_{2,9}(n,l)$ that

$$f_{2,9}(n,l) = \begin{cases} \text{nonzero} & : 3n+l=0 \text{ or } 3n+2l=0 \\ 0 & : \text{ else .} \end{cases} \tag{8.17}$$

We found that there were weak Jacobi forms, e.g. the form $\phi_{0,1}\phi_{0,2}^4 - 2\phi_{0,1}^2\phi_{0,2}^2\phi_{0,3} + \phi_{0,1}^3\phi_{0,3}^2$, that had slow growth at y^3 but not at q^2y^9.

For index 10, we examined the term q^1y^7 of polarity 9. Again, we found that slow growth at y^3 does not guarantee slow growth at q^1y^7. An example is given by $4\phi_{0,2}^2\phi_{0,3}^2 + \phi_{0,2}^5 - \phi_{0,1}\phi_{0,2}^3\phi_{0,3}$ which is slow growth at y^3 but fast growth at q^1y^7. Numerically, we found for slow growing forms that

$$f_{1,7}(n,l) = \begin{cases} \text{nonzero} & : 2n+l=0 \text{ or } 5n+l=0 \\ 0 & : \text{else.} \end{cases} \tag{8.18}$$

For index 11, we examined the term q^1y^7 of polarity 5 and found for all data points (n,l) of our slow growing form that

$$f_{1,7}(n,l) = 0. \tag{8.19}$$

For index 12, we found that the space of weak Jacobi forms with slow growth about its most polar term q^1y^8 has dimension two. This is surprising because the polarity of q^1y^8, 16, is quite higher than the index 12. This might also be surprising since these weak Jacobi forms are fast growth about the term y^4, which has the same polarity as q^1y^8.

Appendix A

CODE FOR IMPLEMENTATION

In the appendix, we present the important parts of the Mathematica code used for the numerical computations of Part II. We include the code for implementing in Mathematica the Gritsenko generating functions for a basis of $J_{0,m}$, the space of weak Jacobi forms of index m, as well as the code for computing the matrix of polar coefficients for this basis.

There are several generating functions for the space $J_{0,m}$ of weak Jacobi forms of weight 0 and index m, as well as several ways of implementing them. One choice of generating functions uses the Eisenstein series $E_{4,1}$, $E_{6,1}$, and certain modular functions as in [76]. Another choice, which we find runs faster, is the generating functions $\phi_{0,1}, \phi_{0,2}, \phi_{0,3}$ of [77]. These functions admit several equivalent formulations.

The following Mathematica code gives an acceptably fast implementation for computing the Fourier-Jacobi expansion of $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{0,3}$, the generators of the Gritsenko basis for $J_{0,m}$. This code uses the formulas for the generators, given in [77, Equation (2.7)].

```
(*This code defines the generating functions ϕ0,1, ϕ0,2, ϕ0,3.
    We take the expansion of y all the way to the polarity bound 4mn-l^2 <
 -m^2. This is necessary because a smaller series expansion of y will return O(y^small) for some relevant terms.*)
ζ00[N_] := Series[Sum[q^(n^2/2) y^n, {n, -Floor[Sqrt[2 N]], Floor[Sqrt[2 N]]}] / (1 + Sum[2 q^(n^2/2),
      {n, 1, Floor[Sqrt[2 N]]}]), {q, 0, N}, {y, 0, Floor[Sqrt[4 m N + m^2]]}]
ζ01[N_] := Series[Sum[(-1)^n q^(n^2/2) y^n, {n, -Floor[Sqrt[2 N]], Floor[Sqrt[2 N]]}] / (1 + Sum[(-1)^n 2 q^(n^2/2),
      {n, 1, Floor[Sqrt[2 N]]}]), {q, 0, N}, {y, 0, Floor[Sqrt[4 m N + m^2]]}]
ζ10[N_] := Series[Sum[q^((n^2 + n)/2) y^(n + 1/2), {n, Floor[(-1 - Sqrt[1 + 8 N])/2], Ceiling[(-1 + Sqrt[1 + 8 N])/2]}] /
    (2 + Sum[2 q^((n^2 + n)/2), {n, 1, Floor[(-1 + Sqrt[1 + 8 N])/2]}]), {q, 0, N}, {y, 0, Floor[Sqrt[4 m N + m^2]]}]
ϕ0,1[N_] := 4 (ζ00[N]^2 + ζ10[N]^2 + ζ01[N]^2)
ϕ0,2[N_] := 2 ((ζ00[N] ζ10[N])^2 + (ζ00[N] ζ01[N])^2 + (ζ10[N] ζ01[N])^2)
ϕ0,3[N_] := 4 (ζ00[N] ζ10[N] ζ01[N])^2
```

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The Mathematica code to compute the list of polar coefficients for the Gritsenko basis of $J_{0,m}$ is given below for the example of $1 \leq m \leq 30$.

```
(*The Do loop generates the polar list PolarList_m for the specified indices m. Input is to the Do loop
 range:{m, smallest index you want to compute, largest index you want to compute}*)
Do[
 (*Output is the basis of J_0,m in terms of ϕ_0,1^a ϕ_0,2^b ϕ_0,3^c.*)
 PowerList = {x, y, z} /.  Solve[x + 2 y + 3 z == m && x ≥ 0 && y ≥ 0 && z ≥ 0, {x, y, z}, Integers];
 (*Output of this evaluation is all (n,l) such that (n,l) is polar for J_0,m. It is ordered from
   most negative polar part to least.
 *)
 nlInit = {n, l} /. Solve[4 n m - l^2 < 0 && n ≥ 0 && 0 ≥ l ≥ -m, {n, l}, Integers] ;
 nlInitPolarity = MapThread[Append, {nlInit, 4 m nlInit[[All, 1]] - Thread[nlInit[[All, 2]]^2]}];
 nlList = nlInitPolarity[[Ordering[nlInitPolarity[[All, Length[nlInitPolarity[[1]]]]]]]];
 nMax = Max[nlList[[All, 1]]];
 Print[AbsoluteTiming[PolarList_m = Table[SeriesCoefficient[
       Expand[ϕ_0,1[nMax]^PowerList[[i, 1]] ϕ_0,2[nMax]^PowerList[[i, 2]] ϕ_0,3[nMax]^PowerList[[i, 3]]]
        //. {x_} :→ x, {q, 0, nlList[[j, 1]]}, {y, 0, nlList[[j, 2]]}] ,
      {i, Length[PowerList]}, {j, Length[nlList]}]][[1]]]
 , {m, 1, 30}]
```

For computing Fourier-Jacobi expansions of $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ to the high orders in q necessary for computing the values $f_{a,b}(n,l)$, we need a faster implementation of these functions. With the implementations above, all series expansion are slow in comparison to the implementation we will give, with $\phi_{0,1}$ the fastest and $\phi_{0,3}$ slowest. The main goal of the code is to avoid products as much as possible, preferring an implementation that emphasizes sums and minimizes the number of products. As is well known, the computational cost of multiplication of series is considerably more expensive than addition of series.

It turns out that partition functions may be used to give much faster implementations of $\phi_{0,2}$ and $\phi_{0,3}$, in conjunction with the formulas below [77, Equation (1.8)].

$$\begin{aligned} \phi_{0,2}(\tau,z) &= \frac{1}{2}\eta(\tau)^{-4}\sum_{m,n\in\mathbb{Z}}(3m-n)(\frac{-4}{m})(\frac{12}{n})q^{\frac{3m^2+n^2}{24}}y^{\frac{m+n}{2}}, \\ \phi_{0,3}(\tau,z) &= \phi_{0,3/2}(\tau,z)^2, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} \phi_{0,3/2}(\tau,z) &= \frac{\theta_1(\tau,2z)}{\theta_1(\tau,z)} \\ &= y^{-1/2}\prod_{n\geq 1}(1+q^{n-1}y)(1+q^ny^{-1})(1-q^{2n-1}y^2)(1-q^{2n-1}y^{-2}). \end{aligned} \tag{A.2}$$

The generating function for the Fourier expansion of $\eta(\tau)^{-4}$ is given by the partitions of four kinds function.

We also discover that the theta decomposition of $\phi_{0,3/2}$ may be written using the partition function $P(q) = \sum_{n=0}^{\infty} p(n)q^n$, where $p(n)$ is the number of partitions of n, which allows us a fast implementation.

Lemma A.1.

$$\begin{aligned}\phi_{0,3/2} = P(q)\Bigg(&\sum_{l\in\mathbb{Z}} q^{6l^2+l} y^{\frac{12l+1}{2}} + \sum_{l\in\mathbb{Z}} q^{6l^2-l} y^{\frac{12l-1}{2}} \\ &- \sum_{l\in\mathbb{Z}} q^{6l^2+5l+1} y^{\frac{12l+5}{2}} - \sum_{l\in\mathbb{Z}} q^{6l^2-5l+1} y^{\frac{12l-5}{2}}\Bigg).\end{aligned} \tag{A.3}$$

Proof. We will prove that the weak Jacobi form $\varphi_{0,6} = \phi_{0,3/2}(\tau, 2z)$ of index 6 has the theta decomposition

$$\begin{aligned}\varphi_{0,6} &= \frac{1}{\eta(\tau)}\theta_{6,1}(\tau,z) + \frac{1}{\eta(\tau)}\theta_{6,-1}(\tau,z) - \frac{1}{\eta(\tau)}\theta_{6,5}(\tau,z) - \frac{1}{\eta(\tau)}\theta_{6,-5}(\tau,z) \\ &= \frac{1}{\eta(\tau)}\sum_{12\mathbb{Z}+1} q^{l^2/24} y^l + \frac{1}{\eta(\tau)}\sum_{12\mathbb{Z}-1} q^{l^2/24} y^l \\ &\quad - \frac{1}{\eta(\tau)}\sum_{12\mathbb{Z}+5} q^{l^2/24} y^l - \frac{1}{\eta(\tau)}\sum_{12\mathbb{Z}-5} q^{l^2/24} y^l.\end{aligned} \tag{A.4}$$

The partition function appears because $P(q) = \sum_{n=0}^{\infty} p(n)q^n = \frac{q^{1/24}}{\eta(\tau)}$. To prove the lemma, we may use Gritsenko's definition of $\phi_{0,3/2}(\tau,z)$ to compute the finitely many polar parts of $h_\mu(\tau)$ for $\mu \in \mathbb{Z}/12\mathbb{Z}$ of the theta decomposition of $\varphi_{0,6}$. Taking the product $\eta(\tau)h_\mu(\tau)$, we then see its Fourier expansion begins at q^0. As in the proof of Proposition 6.3, for every μ, $\eta(\tau)h_\mu(\tau)$ is therefore a modular form of $M_0\big(\Gamma\big(\mathrm{lcm}(24, 4m)\big)\big)$, and so must be a constant. A quick computation then yields that for every μ, the constant is 1, i.e. $h_\mu(\tau) = \frac{1}{\eta(\tau)}$. □

With our novel implementation of $\phi_{0,3/2}$ based on its theta decomposition, $\phi_{0,3}$ is the fastest to expand with $\phi_{0,2}$ the next fastest. This leaves $\phi_{0,1}$ as the slowest one to expand. The Mathematica code for implementing $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ in this faster formulation is given below.

```
(*First, obtain a list of values for Partitions of 4 Kinds from e.g. integer sequence A023003
 from OEIS and store it under the variable name Paritions4KindsValues. Then run the code.*)
DenomEta4[N_] := Expand[q^(-1/6) FromDigits[Reverse[Partitions4KindsValues], q]] /. q^k_ /; k > N :> 0
ϕ0,2[N_] :=
 Expand[
  1/2 DenomEta4[N] Sum[Sum[(3 m - n) JacobiSymbol[-4, m] JacobiSymbol[12, n] q^((3 m^2 + n^2)/24) y^((m + n)/2),
     {n, -Floor[Sqrt[24 N - m^2]], Floor[Sqrt[24 N - m^2]]}], {m, -Floor[Sqrt[8 N]], Floor[Sqrt[8 N]]}]]
ϕ0,3/2[N_] := Sum[PartitionsP[j] q^j, {j, 0, N}] (Sum[q^(6 j^2 + j) y^((12 j + 1)/2) + q^(6 j^2 - j) y^((12 j - 1)/2) -
      q^(6 j^2 + 5 j + 1) y^((12 j + 5)/2) - q^(6 j^2 - 5 j + 1) y^((12 j - 5)/2),
    {j, Ceiling[(-5 - Sqrt[1 + 24 N])/12], Floor[(5 + Sqrt[1 + 24 N])/12]}])
```

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