Rescaling Constraints, BRST Methods, and Refined Algebraic Quantisation

Eric Martínez Pascual

Thesis submitted to the University of Nottingham
for the degree of Doctor of Philosophy

March 2012

Abstract

We investigate the canonical BRST–quantisation and refined algebraic quantisation within a family of classically equivalent constrained Hamiltonian systems that are related to each other by rescaling constraints with nonconstant functions on the configuration space. The quantum constraints are implemented by a rigging map that is motivated by a BRST version of group averaging. Two systems are considered. In the first one we avoid topological built–in complications by considering $\mathbb{R}^4$ as phase space, on which a single constraint, linear in momentum is defined and rescaled. Here, the rigging map has a resolution finer than what can be extracted from the formally divergent contributions to the group averaging integral. Three cases emerge, depending on the asymptotics of the scaling function: (*i*) quantisation is equivalent to that with identity scaling; (*ii*) quantisation fails, owing to nonexistence of self–adjoint extensions of the constraint operator; (*iii*) a quantisation ambiguity arises from the self–adjoint extension of the constraint operator, and the resolution of this purely quantum mechanical ambiguity determines the superselection structure of the physical Hilbert space. The second system we consider is a generalisation of the aforementioned model, two constraints linear in momenta are defined on the phase space $\mathbb{R}^6$ and their rescalings are analysed. With a suitable choice of a parametric family of scaling functions, we turn the unscaled abelian gauge algebra either into an algebra of constraints that (1) keeps the abelian property, or, (2) has a nonunimodular behaviour with gauge invariant structure functions, or, (3) contains structure functions depending on the full configuration space. For cases (1) and (2), we show that the BRST version of group averaging defines a proper rigging map in refined algebraic quantisation. In particular, quantisation case (2) becomes the first example known to the author where structure functions in the algebra of constraints are successfully handled in refined algebraic quantisation. Prospects of generalising the analysis to case (3) are discussed.

Acknowledgments

There have been many people who supported me from the moment I decided to travel to Nottingham to work on this Ph.D. programme. I am going to use these paragraphs to convey in words, though they will be insufficient at times, all my gratitude to them.

In the first place, the most genuine acknowledgement is for my parents; I could never have come this far without their incomparable love, incessant support and endless patience. This thesis is dedicated to them. Edgar and Roxana (alphabetically ordered), my brother and sister, thank you for all your supporting words; they were very important in these years. Special thanks to my brother for helping me to get some references I needed, and quoted, in the final stages of the thesis writing process.

Distinctive appreciation to the person with whom I had the especially good chance to discuss most of the topics touched in this dissertation, to my supervisor Dr Jorma Louko. He not only always found the time to listening to me, but also questioned my points of view in a very stimulating way. Thank you for emphasising to me the difference between a formal solution and a strict solution to a problem. I am appreciative of all the comments he did on the drafts preceding the final form of this dissertation, and I am convinced they boost the readability of it. I am grateful for his motivation, his guidance, his confidence in me, and for his dedicated administrative work related to my Ph.D. programme. I also thank to Prof John Barrett and Dr Kirill Krasnov, members of the quantum gravity group at Nottingham, for enlightening and useful discussion in various topics, and for generating an academically thought–provoking and friendly atmosphere within the group.

I will be forever indebted to Dr Alicia Lopez for many reasons. Thank you for reading this thesis, from the beginning 'R' in the title page to the last 's' in the last page. All your comments really improved the way I expressed my ideas on these pages. Thank you for all your love.

I feel the necessity to mention many good friends I met in Nottingham and in some conference cities, in order of appearance they are: Hector Pacheco, Frank Hellman, Gustavo Niz, Alejandro Satz, Richard Dowdall, Daphne Klotsa, Ryszard Kostecki, Benjamin Bahr, Jamie Vicary, Elisa Manrique, Cecilia Flori, Isabeau Prémont-Schwarz, Kristina Giesel, Mercedes Martin, Bernadette Martínez, John Robertson, Johannes Tambornino, Roberto Pereira, Alan Secker, Henrique Gomes, Winston Fairbairn, Fernando Buitrago, Gianluca Delfino, Carlos Scarinci and Sara Tavares. To all of you guys: thanks for being there.

Special credit to my dear friend Dr Kathryn Taylor, not only for opening the doors of her home to me, but also for proof–reading various sections of this thesis. Thanks to her, many annoying commas were removed and long sentences cut down in the last draft of this thesis. Thank you so much for making the correction process of this work such a fun time.

Finally, and equally important, thanks to all the local communities of organic–coffee producers situated on the mountainous regions of Oaxaca, Mexico. Without your product, those short inspiring moments may have never come.

I have been supported by the funds of CONACYT (Mexico). I also want to thank the Quantum Gravity Network, a research networking programme of the European Science Foundation, and the University of Nottingham for their financial help to attend various conferences inside and outside the United Kingdom.

City of Nottingham, UK. Eric Martínez–Pascual
March, 2012.

Contents

CHAPTER 1

Introduction

In conclusion you see that there is no golden rule
for canonical quantization of constrained systems.

– Kurt Sundermeyer, 1982

There has been a great deal of active research during the last almost five decades in the field of quantisation of constrained systems. Since the vastly influential 87 pages written down by Paul Adrien Maurice Dirac in his 'Lectures of quantum mechanics' [1], this subject has occupied a central position in the development of contemporary theoretical physics and the search for a consistent quantum version of the fundamental forces in nature as known today.

1.1 Classical constrained systems: Historical perspective

We cannot purport to do justice to the history of our subject, either in the short space that has been reserved for this introduction, or in the body of the thesis. Yet, brief mention of a few landmarks in the early history is in order. In 1918, Amalie E. Noether [2] revealed a deep connection between the invariance that some dynamical systems possess under certain transformations and the off–shell vanishing of specific linear combinations of their equations of motion; the occurrence of the former implies the existence of the latter. This result is sometimes referred to as the second Noether theorem [3, 4].

A typical example of the second Noether theorem is found in Maxwell's theory on flat spacetime. Once written in terms of the four–potential A_μ, it is manifestly invariant under $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$, with Λ an arbitrary function on spacetime. This invariance implies the trivial identity $\partial_\mu\big(\partial_\nu F^{\mu\nu}\big) \equiv 0$, where $F^{\mu\nu}$ is the electromagnetic field tensor. An extension of this trivial identity to a more complex one arises by considering free Yang–Mills theory. In this case, the linear combination of equations of motion that vanishes is $D_\mu\big(D_\nu F_a^{\mu\nu}\big) \equiv 0$; these identities are consequence of the invariance of the theory under the non–abelian transformation law $A_\mu^a \to A_\mu^a + (D_\mu \boldsymbol{\theta})^a/g$, where g is a constant and a denotes the internal index of a Lie group parameterised by $\boldsymbol{\theta} = (\theta^a)$. The covariant derivative D (locally) acts on any Lie–valued p–form $T^a_{\nu\ldots\rho}$ as $D_\mu T^a_{\nu\ldots\rho} = \partial_\mu T^a_{\nu\ldots\rho} - g f_{abc} T^b_{\nu\ldots\rho} A^c_\mu$, with f_{abc} the structure constants of the underlying Lie group. Another, much less trivial, example is given by the existence of the Bianchi identities that are implied by the invariance of general relativity under general coordinate transformations.

In this scenario, where the equations that convey the dynamics are not functionally independent, the description of the dynamical system necessarily involves spurious degrees of freedom; these are the source of the obstacles encountered in passing from the Lagrangian to the equivalent Hamiltonian description in these systems. The period of time in which these difficulties were spotted and overcome, mostly as a byproduct in the search for a Hamiltonian description of general relativity, covers from 1930 to 1964. The main contributors during this lapse were León Rosenfeld [5], Peter G. Bergmann and James L. Anderson [6, 7], and P. A. M. Dirac [8, 9, 10, 11]. P. A. M. Dirac's succinct treatise on the subject in 1964 [1] certainly set the stage, the tone, and much of the terminology currently used to describe *singular* or *constrained systems*, as these kind of systems are known nowadays. A historical account of these events can be found in a series of papers by Donald C. Salisbury [12, 13, 14, 15] who remarks that P. G. Bergmann, his Ph.D. supervisor, actually never used the appellation of *Dirac–Bergmann algorithm* to term the series of steps one now follows to construct the Hamiltonian version of singular Lagrangian systems.

In L. Rosenfeld's pioneering work on the subject, it was realised (in a system of fields) that momenta, defined as the partial derivative of the Lagrangian with respect to the corresponding velocities, were not independent of each other and of the configuration variables, but satisfied relations with no time derivatives. These relations are the so–called (*primary*) *constraints*. According to P. G. Bergmann's group in Syracuse, and to P. A. M. Dirac, *any* physically allowed initial conditions must satisfy these constraints. This implies a consistency condition: the constraints must be constant in time, that is, the Poisson bracket of the primary constraints must vanish with the Hamiltonian. From this rule an iterative procedure arises, which ends after a finite number of steps provided that the original Euler–Lagrange equations of motion are self–consistent. Constraints which come into the analysis after the primary constraints were termed by P. G. Bergmann *secondary constraints*. Constraints whose Poisson bracket with each other vanish modulo the constraints themselves are *first–class constraints*, otherwise they are *second–class constraints* in P. A. M. Dirac's terminology.

In this account, the *constraint surface* is defined as the zero locus of all the constraints; the restriction of the symplectic form to it is degenerate. In contrast to the second–class constraints, the first–class constraints generate mappings, *gauge transformations*, from this submanifold of the phase space on itself, so that the first–class constraints of the theory form a Lie algebra on the constraint surface. The orbits of the gauge transformation group that have at least one point on the constraint surface lie in that hypersurface entirely, and form *equivalence classes*. They foliate the constraint surface. Each point on the zero locus of the constraints belongs to exactly one equivalence class, each of which represents one and only one physical situation; all points in the equivalence class are physically the same. On the quotient space of the constraint surface by the gauge orbits, a non–degenerate symplectic structure is recovered; this space is called *reduced phase space*. Dynamical variables on the reduced phase space are automatically constant over each equivalence class. Hence invariant under gauge transformations; that is, commute with all first–class constraints. Such variables are called *observables*, in the sense that their values depend on the actual physical situation, and not on the manner in which we could choose to represent it.

For second–class constraints, P. A. M. Dirac modified the Poisson bracket adding terms bilinear in them in a way that second–class constraints themselves can strictly be considered zero either before, or after, evaluating the modified bracket. In terms of *Dirac brackets*, as they are known, one can express the time evolution and gauge transformations of arbitrary functions on the phase space.

This formulation of constrained systems in phase space terms is universal. The non–trivial classical theoretical framework, quickly revisited in the paragraphs above, permeates into various physical models; together known as *gauge invariant systems*. These models encompass electrodynamics, Yang–Mills type theories, Einstein's general relativity, topological field theories, grand unified theories, superstring, branes and many others manifestly Lorentz invariant dynamical systems. This enormous range of applicability is what confirms that the study of singular systems is interesting in its own right. Furthermore, considering the idea that an ultimate description of physical phenomena is quantum in nature, undoubtedly the quantisation of constrained Hamiltonian dynamics becomes a task worthwhile to accomplish.

1.2 Path integral quantisation and ghost fields

The presence of spurious degrees of freedom in the description of a physical theory, poses a quandary in the process of quantisation. Several proposals to obtain a consistent quantisation that wish to embrace the most general form of constrained systems have been explored. The most popular methods are based either on Hilbert space techniques, where constraint operators play the role of physical state selectors [1]; or, on functional–integral techniques, where the constraints are incorporated by changing the measure of the path integral in order to formally consider only physical paths. The latter is in practice valuable as it is closer to Feynman diagrams machinery [16, 17].

In 1948, based on his doctoral dissertation, Richard Feynman introduced the Lagrangian version of the path integral formalism for non–relativistic mechanical systems [18]. His ultimate goal was to control the infinities with which QED is plagued [19, 20, 21, 22], a task partly achieved later by Freeman Dyson [23, 24]. The sources of divergences in QED include proper self–energy graphs and vertex functions. In addition, when computing the transition amplitude between two states of a gauge invariant system, one faces the problem that the sum inescapably diverges; in phase space terms, because of the existence of equivalence classes. There is not a unique prescription for the propagator of gauge fields. The necessity of imposing conditions (known as *gauge conditions*), to pick one and only one representative within each physical equivalence class, is then imperative. Besides this difficulty, in 1963 R. Feynman emphasised the lack of unitarity to 1–loop in the perturbation theory of the linearised gravity coupled to a scalar field using a Lorentz gauge condition [25]. This complication, however, was not exclusive to general relativity since Yang–Mills theory inherently also contains it[1]. In order to 'cure' the amplitude, R. Feynman suggested the introduction of some fictitious fields, later on known as *ghost fields*, by *subtracting* from the Lagrangian a term which restores a unitary amplitude by cancelling out all unphysical contributions to the intermediate (virtual) states.

The heuristic rules given by R. Feynman to reinstall 1–loop unitarity were derived later by Ludwig D. Faddeev and Victor N. Popov from an action principle [26, 27]. The Lagrangian in the action should contain the original gauge invariant term, the

[1]In Feynman words 'At the suggestion of Gell–Mann I looked at the theory of Yang–Mills with zero mass, [...] and found exactly the same difficulty [...]. So at least there is one good thing: gravity isn't alone in this difficulty' [25].

gauge fixing term, and the new so–called Faddeev–Popov term. The latter, containing the fermionic ghost variables, was seen as a mathematical convenience to express the S–matrix of the theory in the form of an integral of $\exp\left[\frac{i}{\hbar}\times \text{action}\right]$. Ghost fields were hence regarded as a 'measure effect'.

Both, a deeper understanding and the systematic use of ghosts fields were provided by L. D. Faddeev in his proposal for a Hamiltonian path integral in first–class constrained systems [28]. The unconstrained Hamiltonian version of the path integral introduced by R. Feynman [29] had been generalised. The measure in the L. D. Faddeev's integral (also known as BFV path integral) would include Dirac deltas with first–class constraints in their argument, Dirac deltas of the gauge fixing conditions (which turn the constraints into second–class type), and the determinant of the nonsingular matrix formed by the Poisson brackets between the first–class constraints of the theory and the gauge fixing functions. The last factor ensures the independence of the integral on the choice of gauge conditions[2]. Performing the integration of momenta and using Grassmann integrals to recast the nonvanishing determinant, installs the Faddeev–Popov ghost term in the effective action at the configuration space level. The possibility of deriving the Lagrangian path integral from the Hamiltonian one is known in the literature as *Matthews theorem*; see for example [31, 32, 33] and references quoted therein. The corresponding modified path integral in phase space for systems with first– and second–class constraints was derived by Pavao Senjanovic in 1976 [34].

Although the role of ghost fields in the description of gauge invariant theories was already prominent with the advent of L. D. Faddeev and V. N. Popov's results, they were raised to a fundamental level when Carlo Becchi, Alain Rouet and Raymond Stora (BRS) conceived of a global symmetry transformation in the context of the Higgs–Kibble model [35, 36]: an abelian gauge invariant theory minimally coupled to scalar fields with spontaneously symmetry breaking within a certain class of gauge functions. In this symmetry transformation non–ghost fields are transformed by gauge transformations with ghosts for parameters[3]. The global nature of this transformation precludes that more degrees of freedom could be eliminated from the theory. The generalisation of this rigid transformation to non–abelian gauge theories [38], together with the BRST–

[2]The invariance of the BFV path integral under the choice of gauge fixing conditions is commonly known as the Fradkin–Vilkovisky theorem. Subtleties on the validity of this theorem are found in [30], and references quoted therein.

[3]These transformations were subsequently called BRST transformations. The character 'T' in the abbreviation refers to Igor V. Tyutin who discovered analogous transformations in the context of canonical quantisation of gauge theories. This discovery was originally reported in what became one of the most famous unpublished works in the field [37].

invariance of the partition function, imply the so–called Slavnov–Taylor identities [39, 40, 41]. These have proved to be crucially instrumental in the renormalisation process, unitarity and other aspects of gauge theories.

The origins of ghost variables are hence inherently quantum mechanical. It is at this level where they decrease the number of degrees of freedom to its physical number by cancelling out the spurious degrees of freedom that manifest themselves in virtual intermediate processes. The occurrence of ghost variables within the effective action encountered by L. D. Faddeev and V. N. Popov, and their profound relation with the symmetry that remains of the original gauge invariance, suggests their inclusion in the classical formalism from the very beginning. This observation gave rise to a new standpoint in the study of constrained systems: *extended space techniques*.

1.3 Extended space techniques and the BRST symmetry

Motivated by R. Feynman [25], L. D. Faddeev [28], and the contemporaneous discovery of BRST symmetry [35, 36, 38, 37], a prominent group of researchers settled in the Physical Lebedev Institute achieved the task of introducing the ghost fields into the classical scheme before going into the S–matrix of the theory. Developed during the late 70s and early 80s, their BRST approach to gauge systems was addressed from two different, but equivalent [42, 43, 44, 45, 46], points of view. Focused on phase space extensions, one approach is the *Hamiltonian BRST* or *BFV formalism*, abbreviation in honour of Igor Batalin, Efim S. Fradkin and Grigori A. Vilkovisky, authors of the original papers on the subject [47, 48, 49, 50, 51]. Based on configuration space extensions, the other approach is developed in the Lagrangian framework and is known as *field–antifield* or *BV formalism*, the abbreviation due to its developers I. Batalin and G. A. Vilkovisky [52, 53, 54, 55], whose work was based on previous ideas of Jean Zinn–Justin [56], Renata Kallosh [57], and Bernard de Wit and Jan Willem van Holten [58].

In the field–antifield formalism [59, 60] one enlarges the original configuration space in a two–step process. Firstly, among the original basic configuration variables a number of ghosts is introduced. These ghost variables are equal in number to the parameters in the gauge transformation[4]. In this formalism, the original and ghost variables are collectively referred to as *fields*. In the second step, one antifield is introduced for each field. The Grassmann parity of each antifield is opposite to its associated field. An

[4]In cases where there are relations among the gauge transformations (reducible gauge transformations), ghosts of ghosts are also necessary. There actually may exist a ladder of ghost if relations among relations also exist.

additive charge, called *ghost number*, is assigned to each of these fields and antifields. Together field and antifields become the coordinates of an extended configuration space of even dimension.

This super–space is equipped with a Poissonian structure: the antibracket, an odd non–degenerate symplectic form. Concomitantly, phase space concepts such as canonical transformations, can be installed in this super configuration space. The original classical action is extended to a new action. The new action involves fields and antifields and is required to satisfy the *classical master equation*, that is, to have a vanishing antibracket with itself. The BRST transformation of any function on the super configuration space is generated (with the antibracket) by the extended action. The BRST transformation is hence nilpotent of order two. The extended action is by construction BRST invariant and becomes defined up to canonical transformations. A gauge fixing procedure is still needed when extending this theory to its quantum counterpart, usually involving the introduction of more auxiliary fields.

It is the Hamiltonian BRST formalism which will concern us in this thesis [61, 62, 63, 64, 65]. In a similar fashion to the field–antifield formalism, the original number of basic dynamic variables is increased. A number of ghost canonical pairs is added to the original variables in phase space. When only irreducible first–class constraints (independent constraints) are present, the number of ghosts canonical pairs is equal to the number of first–class constraints. If the Lagrange multipliers are to be considered as points in the phase space (nonminimal BRST formalism), the corresponding conjugate momenta are defined as new constraints from the outset. These new constraints and the original constraints form a set with an even number of first–class constraints. In the nonminimal version of the Hamiltonian BRST formalism more ghost canonical pairs are introduced accordingly, having one ghost canonical pair for each first–class constraint. The Grassmann parity of each ghost canonical pair is opposite to the one associated with its correlated constraint; in the case of pure bosonic constraints, only fermionic ghost canonical pairs are defined. To each ghost (resp. ghost–momentum) one assigns a ghost number $+1$ (-1).

The gauge algebra of the theory is determined by the Poisson brackets between first–class constraints. These Poisson brackets render a linear combination of first–class constraints. In Marc Henneaux's terminology [61], if all the coefficients turn out to be constant on the phase space, the gauge algebra is called *closed*, otherwise is called *open algebra*. The gauge algebra (closed or open) and the Jacobi identity of the Poisson brackets ensure the existence of an odd, ghost number $+1$, real, and nilpotent (in the Poisson bracket sense) function Ω on the extended space. This Ω, which generally contains multi–ghost terms higher than three, is defined up to canonical transformations in the super phase space and generates the classical BRST transformation at first order in ghost variables. Observables, or gauge invariant functions, are recognised in this formalism as ghost numbered zero functions which commute (in the Poisson bracket sense) with the BRST generator Ω. The dynamics is generated by a BRST extension of the original Hamiltonian.

One of the conceptual advantages of extended phase space over the original phase space formalism is that it makes first–class constrained systems canonically covariant under the operation of rescaling the constraints γ_a, that is, under

$$\gamma_a \;\mapsto\; \gamma'_a := \Lambda_a{}^b \gamma_b \;,$$

with Λ a pointwise invertible matrix on the phase space. To be more specific, while this transformation does not induce changes in the characterisation of either classical observables or the constraint surface [66, 67, 68], it cannot be recast as a canonical transformation in the ordinary phase space. Nevertheless, the two different BRST charges Ω and Ω' associated to each equivalent set of constraints $\{\gamma_a\}$ and $\{\gamma'_a\}$, respectively, are connected by a canonical transformation on the ghost extended phase space [61, 69]. The quantum consequences of this result have been explored for example in [70, 71, 72]; formally the quantum theories arising from each set of constraints should be unitarily related in a canonical BRST–quantisation scheme.

1.4 Canonical quantisation of constraints and the physical inner product

The most succinct formalisms we have to understand gauge theories are based on extended space techniques. The rich structure of first–class constrained systems, either with an open or closed gauge algebra, can be summarised into one object. In the Lagrangian setting this is the extended action, and in the Hamiltonian setting this corresponds to the BRST charge. Each of these objects generates BRST transformations in its respective space. This point of view, in which ghosts are treated on the same footing as the basic dynamical variables, gives a whole new set of tools to gauge theories and in general to singular systems. *Prima facie*, to implement this standpoint at a quantum level, one must effectively incorporate the ghosts into the quantisation scheme.

With exception of the reducible constraints case, which was fully developed in 1983 [51], by the year of 1978 most of the classical BRST Hamiltonian and its application to path integral quantisation was already consolidated. About the same time, in 1979, Taichiro Kugo and Izumi Ojima introduced the BRST operator quantisation method in the context of Yang–Mills theory [73]. In their approach to the problem of quark confinement in QCD, they proposed the *physical states* to be (usually zero ghost numbered) states in the kernel of the so–called *BRST operator*, $\widehat{\Omega}$. This operator must act on an indefinite inner product space in order to be compatible with its nilpotency ($\widehat{\Omega}^2 = 0$) and hermiticity ($\widehat{\Omega}^\dagger = \widehat{\Omega}$) properties [74]. Within their formalism, physical observables were regarded as hermitian BRST–invariant operators.

One distinctive aspect of the canonical BRST–quantisation over its classical counterpart is that the existence of neither a BRST operator nor BRST observables, with the properties given above, is guaranteed. Another obstacle found in the formalism is that in practice when the inner product on the total BRST state space is restricted to the BRST physical space, an ill–defined inner product results. Hence to supply the physical space with an orthodox probabilistic interpretation, it is mandatory to implement a positive definite inner product by other means. The general resolution to this issue is not a trivial task and several efforts have been made in this direction, they include [65, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90].

Notice that the nilpotency of $\widehat{\Omega}$ introduces a new symmetry: BRST physical states are defined up to a term in the image of the BRST operator. This ambiguity in the definition of BRST physical states is called the *BRST quantum gauge transformation* and all states related by it are BRST gauge–equivalent. The image of $\widehat{\Omega}$ is contained in its kernel. Therefore, one could seek a positive definite inner product on the quotient space of the kernel of $\widehat{\Omega}$ by the image of $\widehat{\Omega}$. Alternatively, one could fix the BRST gauge invariance and evaluate the indefinite inner product of the total space only on suitable gauge–equivalent states. Robert Marnelius and his collaborators worked on the latter [82, 84, 85, 86, 87, 89, 90]. Without altering the original inner product, they singled out specifically chosen representatives in the BRST–invariant equivalence classes which then have a well–defined inner product. Interesting clarifications on the possible failures of R. Marnelius' programme in the presence of Gribov ambiguities have been emphasised in [91, 92].

The aforementioned problem of finding a physical inner product is not specific to the extended space techniques; it is a common feature in all quantum descriptions of first–class constrained systems. In the path integral quantisation, for example, this problem is resolved by modifying the measure of the integral so that only physical paths are considered. In Dirac's approach to quantising constrained systems, the physical space, that is, the set of states annihilated by all the first–class constraints, needs to be turned into a pre–Hilbert space. Again, the restriction of the original inner product –although in this case positive definite– to the physical space produces ill–defined norms.

The implementation of a physical inner product within Dirac's scheme is usually done on a case–by–case basis. The physical inner product may be determined by specific symmetries in the model such as the particular structure of background spacetime metric, the existence of a global time–like Killing vector field, or the property that the quadratic term in the Hamiltonian is projectable on non–degenerate metric on the reduced phase space [66]. However, in a full general case, none of these options is necessarily accessible.

A sophisticated version of Dirac's approach to the quantisation of constrained systems is the *refined algebraic quantisation* scheme. The developers of this scheme were Abhay Ashtekar, Jerzy Lewandowski, Domenico Giulini, Donald Marolf and Thomas Thiemann [93, 94, 95, 96, 97, 98]; they emphasised and suggested a resolution to the ambiguities present in Dirac's approach. Refined algebraic quantisation is based on previous ideas by Atsushi Higuchi [99, 100] and D. Marolf [101, 102]. Successful application to different systems has been proved in [103, 104, 105, 106, 107, 108, 109].

Refined algebraic quantisation broadly comprises two sets of rules. In the first one, the arena where constraint operators are going to act is defined. This first set of steps is shared by an earlier version of the method, simply called algebraic quantisation [110, 111, 112]. The second set of steps in the process guides us to the implementation of constraints as selectors of physical states. Besides the usual obstacles one encounters in any canonical quantisation process [113, 114, 115, 116, 117, 118], in this method one has to input some auxiliary structures such as a Hilbert space $\mathcal{H}_{\rm aux}$ and a test state space $\Phi \subset \mathcal{H}_{\rm aux}$ to give distributional sense to Dirac condition on physical states. The main contribution of this scheme to Dirac's approach is the inclusion of a map called *rigging map*. To some extent, the introduction of this map is an axiomatic way to establish an inner product in the set of physical states which corresponds to the image of the rigging map itself. Instead of listing the axioms that define a rigging map, which will be done in another chapter of this thesis, we mention that the construction of this map is closely related to the group averaging technique [99, 100, 101, 102].

The group averaging technique is a method by which gauge invariant quantum states are obtained by averaging non–invariant states over the gauge group. The proper application of this method permits to define a rigging map, although this is not straightforward for all type of gauge groups; particularly delicate is the sense in which the averaging converges on Φ. For instance, although some control has been provided in [96] for non compact Lie groups, in the physical inner product there may be negative squared norm physical states as shown by Jorma Louko and Alberto Molgado in [105]. In addition to these subtleties, a major obstacle is faced by group averaging techniques with the appearance of nonconstant structure functions at the level of the Poisson brackets between first–class constraints[5].

As pointed out by D. Marolf in [98], one can move into an open algebra regime and concurrently have a closed gauge algebra underlying the gauge theory, by first looking at 'artificially constructed structure functions'. These functions are the result of rescaling the first–class constraints γ_a. Indeed, with the right choice of the coefficients ${\Lambda_a}^b$, in $\gamma'_a := {\Lambda_a}^b\gamma_b$, one can make the Poisson brackets of the rescaled constraints close with nonconstant structure functions. Supposing that at a quantum level we have successfully turned the 'original' constraints γ_a into self–adjoint operators on $\mathcal{H}_{\text{aux}}$, in the construction of the operators associated to the rescaled constraint, new factor ordering problems will be encountered which may prevent their self–adjointness. This is certainly harmful for group averaging techniques because the rescaled constraint operators may not generate a unitary action of the group, and in general we do not know how much the action (if any) will deviate from a unitary one. In addition, D. Giulini and D. Marolf showed that for nonunimodular Lie gauge groups, one needs to proceed with non self–adjoint constraints operators as physical state selectors. The selectors in this case generate an action of the group which ceases to be unitary differing from it by an overall factor related to the nonunimodular function [96]. Similar results have been obtained in geometric quantisation [122, 123].

Comparison with other quantisation methods could provide some insight into the action of a gauge group, on the auxiliary Hilbert space, when open algebras are present. In particular, we mention the proposal by Oleg Yu Shvedov [124], which is based on the canonical BRST–quantisation in the Marnelius' physical inner product. His strategy consists of identifying trivial BRST physical states with test states in $\Phi \subset \mathcal{H}_{\text{aux}}$. The regularised Marnelius' physical inner product between two BRST trivial physical states then resembles the group averaging ansatz which is interpreted as a 'would–be' rigging map. To be precise, when the structure functions are constants of a nonunimodular Lie group, Shvedov's proposal duly reduces to the non–unitary action of the gauge group and the averaging formula in the non–trivial measure adopted by D. Giulini and D. Marolf [96]. Keeping the identification of trivial BRST physical states with elements in Φ, a proposal is given for open gauge algebras by introducing the corresponding BRST operator. Although some technical caveats are provided by O. Y. Shvedov, there is not a complete proof that his would–be rigging map actually satisfies the axioms given in [93] for a general constrained system.

[5] In such a case, self–adjoint constraint operators on $\mathcal{H}_{\text{aux}}$ are incompatible with structure functions in the gauge algebra if one wants to match the reduced space quantisation and Dirac quantisation [66, 67]. It is not clear, however, to what extend reduced phase space quantisation and Dirac constraint quantisation must match, there are several models in the literature in which this is not the case [119, 120, 121].

It would be interesting to test O. Y. Shvedov's proposal in constrained systems with open algebra containing only artificially constructed structure functions. In such models an equivalent set of constraints with closed gauge algebra is present and some control over the group averaging is available. The importance of studying the quantisation of these gauge algebras with artificial structure functions in O. Y. Shvedov's framework is that the relevant technical caveats in order to produce a rigging map can be identified. Related to this, one may clarify at which level of Shvedov's scheme the auxiliary structures of refined algebraic quantisation must be supplemented. Further knowledge in the quantisation of artificial–structure–functions models would also permit the effects of rescaling constraints in the context of refined algebraic quantisation to be investigated. Finally, the lessons learnt from this study may serve as a guide to tackle systems with *authentic* structure functions in their gauge algebra. In this dissertation we begin to develop this plan in a system of a single–constraint. Although rescaling in this case does not change the gauge algebra, its study already contributes to unveil some of the issues present in more complex systems. We also provide some enlightenment with the partial study of an open algebra rescaled version of an abelian two–momentum system of constraints.

In this introduction we have mentioned some methods of quantising constrained systems, this recollection is far from complete and other approaches exist. Examples include reduced phase space quantisation, the programme of geometric quantisation [125], projection–operator method [126, 127, 128, 129], and the master constraint programme [130, 131]. There are interrelationships between them; for instance, in [132, 133, 134] the connection between Dirac's approach and path integrals is remarked. With the use of the abelianisation theorem (see for example Sect. 3.4.2 below or [65, 135]), formal equivalence has been established between canonical BRST–quantisation and Dirac's programme, and between the latter and reduced space quantisation [61, 64, 65]. The same theorem has been extensively exploited to show relations between refined algebraic quantisation, the master constraint programme, and the path integral formulation [136, 137].

1.5 Synopsis of the dissertation

We now turn to a chapter–by–chapter description of this dissertation. Although some of the chapters contain standard information that can be found in the literature, they have been added to make the thesis self–contained. In this way we could not avoid faithful duplications of certain statements, but we add comments which might serve as clarification of the points.

Chapter 2. Classical singular systems

This chapter introduces the subject of this thesis, the study of classical constrained system. To set notation, we first discuss non–singular systems and their conventional description in configuration, as well as in phase space terms. Literature sources include [138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148]. The concepts of non–singularity, reducibility, and regularity conditions are introduced. The relation between local gauge invariance of the action and the failure of non–singularity is presented. A detailed description of singular systems is given. The meaning of first– and second–class constraints, Dirac observables and gauge transformations are described. The published material used in this chapter includes [1, 4, 65, 149, 150, 151, 152, 153, 154].

Chapter 3. Foundations of the classical BRST formalism

We incorporate the concepts introduced in the preceding chapter into the extended phase space. In this chapter the nonminimal Hamiltonian BRST formalism is developed on classical grounds. The construction of the classical BRST symmetry generator, for irreducible and regular first–class constraints, is addressed here; its existence is seen as a corollary of both the gauge algebra and the Jacobi identity. In a similar fashion, the construction of the BRST extended observables is achieved. Some of the sources used in this part of the thesis are [61, 62, 63, 64, 65, 69, 73]. We remark upon the canonical covariance of the theory under rescaling of constraints, and introduce the notation for ghost/antighost variables.

Chapter 4. Quantisation of singular systems

In this chapter two widely accepted methods of quantisation for constrained systems are introduced. The first one is the refined algebraic quantisation formalism [93, 94, 95, 96, 97, 98], widely used in the community of canonical quantum gravity [137, 155, 156]. We introduce the group averaging ansatz using a finite dimensional vector space and a discrete group. The technique is presented also for nonunimodular Lie groups based on [96, 97]. The second quantisation method introduced here is the canonical BRST–quantisation [78, 79, 62, 64, 65, 83] supplemented with a summary of the extensive work on a BRST physical inner product developed by R. Marnelius *et al.* [82, 84, 85, 86, 87, 89, 90]. We discuss the possibility of deriving the group averaging formula from the canonical BRST–quantisation when a suitable anti–hermitian gauge fixing operator is provided.

Chapter 5. Constraint rescaling in refined algebraic quantisation: Momentum constraint

In this chapter we turn to the new results in this thesis [157]. Based on the previous chapters, here we address the canonical quantisation of a class of systems related by rescaling a classical constraint. We focus on a system with a single constraint, so that rescaling does not produce artificial structure functions. However, there are issues in the construction of a self–adjoint rescaled constraint operator. To avoid built–in topological complications in the classical theory, we take the unextended phase space to be $\mathbb{R}^4$ and the constraint to be linear in one of the momenta, but we allow this momentum to be rescaled by a nowhere–vanishing function of the coordinates. From the canonical BRST quantum analysis, the group averaging formula is derived. Once the ghost fermions, inherent to the BRST formalism, are removed, we clarify at which level of the extended phase space quantisation the auxiliary structures must be provided. The refined algebraic quantisation depends on the asymptotic nature of the scaling function. Three cases arise: the quantisation is equivalent to that in which the scaling function is the identity everywhere; quantisation fails; or a quantisation ambiguity arises and leads to a superselection structure of the physical Hilbert space.

Chapter 6. Constraint rescaling in refined algebraic quantisation: Two momentum constraints

In this chapter the first steps to generalise the analysis of Chap. 5 are discussed. We consider a system with two momentum constraints, originally abelian, on the phase space $\mathbb{R}^6$. We admit the rescaling of each constraint by a non–vanishing function of the coordinates. Unless these functions depend on the true degree of freedom only, the Poisson brackets of the rescaled constraints closes with nonvanishing artificial structure functions. To have some control over the infinite number of possible algebras obtained by this rescaling, we provide a specific parameterised family of real–valued scaling functions. Depending on the values taken by the parameters, either the gauge algebra (1) is maintained, or, (2) becomes an algebra of a nonunimodular group with gauge invariant structure functions, or, (3) becomes a full open algebra. Using the group averaging motivated by the regularised BRST inner product, the refined algebraic quantisation of cases (1) and (2) are performed. In particular the second case becomes the first example known to the author where an open algebra is handled in refined algebraic quantisation. Prospects of generalising the two previous results to case (3) are discussed.

Chapter 7. General summary and discussion

The final chapter is devoted to final remarks, conclusions and possible future lines of investigation on the subject of this thesis.

In appendices A and B we write down some technical calculations connected with a pair of theorems quoted in Chap. 5. In the same chapter there occur specific lemmas on the asymptotics of averaging integrals; their proofs are presented in Appendix C. In Appendix D, we derive a crucial formula to explicitly write down a group averaging ansatz for a gauge algebra with gauge invariant structure functions. In Appendix E we collect the basic properties of the gauge group whose open algebra is discussed in Chap. 6. Subsequent to the appendices, we place the References. At the end, in an attempt to improve the readability of the current work, we include a Glossary of Symbols which lists the commonest mathematical characters used throughout the main chapters.

Finally some words on the conventions used in this dissertation are in order. In every part of this thesis, the repeated index implicit Einstein summation is understood unless the contrary is explicitly expressed. The superscript * in front of a variable or function stands for complex conjugation. However, it is also used in the notation for a cotangent manifold $\mathbf{T}^*\mathcal{M}$ of a manifold $\mathcal{M}$, and when it appears in front of the bracket $\{\cdot\,,\cdot\}$ helps to denote the Dirac bracket $\{\cdot\,,\cdot\}^*$. The words '(anti) self–adjoint' and '(anti) hermitian' are used indistinctly. Chapters, as well as sections and subsections, are numbered in arabic numerals, and appendices in uppercase Latin letters. Equations are numbered sequentially within a section or appendix and also contain the number of the chapter. When we refer to the RHS (resp. LHS) of an equation we mean the right– (left–) hand side of it. Footnotes are sequentially numbered within each chapter; when we refer to a footnote we will also make reference to the page on which it is printed. At the end of some sections there is a list of remarks. The ending of each list is announced with a solid black triangle (▲) at the very right of the page. In contrast, a solid black rectangle (▮) signifies the end of a proof. Within the References, following each item, a number/set of numbers is written; it corresponds to the page/pages in the body of the thesis where such item was mentioned.

CHAPTER 2

Structure of the Classical Constrained Systems

When one has put a classical theory into the Hamiltonian form, one is well launched onto the path of getting an accurate quantum theory.

– P. A. M. Dirac, 1964

The principal goal of this chapter is to describe the classical foundations in the Lagrangian and Hamiltonian formalisms in cases where the degrees of freedom are constrained in the Dirac sense [1]. The meaning of non–singularity, reducibility, and regularity conditions will be established. The concepts of first– and second–class constraints, Dirac observables and gauge transformations will be described.

Throughout this work systems with a finite number of discrete degrees of freedom are considered. We assume that their dynamics can be derived from an *action functional*, S, through the Hamilton's variational principle. This assumption does not exclude physical models like dissipative systems with second order *non*–Lagrangian equations of motion. Such systems can either be described by the Euler–Lagrange equations when a suitable multiplier matrix exists or be reformulated in an equivalent first–order form; see for instance Dmtri Maximovich Gitman and Vladislav G. Kupriyanov [158, 159] and references quoted therein. Lagrangian functions depending at most on first derivatives (up to surface terms) are examined. More general considerations as higher derivatives in the Lagrangian or continuum degrees of freedom (fields) are discussed in the excellent monograph by D. M. Gitman and I. V. Tyutin [151]. For simplicity, in this chapter only systems with bosonic degrees of freedom are considered. The BRST symmetry, to be introduced in the next chapter, will require the addition of fermionic variables into the analysis.

2.1 Classical non–singular systems

In this section the Lagrangian and Hamiltonian formalisms for systems that fulfil the non–singularity condition are introduced. There is a detailed account of these ideas in many textbooks in a variety of styles, from the traditional ones such as [139, 141, 142, 144, 147, 148] to more contemporary ones like [140, 143, 145] where a more geometrical point of view is taken.

2.1.1 The starting point: An action and the Lagrangian formalism

For a mechanical system with a finite number n of discrete degrees of freedom, Hamilton's variational principle, or principle of least action, corresponds to a fixed–end–point variational problem for n functions stated as follows: *The motion of a system of particles during the time interval* $[t_0, t_1]$ *is described by the functions* $q^i(t)$, $i = 1, \ldots, n$, *for which the action functional*

$$S[q] := \int_{t_0}^{t_1} \mathrm{d}t\, L(t, q, \dot{q}) \ , \qquad q \equiv (q^i) \tag{2.1.1}$$

is stationary, with L *a Lagrangian of the system.*

A necessary condition for the differentiable functional $S[q]$ to have an extremum at a trajectory $\big(\overline{q}^{\,i}(t)\big)$, among the set of trajectories $\{\big(q^i(t)\big)\}$ fulfilling the boundary conditions $q^i(t_0) \equiv q_0^i$ and $q^i(t_1) \equiv q_1^i$, is that $\overline{q}^{\,i}(t)$ satisfy the Euler–Lagrange equations

$$L_i := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \ , \tag{2.1.2}$$

where L_i is the *Euler derivative of* L *with respect to* q^i.

The Euler–Lagrange equations can also be written as follows:

$$W_{ij}(t, q, \dot{q})\, \ddot{q}^j - V_i(t, q, \dot{q}) = 0 \ , \tag{2.1.3}$$

where the Hessian W_{ij} and the inhomogeneous term V_i are respectively defined by

$$W_{ij} := \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \quad \text{and} \quad V_i := \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\, \dot{q}^j \ . \tag{2.1.4}$$

The equations of motion (2.1.2) are all of second order and functionally independent if the *non–singularity condition* on its Lagrangian,

$$\det(W_{ij}) = \det\left(\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0 \ , \tag{2.1.5}$$

is fulfilled. A system with a Lagrangian of this type is a *non–singular system*.

On the *configuration manifold* $\mathbb{Q}$ the dynamical equations (2.1.2) are considered as n second–order differential equations. Equivalently, on the tangent space $\mathbf{T}\mathbb{Q}$, also termed *velocity phase manifold*, these equations form a set of $2n$ first–order differential equations. Indeed, labelling points in $\mathbf{T}\mathbb{Q}$ by $(q, \dot{q})$ it follows that the Eqs. (2.1.3) can be viewed as

$$W_{ij}(t, q, \dot{q}) \frac{\mathrm{d}\dot{q}^j}{\mathrm{d}t} = V_i(t, q, \dot{q}) \ , \tag{2.1.6a}$$

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \dot{q}^i \ . \tag{2.1.6b}$$

In the non–singular case, given the $2n$ initial conditions $(q_0, \dot{q}_0) \equiv \big(q^i(0), \dot{q}^i(0)\big)$ the motion trajectory is uniquely defined by the Eqs. (2.1.6). For non–singular Lagrangians and in the compact notation where ξ^I denotes $(q^i, \dot{q}^i)$ the equations of motion (2.1.6) can be rewritten in the form $\dfrac{\mathrm{d}\xi^I}{\mathrm{d}t} = f^I(t, \xi)$, $I = 1, \ldots, 2n$. The motion is found by solving for $\xi(t)$ with initial conditions $\xi(0)$.

There is nothing special about the tangent space $\mathbf{T}\mathbb{Q}$, and similar equations to Eqs. (2.1.6) can be written down for the cotangent space as will be shown in the next subsection.

Remarks

1. In solving the Eqs. (2.1.2) one seeks a solution defined on a finite temporal region satisfying the given boundary conditions. The question of whether or not a certain variational problem of this type possesses a solution, does not just reduce to the usual existence theorems for differential equations (like Cauchy–Kowalewski's). These existence theorems consider a solution only defined in the neighbourhood of some point [160], but Eqs. (2.1.2) require solutions 'in the large'; in this regard the existence and uniqueness theorem is due to Sergey Natanovich Bernstein, see for example [146].

2. We have assumed that the extremals are functions[1] C^1 in the interval $[t_0, t_1]$. A generalisation where piecewise C^1 solutions are considered requires the introduction of the so-called *Weierstrass–Erdmann conditions* which correspond to continuity requirements at the point or points where the extremal contains a corner.

3. The equations of motion (2.1.2) are coordinate independent or covariant under coordinate transformations $q = q(q', \dot{q}', t)$. For the primed Lagrangian $L'(q', \dot{q}', t)$ de-

[1]We say that the function $q^i(t)$ is C^k in an interval $[t_0, t_1]$ if it is continuous, with continuous derivatives up to $\dfrac{\mathrm{d}^k q^i}{\mathrm{d}t^k}$, in that interval. We say that $q^i(t)$ is piecewise C^k in $[t_0, t_1]$ if it is continuous, with continuous derivatives up to $\dfrac{\mathrm{d}^k q^i}{\mathrm{d}t^k}$, in that interval, except possibly at a finite number of points.

fined as $L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t)$, the equations of motion read as the Eqs. (2.1.2) with primed coordinates replacing the unprimed ones. This property suggests the possibility of writing down the equations of motion in a coordinate–free way using intrinsically geometric objects belonging to $\mathbf{T}\mathbb{Q}$; a description can be found in Chapter 3 of [145].

4. A Lagrangian uniquely determines the equations of motion (2.1.2); however, the equations of motion do not determine a unique Lagrangian. Two Lagrangians are said to be *equivalent* if they lead to exactly the same equations of motion [145]. If two Lagrangians L_1 and L_2 differ by a constant factor or a total derivative of a function then they are equivalent. However, the converse is not true, *e.g.* $L_1 = \dot{q}^1\dot{q}^2 - q^1q^2$ and $L_2 = \frac{1}{2}\left[(\dot{q}^1)^2 + (\dot{q}^2)^2 - (q^1)^2 - (q^2)^2\right]$ yield the same equations of motion, but $L_2 - L_1$ is neither a total derivative of a function nor a constant.

5. By definition, given a non–singular Lagrangian all its equivalent Lagrangians are also non–singular.

6. The determinant and the rank of the Hessian (W_{ij}) are coordinate independent. Indeed, if $q^i \to q'^i$ is a coordinate transformation with $J^i{}_j$ the nonsingular Jacobian of the transformation, then $W'_{ij} = W_{lm} J^l{}_i J^m{}_j$. ▲

2.1.2 Canonical form of the Euler–Lagrange equations

Considering the total differential of the Lagrangrian L

$$\mathrm{d}L = \frac{\partial L}{\partial t}\mathrm{d}t + \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - \dot{q}^i\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right) ,$$

it follows that

$$\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = -\frac{\partial L}{\partial t}\mathrm{d}t - \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \dot{q}^i\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right) .$$

With the usual definitions of *generalised momenta* and *canonical Hamiltonian* function

$$p_i := \frac{\partial L}{\partial \dot{q}^i}(t, q, \dot{q}) , \tag{2.1.7}$$

$$H := \dot{q}^i p_i - L , \tag{2.1.8}$$

the total differential of the Lagrangian becomes

$$\mathrm{d}H = -\frac{\partial L}{\partial t}\mathrm{d}t - \frac{\partial L}{\partial q^i}\mathrm{d}q^i + \dot{q}^i\mathrm{d}p_i . \tag{2.1.9}$$

For non–singular Lagrangians, by the inverse function theorem [161], the Eqs. (2.1.7) that define the generalised momenta in terms of $(t, q, \dot{q})$ can (locally) be solved uniquely for the velocities. Insertion of the velocities in terms of coordinates and momenta into the RHS of Eq. (2.1.8) ensures the functional dependence $H = H(t, q, p)$. In comparing $\mathrm{d}H(t, q, p)$ with Eq. (2.1.9) the following identity is found:

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) \mathrm{d}p_i - \left(\frac{\partial L}{\partial q^i} + \frac{\partial H}{\partial q^i}\right) \mathrm{d}q^i - \left(\frac{\partial L}{\partial t} + \frac{\partial H}{\partial t}\right) \mathrm{d}t \equiv 0 \ . \tag{2.1.10}$$

Using the independence of coordinates and momenta implied by the independence of coordinates and velocities via the non–singular transformation (2.1.7), one has that the identity (2.1.10) holds when each one of the coefficients in front of the differentials vanishes itself. Therefore the relations

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial \dot{q}^i} \ , \quad \text{and} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{2.1.11}$$

hold after the Euler–Lagrange equations in the form $\dot{p}_i = \dfrac{\partial L}{\partial q^i}$ are used. The Eqs. (2.1.11) are nothing but the Euler–Lagrange equations written as $2n$ first–order differential equations for the so–called *canonical variables* (q, p). These variables serve as local coordinates in the cotangent space manifold $\mathbf{T}^*\mathbb{Q}$ named *phase space manifold*. The ranges of each q and each p are determined by their physical meaning. The Eqs. (2.1.11) are referred to as *Hamilton equations of motion* or *canonical Euler equations*. In the non–singular case, solutions to the Eqs. (2.1.11) yield a local expression for the trajectory $(q(t), p(t))$ in $\mathbf{T}^*\mathbb{Q}$. This trajectory is completely determined by the initial conditions $(q(0), p(0))$.

The transformation from $\mathbf{T}\mathbb{Q}$ and a Lagrangian function $\{(q, \dot{q}), L\}$ to the $\mathbf{T}^*\mathbb{Q}$ and a Hamiltonian function $\{(q, p), H\}$ defined by the formulas (2.1.7) and (2.1.8) is an example of what is called *Legendre transformation*. It becomes one–to–one in the non–singular cases. If the Legendre transformation is applied to the pair $\{(q, p), H\}$, we get back to the pair $\{(q, \dot{q}), L\}$.

Hamilton equations of motion can also be derived from a least action principle that corresponds to the following fixed–end–point variational problem: *The motion of a system of particles during the time interval* $[t_0, t_1]$ *can be described by* $2n$ *functions* $\big(q^i(t), p_i(t)\big)$*, for which the so–called canonical integral (Cornelius Lanczos [138])*

$$S[q, p] := \int_{t_0}^{t_1} \mathrm{d}t \, \big(\dot{q}^i p_i - H(t, q, p)\big) \tag{2.1.12}$$

is stationary.

A necessary condition for the differentiable functional $S[q, p]$ to have an extremum at the trajectory $\big(\overline{q}^i(t), \overline{p}_i(t)\big)$, among the set of trajectories $\big\{\big(q^i(t), p_i(t)\big)\big\}$ fulfilling the boundary conditions $q^i(t_0) \equiv q^i_0$ and $q^i(t_1) \equiv q^i_1$, is that $\big(\overline{q}^i(t), \overline{p}_i(t)\big)$ satisfy the Hamilton equations.

Remarks

1. A dynamical system is said to be a *Hamiltonian dynamical system* if a Hamiltonian function exists such that the dynamics can be expressed as the Eqs. (2.1.11). Not every motion on $\mathbf{T}^*\mathbb{Q}$ is a Hamiltonian dynamical system. Consider the equations of motion $\dot{q} = qp$ and $\dot{p} = -qp$. They possess the explicit solution $q(t) = q_0\, ce^{ct}/\left(p_0 + q_0 e^{ct}\right)$ and $p(t) = p_0\, c/\left(p_0 + q_0 e^{ct}\right)$ with c being $q_0 + p_0$ a constant of motion. This system does not allow a well behaved Hamiltonian [145].

2. Since Hamilton equations are derivable from a fixed–end–point variational problem, adding a total derivative to a Hamiltonian does not modify the dynamical equations.

3. Hamilton's principle for the action functional (2.1.12) required fixing only the positions at the endpoints of the trajectories. Alternatively, one can fix only the momenta; to do this the $\dot{q}p$ term must be replaced by the term $-q\dot{p}$ in the expression (2.1.12). This last step simply corresponds to subtracting the surface term $\int \frac{\mathrm{d}}{\mathrm{d}t}(qp)\,\mathrm{d}t$ from the canonical integral (2.1.12).

4. With regard to the previous remark, an action that gives a more symmetric treatment of the coordinates and momenta boundary data is [65]

$$
\begin{aligned}
\overline{S}\,[q,p] &:= S[q,p] - \frac{1}{2}(p_i(t_0) + p_i(t_1)) \int_{t_0}^{t_1} \frac{\mathrm{d}q^i}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_{t_0}^{t_1} \mathrm{d}t \left[\frac{1}{2}\left(\dot{q}^i p_i - q^i \dot{p}_i\right) - H\right] + \frac{1}{2}\left[q^i(t_0)p_i(t_1) - q^i(t_1)p_i(t_0)\right] \qquad (2.1.13)
\end{aligned}
$$

with $S[q,p]$ that given in Eq. (2.1.12), and corresponding boundary data

$$
\left.\begin{aligned} q^i(t_0) + q^i(t_1) = 2Q^i \\ p_i(t_0) + p_i(t_1) = 2P_i \end{aligned}\right\} \quad \Leftrightarrow \quad \left.\begin{aligned} \delta q^i(t_0) + \delta q^i(t_1) = 0 \\ \delta p_i(t_0) + \delta p_i(t_1) = 0 \end{aligned}\right\} . \qquad (2.1.14)
$$

▲

2.1.3 Poisson brackets and Hamiltonian dynamics

For a system with only bosonic degrees of freedom, if f and g are functions defined on the phase space, the Poisson bracket (PB) between f and g is defined as

$$
\{f, g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} . \qquad (2.1.15)
$$

More than a convenient abbreviation, the PB is a central object of analytical mechanics from the geometric point of view. From its definition, the following properties are fulfilled for any f, g and h functions on the phase space:

(i) Linearity: If a and b are constants $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$

(ii) Antisymmetry: $\{f, g\} = -\{g, f\}$

(iii) Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$

(iv) Jacobi identity: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$

The PBs between the basic canonical variables q^i and p_j become

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = -\{p_j, q^i\} = \delta^i_j, \quad \{p_i, p_j\} = 0. \tag{2.1.16}$$

The time evolution of a well behaved function f on $\mathbb{R} \times \mathbf{T}^*\mathbb{Q}$ along the motion can be written without solving the Hamilton equations as

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t} \ . \tag{2.1.17}$$

Hence the equations of motion themselves (2.1.11) take the more symmetric form

$$\dot{q}^i = \{q^i, H\} \ , \quad \dot{p}_i = \{p_i, H\} \ , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \ . \tag{2.1.18}$$

Remarks

1. Properties (i), (ii) and (iv) of the PBs are the defining properties of a *Lie algebra*. Therefore the space of functions on $\mathbf{T}^*\mathbb{Q}$ has the algebraic structure of a Lie algebra. In the most pedestrian canonical quantisation scheme, a subalgebra of these functions is meant to be mapped to a subalgebra of symmetric operators on some Hilbert space, with the commutator as Lie product.

2. A manifold $\mathcal{M}$ whose functions on it can be paired with a bracket satisfying (i)–(iv) is called a *Poisson manifold* [143]. Therefore the phase space is a Poisson manifold.

3. A *symplectic manifold* is a pair $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a manifold of even dimension and ω is a nondegenerate ($\det \omega \neq 0$) closed ($\mathrm{d}\omega = 0$) 2–form defined on $\mathcal{M}$ [143]. A symplectic manifold $(\mathcal{M}, \omega)$ is always a Poisson manifold $(\mathcal{M}, \{\cdot, \cdot\})$. Indeed, if (x^I) denotes local coordinates in the symplectic manifold $\mathcal{M}$, the Poisson bracket is defined using the components ω_{IJ} of ω in the local basis of 2−forms $\mathrm{d}x^I \wedge \mathrm{d}x^J$ as follows: $\{f, g\} := (\partial_I f)\omega^{IJ}(\partial_J g)$, where ω^{IJ} denotes the inverse of ω_{IJ}. Using Darboux's theorem [143] simplifies the verification of the Jacobi identity.

4. In the notation introduced in the previous remark, local coordinates for $\mathcal{M} = \mathbf{T}^*\mathbb{Q}$ are $(q,p) \equiv (x^I)$, where $I = 1\ldots,2n$, $x^i \equiv q^i$ and $x^{i+n} \equiv p_i$. Locally, one can introduce the symplectic 2–form with contravariant components
$$(\omega^{IJ}) = \begin{pmatrix} \mathbb{O} & \mathbb{1} \\ -\mathbb{1} & \mathbb{O} \end{pmatrix} , \tag{2.1.19}$$
which reduces $\{f,g\} = (\partial_I f)\omega^{IJ}(\partial_J g)$ to the definition (2.1.15), and $\dot{x}^I = \omega^{IJ}\partial_J H$ to the Hamilton equations (2.1.11) except for $\partial L/\partial t = -\partial H/\partial t$.

5. As mentioned before not every motion on $\mathbf{T}^*\mathbb{Q}$ is a Hamiltonian dynamical system; however, the system is Hamiltonian *iff* $\frac{\partial}{\partial t}\{f,g\} = \{\dot{f},g\} + \{f,\dot{g}\}$. See [145] for a proof. ▲

2.1.4 Invariance properties of S: A first look at gauge symmetry

A cornerstone in the formal structure of classical dynamics is the concept of invariance of the action with respect to certain transformations. In specific cases the identification of such invariance may lead to suitable coordinates in which the equations of motion present a more tractable form. Invariance is translated to the concept of symmetry, and this, in turn, to conservation laws or identities among the equations of motion.

Consider an action functional either in Lagrangian Eq. (2.1.1) or Hamiltonian terms Eq. (2.1.12) or Eq. (2.1.13) and assume that neither of them contains explicit time dependence. Since the following claims are independent of the particular form of the action functional, y^α will denote the variables involved, where $\alpha = 1,\ldots,A$, and the action itself will read as
$$S[y] := \int_{t_0}^{t_1} \mathrm{d}t\, G(y,\dot{y}) . \tag{2.1.20}$$
The 'local general Lagrangian' $G(y,\dot{y})$ represents the local $L(q,\dot{q})$ when $y = q$, or $\dot{q}p - H(q,p)$ when $y = (q,p)$, depending on the action principle under consideration[2]. In this common notation for the Lagrangian and Hamiltonian formalisms the equations of motion read
$$\frac{\delta S}{\delta y^\alpha(t)} \equiv \frac{\delta G}{\delta y^\alpha}(y(t)) = 0 , \tag{2.1.21}$$
with $\delta G/\delta y^\alpha$ the *variational derivatives* of G with respect to y^α. The Euler derivatives G_α coincide with $\delta G/\delta y^\alpha$ since G depends at most on $\dot{y}$:
$$\frac{\delta G}{\delta y^\alpha} \equiv \frac{\partial G}{\partial y^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{y}^\alpha}\right) . \tag{2.1.22}$$

[2] In fact, the following results can be extended to integrands of the action functional depending on higher derivatives ([65]: Chap. 3). However, for our purposes it is enough to include only first derivatives.

The derivatives $\delta S/\delta y^\alpha(t)$ are known as *functional derivatives* of the action; they are defined as

$$\delta S = \int \mathrm{d}t\, \frac{\delta S}{\delta y^\alpha(t)}\, \delta y^\alpha(t) \tag{2.1.23}$$

when the variation of S is considered for arbitrary variations $\delta y^\alpha(t)$ that vanish appropriately at the boundary.

We consider transformations parameterised[3] by r arbitrary functions of time ε^a

$$\delta_\varepsilon y^\alpha = R^\alpha_{(0)a}\varepsilon^a + R^\alpha_{(1)a}\dot{\varepsilon}^a + \ldots + R^\alpha_{(s)a}\frac{\mathrm{d}^s\varepsilon^a}{\mathrm{d}t^s}\ , \tag{2.1.24}$$

for some coefficients $R^\alpha_{(m)a}$ dependent of y and $\dot{y}$. These transformations are called *gauge transformations* if they (i) can be prescribed independently at each time t, and (ii) leave invariant $S[y]$. The latter property explicitly means that for any choice of $\varepsilon^a(t)$ one has

$$0 = \delta_\varepsilon S = \int_{t_0}^{t_1} \mathrm{d}t\, \left(\frac{\delta G}{\delta y^\alpha}\delta_\varepsilon y^\alpha + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{y}^\alpha}\delta_\varepsilon y^\alpha\right)\right)\ , \tag{2.1.25}$$

or after some integration by parts

$$0 = \delta_\varepsilon S = \int_{t_0}^{t_1} \mathrm{d}t\, \left(\mathcal{N}_a\varepsilon^a + \frac{\mathrm{d}F}{\mathrm{d}t}\right)\ , \tag{2.1.26}$$

with $\mathcal{N}_a$ given below and $F := \sum_{l=0}^{s}\mathcal{R}_{(l)a}\frac{\mathrm{d}^l\varepsilon}{\mathrm{d}t^l}$. The coefficients $\mathcal{R}_{(l)a}$ in the definition of F are functions whose dependence of y and derivatives of y is irrelevant for our discussion. From this, if $\varepsilon^a(t)$, $\dot{\varepsilon}^a(t)$, ..., $\frac{\mathrm{d}^s\varepsilon^a}{\mathrm{d}t^s}(t)$ vanish at t_0 and t_1, the expression (2.1.26) yields the local relations

$$\mathcal{N}_a := \sum_{l=0}^{s}(-1)^l\frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\delta G}{\delta y^\alpha}R^\alpha_{(l)a}\right) \equiv 0 \tag{2.1.27}$$

known as *Noether identities* or *generalised Bianchi identities.*

The Noether identities correspond to r local relations among the equations of motion. It is important to stress that Eqs. (2.1.21) comprise A differential equations of motion $\frac{\delta G}{\delta y^\alpha} = 0$, for A unknown quantities y^α which one expects to be uniquely determined, provided the initial conditions are given. However, when a gauge symmetry is present y^α are not uniquely determined. Although algebraically independent, the A equations of motion will be related by the r Noether identities (2.1.27). Thus there are not A functionally independent equations of motion, but only $A - r$, leaving r self–governed degrees of freedom. These degrees of freedom correspond to the fact that if $\overline{y}^\alpha(t)$ is a solution to the Eqs. (2.1.21), then so is $\overline{y}'^\alpha(t) = \overline{y}^\alpha(t) + \delta_\varepsilon\overline{y}^\alpha$, where in order to determine

[3] Here it is assumed for simplicity that each parameter is commuting (*i.e.* Grassmann number zero) which leads to ordinary symmetry, in contrast to anti–commuting (*i.e.* Grassmann number one) parameters which lead to a supersymmetry. See for instance the seminal report by Joaquim Gomis *et al.* [59] where a complete analysis of the gauge transformations at the Lagrangian level is performed.

$\overline{y}'^{\alpha}(t)$ from $\overline{y}^{\alpha}(t)$, precisely r arbitrary functions ε^a need to be specified. Then functions $y^{\alpha}(t)$ are not completely determined once initial conditions are given.

Consequences of the existence of invariance under gauge transformations within the Hamiltonian formalism will be discussed in the next section.

Remarks

1. A particular case of the Euler–Lagrange equations (2.1.21) is when the Hamiltonian form of the action Eq. (2.1.13) is in use. Hence, $G = \dot{q}^i\left(p_i - \frac{1}{2}(p_i(t_0) + p_i(t_1))\right) - H(q,p,t)$, and the Eqs. (2.1.21) are indeed reduced to the Hamilton equations
$$\frac{\partial G}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{q}^i}\right) = -\frac{\partial H}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i - \frac{1}{2}(p_i(t_0) + p_i(t_1))\right) = -\frac{\partial H}{\partial q^i} - \dot{p}_i = 0\ ,$$
$$\frac{\partial G}{\partial p_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{p}_i}\right) = \dot{q}^i - \frac{\partial H}{\partial p_i} = 0\ .$$

2. In the particular case where none of the r arbitrary gauge parameters depends on time, the invariance of the action (2.1.25) reduces to
$$\frac{\delta G}{\delta y^{\alpha}} R^{\alpha}_{(0)a} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{y}^{\alpha}} R^{\alpha}_{(0)a}\right) = 0\ .$$
Hence, on–shell, $Q := \frac{\partial G}{\partial \dot{y}^{\alpha}} R^{\alpha}_{(0)a}$ is a conserved quantity known as *Noether charge*.

3. If $\delta_{\eta} y^{\alpha} = T^{\alpha}_{(0)a}\eta^a + T^{\alpha}_{(1)a}\dot{\eta}^a + \ldots + T^{\alpha}_{(s)a}\frac{\mathrm{d}^s\eta^a}{\mathrm{d}t^s}$ and $\delta_{\varepsilon} y^{\alpha}$ Eq. (2.1.24) are gauge transformations, then $\kappa\delta_{\varepsilon}y^{\alpha} + \delta_{\eta}y^{\alpha}$ and $[\delta_{\varepsilon},\delta_{\eta}]y^{\alpha} := \delta_{\varepsilon}(\delta_{\eta}y^{\alpha}) - \delta_{\eta}(\delta_{\varepsilon}y^{\alpha})$ (with $\kappa \in \mathbb{R}$) leave the action invariant. Moreover, the product $[\cdot\,,\cdot]$ is linear, antisymmetric and satisfies the Jacobi identity (*cf.* Sect. 2.1.3), therefore the infinitesimal gauge transformations form a Lie algebra.

4. Transformations of the form $\delta_{\mu} y^{\alpha} = \mu^{\alpha\beta}\frac{\delta G}{\delta y^{\beta}}$, with $\mu^{\alpha\beta} = -\mu^{\beta\alpha}$ arbitrary functions of y and their time derivatives up to some finite order, leave the action invariant provided $\mu^{\alpha\beta}(t_0) = \mu^{\alpha\beta}(t_1) = 0$, *i.e.* $\delta_{\mu}S = \int \mathrm{d}t \left[\frac{\delta G}{\delta y^{\alpha}}\mu^{\alpha\beta}\frac{\delta G}{\delta y^{\beta}}\right] = 0$. These transformations are called *trivial gauge transformations*. Note that a trivial gauge transformation does *not* imply a degeneracy of the equations of motion. The LHS of $\delta_{\mu}S = 0$ identically vanishes due to the antisymmetry of $\mu^{\alpha\beta}$.

5. Let $\delta_{\varepsilon} y^{\alpha}$ be a gauge transformation of the form (2.1.24). A rescaling of the gauge parameters by some matrix depending on the ys, $\varepsilon'^a = \Lambda^a{}_b(y)\varepsilon^b$, redefines the gauge transformation into $\delta_{\varepsilon'} y^{\alpha} = R'^{\alpha}_{(0)a}\varepsilon'^a + \ldots + R'^{\alpha}_{(s)a}\frac{\mathrm{d}^s\varepsilon'^a}{\mathrm{d}t^s}$. From the point of view of Lie algebras, the transformed $\delta_{\varepsilon'}$ is linearly independent of δ_{ε}. Each $R'^{\alpha}_{(m)a}$ is not a linear combination of $R^{\alpha}_{(m)a}$ with coefficients that are real numbers.

However, the Noether identities associated to the rescaled transformation are not independent of those coming from the symmetry $\delta_\varepsilon y^\alpha$. Hence, at a classical level, no new information is contained in the invariance of S under the rescaled gauge transformations. ▲

2.2 Classical singular systems

In the previous section the structure of systems for which the non–singularity condition is satisfied was broadly analysed. In this section, the structure of systems which fail to fulfil Eq. (2.1.5) is exposed. A major emphasis is put on the Hamiltonian formalism based on methods proposed by P. A. M. Dirac [1] and P. G. Bergmann [6, 7]. Modern sources in the analysis of these systems are for example [4, 64, 65, 149, 150, 151, 153].

The point of view that will be adopted within the Sects. 2.2.2 and 2.2.3 is that a sufficiently good understanding in the treatment of constrained systems is obtained by considering only explicitly time–independent Lagrangians $L = L(q, \dot{q})$. The treatment of explicitly time–dependent constraints, either coming from a time–dependent Lagrangian or time–dependent gauge fixing functions, has been reviewed for instance by D. M. Gitman and I. V. Tyutin [151], and S. P. Gavrilov and D. M. Gitman [162].

2.2.1 Gauge invariant systems and the failure of the non–singularity condition

In Sect. 2.1.4 the arbitrariness in the solutions to the equations of motion and its relation with gauge transformations was emphasised. In the current subsection the connection between the existence of Noether identities and the failure of the corresponding Lagrangian to satisfy the non–singularity condition (2.1.5) is described. The fulfilment of the condition (2.1.5) in a Lagrangian system guarantees that the equations of motion will be functionally independent. On the other hand, gauge invariant systems show relations among the equations of motion. Hence gauge invariant systems cannot be non–singular; they are rather a kind of *singular systems*.

Consider the action functional (2.1.1) of a system with gauge invariance. In this case the Noether identities (2.1.27) take the form

$$\mathcal{N}_a := \sum_{l=0}^{s} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} \left(\frac{\delta L}{\delta q^i} R^i_{(l)a} \right) \equiv 0 \ , \tag{2.2.1}$$

with $L = L(t, q, \dot{q})$ a Lagrangian for the system and $\dfrac{\delta L}{\delta q^i} = 0$ the equations of motion. Using the expression (2.1.3) of the dynamical equations and expanding the LHS of the identity in (2.2.1), one can easily identify the terms with the highest number of time derivatives of the generalised coordinates. These terms are

$$(-1)^s W_{ij} R^i_{(s)a}(q,\dot{q}) \, \frac{\mathrm{d}^{s+2}\, q^j}{\mathrm{d}t^{s+2}} \ ,$$

whose coefficients must vanish, therefore $W_{ij}R^i_{(s)a}(q,\dot{q}) = 0$. Since not all the coefficients $R^i_{(s)a}$ in the gauge transformation are identically zero, s is the largest integer for which $R^i_{(s)a} \neq 0$ in Eq. (2.1.24), then these are the components of null eigenvectors of the Hessian W_{ij}; in this way the non–singular condition is not fulfilled for gauge invariant systems

$$\det(W_{ij}) = \det\left(\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0 \ . \tag{2.2.2}$$

So the important result that *a Lagrangian which is invariant under local gauge transformations is necessarily singular* has been obtained. It is worth noting that the converse of the previous statement is not true; there are singular Lagrangians which do not possess a local gauge invariance. In the terminology that will be introduced later on, those systems contain second–class constraints only.

2.2.2 Constrained dynamics: Lagrangian description

The Euler–Lagrange equations for singular Lagrangians will now be examined in more detail . Let the rank of the Hessian be a constant $R_1 \leq n$ throughout the velocity phase manifold $\mathbf{T}\mathbb{Q}$ with n the number of degrees of freedom. Then there exist $n - R_1$ null eigenvectors $u_{\overline{a}}(q,\dot{q})$ of the Hessian

$$W_{ij} u^j_{\overline{a}} = 0, \quad \overline{a} = 1, \ldots, n - R_1 \ , \tag{2.2.3}$$

where some of the null eigenvectors might be encoded in a certain gauge transformation as described before. Contracting $u_{\overline{a}}$ with the Euler–Lagrange equations (2.1.3), $n - R_1$ relations which do not contain accelerations are obtained

$$V_i(q,\dot{q}) u^i_{\overline{a}}(q,\dot{q}) = 0 \ . \tag{2.2.4}$$

These relations between coordinates and velocities, and all other which may occur in the analysis by consistency of (2.2.4), are called *Lagrange constraints*. From the other R_1 equations, which involve accelerations, only those which are independent will be referred to as equations of motion.

One possible situation in the examination of Eqs. (2.2.4) is to have a Lagrangian such that these relations are identically satisfied. Then there are no Lagrange constraints at all. In that case all the Eqs. (2.2.4) are $0 = 0$ for each value of $\overline{a}$, and no more equations beyond Eqs. (2.1.2) to determine the physical trajectory exist. Hence, without loss of generality, the Eqs. (2.1.3) can be used to express the first R_1 accelerations $\ddot{q}^1, \ldots, \ddot{q}^{R_1}$ in the form

$$\ddot{q}^{i'} = f^{i'}\left(q^1, \ldots, q^{R_1}, \dot{q}^1, \ldots, \dot{q}^{R_1} \mid q^{R_1+1}, \ldots, q^n, \dot{q}^{R_1+1}, \ldots, \dot{q}^n, \ddot{q}^{R_1+1} \ldots, \ddot{q}^n\right) \quad (2.2.5)$$

with $i' = 1, \ldots, R_1$. The solutions to the equations of motion are uniquely determined after *fixing* the $n - R_1$ functions of time $q^{R_1+1}(t), \ldots, q^n(t)$ and giving the initial conditions $(q^{i'}(0), \dot{q}^{i'}(0))$. Different choices for the arbitrary functions lead to different solutions. The occurrence of these arbitrary functions of time in the general solution to the equations of motion is a common feature of the dynamics of constrained systems as will be verified.

A more involved situation happens when the LHS of Eqs. (2.2.4) do not identically vanish, determining in this way a number of relations among the qs and $\dot{q}$s. In such a case the following iterative stages have to be taken into account [149]:

Stage 1. [Initial Lagrange constraints] Generally speaking not all of the $n - R_1$ relations (2.2.4) are functionally independent. Let $K_1 \leq n - R_1$ be the number of functionally independent relations. Denote these relations by

$$C_{a_1}(q, \dot{q}) = 0, \qquad a_1 = 1, \ldots, K_1 \quad (2.2.6)$$

which define a $2n - K_1$ dimensional surface Σ_1 in $\mathbf{T}\mathbb{Q}$ to which the motion is constrained. In what follows, at every step, the *regular condition on the Lagrange constraints* is required [149], namely

$$\text{rank}\left(\frac{\partial C_{a_1}}{\partial q^i}, \frac{\partial C_{a_1}}{\partial \dot{q}^i}\right)\Bigg|_{\Sigma_1} = K_1 \ . \quad (2.2.7)$$

The Lagrange constraints C_{a_1} are regular if and only if the rank of the Jacobian matrix formed by partial derivatives of the constraints with respect to coordinates and velocities is maximal on the Lagrange constraint surface[4] Σ_1.

Using the correlations among qs and $\dot{q}$s (2.2.6) the rank R_1 of W_{ij}, originally calculated all over $\mathbf{T}\mathbb{Q}$, can at most decrease to $R_2 \leq R_1$. Therefore more null eigenvectors of W_{ij} might be found, and these in turn introduce more independent constraints than only (2.2.6). Extracting the regular and functionally independent constraints one has altogether at this stage

$$C_{a_2}(q, \dot{q}) = 0 \ , \qquad a_2 = 1, \ldots, K_2 \geq K_1 \ , \quad (2.2.8)$$

[4] This has the consequence that if C_{a_1} obeys (2.2.7) then $C_{a_1}^2$ or $C_{a_1}^{1/2}$, say, will not.

as Lagrange constraints. Relations (2.2.8) define a hypersurface $\Sigma_2 \subset \mathbf{T}\mathbb{Q}$ of lower dimensionality than that of Σ_1. The motion is now restricted to Σ_2. Using the Lagrangian constraints (2.2.8), the rank of W_{ij} must be reconsidered, giving R_3 on Σ_2 with $R_3 \leq R_2 \leq R_1$, then more null eigenvectors of W_{ij} might be found which give rise to more constraints which restrict further the motion. We keep doing this series of operations until the following circumstances are reached: (a) We have the equations of motion (2.1.3) with the motion restricted to some Lagrange constraint surface $\Sigma \subset \mathbf{T}\mathbb{Q}$ of dimensionality $2n - K$ defined by K regular and functionally independent relations, which we refer to as *initial Lagrange constraints*,

$$C_a(q, \dot{q}) = 0 \ , \qquad a = 1, \ldots, K \geq \ldots \geq K_2 \geq K_1 \ . \tag{2.2.9}$$

(b) The rank of W_{ij} reduces on Σ to R with $R \leq \ldots \leq R_2 \leq R_1$. (c) More importantly, given any null eigenvector u of W_{ij} on Σ the equation $V_i(q, \dot{q})\, u^i(q, \dot{q}) = 0$ is identically satisfied when the constraints are used, that is, no more Lagrange constraints in this way arise.

Stage 2. [Consistency condition] Among the initial Lagrange constraints, in the terminology of [149], a constraint is of 'type–A' if it depends on positions only, $A_\alpha(q) = 0$, and of 'type–B' if it also depends on velocities, $B_\beta(q, \dot{q}) = 0$. One separates by algebraic manipulations the maximal number of type–A and type–B constraints such that relations (2.2.9) take the form

$$A_\alpha(q) = 0, \qquad \alpha = 1, \ldots, A, \tag{2.2.10a}$$

$$B_\beta(q, \dot{q}) = 0, \qquad \beta = 1, \ldots, B, \tag{2.2.10b}$$

with $K = A + B$. Then the time derivative of the type–A constraints on Σ (now defined by the Eqs. (2.2.10)) is considered. The result is either zero, that is, $\dot{A}_\alpha = a_{\alpha\alpha'} A_{\alpha'} + b_{\alpha\beta} B_\beta$ or it leads to new constraints which increase the number of regular and functionally independent type–A and/or type–B constraints. In the case where new constraints arise, these constraints together with relations (2.2.10), restrict further the motion on a hypersurface of lower dimensionality than that of Σ. If at this stage a new type–A constraint arises, one takes its time derivative evaluated at the new hypersurface, and so on. This procedure ends when the time derivatives of all type–A constraints vanish on the latest hypersurface defined by the Lagrange constraints, that is, one ends with a set of Lagrange constraints

$$A_{\alpha'}(q) = 0 \ , \qquad \alpha' = 1, \ldots, A' \geq A \ , \tag{2.2.11a}$$

$$B_{\beta'}(q, \dot{q}) = 0 \ , \qquad \beta' = 1, \ldots, B' \geq B \ , \tag{2.2.11b}$$

where the equations $\dot{A}_{\alpha'} = 0$ are obeyed identically on the hypersurface Σ' defined by (2.2.11).

One adds to the analysis the second time derivative of (2.2.11a) and the first time derivative of (2.2.11b)

$$\ddot{A}_{\alpha'}(q) = \frac{\partial^2 A_{\alpha'}}{\partial q^i \partial q^j}\,\dot{q}^i\dot{q}^j + \frac{\partial A_{\alpha'}}{\partial q^i}\,\ddot{q}^i = 0\ , \qquad \alpha' = 1,\ldots,A' \geq A\ , \tag{2.2.12a}$$

$$\dot{B}_{\beta'}(q,\dot{q}) = \frac{\partial B_{\beta'}}{\partial q^i}\,\dot{q}^i + \frac{\partial B_{\beta'}}{\partial \dot{q}^i}\ddot{q}^i = 0\ , \qquad \beta' = 1,\ldots,B' \geq B\ . \tag{2.2.12b}$$

After using the equations of motion and the constraints (2.2.11), more independent equations of motion than the initial ones and/or more constraints of any of the two types described above may appear. To those new type–A and –B constraints, if any, the same treatment of differentiation needs to be applied. This will possibly generate new equations of motion and/or more constraints.

Stage 3. [End of the process] For a system with a finite number of degrees of freedom, for which consistent equations of motion are possible (see the remarks below), this iterative process ends after a finite number of steps. At the end one will have: (a) A finite number of equations for the accelerations

$$\sum_{j=1}^{n} \mathscr{W}_{ij}(q,\dot{q})\,\ddot{q}^j = \mathscr{V}_i(q,\dot{q})\ , \qquad i = 1,\ldots,\mathscr{R} \leq n\ . \tag{2.2.13}$$

(b) A set of type–A and type–B Lagrange constraints

$$\mathscr{A}_\mu(q) = 0\ , \qquad \mu = 1,\ldots,\mathscr{A} \geq \ldots \geq A\ , \tag{2.2.14a}$$

$$\mathscr{B}_\nu(q,\dot{q}) = 0\ , \qquad \nu = 1,\ldots,\mathscr{B} \geq \ldots \geq B\ , \tag{2.2.14b}$$

which define the hypersurface $\overline{\Sigma}$ in $\mathbf{T}\mathbb{Q}$ on which the motion is restricted to. (c) All the consistency conditions on this set of constraints are satisfied identically on $\overline{\Sigma}$.

Stage 3 implies that the first time derivative of type–A constraints (2.2.14a), that is, $\left(\partial\mathscr{A}_\mu/\partial q^i\right)\,\dot{q}^i = 0$, must be contained in the type–B constraints (2.2.14b) that involve $\dot{q}$. Hence implying $\mathscr{B} \geq \mathscr{A}$. The second time derivative of type–A constraints (2.2.14a) as well as the first time derivative of type–B constraints (2.2.14b) must be contained in the equations of motion (2.2.13). Hence $\mathscr{R} \geq \mathscr{B} \geq \mathscr{A}$. In conclusion the constraints (2.2.14) become an invariant system of the differential equations (2.2.13).

Finally, the nature of the solutions to the system of equations (2.2.13) and (2.2.14) will be analysed. Among the $\mathscr{R}$ equations of motion (2.2.13), $\mathscr{B}$ equations are consequence of the consistency conditions on the Lagrange constraints (2.2.14). By construction, the set of constraints (2.2.14b) can be split into the first $\mathscr{A}$, $\mathscr{B}_\mu = \dot{\mathscr{A}}_\mu = 0$, and the remaining $\mathscr{B} - \mathscr{A}$ ones, denoted by $\mathscr{B}'_\rho$ with $\rho = \mathscr{A} + 1, \dots, \mathscr{B}$. Since this set of constraints is regular and functionally independent, the Jacobian

$$\left(\frac{\partial \mathscr{B}_\nu}{\partial q^i}, \frac{\partial \mathscr{B}_\nu}{\partial \dot{q}^i}\right) = \begin{pmatrix} \frac{\partial \mathscr{B}_\mu}{\partial q^i} & \frac{\partial \mathscr{B}_\mu}{\partial \dot{q}^i} \\ \frac{\partial \mathscr{B}'_\rho}{\partial q^i} & \frac{\partial \mathscr{B}'_\rho}{\partial \dot{q}^i} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathscr{B}_\mu}{\partial q^i} & \frac{\partial \mathscr{A}_\mu}{\partial q^i} \\ \frac{\partial \mathscr{B}'_\rho}{\partial q^i} & \frac{\partial \mathscr{B}'_\rho}{\partial \dot{q}^i} \end{pmatrix}$$

is of maximal rank, namely $\mathscr{B}$. One may arrange the first $\mathscr{B}$ equations of motion (2.2.13) read as the first time derivative of $\mathscr{B}_\nu$

$$\dot{\mathscr{B}}_\mu \equiv \ddot{\mathscr{A}}_\mu = \frac{\partial^2 \mathscr{A}_\mu}{\partial q^i \partial q^j}\, \dot{q}^i \dot{q}^j + \frac{\partial \mathscr{A}_\mu}{\partial q^i}\, \ddot{q}^i = 0 \;, \qquad \mu = 1, \dots, \mathscr{A} \;, \tag{2.2.15a}$$

$$\dot{\mathscr{B}}_\rho \equiv \dot{\mathscr{B}}'_\rho = \frac{\partial \mathscr{B}'_\rho}{\partial q^i}\, \dot{q}^i + \frac{\partial \mathscr{B}'_\rho}{\partial \dot{q}^i}\ddot{q}^i = 0 \;, \qquad \rho = \mathscr{A} + 1, \dots, \mathscr{B} \;. \tag{2.2.15b}$$

The remaining equations of motion are denoted by

$$\sum_{j=1}^{n} \mathscr{W}'_{i'j}(q, \dot{q})\ddot{q}^j = \mathscr{V}'_{i'}(q, \dot{q}) \;, \qquad i' = \mathscr{B} + 1, \dots, \mathscr{R} \;. \tag{2.2.15c}$$

The dynamical content encoded in the Eqs. (2.2.13) and Eqs. (2.2.14) has been translated, after the usage of consistency conditions, into the system of equations (2.2.15) and (2.2.14). These equations make explicit that $\mathscr{B}$ of the $\mathscr{R}$ equations of motion (2.2.13) are a consequence of the consistency condition. The type–A constraints (2.2.14a) can be used to solve, say, the first $\mathscr{A}$ coordinates, $q^\mu = q^\mu(q^m)$, $m = \mathscr{A} + 1, \dots, n$, in terms of the other $n - \mathscr{A}$. Note that $\dot{\mathscr{A}}_\mu = \ddot{\mathscr{A}}_\mu = 0$ are automatically satisfied when these relations among the coordinates are employed. Substitute $q^\mu(q^m)$ into $\mathscr{B}'_\rho(q, \dot{q}) = 0$ to obtain $\mathscr{B}''_\rho(q^m, \dot{q}^m) = 0$ and the $\mathscr{R} - \mathscr{A}$ equations of motion

$$\sum_{m=\mathscr{A}+1}^{n} \frac{\partial \mathscr{B}''_\rho}{\partial q^m}\, \dot{q}^m + \frac{\partial \mathscr{B}''_\rho}{\partial \dot{q}^m}\ddot{q}^m = 0 \;, \qquad \rho = \mathscr{A} + 1, \dots, \mathscr{B} \;, \tag{2.2.16a}$$

$$\sum_{m=\mathscr{A}+1}^{n} \mathscr{W}''_{i'm}\ddot{q}^m - \mathscr{V}''_{i'} = 0 \;, \qquad i' = \mathscr{B} + 1, \dots, \mathscr{R} \;. \tag{2.2.16b}$$

The last set of equations corresponds to (2.2.15c) after the substitution of $q^\mu(q^m)$. So one has $(\mathscr{B} - \mathscr{A}) + (\mathscr{R} - \mathscr{B}) = \mathscr{R} - \mathscr{A}$ equations for the $n - \mathscr{A}$ unknowns q^m, with $\mathscr{R} - \mathscr{A} \leq n - \mathscr{A}$. Then solving for the first $\mathscr{R} - \mathscr{A}$ unknowns

$$\ddot{q}^{m'} = g^{m'}\left(q^{m'}, \dot{q}^{m'} \;\mid\; q^{m''}, \dot{q}^{m''}, \ddot{q}^{m''}\right), \qquad m' = \mathscr{A} + 1, \dots, \mathscr{R} \;, \tag{2.2.17}$$

where $m'' = \mathscr{R} + 1, \dots, n$. Equations (2.2.16) show that if the type–B constraints $\mathscr{B}''_\rho(q^m, \dot{q}^m) = 0$ are satisfied at time $t = 0$, they will be satisfied at any later time. The solutions to (2.2.17) depend on the $n - \mathscr{R}$ (which could be zero) number of arbitrary functions $q^{m''}(t)$. *Fixing* the arbitrary functions $q^{m''}(t)$ and giving the initial values $(q^m(0), \dot{q}^m(0))$ consistently with the type–B constraints $\mathscr{B}''_\rho(q^m, \dot{q}^m) = 0$, the trajectory $(q^m(t), \dot{q}^m(t))$ will be uniquely defined. Substitution of $q^m(t)$ into $q^\mu(q^m(t)) = q^\mu(t)$ defines the physical motion of the system $q^i(t) = (q^\mu(t), q^m(t))$ in $\mathbb{Q}$.

Remarks

1. At every stage in the analysis of the subsection 2.2.2, singular Lagrangians that do not lead to inconsistencies have been assumed. Excluded singular Lagrangians are for instance $L = \dot{q} - q$, which yields $1 = 0$ as Euler–Lagrange equation, or, $L = \frac{1}{2}\, q^1 \left((\dot{q}^2)^2 + 1\right) + q^2 q^3$ which yields $(q^1 \dot{q}^2)^{\cdot} - q^3 = 0$, $(\dot{q}^2)^2 + 1 = 0$ and $q^2 = 0$. These two particular examples show that the corresponding action functionals do not have a stationary point.

2. It was also assumed at various stages that some algebraic manipulations can be carried out, for instance, several equations were written to expose some variables as explicit functions of the others. Although this is possible in principle, it might be difficult and disadvantageous at times. In any case the 'algorithm' described above tells us what to expect to happen in a general case although the equations may vary.

3. A set of gauge transformations is known as complete [163] when the number of independent arbitrary functions in the gauge transformations (2.1.24) is equal to the number of arbitrary functions in the general solution of the Lagrangian equations of motion. ▲

2.2.3 Constrained dynamics: Hamiltonian description

The transition from the Lagrangian to the Hamiltonian formulation of classical mechanics is carried out by means of a Legendre transformation over the velocities. In constrained dynamics, the failure of the non–singularity condition (2.1.5) is translated in the noninvertibility of the velocities as functions of coordinates and momenta from the equations

$$p_i = \frac{\partial L}{\partial \dot{q}^i}\,(q, \dot{q})\ , \qquad i = 1, \ldots, n\ . \tag{2.2.18}$$

Instead, from the definition of momenta, relations among qs and ps emerge. These relations suggest a similar iterative process as the one applied to the initial Lagrange constraints. In the Hamiltonian setting, the corresponding set of steps is known as *Dirac–Bergmann algorithm*; this will be described below.

Stage 1. [Primary constraints] Let the rank of the matrix $\left(\frac{\partial p_i}{\partial \dot{q}^j}\right) = \left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right)$ be $n - M'$. This number not only determines that from (2.2.18) $n - M'$ velocities are expressible in terms of qs, ps and the remaining M' velocities; but also that there exist M' relations among qs and ps. These M' relations are called *primary constraints*, a term coined by P. G. Bergmann [7]. The word 'primary' refers to the fact that the equations of motion are not used to obtain them, but they arise directly from the definition of momenta. Primary constraints restrict the motion to a surface Γ_1 of dimension $2n - M'$ in the phase space $\mathbf{T}^*\mathbb{Q}$. In principle, directly from (2.2.18) $M > M'$ (redundant) relations may arise; they are denoted as follows:

$$\phi_m(q, p) = 0 , \qquad m = 1, \ldots, M > M' . \tag{2.2.19}$$

The separation of the primary constraints (2.2.19) into M' *essential constraints*, which are those that completely define Γ_1, and $M - M'$ *dependent constraints*, which hold as consequences of the first ones, might *not* be achievable.

In what follows, it will be assumed that the relations (2.2.19), as well as and any other set of constraints that can occur through consistency conditions of these ones, can be separated into essential and dependent constraints. In other words, it will be assumed that the relations (2.2.19) satisfy the so–called *regularity condition*. Strictly speaking, the regularity condition corresponds to the following requirement [65]: The $(2n - M')$–dimensional constraint surface Γ_1 (redundantly) defined by $\phi_m = 0$ should be coverable by open regions, on each of which the constraint functions ϕ_m can be split into M' essential constraints $\phi_{m'}$, $m' = 1, \ldots, M'$, determined by

$$\text{rank}\left(\frac{\partial \phi_{m'}}{\partial q^i}, \frac{\partial \phi_{m'}}{\partial p_i}\right)\Bigg|_{\Gamma_1} = M', \tag{2.2.20}$$

and $M - M'$ dependent constraints $\phi_{m''}$, $m'' = M' + 1, \ldots, M$, which hold as a consequence of the others, that is, $\phi_{m'} = 0 \;\Rightarrow\; \phi_{m''} = 0$.

A set of constraints $\{\phi_m\}$ that fulfils the regularity condition satisfies the following two theorems. The corresponding proofs can be found in [65].

Theorem 2.2.1 *If a (smooth) phase space function G vanishes on the surface $\phi_m = 0$, then $G = g^m \phi_m$ for some functions g^m.*

Theorem 2.2.2 *If $\lambda_i \delta q^i + \mu^i \delta p_i = 0$ for arbitrary variations of δq^i, δp_i tangent to the constraint surface then, on the constraint surface,*

$$\lambda_i = u^m \frac{\partial \phi_m}{\partial q^i} \tag{2.2.21a}$$

$$\mu^i = u^m \frac{\partial \phi_m}{\partial p_i} \tag{2.2.21b}$$

for some u^m. In the presence of redundant constraints, like $\phi_{m''} = 0$, the functions u^m exist but are not unique.

In the Lagrangian formalism, the Lagrange constraints were required to be an invariant system of the equations of motion. In order to bring the equations of motion into the Hamiltonian setting of constrained systems, a closer examination of the canonical Hamiltonian (2.1.8) is necessary. By direct calculation one can see that the variation *on* the constraint surface Γ_1 of $H = \dot{q}^i p_i - L$, induced by arbitrary independent variations of positions and velocities, leads to the conclusion that H is only a function of qs and ps. In other words, the canonical Hamiltonian (2.1.8) is only well defined on Γ_1. The most general extension of H to the whole phase space comes from adding to it an arbitrary function that vanishes on the constraint surface. In view of the Theorem 2.2.1, such a function must be a linear combination of constraints. Adding this specific combination to the Hamiltonian (2.1.8) gives rise to the *primary Hamiltonian* or *total Hamiltonian*

$$H^{(1)} := H + u^m \phi_m \ , \qquad m = 1, \dots, M \ . \tag{2.2.22}$$

From the comparison between the total variations (valid only on Γ_1) $\mathrm{d}H = \frac{\partial H}{\partial q^i}\mathrm{d}q^i + \frac{\partial H}{\partial p_i}\mathrm{d}p_i$ and $\mathrm{d}H = \dot{q}^i \mathrm{d}p_i - \frac{\partial L}{\partial q^i}\mathrm{d}q^i$, one obtains

$$\left(\frac{\partial H}{\partial q^i} + \frac{\partial L}{\partial q^i}\right)\mathrm{d}q^i + \left(\frac{\partial H}{\partial p_i} - \dot{q}^i\right)\mathrm{d}p_i = 0 \ . \tag{2.2.23}$$

The Theorem 2.2.2 implies that on the primary constraint surface Γ_1 the following equations hold:

$$\dot{q}^i = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_i} \ , \tag{2.2.24a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - u^m \frac{\partial \phi_m}{\partial q^i} \ . \tag{2.2.24b}$$

In combination, the equations (2.2.19) and (2.2.24) are equivalent to the Euler–Lagrange equations (2.1.2) when the determinant of the Hessian vanishes. The Eqs. (2.2.24) are referred to as the *Hamilton equations of motion for constrained systems.*

A more symmetric way of writing the Eqs. (2.2.24) is by using the PBs defined in (2.1.15):

$$\dot{q}^i = \{q^i, H\} + u^m \{q^i, \phi_m\} \approx \{q^i, H^{(1)}\} \ , \tag{2.2.25a}$$

$$\dot{p}_i = \{p_i, H\} + u^m \{p_i, \phi_m\} \approx \{p_i, H^{(1)}\} \ , \tag{2.2.25b}$$

where the symbol '≈' (which is read as 'weakly zero') means that the function on its LHS is equal to the function on its RHS on the constraint surface only. Hence $\phi_m \approx 0$, that is, the quantity ϕ_m is numerically restricted to be zero but it does not identically vanish throughout the phase space.

The time evolution along the motion of an arbitrary function $F(q,p)$ on $\mathbf{T}^*\mathbb{Q}$ can be written as

$$\dot{F} = \{F, H\} + u^m\{F, \phi_m\} \approx \{F, H^{(1)}\} \ . \tag{2.2.26}$$

Stage 2. [Consistency condition] The consistency condition, as presented in the Lagrangian setting, meant that the time evolution of the constraints is bound to vanish on the constraint surface. In the Hamiltonian formalism this is achieved using the Eq. (2.2.26) for each of the ϕ_m and one should have $\dot{\phi}_m \approx 0$. This condition gives the following set of M algebraic inhomogeneous equations on Γ_1 for the M unknown us:

$$\{\phi_m, H\} + u^n\{\phi_m, \phi_n\} = 0 \ , \qquad m, n = 1, \ldots, M \ . \tag{2.2.27}$$

Without loss of generality the following two cases are distinguishable:

Case 1. The determinant of the matrix formed by the primary constraints is nonzero on Γ_1

$$\det\left(\{\phi_m, \phi_n\}\right)|_{\Gamma_1} \neq 0 \ . \tag{2.2.28}$$

In this case, from Eqs. (2.2.27), all the us can be explicitly known as functions of coordinates and momenta on the constraint surface Γ_1. The primary Hamiltonian (2.2.22) then becomes

$$H^{(1)} = H - \phi_m\{\phi, \phi\}^{-1}_{mn}\{\phi_n, H\} \ , \tag{2.2.29}$$

where $\{\phi, \phi\}^{-1}_{mn}$ stands for the inverse matrix of $(\{\phi_m, \phi_n\})$. The solution to the equations of motion (2.2.25) is well defined, that is, given the initial values $(q^i(0), p_i(0))$, consistent with the primary constraints (2.2.19), the motion trajectory $(q^i(t), p_i(t))$ has no arbitrariness.

Case 2. The determinant of the matrix formed by the primary constraints vanishes on Γ_1. In this case the matrix $(\{\phi_m, \phi_n\})$ is assumed to have a constant rank r_1 on the constraint surface Γ_1

$$\text{rank}\left(\{\phi_m, \phi_n\}\right)|_{\Gamma_1} = r_1 < M \ . \tag{2.2.30}$$

From the Eqs. (2.2.27) one has that $M - r_1$ us remain undetermined. There are $v_{\underline{a}}(q,p)$ independent null eigenvectors of the matrix $(\{\phi_m, \phi_n\})$,

$$v^n_{\underline{a}}\ \{\phi_n, \phi_m\}|_{\Gamma_1} = 0 \ , \qquad \underline{a} = 1, \ldots, M - r_1 \ .$$

Contracting $v^n_{\underline{a}}$ from the left with Eq. (2.2.27) originates the following relations:

$$v^n_{\underline{a}}\{\phi_n, H\}|_{\Gamma_1} = 0 \; ; \tag{2.2.31}$$

these equations may or may not be identically obeyed on Γ_1. If they are not, then more constraints have appeared. Among these new constraints, one selects those which are independent of $\phi_m \approx 0$. Let these be M_1 in number. By assumption, these new constraints (together with the primary constraints (2.2.19)) satisfy the regularity condition. They are denoted by

$$\phi_{M+m_1}(q,p) = 0 \qquad m_1 = 1\;,\ldots, M_1\;. \tag{2.2.32}$$

On the constraint surface defined by both $\phi_m = 0$ and $\phi_{M+m_1} = 0$ fewer variables are independent. On this surface the rank of the matrix $(\{\phi_m, \phi_n\})$ can at most decrease giving rise to more independent null eigenvectors. Such eigenvectors bring more constraints of the form (2.2.31). This particular process of generating more constraints at this level ends when the following situation is reached: (a) The motion is restricted to the constraint surface Γ_2 defined by

$$\phi_{\mu_1} = 0\;, \qquad \mu_1 = 1,\ldots, M, M+1,\ldots, M+K_1 \equiv J_1\;, \tag{2.2.33}$$

with $K_1 \geq M_1$. The matrix $(\{\phi_n, \phi_m\})$ is assumed to have a constant rank ϱ_1 on Γ_2. (b) Any null eigenvector w of the matrix $(\{\phi_n, \phi_m\})$ makes $w^n\{\phi_n, H\} = 0$ to be identically satisfied on Γ_2. The K_1 relations are called *second–stage constraints* [151].

For consistency one has to require that the K_1 second–stage constraints are preserved in time. Using the equations of motion (2.2.26) the following is obtained:

$$\{\phi_{M+k_1}, H\} + u^m\{\phi_{M+k_1}, \phi_m\} = 0\;, \quad (m = 1,\ldots, M;\; k_1 = 1,\ldots, K_1)\;, \tag{2.2.34}$$

a condition to be fulfilled on Γ_2 for each k_1. Analysing (2.2.34) in the same way in which (2.2.27) was analysed, some of the us left undetermined at the previous step may be specified and/or more functionally independent constraints be obtained. These new constraints restrict further the motion to the constraint surface defined by

$$\phi_{\mu_2} = 0\;, \qquad \mu_2 = 1,\ldots, M+K_1+K_2 = J_2\;, \tag{2.2.35}$$

with M primary constraints, K_1 second–stage constraints, and K_2 so–called *third–stage constraints*. Requiring the consistency condition on the third–stage constraints, again, more equations for the unknown us operating at this point may appear and a fourth–generation of constraints may arise. One keeps doing this process until the stage described below is reached.

Stage 3. [End of the process] This process of getting new equations for the us and new independent constraints among qs and ps ends after a finite number of steps. At the end, the situation should be the following: (a) There are M primary constraints and a total of K *secondary constraints*, as second– third– etc. stage constraints are known. All the constraints collectively denoted by

$$\phi_\mu(q,p) \approx 0 \ , \qquad \mu = 1, \ldots, M, M+1, \ldots, M+K = J \ . \tag{2.2.36}$$

The system of constraints (2.2.36) defines the constraint surface Γ with respect to which the symbol $\approx$ is taken. The consistency condition on any of the constraints (2.2.36) becomes

$$\{\phi_\mu, H\} + u^m\{\phi_\mu, \phi_m\} \approx 0 \ , \qquad m = 1, \ldots, M \ . \tag{2.2.37}$$

(b) Any null eigenvector w of the rectangular matrix $(\{\phi_\mu, \phi_m\})$ on Γ makes

$$w^\mu\{\phi_\mu, H\}|_\Gamma = 0$$

to be identically satisfied.

Stage 3 is the point where the set of constraints (2.2.36) becomes an invariant system for the Hamilton equations of motion (2.2.25). At this stage some or all of the us may be known as functions of coordinates and momenta on Γ; in the case where some of the us remains unknown, there exists arbitrariness in the equations of motion (2.2.25). Only after *fixing* these unknown us, the motion becomes uniquely defined on the constraint surface Γ given the compatible initial conditions.

Remarks

1. As a curiosity one may mention that although the term 'constraints' for the relations (2.2.36) is the most common in the literature, one may also find the term 'links' [28].

2. According to Olivera Misčović and Jorge Zanelli [154], a set of constraints $\varphi_n \approx 0$, $n = 1, \ldots, r$, which defines the surface $\Gamma \in \mathbf{T}^*\mathbb{Q}$, is a *regular set* if and only if the Jacobian matrix of the derivatives of the constraints with respect to the phase space variables is of maximal rank r on Γ (*cf.* Eq. (2.2.20)). With this definition and the one given in this thesis –paradoxically– a set of constraints that satisfies the regularity condition can *or* cannot be a regular set of constraints in the sense of J. Zanelli. For instance, in a two dimensional phase space $\mathbf{T}^*\mathbb{Q} = \{(q,p)\}$ the constraint $\phi = p = 0$ satisfies the regularity condition and is also a regular set

of constraints, Eq. (2.2.20) is satisfied; however, the set of constraints formed by $\phi_1 = p = 0$ and $\phi_2 = p^2 = 0$ satisfies the regularity condition (ϕ_1 is an essential constraint and ϕ_2 a dependent one), but it is *not* a regular set of constraints in J. Zanelli's sense. The corresponding Jacobian matrix has rank one on the constraint surface. A set of constraints which only contains essential constraints and satisfies the regularity condition is regular in J. Zanelli's sense.

3. Given the constraints that fulfil the regularity condition, their explicit separation into essential and dependent constraints is not necessary in the analysis. All that is required is to choose the constraints in such a way that the split is in principle achievable.

4. The Hamilton equations (2.2.24) and the primary constraints (2.2.19) can be derived from a fixed–end–point variational problem where coordinates, momenta and us are independently varied. An action functional is

$$S[q,p,u] := \int_{t_0}^{t_1} \mathrm{d}t \, \left(\dot{q}^i p_i - H - u^m \phi_m\right) \tag{2.2.38}$$

where the sum is solely made over the primary constraints. The boundary data restriction for this action is $\delta q^i(t_0) = \delta q^i(t_1) = 0$. Alternatively, a more symmetric treatment of the boundary data can be given if one considers instead of (2.2.38) the action (*cf.* Eq. (2.1.13))

$$\begin{aligned} \overline{S}[q,p,u] &:= S[q,p,u] - \frac{1}{2}(p_i(t_0) + p_i(t_1)) \int_{t_0}^{t_1} \frac{\mathrm{d}q^i}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \mathrm{d}t \left[\frac{1}{2}\left(\dot{q}^i p_i - q^i \dot{p}_i\right) - H - u^m \phi_m\right] \\ &\quad + \frac{1}{2}\left[q^i(t_0)p_i(t_1) - q^i(t_1)p_i(t_0)\right] \ , \end{aligned} \tag{2.2.39}$$

then the corresponding boundary data is (2.1.14).

5. The variational principle based on the Lagrangian (2.1.1) is equivalent to the variational principle based on any of the Hamiltonian actions (2.2.38) or (2.2.39). Solving the equations $\delta S/\delta p^i = 0$ and $\delta S/\delta u^m = 0$ for p_i and u^m from (2.2.38) or (2.2.39) one recovers the Lagrangian action (2.1.1).

6. There is a relation between the Lagrange and the Hamilton constraints. An account of this can be found in [4] under the assumption that the set of constraints satisfies the regularity condition and consists of only essential constraints.

▲

2.2.4 First– and second–class functions

The notation used to designate all the constraints (2.2.36) suggests that the importance of their classification into primary and secondary is of minor importance in the theory. The classification of constraints into *first–class* and *second–class* due to P. A. M. Dirac [1], which applies to all functions on the phase space, has proved to play a central role in the theory of constrained systems.

In Dirac's terminology, a function F on $\mathbf{T}^*\mathbb{Q}$ is said to be first–class if its Poisson bracket with every constraint vanishes on the constraint surface defined by (2.2.36), that is,

$$\{F, \phi_\mu\} \approx 0 \ , \qquad \mu = 1, \ldots, J \ . \tag{2.2.40}$$

A function on $\mathbf{T}^*\mathbb{Q}$ that is not first–class is called second–class. A property of the first–class functions is that they are preserved under the PB, that is, the PB between two first–class functions is first–class.

Let $\{v_a\}$ be a set of linearly independent solutions to the homogeneous system of equations associated to (2.2.37),

$$\{\phi_\mu, \phi_m\} v_a^m \approx 0 \ . \tag{2.2.41}$$

The most general solution to the Eqs. (2.2.37) is hence of the form $u^m = U^m + \lambda^a v_a^m$, with U^m a particular solution of the Eqs. (2.2.37). In the extreme case where none of the us could be specified as a function of coordinates and momenta, one has that $u^m = \lambda^a v_a^m$ is the most general solution to the system of equations (2.2.37) with λ^a totally arbitrary coefficients. After the substitution of the most general solution to the Eqs. (2.2.37) into the primary Hamiltonian, one has

$$H^{(1)} = H + U^m \phi_m + \lambda^a \phi_a \ , \tag{2.2.42}$$

with

$$\phi_a := v_a^m \phi_m \ . \tag{2.2.43}$$

In the expression (2.2.42) there is an explicit separation of the part of u^m that remains unknown, $\lambda^a v_a^m$, from the part that is fixed by the consistency conditions, U^m. The *Lagrange multipliers* λ^a indicate the arbitrariness in the solution to the equations of motion.

The equations (2.2.37) and (2.2.41) imply that $H^{(1)}$ (2.2.22) and ϕ_a (2.2.43) both are first–class functions. From these observations we can also draw the important conclusion that there are as many undetermined combinations of u^m as there are first–class primary constraints. Moreover, $\{\phi_a\}$ is a complete set of first–class primary constraints in the sense that any first–class primary constraint is a linear combination of the ϕ_a. Indeed, $\lambda^a v_a^m$ is the most general solution of the homogeneous system of equations associated to (2.2.37) on Γ.

Remarks

1. The distinction between first– and second–class constraints is defined without ambiguities when the regularity condition is fulfilled. When this condition has not been satisfied, one can turn any second–class constraint into a first–class one. Indeed, let the relations $\chi_\alpha = 0$ form a set of essential second–class constraints, that is, $\det(\{\chi_\alpha, \chi_\beta\}) \not\approx 0$. Replacing χ_α by $\overline{\chi}_\alpha := \chi_\alpha^2$, it follows that $\{\overline{\chi}_\alpha, \overline{\chi}_\beta\} \approx 0$. The functions $\overline{\chi}_\alpha$ hence form a set of first–class constraints. However, the set $\{\overline{\chi}_\alpha\}$ does not fulfil the condition (2.2.20). ▲

2.2.5 First–class constraints and Hamiltonian gauge transformations

A physical state at a certain time is uniquely defined once the canonical pairs are specified; however, this is not a one–to–one relation as it is clearly shown in the cases where a certain symmetry underlies the theory.

In the presence of first–class primary constraints the equations of motion contain arbitrary functions of time λ^a; nevertheless, on classical grounds given a physical state at time $t = t_0$ the equations of motion should fully determine the physical state at any other time. Thus any ambiguity in the value of the canonical pairs at $t \neq t_0$ should be physically irrelevant. The presence of first–class constraints allows, for example, that two different trajectories fulfilling the equations of motion evolve from *one* given initial state; each trajectory corresponding to a definite choice of the arbitrary functions of time λ^a. However, these two trajectories must be physically equivalent. The transformation that mediates between such histories is called *Hamiltonian gauge transformation.*

Using the primary Hamiltonian (2.2.42) and the Hamilton equations (2.2.26), the difference between the values of a dynamical variable $F(q,p)$ at a time $t = t_0 + \delta t$, corresponding to two different choices λ^a and $\widetilde{\lambda}^a$, can be written down. At order δt this difference takes the form

$$\delta_\epsilon F = \epsilon^a \{F, \phi_a\} \approx \{F, \epsilon^a \phi_a\} \ , \tag{2.2.44}$$

with $\epsilon^a := \left(\lambda^a(t_0) - \widetilde{\lambda}^a(t_0)\right)\delta t$. The values F_λ and $F_{\widetilde{\lambda}}$ are infinitesimally related by the Eq. (2.2.44). The generators that occur in the transformation (2.2.44) are all the independent first–class primary constraints ϕ_a. However, these are not the only gauge generators. For instance, it can be seen [4] that also $\dot{\phi}_a$ generates Hamiltonian gauge transformations if we go to the next order $(\delta t)^2$ in the change $\delta_\epsilon F$. In addition, both the PB between any two first–class primary constraints and the PB of any first–class primary constraint with $H + U^m\phi_m$ generate a gauge transformation [65]. These results suggest that some first–class secondary constraints may also be regarded as a gauge generator.

Although from the above considerations it cannot be inferred that all first–class secondary constraints are gauge generators, in physical applications each first–class (primary or secondary) constraint is found to be a gauge generator. In what follows it will be posited that *all first–class constraints generate gauge transformations* or equivalently that the so–called *Dirac conjecture* is satisfied. Under this assumption, the generator of all the Hamiltonian gauge transformations is $G := \epsilon^a\gamma_a$, with γ_a all the first class constraints of the theory.

Not all the functions on the constraint surface Γ are classical observables. Only those functions whose time evolution is not affected by the arbitrariness in the choice of λ^a must be regarded as classical observables. Equivalently, a classical observable can be described as a phase space function that has weakly vanishing PB with all the first–class constraints

$$\{A, \gamma_a\} = A_a{}^b(q,p)\gamma_b \; . \tag{2.2.45}$$

A *classical observable* is hence a function on the constraint surface that is gauge invariant.

The concept of independent gauge transformations can be inferred from the concept of independent first–class constraints. Besides the regularity condition on a set of constraints, which involves the possibility of separating the constraints into those which are essential to describe the surface $\Gamma \subset \mathbf{T}^*\mathbb{Q}$ and those which are dependent of the first ones, one can speak of the *reducibility condition* [61, 65, 152]. Reducibility makes reference to linear dependence among the constraints. One says that the first– and second–class constraints (γ_a) and (χ_α) obey the reducibility condition if they satisfy the following two requirements:

(i) There exists a set of functions of the phase space different from zero on the constraint surface, $Z^a_{\bar{a}} \not\approx 0$ and $Z^\alpha_{\bar{\alpha}} \not\approx 0$, such that

$$Z^a_{\bar{a}}\,\gamma_a = 0 \; , \qquad (a = 1, \ldots, A;\, \bar{a} = 1, \ldots, \overline{A}) \; , \tag{2.2.46a}$$

$$Z^\alpha_{\bar{\alpha}}\,\chi_\alpha = 0 \; , \qquad (\alpha = 1, \ldots, B;\, \bar{\alpha} = 1, \ldots \overline{B}) \; . \tag{2.2.46b}$$

(ii) The matrix $(\{\chi_\alpha, \chi_\beta\})$ is of maximal rank $B - \overline{B}$ on the constraint surface,

$$\text{rank}\;(\{\chi_\alpha, \chi_\beta\})|_\Gamma = B - \overline{B} \; . \tag{2.2.46c}$$

The condition (ii) has been included to shorten the discussion on the second–class constraint sector; this implies that there are exactly $B - \overline{B}$ independent second–class constraints. Indeed, if (2.2.46c) is satisfied, there is a subset of the constraints χ_α, name them χ_A, $A = 1, \ldots, B - \overline{B}$, such that $\det(\{\chi_A, \chi_{A'}\}) \not\approx 0$ which are independent (otherwise, the matrix $(\{\chi_A, \chi_{A'}\})$ would possess a null eigenvector). An equivalent way to state (ii) is by saying that rank $(Z^{\alpha}_{\overline{\alpha}}) = \overline{B} < B$. Similarly, if the relations (2.2.46a) are all independent, one has exactly $A - \overline{A}$ independent gauge generators.

In the situation where there is no dependence of the type (2.2.46a) and (2.2.46b) among the constraints, it is said that $\{\phi_m\}$ is an *irreducible set of constraints*. From this definition, the gauge transformations are linearly independent if and only if the constraints that generate them are irreducible.

Remarks

1. Given a complete set of gauge transformations at a Lagrangian level (see the Remarks of Sect. 2.2.2), there are very precise relations with the corresponding Hamiltonian gauge transformations. If regular and irreducible first–class constraints are present in a system that fulfils the Dirac conjecture, the following claims can be proved [4, 7, 65, 164, 163]: (i) The number of independent Lagrangian gauge symmetries (2.1.24) is equal to the number of independent first–class primary constraints; (ii) the number of constraint–stages at the end of the Dirac algorithm (primary, second–stage, third–stage, etc.) is equal to the order of the highest time derivative of the gauge parameter in (2.1.24) plus one; (iii) if one counts independently gauge parameters and their time derivatives in (2.1.24), the final number is equal to the total number of first–class constraints. In general, the specific relations between the gauge parameters in the Lagrangian gauge transformations (2.1.24) and the corresponding ones in the generator $G := \epsilon^a \gamma_a$ of the Hamiltonian setting are model dependent.

2. Second–class constraints cannot be interpreted as gauge generators, or in general, generators of any transformation of physical relevance. The reason is that, by definition, second–class constraints do not preserve all the constraints ϕ_μ and thus permits dependence on the Lagrange multipliers in the transformation.

3. **Dirac Bracket**. Using the advantage that PBs between independent second–class constraints form an invertible matrix $(\Delta_{\alpha\beta}) := (\{\chi_\alpha, \chi_\beta\})$, the Dirac bracket is defined as

$$\{F, G\}^* := \{F, G\} + \{F, \chi_\alpha\}\Delta^{\alpha\beta}\{\chi_\beta, G\} \tag{2.2.47}$$

where $\Delta^{\alpha\gamma}\Delta_{\gamma\beta} = \delta^\alpha_\beta$. The Dirac bracket obeys the properties satisfied by the PB (*cf.* Sect. 2.1.3). In addition it also fulfils

$$\{\chi_\alpha, F\}^* = 0 \quad \text{for any } F \in C^\infty(\mathbf{T}^*\mathbb{Q}), \tag{2.2.48a}$$

$$\{F, G\}^* \approx \{F, G\} \quad \text{for first–class } G \text{ and arbitrary } F, \tag{2.2.48b}$$

$$\{R, \{F, G\}^*\}^* \approx \{R, \{F, G\}\} \quad \text{for first–class } F \text{ and } G \text{ and arbitrary } R. \tag{2.2.48c}$$

From (2.2.48a) the second–class constraints can strictly be set to zero either before or after evaluating the Dirac bracket. Since $H^{(1)}$ is a first–class function, from (2.2.48b) the Dirac bracket can be used to express the time evolution of an arbitrary function F in the Eq. (2.2.26). Similarly, using the Eq. (2.2.48b) the Hamiltonian gauge transformations can be expressed using the Dirac bracket.

4. In the reducible case, one could encounter that not all the linear combinations of first–class constraints (2.2.46a) are independent. It may happen that the coefficients $Z^a_{\bar{a}} \equiv Z^a_{a_1}$, with a_1 taking the same values as $\bar{a}$, are also correlated. In fact there may exist a tower of reducibility relations of the following type:

$$Z^{a_1}_{a_2} Z^a_{a_1}\big|_\Gamma = 0, \quad Z^{a_2}_{a_3} Z^{a_1}_{a_2}\big|_\Gamma = 0, \quad \ldots, \quad Z^{a_{L-2}}_{a_{L-1}} Z^{a_L}_{a_{L-1}}\Big|_\Gamma = 0, \tag{2.2.49}$$

where $a_s = 1, \ldots, A_s$, for any $s = 1 \ldots, L$, and $Z^{a_s}_{a_{s-1}} \not\approx 0$. If this is the case the set of constraints $\{\phi_\mu = (\gamma_a, \chi_\alpha)\}$ is said to be $L-$*stage reducible* in the first–class constraints. This situation is met in models like $p-$form gauge theories [165] or superstring theory [152, 166].

5. In practice it is sometimes desirable to explicitly eliminate the arbitrariness introduced by the independent first–class constraints such that the correspondence between values of canonical pairs and physical states becomes one–to–one. In order to have equations of the form (2.2.37) that impose restrictions on the arbitrary Lagrange multipliers, some *ad–hoc* supplementary conditions must be introduced. These supplementary conditions are called *gauge fixing constraints* or simply *gauge conditions* [11]. For an irreducible set of constraints, the number of independent gauge conditions must be equal to the number of first–class constraints such that all together form a set of second–class constraints. Thus after the gauge fixing procedure has been done there are not first–class constraints left; hence no more arbitrariness in the equations of motion does exist and one can pass to the construction of the Dirac bracket. The gauge fixing procedure is very interesting in its own right [167] and its application to Feynman path integral [26, 28, 34, 168] is conventional in particle physics phenomenology.

6. When only m' second–class constraints are present in the theory, no arbitrariness exists in the Hamiltonian. Given the initial conditions compatible with the constraints permits the equations of motion (2.2.26) to uniquely determine the motion trajectory. A set of canonical variables that satisfies the constraints determines one and only one physical state. Therefore the number of *physical* or *true degrees of freedom* corresponds to the number of independent canonical variables divided by two, that is, $(2n - m')/2$. If the m' second–class constraints come from a gauge fixing procedure of m irreducible first–class constraints, after the implementation of m gauge fixing conditions, one has $(n - m)$ true degrees of freedom. Hence, in the case of a system with m and m' irreducible first– and second–class constraints, respectively, one has that $(2n - 2m - m')/2$ is the number of true degrees of freedom. ▲

CHAPTER 3

Foundations of the Classical BRST Formalism

BRST symmetry could have been discovered within a strictly classical context by mathematicians dealing with the geometry of phase space, had they only been willing to extend their analysis to Grassmann variables.

– Marc Henneaux, 1988

A valuable notion in gauge theories is BRST symmetry. In the path integral formalism this is what is left of gauge invariance after a gauge–fixing procedure has been implemented. However, its analysis is not limited to the quantum regime; classically, it corresponds to a symmetry in a certain enlarged phase space. This enlarged manifold includes Grassmann variables besides the canonical pairs that label points in $\mathbf{T}^*\mathbb{Q}$ which are first–class constrained by $\gamma_a \approx 0$.

In this chapter the construction of the classical BRST–symmetry generator, or simply BRST generator, is addressed. Within the Hamiltonian formalism, two equivalent ways to build and show the existence of the BRST generator are available. One of them strongly relies on the abstract graded algebraic structures implemented by the introduction of Grassmann variables (ghosts) in the phase space $\mathbf{T}^*\mathbb{Q}$ [63, 169, 170, 171, 172]. This method is based upon the following fact [65]: Whenever there is a co–isotropic surface Γ embedded in a manifold $\mathbf{T}^*\mathbb{Q}$, it is possible to construct a nilpotent derivation s, the BRST differential. This differential acts on an appropriate graded algebra containing $C^\infty(\mathbf{T}^*\mathbb{Q})$ and is such that the classical cohomolgy of s at ghost number zero,

$H^0(s)$, corresponds to functions of $C^\infty(\Gamma)$ constant along the gauge orbits (classical observables). By co–isotropic surface Γ is meant the first–class constraint surface and tangent vector fields, associated to the constraint functions, which close on Γ defining gauge orbits. The other method to build the BRST generator has been described in the excellent works [61, 62]. This one relies on the symplectic structure of the ghost extended phase space and the iterative use of the Jacobi identity applied to PBs of the first–class constraints. Since this method can be implemented using the notation and concepts already introduced in the previous chapter, it will be summarised here. For our purposes this approach will be advantageous from the computational viewpoint.

To be definite, in this chapter and the rest of this work the following assumptions on the constrained systems are made: (i) The constrained phase space $\mathbf{T}^*\mathbb{Q}$, which is also referred to as *big phase space* or *original phase space*, only contains bosonic variables, hence only bosonic constraints will be present; (ii) only first–class constrained systems will be considered; (iii) Dirac conjecture applies on the systems; (iv) the regularity condition is satisfied by the constraints with only essential constraints locally present; (v) only irreducible constraints are adopted. If second–class constraints occurred, it is assumed that they were consistently set to zero via the Dirac brackets to the point where the remaining variables form canonical pairs, and the first–class constraints satisfy $(i)-(v)$. Therefore in what follows the Poisson brackets can be considered Dirac brackets if necessary.

3.1 Extended Hamiltonian and its gauge symmetries

In first–class constrained systems where the Dirac conjecture applies is useful to introduce the so–called *extended Hamiltonian*

$$H_E := H + \lambda^a \gamma_a \ , \qquad a = 1, \ldots, m \ . \tag{3.1.1}$$

The function H_E is obtained from adding to the canonical Hamiltonian (2.1.8) all first–class constraints. Therefore H_E contains all the information concerning gauge transformations. For gauge–invariant dynamical variables, the evolution predicted by the canonical H, the primary $H^{(1)}$, or the extended H_E Hamiltonian coincide on the constraint surface.

The extended action functional

$$S_E[q,p,\lambda] := \int_{t_0}^{t_1} \mathrm{d}t \ \left(\dot{q}^i p_i - H_E\right) = \int_{t_0}^{t_1} \mathrm{d}t \ \left(\dot{q}^i p_i - H - \lambda^a \gamma_a\right) \tag{3.1.2}$$

induces the equations of motion

$$\dot{F} = \{F, H\} + \lambda^a \{F, \gamma_a\} \approx \{F, H_E\} \ , \quad (3.1.3a)$$

$$\gamma_a(q, p) \approx 0 \ . \quad (3.1.3b)$$

When F is substituted by each one of the canonical variables, the expressions (3.1.3) are *not* equivalent to the equations of motion (2.2.25) produced by the action functional (2.2.38). In other words, the Eqs. (3.1.3) are not equivalent to the original Euler–Lagrange equations. Although the introduction of H_E changes the equations of motion, it does not alter the time evolution of gauge–invariant functions. Therefore, the extended formalism describes the same physical system. It simply contains additional pure gauge variables, new Lagrange multipliers, and consequently, also additional gauge invariance [163].

The action (3.1.2) is invariant (up to a boundary term) under

$$\delta_\epsilon q^i = \epsilon^a \{q^i, \gamma_a\} \ , \quad (3.1.4a)$$

$$\delta_\epsilon p_i = \epsilon^a \{p_i, \gamma_a\} \ , \quad (3.1.4b)$$

$$\delta_\epsilon \lambda^a = \dot{\epsilon}^a - \lambda^b \epsilon^c f_{bc}{}^a - \epsilon^b H_b{}^a \ . \quad (3.1.4c)$$

The so–called structure functions $f_{ab}{}^c(q, p)$ and $H_a{}^b(q, p)$ respectively come from the first–class property of the constraints and the consistency condition on the first–class constraints,

$$\{\gamma_a, \gamma_b\} = f_{ab}{}^c \gamma_c \ , \quad (3.1.5)$$

$$\{H, \gamma_a\} = H_a{}^b \gamma_b \ . \quad (3.1.6)$$

The PB (3.1.5) between any two first–class constraints is a linear combination of the first–class constraints. In contrast, when the structure functions $f_{ab}{}^c$ explicitly involve phase space variables the commutator between two infinitesimal transformations generated by γ_a, $\delta_\epsilon F = \{F, \epsilon^a \gamma_a\}$ and $\delta_\eta F = \{F, \eta^a \gamma_a\}$, is not necessarily another infinitesimal transformation of the same kind. The commutator does not close on $\mathbf{T}^*\mathbb{Q}$. Indeed

$$\delta_\epsilon \delta_\eta F - \delta_\eta \delta_\epsilon F = \epsilon^b \eta^a f_{ab}{}^c \{F, \gamma_c\} + \epsilon^b \eta^a \{F, f_{ab}{}^c\} \gamma_c \ , \quad (3.1.7)$$

from which an extra term proportional to the constraints appears. Henceforth, when the gauge algebra (3.1.5) closes with structure constants (resp. functions) the corresponding algebra of constraints shall be called *closed* (*open*) *gauge algebra*. It only is on the constraint surface Γ that the algebra of transformations $\delta_\epsilon F = \{F, \epsilon^a \gamma_a\}$ always closes and generates an $m-$dimensional submanifold.

In what follows a pair of extensions of the original phase space $\mathbf{T}^*\mathbb{Q}$, first–class constrained by $\gamma_a = 0$, will be done. The first of them concerns with the introduction of the m arbitrary Lagrange multipliers λ^a as coordinates and π_a as their conjugate momenta. The symplectic structure of $\mathbf{T}^*\mathbb{Q}$ becomes trivially enlarged by the inclusion of

$$\{\lambda^a, \lambda^b\} = 0 \ , \quad \{\lambda^a, \pi_b\} = \delta^a_b \ , \quad \{\pi_a, \pi_b\} = 0 \ . \tag{3.1.8}$$

For simplicity all canonical pairs in the original phase space have been assumed to be bosonic. Therefore the corresponding first–class constraints also obey this statistics, and in order to have a bosonic action functional (3.1.2), the associated Lagrange multipliers must also be variables with even Grassmann number (bosons).

In order not to affect the dynamical content of theory, the following constraints need to be imposed:

$$\pi_a = 0 \ , \qquad a = 1, \ldots, m \ . \tag{3.1.9}$$

The constraints (3.1.9) generate on the Lagrange multipliers the following transformation:

$$\delta\lambda^a = \epsilon^b\{\lambda^a, \pi_b\} = \epsilon^a \quad \Leftrightarrow \quad \lambda'^a = \lambda^a + \epsilon^a \ , \tag{3.1.10}$$

this expresses the arbitrariness in choosing λ^a. Note that the *ad–hoc* introduced constraints $\pi_a \approx 0$ neither generate the gauge transformation (3.1.4c) nor change the first–class constraint structure of $\gamma_a = 0$. There are no Lagrange multipliers in the original constraints. Collectively $\gamma_a \approx 0$ and $\pi_a \approx 0$ form a set of first–class constraints in the Lagrange–multiplier enlarged phase space $\mathbf{T}^*_\lambda\mathbb{Q}$. The relations (3.1.5) and (3.1.6) are trivially extended to

$$\{G_\alpha, G_\beta\} = f_{\alpha\beta}{}^\gamma G_\gamma \ , \tag{3.1.11}$$

$$\{H, G_\alpha\} = H_\alpha{}^\beta\, G_\beta \ , \qquad \alpha, \beta, \gamma = 1, \ldots, 2m \ , \tag{3.1.12}$$

in the $2(n+m)$–dimensional space $\mathbf{T}^*_\lambda\mathbb{Q}$. The functions G_α correspond to the $2m$ first–class constraints (γ_a, π_a). The structure functions $f_{\alpha\beta}{}^\gamma$ and $H_\alpha{}^\beta$ correspond to $f_{ab}{}^c$ and $H_a{}^b$ when all the indices are associated to the original constraints $\gamma_a \approx 0$ and are set to zero otherwise.

The formalism in which the constraints $\pi_a \approx 0$ are introduced is known as the *non–minimal BRST formalism.*

Remarks

1. The action (3.1.2) is invariant up a term which identically vanishes if the arbitrary parameters $\epsilon^a(t)$ vanish at the boundaries: $\epsilon^a(t_0) = \epsilon^a(t_1) = 0$.

2. Gauge transformations (3.1.4) also leave invariant the extended action functional

$$\begin{aligned}\overline{S}_E[q,p,\lambda] &:= S_E[q,p,\lambda] - \frac{1}{2}(p_i(t_0)+p_i(t_1))\int_{t_0}^{t_1}\frac{\mathrm{d}q^i}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int_{t_0}^{t_1}\mathrm{d}t\left[\frac{1}{2}\left(\dot{q}^i p_i - q^i\dot{p}_i\right) - H - \lambda^a\gamma_a\right]\\ &\quad+\frac{1}{2}\left[q^i(t_0)p_i(t_1) - q^i(t_1)p_i(t_0)\right]\ . \qquad (3.1.13)\end{aligned}$$

A fixed–end–point variational problem which uses this action and the boundary data (2.1.14) gives the equations of motion (3.1.3) ▲

3.2 Tensorial structures present in the formalism

In the construction of the BRST generator, one encounters totally antisymmetric quantities[1] of the form $F^{\alpha_1\ldots\alpha_p}$ for some $p\in\mathbb{N}$. The action of contracting any of these quantities with the constraints G_α defines the following operator:

$$\delta : F^{\alpha_1\ldots\alpha_p} \longmapsto (\delta F)^{\alpha_1\ldots\alpha_{p-1}} := F^{\alpha_1\ldots\alpha_{p-1}\alpha_p}G_{\alpha_p}\ , \qquad (3.2.1)$$

and immediately, by the bosonic nature of the constraints and the antisymmetry of $F^{\alpha_1\ldots\alpha_p}$ in their indices, it follows that $\delta^2 F^{\alpha_1\ldots\alpha_p} = F^{\alpha_1\ldots\alpha_{p-1}\alpha_p}G_{\alpha_p}G_{\alpha_{p-1}} \equiv 0$. The following theorem establishes the converse [61]:

Theorem 3.2.1 *Let F be an antisymmetric tensor of rank p. If $\delta F = 0$, then there exists an antisymmetric tensor K of rank $p+1$ such that $F=\delta K$.*

This theorem is a prelude of the cohomological structure of the theory, similar to the De Rham cohomology of forms on a differentiable manifold. The theorem is compatible with the irreducibility condition imposed at the beginning. Indeed, $\delta F = 0$ is a linear combination $F^{\cdots\alpha}G_\alpha$ of first–class constraints required to vanish, then (by irreducibility) no coefficient functions different from zero on the constraint surface exist that satisfy the equality, the Theorem 3.2.1 ensures the existence of coefficients which do vanish on the constraint surface, namely $F^{\cdots\alpha} = K^{\cdots\alpha\beta}G_\beta$. So, the Theorem 3.2.1 is complementary to the irreducibility condition in the following sense: For a set of irreducible constraints $G_\alpha\approx 0$, in the linear combination $Z^\alpha G_\alpha = 0$ the existence of $Z^\alpha \not\approx 0$ is avoided, but $Z^\alpha = Y^{[\alpha\beta]}G_\beta \approx 0$ [2].

[1] In the case where also odd variables are considered in the first–class constrained manifold $\mathbf{T}^*\mathbb{Q}$, there are quantities which do not have a definite symmetry in their indices.

[2] The square brackets at the indices level denote total antisymmetrisation. Here the convention of the factor of $1/p!$ inside the definition of antisymmetrised indices is taken, *e.g.* $F^{[\alpha\beta\gamma]} := \frac{1}{3!}\left(F^{\alpha\beta\gamma}+F^{\gamma\alpha\beta}+F^{\beta\gamma\alpha}-F^{\alpha\gamma\beta}-F^{\gamma\beta\alpha}-F^{\beta\alpha\gamma}\right)$.

Besides the homogeneous equations $\delta F = 0$ some inhomogeneous equations $\delta E = B$ are also found in the BRST formalism. The following result proves to be useful in their analysis [61] :

Theorem 3.2.2 *A necessary and sufficient condition for the existence of solutions to the inhomogeneous equations*

$$\delta E = B \ , \tag{3.2.2}$$

in the unknown E, is that $\delta B = 0$.

The above theorem says that the tensorial equation $E^{[\alpha_1 \ldots \alpha_{p-1}\alpha_p]}G_{\alpha_p} = B^{[\alpha_1 \ldots \alpha_{p-1}]}$, has solutions if and only if $B^{[\alpha_1 \ldots \alpha_{p-1}]}G_{\alpha_{p-1}} = 0$. More about the solutions to the Eq. (3.2.2) is indicated in the following theorem [61]:

Theorem 3.2.3 *When $\delta B = 0$ holds, i.e. when the Eq. (3.2.2) has a particular solution, the general solution to $\delta E = B$ is given by*

$$E = E_0 + \delta K \tag{3.2.3}$$

where E_0 is a particular solution to the Eq. (3.2.2) and K is an arbitrary tensor of appropriate rank.

The above three theorems are the fundamental building blocks in the construction of the BRST generator, their proofs can be found in [61]. They are based on the irreducibility and regularity conditions assumed for the first–class constraints G_α –not to mention the smoothness of each component of the tensors involved.

3.3 The ladder of higher structure functions

In this section, applying the Jacobi identity (see Sect. 2.1.3) to the Eq. (3.1.11) and the results presented in the previous section, the so–called higher structure functions will be iteratively built.

In order to make the construction look more natural a new notation is necessary. It is conventional to denote

$$\overset{(0)}{U}_\alpha(q,p) := G_\alpha(q,p) \ , \tag{3.3.1}$$

$$\overset{(1)}{U}_{\alpha\beta}{}^{\gamma}(q,p) := -\frac{1}{2} f_{\alpha\beta}{}^{\gamma}(q,p) \ , \tag{3.3.2}$$

and call them zeroth– and first–order structure functions, respectively. With these new symbols one has that the PBs (3.1.11) read

$$\{\overset{(0)}{U}_\alpha, \overset{(0)}{U}_\beta\} = -2\overset{(1)}{U}_{\alpha\beta}{}^{\gamma}\,\overset{(0)}{U}_\gamma \ , \tag{3.3.3}$$

with $\overset{(1)}{U}_{\alpha\beta}{}^{\gamma} = \overset{(1)}{U}_{[\alpha\beta]}{}^{\gamma}$ totally antisymmetric in the covariant indices.

From the Jacobi identity one has

$$\{\{\overset{(0)}{U}_{[\alpha_1}, \overset{(0)}{U}_{\alpha_2}\}, \overset{(0)}{U}_{\alpha_3]}\} = \frac{1}{3} \sum_{\text{cyclic perm}} \{\{\overset{(0)}{U}_{\alpha_1}, \overset{(0)}{U}_{\alpha_2}\}, \overset{(0)}{U}_{\alpha_3}\} = 0 \,, \tag{3.3.4}$$

where the antisymmetry in the indices (α_1, α_2) is understood. Using the gauge algebra (3.3.3) one gets from this expression

$$\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1} \overset{(0)}{U}_{\beta_1} = 0 \,, \tag{3.3.5}$$

where

$$\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1} := \{\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta_1}, \overset{(0)}{U}_{\alpha_3]}\} + 2\,\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\alpha}\,\overset{(1)}{U}_{\alpha_3]\alpha}{}^{\beta_1} \,. \tag{3.3.6}$$

Note that the Eq. (3.3.5) has the structure of the homogeneous equation $\delta\overset{(1)}{D} = 0$, then by the Theorem 3.2.1 there exists an antisymmetric tensor $K := 2\overset{(2)}{U}$ such that $\overset{(1)}{D} = \delta K = 2\delta\overset{(2)}{U}$. In local coordinates this is

$$\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1} = 2\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2]}\,\overset{(0)}{U}_{\beta_2} \,. \tag{3.3.7}$$

This expression defines the so–called second–order structure functions $\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2]}$. Note, however, that by the Theorem 3.2.3, these functions are only defined up to a tensor of the form $\overset{(2)}{M}_{[\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2\beta_3]}\,\overset{(0)}{U}_{\beta_3}$. Indeed, $\overset{(2)}{U}$ would denote a particular solution to $2\,\delta\overset{(2)}{U} = \overset{(1)}{D}$ and $\overset{(2)}{U} + \delta\overset{(2)}{M}$ the most general solution to the Eq. (3.3.7). From now on, this ambiguity –and all others of this kind that appear later– will be removed by choosing a vanishing exact term[3] $\delta\overset{(2)}{M} \equiv 0$. In general, the second–order structure functions are functions only on the original phase space $\mathbf{T}^*\mathbb{Q}$ since the sector that includes the Lagrange multipliers belongs to the abelian sector within the gauge algebra (3.1.11). The first–order structure functions $\overset{(1)}{U}_{[\alpha_1\alpha_2]}{}^{\beta_1}$ identically vanish when the indices take the values corresponding to the abelian constraints $\pi_a = 0$.

The existence of the second–order structure functions $\overset{(2)}{U}$ is a consequence of the Jacobi identity and the use of the Theorem 3.2.1. This suggests an iterative process. In the same fashion as the equation that defines $\overset{(1)}{U}$ was treated, namely (3.3.4), the antisymmetrised PB between the equation that defines $\overset{(2)}{U}$ and the constraints $\overset{(0)}{U}$ is considered

$$\{\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3}{}^{\beta_1}, -\overset{(0)}{U}_{\alpha_4]}\} = 2\{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2]}\,\overset{(0)}{U}_{\beta_2}, -\overset{(0)}{U}_{\alpha_4]}\} \,. \tag{3.3.8}$$

[3]In the case where fermionic and bosonic variables coexist in the original phase space $\mathbf{T}^*\mathbb{Q}$, this ambiguity helps to adjust the second–order structure functions to have ghost number zero (see below this concept).

The minus sign in front of $\overset{(0)}{U}_{\alpha_4}$ has been inserted for convenience. The RHS of this equation, where the antisymmetrisation in the lower indices concerns only the α_i, clearly corresponds to a linear combination of the constraints, explicitly

$$2\{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2]}\overset{(0)}{U}_{\beta_2}, -\overset{(0)}{U}_{\alpha_4]}\} = -2\left(\{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2]}, \overset{(0)}{U}_{\alpha_4]}\} + 2\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta]}\overset{(1)}{U}_{\alpha_4]\beta}{}^{\beta_2}\right)\overset{(0)}{U}_{\beta_2}. \tag{3.3.9}$$

Note the lack of antisymmetrisation in the indices (β_1, β_2) in the second term of the RHS in this expression, this will be recovered when the LHS of (3.3.8) is considered. The LHS of the Eq. (3.3.8) has the following structure in the lower indices:

$$\begin{aligned} X_{[\alpha_1\alpha_2\alpha_3\alpha_4]} &= X_{[[\alpha_1\alpha_2\alpha_3]\alpha_4]} \\ &= \frac{1}{4}\left(X_{[\alpha_1\alpha_2\alpha_3]\alpha_4} + X_{[\alpha_1\alpha_3\alpha_4]\alpha_2} + X_{[\alpha_1\alpha_4\alpha_2]\alpha_3} + X_{[\alpha_2\alpha_4\alpha_3]\alpha_1}\right) \end{aligned} \tag{3.3.10}$$

with

$$X_{[\alpha_1\alpha_2\alpha_3]\alpha_4} := \{\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1}, -\overset{(0)}{U}_{\alpha_4}\}\ . \tag{3.3.11}$$

Substitution of the explicit expression for $\overset{(1)}{D}$, *cf.* Eq. (3.3.6), into the Eq. (3.3.11) gives an explicit formula for the RHS of the Eq. (3.3.10) in terms of (symbolically) $\{\overset{(1)}{U}, \overset{(0)}{U}\}\overset{(1)}{U}$ and the nested PBs $\{\{\overset{(1)}{U}, \overset{(0)}{U}\}, \overset{(0)}{U}\}$. The introduction of the identity

$$\begin{aligned} &\{\{\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta}, \overset{(0)}{U}_{\alpha_3}\}, -\overset{(0)}{U}_{\alpha_4}\} + \{\{\overset{(1)}{U}_{\alpha_2\alpha_1}{}^{\beta}, \overset{(0)}{U}_{\alpha_4}\}, -\overset{(0)}{U}_{\alpha_3}\} = \\ &\qquad 2\left(\{\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta}, \overset{(1)}{U}_{\alpha_3\alpha_4}{}^{\gamma}\}\overset{(0)}{U}_{\gamma} + \overset{(1)}{U}_{\alpha_3\alpha_4}{}^{\gamma}\{\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta}, \overset{(0)}{U}_{\gamma}\}\right)\ , \end{aligned} \tag{3.3.12}$$

makes the LHS of the Eq. (3.3.8) acquire the unexpected form of a linear combination of first–class constraints, namely

$$\begin{aligned} \{\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3}{}^{\beta_1}, -\overset{(0)}{U}_{\alpha_4]}\} &= \{\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta_1}, \overset{(1)}{U}_{\alpha_2\alpha_4]}{}^{\beta_2}\}\overset{(0)}{U}_{\beta_2} + 6\,\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(2)}{U}_{\alpha_3\alpha_4]\beta}{}^{[\beta_1\beta_2]}\overset{(0)}{U}_{\beta_2} \\ &\quad - 4\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_2\beta]}\overset{(1)}{U}_{\alpha_4]\beta}{}^{\beta_1}\overset{(0)}{U}_{\beta_2}\ . \end{aligned} \tag{3.3.13}$$

The expression (3.3.12) is implied by the Jacobi identity and the gauge algebra (3.3.3). The RHS of the Eq. (3.3.13) contains, besides the antisymmetrisation in the α_i, antisymmetrisations in the indices $(\alpha_3, \alpha_4, \beta)$ in the second term and in the lower indices (α_4, β) of the third term.

The substitution of Eq. (3.3.9) and Eq. (3.3.13) into the RHS and LHS of the Eq. (3.3.8), respectively, gives

$$\begin{aligned} 0 = &\{\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta_1}, \overset{(1)}{U}_{\alpha_2\alpha_4]}{}^{\beta_2}\}\overset{(0)}{U}_{\beta_2} + 6\,\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(2)}{U}_{\alpha_3\alpha_4]\beta}{}^{[\beta_1\beta_2]}\overset{(0)}{U}_{\beta_2} - 4\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_2\beta]}\overset{(1)}{U}_{\alpha_4]\beta}{}^{\beta_1}\overset{(0)}{U}_{\beta_2} + \\ &+ 2\,\{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2]}, \overset{(0)}{U}_{\alpha_4]}\}\overset{(0)}{U}_{\beta_2} + 4\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta]}\overset{(1)}{U}_{\alpha_4]\beta}{}^{\beta_2}\overset{(0)}{U}_{\beta_2}\ . \end{aligned} \tag{3.3.14}$$

A final manipulation of indices to get all (β_1, β_2) antisymmetrised, together with the factorisation of $\overset{(0)}{U}_{\beta_2}$, render the remarkable expression

$$\overset{(2)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{[\beta_1\beta_2]}\,\overset{(0)}{U}_{\beta_2} = 0\ , \tag{3.3.15}$$

with

$$\begin{aligned}\overset{(2)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{[\beta_1\beta_2]} :=\ & -\{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2]}, \overset{(0)}{U}_{\alpha_4]}\} - \frac{1}{2}\{\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{[\beta_1}, \overset{(1)}{U}_{\alpha_3\alpha_4]}{}^{\beta_2]}\} - \\ & - 3\,\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(2)}{U}_{\alpha_3\alpha_4]\beta}{}^{[\beta_1\beta_2]} + 4\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{\beta[\beta_1}\,\overset{(1)}{U}_{\alpha_4]\beta}{}^{\beta_2]}\ .\end{aligned} \tag{3.3.16}$$

In this expression an extra antisymmetrisation takes place in the indices $(\alpha_3, \alpha_4, \beta)$ within the third term of the RHS.

The rich structure of open gauge algebras starts to be revealed with the non–trivial identity (3.3.15) which has the form $\delta\overset{(2)}{D} = 0$, with $\overset{(2)}{D}$ a sixth rank antisymmetric tensor. Similarly to (3.3.5) the expression (3.3.15) implies, by the Theorem 3.2.1, the existence of a seventh rank antisymmetric tensor $K := 3\overset{(3)}{U}$ such that $\overset{(2)}{D} = \delta K = 3\delta\overset{(3)}{U}$. In local coordinates

$$\overset{(2)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{[\beta_1\beta_2]} = 3\,\overset{(3)}{U}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{[\beta_1\beta_2\beta_3]}\,\overset{(0)}{U}_{\beta_3}\ . \tag{3.3.17}$$

This equation defines the so–called third–order structure functions $\overset{(3)}{U}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{[\beta_1\beta_2\beta_3]}$, again, functions on the original phase space $\mathbf{T}^*\mathbb{Q}$.

What is remarkable from this point is that one can repeat the above construction, step by step, generating in this way higher structure functions. Let us sketch the following step and, at the same time, give the corresponding explicit equations. Each step starts commuting –in the PB sense– the equation that defined the latest structure function, in this case (3.3.17), with the constraints and antisymmetrising it:

$$\{\overset{(2)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4}{}^{[\beta_1\beta_2]}, \overset{(0)}{U}_{\alpha_5]}\} = 3\,\{\overset{(3)}{U}_{[\alpha_1\alpha_2\alpha_3\alpha_4}{}^{[\beta_1\beta_2\beta_3]}\,\overset{(0)}{U}_{\beta_3}, \overset{(0)}{U}_{\alpha_5]}\}\ . \tag{3.3.18}$$

The RHS is obviously a linear combination of the first–class constraints. A more difficult task is to show that this is also the case for the LHS of the equation; with the use of the Jacobi identity, the gauge algebra (3.3.3) and some manipulation of the indices, one can show that (3.3.18) is equivalent to

$$\overset{(3)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5]}{}^{[\beta_1\beta_2\beta_3]}\,\overset{(0)}{U}_{\beta_3} = 0\ , \tag{3.3.19}$$

with

$$\begin{aligned}\overset{(3)}{D}_{[\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5]}{}^{[\beta_1\beta_2\beta_3]} :=\ & \{\overset{(3)}{U}_{[\alpha_1\alpha_2\alpha_3\alpha_4}{}^{[\beta_1\beta_2\beta_3]}, \overset{(0)}{U}_{\alpha_5]}\} - \{\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{[\beta_1\beta_2}, \overset{(1)}{U}_{\alpha_4\alpha_5]}{}^{\beta_3]}\} \\ & + 4\,\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(3)}{U}_{\alpha_3\alpha_4\alpha_5]\beta}{}^{[\beta_1\beta_2\beta_3]} + 6\,\overset{(2)}{U}_{[\alpha_1\alpha_2\alpha_3}{}^{\beta[\beta_1}\overset{(2)}{U}_{\alpha_4\alpha_5]\beta}{}^{\beta_2\beta_3]} + \\ & + 6\,\overset{(3)}{U}_{[\alpha_1\alpha_2\alpha_3\alpha_4}{}^{\beta[\beta_1\beta_2}\,\overset{(1)}{U}_{\alpha_5]\beta}{}^{\beta_3]}\ .\end{aligned} \tag{3.3.20}$$

In this expression an additional antisymmetrisation is implied in the lower indices $(\alpha_3, \alpha_4, \alpha_5, \beta)$, $(\alpha_4,\ \alpha_5, \beta)$ and (α_5, β) within the third, fourth and fifth terms, respectively. Equation (3.3.19) can be treated in a similar fashion to equations (3.3.5) and (3.3.15). The existence of the fourth–order structure functions $\overset{(4)}{U}$ is inferred from the Eq. (3.3.19); the structure functions $\overset{(4)}{U}$ are defined up to a tensor of the form $\delta \overset{(4)}{M}$ which is set to vanish. Locally the Eq. (3.3.19) implies

$$\overset{(3)}{D}{}_{[\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5]}{}^{[\beta_1\beta_2\beta_3]} = 4\,\overset{(4)}{U}{}_{[\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5]}{}^{[\beta_1\beta_2\beta_3\beta_4]}\,\overset{(0)}{U}_{\beta_4}\ . \tag{3.3.21}$$

For a system with a finite number of constraints, say $2m$, necessarily there is a point from which all the structure functions identically vanish. Indeed, since each $\overset{(k)}{U}$ is antisymmetric in all its $(k+1)$ lower indices, and they in turn also labelled the constraints themselves, one has that $\overset{(k)}{U} = \overset{(k+1)}{U} = \ldots = 0$ for $k \geq 2m$, just like $(p+1)$–forms on a differential manifold of dimension $2m$ identically vanish when $p \geq 2m$. For $k \in \{1, \ldots, 2m-2\}$ one finds by induction [62] the equation

$$\overset{(k)}{D}{}_{[\alpha_1\ldots\alpha_{k+2}]}{}^{[\beta_1\ldots\beta_k]}\,\overset{(0)}{U}_{\beta_k} = 0\ . \tag{3.3.22}$$

This relation ensures the existence of the next–order structure functions which should be obtained from the equation $\overset{(k)}{D} = (k+1)\,\delta\overset{(k+1)}{U}$, or equivalently

$$\overset{(k)}{D}{}_{[\alpha_1\ldots\alpha_{k+2}]}{}^{[\beta_1\ldots\beta_k]} = (k+1)\,\overset{(k+1)}{U}{}_{[\alpha_1\ldots\alpha_{k+2}]}{}^{[\beta_1\ldots\beta_k\beta_{k+1}]}\,\overset{(0)}{U}_{\beta_{k+1}}\ , \tag{3.3.23}$$

with

$$\begin{aligned}\overset{(k)}{D}{}_{[\alpha_1\ldots\alpha_{k+2}]}{}^{[\beta_1\ldots\beta_k]} = {} & \frac{1}{2}\sum_{p=0}^{k}(-)^{(kp+1)}\{\overset{(p)}{U}{}_{[\alpha_1\ldots\alpha_{p+1}}{}^{[\beta_1\ldots\beta_p}, \overset{(k-p)}{U}{}_{\alpha_{p+2}\ldots\alpha_{k+2}]}{}^{\beta_{p+1}\ldots\beta_k]}\} \\ & - \sum_{p=0}^{k-1}(-)^{k(p+1)}(p+1)(k-p+1)\,\overset{(p+1)}{U}{}_{[\alpha_1\ldots\alpha_{p+2}}{}^{[\beta_1\ldots\beta_p\beta}\,\overset{(k-p)}{U}{}_{\alpha_{p+3}\ldots\alpha_{k+2}]\beta}{}^{\beta_{p+1}\ldots\beta_k]}\ .\end{aligned} \tag{3.3.24}$$

In the RHS of this expression additional antisymmetrisations in the indices $(\alpha_{p+3}, \ldots, \alpha_{k+2}, \beta)$ within the terms of the second summation are implied. In addition, within these terms the antisymmetrisation in the upper indices concerns only the β_i, that is, the upper index β plays no role in the antisymmetrisation. Note that $\overset{(k)}{D}$ can always be calculated from structure functions of order lower than $k+1$, which are supposed to be obtained in a previous step. In this way the ladder of structure functions can be iteratively constructed from three purely classical equations: the Jacobi identity, the gauge algebra and the remarkable identity (3.3.22).

3.4 The super phase space and the BRST generator

The very rich structure, shared by all gauge theories, presented in the previous section completely emerges from two fundamental equations: the gauge algebra and the Jacobi identity. Moreover, in contrast to this simplicity, it turns out that the emergent structure functions are at the heart of the BRST symmetry. The identities already found are the essence in the proof of the existence of a BRST generator. The BRST generator corresponds to a real–valued function that is nilpotent (with respect to some symplectic structure), of odd Grassmann parity and +1 ghost numbered. A BRST charge generates a transformation which resembles the gauge one with the bosonic gauge parameters replaced by fermionic variables. These fermions were systematically introduced by L. D. Faddeev and N. V. Popov within their review of path integral formalism for gauge theories [26, 28]; since then, these variables are called Faddeev–Popov ghosts, or simply *ghosts*. In the present context, they will help to condense the identities found in the previous section to build the BRST symmetry generator.

Within Sect. 3.2 it was mentioned that the original phase space $\mathbf{T}^*\mathbb{Q}$ was going to suffer a couple of enlargements. The first one was the passage from $\mathbf{T}^*\mathbb{Q}$ to $\mathbf{T}^*_\lambda\mathbb{Q}$ by adding the m Lagrange multipliers (corresponding to the m first–class constraints) and their conjugate momenta. The second extension to $\mathbf{T}^*\mathbb{Q}$ is a bit more radical, it corresponds to an enlargement with the fermionic variables η^α ($\alpha = 1, \ldots, 2m$) and their associate conjugate momenta $\mathcal{P}_\alpha$. These fermionic canonical pairs supplement the symplectic structure of $\mathbf{T}^*_\lambda\mathbb{Q}$ with the *symmetric* PBs

$$\{\eta^\alpha, \eta^\beta\} = 0 \ , \quad \{\eta^\alpha, \mathcal{P}_\beta\} = \{\mathcal{P}_\beta, \eta^\alpha\} = -\delta^\alpha_\beta \ , \quad \{\mathcal{P}_\alpha, \mathcal{P}_\beta\} = 0 \ . \tag{3.4.1}$$

Notice the minus sign in front of the Kronecker delta. The fermionic nature of these degrees of freedom is expressed by saying that their Grassmann number is 1, in symbols

$$\epsilon(\eta^\alpha) := 1 \ , \quad \epsilon(\mathcal{P}_\alpha) := 1 \ , \tag{3.4.2}$$

or equivalently, that they are anticommuting $c-$numbers

$$[\eta^\alpha, \eta^\beta]_+ = [\eta^\alpha, \mathcal{P}_\beta]_+ = [\mathcal{P}_\alpha, \mathcal{P}_\beta]_+ = 0 \ , \tag{3.4.3}$$

with $[A, B]_+ := AB + BA$; while for the rest of canonical variables one has

$$\epsilon(q^i) = \epsilon(p_i) := 0 \ , \quad \epsilon(\lambda^a) = \epsilon(\pi_a) := 0 \ , \tag{3.4.4}$$

that is, they are commuting $c-$numbers.

A consistent symplectic structure is given to the whole ghost–extended phase space or *super phase space* –as it is sometimes called– by means of the generalised PB. This bracket is reduced to the fermionic sector (3.4.1) when only ghosts and their conjugate momenta are considered, and to the bosonic sector (2.1.16) and (3.1.8) when the rest of the canonical pairs are taken into account. For two functions f and g on the super phase space, with ϵ_f the parity of f, the generalised PB reads as

$$\{f,g\} := \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} + \frac{\partial f}{\partial \lambda^a}\frac{\partial g}{\partial \pi_a} - \frac{\partial f}{\partial \pi_a}\frac{\partial g}{\partial \lambda^a}\right) + (-)^{\epsilon_f}\left(\frac{\partial^\ell f}{\partial \eta^\alpha}\frac{\partial^\ell g}{\partial \mathcal{P}_\alpha} + \frac{\partial^\ell f}{\partial \mathcal{P}_\alpha}\frac{\partial^\ell g}{\partial \eta^\alpha}\right) . \tag{3.4.5}$$

We have decided to use the same symbol for the extended PB as the one used for the PB in $\mathbf{T}^*_\lambda\mathbb{Q}$ since we think no confusion arises. The terms in the first parenthesis on the RHS of (3.4.5) correspond to the bosonic sector of the generalised PB, and the terms in the second parenthesis to the fermionic sector. The symbols $\partial^\ell/\partial\eta^\alpha$ and $\partial^\ell/\partial\mathcal{P}_\alpha$ denote left partial derivatives with respect to η^α and $\mathcal{P}_\alpha$, respectively. On a function f of the super phase space, the left derivative acts as 'coming from the left' and each time it 'jumps' a factor, a minus sign might appear depending on the parity of that factor. If f and g are any two functions on the super phase space and at least f has a well defined parity, one has[4]

$$\frac{\partial^\ell (fg)}{\partial \eta^\alpha} = \frac{\partial^\ell f}{\partial \eta^\alpha} g + (-)^{\epsilon_f} f \frac{\partial^\ell g}{\partial \eta^\alpha} . \tag{3.4.6}$$

In general, the parity of a function f is not necessarily well defined but f can always be decomposed into the sum of a commuting (or even) part and an anticommuting (or odd) part: $f = f_E + f_O$. Indeed, this is evident when f is expanded in powers of fermionic variables, the even (odd) component only contains the even (odd) powers of the fermionic variables in the expansion.

The generalised PB (3.4.5) obeys generalised versions of the properties listed in Sect. 2.1.3 for the ordinary PB. For f, g and h functions on the super phase space with well defined Grassmann numbers ϵ_f, ϵ_g and ϵ_h, respectively one has

(*i*) Linearity: If a and b are constants $\{af + bg, h\} = a\{f,h\} + b\{g,h\}$,

(*ii*) Antisymmetry: $\{f,g\} = -(-)^{\epsilon_f\epsilon_g}\{g,f\}$,

(*iii*) Leibniz rule: $\{f,gh\} = \{f,g\}h + (-)^{\epsilon_f\epsilon_g} g\{f,h\}$,

[4] In a similar fashion one can define right partial derivatives $\frac{\partial^r}{\partial\eta^\alpha}$ and $\frac{\partial^r}{\partial\mathcal{P}_\alpha}$ which act as 'coming from the right'. The relation between the left and right derivatives is

$$\frac{\partial^r f}{\partial \eta^\alpha} = (-)^{(\epsilon_f+1)}\frac{\partial^\ell f}{\partial \eta^\alpha} .$$

In the case where the derivative is taken with respect to any bosonic degree of freedom there is no difference between these two kind of derivatives.

(*iv*) Jacobi identity: $\{\{f,g\},h\} + (-)^{\epsilon_h(\epsilon_f+\epsilon_g)}\{\{h,f\},g\} + (-)^{\epsilon_f(\epsilon_g+\epsilon_h)}\{\{g,h\},f\} = 0.$

In addition, the concept of parity in the functions introduces two more properties:

(*v*) Parity of the super–PB: $\epsilon(\{f,g\}) = \epsilon_f + \epsilon_g$,

(*vi*) Complex conjugation of super–PB: $\big(\{f,g\}\big)^* = -\{g^*,f^*\}$.

The last property comes from the complex conjugation conventions

$$(q^i)^* = q^i \ , \quad p_i^* = p_i \ , \quad (\lambda^a)^* = \lambda^a \ , \quad \pi_a^* = \pi_a \ , \quad (\eta^\alpha)^* = \eta^\alpha \ , \quad \mathcal{P}_\alpha^* = -\mathcal{P}_\alpha \ , \tag{3.4.7}$$

which, in particular, are consistent with the pure fermionic PBs (3.4.1). Moreover, from this convention one has for example that monomials $\eta^\alpha\eta^\beta$ and $\mathcal{P}_\alpha\mathcal{P}_\beta$ are both imaginary; in general

$$(\eta^{\alpha_1}\dots\eta^{\alpha_n})^* = (-)^{n(n-1)/2}(\eta^{\alpha_1}\dots\eta^{\alpha_n}) \ , \tag{3.4.8a}$$
$$(\mathcal{P}_{\alpha_1}\dots\mathcal{P}_{\alpha_n})^* = (-)^{n(n+1)/2}(\mathcal{P}_{\alpha_1}\dots\mathcal{P}_{\alpha_n}) \ . \tag{3.4.8b}$$

So a monomial that mixes ghosts and ghost–momenta in which the number of ghost–momenta is equal to the number of either ghost or ghosts plus one, is real.

In addition to this extended symplectic structure and complex conjugation (involution) operation, the super phase space is also equipped with the notion of *ghost numbers*. In the BRST formalism the ghost number on each ghost and ghost–momentum is defined as follows:

$$\text{gh}(\eta^\alpha) := 1 \quad \text{and} \quad \text{gh}(\mathcal{P}_\alpha) := -1 \ , \tag{3.4.9a}$$

whereas zero ghost number is attached to the basic even conjugate pairs

$$\text{gh}(q^i) = \text{gh}(p_i) := 0 \quad \text{and} \quad \text{gh}(\lambda^a) = \text{gh}(\pi_a) := 0 \ . \tag{3.4.9b}$$

This numbering is generalised to any monomial in the super phase space by the rule $\text{gh}(AB) = \text{gh}(A) + \text{gh}(B)$ and only the sum of monomials with the same ghost number does specify polynomials of definite ghost number. Therefore, a monomial that mixes ghosts and ghost–momenta which contains one more ghost than the number of ghost–momenta, has ghost number $+1$.

In order to lift the higher structure functions to the super phase space, the following definitions are useful:

$$\overset{(k)}{U}{}^{[\beta_1\dots\beta_k]} := \eta^{\alpha_{k+1}}\dots\eta^{\alpha_1}\,\overset{(k)}{U}{}_{\alpha_1\dots\alpha_{k+1}}{}^{[\beta_1\dots\beta_k]} \ , \tag{3.4.10a}$$
$$\overset{(k)}{D}{}^{[\beta_1\dots\beta_k]} := \eta^{\alpha_{k+2}}\dots\eta^{\alpha_1}\,\overset{(k)}{U}{}_{\alpha_1\dots\alpha_{k+2}}{}^{[\beta_1\dots\beta_k]} \ . \tag{3.4.10b}$$

In these equations, the antisymmetry in the lower indices of $\overset{(k)}{U}$ and $\overset{(k)}{D}$ is automatically taken into account due to the anticommuting nature of the ghosts. Moreover, these definitions make the general identity (3.3.23) look like

$$\overset{(k)}{D}{}^{[\beta_1\dots\beta_k]} = (k+1)\overset{(k+1)}{U}{}^{[\beta_1\dots\beta_k\beta_{k+1}]}\overset{(0)}{U}_{\beta_{k+1}} \, . \tag{3.4.11}$$

These conventions make the explicit expression for $\overset{(k)}{D}$, *cf.* Eq. (3.3.24), look like

$$\begin{aligned}\overset{(k)}{D}{}^{[\beta_1\dots\beta_k]} =& \frac{1}{2}\sum_{p=0}^{k}(-)^{(k-p)}\{\overset{(p)}{U}{}^{[\beta_1\dots\beta_p}, \overset{(k-p)}{U}{}^{\beta_{p+1}\dots\beta_k]}\} - \\ & - \sum_{p=0}^{k-1}(-)^{(k-p)}(p+1)\overset{(p+1)}{U}{}^{[\beta_1\dots\beta_p\beta}\frac{\partial^\ell \overset{(k-p)}{U}{}^{\beta_{p+1}\dots\beta_k]}}{\partial\eta^\beta} \, ,\end{aligned} \tag{3.4.12}$$

where the antisymmetrisation in the terms of the second summation in the RHS excludes β, and the PBs in the first summation refer to the generalised ones.

With all these preparations, one can now pass to the main theorem of this chapter. The following theorem describes the way in which the gauge invariance is merged in the super phase space [63].

Theorem 3.4.1 [Existence of the BRST generator] *To any first–class constrained system, one can associate an odd Grassmann parity BRST generator function* Ω *characterised by*

$$\mathrm{gh}(\Omega) = 1 \, , \tag{3.4.13a}$$

$$\Omega = \eta^\alpha \overset{(0)}{U}_\alpha + \text{ terms that vanish with the ghost–momenta,} \tag{3.4.13b}$$

$$\{\Omega, \Omega\} = 0 \, , \tag{3.4.13c}$$

$$\Omega^* = \Omega \, . \tag{3.4.13d}$$

The solution to the conditions (3.4.13) corresponds to the finite sum:

$$\begin{aligned}\Omega :=& \sum_{k=0}\overset{(k)}{U}{}^{\beta_1\dots\beta_k}\mathcal{P}_{\alpha_1}\dots\mathcal{P}_{\alpha_k} \\ =& \, \eta^\alpha\overset{(0)}{U}_\alpha + \eta^{\alpha_2}\eta^{\alpha_1}\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta_1}\mathcal{P}_{\beta_1} + \eta^{\alpha_3}\eta^{\alpha_2}\eta^{\alpha_1}\overset{(2)}{U}_{\alpha_1\alpha_2\alpha_3}{}^{\beta_1\beta_2}\mathcal{P}_{\beta_2}\mathcal{P}_{\alpha_1} + \dots \, ,\end{aligned} \tag{3.4.14}$$

where for consistency one sets $\overset{(0)}{U} := \eta^\alpha\overset{(0)}{U}_\alpha$ as the first term in the summation.

The odd Grassmann parity condition on the function Ω is automatically fulfilled by (3.4.14) since each of the monomials in the sum is anticommuting. Moreover, since each of the monomials in (3.4.14) contains one extra ghost compared with the number of ghosts–momenta, one concludes that Ω itself is ghost numbered $+1$. By construction, all the terms in Ω vanish when the ghost–momenta are set to zero, except for the first one, which has the form required by (3.4.13b). The nilpotency of Ω with respect to the super–PB (3.4.13c) is a non–trivial condition due to the odd Grassmann nature of Ω. By direct calculation one has that

$$\begin{aligned}\{\Omega,\Omega\} = \ & \{\overset{(0)}{U},\overset{(0)}{U}\} + 2\{\overset{(0)}{U},\overset{(1)}{U}{}^{\beta_1}\mathcal{P}_{\beta_1}\} + 2\{\overset{(0)}{U},\overset{(2)}{U}{}^{\beta_1\beta_2}\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2}\} + 2\{\overset{(0)}{U},\overset{(3)}{U}{}^{\beta_1\beta_2\beta_3}\mathcal{P}_{\beta_3}\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_1}\} + \\ & \ldots + \{\overset{(1)}{U}{}^{\beta_1}\mathcal{P}_{\beta_1},\overset{(1)}{U}{}^{\gamma_1}\mathcal{P}_{\gamma_1}\} + 2\{\overset{(1)}{U}{}^{\beta_1}\mathcal{P}_{\beta_1},\overset{(2)}{U}{}^{\gamma_1\gamma_2}\mathcal{P}_{\gamma_1}\mathcal{P}_{\gamma_2}\} + \ldots \ . \qquad (3.4.15)\end{aligned}$$

The RHS of this expression is a polynomial in powers of ghost–momenta which vanishes as it will be proved.

After some algebra, at order zero one obtains the coefficients

$$\{\overset{(0)}{U},\overset{(0)}{U}\} - 2\overset{(1)}{U}{}^{\beta}\frac{\partial^\ell \overset{(0)}{U}}{\partial\eta^\beta} = \eta^{\alpha_2}\eta^{\alpha_1}\left(-2\,\overset{(1)}{U}{}_{\alpha_1\alpha_2}{}^{\beta_1}\overset{(0)}{U}{}_{\beta_1} - 2\,\overset{(1)}{U}{}_{\alpha_2\alpha_1}{}^{\beta_1}\,\overset{(0)}{U}{}_{\beta_1}\right) \equiv 0 \ . \qquad (3.4.16)$$

The coefficients at order zero in ghost–momenta vanish.

At first order in ghost–momenta one finds the corresponding coefficients to be

$$\begin{aligned}&2\{\overset{(0)}{U},\overset{(1)}{U}{}^{\beta_1}\} - 2\overset{(1)}{U}{}^{\beta}\frac{\partial^\ell \overset{(1)}{U}{}^{\beta_1}}{\partial\eta^\beta} + 4\overset{(2)}{U}{}^{\beta_1\beta}\frac{\partial^\ell \overset{(0)}{U}}{\partial\eta^\beta} = \\ &\qquad 2\eta^{\alpha_3}\eta^{\alpha_2}\eta^{\alpha_1}\left(\{\overset{(1)}{U}{}_{[\alpha_1\alpha_2}{}^{\beta_1},\overset{(0)}{U}{}_{\alpha_3]}\} + 2\overset{(1)}{U}{}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(1)}{U}{}_{\alpha_3]\beta}{}^{\beta_1} - 2\overset{(2)}{U}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1\beta}\,\overset{(0)}{U}{}_{\beta}\right) \ , \\ &\qquad\qquad (3.4.17)\end{aligned}$$

where the term in parenthesis on the RHS of this equation is nothing but $\overset{(1)}{D} - 2\,\delta\overset{(2)}{U}$, *cf.* Eq. (3.3.6), which by virtue of the Eq. (3.3.7) vanishes. Therefore the coefficients at zeroth and first power of ghost–momenta vanish on the RHS of (3.4.15).

Going to next order the corresponding coefficients are

$$\begin{aligned}&2\{\overset{(0)}{U},\overset{(2)}{U}{}^{\beta_1\beta_2}\} - \{\overset{(1)}{U}{}^{\beta_1},\overset{(1)}{U}{}^{\beta_2}\} - 2\overset{(1)}{U}{}^{\beta}\frac{\partial^\ell \overset{(2)}{U}{}^{\beta_1\beta_2}}{\partial\eta^\beta} + 4\overset{(2)}{U}{}^{\beta_1\beta}\frac{\partial^\ell \overset{(1)}{U}{}^{\beta_2}}{\partial\eta^\beta} - 6\overset{(3)}{U}{}^{\beta_1\beta_2\beta}\frac{\partial^\ell \overset{(0)}{U}}{\partial\eta^\beta} = \\ &2\eta^{\alpha_4}\eta^{\alpha_3}\eta^{\alpha_2}\eta^{\alpha_1}\left(\{\overset{(0)}{U}{}_{[\alpha_4},\overset{(2)}{U}{}_{\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2]}\} - \frac{1}{2}\{\overset{(1)}{U}{}_{[\alpha_1\alpha_2}{}^{[\beta_1},\overset{(1)}{U}{}_{\alpha_2\alpha_4]}{}^{\beta_2]}\} - \right. \\ &\left. -3\,\overset{(1)}{U}{}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(2)}{U}{}_{\alpha_3\alpha_4]\beta}{}^{[\beta_1\beta_2]} + 4\,\overset{(2)}{U}{}_{[\alpha_1\alpha_2\alpha_3}{}^{\beta[\beta_1}\,\overset{(1)}{U}{}_{\alpha_4]\beta}{}^{\beta_2]} - 3\overset{(3)}{U}{}_{[\alpha_1\alpha_2\alpha_3\alpha_4]}{}^{\beta_1\beta_2\beta}\,\overset{(0)}{U}{}_{\beta}\right) \ , \quad (3.4.18)\end{aligned}$$

where the term in parenthesis on the RHS is nothing but $\overset{(2)}{D} - 3\,\delta\overset{(3)}{U} = 0$, *cf.* Eqs. (3.3.17) and (3.3.16).

By induction one can prove that the coefficient of the kth power in ghost–momenta on the RHS of (3.4.15) corresponds to a multiple of $\overset{(k)}{D} - (k+1)\delta\overset{(k+1)}{U}$ which by virtue of equation (3.3.23) vanishes identically. Therefore $\{\Omega,\Omega\} = 0$.

Finally, Ω given in (3.4.14) is a real function on the super phase space. Indeed, by construction the higher structure functions $\overset{(n)}{U}{}_{\cdots}^{\cdots}$ are all real and it has been mentioned that the precise combination of ghost and ghost–momenta that is contracted with these functions is real (see paragraph after Eq. (3.4.8)), hence $\Omega^* = \Omega$. In conclusion Ω in (3.4.14) is the BRST generator cited in the Theorem 3.4.1.

Remarks

1. The BRST generator can also be viewed as the 'generating function' of the equations that define the high order structure functions of the theory. These functions appear as coefficients in the expansion (3.4.14). Postulating nilpotency with respect to the generalised–PB, the equations (3.3.23) are found at each order in the ghost–momenta.

2. The BRST generator only depends on the gauge algebra and not on the dynamics of the theory, *i.e.* Ω is independent of the form of the Hamiltonian. ▲

3.4.1 Ghost/Antighost notation and conventions

Before going into specific examples of BRST charges, in this section some notation included in the standard literature is introduced. The super phase space contains the original variables (q^i, p_i), the Lagrange multipliers and their conjugate momenta (λ^a, π_a), as well as ghost conjugate pairs $(\eta^\alpha, \mathcal{P}_\alpha)$. The number of fermionic conjugate pairs doubles the number of the original constraints $\gamma_a \approx 0$ $(a = 1, \ldots, m)$. It can be inferred from the gauge algebra (3.1.11) on the super phase space that the ghosts associated to the *ad hoc* introduced constraints $\pi_a \approx 0$, chosen to be η^{m+a}, only appear linearly in the BRST charge (3.4.14); see also Eqs. (3.4.24) and Eq. (3.4.28) below.

Following [61, 65], the variables $i\eta^{m+a}$ and $-i\mathcal{P}_{m+a}$ will be denoted by ρ^a and $\bar{C}_a$, respectively. The variables $\bar{C}_a$ (resp. ρ^a) are known as *antighosts* (*antighost–momenta*). In this notation from the PBs (3.4.1) one has

$$\{\eta^{m+a}, \mathcal{P}_{m+b}\} = -\delta^a_b = \{\rho^a, \bar{C}_b\} \ ; \qquad i\eta^{m+a} \equiv \rho^a, \ -i\mathcal{P}_{m+a} \equiv \bar{C}_a \ . \tag{3.4.19a}$$

From (3.4.7) one can read the reality conditions

$$(\bar{C}_a)^* = \bar{C}_a \ , \quad (\rho^a)^* = -\rho^a \ . \tag{3.4.19b}$$

The corresponding ghost numbers are

$$\mathrm{gh}(\bar{C}_a) \equiv -1 \ , \quad \mathrm{gh}(\rho^a) \equiv 1 \ . \tag{3.4.19c}$$

Ghosts canonical pairs associated to the original m first–class constraints retain their name within these conventions; however, in order to avoid any confusion with the set of *all* ghosts and ghost–momenta a different notation for them will be introduced. The variables η^a and $\mathcal{P}_a$ will be denoted by C^a and $\bar{\rho}_a$, respectively. Their PBs are hence

$$\{\eta^a, \mathcal{P}_b\} = -\delta^a_b = \{C^a, \bar{\rho}_b\} \ ; \qquad \eta^a \equiv C^a, \ \mathcal{P}_a \equiv \bar{\rho}_a \ , \tag{3.4.20a}$$

having vanishing super PBs with any other basic variable. The reality conditions for C_a and $\bar{\rho}_a$ are

$$(C^a)^* = C^a \ , \quad (\bar{\rho}_a)^* = -\bar{\rho}_a \ , \tag{3.4.20b}$$

and the corresponding ghost numbers are

$$\mathrm{gh}(C^a) \equiv 1 \ , \quad \mathrm{gh}(\bar{\rho}_a) \equiv -1 \ . \tag{3.4.20c}$$

In a few words, all fermionic momenta $(\bar{\rho}, \rho)$ are imaginary and their conjugate pairs $(C, \bar{C})$ real. The ghost number of ρ and C (resp. $\bar{\rho}$ and $\bar{C}$) is 1 (-1).

In this notation, a general BRST charge (3.4.14) is given by

$$\Omega = \Omega^{\,\mathrm{min}} - i\rho^a \pi_a \ . \tag{3.4.21}$$

This is the explicit split of the nonminimal BRST charge Ω into the sector corresponding to the phase space $\mathbf{T}^*\mathbb{Q} \times \{(C^a, \bar{\rho}_a)\}$, $\Omega^{\,\mathrm{min}} = \Omega^{\,\mathrm{min}}(q, p, C, \bar{\rho})$, also called minimal sector, and the sector corresponding to the Lagrange multipliers $-i\rho^a\pi_a$, also called non–minimal sector.

3.4.2 Abelian gauge algebra and the abelianisation theorem

The simplest possible gauge algebra is the abelian one

$$\{G_\alpha, G_\beta\} = 0 \ . \tag{3.4.22}$$

This case will be encountered whenever the gauge symmetry (at the Hamiltonian level) consists of a single parameter.

The gauge algebra (3.4.22) is relevant because it *locally* covers the most general situation (3.1.11). In the gauge algebra (3.1.11) half of the constraints is already abelian, namely the m constraints $\pi_a \approx 0$; the rest of the constraints $\gamma_a \approx 0$ in general obeys the non–abelian algebra (3.1.5). According to the *abelianisation theorem* [65, 135, 173], given a set of first–class constraints $\{\gamma_a\}$ on $\mathbf{T}^*\mathbb{Q}$ it is always possible to locally find m new equivalent set of constraints $\{\Upsilon_a\}$ that defines the same constraint surface so that they are abelian, that is, $\{\Upsilon_a, \Upsilon_b\} = 0$ all over $\mathbf{T}^*\mathbb{Q}$.

A direct way to abelianise the set of constraints $\{\gamma_a\}$ is via the constraint resolution. Under the assumption of irreducibility and regularity, one can always (locally) resolve the m constraints $\gamma_a(q,p) \approx 0$ for m ps, that is,

$$p_a = g_a(\underline{p}, q) \ , \qquad a = 1, \ldots, m \ ,$$

where $\underline{p}$ denotes the remaining ps. By construction, the constraints

$$\Upsilon_a := p_a - g_a(\underline{p}, q) \tag{3.4.23}$$

define the same surface that $\gamma_a \approx 0$ define. Two aspects of the new constraints Υ_a are important. First, by direct calculation one can see that the PBs $\{\Upsilon_a, \Upsilon_b\}$ are independent of p_a. Second, on the constraints surface, that is, when $p_a = g_a(\underline{p}, q)$ is used, $\{\Upsilon_a, \Upsilon_b\} = 0$ (this is the first–class property). Since each function g_a is assumed to be well defined *all over* the values taken by $(\underline{p}, q)$, then $\{\Upsilon_a, \Upsilon_b\}$ must identically vanish on all $\mathbf{T}^*\mathbb{Q}$.

In order to construct a BRST generator associated to the gauge algebra (3.4.22), one recognises $\overset{(1)}{U}{}_{\alpha\beta}{}^{\gamma} \equiv 0$, then $\overset{(1)}{D}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta}$ automatically vanish (*cf.* Eq. (3.3.6)) and $\delta\overset{(1)}{D} = 0$ is trivial. No equations that could imply the existence of higher structure functions arise. Therefore, in the case of an abelian set of constraints (or locally in any gauge algebra) the BRST generator reads as

$$\Omega = \eta^{\alpha}\, G_{\alpha} = \eta^{a}\gamma_a + \eta^{m+a}\pi_a \ , \quad a = 1, \ldots, m \ , \tag{3.4.24}$$

or using the conventions (3.4.19) and (3.4.20)

$$\Omega = C^{a}\gamma_a - i\rho^{a}\pi_a \ , \quad a = 1, \ldots, m \ . \tag{3.4.25}$$

Observe that to obtain this result some ambiguity at the level of the gauge algebra was removed beforehand. In general, the PBs (3.3.3) are invariant under the redefinition of structure functions $\overset{(1)}{U}{}_{\alpha_1\alpha_2}{}^{\beta_1} \to \overset{(1)}{U}{}_{\alpha_1\alpha_2}{}^{\beta_1} + \overset{(1)}{M}{}_{\alpha_1\alpha_2}{}^{[\beta_1\beta_2]}\,\overset{(0)}{U}{}_{\beta_2}$. In the abelian case, the gauge algebra implies $\overset{(1)}{U}{}_{\alpha_1\alpha_2}{}^{\beta_1} = 0$, but it does not fix the arbitrariness induced by $\overset{(1)}{M}{}_{\alpha_1\alpha_2}{}^{[\beta_1\beta_2]}$. Such arbitrariness could have been fixed with nonzero functions on the super phase space. In such a situation higher structure functions can actually exist in the BRST generator of abelian constraints. However, this vagueness is removed by setting $\overset{(1)}{M} \equiv 0$ in all cases. This ambiguity is not exclusive of the first–order structure functions, similar arbitrariness arises at higher levels in the structure–functions ladder as it was pointed out just after writing equation (3.3.7).

Remarks

1. Although the abelianisation theorem is useful in proving local properties of constrained systems, it may not be practically easy to find an equivalent abelian

constraints Υ_a to the non–abelian first–class γ_a. The passage from γ_a to Υ_a might spoil manifest symmetries involved in the theory. ▲

3.4.3 Constraints that form a closed gauge algebra

Originally the BRST symmetry was discovered in first–class constrained systems whose gauge algebra forms a closed gauge algebra [35, 36, 37, 38]. When this is the case, one can set the first–order structure functions to be the *constants* $\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta_1}$ and $\overset{(1)}{M} \equiv 0$, hence the Jacobi identity of the PBs requires

$$0 = \overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1}\, \overset{(0)}{U}_{\beta_1} = 2\, \overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\alpha}\, \overset{(1)}{U}_{\alpha_3]\alpha}{}^{\beta_1}\, \overset{(0)}{U}_{\beta_1} \tag{3.4.26}$$

to be valid, *cf.* Eq. (3.3.6). In the rightmost expression one has, for given values of $(\alpha_1, \alpha_2, \alpha_3)$, a linear combination (with constant coefficients) of the constraints $Z^{\beta_1}\overset{(0)}{U}_{\beta_1}$ required to vanish. Therefore by irreducibility and the Theorem 3.2.1 one knows that $Z^{\beta_1} = Y^{[\beta_1\beta_2]}\,\overset{(0)}{U}_{\beta_2}$ over the whole super phase space and in particular on the constraint surface $\overset{(0)}{U}_{\beta_2} = 0$, hence the constants Z^{β_1} must vanish, implying for the structure constants $\overset{(1)}{U}_{[\alpha_1\alpha_2}{}^{\alpha}\, \overset{(1)}{U}_{\alpha_3]\alpha}{}^{\beta_1} = 0$ or

$$\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\alpha}\, \overset{(1)}{U}_{\alpha_3\alpha}{}^{\beta_1} + \overset{(1)}{U}_{\alpha_3\alpha_1}{}^{\alpha}\, \overset{(1)}{U}_{\alpha_2\alpha}{}^{\beta_1} + \overset{(1)}{U}_{\alpha_2\alpha_3}{}^{\alpha}\, \overset{(1)}{U}_{\alpha_1\alpha}{}^{\beta_1} = 0\ . \tag{3.4.27}$$

In other words, irreducibility implies in this case $\overset{(1)}{D}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1} \equiv 0$. Therefore $\delta\overset{(1)}{D} = 0$ becomes trivial again. No equation that could imply the existence of higher structure functions arises. Even if one insists on the equation $\overset{(1)}{D} = 2\delta\overset{(2)}{U}$, it reduces to $0 = \delta\overset{(2)}{U}$ from which $\overset{(2)}{U}$ is at most $\delta\overset{(2)}{M}$, but such solutions have already been discarded by setting $\overset{(2)}{M} = 0$. In conclusion the BRST charge for a gauge algebra with structure constants $\overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta_1}$ can be taken to be of the form

$$\Omega = \eta^{\alpha}\overset{(0)}{U}_{\alpha} + \eta^{\alpha_2}\eta^{\alpha_1}\, \overset{(1)}{U}_{\alpha_1\alpha_2}{}^{\beta_1}\, \mathcal{P}_{\beta_1}\ , \tag{3.4.28a}$$

$$= \eta^{a}\gamma_a + \frac{1}{2}\eta^a\eta^b\, f_{ab}{}^{c}\,\mathcal{P}_c + \eta^{m+a}\pi_a\ , \tag{3.4.28b}$$

where (3.3.2) has been used and $f_{ab}{}^{c}$ correspond to the structure constants of the original gauge algebra (3.1.5). Using the conventions (3.4.19) and (3.4.20) one alternatively has

$$\Omega = C^a\gamma_a + \frac{1}{2}C^aC^b\, f_{ab}{}^{c}\,\bar{\rho}_c - i\rho^a\pi_a\ , \quad (a = 1, \ldots, m)\ . \tag{3.4.29}$$

From here it becomes evident the abelian role that the constraints $\pi_a = 0$ play.

Remarks

1. One says that a set of constraints and of associated structure functions is of *rank* s if all structure functions of order strictly greater than s vanish $\overset{(p)}{U}{}^{\beta_1\ldots\beta_p} \equiv 0$ $p > s$.

In this sense an abelian set of constraints can be chosen to be of rank 0, while a set of constraints forming a closed gauge algebra can be chosen to be of rank 1.

2. By simple inspection of the explicit formula for $\overset{(k)}{D}$, *cf.* Eq. (3.3.24), one realises that if all the structure functions of order k are zero, with $s < k \leq (2s+1)$, then $\overset{(2s+1)}{D}$ vanishes, implying that one can choose $\overset{(2s+2)}{U} = 0$. This in turn implies that $\overset{(2s+2)}{D}$ also vanishes, so $\overset{(2s+3)}{U} = 0$ can be taken, and so on. Since by hypothesis $\overset{(s+1)}{U} = \ldots = \overset{(2s+1)}{U} = 0$, one concludes that in these cases the rank can be taken to be s.

3. In the general case of nonconstant structure functions in the gauge algebra, the sum of the first two terms in the expansion of Ω as in (3.4.28) may not be enough to establish the nilpotency property. Higher order structure functions may be needed in Ω. ▲

3.5 Rescaling constraints and about the uniqueness of the BRST generator

The gauge algebra and the consequences of the Jacobi identity determine the form of the BRST generator. However, the gauge algebra does not determine the constraint surface on which the motion takes place. The constraint surface is given by the zero locus of the constraints, $G_\alpha = 0$. Since two different sets of constraints can define the same constraint surface, it results immediate that the form of the BRST generator is not intrinsic to the constraint surface.

A constraint surface defined by $G_\alpha = 0$, can alternatively be described in terms of $G'_\alpha = 0$ with

$$G'_\alpha = \Lambda_\alpha{}^\beta(q, p, \lambda, \pi) G_\beta \ , \tag{3.5.1}$$

where the *rescaling matrix* $\Lambda_\alpha{}^\beta$ is invertible at each point on the phase space and may vary from point to point on it. The transformations (3.5.1) are not canonical ones in the phase space $\mathbf{T}^*_\lambda\mathbb{Q}$. The PBs of the new constraints G'_α may drastically vary from the one associated to the original constraints G_α; for instance, one can have G_α forming an abelian gauge algebra and –on the other hand– G'_α obeying an open gauge algebra. A complete understanding of the equivalence between these two sets of constraints is found beyond the structures in $\mathbf{T}^*_\lambda\mathbb{Q}$. Although in the previous section the ghost–extension was introduced as a mere convenience –all structure functions were gathered into a single object– this procedure is not only mathematically advantageous, but the insertion of ghosts and ghost–momenta implements (3.5.1) as an (even) canonical transformation in the super phase space. This is established in the following theorem [61, 63, 69, 70, 71, 72].

Theorem 3.5.1 *Let Ω and Ω' be two BRST generators associated with the same constraint surface. Then, Ω and Ω' are related by a canonical transformation in the super phase space.*

The proof of this theorem for infinitesimal rescaling is discussed in [61]. Its validity for invertible linear transformations (3.5.1) which are in the connected component of the identity (positive determinant) is then proved. The general case with both positive and negative determinants of ${\Lambda_\alpha}^\beta$ is considered in [63]. An application of this theorem to the case of constraints linear in momenta is reported in [69, 71], where an explicit generating functional of the canonical transformation is presented.

This theorem ensures that the ambiguity in the structure functions is harmless in the classical theory. Two BRST generators corresponding to two different elections in the description of the same constraint surface are canonically related in the super phase space. The introduction of ghosts at a classical level makes manifest the canonical covariance of the structure of co–isotropic surfaces.

3.6 BRST observables

So far the implications of the Jacobi identity on the PBs (3.1.12) that involve the Hamiltonian have not been explored. It will result that treating these PBs in the same fashion as the PBs between constraints, introduces the notion of BRST observables.

Theorem 3.6.1 [Existence of BRST observables] *Given a Dirac observable A_0 on $\mathbf{T}^*\mathbb{Q}$, there is a BRST extension or BRST associated observable defined as an even Grassmann parity function A on the super phase space that satisfies the following conditions:*

$$\mathrm{gh}(A) = 0 = \mathrm{gh}(A_0), \tag{3.6.1a}$$

$$A = A_0 \quad \text{when} \quad \eta^\alpha = 0 = \mathcal{P}_\alpha, \tag{3.6.1b}$$

$$\{A, \Omega\} = 0, \tag{3.6.1c}$$

$$A^* = A. \tag{3.6.1d}$$

Let A_0 be a Dirac observable on $\mathbf{T}^*\mathbb{Q}$. It trivially can be lifted to $\mathbf{T}^*_\lambda\mathbb{Q}$ as follows:

$$\{A_0, G_\alpha\} = {A_\alpha}^\beta(q,p)\, G_\beta\ , \tag{3.6.2}$$

where $A_\alpha{}^\beta$ corresponds to $A_a{}^b(q,p)$ given in (2.2.45) when both indices are associated to the first–class constraints γ_a and zero otherwise. The following ansatz will be used to define the BRST extension of a Dirac observable:

$$A := \sum_{k=0} \overset{(k)}{A}{}^{\beta_1\ldots\beta_k}\, \mathcal{P}_{\beta_k}\ldots\mathcal{P}_{\beta_1} \tag{3.6.3a}$$

$$\equiv \overset{(0)}{A} + \eta^{\alpha_1}\overset{(1)}{A}{}_{\alpha_1}{}^{\beta_1}\mathcal{P}_{\beta_1} + \eta^{\alpha_2}\eta^{\alpha_1}\overset{(2)}{A}{}_{\alpha_1\alpha_2}{}^{\beta_1\beta_2}\,\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_1} + \ldots\ , \tag{3.6.3b}$$

with $\overset{(0)}{A}(q,p)$ being the Dirac observable $A_0(q,p)$, and the coefficients $\overset{(k)}{A}{}_{\alpha\ldots}{}^{\beta\ldots}$ functions to be determined on phase space.

As desired, each term in the finite sum (3.6.3) is of ghost number zero. Indeed, first

$$\mathrm{gh}(\overset{(k)}{A}{}_{[\alpha_1\ldots\alpha_k]}{}^{[\beta_1\ldots\beta_k]}) = 0,$$

and, second, the ghost number of any monomial that mixes ghost and ghost–momenta in equal number vanishes. Therefore (3.6.1a) is fulfilled. By construction (3.6.1b) is automatically satisfied. The function A is of even Grassmann parity and satisfies (3.6.1d) since any monomial with the same number of ghost and ghost–momenta is real and has even Grassmann parity, just as it happens to be with each coefficient $\overset{(k)}{A}{}_{\ldots}{}^{\cdots}(q,p)$. Finally, the existence of the coefficients of higher powers of ghost–momenta in A such that (3.6.1c) is fulfilled will be shown.

By direct calculation one can see that the coefficients corresponding to zeroth order in ghost–momenta on the LHS of (3.6.1c) are

$$\{\overset{(0)}{A},\overset{(0)}{U}\} - \overset{(1)}{A}{}^{\beta}\frac{\partial^\ell \overset{(0)}{U}}{\partial\eta^\beta} = \eta^{\alpha_1}\left(A_{\alpha_1}{}^{\beta_1} - \overset{(1)}{A}{}_{\alpha_1}{}^{\beta_1}\right)\overset{(0)}{U}_{\beta_1} \tag{3.6.4}$$

which suggests the definition $\overset{(1)}{A}{}_\alpha{}^\beta := A_\alpha{}^\beta$ in order to make the term in parenthesis vanish. With this identification the Eq. (3.6.2) is rewritten as

$$\{\overset{(0)}{A},\overset{(0)}{U}_{\alpha_1}\} = \overset{(1)}{A}{}_{\alpha_1}{}^{\beta_1}\,\overset{(0)}{U}_{\beta_1}\ . \tag{3.6.5}$$

Up to a term $\overset{(1)}{B}{}_{\alpha_1}{}^{[\beta_1\beta_2]}\,\overset{(0)}{U}_{\beta_2}$, which is to set to be zero, the Eq. (3.6.5) defines the coefficients of the terms with linear ghost–momenta in the sum (3.6.3).

The coefficients of the first order in ghost–momenta within $\{A,\Omega\}$ are

$$\begin{aligned}\{\overset{(0)}{A},\overset{(1)}{U}{}^{\beta_1}\} - \{\overset{(1)}{A}{}^{\beta_1},\overset{(0)}{U}\} + \overset{(1)}{U}{}^{\beta}\frac{\partial^\ell\overset{(1)}{A}{}^{\beta_1}}{\partial\eta^\beta} - \overset{(1)}{A}{}^{\beta}\frac{\partial^\ell\overset{(1)}{U}{}^{\beta_1}}{\partial\eta^\beta} + 2\overset{(2)}{A}{}^{\beta_1\beta}\frac{\partial^\ell\overset{(0)}{U}}{\partial\eta^\beta} = \eta^{\alpha_2}\eta^{\alpha_1}\Big(2\overset{(2)}{A}{}_{[\alpha_1\alpha_2]}{}^{\beta_1\beta}\,\overset{(0)}{U}_\beta \\ + \{\overset{(0)}{A},\overset{(1)}{U}{}_{[\alpha_1\alpha_2]}{}^{\beta_1}\} + \{\overset{(1)}{A}{}_{[\alpha_1}{}^{\beta_1},\overset{(0)}{U}_{\alpha_2]}\} + \overset{(1)}{U}{}_{[\alpha_1\alpha_2]}{}^{\beta}\overset{(1)}{A}{}_\beta{}^{\beta_1} + 2\overset{(1)}{A}{}_{[\alpha_1}{}^{\beta}\,\overset{(1)}{U}{}_{\alpha_2]\beta}{}^{\beta_1}\Big)\ .\end{aligned} \tag{3.6.6}$$

The BRST invariance of A (3.6.1c) requires that the term in parenthesis vanishes. Defining

$$\overset{(1)}{C}{}_{[\alpha_1\alpha_2]}{}^{\beta_1} := -\left(\{\overset{(0)}{A},\overset{(1)}{U}{}_{[\alpha_1\alpha_2]}{}^{\beta_1}\} + \{\overset{(1)}{A}{}_{[\alpha_1}{}^{\beta_1},\overset{(0)}{U}{}_{\alpha_2]}\} + \overset{(1)}{U}{}_{[\alpha_1\alpha_2]}{}^{\beta}\overset{(1)}{A}{}_{\beta}{}^{\beta_1} + 2\overset{(1)}{A}{}_{[\alpha_1}{}^{\beta}\overset{(1)}{U}{}_{\alpha_2]\beta}{}^{\beta_1}\right) , \tag{3.6.7}$$

the term in parenthesis on the RHS of (3.6.6) can be rewritten as $2\,\delta\overset{(2)}{A} - \overset{(1)}{C}$. This combination vanishes if $\overset{(1)}{C} = 2\,\delta\overset{(2)}{A}$. In order to guarantee that such $\overset{(2)}{A}$ actually exists, one needs to prove (see Theorem 3.2.2) that $\delta\overset{(1)}{C} = 0$. This can be done by considering the antisymmetrised PB of (3.6.5) with minus the constraints $\overset{(0)}{U}_{\alpha_2}$ and using the Jacobi identity. In the same way that the existence of second–order structure functions $\overset{(2)}{U}$ was proved, a direct calculation shows in this case that

$$\begin{aligned} 0 &= \{\{\overset{(0)}{A},\overset{(0)}{U}_{\alpha_1}\},-\overset{(0)}{U}_{\alpha_2}\} + \{\{-\overset{(0)}{U}_{\alpha_2},\overset{(0)}{A}\},\overset{(0)}{U}_{\alpha_1}\} + \{\{\overset{(0)}{U}_{\alpha_1},-\overset{(0)}{U}_{\alpha_2}\},\overset{(0)}{A}\} \\ &\equiv \left(\{\{\overset{(0)}{A},\overset{(0)}{U}_{\alpha_1}\},-\overset{(0)}{U}_{\alpha_2}\}\right)_A = 2\overset{(1)}{C}{}_{[\alpha_1\alpha_2]}{}^{\beta_1}\overset{(0)}{U}_{\beta_1} , \end{aligned} \tag{3.6.8}$$

which corresponds to the relation $\delta\overset{(1)}{C} = 0$. Therefore there exists $K := 2\overset{(2)}{A}$ such that $\overset{(1)}{C} = 2\,\delta\overset{(2)}{A}$, that is,

$$\overset{(1)}{C}{}_{[\alpha_1\alpha_2]}{}^{\beta_1} = 2\overset{(2)}{A}{}_{[\alpha_1\alpha_2]}{}^{[\beta_1\beta_2]}\overset{(0)}{U}_{\beta_2} . \tag{3.6.9}$$

This relation defines $\overset{(2)}{A}{}_{[\alpha_1\alpha_2]}{}^{[\beta_1\beta_2]}$ up to a term $\overset{(2)}{B}{}_{[\alpha_1\alpha_2]}{}^{[\beta_1\beta_2\beta_3]}\overset{(0)}{U}_{\beta_3}$ which is set to be zero. Simultaneously it makes the linear terms in the ghost–momenta within $\{A,\Omega\}$ vanish.

The coefficients of the terms with quadratic ghost–momenta within $\{A,\Omega\}$ are

$$\begin{aligned} &\{\overset{(0)}{A},\overset{(2)}{U}{}^{\beta_1\beta_2}\} - \{\overset{(1)}{A}{}^{\beta_1},\overset{(1)}{U}{}^{\beta_2}\} + \{\overset{(2)}{A}{}^{\beta_1\beta_2},\overset{(0)}{U}\} + \overset{(1)}{U}{}^{\beta}\frac{\partial^{\ell}\overset{(2)}{A}{}^{\beta_1\beta_2}}{\partial\eta^{\beta}} + 2\overset{(2)}{U}{}^{\beta_1\beta}\frac{\partial^{\ell}\overset{(1)}{A}{}^{\beta_2}}{\partial\eta^{\beta}} - \overset{(1)}{A}{}^{\beta}\frac{\partial^{\ell}\overset{(2)}{U}{}^{\beta_1\beta_2}}{\partial\eta^{\beta}} \\ &+ 2\overset{(2)}{A}{}^{\beta_1\beta}\frac{\partial^{\ell}\overset{(1)}{U}{}^{\beta_2}}{\partial\eta^{\beta}} - 3\overset{(3)}{A}{}^{\beta_1\beta_2\beta}\frac{\partial^{\ell}\overset{(0)}{U}}{\partial\eta^{\beta}} = \eta^{\alpha_3}\eta^{\alpha_2}\eta^{\alpha_1}\left(\overset{(2)}{C}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1\beta_2} - 3\overset{(3)}{A}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta_1\beta_2\beta}\overset{(0)}{U}_{\beta}\right) , \end{aligned} \tag{3.6.10}$$

where antisymmetrisation occurs in the indices (β_1,β_2) and (β_1,β_2,β) in the first and second terms inside the parenthesis. In Eq. (3.6.10) the following tensor has been defined:

$$\begin{aligned} \overset{(2)}{C}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2]} := &\{\overset{(0)}{A},\overset{(2)}{U}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{[\beta_1\beta_2]}\} - \{\overset{(1)}{A}{}_{[\alpha_1}{}^{[\beta_1},\overset{(1)}{U}{}_{\alpha_2\alpha_3]}{}^{\beta_2]}\} + \{\overset{(2)}{A}{}_{[\alpha_1\alpha_2}{}^{[\beta_1\beta_2]},\overset{(0)}{U}{}_{\alpha_3]}\} + \\ &+ 2\overset{(1)}{U}{}_{[\alpha_1\alpha_2}{}^{\beta}\overset{(2)}{A}{}_{\alpha_3]\beta}{}^{[\beta_1\beta_2]} - 2\overset{(2)}{U}{}_{[\alpha_1\alpha_2\alpha_3]}{}^{\beta[\beta_1}\overset{(1)}{A}{}_{\alpha_3]}{}^{\beta_2]} - \\ &- 3\overset{(1)}{A}{}_{[\alpha_1}{}^{\beta}\overset{(2)}{U}{}_{\alpha_2\alpha_3]\beta}{}^{[\beta_1\beta_2]} - 4\overset{(2)}{A}{}_{[\alpha_1\alpha_2}{}^{\beta[\beta_1}\overset{(1)}{U}{}_{\alpha_3]\beta}{}^{\beta_2]} , \end{aligned} \tag{3.6.11}$$

where an additional antisymmetrisation is present in (α_3,β) within the fourth and last terms as well as in $(\alpha_2,\alpha_3,\beta)$ in the sixth term. The expression in parenthesis in (3.6.10) has the form $\overset{(2)}{C} - 3\,\delta\overset{(3)}{A}$ which vanishes if there exists the appropriate tensor $\overset{(3)}{A}$ such that

$\overset{(2)}{C} = 3\,\delta\overset{(3)}{A}$. This is equivalent to show that $\delta\overset{(2)}{C} = 0$. As expected this condition is proved when one considers the antisymmetrised PB of the equation that define $\overset{(2)}{A}$ (3.6.9) with the constraints $\overset{(0)}{U}$ using the explicit form of $\overset{(1)}{C}$.

By induction one can show that the coefficients of the kth power in ghost–momenta within $\{A, \Omega\}$ correspond to $\overset{(k)}{C} - (k+1)\delta\overset{(k+1)}{A}$, in local coordinates they are

$$\eta^{\alpha_1} \dots \eta^{\alpha_k}\eta^{\alpha_{k+1}} \left(\overset{(k)}{C}_{[\alpha_1 \dots \alpha_k \alpha_{k+1}]}{}^{\beta_1 \dots \beta_k} - (k+1)\overset{(k+1)}{A}{}_{[\alpha_1 \dots \alpha_k \alpha_{k+1}]}{}^{\beta_1 \dots \beta_k \beta} \overset{(0)}{U}_{\beta} \right) . \tag{3.6.12}$$

Indeed the terms in parenthesis vanish since $\delta\overset{(k)}{C} = 0$ as a consequence of the Jacobi identity, the equation that defines $\overset{(k)}{A}$, namely $\overset{(k-1)}{C} = k\,\delta\overset{(k)}{A}$, and the explicit form of $\overset{(k)}{C}$ which corresponds to

$$\begin{aligned} \eta^{\alpha_{k+1}} \dots \eta^{\alpha_1} \overset{(k)}{C}_{\alpha_1 \dots \alpha_{k+1}}{}^{\beta_1 \dots \beta_k} = &\sum_{p=0}^{k} (-)^{k-p} \{\overset{(p)}{A}{}^{\beta_1 \dots \beta_p}, \overset{(k-p)}{U}{}^{\beta_{p+1} \dots \beta_k}\} \\ &+ \sum_{p=0}^{k} (-)^{k} (p+1) \overset{(p+1)}{U}{}^{\beta_1 \dots \beta_p \beta} \frac{\partial^{\ell}\, \overset{(k-p)}{A}{}^{\beta_{p+1} \dots \beta_k}}{\partial \eta^{\beta}} \\ &- \sum_{p=0}^{k-1} (-)^{k-p} (p+1) \overset{(p+1)}{A}{}^{\beta_1 \dots \beta_p \beta} \frac{\partial^{\ell}\, \overset{(k-p)}{U}{}^{\beta_{p+1} \dots \beta_k}}{\partial \eta^{\beta}} . \end{aligned} \tag{3.6.13}$$

This process to construct a BRST observable from a gauge invariant function can be applied to the Hamiltonian of the system. Denoting the structure functions $H_\alpha{}^{\beta}$ on the RHS of (3.1.12) as $\overset{(1)}{H}_\alpha{}^{\beta}$, the corresponding BRST extended Hamiltonian reads

$$H_{BRST} = \overset{(0)}{H} + \eta^{\alpha_1} \overset{(1)}{H}_{\alpha_1}{}^{\beta_1} \mathcal{P}_{\beta_1} + \eta^{\alpha_2}\eta^{\alpha_1} \overset{(2)}{H}_{\alpha_1 \alpha_2}{}^{\beta_1 \beta_2} \mathcal{P}_{\beta_2}\mathcal{P}_{\beta_1} + \dots \, . \tag{3.6.14}$$

The function $\overset{(0)}{H}$ corresponds to the Hamiltonian function H on the original phase space, *cf.* Eq. (2.1.8), and the higher structure functions $\overset{(k)}{H}$ are iteratively obtained from the equations $\overset{(k)}{C} = (k+1)\,\delta\overset{(k+1)}{H}$. Each $\overset{(k)}{C}$ is obtained from higher structure functions of the gauge algebra $\overset{(k)}{U}$ and higher coefficients of the Hamiltonian $\overset{(k)}{H}$ up to kth order. Since $\overset{(0)}{H}$ does not depend on the Lagrange multipliers, in the ghost/antighost notation of Sect. 3.4.1, the BRST extended Hamiltonian is expressed as follows:

$$H_{BRST} = \overset{(0)}{H} + C^{a_1} \overset{(1)}{H}_{a_1}{}^{b_1} \bar{\rho}_{b_1} + C^{a_2}C^{a_1} \overset{(2)}{H}_{a_1 a_2}{}^{b_1 b_2} \bar{\rho}_{b_2}\bar{\rho}_{b_1} + \dots \, . \tag{3.6.15}$$

Remarks

1. Given a function K defined on the super phase space satisfying $\epsilon(K) = 1$, $\mathrm{gh}(K) = -1$, and $K^* = -K$, the new function

$$A' := A + \{\Omega, K\} \tag{3.6.16}$$

is a BRST observable provided A is so. The ghost number zero component of A' corresponds to $A'_0 = A_0 + K^\alpha(q,\lambda;p,\pi)G_\alpha$, *i.e.* at the lowest order of ghosts, one recovers the equivalence between two Dirac observables A_0 and A'_0.

2. If A and B are, respectively, BRST invariant extensions of the gauge invariant functions A_0 and B_0 on the original phase space, the functions AB and $\{A,B\}$ are, respectively, BRST invariant extensions of A_0B_0 and $\{A_0,B_0\}$.

3. Note that each first–class constraint G_α is gauge invariant. One possible choice for its BRST extension is

$$G_\alpha^{BRST} := \{-\mathcal{P}_\alpha, \Omega\} = G_\alpha + 2\,\eta^\beta \overset{(1)}{U}{}_{\alpha\beta}{}^\gamma \mathcal{P}_\gamma + \ldots \tag{3.6.17}$$

which explicitly shows that G_α^{BRST} is BRST invariant $\{G_\alpha^{BRST}, \Omega\} = 0$.

4. Both the BRST invariant extension H_{BRST} of the Hamiltonian and the Hamiltonian H_0 produce the same equations of motion on the constraint surface. ▲

CHAPTER 4

Quantisation of Singular Systems

Quantization is by no means unique and should be regarded as a theoretical way to guess the true theory.

– Sergei V. Shabanov, 2000

There is no golden rule to quantise constrained systems. Historically, the first method to quantise *un*constrained systems was the so–called canonical quantisation. This approach consists of a series of ansatz to fundamentally describe a system with the use of Hilbert–space methods [174]. The mathematical side of the programme consists on the following: Find a 'suitable' Lie homomorphism from a 'suitable' Lie subalgebra $\mathcal{F} \subset C^{\infty}(\mathbf{T}^*\mathbb{Q})$ to the Lie algebra of self–adjoint operators –with some common dense domain $\mathcal{D}$– acting on a Hilbert space $\mathcal{H}$ [118]. The Lie algebra structure on the classical side is given by the PB, whereas on the quantum side by the usual commutator. A somewhat general obstruction is however encountered in this process and is emphasised in Groenewold's theorem [113, 117]: It is not possible to consistently quantise the Poisson algebra of all polynomials in the canonical variables (q^i, p_i) of $\mathbf{T}^*\mathbb{Q} = \mathbb{R}^{2n}$ as symmetric operators on $\mathcal{H}$ under the condition that $(\widehat{q}^i, \widehat{p}_i)$ are irreducibly represented[1]. When the last condition is relaxed[2], the Lie algebra pre–quantisation of all C^{∞}–functions on $\mathbb{R}^{2n}$ exists (see [118]: Theorem 1).

[1]However, the quantisation of the Poisson algebra of the torus in which a suitable irreducible requirement is imposed can be achieved [115, 116].

[2]Quantisation without the irreducibility representation postulate is called *pre–quantisation*.

Besides the intrinsic problems already present in the canonical quantisation of unconstrained systems, the existence of gauge transformations make the process more intricate. The classically allowed states lie on the constraint surface Γ in a redundant fashion: one physical state of the classical system is labelled by many points on Γ, all of them connected by a gauge transformation, all of them lying on the same gauge orbit. Basically two paths can be taken to tackle the quantisation of these systems, the fundamental difference between the two paths depends on the order in which the implementation of constraints takes place. The constraints are implemented either after or before the quantisation. P. A. M. Dirac put into practice the first option [1] translating the whole problem of finding physical states into the linear spaces regime. The second option faces the endeavour to explicitly extract the true degrees of freedom to quantise by first constructing the reduced phase space. In general, there is not an equivalence between these two procedures as exemplified in [121]. However, in some cases the insistence on the compatibility between the two approaches is used to construct one of the quantisations when the other is known [66, 67, 68].

In the present chapter, two widely accepted methods which follow Dirac's strategy to quantise constrained systems are exhibited. Firstly, refined algebraic quantisation (RAQ) is reviewed [93, 94, 95, 96, 97, 98]. This is a widely used method in the community of quantum gravity [155]. Secondly, the canonical BRST–quantisation is revisited [61, 65, 73, 78, 79, 83] . This corresponds to a widely accepted quantisation method among the string theorists' community. We supplement the canonical BRST–quantisation with the Batalin–Marnelius–Shvedov (BMS) proposal for a physical inner product [82, 84, 85, 86, 87, 89, 90, 124], so that, the contact between RAQ under certain circumstances and BRST–quantisation is made. Both quantisation methods will become our main tools to give some light on the quantisation of rescaled constraints in the subsequent chapters.

4.1 Refined algebraic quantisation

In Dirac's method to quantise constrained systems one realises all dynamical variables (gauge invariant and non–invariant ones) as operators on some linear space of states. The physical states are selected by means of subsidiary conditions involving the quantum constraints. RAQ is a precise formulation of Dirac's programme and its subtleties. In RAQ one establishes a Hilbert space –the *auxiliary Hilbert space* $\mathcal{H}_{\text{aux}}$– on which the constraints will act as operators. The observables and physical state condition are interpreted in terms of distributions on some dense subspace $\Phi \subset \mathcal{H}_{\text{aux}}$. Within this scheme, the concept of a *rigging map* (η) is axiomatically introduced to establish an inner product between physical states. The *physical Hilbert space* $\mathcal{H}_{\text{phys}}$ is constructed by completion. The rigging map is closely related to the 'group averaging' technique [99, 100, 98, 108]. The axiomatic prescription to construct a physical inner product in this way is the most significant contribution from RAQ to tighten one of the loose ends in Dirac's strategy. In this section these procedures will be detailed.

The RAQ scheme corresponds to an improved version of what is known as *algebraic quantisation* [110, 112], which is briefly described in the following subsection.

4.1.1 Algebraic quantisation

The principle to find a physical inner product in algebraic quantisation is to represent a relevant set of real classical observables by self–adjoint operators acting on the kernel of the quantum constraints, that is, on the physical state space. Consider a classical system with phase space $\mathbf{T}^*\mathbb{Q}$ constrained by m first–class constraints $\gamma_a \approx 0$. The main steps in algebraic quantisation [110, 112] can be broadly divided into two parts: In the first one, steps (1) to (4), the arena where the constraint operators are going to act is established; this set of steps is independent of the constraints. The second part, steps (5) to (7), regards with the implementation of the quantum constraints and the search of a physical inner product compatible with the real/self–adjoint conditions of relevant observables. As a list one has

(1) Select a subspace $\mathscr{S}$ of the vector space of all smooth, complex–valued functions on $\mathbf{T}^*\mathbb{Q}$ subject to the following conditions:

- (i) $\mathscr{S}$ is large enough so that any sufficiently regular functions on the phase space can be obtained as (possibly a suitable limit of) a sum of products of elements[3] $f^{(i)}$ in $\mathscr{S}$. The unit function '1' should also be included in $\mathscr{S}$.
- (ii) $\mathscr{S}$ is closed under the Poisson bracket.
- (iii) $\mathscr{S}$ is closed under complex conjugation.
- (iv) Each element in $\mathscr{S}$, called *elementary classical variable*, is to have an unambiguous quantum analogue[4]

(2) Associate with each element $f^{(i)}$ in $\mathscr{S}$ an abstract operator $\widehat{f}^{(i)}$. Construct the free associative algebra ([175]: Chap. 20) generated by these elementary quantum

[3] One says that $\mathscr{S}$ is (locally) complete if and only if the gradients of the functions $f^{(i)}$ in $\mathscr{S}$ span the cotangent space of $\mathbf{T}^*\mathbb{Q}$ at each point.

[4] For our proposes the set of all complex–valued functions on $\mathbf{T}^*\mathbb{Q}$ which are either independent of momenta or linear in them are adequate as elementary variables [110]. An algebraic definition of variables with unambiguous quantum analogue can be found in [114, 175].

operators. Impose on it the canonical relations $[\widehat{f}^{(i)}, \widehat{f}^{(j)}] = i\hbar \widehat{\{f^{(i)}, f^{(j)}\}}$, and, if necessary, also a set of (anti–commutation) relations that captures the algebraic identities by the elementary classical variables (for details see [112]: Sec. II). Denote the resulting algebra by $\mathcal{A}_{\text{aux}}$.

(3) Introduce an involution operation[5] $\star$ on $\mathcal{A}_{\text{aux}}$ by requiring that if two elementary classical variables $f^{(i)}$ and $f^{(j)}$ are related by $f^{(i)*} = f^{(j)}$, then $\widehat{f}^{(i)\star} = \widehat{f}^{(j)}$ in $\mathcal{A}_{\text{aux}}$. Denote the resulting $\star$–algebra by $\mathcal{A}^{(\star)}_{\text{aux}}$.

(4) Ignoring the $\star$–relations, construct a linear representation of the abstract algebra $\mathcal{A}_{\text{aux}}$ via linear operators on some complex vector space $\mathcal{V}$.

Representing the associative free algebra $\mathcal{A}_{\text{aux}}$ on a complex vector space $\mathcal{V}$ is not in conflict with the no–go theorem by Groenewold. The linear representation does not correspond to a homomorphism of Lie algebras.

In the second part of the programme one enables the implementation of constraints as selectors of physical states. A requirement that reflects the gauge generator status of the constraints.

(5) Obtain explicit operators $\widehat{\gamma}_a$ on $\mathcal{V}$ representing the quantum constraints. In general, a choice of factor ordering has to be made at this stage. Physical states lie in the kernel $\mathcal{V}_{\text{phys}}$ of these operators

$$\widehat{\gamma}_a|\psi\rangle = 0 \; . \tag{4.1.1}$$

(6) Extract the physical subalgebra, $\mathcal{A}_{\text{phys}}$, of operators that leave $\mathcal{V}_{\text{phys}}$ invariant. Not all operators in $\mathcal{A}_{\text{aux}}$ are of this kind. An operator $\widehat{A}$ in $\mathcal{A}_{\text{aux}}$ will leave $\mathcal{V}_{\text{phys}}$ invariant if and only if $\widehat{A}$ weakly commutes with the constraints, that is,

$$[\widehat{A}, \widehat{\gamma}_a] = \widehat{A}_a{}^b\, \widehat{\gamma}_b \; . \tag{4.1.2}$$

Operators that satisfy this condition are the *Dirac quantum observables*. Introduce an involution operation on the physical algebra $\mathcal{A}_{\text{phys}}$. At this level the $\star$–relation in $\mathcal{A}^{(\star)}_{\text{aux}}$ can be used to induce a $\star$–relation in $\mathcal{A}_{\text{phys}}$. Denote the resulting $\star$–algebra of physical operators by $\mathcal{A}^{(\star)}_{\text{phys}}$.

(7) Induce on $\mathcal{V}_{\text{phys}}$ a Hermitian inner product so that the $\star$–relations on $\mathcal{A}^{(\star)}_{\text{phys}}$ are represented as adjoint relations on the resulting Hilbert space. In other words, whenever $\widehat{f}^{(i)\star} = \widehat{f}^{(j)}$ in $\mathcal{A}^{(\star)}_{\text{phys}}$, then the inner product on physical states should

[5] Remember, an involution operation on $\mathcal{A}_{\text{aux}}$ is a map $\star : \mathcal{A}_{\text{aux}} \to \mathcal{A}_{\text{aux}}$ that satisfies three conditions: (i) $(F + \alpha G)^\star = F^\star + \alpha^* G^\star$, $\alpha \in \mathbb{C}$; (ii) $(FG)^\star = G^\star F^\star$; and (iii) $(F^\star)^\star = F$.

be chosen such that the corresponding explicit operators in the representation satisfy $\widehat{f}^{(i)\dagger} = \widehat{f}^{(j)}$, where † is the Hermitian adjoint with respect to the physical inner product.

The last step is the requirement that real classical observables become symmetric quantum operators with respect to the physical inner product. Although at first sight this sounds rather elementary, it is subtle and of large scope. In general, there is no *a priori* guarantee that the physical vector space would admit an inner product satisfying the required reality conditions. If it does not, the vector space $\mathcal{V}$ can always be reconsidered and the process be restarted. The successful application of the broad guidelines listed above has been reported in various examples [104, 105, 107, 111, 112, 176].

In order to arrive at RAQ, the list of ansatz above requires a fine–tuning at the level of the implementation of reality conditions. In both scenarios, algebraic quantisation and its refined version, the compatibility between the reality and hermitian conditions with respect to the physical inner product is required.

4.1.2 From algebraic to refined algebraic quantisation

In this section refined algebraic quantisation programme is introduced. This is done based on [93] and the complementary material by D. Giulini, A. Gomberoff, J. Louko, D. Marolf, A. Molgado, I. A. Morrison and C. Rovelli [89, 96, 97, 98, 103, 104, 105, 106, 107, 108, 109].

One applies RAQ to quantise a classical system with phase space $\mathbf{T}^*\mathbb{Q}$ first–class constrained by $\gamma_a \approx 0$. This scheme deviates from the algebraic quantisation after the third step (see above). In step (4) rather than ignoring the $\star$–relations in the linear representation of $\mathcal{A}^{(\star)}_{\text{aux}}$, these are realised as adjoint relations on a Hilbert space. Step (4) now reads as follows:

(4) Construct a linear $\star$–representation R of the abstract algebra $\mathcal{A}^{(\star)}_{\text{aux}}$ via linear operators on an auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$, that is,

$$R(\widehat{A}^{\star}) = R(\widehat{A})^{\dagger}\ , \quad \forall\, \widehat{A} \in \mathcal{A}^{(\star)}_{\text{aux}}\ , \tag{4.1.3}$$

where † denote Hermitian conjugation with respect to the inner product $(\cdot\,,\cdot)_{\text{aux}}$ defined in $\mathcal{H}_{\text{aux}}$.

It should be recalled at this point that only for densely defined operators[6], $\widehat{T}$, on a

[6] A linear operator $\widehat{T}$ on $\mathcal{H}$ is said to be *densely defined* if its domain is dense, *i.e.* $\overline{D(\widehat{T})} = \mathcal{H}$ where the overline in this case denotes closure. Any bounded operator is densely defined.

Hilbert space $\mathcal{H}_{\text{aux}}$, a precise definition of its adjoint $\widehat{T}^{\dagger}$ can be assigned ([177]: Chap. VIII). Therefore for each abstract operator $\widehat{A} \in \mathcal{A}^{(\star)}_{\text{aux}}$, $R(\widehat{A})$ must be densely defined in $\mathcal{H}_{\text{aux}}$.

In the refined version of algebraic quantisation, $\mathcal{H}_{\text{aux}}$ is a space on which the quantum constraints act. The remaining steps address the implementation of these constraints as physical state selectors, this includes an exhaustively mathematical reconsideration of (4.1.1) via generalised vectors. In general, the space of physical states will *not* necessarily be a subspace of $\mathcal{H}_{\text{aux}}$. The next step establishes the debatable adjoint properties that the constraint operators should satisfy[7]. The explicit form of the constraint operators is also essential in the definition of observables later on.

(5a) Represent the constraints γ_a as self–adjoint operators $\widehat{\gamma}_a$ (or their exponentiated action, representing the finite gauge transformations, as unitary operators $\widehat{U}(g)$) on $\mathcal{H}_{\text{aux}}$.

Regarding Dirac condition (4.1.1) as an eigenvalue equation, with $\widehat{\gamma}_a$ self–adjoint constraint operators on $\mathcal{H}_{\text{aux}}$, it may happen that too few solutions on $\mathcal{H}_{\text{aux}}$ exist to construct any reasonable Hilbert space from them. Even more, when zero lies in the continuum part of the spectrum of one or more of the constraint operators $\widehat{\gamma}_a$, the solutions to (4.1.1) are not normalisable in the $(\cdot\,,\cdot)_{\text{aux}}$ [178]. Hence, the belief that restricting the auxiliary inner product $(\cdot\,,\cdot)_{\text{aux}}$ to some subspace to obtain a physical scalar product may result hopeless. One then looks for solutions of the Dirac condition in the algebraic dual Φ' of some dense subspace $\Phi \subset \mathcal{H}_{\text{aux}}$. The space $\Phi' \supset \mathcal{H}_{\text{aux}}$ is the space to which the generalised vectors belong, they turn out to be none other than *linear functionals* defined on the linear dense subspace Φ whose elements are called *test states*. The inclusion of a dense subspace Φ into the process is not an exclusive practice in quantisation of constrained systems, in fact, the most rigorous way to make sense of the bra–ket Dirac notation in quantum mechanics is through the triplet structure $\Phi \subset \mathcal{H}_{\text{aux}} \subset \Phi'$; for a complete exposition see for instance [179, 180, 181].

For the case in which the exponentiation of all constraints $\widehat{\gamma}_a$ defines a unitary action $\widehat{U}(g)$ of the gauge group on $\mathcal{H}_{\text{aux}}$, the condition (4.1.1) for gauge invariance, can be rephrased as

$$\widehat{U}|\psi\rangle = |\psi\rangle \ , \tag{4.1.4}$$

[7] Under the hypothesis that all physical predictions obtained from Dirac quantisation should coincide with reduced phase space quantisation, and that the whole quantisation process should be invariant under rescaling constraints transformations of the kind (4.1.23) in $\mathbf{T}^*\mathbb{Q}$, in a series of papers by K. V. Kuchar and P. Hájíček [66, 67, 68] there has been debated that there is no physical motivation to require self–adjoint constraint operators at a kinematical level.

that is, $\widehat{U}$ acts trivially on the physical states[8]. If $\widehat{U}(g)$ is accessible, seeking physical states prevents one dealing with unbounded operators such as $\widehat{\gamma}_a$.

As a general rule, and as a consequence of the fact that zero may be part of the continuum spectrum of constraints, the discrete spectrum of $\widehat{U}(g)$ need not contain one. This forces one to reconsider (4.1.4) on the algebraic dual $\widetilde{\Phi}'$ of some linear dense subspace $\widetilde{\Phi} \subset \mathcal{H}_{\text{aux}}$ of test states. Some comments on this dense subspace will be addressed below.

Basic invariance conditions on Φ, an important piece in this construction, are stated in the next step.

(5b) Choose a 'suitable' dense subspace $\Phi \subset \mathcal{H}_{\text{aux}}$ which is left invariant by the constraints $\widehat{\gamma}_a$,

$$\widehat{\gamma}_a \, \Phi \subset \Phi \tag{4.1.5a}$$

and let $\mathcal{A}^{(\star)}_{\text{phys}}$ be the $\star-$algebra of operators on $\mathcal{H}_{\text{aux}}$ which commute with the constraints $\widehat{\gamma}_a$ and such that, for $\widehat{A} \in \mathcal{A}^{(\star)}_{\text{phys}}$, both $\widehat{A}$ and $\widehat{A}^\dagger$ are defined on Φ and map Φ to itself

$$\widehat{A}\, \Phi \subset \Phi \;, \qquad \widehat{A}^\dagger \, \Phi \subset \Phi \;. \tag{4.1.5b}$$

The $\star-$operation in $\mathcal{A}^{(\star)}_{\text{phys}}$ is that induced by the adjoint operation $\dagger$ for operators on $\mathcal{H}_{\text{aux}}$.

This step in the quantisation programme is crucial. The dense subspace Φ becomes an essential component in this strategy of quantisation. Exactly what linear dense subspace Φ is 'suitable' depends on the theory under discussion. Its invariance properties (4.1.5) are precisely under the physically relevant operators: constraints and observables; then, some physical input is generally required in the choice of Φ. For instance, the subspace Φ should be large so that $\mathcal{A}^{(\star)}_{\text{phys}}$ contains 'enough' physically interesting operators while it must also be sufficiently small that its algebraic dual Φ' contains enough physical states. The choice of Φ has an effect on the set of observables, they *must* leave invariant Φ. It is important to emphasise that the selection of Φ affects the rest of the process, its impact even permeates the definition of physical inner product which is established almost at the end of the procedure. The states in Φ must ensure the convergence of the physical inner product. Finally, the existence of superselection rules on the physical Hilbert space, to be constructed, can strongly depend on the choice of the dense linear subspace Φ as shown in [94].

---

[8] This condition is valid at least for gauge groups with unimodular Lie group structure. When the gauge group corresponds to a nonunimodular one (${f_{ab}}^b \neq 0$ in the gauge algebra), the condition (4.1.1) varies according to $\widehat{\gamma}_a|\psi\rangle = -\frac{i}{2}\,{f_{ab}}^b|\psi\rangle$ which makes the RHS of (4.1.4) to be recast in the form $\Delta^{1/2}(g)\widehat{U}|\psi\rangle = |\psi\rangle$ (see [96] and section 4.1.3 below).

In the presence of a unitary action $\widehat{U}$ of the gauge group on $\mathcal{H}_{\text{aux}}$ one can formulate a similar condition to (4.1.5a) for $\widetilde{\Phi}$. A 'suitable' linear dense subspace $\widetilde{\Phi}$ must be invariant under the action of $\widehat{U}$

$$\widehat{U}\,\widetilde{\Phi} \subset \widetilde{\Phi}\ . \tag{4.1.6}$$

The algebra $\mathcal{A}^{(\star)}_{\text{phys}}$ corresponds to the $\star-$algebra of operators on $\mathcal{H}_{\text{aux}}$ that commute with $\widehat{U}$ such that for $\widehat{A} \in \mathcal{A}^{(\star)}_{\text{phys}}$, both $\widehat{A}$ and $\widehat{A}^\dagger$ are defined on $\widetilde{\Phi}$ and map $\widetilde{\Phi}$ to itself.

In cases where both formulations (4.1.1) and (4.1.4) of gauge invariance are available, it may be found that $\Phi \neq \widetilde{\Phi}$. It may well happen that there is a dense subspace $\Phi \subset \mathcal{H}_{\text{aux}}$ invariant under the action of constraints, *cf.* Eq. (4.1.5a), but $\widehat{U}\,\Phi \not\subset \Phi$. Therefore, the set of physical observables $\mathcal{A}^{(\star)}_{\text{phys}}$ when one considers $\widehat{U}$ to select physical states needs not to coincide with the one obtained when $\widehat{\gamma}_a$ are the physical–state selectors. $\mathcal{A}^{(\star)}_{\text{phys}}$ in the former consideration has to leave invariant $\widetilde{\Phi}$ and not Φ.

In the next step, a key ingredient that ultimately will define a physical inner product and simultaneously solve the constraints is introduced. This is the *rigging map*.

(5c) Find an anti–linear map $\eta : \Phi \to \Phi'$ that satisfies the following requirements:

(*i*) For every $\psi \in \Phi$, $\eta(\psi)$ is a solution of the constraints[9]

$$\widehat{\gamma}_a \eta(\psi)[\chi] \equiv \eta(\psi)[\widehat{\gamma}_a(\chi)] = 0 \qquad \forall\, \chi \in \Phi\ . \tag{4.1.7a}$$

(*ii*) The map η is real and positive in the sense that, for all $\psi, \chi \in \Phi$,

$$\eta(\psi)[\chi] = \left(\eta(\chi)[\psi]\right)^*\ , \tag{4.1.7b}$$

$$\eta(\psi)[\psi] \geqslant 0\ . \tag{4.1.7c}$$

(*iii*) The map η commutes with the action of any $\widehat{A} \in \mathcal{A}^{(\star)}_{\text{phys}}$, that is $\eta(\widehat{A}(\psi)) = \widehat{A}\eta(\psi)$ for all $\psi \in \Phi$, or, equivalently that for all $\psi \in \Phi$

$$\eta\big(\widehat{A}(\psi)\big)[\chi] = \eta(\psi)[\widehat{A}^\dagger(\chi)] \qquad \forall\, \chi \in \Phi\ . \tag{4.1.7d}$$

[9] An operator $\widehat{T}$ densely defined on Φ, such that together with its adjoint leave invariant Φ, has a natural extension to Φ'. Given a generalised vector $F \in \Phi'$, one has (with abusing of notation) that $\widehat{T}(F) \equiv \widehat{T}F$ is another functional in Φ' such that $\widehat{T}F : \Phi \to \mathbb{C}$. The new functional $\widehat{T}F$ on elements in Φ is defined as follows:

$$(\widehat{T}F)[\psi] := F[\widehat{T}^\dagger(\psi)] \qquad \forall\, \psi \in \Phi\ .$$

Let $\widehat{T}$ be self–adjoint with domain $D(\widehat{T}) = D(\widehat{T}^\dagger) \supset \Phi$, then $\widehat{T}^\dagger\,\Phi \subset \Phi$ and for every $\psi \in \Phi$ one has $\widehat{T}^\dagger(\psi) = \widehat{T}(\psi)$. In this case

$$(\widehat{T}F)[\psi] = F[\widehat{T}^\dagger(\psi)] \equiv F[\widehat{T}(\psi)] \qquad \forall\, \psi \in \Phi\ .$$

It is due to the adjoint on the RHS of this expression that the image of η, $\mathrm{Im}(\eta) \subset \Phi'$, will carry an anti–linear representation of $\mathcal{A}^{(\star)}_{\mathrm{phys}}$, and at the same time, it will be an invariant domain of physical observables.

The topology of Φ is that induced by the inclusion in $\mathcal{H}_{\mathrm{aux}}$. Whereas, the topology of Φ' is that of pointwise convergence, that is, a sequence $\{F_n\} \in \Phi'$ converges to $F \in \Phi'$ if and only if $F_n[\psi] \to F[\psi]$ for all $\psi \in \Phi$. So that one can identify $\mathcal{H}_{\mathrm{aux}} \subset \Phi'$ since vectors in $\mathcal{H}_{\mathrm{aux}}$ define linear functionals on the subset Φ by taking inner products.

Once again, having the unitary action $\widehat{U}$ of (a unimodular) group on $\mathcal{H}_{\mathrm{aux}}$, the requirement (4.1.7a) on a rigging map can be reformulated as follows: Each generalised vector, image of η, must be invariant under the unitary action $\widehat{U}\eta(\psi) = \eta(\psi)\,\forall\,\psi \in \widetilde{\Phi}$, or

$$\widehat{U}\eta(\psi)[\chi] = \eta(\psi)[\chi] \qquad \forall\,\psi, \chi \in \widetilde{\Phi}\ . \tag{4.1.8}$$

When the gauge group is nonunimodular the corresponding square root of its modular function should enter multiplying the LHS of this expression (see footnote 8 in page 76).

By construction the generalised vectors $\eta(\psi)$, with $\psi \in \Phi$ or $\psi \in \widetilde{\Phi}$ depending on the formulation, span the space of solutions of the constraints. In the next step of RAQ an inner product in such a space is introduced.

(5d) Define an inner product on the vector space $\mathrm{span}\{\eta(\psi)\}$ through

$$(\eta(\chi), \eta(\psi))_{\mathrm{RAQ}} := \eta(\psi)[\chi]\ . \tag{4.1.9}$$

Note that the positions of χ and ψ must be opposite on the two sides of (4.1.9) due to the anti–linear nature of η. The requirements (4.1.7b) and (4.1.7c) guarantee that $(\cdot\,,\cdot)_{\mathrm{RAQ}}$ is a Hermitian and positive definite inner product. The physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$ is defined by the Cauchy completion of the image of η in this inner product; therefore, $\mathrm{Im}(\eta)$ is by construction an invariant common dense subspace of $\mathcal{H}_{\mathrm{phys}}$ for the (strong) physical observables in $\mathcal{A}^{(\star)}_{\mathrm{phys}}$ (see eq. (4.1.7d)).

One could think that $\mathcal{H}_{\mathrm{phys}}$, the closure of $\mathrm{Im}(\eta)$, may have elements which rigorously are not solutions of the Dirac condition (4.1.1) in the RAQ sense. However, one can verify that this is not the case: $\mathcal{H}_{\mathrm{phys}}$ is a subspace of Φ' [97]. Consider the mapping $\sigma : \mathcal{H}_{\mathrm{phys}} \to \Phi' : F \mapsto \sigma F$ defined by

$$(\sigma F)[\psi] := (F, \eta(\psi))_{\mathrm{RAQ}}\ ,$$

we have that σF vanishes only if $(\sigma F)[\psi] = 0$, $\forall\,\psi \in \Phi$, or equivalently, only if F is orthogonal to $\mathrm{Im}(\eta)$. But the space $\mathrm{Im}(\eta)$ is by construction a dense subspace in $\mathcal{H}_{\mathrm{phys}}$. Then $F = 0$, that is, σ is an embedding of linear spaces.

The convergence of the physical inner product (4.1.9) on Φ (or $\widetilde{\Phi}$) in this context means that $\eta(\psi)[\chi]$ is *actually* an element in $\mathbb{C}$ for all $\psi, \chi \in \Phi$ ($\in \widetilde{\Phi}$). The fulfilment of such requirement is subtle. It often happens that $\eta(\psi)[\chi]$ diverges on some interesting domain and, when this is so in a 'tractable' fashion, one can construct a renormalised version of rigging map. This technique was effectively used in [93] within the context of loop approach to quantum gravity to construct a Hilbert space of states invariant under the group of diffeomorphisms of a spacelike surface Σ; this was not pathological or exclusive of that theory, one can find more examples of successful application of a renormalised rigging map. For instance, when the gauge group is $SO_c(n,1)$, $n > 1$, acting on square–integrable functions on $n+1$ dimensional Minkowski space $L^2(M^{n,1}, \mathrm{d}^n x)$, whose support extends outside the light cone, A. Gomberoff and D. Marolf [103] showed the existence of a rigging map whose divergence could be factorised out allowing a renormalised redefinition of rigging map. In a similar style, using a simplified version of the single constrained Ashtekar–Horowitz model [119] by D. G. Boulware [182], it was shown by J. Louko and A. Molgado [107] that for a class of special potentials in the constraint function, the rigging map they propose 'nicely' diverges in a way that a renormalised version can be introduced. Later on in this thesis, we will show another example (based on a single rescaled constraint) where some renormalisation process is required to obtain a convergent rigging map. In all these examples the rigging map is obtained using the group averaging technique as an ansatz, this will be the topic in the following section.

In summary, algebraic quantisation and its refined version are based on an abstract free associative $\star-$algebra of operators $\mathcal{A}^{(\star)}_{\mathrm{aux}}$ generated by the quantum analogue of elementary classical variables, where the basic canonical commutation relations have been imposed. In contrast to algebraic quantisation, where one seeks the kernel of the constraint operators on a linear space $\mathcal{V}$ lacking of inner product, in the refined version, one represents constraints as self–adjoint operators on an auxiliary Hilbert space $\mathcal{H}_{\mathrm{aux}}$. The physical space $\mathcal{H}_{\mathrm{phys}}$ is written in terms of generalised vectors. In addition, whereas in algebraic quantisation one searches for a Hermitian inner product in $\mathcal{V}_{\mathrm{phys}}$ that translates $\star-$adjoint into $\dagger-$Hermitian relations for the physical observables in $\mathcal{A}^{(\star)}_{\mathrm{phys}}$, in the refined version these relations will descend from $\mathcal{H}_{\mathrm{aux}}$ to the physical Hilbert space through the rigging map and its properties. The final result after applying RAQ is highly sensitive on three main inputs: the choice of $\mathcal{H}_{\mathrm{aux}}$, the realisation of the constraints as operators on it, and the choice of the dense linear subspace $\Phi \subset \mathcal{H}_{\mathrm{aux}}$ whose algebraic dual hosts Dirac physical states. The success of the programme depends on the three inputs mentioned above and any slight variation in one or more of these ingredients might result in a complete different $\mathcal{H}_{\mathrm{phys}}$ (if any).

4.1.3 Group averaging ansatz

In order to discuss quantities of physical relevance such as probabilities, expectation values and transition amplitudes, the physical space must be endowed with an inner product. Nevertheless, none of the Dirac's principles guides us how to obtain such an inner product. This is where the rigging map, when it exists, plays a central role and complements Dirac's approach.

A possible path one can follow to construct a rigging map when the gauge group corresponds to a Lie group, is prescribed by the so–called *group averaging technique*. In short, this method consists in the construction of gauge invariant quantum states, $\eta(\psi)$, by averaging non–invariant states over the gauge group [99, 100]. In the following paragraphs this idea is developed for a variety of representative groups.

A. Finite Group

The main idea of this method can be seen in the following situation: Let $G \equiv \{g_1, \ldots, g_n\}$ be a finite group and ρ a unitary representation of it on an inner product (complex) linear vector space $(V, (\cdot\,,\cdot)_{\text{aux}})$ of finite dimension. Each $\rho(g_k)$ has a natural action on elements in the dual space V^*, namely, for $\alpha \in V^*$ one has $\rho(g_k)\alpha : V \to \mathbb{C}$ with $(\rho(g_k)\alpha)[v] := \alpha[\rho(g_k^{-1})v]$, where g^{-1} is the inverse of g under the group product. By Reisz theorem, all linear functionals act as $\alpha[v] = (u, v)_{\text{aux}}$ for all $v \in V$ and u a fixed vector in V; this is the well known isomorphism between V and V^*. In this construction, one can always build the following linear functionals:

$$\eta(u)[v] := \frac{1}{n}\sum_{j=1}^{n} (\rho(g_j)u, v)_{\text{aux}} \qquad \forall\, v \in V \ . \tag{4.1.10}$$

Due to the factor $1/n$ if the vector $u \in V$ is invariant under the group action, then $\eta(u) \in V^*$ is *the* functional associated to *the* vector $u \in V$ prescribed by Reisz theorem. The mapping $\eta : V \to V^*$ is anti–linear due to the anti–linearity of $(\cdot\,,\cdot)_{\text{aux}}$ in the first slot.

Functionals defined in (4.1.10) have various desirable properties. First, they are $G-$invariant

$$\begin{aligned}(\rho(g_k)\eta(u))[v] = \eta(u)[\rho(g_k^{-1})v] &= \frac{1}{n}\sum_{j=1}^{n} \left(\rho(g_j)u, \rho(g_k^{-1})v\right)_{\text{aux}} = \frac{1}{n}\sum_{j=1}^{n} (\rho(g_k g_j)u, v)_{\text{aux}} \\ &= \frac{1}{n}\sum_{j'=1}^{n} \left(\rho(g_{j'})u, v\right)_{\text{aux}} = \eta(u)[v] \ , \qquad \forall\, v \in V \ ,\end{aligned}$$

where the summation in the second line was performed effectively again over the whole group, so that one only gets $\eta(u)[v]$. Therefore the invariance of $\eta(u)$ in the dual space is accomplished by *averaging* non–invariant vectors $(\rho(g_j)u)$ over the group. The arithmetic mean of scalar products with the arguments being elements in the orbit[10] of u in one slot and v in the other is performed. Second, using the unitarity of the representation and the fact that it results irrelevant which group elements happen to be at each term in the sum (4.1.10), either g_j or its inverse, one has $\eta(u)[v] = (\eta(v)[u])^*$. Third, η satisfies positivity in the sense of $\eta(u)[u] \geq 0$. Indeed, note that $P := \frac{1}{n}\sum_j \rho(g_j)$ is a projection operator,

$$P^2 = \frac{1}{n^2}\sum_{jk}\rho(g_j g_k) = \frac{1}{n^2}\Big(\sum_k \rho(g_1 g_k) + \cdots + \sum_k \rho(g_n g_k)\Big) = \frac{1}{n}\sum_{k'}\rho(g_{k'}) = P \ ,$$

so in this case the averaging yields a projection to an invariant subspace of V, $V_{\text{phys}} := \{Pu : \ u \in V\}$. For any $v \in V_{\text{phys}}$ one has $\rho(g)v = v$, then 1 is contained in the discrete spectrum of $\rho(g)$, and $V_{\text{phys}} \subset V$. Therefore, $\eta(u)[u] = (Pu, u)_{\text{aux}} = \big(P^2 u, u\big)_{\text{aux}} = (Pu, Pu)_{\text{aux}} \geq 0$ after using $P^\dagger = P$ and the positivity of the auxiliary inner product.

B. Unimodular and nonunimodular Lie groups

The above observation opens the possibility to construct gauge invariant generalised vectors in the algebraic dual $\widetilde{\Phi}'$ by averaging non–invariant test states belonging to $\widetilde{\Phi}$ when a unitary representation of a Lie gauge group $\widehat{U}(g)$ is accessible. For a continuous (Lie) gauge group G one should replace the sum (4.1.10) by an integral. For the integral, a volume form is needed; however, an infinite number of them exists in this case[11].

Let $\mathrm{d}_L g$ be a left–invariant measure on G, that is, a measure that under the integration symbol obeys

$$\mathrm{d}_L(hg) = \mathrm{d}_L g \tag{4.1.11}$$

for all fixed $h \in G$. In analogy to (4.1.10) one defines the antilinear mapping $\eta : \widetilde{\Phi} \to \widetilde{\Phi}'$ by its action on test states as follows:

$$\eta(\psi)[\chi] := \int_G \mathrm{d}_L g \ \Big(\widehat{U}(g)\psi, \chi\Big)_{\text{aux}} \ , \quad \forall\, \psi, \chi \in \widetilde{\Phi} \ . \tag{4.1.12}$$

[10]The orbit of u consists of the images of u under the action of G, $\mathcal{O} = \{\rho(g_k)u : \ k = 1, \ldots, n\} \subset V$

[11]Every Lie group is orientable. Every Lie group has a left–invariant volume form that is uniquely defined up to a positive constant which is the source of an infinite number of volume forms. It is only in the compact case that one can use a volume normalisation to single out a unique top form. For a compact Lie group, the unique left–invariant volume form with the property $\int_G \mathrm{d}g \equiv 1$ is called *Haar measure on G*. It turns out that the Haar measure is also right–invariant. The map $f \mapsto \int_G \mathrm{d}g\, f$ is called *Haar integral*. For details see for instance [183, 184, 185, 186].

Here, instead of the fraction $1/n$ on the RHS of (4.1.10), the $1/\int_G \mathrm{d}_L g$ should be placed; however, in general G need not to be compact and such a factor may vanish, so (4.1.12) dispenses with this pre–factor. Although it is not indicated, in the Schrödinger representation, within (4.1.12) one has integrals coming from inside the definition of $(\cdot\,,\cdot)_{\text{aux}}$ and they explicitly involve the domain $\widetilde{\Phi}$ since $\psi, \chi \in \widetilde{\Phi}$. Assuming for the moment the delicate point that (4.1.12) converges in absolute value for all $\psi, \chi \in \widetilde{\Phi}$, one wish to prove that it solves the constraints, is hermitian and positive definite as it was done for the finite group case.

The action of $\widehat{U}(h)$ on any linear functional $\eta(\psi)$ gives

$$\begin{aligned}(\widehat{U}(h)\eta(\psi))[\chi] = \eta(\psi)[\widehat{U}(h^{-1})\chi] &= \int_G \mathrm{d}_L g \left(\widehat{U}(g)\psi, \widehat{U}(h^{-1})\chi\right)_{\text{aux}} \\ &= \int_G \mathrm{d}_L g \left(\widehat{U}(hg)\psi, \chi\right)_{\text{aux}} \int_{L_h G \equiv G} \mathrm{d}_L(h^{-1}k) \left(\widehat{U}(k)\psi, \chi\right)_{\text{aux}} \\ &= \int_G \mathrm{d}_L(h^{-1}k) \left(\widehat{U}(k)\psi, \chi\right)_{\text{aux}} = \eta(\psi)[\chi] \ ,\end{aligned} \tag{4.1.13}$$

where the left–invariance of $\mathrm{d}_L g$ became crucial in the second line. One can anticipate from the analysis on (4.1.10) that to prove the reality condition (4.1.7b), the invariance of $\mathrm{d}_L g$ with respect to $g \mapsto g^{-1}$ needs to be guaranteed. However, $\mathrm{d}_L g$ is not necessarily invariant under this change, neither is its coequal right–invariant ($\mathrm{d}_R(gh) = \mathrm{d}_R(g)$) measure $\mathrm{d}_R g$. The relation between these two measures being

$$\mathrm{d}_L g = \Delta^{-1}(g)\, \mathrm{d}_R g \ , \tag{4.1.14}$$

where $\Delta(g)$ is the so–called *modular function* [96]. This function corresponds to a homomorphism Δ from the group G to the positive real numbers. For finite dimensional Lie groups the modular function is $\Delta(g) := \det(\mathrm{Ad}_g)$, where Ad denotes the adjoint representation. The changes induced on $\mathrm{d}_L g$ and $\mathrm{d}_R g$ by $g \mapsto g^{-1}$ inside group integrals respectively are

$$\mathrm{d}_L g \mapsto \mathrm{d}_L(g^{-1}) = \Delta(g)\, \mathrm{d}_L g \ , \tag{4.1.15a}$$

$$\mathrm{d}_R g \mapsto \mathrm{d}_R(g^{-1}) = \Delta^{-1}(g)\, \mathrm{d}_R g \ , \tag{4.1.15b}$$

So, defining [96]

$$\mathrm{d}_0 g := \Delta^{1/2}(g)\mathrm{d}_L g = \Delta^{-1/2}(g)\mathrm{d}_R g \tag{4.1.16}$$

one obtains an invariant measure under the desired change, $\mathrm{d}_0 g = \mathrm{d}_0(g^{-1})$. This measure is called the *symmetric measure*. However, under left and right translation this symmetric measure behaves as

$$\mathrm{d}_0(hg) := \Delta^{1/2}(h)\, \mathrm{d}_0 g \ , \tag{4.1.17a}$$

$$\mathrm{d}_0(gh) := \Delta^{-1/2}(h)\, \mathrm{d}_0 g \ . \tag{4.1.17b}$$

It seems then natural to separately deal with two different cases:

B.1. Unimodular case. This case is encountered when the identity $\Delta(g) \equiv 1$, for all $g \in G$, is fulfilled. Left– and right–invariant measures collapse into the symmetric measure (4.1.16)

$$\mathrm{d}_L g = \mathrm{d}_R g = \mathrm{d}_0 g$$

and the group averaging formula (4.1.12) becomes

$$\eta(\psi)[\chi] := \int_G \mathrm{d}_0 g \left(\widehat{U}(g)\psi, \chi\right)_{\mathrm{aux}} , \tag{4.1.18a}$$

the RHS being invariant under $g \mapsto g^{-1}$. This invariance and the unitarity of $\widehat{U}$ make of

$$\eta(\psi)[\chi] = \int_G \mathrm{d}_0 g \left(\psi, \widehat{U}(g)\chi\right)_{\mathrm{aux}} \tag{4.1.18b}$$

an equivalent definition for η (see [96]: Eq. (2.3), or [104]: Eq. (4.3)).

By the calculation (4.1.13), the left–invariance of $\mathrm{d}_0 g$ implies that (4.1.18a) satisfies the constraints in the sense of (4.1.8). Unitarity of $\widehat{U}$ and invariance of $\mathrm{d}_0 g$ under $g \mapsto g^{-1}$ (Eqs. (4.1.15) with $\Delta \equiv 1$) ensure the reality condition $\eta(\psi)[\chi] = (\eta(\chi)[\psi])^*$. In contrast to the discrete group averaging (4.1.10), $\eta(\psi)[\psi]$ need not to be positive definite as it is exemplified by J. Louko and A. Molgado [105] when the non–compact group $SL(2,\mathbb{R})$ in the $(1,1)$ oscillator representation is considered. If, however, property (4.1.7c) turned out to be satisfied, being η not identically vanishing, the group averaging formula (4.1.18a) produces a rigging map when η commutes with the observables. These conclusions do not change if one instead uses the mapping given by (4.1.18b).

B.2. Nonunimodular case. For nonunimodular groups, defined by $\Delta(g) \not\equiv 1$, the Eq. (4.1.16) relates $\mathrm{d}_L g$ and $\mathrm{d}_R g$ to the measure $\mathrm{d}_0 g$. Using this more symmetric measure into the group averaging formula (4.1.12) one has

$$\widetilde{\eta}(\psi)[\chi] := \int_G \mathrm{d}_0 g \left(\widehat{U}(g)\psi, \chi\right)_{\mathrm{aux}} , \tag{4.1.19a}$$

or equivalently (see [96]: Eq. (2.6), or [106]: Eq. (5.2))

$$\widetilde{\eta}(\psi)[\chi] = \int_G \mathrm{d}_0 g \left(\psi, \widehat{U}(g)\chi\right)_{\mathrm{aux}} . \tag{4.1.19b}$$

Performing a similar calculation to (4.1.13), functionals of the form (4.1.19a) are quasi–invariant under the unitary action of G

$$(\widehat{U}(h)\widetilde{\eta}(\psi))[\chi] = \Delta^{-1/2}(h)\, \widetilde{\eta}(\psi)[\chi] , \tag{4.1.20}$$

where the equalities $\mathrm{d}_0(h^{-1}g) = \Delta^{1/2}(h^{-1})\mathrm{d}_0 g = \Delta^{-1/2}(h)\mathrm{d}_0 g$ were used on the way. From the unitarity of $\widehat{U}$ and the invariance of $\mathrm{d}_0 g$ under the change $g \mapsto g^{-1}$, one has that $\widetilde{\eta}$ satisfies the reality condition (4.1.7b). Again, $\widetilde{\eta}(\psi)[\psi]$ need not to be positive definite. If, however, by other means (4.1.7c) is shown to be fulfilled, and $\widetilde{\eta}$ is not identically vanishing, one has that $\widetilde{\eta}$ satisfies the rigging map axioms with the $G-$quasi–invariance (4.1.20). These conclusions do not change if one instead uses the mapping given by (4.1.19b). The typical extra factor $\Delta^{-1/2}(h)$ in the quasi–invariance (4.1.20) (see for instance [106]: Eq. (5.4)) arises after using the equality (4.1.17b) in a calculation similar to (4.1.13). As explained in [187, 188] such factor comes after insisting on regaining the correspondence between Dirac's and reduced phase space quantisations.

Remarks

1. From (4.1.8) and the action of operators on the dual $\widetilde{\Phi}'$ (footonote 9 in page 77), by gauge invariance we mean [106]

$$\eta(\psi)[\widehat{U}(g^{-1})\chi] = \eta(\psi)[\chi], \quad \forall\, \psi, \chi \in \widetilde{\Phi} \tag{4.1.21}$$

In contrast in Ref. [96], where Eq. (4.1.19b) is used as group averaging formula, by gauge invariance is meant

$$\eta(\psi)[\widehat{U}(g)\chi] = \eta(\psi)[\chi]\ .$$

Note the change in the argument of $\widehat{U}$ on the LHS with respect to our formula (4.1.21). In [96] for nonunimodular gauge groups the gauge quasi–invariance reads

$$\widetilde{\eta}(\psi)[\widehat{U}(g)\chi] = \Delta^{+1/2}(g)\,\widetilde{\eta}(\psi)[\chi]$$

instead of our (4.1.20), the reason being the use of g instead of g^{-1} in the formulas. See also footnote 1 within Ref. [106].

2. Compact Lie groups are unimodular [184, 189]. The symbol $\mathrm{d}g$ has been reserved for its (unique) normalisable left– and right–invariant volume form. Hence the expression (4.1.18) can be used as an ansatz of a rigging map in this case. In fact

$$P := \int_G \mathrm{d}g\, \widehat{U}(g) \tag{4.1.22}$$

converges as an operator on $\mathcal{H}_{\text{aux}}$, projecting onto just the set of states that solve the constraints as in the finite group case; that is, P projects onto states in the trivial representation of G, so 1 is contained in the discrete part of the spectrum of P. These results implies that for a compact gauge groups, positivity is always guaranteed.

3. After non–compact Lie gauge groups the next level of difficulty in the construction of a rigging map comes by considering first–class constraints which not even form a 'genuine' Lie algebra, *i.e.* constrained systems with nonconstant structure functions on their gauge algebra. ▲

4.1.4 Rescaling constraints and RAQ: A first discussion

It has been emphasised that in any constrained system the constraint surface, on which the classical motion takes place, can be defined by two different sets of equivalent constraints $\{\gamma_a\}$ and $\{\gamma'_a\}$. The explicit functional form of the constraints in each set is not relevant, but the surface they define by their zero locus. For instance, the surface $\Gamma \subset \mathbf{T}^*\mathbb{Q}$ defined by $\{\gamma_a\}$ is also the one defined by the anholonomic basis [66]

$$\gamma'_a(q,p) = \Lambda_a{}^b(q,p)\gamma_b \tag{4.1.23}$$

whenever the rescaling matrix Λ is invertible at each point on the phase space. Under this transformation, the structure functions undergo an inhomogeneous transformation

$$f'_{ab}{}^c = \left(\Lambda_{[a}{}^d \Lambda_{b]}{}^f f_{df}{}^e - 2\,\Lambda_{[a}{}^d \Lambda_{b],d}{}^e + \{\Lambda_a{}^d, \Lambda_b{}^e\}\gamma_d\right)\left(\Lambda^{-1}\right)_e{}^c . \tag{4.1.24}$$

Here $\cdot_{,a} \equiv \{\cdot\,,\gamma_a\}$ is the directional derivative along the constraint field, and the square brackets means total antisymmetrisation in the indices (see footnote 2 in page 49). By the abelianisation theorem, at least locally, one can always use a transformation (4.1.23) to reduce the corresponding structure functions to zero [135]. Or the other way around, given an abelian set of constraints, $f_{ab}{}^c \equiv 0$, a transformation (4.1.23) might induce drastic changes in the gauge algebra. These changes include turning an abelian gauge algebra into an algebra which closes with structure functions defined on the phase space.

Although these rescalings at the constraints level may imply changes in the classification of the gauge algebra, from a closed to an open one, the classical observables of the theory remain invariant under such redefinitions. In other words, to rescale constraints must be viewed as a change of basis at the level of the gauge generators under which the physical theory is invariant. The nature of these rescalings is well understood at a classical level. In general, a transformation (4.1.23) is not a canonical one in the phase space $\mathbf{T}^*\mathbb{Q}$, but it is in a ghost–extended phase space (see Sect. 3.5).

At a quantum level the effects of a general constraint rescaling transformation is much more involved. Formally, one would expect that in a canonical quantisation of the ghost–extended phase space the quantum theories emerging from one and the other set of constraints are connected by a unitary transformation ([174]: Chap. 26). Interpretation of Dirac quantisation in the RAQ–sense may make this unitarity be lost since there is not a canonical relation between the corresponding classical theories on the original phase space. At the time this thesis was written and to the knowledge of the author, an ultimate comprehension of this issue is far from clear in the most general case.

The RAQ is complete once a rigging map has been successfully implemented. A somewhat general expression for η is only available when the gauge group has the structure of a (non) unimodular Lie group with a unitary representation on an auxiliary Hilbert space. In terms of the constraint algebra, this is translated into the cases where structure constants, rather than functions, are present. In cases where the gauge group forms an open algebra, a rigging map must be constructed from scratch, even the necessity of reconsidering what is meant by gauge invariant states might be a requisite[12]. Hence, if one does not even have a tentative formula for the rigging map in the general case, or, the abstract axioms that define it, it results too premature for the RAQ scheme to give a complete answer about the implications of a general rescaling constraint transformation (4.1.23). The relation between the quantised theory using, on one side, the 'original' constraints and, on the other, the rescaled version of them is hence not clear.

In order to gain some insight into the open–constraint–algebra territory, one may still investigate the situation suggested by Marolf [98]: Consider a system with an underlying genuine Lie gauge group G generated by $\{\gamma_a\}$, that is, generated by a closed gauge algebra. Perform a RAQ on it with special emphasis on the rigging map, then using this result as a guide, try to construct the corresponding RAQ of the rescaled–constraints version. Two comments take place here. First, if the original quantum constraints were self–adjoint, the scaled constraints will in general not be so. Second, if Φ is some dense subspace for which the RAQ was successfully applied in the original constraint setting, having in particular $\widehat{\gamma}_a\Phi \subset \Phi$, the dense subspace Φ in general will fail to be invariant under the rescaled constraints version: $\widehat{\gamma}'_a\Phi \not\subset \Phi$. This indicates that the whole RAQ for the rescaled constraints will render in general a different $\mathcal{H}_{\text{phys}}$. The task would then be to prove a unitarity relation between the two different physical Hilbert spaces.

In a modest model, in the next chapter, we will discuss the issues related to self–adjointness of rescaled constraints. It will be found that the self–adjointness of the rescaled constraint operator is conditioned by the nature of the scaling functions, the

[12] The first step towards this redefinition can be read from the nonunimodular Lie gauge group cases, where instead of full invariance of states ($\widehat{\gamma}_a|\psi\rangle = 0$) a quasi–invariance is needed ($\widehat{\gamma}_a|\psi\rangle = (-i/2)\ \mathrm{tr}(\mathrm{ad}_a)|\psi\rangle$).

consequences at the level of the rigging map will be analysed. When the scaling functions do not affect the self–adjointness of the rescaled constraints, in Chap. 6 using tractable examples, we will show that the rescaled constraint quantum theory can be mapped to that in which the scaling functions are the identity.

Another route one can follow in the investigation of the quantum effects introduced by (4.1.23), is using the fact that such transformation is canonical in the super phase space. At a formal level, hence, there should be a unitary connection between both quantum theories: that based on γ_a and the other based on γ'_a. However, as it will be seen in the next section, the canonical quantisation of the super phase space must be achieved on an indefinite inner product space[13] $\mathcal{V}_{\mathrm{BRST}}$. One cannot attach an authentic Hilbert space structure to $\mathcal{V}_{\mathrm{BRST}}$, and the unitary transformation that connects the two quantum descriptions of the system must then be taken with a grain of salt. This unitarity is with respect to an inner product which is not the physical one. Nevertheless, the research about the quantum effects induced by (4.1.23) using BRST methods has been performed for certain constrained systems [69, 70, 71, 72]. Canonical BRST–quantisation of the super phase space will be the central topic in the following section.

4.2 Canonical BRST–quantisation

There is no unique way of setting up a canonical BRST–quantisation of a super phase space, just as there is no unique quantisation of the original phase space. In this section the canonical BRST–quantisation developed is the one outlined in [61, 64, 65, 83], supplemented with a summary of the extensive work by R. Marnelius *et al.* [78, 79, 82, 84, 85, 89, 90]. The connection between BRST methods and RAQ is based on the proposal by O. Y. Shvedov [124] and exemplified in the subsequent chapters.

4.2.1 States and operators: Formal considerations

Canonical BRST–quantisation as presented in [61, 64, 65, 83] aims to preserve as much as possible the classical super–Lie algebra structure of functions on the super phase space.

[13] Let $(V, (\cdot\,,\cdot))$ be an inner product space, for each $v \in V$ either $(v,v) > 0$, or $(v,v) < 0$, or $(v,v) = 0$. Correspondingly v is said to be positive, negative or neutral. It is clear that the zero vector is neutral. If V contains positive as well as negative elements, one says that V is an *indefinite inner product space*. Every indefinite inner product space contains non–zero neutral vectors. The inner product is said to be *semi–definite* on V, if it is not indefinite; a semi–definite inner product may be either *positive inner product* when $(v,v) \geq 0\ \forall v \in V$, or *negative inner product* when $(v,v) \leq 0\ \forall v \in V$. An inner product is said to be *definite*, if $(v,v) = 0$ implies $v = 0$. Every definite inner product is semi–definite. Hence, a definite inner product is positive (resp. negative) when $(v,v) \geq 0$ $((v,v) \leq 0)$ $\forall\, v \in V$ and the equality only holds when $v = 0$ [74].

The quantum parallels to these functions are meant to be operators acting on some vector space, the set of operators endowed with a *generalised* or *graded commutator* $[\cdot,\cdot]$. For $\widehat{f},\widehat{g}$ operators corresponding to classical variables functions f and g, with definite Grassmann parities ϵ_f and ϵ_g, respectively, the graded commutator is defined as

$$[\widehat{f},\widehat{g}\,] := \widehat{f}\,\widehat{g} - (-)^{\epsilon_f\epsilon_g}\widehat{g}\,\widehat{f}\ . \tag{4.2.1}$$

This bracket is equal to the standard commutator $[\widehat{f},\widehat{g}\,] = \widehat{f}\,\widehat{g} - \widehat{g}\,\widehat{f}$ unless the operators $\widehat{f}$ and $\widehat{g}$ are both odd in which case it is equal to the anticommutator $[\widehat{f},\widehat{g}\,]_+ = \widehat{f}\,\widehat{g} + \widehat{g}\,\widehat{f}$.

This process will obviously find the same Groenewold's obstructions that any canonical quantisation method faces. The author suggests that this initial stage in the canonical BRST–quantisation can be rephrased in steps similar to (1)–(3) of (refined) algebraic quantisation (*cf.* Sect. 4.1.1) but applied to the super phase space. One possible linear subspace $\mathscr{S}_{\mathrm{BRST}}$ of all the smooth complex–valued functions on the super phase space, is that spanned by the basic canonical variables $\{q,\lambda,\eta;p,\pi,\mathcal{P}\}$ and 1. The Lagrange multipliers λ^a and their conjugate momenta being included. The set $\mathscr{S}_{\mathrm{BRST}}$ is closed under the super PB, *cf.* Eqs. (2.1.16), (3.1.8) and (3.4.1). It is also closed under complex conjugation, *cf.* Eqs. (3.4.7). In this point of view, $\mathcal{B}_{\mathrm{aux}}$ will denote the free associative algebra generated by the elementary abstract quantum operators $\{\widehat{1},\widehat{q},\widehat{\lambda},\widehat{\eta};\widehat{p},\widehat{\pi},\widehat{\mathcal{P}}\}$ where the (graded) commutation relations

$$[\widehat{q}^{\,i},\widehat{q}^{\,j}] = 0\ ,\quad [\widehat{q}^{\,i},\widehat{p}_j] = -[\widehat{p}_j,\widehat{q}^{\,i}] = i\hbar\,\delta^i_j\ ,\quad [\widehat{p}_i,\widehat{p}_j] = 0\ , \tag{4.2.2a}$$

$$[\widehat{\lambda}^a,\widehat{\lambda}^b] = 0\ ,\quad [\widehat{\lambda}^a,\widehat{\pi}_b] = -[\widehat{\pi}_b,\widehat{\lambda}^a] = i\hbar\,\delta^a_b\ ,\quad [\widehat{\pi}_a,\widehat{\pi}_b] = 0\ , \tag{4.2.2b}$$

$$[\widehat{\eta}^\alpha,\widehat{\eta}^\beta] = 0\ ,\quad [\widehat{\eta}^\alpha,\widehat{\mathcal{P}}_\beta] = [\widehat{\mathcal{P}}_\beta,\widehat{\eta}^\alpha] = -i\hbar\,\delta^\alpha_\beta\ ,\quad [\widehat{\mathcal{P}}_\alpha,\widehat{\mathcal{P}}_\beta] = 0\ , \tag{4.2.2c}$$

together with others that capture algebraic identities satisfied by the elementary classical variables (if any), have been input. An involution $\star$–operation on $\mathcal{B}_{\mathrm{aux}}$ is introduced based on the reality conditions (3.4.7), hence

$$(\widehat{q}^{\,i})^\star = \widehat{q}^{\,i}\ ,\quad (\widehat{p}_i)^\star = \widehat{p}_i\ ,\quad (\widehat{\lambda}^a)^\star = \widehat{\lambda}^a\ ,\quad (\widehat{\pi}_a)^\star = \widehat{\pi}_a\ ,\quad (\widehat{\eta}^\alpha)^\star = \widehat{\eta}^\alpha\ ,\quad (\widehat{\mathcal{P}}_\alpha)^\star = -\widehat{\mathcal{P}}_\alpha\ , \tag{4.2.3}$$

so that $\mathcal{B}^{(\star)}_{\mathrm{aux}}$ is constructed.

In $\mathcal{B}^{(\star)}_{\mathrm{aux}}$ the abstract *ghost number operator*

$$\widehat{G}\ := \frac{i}{2}\sum_{\alpha=1}^{2m}\left(\widehat{\eta}^\alpha\widehat{\mathcal{P}}_\alpha - \widehat{\mathcal{P}}_\alpha\widehat{\eta}^\alpha\right),\quad \widehat{G}^\dagger = -\widehat{G}\ , \tag{4.2.4}$$

is found. This is an anti–hermitian operator that satisfies (under $\hbar\equiv 1$)

$$[\widehat{G},\widehat{\eta}^\alpha] = \widehat{\eta}^\alpha\ ,\quad [\widehat{G},\widehat{\mathcal{P}}_\alpha] = -\widehat{\mathcal{P}}_\alpha\ ,\quad [\widehat{G},\widehat{z}\,] = 0\ ,\ \text{ with } \widehat{z} = \widehat{q},\widehat{\lambda},\widehat{p},\ \text{or } \widehat{\pi}\ , \tag{4.2.5}$$

The BRST operator $\widehat{\Omega}$ is also included in $\mathcal{B}^{(\star)}_{\text{aux}}$. The graded commutation between $\widehat{\Omega}$ and $\widehat{G}$ is required to be

$$[\widehat{G}, \widehat{\Omega}\,] = \widehat{\Omega}\ , \tag{4.2.6}$$

which reflects the classical nature of Ω of being a $+1$ ghost numbered function. In addition Ω is nilpotent in PB sense (*cf.* Eq. (3.4.13c)) and real (*cf.* Eq. (3.4.13d)); accordingly, the operator $\widehat{\Omega}$ is required to satisfy the conditions

$$[\widehat{\Omega}, \widehat{\Omega}\,] = 2\,\widehat{\Omega}^2 = 0\ , \tag{4.2.7a}$$

$$\widehat{\Omega}^\star = \widehat{\Omega}\ . \tag{4.2.7b}$$

The next step in this quantisation process is to construct a linear $\star$–representation of the abstract $\mathcal{B}^{(\star)}_{\text{aux}}$ via linear operators on some vector space $\mathcal{V}_{\text{BRST}}$ with inner product $(\cdot\,,\cdot)_{\text{BRST}}$ such that the $\star$–relations become $\dagger$–relations.

Unlike the situation in the classical case, where the existence of a function Ω that satisfies (3.4.13) is guaranteed by the Theorem 3.4.1, the existence of an operator $\widehat{\Omega}$ acting on a vector space $\mathcal{V}_{\text{BRST}}$ with the properties (4.2.6) and (4.2.7) is more subtle. The question of ordering of factors becomes crucial. In what follows it will be assumed that a BRST operator $\widehat{\Omega}$ satisfying these properties can be represented as linear operator on $\mathcal{V}_{\text{BRST}}$.

The first fundamental difference in a canonical BRST–quantisation with respect to a ghost–free quantisation is that the operators in $\mathcal{B}^{(\star)}_{\text{aux}}$ cannot be represented on a definite inner product space, neither positive nor negative (*cf.* footnote 13 in page 87 for definitions), specially if one requires a non–trivial representation of the BRST operator. Indeed, the characteristic properties of nilpotency (4.2.7a) and symmetry (4.2.7b) imply

$$\left(\widehat{\Omega}\Psi, \widehat{\Omega}\Upsilon\right)_{\text{BRST}} = \left(\Psi, \widehat{\Omega}^\dagger\widehat{\Omega}\Upsilon\right)_{\text{BRST}} = \left(\Psi, \widehat{\Omega}^2\Upsilon\right)_{\text{BRST}} \equiv 0 \quad \forall\,\Psi, \Upsilon \in \mathcal{V}_{\text{BRST}}\ .$$

Doing $\Psi = \Upsilon$, the leftmost expression would imply $\widehat{\Omega}\Psi = 0$ for all Ψ in the domain of $\widehat{\Omega}$ whenever a definite inner product $(\cdot\,,\cdot)_{\text{BRST}}$ is used.

As a consequence of the observation above, it is sometimes (wrongly) claimed that the vector space $\mathcal{V}_{\text{BRST}}$ is then indefinite. However, from an inner product space which is not definite one can only say that $(v, v) = 0$ does not imply $v = 0$; nevertheless, this does not mean that positive as well as negative vectors exist in such a space. Another ingredient is needed to ensure that $\mathcal{V}_{\text{BRST}}$ is indefinite (or equivalently, not semi–definite): the nondegeneracy of the inner product[14].

[14] An arbitrary inner product space V is said to be *nondegenerate inner product space* iff the zero vector is the only perpendicular vector to all elements in V. [74]

Let $\mathcal{V}_{\mathrm{BRST}}$ be a nondegenerate and not definite inner product space. For a general inner product space V, a subspace W may have non–zero intersection with its orthogonal companion[15] $W^{\perp}$. The subspace $W \cap W^{\perp} \equiv W^0$ is called the *isotropic part of* W. The isotropic part of the whole space is $V \cap V^{\perp} = V^{\perp} = V^0$. By definition, for a nondegenerate $V = \mathcal{V}_{\mathrm{BRST}}$ one has that $\mathcal{V}_{\mathrm{BRST}}^{\perp} = \{0\} = \mathcal{V}_{\mathrm{BRST}}^0$, that is, the isotropic part of $\mathcal{V}_{\mathrm{BRST}}$ consists of only the zero vector, which is obviously a neutral vector. There are more neutral vectors in $\mathcal{V}_{\mathrm{BRST}}$, namely all elements in the image of $\widehat{\Omega}$, $\mathrm{Im}(\widehat{\Omega})$, which are not in the isotropic part of $\mathcal{V}_{\mathrm{BRST}}$. Not all neutral elements of $\mathcal{V}_{\mathrm{BRST}}$ are in $\mathcal{V}_{\mathrm{BRST}}^0$. On the other hand, the contrapositive version of the Lemma 4.4 by J. Bognár [74] reads

Lemma 4.1 *If the isotropic part V^0 of V does not consist of all neutral elements in V, V is a not semi–definite inner product space .*

Therefore, one has that $\mathcal{V}_{\mathrm{BRST}}$ is not semi–definite, that is, it is indefinite.

In conclusion, the following theorem has been proved:

Theorem 4.2.1 *To non–trivially represent a BRST operator $\widehat{\Omega}$, an inner product space $\mathcal{V}_{\mathrm{BRST}}$ with a non–definite inner product $(\cdot\,,\cdot)_{\mathrm{BRST}}$ is required. If the inner product is nondegenerate, hence the BRST state space $\mathcal{V}_{\mathrm{BRST}}$ is necessarily an indefinite inner product space. Elements of $\mathcal{V}_{\mathrm{BRST}}$ are named BRST states and accordingly they are either positive, or negative, or neutral.*

An immediate consequence of this theorem is that it prohibits the BRST state space $\mathcal{V}_{\mathrm{BRST}}$ from having a genuine Hilbert space structure. On purely operational grounds, densely defined operators acting on $\mathcal{V}_{\mathrm{BRST}}$ are meaningless, unless some topology is input from the outset. Hence, operators will be treated formally in the BRST quantum analysis.

In what follows it will be assumed that the vector space $\mathcal{V}_{\mathrm{BRST}}$ splits as a sum of eigenspaces of the ghost number operator $\widehat{G}$ with definite real ghost number p

$$\mathcal{V}_{\mathrm{BRST}} = \bigoplus_p \mathcal{V}_p \ , \quad \widehat{G}\Psi_p = p\,\Psi_p \ , \quad (\forall\,\Psi_p \in \mathcal{V}_p) \ . \tag{4.2.8}$$

This property holds in particular for the Schrödinger representation as will be shown in the models we consider in the following chapters. Although it may sound strange that the anti–hermitian ghost number operator (4.2.4) has pure real eigenvalues, this is fully

[15] The *orthogonal companion* of any subset $\mathcal{U} \subset V$ is the subspace $\mathcal{U}^{\perp} := \{v \in V : (v,u) = 0\ \forall\, u \in \mathcal{U}\}$. The vector space V is a *degenerate inner product space* when is not nondegenerate [74]. The vector space V is a nondegenerate inner product space iff its orthogonal companion only consists of the zero vector.

consistent in the presence of an *indefinite metric*: it holds for any system with bosonic and/or fermionic constraints when the corresponding ghosts are quantised in the Fock representation [65].

Any operator $\widehat{A} \in \mathcal{B}^{(\star)}_{\mathrm{aux}}$, which abstractly corresponds to a polynomial in ghost and ghost–momentum operators with zero ghost numbered operator coefficients, has the decomposition

$$\widehat{A} = \sum_g \widehat{A}_g \ , \quad [\widehat{G}, \widehat{A}_g\,] = g\widehat{A}_g \ , \ g \in \mathbb{Z} \tag{4.2.9}$$

with g an integer by virtue of (4.2.5).

Based on the nondegenerate nature of $(\cdot\,,\cdot)_{\mathrm{BRST}}$ and the ghost decomposition (4.2.9) the following theorem, fully proved in [65], holds:

Theorem 4.2.2

(a) The scalar product of two states Ψ_p and $\Psi_{p'}$ with respective ghost number p and p' vanishes if $p + p' \neq 0$

$$\left(\Psi_p, \Psi_{p'}\right)_{\mathrm{BRST}} = 0, \quad p + p' \neq 0 \ . \tag{4.2.10}$$

(b) The ghost number of states Ψ_p is either integer or half integer.

If the system consists of an even number of constraints, as in the non–minimal BRST formalism, there is no fractionalisation of the ghost number, only integers are present [65]. BRST states with non–vanishing ghost number are neutral in $\mathcal{V}_{\mathrm{BRST}}$.

As in Dirac's strategy a prescription to extract physical states out of the BRST states in $\mathcal{V}_{\mathrm{BRST}}$ must be implemented. Following the original proposal by T. Kugo and I. Ojima [73], $\widehat{\Omega}$ becomes the BRST physical state selector

$$\widehat{\Omega}\Psi_{\mathrm{phys}} = 0 \tag{4.2.11a}$$

which is complemented with a zero ghost number condition on physical states

$$\widehat{G}\Psi_{\mathrm{phys}} = 0 \ . \tag{4.2.11b}$$

The quantum BRST observables are defined by ghost number zero, BRST invariance, and hermiticity,

$$[\widehat{G}, \widehat{A}\,] = 0 \ , \tag{4.2.12a}$$

$$[\widehat{A}, \widehat{\Omega}\,] = 0 \ , \tag{4.2.12b}$$

$$\widehat{A}^\star = \widehat{A} \ . \tag{4.2.12c}$$

Due to the nilpotency of $\widehat{\Omega}$, solutions to the Eq. (4.2.11a) need further identification. Any state of the form $\widehat{\Omega}X$ obeys (4.2.11a), these are termed *BRST–exact states.* These states have the following properties: (*i*) They have zero norm; (*ii*) they have vanishing inner product with any physical state; and, (*iii*) the 'expectation value' of any BRST quantum observable between a BRST–exact state and a physical state vanishes. These characteristics suggest that two physical states differing by a BRST–exact state must be identified. The transformation

$$\Psi \;\rightarrow\; \Psi' := \Psi + \widehat{\Omega}X \tag{4.2.13}$$

is sometimes called *quantum BRST gauge transformation* [124], and states related by a BRST gauge transformation are called *BRST gauge–equivalent.*

Remarks

1. One can infer directly from the condition (4.2.12b) that BRST observables map physical states onto physical states.

2. The formulation of the statement that $\{K, \Omega\}$ is classically a trivial BRST observable (see Eq. (3.6.16)) finds its quantum counterpart. Given an odd operator $\widehat{K}$ of ghost number minus one, $[\widehat{G}, \widehat{K}] = -\widehat{K}$, and anti–hermitian, $\widehat{K}^\dagger = -\widehat{K}$, the operator
$$\widehat{A}' := \widehat{A} + i[\,\widehat{\Omega}, \widehat{K}\,] \tag{4.2.14}$$
constructed from the BRST quantum observable $\widehat{A}$ is also a quantum observable. $\widehat{A}'$ is BRST closed, $[\widehat{A}', \widehat{\Omega}] = 0$, has ghost number zero, $[\widehat{G}, \widehat{A}'] = 0$, and is self–adjoint, $\widehat{A}^\dagger = \widehat{A}$.

3. It is not difficult to see that the trivial observables $[\,\widehat{\Omega}, \widehat{K}\,]$ have vanishing matrix elements between physical states
$$\left(\Psi^{(1)}_{\text{phys}}, [\,\widehat{\Omega}, \widehat{K}\,]\Psi^{(2)}_{\text{phys}}\right)_{\text{BRST}} \equiv 0$$
as a consequence of the self–adjointness of $\widehat{\Omega}$.

4. If one considers the formal expressions of the quantum counterpart of the BRST extensions to the constraints G_α, equation (3.6.17)
$$\widehat{G}^{BRST}_\alpha := i[\widehat{\mathcal{P}}_\alpha, \widehat{\Omega}]\ , \tag{4.2.15}$$
then $\widehat{G}^{BRST}_\alpha \Psi_{\text{phys}} \equiv 0$ modulo a BRST–exact state. Hence, the BRST quantum constraints act trivially on the space of equivalence classes of physical states.

5. The condition (4.2.11b) seems to be satisfactory in the non–minimal BRST formalism. The inclusion of Lagrange multipliers and their conjugate momenta, doubles the number of original first class constraints of the theory from m to $2m$. Hence, no fractionalisation of ghost number is present. Moreover, this condition will become important to us in order to make a connection between the canonical BRST–quantisation and RAQ. ▲

4.2.2 Physical inner product: Batalin–Marnelius–Shvedov proposal

One could say that the physical condition (4.2.11a) in the BRST formalism needs a similar treatment as the one applied to the Dirac condition (4.1.1) in RAQ. However, an interpretation of the BRST condition (4.2.11a) in terms of generalised vectors does not seem directly achievable. First, unlike the space of states $\mathcal{H}_{\text{aux}}$ present in RAQ, the vector space $\mathcal{V}_{\text{BRST}}$ neither is a pre–Hilbert space nor some topology to it has been attached, so there is no direct notion of distance between BRST states, which in turns does not permit a proper definition of a dense test state space $\Phi_{\text{BRST}} \subset \mathcal{V}_{\text{BRST}}$ where $\widehat{\Omega}$ could act on. Second, $\widehat{\Omega}$ itself has not been densely defined, the self–adjointness property of $\widehat{\Omega}$ was only done at a formal level. Third, a Gelfand triple structure out of $\mathcal{V}_{\text{BRST}}$ and a dense subset of the domain of $\widehat{\Omega}$ looks then by no means straightforward.

Instead of pursuing the interesting challenge that would signify to give a consistent distributional interpretation to the BRST physical condition and then seek an inner product which pairs BRST physical states, here it has been opted for following Batalin–Marnelius and Shvedov ideas on the issue [82, 84, 85, 89, 90, 124]. In practice, physical states Ψ_{phys} do not have a well defined norm (neither positive, or negative) in the BRST inner product $(\cdot\,,\cdot)_{\text{BRST}}$. The physical conditions (4.2.11) forbid vector states, typically wave functions on (super) configuration space, to depend on *all* configuration variables (ghost among them), therefore BRST physical states have ill–defined square norm. To be precise, in a Schrödinger representation, the inner product $(\cdot\,,\cdot)_{\text{BRST}}$ involves an integration over all ghosts and configuration variables. Hence, for BRST physical states, norms proportional to the meaningless $\infty \cdot 0$ are usual. The infinity coming from an integration over the bosonic Lagrange multipliers, when they are assumed to take values all over the real line, while zero results from integration over the fermionic ghosts. If by any reason the integration of Lagrange multipliers is performed over a compact set, BRST physical states are neutral vectors whose probabilistic interpretation is empty.

In order to define a positive definite inner product in the physical subspace, Batalin–Marnelius–Shvedov's strategy uses the room that is left by the arbitrariness in the physical states due to the existence of BRST–exact states (4.2.13). A gauge fixing procedure within the BRST formalism is introduced. A gauge transformed physical state might contain ghosts and other non–physical configuration variables. The main idea is to evaluate the inner product $(\cdot\,,\cdot)_{\mathrm{BRST}}$ between two physical states and obtain a regularised result by selecting specifically chosen gauge–equivalent physical states which have a well–defined inner product. In the process of choosing those representatives is where the gauge fixing plays a central role. This ansatz of constructing a positive definite inner product on the space of physical states from $(\cdot\,,\cdot)_{\mathrm{BRST}}$ by only choosing suitable gauge–equivalent physical states, as one may suspect, might exhibit Gribov obstructions due to possible non–trivial topology of the gauge orbits in the configuration space; these have already been pointed out by N. Düchting, F. G. Scholz, S. V. Shabanov, and T. Strobl in [92, 91].

Batalin–Marnelius–Shvedov's ansatz to construct a positive definite inner product on the physical state space relies on the following observation [82]: Given a physical state, Ψ^0_{phys}, the transformed states

$$\Psi_{\mathrm{phys}} := e^{[\widehat{\Omega},\widehat{K}]}\,\Psi^0_{\mathrm{phys}}\ , \tag{4.2.16}$$

with $\widehat{K}$ an operator with properties to be specified, yield (at least up to global issues) the whole class of gauge equivalent states to Ψ^0_{phys}. Indeed, formally, for a general $\widehat{K}$ the $\exp\,[\,\widehat{\Omega},\widehat{K}\,]$ differs from the identity by a BRST–exact operator

$$\exp\,[\widehat{\Omega},\widehat{K}] = \mathbb{1} + [\widehat{L},\widehat{\Omega}]\ , \tag{4.2.17}$$

where

$$\widehat{L} := (-)^{\epsilon_K+1}\left(\sum_{k=0}\frac{1}{(k+1)!}[\widehat{\Omega},\widehat{K}]^k\right)\widehat{K}\ \equiv\ (-)^{\epsilon_K+1}\left(\frac{\exp([\widehat{\Omega},\widehat{K}])-\mathbb{1}}{[\widehat{\Omega},\widehat{K}]}\right)\widehat{K}\ .$$

Hence, Ψ_{phys} on the LHS of (4.2.16) is BRST gauge–equivalent to Ψ^0_{phys}

$$\Psi_{\mathrm{phys}} := \Psi^0_{\mathrm{phys}} + \widehat{\Omega}X\ , \tag{4.2.18}$$

where $X := \widehat{L}\Psi^0_{\mathrm{phys}}$.

In Batalin–Marnelius–Shvedov's proposal it is supposed that the gauge fixing fermion $\widehat{K}$ is an odd hermitian operator of ghost number -1 [85, 87, 86, 89, 90, 124]. As a consequence, $\widehat{V} := \exp\,[\widehat{\Omega},\widehat{K}]$ becomes (formally) self–adjoint, so that the inner product between two physical states (4.2.16), or equivalently (4.2.18), becomes

$$\big(\Psi_{\mathrm{phys}},\Psi'_{\mathrm{phys}}\big)_{\mathrm{BM}} := \Big(\Psi^0_{\mathrm{phys}},\,e^{2[\widehat{\Omega},\widehat{K}]}\,\Psi'^0_{\mathrm{phys}}\Big)_{\mathrm{BRST}}\ . \tag{4.2.19}$$

In the scheme provided in [90], the space of states to which Ψ^0_{phys} belongs to is characterised either to be the set of physical states obtained in the Dirac approach, or, to correspond to states with no dependence on Lagrange multipliers and ghosts (these states are also called trivial ghost fixed solutions or *trivial BRST invariant states*). Since BRST–exact states have vanishing inner product with physical states, $\left(\Psi^0_{\text{phys}}, \widehat{\Omega} Y\right)_{\text{BRST}} = 0$, formally the RHS of the Eq. (4.2.19) corresponds to the ill–defined inner product $\left(\Psi^0_{\text{phys}}, \Psi'^0_{\text{phys}}\right)_{\text{BRST}}$. Nevertheless, the inclusion of $e^{2[\widehat{\Omega},\widehat{K}]}$ is chosen to bring back ghosts and Lagrange multipliers that may turn $\left(\Psi_{\text{phys}}, \Psi'_{\text{phys}}\right)_{\text{BM}}$ into a well defined inner product. Hence, the reason for introducing the intermediate states Ψ^0_{phys} is twofold: First, these states are much simpler than the states Ψ_{phys}, and, second, the introduction of $e^{[\widehat{K},\widehat{\Omega}]}$ will supply extra terms in the measure involved in the definition of $(\cdot\,,\cdot)_{\text{BRST}}$ that will act as a regulator to make the inner product between the states Ψ_{phys} and Ψ'_{phys} well defined.

A consequence of the hermitian nature of $e^{[\widehat{\Omega},\widehat{K}]}$ is that the inner product (4.2.19) has the seemingly desirable property

$$\left(\Psi_{\text{phys}}, \Psi'_{\text{phys}}\right)_{\text{BM}} = \left(\Psi'_{\text{phys}}, \Psi_{\text{phys}}\right)^*_{\text{BM}} , \tag{4.2.20}$$

a relation that originated the choice of a *hermitian* gauge fixing fermion in Marnelius' work, however, it brings issues over the Lagrange multipliers which were drawn by specific examples ([84]: Sect. 9).

An additional imaginary factor in the exponential argument on the RHS of (4.2.19), say inserted through $\widehat{K} = i\widehat{\varrho}/2$ with $\widehat{\varrho} = \widehat{\varrho}^\dagger$, would formally spoil (4.2.20). This was already noted in the original proposal by R. Marnelius and M. Ögren ([82]: Eq. (3.2)), where

$$\left(\Psi^0_{\text{phys}}, \Psi'^0_{\text{phys}}\right)^{\varrho}_{\text{BRST}} := \left(\Psi^0_{\text{phys}}, e^{i[\widehat{\Omega},\widehat{\varrho}]}\, \Psi'^0_{\text{phys}}\right)_{\text{BRST}} \tag{4.2.21}$$

was defined as an ansatz for a physical inner product, with $i\widehat{\varrho}$ an odd *anti*–hermitian gauge fixing fermion of ghost number -1. An additional phase factor was therefore added in that reference in order to recover the hermitian property (4.2.20) of the inner product (4.2.21).

The RHS of (4.2.21) also corresponds to the ill–defined inner product between Ψ^0_{phys} and Ψ'^0_{phys}, the (formal) anti–hermiticity and odd parity the gauge fixing fermion $i\widehat{\varrho}$ do not play any role in this respect. *In the subsequent chapters we argue, through some examples, that this anti–hermitian choice for the gauge fixing fermion is good enough in order to derive the group averaging formula from the canonical BRST–quantisation.* When the ghost variables are integrated out from (4.2.21), we will recover the hermiticity property at the level of the physical inner product in the RAQ context.

Remarks

1. To define the notion of convergence in a BRST state space, it has been proposed

to merge $\mathcal{V}_{\mathrm{BRST}}$ into some genuine Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$ such that the BRST indefinite inner product is defined through $(\Psi, \Upsilon)_{\mathrm{BRST}} := (\Psi, \mathfrak{J}\Upsilon)$, where $\mathfrak{J}$ is a self–adjoint operator, with the property $\mathfrak{J}^2 = \mathbb{1}$. In this point of view, the BRST state space is said to be a Krein space [74, 80]. Topology in a Krein space is defined with the help of the norm generated by the positive definite inner product in the Hilbert space $\mathcal{H}$. ▲

CHAPTER 5

Constraint Rescaling in Refined Algebraic Quantisation: Momentum Constraint

The infinity would be due to the volume of the gauge group, and a finite inner product would be obtained by dividing the infinite inner product by the volume of the gauge group.

– Atsushi Higuchi, 1991

In this chapter the quantisation of a system with a single constraint will be discussed. We address the specific question of rescaling a classical constraint by somewhat general functions on the configuration space. Using an anti–hermitian gauge fixing fermion, the hybrid BRST–RAQ point of view developed in [124] is employed to quantise the system. We recognise and provide the additional auxiliary structures to make technically rigorous statements on the physical quantum sector. The chapter is divided into two parts.

In the first part, Sect. 5.1, the unscaled–constrained system is described. It consists of two degrees of freedom (θ, x) and an innocuous constraint $\gamma := p_\theta \approx 0$. The machinery of canonical BRST–quantisation as developed in the previous chapter is applied to this system. The analysis is relevant for the following reasons: First, in this model we show that is possible to obtain a tentative group averaging formula using BRST methods. A suitable choice of the gauge fixing operator and the use of trivial BRST physical states remove the fermionic variables form the Batalin–Marnelius' physical inner product (4.2.21). Second, given this group averaging formula motivated by BRST–quantisation, we clarify at which level of the process one must provide the auxiliary additional structures to perform the RAQ. An auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ and a test states space $\Phi \subset \mathcal{H}_{\text{aux}}$ are specified. Third, the methodology used in this simple example will serve as a guide to tackle the rescaled–constrained system.

In the second part, Sects. 5.2 to 5.6, the system with the two degrees of freedom (θ, x) constrained by the rescaled constraint $\phi := M(\theta, x)\, p_\theta \approx 0$ is considered. Using as a guide the analysis of the *un*scaled–constrained system, a BRST motivated group averaging formula is obtained after a specific choice of the gauge fixing fermion. This sesquilinear from is supplemented with the corresponding auxiliary structures to perform the RAQ. Three cases arise depending on the asymptotic nature of the scaling function, these are: (i) Refined algebraic quantisation is equivalent to that developed in Sect. 5.1; (ii) refined algebraic quantisation fails, mainly due to the nonexistence of self–adjoint extensions of the constraint operator; (iii) a quantisation ambiguity arises from the choice of a self–adjoint extension to the constraint operator, its resolution determines a superselection structure of the physical Hilbert space.

5.1 Momentum constraint system

This section has a methodological character. Here we review the system with configuration space $\mathbb{Q} \equiv \mathbb{R}^2 = \{(\theta, x)\}$ and phase space $\mathbf{T}^*\mathbb{Q} = \mathbf{T}^*\mathbb{R}^2 = \{(\theta, x, p_\theta, p_x)\} \simeq \mathbb{R}^4$ subject to the trivial constraint

$$\gamma := p_\theta \approx 0 \ . \tag{5.1.1}$$

The constraint γ generates a translational gauge symmetry along the θ–direction. The constraint surface is hence $\Gamma := \{(\theta, x, 0, p_x)\} \simeq \mathbb{R}^3$. The gauge orbits are straight lines parallel to the θ–axis. The vector field associated to the constraint (5.1.1), $Y := \partial_\theta$, is complete on the phase space. In particular, on the configuration space $\mathbb{R}^2$, any curve $\varsigma^{(p)}(t) = (t + a, b)$, well defined for all $t \in \mathbb{R}$, is an integral curve of Y with generic starting point $\varsigma(0) = p = (a, b)$. Accordingly, Y defines a global flow (one–parameter group action) on $\mathbb{R}^2$, that is, a smooth left action of th group $(\mathbb{R}, +)$ on $\mathbb{R}^2$ denoted by $\varsigma : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 : (t, p) \mapsto \varsigma(t, p)$ such that

$$\varsigma(t, \varsigma(s, p)) = \varsigma(t + s, p) \ , \tag{5.1.2a}$$

$$\varsigma(0, p) = p \ , \tag{5.1.2b}$$

for all $s, t \in \mathbb{R}$ and $p \in \mathbb{R}^2$. For a fixed $p = (a, b) \in \mathbb{R}^2$, ς is defined via the integral curve $\varsigma^{(p)}$ as $\varsigma : \mathbb{R} \times \{p\} \to \mathbb{R}^2 : (t, p) \mapsto \varsigma(t, p) := \varsigma^{(p)}(t)$. For each $t \in \mathbb{R}$ ς is defined via the diffeomorphism $\varsigma_t : \mathbb{R}^2 \to \mathbb{R}^2 : (\theta, x) \mapsto \varsigma_t((\theta, x)) := (t + \theta, x)$ as

$\varsigma : \{t\} \times \mathbb{R}^2 \to \mathbb{R}^2 : (t,p) \mapsto \varsigma(t,p) := \varsigma_t(p)$. Having a constraint that defines a complete vector field on the configuration space becomes utterly important at a quantum level in the Schrödinger representation.

We assume that there is no true Hamiltonian, although its inclusion would be straightforward: any H independent of θ preserves the translational symmetry.

5.1.1 Canonical BRST analysis

Using the ghost/antighost notation and conventions established in Sect. 3.4.1, the super phase space is $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta, \mathcal{P})\} = \{(\theta, x, \lambda, p_\theta, p_x, \pi)\} \times \{(C, \bar{C}, \bar{\rho}, \rho)\}$. The reality conditions on the canonical pairs are

$$\theta^* = \theta\ , \quad x^* = x\ , \quad \lambda^* = \lambda\ , \quad C^* = C\ , \quad \bar{C}^* = \bar{C}^*\ , \tag{5.1.3a}$$

$$p_\theta^* = p_\theta\ , \quad p_x^* = p_x\ , \quad \pi^* = \pi\ , \quad \bar{\rho}^* = -\bar{\rho}\ , \quad \rho^* = -\rho\ , \tag{5.1.3b}$$

The non–trivial part of the canonical symplectic structure on $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta, \mathcal{P})\}$ reads

$$\{\theta, p_\theta\} = 1\ , \quad \{x, p_x\} = 1\ , \quad \{\lambda,\ \pi\} = 1\ , \qquad \text{(bosonic)}\ , \tag{5.1.4a}$$

$$\{C, \bar{\rho}\} = -1\ , \quad \{\bar{C}, \rho\} = -1\ , \qquad \text{(fermionic)}\ . \tag{5.1.4b}$$

In addition, $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta, \mathcal{P})\}$ acquires the extra constraint $\pi \approx 0$. Both γ and π form an abelian first–class constraint set.

In view of the results provided in Sect. 3.4.2, *cf.* Eq. (3.4.25), the nonminimal BRST generator and the ghost number function in this system read

$$\Omega = p_\theta C - i\rho\pi\ , \tag{5.1.5}$$

$$G = i(C\bar{\rho} + \rho\bar{C})\ . \tag{5.1.6}$$

These key functions on the super phase space can be built from $\theta, x, \lambda, C, \bar{C}$ and their conjugate pairs, then we let all these ten variables to span the subspace $\mathscr{S}_{\mathrm{BRST}}$. Each element in $\mathscr{S}_{\mathrm{BRST}}$ is an elementary classical variable.

The abstract (free associative) algebra of quantum operators $\mathcal{B}_{\mathrm{aux}}$ is constructed from the basic operators once the graded commutation relations

$$[\widehat{\theta}, \widehat{p}_\theta] = [\widehat{x}, \widehat{p}_x] = [\widehat{\lambda}, \widehat{\pi}\,] = i\ , \tag{5.1.7a}$$

$$[\widehat{C}, \widehat{\bar{\rho}}\,] = [\widehat{\bar{C}}, \widehat{\rho}] = -i\ , \tag{5.1.7b}$$

have been input; these relations are in correspondence with i times the basic super–PBs (5.1.4). Based on the reality conditions of the super manifold $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta, \mathcal{P})\}$ (5.1.3),

$\mathcal{B}_{\rm aux}^{(\star)}$ is built from the inclusion of the $\star$–relations

$$\widehat{\theta}^{\star} = \widehat{\theta}\ , \quad \widehat{x}^{\star} = \widehat{x}\ , \quad \widehat{\lambda}^{\star} = \widehat{\lambda}\ , \quad \widehat{C}^{\star} = \widehat{C}\ , \quad \widehat{\bar{C}}^{\star} = \widehat{\bar{C}}\ , \tag{5.1.8a}$$

$$\widehat{p}_{\theta}^{\star} = \widehat{p}_{\theta}\ , \quad \widehat{p}_{x}^{\star} = \widehat{p}_{x}\ , \quad \widehat{\pi}^{\star} = \widehat{\pi}\ , \quad \widehat{\bar{\rho}}^{\star} = -\widehat{\bar{\rho}}\ , \quad \widehat{\rho}^{\star} = -\,\widehat{\rho}\ . \tag{5.1.8b}$$

The Schrödinger representation in which the wave functions depend on the bosonic coordinates (θ, x, λ) and the fermionic momenta $(\bar{\rho}, \rho)$ is chosen. Due to the anticommutative nature of the Grassmann variables $(\bar{\rho}, \rho)$, in this representation any BRST wave function can be expanded as

$$\Psi(\theta, x, \lambda, \bar{\rho}, \rho) = \psi(\theta, x, \lambda) + \Psi^{1}(\theta, x, \lambda)\bar{\rho} + \Psi_{1}(\theta, x, \lambda)\rho + \Psi_{1}^{1}(\theta, x, \lambda)\bar{\rho}\rho\ , \tag{5.1.9}$$

where ψ, Ψ^1, Ψ_1 and Ψ_1^1 are complex–valued functions. The action of the fundamental operators on the BRST states (5.1.9) reads

$$\widehat{\theta}\,\Psi := \theta\Psi\ , \qquad \widehat{p}_{\theta}\,\Psi := -i\frac{\partial\Psi}{\partial\theta}\ , \tag{5.1.10a}$$

$$\widehat{x}\,\Psi := x\Psi\ , \qquad \widehat{p}_{x}\,\Psi := -i\frac{\partial\Psi}{\partial x}\ , \tag{5.1.10b}$$

$$\widehat{\lambda}\,\Psi := \lambda\Psi\ , \qquad \widehat{\pi}\,\Psi := -i\frac{\partial\Psi}{\partial\lambda}\ , \tag{5.1.10c}$$

$$\widehat{C}\,\Psi := -i\frac{\partial^{\ell}\Psi}{\partial\bar{\rho}}\ , \qquad \widehat{\bar{\rho}}\,\Psi := \bar{\rho}\Psi\ , \tag{5.1.10d}$$

$$\widehat{\bar{C}}\,\Psi := -i\frac{\partial^{\ell}\Psi}{\partial\rho}\ , \qquad \widehat{\rho}\,\Psi := \rho\Psi\ , \tag{5.1.10e}$$

where the superscript ℓ on the fermionic derivative stands for left derivative, see Sect. 3.4. This choice is compatible with the graded commutation relations (5.1.7).

In order to promote the $\star$–relations (5.1.8) into hermitian conditions, a sesquilinear form must be introduced. Typically for two BRST states Ψ and Υ, one pairs them as follows [75]:

$$(\Psi, \Upsilon)^{c}_{\rm BRST} := c\int {\rm d}\lambda\,{\rm d}\theta\,{\rm d}x\,{\rm d}\bar{\rho}\,{\rm d}\rho\ \Psi^{*}(\theta, x, \lambda, \bar{\rho}, \rho)\Upsilon(\theta, x, \lambda, \bar{\rho}, \rho)\ , \tag{5.1.11}$$

where c is a nonzero constant that may *a priori* take complex values. As usual the integral over fermionic variables has the properties

$$\int {\rm d}\bar{\rho}\,{\rm d}\rho = \int {\rm d}\bar{\rho}\,{\rm d}\rho\ \bar{\rho} = \int {\rm d}\bar{\rho}\,{\rm d}\rho\ \rho = 0\ , \quad \int {\rm d}\bar{\rho}\,{\rm d}\rho\ \rho\bar{\rho} = 1\ . \tag{5.1.12}$$

The sesquilinear form (5.1.11) has remarkable properties independently of the value taken by c. First, $(\cdot\,,\cdot)^{c}_{\rm BRST}$ is compatible with the $\star$–relations (5.1.8), in the sense that fermionic momenta operators $\widehat{\bar{\rho}}$ and $\widehat{\rho}$ are anti–hermitian and all the other fundamental operators in (5.1.10) are hermitian. Second, the nilpotent BRST operator and the ghost number operator

$$\widehat{\Omega} := \widehat{p}_\theta \, \widehat{C} - i\widehat{\rho}\,\widehat{\pi} \;, \tag{5.1.13}$$
$$\widehat{G} := i(\widehat{\rho}\,\widehat{\bar{C}} - \widehat{\bar{\rho}}\,\widehat{C}) \;, \tag{5.1.14}$$

whose action on BRST wave functions can be directly inferred from (5.1.10), are respectively and in reference to $(\cdot\,,\cdot)^c_{\rm BRST}$, hermitian and anti–hermitian. Third, from the hermiticity of $\widehat{\Omega}$ it follows that the product $(\cdot\,,\cdot)^c_{\rm BRST}$ between physical states depends on the states only through their gauge–equivalence class (4.2.13).

We emphasise here that the domain of each operator remains unspecified throughout the BRST analysis, hence all (anti) hermiticity considerations are left formal until we have access to a genuine Hilbert space.

When c is chosen to be an imaginary number, $c = i\alpha$ with $\alpha \in \mathbb{R}$, the sesquilinear form (5.1.11) fulfils[1]

$$(\Psi, \Upsilon)^{i\alpha}_{\rm BRST} = (\Upsilon, \Psi)^{i\alpha}_{\rm BRST}{}^* \;, \quad \alpha \in \mathbb{R} \;. \tag{5.1.15}$$

Hence $(\cdot,\cdot)^{i\alpha}_{\rm BRST}$ is a hermitian inner product on the BRST state space

$$\mathcal{V}_{\rm BRST} := \{\Psi = \psi + \Psi^1\bar{\rho} + \Psi_1\rho + \Psi^1_1\bar{\rho}\rho\} \;. \tag{5.1.16}$$

In the inner product $(\cdot\,,\cdot)^{i\alpha}_{\rm BRST}$ natural definitions of positive and negative vectors arise; see footnote 13 in page 87. The vector space $(\mathcal{V}_{\rm BRST}, (\cdot\,,\cdot)^{i\alpha}_{\rm BRST})$ becomes an infinite dimensional and indefinite inner product space containing positive as well as negative vectors.

In contrast, when c is real, $c = \alpha \in \mathbb{R}$, the sesquilinear form (5.1.11) satisfies

$$(\Psi, \Upsilon)^{\alpha}_{\rm BRST} = -\,(\Upsilon, \Psi)^{\alpha}_{\rm BRST}{}^* \;, \quad \alpha \in \mathbb{R} \;, \tag{5.1.17}$$

defining a skew–hermitian inner product in $\mathcal{V}_{\rm BRST}$. Replacing each value of the inner product (5.1.17) by $(\Psi, \Upsilon)^{\alpha}_{\rm BRST}{}' := i\,(\Psi, \Upsilon)^{\alpha}_{\rm BRST}$, a hermitian inner product space is recovered. Hence, the definitions of positive and negative vector become available. It is obvious that any operator which is self–adjoint in the space $(\mathcal{V}_{\rm BRST}, (\cdot\,,\cdot)^{\alpha}_{\rm BRST})$, so is in the vector space $(\mathcal{V}_{\rm BRST}, (\cdot\,,\cdot)^{\alpha}_{\rm BRST}{}')$. Therefore, if one opts for $c = \alpha$ in (5.1.11),

[1] For two general BRST states of the form (5.1.9), using (5.1.12), it is easy to see that

$$\int \mathrm{d}\bar{\rho}\,\mathrm{d}\rho\;\Psi^*\Upsilon = -\left(\int \mathrm{d}\bar{\rho}\,\mathrm{d}\rho\;\Upsilon^*\Psi\right)^* \;.$$

a nilpotent and self–adjoint BRST operator $\widehat{\Omega}$ defined on $(\mathcal{V}_{\mathrm{BRST}}, (\cdot\,,\cdot)^{\alpha}_{\mathrm{BRST}})$ will automatically be defined (with the same properties) on the hermitian inner product space $(\mathcal{V}_{\mathrm{BRST}}, (\cdot\,,\cdot)^{\alpha}_{\mathrm{BRST}}{}')$, which in turn must be an indefinite inner product space.

To sum up: In the momentum constraint model, independently of which of the two choices $c = i\alpha$ or $c = \alpha$ is done, $\widehat{\Omega}$ can always be represented on an indefinite inner product state space (*cf.* Theorem 4.2.1). Later on, it will be recognised that the choice $c = \alpha \equiv 1$ is more suitable in deriving the group averaging formula for the system of a single momentum constraint.

Now, the hermitian and nilpotent BRST operator (5.1.13) together with the ghost number operator (5.1.14), both written in a $(\bar{\rho}C, \rho\bar{C})$–order, trivially annihilate ghost–free states of the form[2]

$$\Psi^0_{\mathrm{phys}} = \psi(\theta, x)\ , \quad \Upsilon^0_{\mathrm{phys}} = \chi(\theta, x)\ , \tag{5.1.18}$$

where the λ–independence follows from the nonminimal part of the BRST condition (4.2.11a). The sesquilinear form (5.1.11) between two states of the type (5.1.18) is ill–defined for any choice of c, hence some regularisation is needed. Using the skew–hermitian inner product $(\cdot\,,\cdot)^{c}_{\mathrm{BRST}}$, $c = \alpha \equiv 1$, a regularised scalar product from (4.2.21) will be constructed. A quite surprising result that will be derived below is that although $(\cdot\,,\cdot)^{\alpha=1}_{\mathrm{BRST}}$ is skew–hermitian (5.1.17) for BRST quantum states that contain fermionic information, the regularised inner product will be hermitian once the fermionic variables have been integrated out. In order not to clutter up the notation, in the rest of this section the superscript $\alpha = 1$ in $(\cdot\,,\cdot)^{\alpha=1}_{\mathrm{BRST}}$ will be omitted.

The regularised version of inner product will be obtained from (4.2.21), where we choose the anti–hermitian gauge fixing fermion to be $i\widehat{\varrho} \equiv -\widehat{\lambda}\widehat{\bar{\rho}}$. It follows that when acting on general BRST states (5.1.9), $i[\widehat{\Omega}, \widehat{\varrho}] = i\lambda\widehat{p}_\theta + \rho\bar{\rho}$. Then, after an elementary integration over the fermionic momenta

$$\big(\Psi^0_{\mathrm{phys}}, \Upsilon^0_{\mathrm{phys}}\big)^{\varrho}_{\mathrm{BRST}} = \int \mathrm{d}\lambda\, \mathrm{d}\theta\, \mathrm{d}x\ \psi^*(\theta, x)\big[\exp\left(i\lambda\widehat{p}_\theta\right)\chi\big](\theta, x)\ . \tag{5.1.19}$$

This formula comes with some comments. First, a remarkable consequence of choosing the gauge fixing fermion $i\widehat{\varrho}$ as anti–hermitian is that $\widehat{V} = \exp\big(i[\widehat{\Omega}, \widehat{\varrho}]\big)$ is unitary and the operator $\exp\left(i\lambda\widehat{p}_\theta\right)$ in (5.1.19) can be made unitary once the fermions no longer play any role in the scheme. If the gauge fixing fermion were chosen to be hermitian, say $i\widehat{\lambda}\widehat{\bar{\rho}}$, then both $\widehat{V}$ and $\exp(i\lambda\widehat{p}_\theta)$ would be hermitian. In order to bring back unitarity, λ should be regarded as imaginary bringing some issues with it already pointed out in [84,

[2] In a $(C\bar{\rho}, \bar{C}\rho)$–order, ghosts and antighosts at the left, the corresponding BRST physical states do match with Dirac physical states.

85]. Second, the expression (5.1.19) resembles the structure of the averaging formula over the translational group provided the operator $\exp(i\lambda\widehat{p}_\theta)$ and the λ–integration can be appropriately defined. Third, although Eq. (5.1.19) still needs a technical precise definition in order to be used in RAQ, it can be considered a starting point in the construction of a physical inner product in this scheme. One needs to make an accurate definition of the class of functions that enter into the expression (5.1.19), as well as, the range of values taken by the Lagrange multipliers, such that both simultaneously permit this integral to be well defined. Fourth, in contrast to the formal expression (4.2.21), once the integral (5.1.19) becomes well defined as a group averaging within RAQ, hermiticity at the level of the inner product will be recovered in the sense of the condition (4.1.7b). Fifth, in the literature, alternative ghost/antighost conventions can be used, for instance, where $(C^a, \bar{\rho}_a)$ are real and $(\bar{C}_a, \rho^a)$ are purely imaginary [124]; in this case some factors of i in the super phase space symplectic form are added and, at a quantum level, they imply the need of c real in order to make the sesquilinear form (5.1.11) hermitian. Nevertheless the conclusion drawn about hermitian versus anti–hermitian choice for $i\widehat{\varrho}$ is independent of which convention is adopted.

5.1.2 Refined algebraic quantisation

In this section, the expression (5.1.19) will be used to construct a rigging map within RAQ. The task is to supplement the required structures in order to interpret (5.1.19) as a group averaging formula.

Following the steps listed in Sect. 4.1.2, we notice that the relevant classical functions on the original phase space $\mathbf{T}^*\mathbb{Q}$ are constructed from the canonical pairs (θ, p_θ) and (x, p_x). Hence let $\mathscr{S} :=$ span$\{1, \theta, x, p_\theta, p_x\}$ be the space mentioned in the step (1) of RAQ. The algebra $\mathcal{A}^{(\star)}_{\text{aux}}$ of abstract operators is constructed as prescribed in steps (2) and (3). The auxiliary Hilbert space $\mathcal{H}_{\text{aux}}$ is taken to be the space of square–integrable functions on the classical configuration space $\mathbb{R}^2 = \{(\theta, x)\}$,

$$\mathcal{H}_{\text{aux}} := L^2(\mathbb{R}^2, \mathrm{d}\theta\mathrm{d}x) \ . \tag{5.1.20}$$

The space $\mathcal{H}_{\text{aux}}$ is endowed with the positive definite auxiliary inner product

$$(\psi, \chi)_{\text{aux}} := \int_{\mathbb{R}^2} \mathrm{d}\theta \, \mathrm{d}x \ \psi^*(\theta, x)\chi(\theta, x) \ . \tag{5.1.21}$$

The basic quantum coordinates operators ($\widehat{\theta}$ and $\widehat{x}$) act by multiplication, whereas the momentum operators ($\widehat{p}_\theta$ and $\widehat{p}_x$) act by differentiation on element of $\mathcal{H}_{\text{aux}}$. Therefore, all elementary operators, including the constraint operator, become essentially self–adjoint on the dense subspace $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions of compact support. Therefore

$$\widehat{U}(\lambda) := \exp(i\lambda\widehat{p}_\theta) \tag{5.1.22}$$

is an element of a one–parameter group of unitary operators $\{\widehat{U}(\lambda) : \lambda \in \mathbb{R}\}$. Let the test state space be $\Phi := C_0^\infty(\mathbb{R}^2)$, the action of the gauge group on it reads

$$(\widehat{U}(\lambda)\, f)(\theta, x) = f(\theta + \lambda, x) = \big(f \circ \varsigma_\lambda\big)(\theta, x) \ , \quad \forall\, f \in \Phi \ , \tag{5.1.23}$$

which shows that $\widehat{U}(\lambda)\Phi \subset \Phi$ for all $\lambda \in \mathbb{R}$.

We have specified the auxiliary structures $\mathcal{H}_{\rm aux}$ and Φ, and the unitary action of the gauge group on $\mathcal{H}_{\rm aux}$. It remains to specify the antilinear rigging map $\eta : \Phi \to \Phi'$, from Φ to its topological dual Φ'. The sesquilinear form (5.1.19) provides a group averaging in the present context if, firstly, the trivial BRST invariant states (5.1.18) are identified with genuine elements of $\Phi = C_0^\infty(\mathbb{R}^2)$, and, secondly, if (5.1.19) can be properly defined. In RAQ terms, we then have that (5.1.19) becomes

$$(f, g)_{\rm ga} := \int {\rm d}\lambda\ \Big(f, \widehat{U}(\lambda) g\Big)_{\rm aux} \equiv \int {\rm d}\lambda\ F(\lambda) \ , \tag{5.1.24}$$

with the $\lambda-$interval of integration to be determined and

$$F(\lambda) := \int_{\mathbb{R}^2} {\rm d}\theta\, {\rm d}x\ f^*(\theta, x)\, g(\theta + \lambda, x). \tag{5.1.25}$$

The expression (5.1.24) can be thought as the averaging over the translation group if the range of λ can be extended over the whole real line. The function $F(\lambda)$ represents a convergent integral for each $\lambda \in \mathbb{R}$, moreover, F is a continuous compactly supported function over the real line ([190]: Chap. VI). So F is Lebesgue integrable over all $\mathbb{R}$ and the range of integration in (5.1.24) is taken to be the full real axis; we seek hence for a rigging map of the form

$$\eta(f)[g] := (f, g)_{\rm ga} = \int_{\mathbb{R}^3} {\rm d}\lambda\, {\rm d}\theta\, {\rm d}x\ f^*(\theta, x)\, g(\theta + \lambda, x) \ . \tag{5.1.26}$$

The above structure determines the algebra of observables $\mathcal{A}^{(\star)}_{\rm phys}$ as those operators which together with their adjoints leave invariant the chosen test state space Φ. Observables should contain Φ within their domains and commute with $\widehat{U}(\lambda)$ for all $\lambda \in \mathbb{R}$.

From the discussion developed in Sect. 4.1.3, *cf.* Eq. (4.1.18b), the mapping given in Eq. (5.1.26) is a good candidate for a rigging map. What remains to be proven is that it actually satisfies the conditions listed in Sect. 4.1.2.

From the following chain of equalities:

$$\begin{aligned}\eta(f)[\widehat{U}^\dagger(\lambda) g] &= \int_{\mathbb{R}} {\rm d}\lambda'\ \Big(f, \widehat{U}^\dagger(\lambda)\widehat{U}(\lambda') g\Big)_{\rm aux} = \int_{\mathbb{R}} {\rm d}\lambda'\ \Big(f, \widehat{U}(\lambda' - \lambda) g\Big)_{\rm aux} \\ &= \int_{\mathbb{R}} {\rm d}\widetilde{\lambda}\ \Big(f, \widehat{U}(\widetilde{\lambda}) g\Big)_{\rm aux} \ ,\end{aligned}$$

the map η solves the constraints in the sense of (4.1.8). From the hermiticity of $(\cdot\,,\cdot)_{\text{aux}}$, the unitarity of $\widehat{U}$ and the reflection property of the integral, one can easily see that η is real as in (4.1.7b). To prove positivity of η, one uses the Fourier transform

$$f(\theta,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}k\, \widetilde{f}(k,x)e^{-ik\theta}\ , \tag{5.1.27}$$

with $\widetilde{f}(k,x)$ belonging to the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ [191]. In the momentum space, the mapping (5.1.26) is hence nothing but

$$\eta(f)[g] = 2\pi\int_{\mathbb{R}} \mathrm{d}x\ \widetilde{f}^{\,*}(0,x)\,\widetilde{g}(0,x)\ , \tag{5.1.28}$$

which immediately yields that η is positive in the sense of (4.1.7c). Finally, let $\widehat{A}\in\mathcal{A}^{(\star)}_{\text{phys}}$, since $\widehat{U}(\lambda)$ commutes with $\widehat{A}$ and $\widehat{A}^\dagger$ on Φ, for all $\lambda\in\mathbb{R}$, it results that η intertwines with observables. Therefore, the prescription (5.1.26) is a rigging map.

The space $\text{span}\{\eta(f):\ f\in\Phi\}$ solves the constraint (5.1.1) and by construction will be a dense subspace of $\mathcal{H}_{\text{phys}}$. The functional $\eta(f)$ in the momentum space reads

$$\eta(f) = 2\pi\int_{\mathbb{R}^2} \mathrm{d}k\,\mathrm{d}x\ \delta(k)\widetilde{f}^{\,*}(k,x)\quad \forall\, f\in\Phi\ , \tag{5.1.29}$$

so, the physical inner product defined by (4.1.9) takes the form

$$(\eta(g),\eta(f))_{\text{RAQ}} = \eta(f)[g] = 2\pi\int_{\mathbb{R}} \mathrm{d}x\ \widetilde{f}^{\,*}(0,x)\,\widetilde{g}(0,x)\ , \tag{5.1.30}$$

which is a non–trivial, hermitian, positive definite bilinear form. The physical space $\mathcal{H}_{\text{phys}}$ is constructed by the Cauchy completion $\overline{\text{Im}(\eta)}$. The averaging procedure projected out the $\theta-$dependence of the wave functions, and the physical Hilbert space became $L^2(\mathbb{R},\mathrm{d}x)$ as one would expect on intuitive basis.

Remarks

1. The rigging map (5.1.26) is defined up to a real constant a. The rescaled $\eta_a := a\eta$ multiplies with an overall factor the physical inner product and leads to an equivalent physical Hilbert space. This freedom can be traced back to the fact that the measure of the non–compact translational group is uniquely defined only up to a multiplicative factor (see footnote 11 in page 81). ▲

5.2 Rescaled momentum constraint

We now turn our attention to the rescaled version of the classical constraint (5.1.1), namely

$$\phi := M(\theta,x)\,p_\theta \approx 0\ . \tag{5.2.1}$$

This constraint is defined on the phase space $\mathbf{T}^*\mathbb{Q} = \{(\theta, x, p_\theta, p_x)\} \simeq \mathbb{R}^4$. For a system with a single–constraint, issues of the PB algebra play no role. The rescaled constraint ϕ shares the abelian algebra of γ (5.1.1). In order to maintain the regularity property and the polynomial structure in momenta of the unscaled constraint, we require the real–valued function M to be smooth and nowhere vanishing on the configuration space. We may assume without loss of generality that M is positive. Hence $\Gamma := \{(\theta, x, 0, p_x)\} \simeq \mathbb{R}^3$, the constraint surface, is preserved by the scaling.

The generator of gauge transformations on Γ is the restriction of the Hamiltonian vector field of ϕ,

$$X := M(\theta, x)\, \partial_\theta \ . \tag{5.2.2}$$

The integral curves of X have constant x and p_x, but they connect any two given values of θ. The reduced phase space is hence $\Gamma_{\text{red}} = \{(x, p_x)\} \simeq \mathbb{R}^2$.

If we wish to view the gauge transformations as maps on Γ, rather than just as maps of individual initial points in Γ, a subtlety arises. The gauge transformation with the (finite) parameter λ is the exponential map of λX, $\exp(\lambda X)$. If M satisfies

$$\int_{-\infty}^{0} \frac{\mathrm{d}\theta}{M(\theta, x)} = \infty = \int_{0}^{\infty} \frac{\mathrm{d}\theta}{M(\theta, x)} \tag{5.2.3}$$

for all x, then X is a complete vector field, and the family $\{\exp(\lambda X) : \lambda \in \mathbb{R}\}$ is a one–parameter group of diffeomorphisms $\Gamma \to \Gamma$ [185]. If the conditions (5.2.3) do not hold for all x, then X is incomplete. It is still true that the action of $\exp(\lambda X)$ on any *given* initial point in Γ is well defined for sufficiently small $|\lambda|$; however, there are no values of $\lambda \neq 0$ for which both of $\exp(\pm\lambda X)$ are defined as maps $\Gamma \to \Gamma$, since at least one of them will try to move points past the infinity. It is this classical subtlety whose quantum mechanical counterpart will be at the heart of our quantisation results.

To have an incomplete X does not require an exceptional function M. Take the smooth and nonvanishing $M(\theta, x) = e^\theta$, then the equations that define the integral curve on the configuration space, with starting point $p = (0, 1) \in \mathbb{R}^2$, are

$$\theta(t) = \ln\left(\frac{1}{1-t}\right) \ , \quad x(t) = 1 \ , \tag{5.2.4}$$

which are only well defined for $t < 1$, and *not* for all $t \in \mathbb{R}$. In contrast, from the trivial case $M(\theta, x) \equiv 1$ a complete vector field X arises; this corresponds to the Hamiltonian vector field of the unscaled constraint. Henceforth, we refer to M as the *scaling function*.

As before, there is no true Hamiltonian although its inclusion would be straightforward.

5.2.1 Canonical BRST analysis

The BRST canonical structure of the theory lies on the super phase space $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta,\mathcal{P})\} = \{(\theta,x,\lambda,p_\theta,p_x,\pi)\} \times \{(C,\bar{C},\bar{\rho},\rho)\}$, where the non–trivial part of the symplectic structure reads as in Eq. (5.1.4). In the ghost extended phase space, not only the rescaled constraint (5.2.1) is defined but also the *ad hoc* constraint $\pi \approx 0$ is introduced. Both forming an abelian set of first–class constraints.

The nonminimal BRST generator (see Sect. 3.4.2) and ghost number function of the system read

$$\Omega = M(\theta,x)\,p_\theta C - i\rho\,\pi \ , \tag{5.2.5}$$

$$G = i(C\bar{\rho} + \rho\,\bar{C}) \ . \tag{5.2.6}$$

We shall proceed to the canonical BRST–quantisation. The subspace $\mathscr{S}_{\mathrm{BRST}}$ of the vector space of all smooth, complex–valued functions on the super phase space, and the (free associative) $\star-$algebra of abstract operators $\mathcal{B}^{(\star)}_{\mathrm{aux}}$ are built as in Sect. 5.1.1; relations (5.1.7) and (5.1.8) are taken into account then. We realise the basic operators as in (5.1.10) on BRST wave functions in the Schrödinger representation (5.1.9). To finally set the arena on which the BRST operator will act, we endow the BRST vector space with the sesquilinear form (5.1.11), where $c = 1$ is taken for definiteness.

The BRST physical quantum states satisfy

$$\widehat{\Omega}\Psi = 0 \ , \quad \widehat{G}\Psi = 0 \ , \tag{5.2.7}$$

where the ghost number operator $\widehat{G}$ coincide with (5.1.14) and the nilpotent BRST operator in a $(\bar{\rho}C, \rho\bar{C})-$order is given by

$$\widehat{\Omega} := \widehat{\phi}\,C - i\widehat{\rho}\,\widehat{\pi} \ . \tag{5.2.8}$$

To make compatible the reality of the classical BRST generator Ω at a quantum level, hermiticity of $\widehat{\Omega}$ must be ensured. The only non–trivial ordering issue in $\widehat{\Omega}$ is that of the purely bosonic factor $\widehat{\phi}$ whose resolution is specified by the symmetric ordering

$$\widehat{\phi} := -i\big(M\partial_\theta + \tfrac{1}{2}(\partial_\theta M)\big). \tag{5.2.9}$$

To connect the BRST quantisation to a formalism that only involves bosonic variables, we first recognise that there are ghost–free BRST invariant states of the form

$$\Psi^0_{\mathrm{phys}} = \psi(\theta,x) \ , \quad \Upsilon^0_{\mathrm{phys}} = \chi(\theta,x) \ . \tag{5.2.10}$$

These are trivially annihilated by both the BRST and the ghost number operators. However, it is not possible simply to drop all the powers of the fermions from the quantum states keeping the BRST inner product (5.1.11); the sesquilinear form (5.1.11) evaluated at the ghost–free states (5.2.10) is ill–defined due to the fermionic and Lagrange multiplier integration. There is however the option to evaluate $(\cdot\,,\cdot)_{\mathrm{BRST}}$ on suitable gauge–equivalent states as proposed by the regularised inner product (4.2.21). Accordingly, we then provide the anti–hermitian gauge fixing fermion $i\widehat{\varrho} := -\widehat{\lambda}\widehat{\bar{\rho}}$ that mixes bosonic nonminimal and fermionic minimal sectors of the theory. From this choice, it follows that $i[\widehat{\Omega}, \widehat{\varrho}] = i\lambda\widehat{\phi} + \rho\bar{\rho}$, then after a basic integration over all ghost–momenta we obtain

$$\left(\Psi^0_{\mathrm{phys}}, \Upsilon^0_{\mathrm{phys}}\right)^{\varrho}_{\mathrm{BRST}} = \int \mathrm{d}\lambda\, \mathrm{d}\theta\, \mathrm{d}x\; \psi^*(\theta, x)\left[\exp\left(i\lambda\widehat{\phi}\,\right)\chi\right](\theta, x)\;. \tag{5.2.11}$$

When one considers the arbitrary Lagrange multiplier to be the parameter of the gauge group, this expression resembles a group averaging formula over the group generated by $\widehat{\phi}$ provided we can give a rigorous interpretation of $\widehat{U}(\lambda) := \exp(i\lambda\widehat{\phi}\,)$ on some states $\chi(\theta, x)$. The task for the rest of this chapter is to supply the appropriate auxiliary structures to fit (5.2.11) within RAQ.

5.3 Refined algebraic quantisation: Auxiliary structures

We will now give a precise meaning to the sesquilinear form (5.2.11). As in the previous section, we take the auxiliary Hilbert space to be the space of square–integrable functions on the configuration space (5.1.20). The positive definite scalar product between two auxiliary states is given by (5.1.21).

We now want to capitalise on the formal symmetric ordered definition we gave to the quantum constraint (5.2.9) and obtain a family of operators $\{\widehat{U}(\lambda)\}$ by exponentiation,

$$\widehat{U}(\lambda) := \exp(i\lambda\widehat{\phi}\,)\;, \tag{5.3.1}$$

hence we need to provide a dense definition of $\widehat{\phi}$. An inner product on the physical Hilbert space could be found afterwards by a suitable interpretation of the sesquilinear form, *cf.* Eq. (5.2.11),

$$(\psi, \chi)_{\mathrm{ga}} := \int \mathrm{d}\lambda\, \left(\psi, \widehat{U}(\lambda)\chi\right)_{\mathrm{aux}}. \tag{5.3.2}$$

The operator $\widehat{\phi}$ is symmetric on the dense linear subspace of smooth functions of compact support $C_0^\infty(\mathbb{R}^2) \subset \mathcal{H}_{\mathrm{aux}}$. If $\widehat{\phi}$ has self–adjoint extensions on $\mathcal{H}_{\mathrm{aux}}$, a choice of the self–adjoint extension in (5.3.1) defines $\{\widehat{U}(\lambda) :\, \lambda \in \mathbb{R}\}$ as a one–parameter group of unitary operators, and we can look for an interpretation for (5.3.2) as the group averaging sesquilinear form in RAQ. So, it is clear the need to analyse the self–adjoint extensions of $\widehat{\phi}$.

The basic criterion for self–adjointness, or deficiency indices theorem by von Neumann[3] is proved to be of paramount usefulness in this case. The existence of self–adjoint extensions of any symmetric operator $\widehat{T}$ on a Hilbert space $\mathcal{H}$ is determined by the deficiency indices (n_+, n_-), that is, the dimensions $n_\pm$ of the subspaces of $\mathcal{H}$ satisfying $\widehat{T}\psi = \pm i\psi$. There are three different outputs [191, 193]: (a) $n_+ = n_- = 0$, then $\widehat{T}$ is self–adjoint (this is a necessary and sufficient condition); (b) $n_+ = n_- =: d \geq 1$, if $d < \infty$ then any maximal symmetric extension[4] of $\widehat{T}$ is self–adjoint, that is, $\widehat{T}$ has self–adjoint extensions. The situation is more complicated if d is infinite, then some maximal symmetric extensions of $\widehat{T}$ are self–adjoint and some are not. (c) $n_+ \neq n_-$, then $\widehat{T}$ has no self–adjoint extension.

Then the existence of self–adjoint extensions of $\widehat{\phi}$ boils down just to the counting of solutions of the equations $\widehat{\phi}\psi = \pm i\psi$ that have a finite norm in the auxiliary inner product!. The (weak) solutions to the differential equation

$$-\, i\big(M\partial_\theta + \tfrac{1}{2}(\partial_\theta M)\big)\psi = \pm i\psi \tag{5.3.3}$$

are

$$\psi_\pm(\theta, x) = \frac{F_\pm(x)}{\sqrt{M(\theta, x)}} \exp\big[\mp \sigma_x(\theta)\big] \,, \tag{5.3.4}$$

where

$$\sigma_x(\theta) := \int_0^\theta \frac{\mathrm{d}\theta'}{M(\theta', x)} \,, \tag{5.3.5}$$

and the complex valued functions $F_\pm(x)$ are arbitrary. The corresponding norms of the solutions $\psi_\pm$ are respectively

$$\begin{aligned} I_\pm &= \int_{\mathbb{R}^2} \mathrm{d}\theta\, \mathrm{d}x\, \frac{|F_\pm(x)|^2 \exp\big[\mp 2\sigma_x(\theta)\big]}{M(\theta, x)} \\ &= \mp\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x\, |F_\pm(x)|^2 \left[\lim_{\theta\to\infty} e^{\mp 2\sigma_x(\theta)} - \lim_{\theta\to-\infty} e^{\mp 2\sigma_x(\theta)}\right] \,. \end{aligned} \tag{5.3.6}$$

---

[3] This corresponds to the Theorem X.2 and its corollary cited in [191] and originally proved by von Neumann in the late 1920's. This theorem has as special case the Theorem VIII.3 printed in [177]. A beautiful pedagogical introduction to these results and their application to quantum mechanics can be found in [192].

[4] A symmetric operator $\widehat{T}$ is maximal if it has no proper symmetric extensions, *i.e.* if the relation $\widehat{T} \subset \widehat{T}'$ for a symmetric $\widehat{T}'$ implies $\widehat{T} = \widehat{T}'$. Any self–adjoint operator is maximal, but there are maximal symmetric operators which are not self–adjoint; for instance, the differential expression $i\dfrac{\mathrm{d}}{\mathrm{d}\theta}$ with domain

$$D := \{\psi, \psi' \in L^2(0,\infty) :\ \psi \in \mathrm{ac}(0,\infty) \text{ and } \psi(0) = 0\} \,,$$

where 'ac' stands for absolutely continuous, is a maximal symmetric operator which is not self–adjoint [193].

Only positive definite scaling functions are under consideration now, then $\sigma_x(\theta)$ is, for almost every $x \in \mathbb{R}$, a strictly increasing function in θ whose range is either $(-\infty, +\infty)$, $(-\infty, \sigma_x^{(1)})$, $(\sigma_x^{(a)}, \sigma_x^{(b)})$, or $(\sigma_x^{(2)}, +\infty)$ as θ varies from $-\infty$ to ∞. Here $\sigma_x^{(a,b)}$ and $\sigma_x^{(1,2)}$ are real constants as we vary θ at a constant x. Then there are qualitatively three different cases, depending on the asymptotics of $\sigma_x(\theta)$ as $\theta \to \pm\infty$: the range of σ_x is either the real line, or a semi–infinite subset, or a compact set of the real line. We shall analyse these situations in the following section.

5.4 Self–adjointness of the constraint operator and types of scaling functions

The evaluation performed in this section can be repeated, finding no new phenomena, for the case of negative definite scaling function.

[Scaling functions of type I] Suppose that the range of σ_x is the whole real line

$$\sigma_x(\theta) \to \pm\infty \quad \text{as} \quad \theta \to \pm\infty \quad \text{for a.e. } x \; , \tag{5.4.1}$$

where 'a.e.' stands for almost everywhere in the Lebesgue measure on $\mathbb{R}$. Then from Eq. (5.3.6) every nonzero $\psi_\pm$ (5.3.4) has infinite norm, for ψ_+ because of the behaviour at $\theta \to -\infty$ and for ψ_- because of the behaviour at $\theta \to \infty$. The deficiency indices are $(0, 0)$ and $\widehat{\phi}$ is self–adjoint. The operator $\widehat{U}(\lambda)$ is unitary, and it acts on the wave functions by the exponential map of the vector field X (5.2.2), explicitly

$$\big(\widehat{U}(\lambda)\psi\big)(\theta, x) = \frac{\sqrt{M\big(\sigma_x^{-1}(\sigma_x(\theta) + \lambda), x\big)}}{\sqrt{M(\theta, x)}}\, \psi\big(\sigma_x^{-1}(\sigma_x(\theta) + \lambda), x\big) \; . \tag{5.4.2}$$

This expression can formally be obtained by iterative applications of $\widehat{\phi}$ on ψ. In this calculation a useful expression for $\widehat{\phi}$ acting on ψ is

$$\big(\widehat{\phi}\psi\big)(\theta, x) = -i\frac{1}{\sqrt{M(\theta, x)}}\big(M\partial_\theta(\sqrt{M}\,\psi)\big)(\theta, x) \; ,$$

which is equivalent to the action of $\widehat{\phi}$ (5.2.9).

[Scaling functions of type II] Suppose a semi–infinite range of σ_x, that is, the condition (5.4.1) holds either with the upper signs or with the lower signs but not both. If the condition (5.4.1) holds for the upper signs, say $\sigma_x(\theta) \to \infty$ as $\theta \to \infty$ but $\sigma_x(\theta) \to \sigma_x^{(2)}$ as $\theta \to -\infty$, then from (5.3.6) every nonzero ψ_- has again an infinite norm; however, any $F_+ \in L^2(\mathbb{R})$ whose support is in the set where (5.4.1) with the lower signs fails will give a square–integrable ψ_+. The deficiency indices are hence $(\infty, 0)$. Similarly, if (5.4.1) holds for the lower signs, the deficiency indices are $(0, \infty)$. Then the quantum constraint (5.2.9) has no self–adjoint extension in either case, and (5.4.2) does not provide a definition of $\widehat{U}(\lambda)$. At the level of formula (5.4.2), the problem is that σ_x^{-1} is not well defined even for a.e. x.

The scaling function $M(\theta, x) = e^{\theta}$, which defines an incomplete X (5.2.2), is also an example of scaling function of type II; the corresponding $\sigma(\theta)$ holds the condition (5.4.1) only for the lower signs, but $\sigma(\theta) \to 0$ as $\theta \to \infty$. Within the gauge part of the argument in the functions at the RHS of Eq. (5.4.2), $\sigma^{-1}(\sigma(\theta) + \lambda)$ is not defined for $\lambda > e^{-\theta}$.

[Scaling functions of type III] Suppose a compact range of σ_x, that is, the condition (5.4.1) holds with neither upper nor lower signs; for instance, $\sigma_x \to \sigma_x^{(a)}$ as $\theta \to -\infty$ and $\sigma_x \to \sigma_x^{(b)}$ as $\theta \to \infty$. Reasoning as with scaling functions of type II, it can be seen that the values for both $I_\pm$ are finite in this case, showing that the deficiency indices are (∞, ∞). The quantum constraint $\widehat{\phi}$ has an infinity of self–adjoint extensions, and each of them defines $\{\widehat{U}(\lambda) : \lambda \in \mathbb{R}\}$ as a one–parameter group of unitary operators. Formula (5.4.2) has again a problem in that σ_x^{-1} is not defined, but the self–adjoint extension of $\widehat{\phi}$ provides a rule by which the probability that is pushed beyond $\theta = \pm\infty$ by (5.4.2) will re–emerge from $\mp\infty$. The group $\{\widehat{U}(\lambda) : \lambda \in \mathbb{R}\}$ may be isomorphic to either $\mathbb{R}$ or $U(1)$.

A concrete example of a scaling function of type III is given by $M(\theta, x) = (\theta^2 + 1)/(x^2 + 1)$, for which $\sigma_x(\theta) = (x^2 + 1)\arctan(\theta)$. The limit values for σ_x as $\theta \to \pm\infty$ are $\pm\frac{\pi}{2}(x^2 + 1)$. The corresponding norms for $\psi_\pm$,

$$I_\pm = \int_{-\infty}^{+\infty} |F_\pm(x)|^2 \sinh\mu(x)\,\mathrm{d}x \quad \text{with} \quad \mu(x) = \pi(x^2 + 1)\ ,$$

are finite for suitable $F_\pm$, *e.g.* when $F_\pm = \dfrac{e^{-x^2/2}}{\sqrt{\sinh\mu(x)}}$ we have $I_\pm = \sqrt{\pi}$. The expression for $\sigma_x^{-1}(\sigma_x(\theta) + \lambda)$ becomes a periodic function in this case, namely $\tan(\arctan(\theta) + \lambda)$. The Eq. (5.4.2) specifies the translation for compactly supported functions and only for sufficiently small $|\lambda|$, one way to specify what happens to them beyond $\pm\infty$ by a unitary action is as follows: one requires at the RHS of (5.4.2) that what shows up after $\pm\frac{\pi}{2}(x^2+1)$ (equivalently $\theta \to \pm\infty$) enters at the other end $\mp\frac{\pi}{2}(x^2 + 1)$ (equivalently $\theta \to \mp\infty$). In other words, from all of the infinity possible extended domains of $\widehat{\phi}$, we characterise a subfamily of them by the boundary conditions that match the wave functions (possibly up to a phase) at the endpoints in the range of σ_x.

This qualitative classification in the scaling functions permit us to proceed only with those of types I and III. Only for them a unitary action (5.4.2) can be achieved. In sections 5.5 and 5.6 below, we address the integral (5.3.2) for these two types.

5.5 Refined algebraic quantisation: Scaling functions of type I

For scaling functions of type I, the multiplication law in the group $\{\widehat{U}(\lambda) : \lambda \in \mathbb{R}\}$ is the addition in λ. We hence take the range of integration in (5.3.2) to be the full real axis.

It is convenient to map $\mathcal{H}_{\text{aux}}$ into $\widetilde{\mathcal{H}}_{\text{aux}} := L^2(\mathbb{R}^2, \mathrm{d}\Theta\, \mathrm{d}x)$ by the Hilbert space isomorphism

$$
\begin{aligned}
\mathcal{H}_{\text{aux}} &\to \widetilde{\mathcal{H}}_{\text{aux}} \,, \\
\psi &\mapsto \widetilde{\psi} \,, \\
\widetilde{\psi}(\Theta, x) &:= \sqrt{M\big(\sigma_x^{-1}(\Theta), x\big)}\, \psi\big(\sigma_x^{-1}(\Theta), x\big) \,,
\end{aligned}
\tag{5.5.1}
$$

where the last line is well defined for a.e. x, this isomorphism suggests[5] $\Theta \equiv \sigma_x(\theta)$. Working in $\widetilde{\mathcal{H}}_{\text{aux}}$, the auxiliary inner product reads

$$
\left(\widetilde{\psi}, \widetilde{\chi}\right)_{\widetilde{\text{aux}}} := \int_{\mathbb{R}^2} \mathrm{d}\Theta\, \mathrm{d}x\, \widetilde{\psi}^*(\Theta, x)\, \widetilde{\chi}(\Theta, x) \,. \tag{5.5.2}
$$

It can be seen that the group averaging sesquilinear form (5.3.2) is reduced to

$$
\left(\widetilde{\psi}, \widetilde{\chi}\right)_{\widetilde{\text{ga}}} := \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \left(\widetilde{\psi}, \widetilde{U}(\lambda)\widetilde{\chi}\right)_{\widetilde{\text{aux}}} \,, \tag{5.5.3}
$$

where

$$
\left(\widetilde{U}(\lambda)\widetilde{\psi}\right)(\Theta, x) = \widetilde{\psi}(\Theta + \lambda, x) \,, \tag{5.5.4}
$$

which is equivalent to (5.4.2) times $\sqrt{M(\theta, x)}$. The system has thus been mapped to that in which M is the constant function 1.

RAQ in $\widetilde{\mathcal{H}}_{\text{aux}}$ can now be carried out as for the closely related system discussed in the previous section (see also Section IIB of [93]). We can choose smooth functions of compact support on $\mathbb{R}^2 = \{(\Theta, x)\}$ as the dense linear subspace of $\widetilde{\mathcal{H}}_{\text{aux}}$ on which (5.5.3)

[5] At a classical level this can be thought as a point transformation on the original configuration space, that once elevated to the phase space [141], implies $P_\Theta = M(\theta, x)p_\theta$ as a change in the corresponding momenta.

is well defined. The averaging projects out the Θ–dependence of the wave functions, and the physical Hilbert space is $L^2(\mathbb{R}, \mathrm{d}x)$. The technical steps are identical as in the Sect. 5.1.2 and we will not repeat them here.

5.6 Refined algebraic quantisation: Scaling functions of type III

For scaling functions of type III, the sets in which the conditions (5.4.1) fail for the upper and lower signs can be arbitrary sets of positive measure. In the immediate subsection, we first make two assumptions that allow the action of the gauge group to be written down in an explicit form. Then in subsequent subsections, we consider two special cases where we are able to extract from the group averaging formula (5.3.2) families of sesquilinear forms that provide RAQ rigging maps.

5.6.1 Subfamily of classical rescalings and quantum boundary conditions

The first assumption we make is at a classical level. We assume that (5.4.1) fails for all x for both signs, so that the formula

$$N(x) := 2\pi \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{M(\theta, x)} \right)^{-1} \tag{5.6.1}$$

defines a function $N : \mathbb{R} \to \mathbb{R}_+$. It follows that we can map $\mathcal{H}_{\mathrm{aux}}$ to $\widetilde{\mathcal{H}}_{\mathrm{c}} := L^2(I \times \mathbb{R}, \mathrm{d}\omega\, \mathrm{d}x)$, where $I = [0, 2\pi]$, by the Hilbert space isomorphism

$$\begin{aligned} \mathcal{H}_{\mathrm{aux}} &\to \widetilde{\mathcal{H}}_{\mathrm{c}} \ , \\ \psi &\mapsto \psi_c \ , \\ \psi_c(\omega, x) &:= \sqrt{\frac{M\big(\tilde{\sigma}_x^{-1}(\omega/N(x)), x\big)}{N(x)}}\, \psi\big(\tilde{\sigma}_x^{-1}(\omega/N(x)), x\big) \ , \end{aligned} \tag{5.6.2}$$

where

$$\tilde{\sigma}_x(\theta) := \int_{-\infty}^{\theta} \frac{\mathrm{d}\theta'}{M(\theta', x)} \ , \tag{5.6.3}$$

so that $\tilde{\sigma}_x^{-1} : [0, \sigma_x^{(b)} - \sigma_x^{(a)}] \to \mathbb{R} : \dfrac{\omega}{N(x)} \mapsto \tilde{\sigma}_x^{-1}\big(\omega/N(x)\big)$. The auxiliary inner product in $\widetilde{\mathcal{H}}_{\mathrm{c}}$ reads

$$(\psi_c, \chi_c)_c := \int_{I\times\mathbb{R}} \mathrm{d}\omega\, \mathrm{d}x\ \psi_c^*(\omega, x)\, \chi_c(\omega, x) \ , \tag{5.6.4}$$

and $\widehat{\phi}$ (5.2.9) is mapped to

$$\widehat{\phi}_c := -iN(x)\, \partial_\omega \ , \tag{5.6.5a}$$

explicitly

$$\left(\widehat{\phi}_c\psi_c\right)(\omega,x) = \sqrt{\frac{M\left(\tilde{\sigma}_x^{-1}(\omega/N(x)),x\right)}{N(x)}}\left(\widehat{\phi}\psi\right)\left(\tilde{\sigma}_x^{-1}(\omega/N(x)),x\right) \ . \tag{5.6.5b}$$

We work from now on in $\widetilde{\mathcal{H}}_{\rm c}$, dropping the subscript c from the wave functions.

The second assumption we make is at a quantum level. We consider those self–adjoint extensions of $\widehat{\phi}_c$ whose domain consists of wave functions with boundary conditions at $\omega = 0$ and $\omega = 2\pi$ that do not couple different values at x. The self–adjointness analysis then reduces to that of the momentum operator on an interval [191, 193], independently at each x. Concisely, the domains of self–adjointness are

$$D_\alpha := \left\{\psi, \partial_\omega\psi \in \widetilde{\mathcal{H}}_{\rm c} \mid \psi(\cdot,x) \in \text{ac}(0,2\pi) \text{ and } \psi(0,x) = e^{i2\pi\alpha(x)}\psi(2\pi,x),\ \forall x\right\}, \tag{5.6.6}$$

where $\text{ac}(0,2\pi)$ denotes absolutely continuous functions of ω and the function $\alpha:\mathbb{R}\to\mathbb{R}$ specifies the phase shift between $\omega = 0$ and $\omega = 2\pi$ at each x.

Under these assumptions, the remaining freedom in the classical scaling function $M:\mathbb{R}^2\to\mathbb{R}^2$ is encoded in the function $N:\mathbb{R}\to\mathbb{R}_+$, while the remaining freedom in the self–adjoint extension of $\widehat{\phi}_c$ (5.6.5) is encoded in the function $\alpha:\mathbb{R}\to\mathbb{R}$. Note that no smoothness assumptions about either function are needed at this stage.

On $\widetilde{\mathcal{H}}_{\rm c}$, the action of $\widehat{U}_c(\lambda) := \exp\left(i\lambda\widehat{\phi}_c\right)$ takes now a simple form in a Fourier decomposition adapted to D_α. We write each $\psi\in\widetilde{\mathcal{H}}_{\rm c}$ in the unique decomposition

$$\psi(\omega,x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} e^{i[n-\alpha(x)]\omega}\,\psi_n(x) \ , \tag{5.6.7}$$

where each ψ_n is in $L^2(\mathbb{R},\mathrm{d}x)$. It follows that

$$\left(\psi,\chi\right)_c = \sum_{n\in\mathbb{Z}}\left(\psi_n,\chi_n\right)_{\mathbb{R}} \ , \tag{5.6.8}$$

where $(\cdot\,,\cdot)_{\mathbb{R}}$ is the inner product in $L^2(\mathbb{R},\mathrm{d}x)$. The action of $\widehat{U}_c(\lambda)$ reads[6]

$$\begin{aligned}\left(\widehat{U}_c(\lambda)\psi\right)(\omega,x) &= \psi(\omega+N(x)\lambda,x) \ ,\\ &= \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} e^{i[n-\alpha(x)]\omega}\left(\widehat{U}_c(\lambda)\psi\right)_n(x) \ ,\end{aligned} \tag{5.6.9a}$$

with

$$\left(\widehat{U}_c(\lambda)\psi\right)_n(x) := e^{iR_n(x)\lambda}\psi_n(x) \ , \tag{5.6.9b}$$

where for each $n\in\mathbb{Z}$ the function $R_n:\mathbb{R}\to\mathbb{R}$ is defined by

$$R_n(x) = [n-\alpha(x)]\,N(x) \ . \tag{5.6.10}$$

[6]The operator $\widehat{\phi}_c = -iN(x)\partial_\omega$ can be thought as the quantum analogue of the classical constraint $N(x)p_\omega \approx 0$; therefore, the first line in the expression (5.6.9a) indicates that scaling the constraint $p_\omega\approx 0$ by a function that only depends on the true degree of freedom $N(x)$, yields the expected scaling in the translation group parameter: $\psi(\omega+\lambda,x)\mapsto\psi(\omega+N(x)\lambda,x)$.

5.6.2 Test space, observables and rigging map candidates

Let $\widetilde{\Phi}$ be the dense linear subspace of $\widetilde{\mathcal{H}}_c$ where the states have the form (5.6.7) such that every ψ_n is smooth with compact support and only finitely many of them are nonzero for each $\psi \in \widetilde{\Phi}$. From (5.6.9) we see that $\widetilde{\Phi}$ is invariant under $\widehat{U}_c(\lambda)$ for each λ. We adopt $\widetilde{\Phi}$ as the RAQ test space of 'sufficiently well–behaved' auxiliary states.

Given $\widetilde{\mathcal{H}}_c$, $\widetilde{\Phi}$ and $\widehat{U}_c(\lambda)$, the RAQ observables are operators $\widehat{A}$ on $\widetilde{\mathcal{H}}_c$ such that the domains of $\widehat{A}$ and $\widehat{A}^\dagger$ include $\widetilde{\Phi}$, $\widehat{A}$ and $\widehat{A}^\dagger$ map $\widetilde{\Phi}$ to itself and $\widehat{A}$ commutes with $\widehat{U}_c(\lambda)$ on $\widetilde{\Phi}$ for all λ. We denote the algebra of the observables by $\mathcal{A}^{(\star)}_{\text{phys}}$.

We now seek the anti–linear map $\eta : \widetilde{\Phi} \to \widetilde{\Phi}'$, from the test state space to its algebraic dual, such that the conditions of reality (4.1.7b) and positivity (4.1.7c) are satisfied. The map η must also solve the constraints in the sense of (4.1.8) and intertwine with the observables (4.1.7d). The physical Hilbert space $\mathcal{H}_{\text{phys}}$ is then the completion of the image of η in the inner product (4.1.9)

$$(\eta(g), \eta(f))_{\text{RAQ}} := \eta(f)[g] \ , \quad f, g \in \widetilde{\Phi} \tag{5.6.11}$$

and the properties of η and $\mathcal{A}^{(\star)}_{\text{phys}}$ imply that η induces an anti–linear representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ on $\mathcal{H}_{\text{phys}}$, with the image of η as the dense domain.

Observe that $\sigma \circ \tilde{\sigma}^{-1} : \left[0, \sigma_x^{(b)} - \sigma_x^{(a)}\right] \to \left[\sigma_x^{(a)}, \sigma_x^{(b)}\right] : y \mapsto y + \sigma_x^{(a)}$ implies the identity

$$\sigma^{-1}\left(\sigma\left(\tilde{\sigma}^{-1}(\omega/N)\right) + \lambda\right) = \tilde{\sigma}^{-1}\left((\omega/N) + \lambda\right) . \tag{5.6.12}$$

The change of variable $\theta \equiv \tilde{\sigma}^{-1}(\omega/N)$ into the sesquilinear form (5.3.2), the isomorphism (5.6.2), and the identity (5.6.12), show that on $\widetilde{\mathcal{H}}_c$ the group averaging ansatz reads

$$\int \mathrm{d}\lambda \left(f, \widehat{U}_c(\lambda) g\right)_c = \int \mathrm{d}\lambda \int_{I\times\mathbb{R}} \mathrm{d}\omega \mathrm{d}x \ f^*(\omega, x)\, g(\omega + N(x)\lambda, x) \ , \tag{5.6.13a}$$

$$= \int \mathrm{d}\lambda \int_{\mathbb{R}} \mathrm{d}x \ \sum_n f_n^*(x)\, g_n(x)\, e^{iR_n(x)\lambda} \ , \tag{5.6.13b}$$

for all $f, g \in \widetilde{\Phi}$ and the integration over λ to be specified. The second line in this expression is obtained by direct substitution of (5.6.7) and (5.6.9) into the first line. To define a rigging map we reconsider the group averaging proposal in a renormalised form; namely,

$$\eta(f)[g] := \lim_{L\to\infty} \frac{1}{\rho(L)} \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x\, f_n^*(x) g_n(x) \int_{-L}^{L} \mathrm{d}\lambda\, e^{iR_n(x)\lambda} \ , \tag{5.6.14}$$

where the LHS follows from (5.6.13) after interchanging sums and integrals, justified by the assumptions about $\widetilde{\Phi}$. The normalisation function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ has been included in order to seek a finite answer in cases where the limit would otherwise diverge.

The existence of the limit in (5.6.14) depends delicately on the zero sets and the stationary point sets of the functions R_n. In what follows we introduce conditions that make the limit controllable.

5.6.3 Functions N and α smooth, α with integer–valued intervals

We assume that α and N are smooth functions. What will play a central role are the integer value sets of α and the stationary point sets of the functions $\{R_n : n \in \mathbb{Z}\}$. To control the stationary point sets, we assume that R_n satisfy the following technical condition:

(i) The stationary point set of each R_n is either empty or the union of at most countably many isolated points, at most countably many closed intervals and at most two closed half–lines, such that any compact subset of $\mathbb{R}$ contains at most finitely many of the isolated points and at most finitely many of the finite intervals.

To control the integer value set of α, we assume in this subsection the following condition:

(ii) α takes an integer value on at least one interval.

It follows from (ii) that at least one R_n takes the value zero on an interval. Note that (i) and (ii) include the special case where α takes an integer value everywhere, and the very special case where this integer value is zero.

The group averaging formula (5.6.14) takes the form

$$\eta(f)[g] = \lim_{L\to\infty} \frac{2L}{\rho(L)} \sum_{n\in\mathbb{Z}} \left(\int_{J_n} \mathrm{d}x\, f_n^*(x) g_n(x) + \int_{\mathbb{R}\setminus J_n} \mathrm{d}x\, f_n^*(x) g_n(x)\, \frac{\sin\left[LR_n(x)\right]}{LR_n(x)} \right), \tag{5.6.15}$$

where $J_n \subset \mathbb{R}$ is the union of all open intervals contained in the zero set of R_n, that is, in the solution set of $\alpha(x) = n$. The function N is nonvanishing due to the scaling function properties. Setting $\rho(L) = 2L$, the second term in (5.6.15) vanishes by dominated convergence, and from the first term we obtain the map $\eta_\infty : \widetilde{\Phi} \to \widetilde{\Phi}'$,

$$\big(\eta_\infty(f)\big)[g] = \sum_{n\in\mathbb{Z}} \int_{J_n} \mathrm{d}x\, f_n^*(x)\, g_n(x)\ . \tag{5.6.16}$$

We have the following theorem.

Theorem 5.6.1 *The map η_∞ is a rigging map, with a non–trivial image.*

Proof. The map η_∞ solves the constraints. Note that η_∞ is the evaluation of the integral specifically *on* the set $J_n \subset \mathbb{R}$, where $R_n(x) \equiv 0$, hence the exponential

$e^{-iR_n(x)\lambda}$ in $(\widehat{U}_c^\dagger(\lambda)g)_n(x)$, see Eq. (5.6.9b), contributes nothing to the integral, so that $\eta_\infty(f)[\widehat{U}_c^\dagger(\lambda)g]$ assigns back $\eta_\infty(f)[g]$. Positivity and reality rigging map axioms are immediate. We verify the intertwining property (4.1.7d) in Appendix A. ∎

Group averaging has thus yielded a genuine rigging map η_∞ after a suitable renormalisation. The Hilbert space $\mathcal{H}_\infty$ is separable and carries a non–trivial representation of $\mathcal{A}^{(\star)}_{\text{phys}}$. Comparison of (5.6.8) and (5.6.16) shows that $\mathcal{H}_\infty$ can be (antilinearly) embedded in $\widetilde{\mathcal{H}}_\text{c}$ as a Hilbert subspace, such that η_∞ extends into the projection $L^2(\mathbb{R}, \mathrm{d}x) \to L^2(J_n, \mathrm{d}x)$ in each of the components in (5.6.7).

Note that the function N does not appear in η_∞ (5.6.16), and the discussion in Appendix A shows that the representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ on the image of η_∞ does not depend on N either. The quantum theory has turned out completely independent of the choice of scaling function, even when the scaling function may vary non–trivially over the intervals J_n that contribute in (5.6.16).

In the special case where $\alpha(x) = 0$ for all x, where the only nonempty zero set of $R_n(x) = nN(x)$ at $n = 0$ is nothing but $\mathbb{R}$, from (5.6.16) we obtain

$$\eta(f)[g] = (f_0, g_0)_{\mathbb{R}} \ . \tag{5.6.17}$$

Embedding $\mathcal{H}_\infty$ antilinearly as a Hilbert subspace of $\mathcal{H}_c$ as above, that is, η_∞ extends into the (antilinear) projection to the $n = 0$ sector in $\widetilde{\mathcal{H}}_\text{c}$. When N is a constant function, $N(x) = N_0$ for all x, we can recover this extension of η_∞ directly, without introducing a test space, by noticing that the quantum gauge group $\{U_c(\lambda) \mid \lambda \in \mathbb{R}\} \simeq \mathrm{U}(1)$ is compact and taking the group averaging formula to read

$$\eta(f)[g] = \frac{N_0}{2\pi} \int_0^{2\pi/N_0} \mathrm{d}\lambda \left(f, \widehat{U}_c(\lambda)g\right)_c \ , \tag{5.6.18}$$

so that the integration is over $\mathrm{U}(1)$ exactly once. However, if N is not constant, this shortcut is not available because the quantum gauge group is still isomorphic to $\mathbb{R}$ rather than $\mathrm{U}(1)$.

5.6.4 Functions N and α smooth and generic

In subsection 5.6.3, the quantum theory arose entirely from the integer value intervals of α. We now continue to assume that α and N are smooth, the technical stationary point condition (i) holds and α takes an integer value somewhere, but we take the integer value set of α to consist of isolated points. We first replace condition (ii) by the following:

(ii′) The integer value set of α is non–empty, at most countable and without accumulation points, and α has a nonvanishing derivative of some order at each integer value.

Second, we introduce the following notation for the zeroes of R_n. Let p be the order of the lowest nonvanishing derivative of α (and hence also of R_n) at a zero of R_n. For odd p, we write the zeroes as x_{pnj}, where the last index enumerates the solutions with given p and n. For even p, we write the zeroes as $x_{p\epsilon nj}$, where $\epsilon \in \{1, -1\}$ is the sign of the pth derivative of α and the last index enumerates the zeroes with given p, ϵ and n. Let $\mathcal{P}$ be the value set of the first index of the zeroes $\{x_{pnj}\}$ and $\{x_{p\epsilon nj}\}$. Given this notation, we assume:

(iii) If $p \in \mathcal{P}$, then $\mathcal{P}$ contains no factors of p smaller than $p/2$.

Before examining the group averaging formula (5.6.14) under these assumptions, we use the assumptions to define directly a family of rigging maps as follows. For each odd $p \in \mathcal{P}$ we define the mapping $\eta_p : \widetilde{\Phi} \to \widetilde{\Phi}'$, and for each even $p \in \mathcal{P}$ and $\epsilon \in \{1, -1\}$ for which the set $\{x_{p\epsilon nj}\}$ is non–empty, we define the map $\eta_{p\epsilon} : \widetilde{\Phi} \to \widetilde{\Phi}'$, by the formulas

$$\big(\eta_p(f)\big)[g] = \sum_{nj} \frac{f_n^*(x_{pnj})\, g_n(x_{pnj})}{\big|\alpha^{(p)}(x_{pnj}) N(x_{pnj})\big|^{1/p}} , \tag{5.6.19a}$$

$$\big(\eta_{p\epsilon}(f)\big)[g] = \sum_{nj} \frac{f_n^*(x_{p\epsilon nj})\, g_n(x_{p\epsilon nj})}{\big|\alpha^{(p)}(x_{p\epsilon nj}) N(x_{p\epsilon nj})\big|^{1/p}} . \tag{5.6.19b}$$

These maps are rigging maps, with properties given in the following theorem.

Theorem 5.6.2 *Under the assumptions* (i), (ii′) *and* (iii), *we have that*

1. *Each η_p and $\eta_{p\epsilon}$ is a rigging map, with a non–trivial image.*

2. *The representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ on the image of each η_p and $\eta_{p\epsilon}$ is irreducible.*

Proof.

1. All the rigging map axioms except the intertwining property (4.1.7d) are immediate from (5.6.19). In particular, each η_p (resp. $\eta_{p\epsilon}$) solves the constraint in the sense of (4.1.8), since in the RHS of (5.6.19a) ((5.6.19b)) the terms in the sum are evaluated at the points where $R_n = 0$, so $(\widehat{U}_c^\dagger(\lambda)g)_n(x_{pnj}) = g_n(x_{pnj})$ (*cf.* Eq. (5.6.9b)) for p even or odd, implying that $\eta_p(f)[\widehat{U}_c^\dagger(\lambda)g]$ ($\eta_{p\epsilon}(f)[\widehat{U}_c^\dagger(\lambda)g]$) assigns back $\eta_p(f)[g]$ ($\eta_{p\epsilon}(f)[g]$) for all λ. The relevance on p even or odd enters when we verify (4.1.7d) in Appendix A.

2. The proof is an almost verbatim transcription of that given for a closely similar system in Appendix C of [107]. We review the details in Appendix B. ∎

The rigging maps (5.6.19) thus yield a family of quantum theories, one from each η_p and $\eta_{p\epsilon}$. Each of the physical Hilbert spaces is either finite–dimensional or separable and carries a non–trivial representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ that is irreducible on its dense domain. Functions $f \in \widetilde{\Phi}$ whose only nonvanishing component f_n is non–negative and is positive only near a single zero of R_n provide the Hilbert spaces with a canonical orthonormal basis.

From Appendix B, we see that the representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ on the image of each η_p and $\eta_{p\epsilon}$ is not just irreducible but has the following stronger property, which one might call *strong irreducibility*: given any two vectors v and v' in the canonical orthonormal basis, there exists an element of $\mathcal{A}^{(\star)}_{\text{phys}}$ that annihilates all the basis vectors except v and takes v to v'. The upshot of this is that the function N plays little role in the quantum theory, despite appearing in the rigging map formulas (5.6.19). The Hilbert spaces and their canonical bases are determined by the function α up to the normalisation of the individual basis vectors (B.4), and the representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ is so 'large' that the normalisation of the individual basis vectors, determined by N, is of limited consequence. In particular, the representation of $\mathcal{A}^{(\star)}_{\text{phys}}$ on any Hilbert space with dimension $n_0 < \infty$ is isomorphic to the complex $n_0 \times n_0$ matrix algebra, independently of N.

Note that the images of any two rigging maps (5.6.19) have trivial intersection in $\widetilde{\Phi}'$. The 'total' RAQ Hilbert space decomposes as a direct sum of orthogonal spaces

$$\mathcal{H}^{\text{tot}}_{\text{phys}} := \left(\bigoplus_{p \text{ odd}} \mathcal{H}^{p}_{\text{phys}} \right) \oplus \left(\bigoplus_{p \text{ even}, \epsilon} \mathcal{H}^{p\epsilon}_{\text{phys}} \right) . \tag{5.6.20}$$

In view of Theorem 5.6.2, under the action of $\mathcal{A}^{(\star)}_{\text{phys}}$ vectors in $\mathcal{H}^{p}_{\text{phys}}$ (resp. $\mathcal{H}^{p\epsilon}_{\text{phys}}$) are transformed into vectors in $\mathcal{H}^{p}_{\text{phys}}$ ($\mathcal{H}^{p\epsilon}_{\text{phys}}$) in an irreducible way. This means that $\mathcal{H}^{p}_{\text{phys}}$ and $\mathcal{H}^{p\epsilon}_{\text{phys}}$ can be regarded as exhaustive superselection sectors in $\mathcal{H}^{\text{tot}}_{\text{phys}}$, there are no further superselection sectors.

Under the assumption of N and α smooth functions, we wish to relate these quantum theories to the renormalised group averaging formula (5.6.14), which under the assumption (ii′) takes the form

$$\eta(f)[g] = \lim_{L\to\infty} \frac{2}{\rho(L)} \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathrm{d}x \, f_n^*(x) g_n(x) \, \frac{\sin\left[L R_n(x)\right]}{R_n(x)} \, . \tag{5.6.21}$$

Note that the integral over x in (5.6.21) is well defined because the zeroes of the denominator are isolated and the integrand does not diverge at them. However, as $L \to \infty$

each term in the sum of (5.6.21)

$$I_n(L) := \int_{-\infty}^{\infty} \mathrm{d}x \, f_n^*(x) g_n(x) \, \frac{\sin\big[L R_n(x)\big]}{R_n(x)} \ , \tag{5.6.22}$$

diverges. The viability to extract from (5.6.21) a genuine rigging map relies on the possibility to 'nicely' isolate the divergent terms coming from $J(L) := \sum_{n\in\mathbb{Z}} I_n(L)$ as $L \to \infty$. The major contribution to the divergence is seeded at the zeroes of R_n.

Suppose first that $\mathcal{P} = \{1\}$. Then all zeroes of R_n are of order $p = 1$. The lemmas of Appendix C then show that (5.6.21) is well defined and equals $\eta_1(f)[g]$ provided $\rho(L)$ has been chosen as 2π and the assumptions on N are modestly strengthened, in particular to preclude any R_n from taking a constant value on any interval.

Suppose then that $\mathcal{P} \neq \{1\}$ and we again set $\rho(L) = 2\pi$. Suppose further that the assumptions on N are again modestly strengthened so that the conditions of Appendix C hold, and suppose that condition (iii) above is strengthened to the following:

(iii′) If $p \in \mathcal{P}$, then $\mathcal{P}$ contains no factors of p.

Lemmas of Appendix C then show that (5.6.21) contains contributions that diverge in the $L \to \infty$ limit; however, these divergences come in well–defined inverse fractional powers of L such that the coefficient of each $L^{(p-1)/p}$ is proportional to $\eta_p(f)[g]$ for odd p and to $\eta_{p,1}(f)[g] + \eta_{p,-1}(f)[g]$ for even p.

When $\mathcal{P} = \{1\}$, we may hence regard the rigging map η_1 as arising from (5.6.21) with only minor strengthening of our technical assumptions. When $\mathcal{P} \neq \{1\}$, we may regard the rigging maps η_p and $\eta_{p,1} + \eta_{p,-1}$ as arising from (5.6.21) by peeling off and appropriately renormalising the various divergent contributions, but only after strengthening the assumptions so that some generality is lost, and even then the two signs of ϵ are recovered only in a fixed linear combination but not individually.

Remarks

1. Another, qualitatively different self–adjointness domains for $\widehat{\phi}_c$ are

   $$\mathfrak{D}_\alpha := \Big\{\psi, \partial_\omega\psi \in \widetilde{\mathcal{H}}_c \mid \psi(\cdot, x) \in \mathrm{ac}(0, 2\pi) \text{ and } \psi(0, x+a) = e^{i2\pi\alpha(x)}\psi(2\pi, x), \forall x\Big\} ,$$

   which are excluded by our second assumption written down after Eq. (5.6.5b).

2. It may be possible to find assumptions that interpolate between those in sections 5.6.3 and 5.6.4, allowing both a superselection sector that comes from integer–valued intervals of α and superselection sectors that come from isolated zeroes of α.

In formula (5.6.15), the task would be to provide a peeling argument in the L–dependence of the second term, that is, one would need to supply an analysis of the $1/L$ expansion that can provide the algebraic properties of the coefficients from which possibly different rigging maps could be obtained. In the observable analysis of Appendix A, the task would be to provide a similar argument in the small $|s|$ behaviour of the integrands in (A.3b).

3. Our quantum theories arise from the integer value set of α, as established in the assumption (ii) or (ii′). Neither the averaging formulas nor the observable analysis of Appendix A suggest ways to proceed when α takes no integer values, no zeroes of R_n do exist. In (5.6.21), the challenge would be to recover, from the oscillatory L–dependence, a map that satisfies the positivity condition (4.1.7c). A similar oscillatory dependence on λ offers the same challenge in the observable formula (A.2). ▲

CHAPTER 6

Constraint Rescaling in Refined Algebraic Quantisation: Two Momentum Constraints

Can anything be sadder than work left unfinished? Yes, work never begun.

– Christina G. Rossetti

In the previous chapter a system with a single constraint was considered throughout, gauge transformations formed an abelian Lie group before and after the constraint rescaling. This chapter is devoted to mentioning some of the quantum obstacles, at the levels of BRST and RAQ, found after the rescaling of two constraints. The two–constrained system under consideration is an extension of the single–constrained system introduced in the Sect. 5.1. The original phase space $\mathbf{T}^*\mathbb{Q}$ corresponds to $\mathbf{T}^*\mathbb{R}^3$ with points labelled by $(\theta, \varphi, x, p_\theta, p_\varphi, p_x)$, the *un*scaled constraints are $\gamma_1 := p_\theta$ and $\gamma_2 := p_\varphi$. For completeness, we develop the BRST analysis of this system. The group averaging formula is derived form the canonical BRST–quantisation, we implement this expression as a mathematical precise rigging map in RAQ.

The constraints γ_1 and γ_2 are rescaled by real–valued, positive definite, and well behaved scaling functions $M(\theta, \varphi, x)$ and $N(\theta, \varphi, x)$, respectively. In the most general case, the rescaled–constrained system exhibits an open gauge algebra. The canonical BRST–quantisation of the rescaled constraints is performed. The quantum states that are found to fulfil BRST and zero ghost number conditions are the trivial BRST invariant physical states. Using a specific anti–hermitian gauge fixing operator, we construct the regularised inner product (4.2.21). The intricate structure of the final expression for the regularised inner product does not lead to a simple integral in the ghost–momenta. Only in cases where the structure functions are gauge invariant a ghost–free expression can be explicitly obtained; this includes the cases where the structure functions are constant everywhere.

To gain some control over the infinite number of possible algebras obtained by rescaling the constraints γ_a, we use a specific parameterised family of real–valued scaling functions. Depending on the values taken by the parameters, either the original abelian gauge algebra: (1) is maintained, or, (2) corresponds to the algebra of a nonunimodular group with gauge invariant structure functions, or, (3) is a full open algebra, the structure functions depending on all the configuration variables, gauge and non–gauge invariant.

The RAQ of cases (1) and (2) are analysed in full. We map each of them to the case where the scaling functions are the identity function, recovering then the physical Hilbert space of the unscaled constrained system. In particular the resolution of (2) signifies the first example known to the author where a constrained system with structure functions is handled by RAQ. Our results for case (3) remain incomplete, although we have a formal expression for the 'group averaging ansatz' coming from the BRST regularised inner product, this still includes the ghost–momenta variables.

6.1 Two momentum constraints system

In this section we briefly comment on a generalisation of the system introduced in Sect. 5.1 to the case where two trivial momentum constraints are present. The configuration space of the system is $\mathbb{Q} = \mathbb{R}^3 = \{(\theta, \varphi, x)\}$, whereas the corresponding phase space is $\mathbf{T}^*\mathbb{Q} = \mathbf{T}^*\mathbb{R}^3 = \{(\theta, \varphi, x, p_\theta, p_\varphi,\ p_x)\} \simeq \mathbb{R}^6$. The following two trivial constraints are considered

$$\gamma_1 := p_\theta \approx 0 \ , \tag{6.1.1a}$$

$$\gamma_2 := p_\varphi \approx 0 \ . \tag{6.1.1b}$$

The corresponding gauge algebra is abelian

$$\{\gamma_1, \gamma_2\} = 0 \ . \tag{6.1.2}$$

So two independent translational gauge symmetries, one along the $\theta-$direction and the other along the $\varphi-$direction, are present. The gauge orbits on the surface constraint $\Gamma := \{(\theta, \varphi, x, 0, 0, p_x)\} \simeq \mathbb{R}^4$ are bi–dimensional planes parallel to the θ–φ plane. The vector fields $Y_1 := \partial_\theta$ and $Y_2 := \partial_\varphi$ are complete on the configuration space $\mathbb{R}^3$. The inclusion of a true Hamiltonian would be straightforward: any H independent of the variables θ and φ is gauge invariant.

6.1.1 Canonical BRST analysis

We wish to construct the super phase space $\mathbf{T}^*_\lambda\mathbb{Q} \times \{(\eta, \mathcal{P})\}$. Points in this manifold are labelled by $(\theta, \varphi, x, \lambda^a, C^a, \bar{C}_a, p_\theta, p_\varphi, p_x, \pi_a, \bar{\rho}_a, \rho^a)$, with $a = 1, 2$. The notation and conventions of Sect. 3.4.1 will be used. The non–trivial part of the symplectic structure on the super phase space reads

$$\begin{aligned} &\{\theta, p_\theta\} = 1 \ , \qquad \{\varphi, p_\varphi\} = 1 \ , \qquad \{x, p_x\} = 1 \ , \quad \{\lambda^a, \pi_b\} = \delta^a_b \ , \\ &\{C^a, \bar{\rho}_b\} = -\delta^a_b \ , \quad \{\bar{C}_a, \rho^b\} = -\delta^a_b \ . \end{aligned} \tag{6.1.3}$$

The pair of fermionic variables $(C^a, \bar{\rho}_a)$ are associated to the constraints (6.1.1), while $(\bar{C}_a, \rho^a)$ are associated to the nonminimal sector of constraints $\pi_a \approx 0$. Together γ_a and π_a form an abelian first–class set of constraints.

The nonminimal BRST generator and the ghost number function are

$$\Omega = p_\theta C^1 + p_\varphi C^2 - i\rho^a \pi_a \ , \tag{6.1.4a}$$

$$G = i(C^a \bar{\rho}_a + \rho^a \bar{C}_a) \ . \tag{6.1.4b}$$

Both functions can be built from the basic canonical variables in the super phase space, then we let those to span the subspace $\mathscr{S}_{\mathrm{BRST}}$. Each element in $\mathscr{S}_{\mathrm{BRST}}$ is an elementary classical variable.

Quantum mechanically, the abstract (free associative) algebra of quantum operators $\mathcal{B}_{\mathrm{aux}}$ is constructed from the basic operators once the graded commutation relations, which are in correspondence with i times the super–PBs (6.1.3), have been input. The reality conditions of points in the super manifold are

$$q^* = q \ , \qquad (\lambda^a)^* = \lambda_a \ , \qquad (C^a)^* = C^a \ , \qquad \bar{C}^*_a = \bar{C}^*_a \ , \tag{6.1.5a}$$

$$p^* = p \ , \qquad \pi^*_a = \pi_a \ , \qquad \bar{\rho}^*_a = -\bar{\rho}_a \ , \qquad \rho^*_a = -\,\rho_a \ . \tag{6.1.5b}$$

where q and p collectively denote coordinates (θ, φ, x) and momenta $(p_\theta, p_\varphi, p_x)$, respectively. The algebra $\mathcal{B}^{(\star)}_{\mathrm{aux}}$ is built from the inclusion of the corresponding $\star$–relations. All bosonic variables, together with ghosts and antighosts, are meant to be hermitian, while fermionic momenta $\widehat{\rho}^a$ and $\widehat{\bar{\rho}}_a$ are anti–hermitian.

The representation of the basic operators on some vector space $\mathcal{V}_{\mathrm{BRST}}$ is the next step in the canonical BRST–quantisation. We choose as before the Schrödinger representation in which the wave functions depend on the bosonic coordinates (q, λ^a) and the fermionic momenta $(\bar{\rho}_a, \rho^a)$. In this representation, any BRST wave function can be expanded as a polynomial in the ghost–momenta

$$\begin{aligned}\Psi(q, \lambda, \bar{\rho}, \rho) = \psi(q, \lambda) + \Psi^a(q, \lambda)\, \bar{\rho}_a + \tfrac{1}{2}\Psi^{[ab]}(q, \lambda)\, \bar{\rho}_a \bar{\rho}_b + \tfrac{1}{2}\Psi_c^{[ab]}\, \bar{\rho}_a \bar{\rho}_b \rho^c \\ + \tfrac{1}{2}\Psi^a_{[bc]}(q, \lambda)\, \bar{\rho}_a \rho^b \rho^c + \Psi_a(q, \lambda)\, \rho^a + \tfrac{1}{2}\Psi_{[ab]}(q, \lambda)\, \rho^a \rho^b \\ + \Psi^{12}_{12}(q, \lambda)\, \bar{\rho}_1 \bar{\rho}_2\, \rho^1 \rho^2 \ . \end{aligned} \tag{6.1.6}$$

with the coefficients being complex–valued functions. The action of the fundamental operators reads

$$\widehat{q}\,\Psi := q\Psi \ , \qquad \widehat{p}\,\Psi := -i\frac{\partial \Psi}{\partial q} \ , \tag{6.1.7a}$$

$$\widehat{\lambda}^a\,\Psi := \lambda^a \Psi \ , \qquad \widehat{\pi}_a\,\Psi := -i\frac{\partial \Psi}{\partial \lambda^a} \ , \tag{6.1.7b}$$

$$\widehat{C}^a\,\Psi := -i\frac{\partial^\ell \Psi}{\partial \bar{\rho}_a} \ , \qquad \widehat{\bar{\rho}}_a\,\Psi := \bar{\rho}_a \Psi \ , \tag{6.1.7c}$$

$$\widehat{\bar{C}}_a\,\Psi := -i\frac{\partial^\ell \Psi}{\partial \rho^a} \ , \qquad \widehat{\rho}^a\,\Psi := \rho^a\,\Psi \ , \tag{6.1.7d}$$

where $\widehat{p}$ denotes the momentum operators $\widehat{p}_\theta$, $\widehat{p}_\varphi$ and $\widehat{p}_x$. The superscript ℓ on the fermionic derivative stands for left derivative. This choice is compatible with the basic graded commutation relations

$$[\widehat{\theta}, \widehat{p}_\theta] = i \ , \quad [\widehat{\varphi}, \widehat{p}_\varphi] = i \ , \quad [\widehat{x}, \widehat{p}_x] = i \ , \quad [\widehat{\lambda}^a, \widehat{\pi}_b\,] = i\delta^a_b \ , \tag{6.1.8a}$$

$$[\widehat{C}^a, \widehat{\bar{\rho}}_b\,] = -i\delta^a_b \ , \quad [\widehat{\bar{C}}_b, \widehat{\rho}^a\,] = -i\delta^a_b \ . \tag{6.1.8b}$$

The sesquilinear form that we introduce to realise $\star$−relations as hermitian ones is the following:

$$(\Psi, \Upsilon)^c_{\mathrm{BRST}} := c\int \mathrm{d}^2\lambda\, \mathrm{d}^3 q\, \mathrm{d}^2\bar{\rho}\, \mathrm{d}^2\rho\ \Psi^*(q, \lambda, \bar{\rho}, \rho)\Upsilon(q, \lambda, \bar{\rho}, \rho) \ , \tag{6.1.9}$$

with c a nonzero constant which in general takes complex values, $\mathrm{d}^2\lambda \equiv \mathrm{d}\lambda^1\mathrm{d}\lambda^2$, $\mathrm{d}^3 q \equiv \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}x$ and $\mathrm{d}^2\bar{\rho}\, \mathrm{d}^2\rho \equiv \mathrm{d}\bar{\rho}_2 \mathrm{d}\bar{\rho}_1\, \mathrm{d}\rho^2 \mathrm{d}\rho^1$. The integrals over fermionic variables are different from zero when the integrand contains linearly all the fermionic momenta, that is,

$$\int \mathrm{d}^2\rho\, \mathrm{d}^2\bar{\rho}\ f(q, \lambda)\, \bar{\rho}_1 \bar{\rho}_2 \rho^1 \rho^2 = f(q, \lambda) \ . \tag{6.1.10}$$

Independently of the value of c, the sesquilinear form (6.1.9) shares some properties with its lower dimension version (5.1.11). First, it is compatible with the $\star$−relations in the sense that all basic operators (6.1.7) become hermitian with the exception of the fermionic momenta which are realised anti–hermitian. Second, the nilpotent BRST operator and the ghost number operator.

$$\widehat{\Omega} := \widehat{p}_\theta \, \widehat{C}^{\,1} + \widehat{p}_\varphi \, \widehat{C}^{\,2} - i\widehat{\rho}^{\,a}\, \widehat{\pi}_a \ , \tag{6.1.11}$$

$$\widehat{G} := i(\widehat{\rho}^{\,a}\widehat{\bar{C}}_a - \widehat{\bar{\rho}}^{\,a}\widehat{C}_a) \ , \tag{6.1.12}$$

are hermitian and anti–hermitian respectively. Third, the product $(\cdot\,,\cdot)^c_{\mathrm{BRST}}$ between physical states depends on the states only through a gauge–equivalent class (4.2.13) as it follows from the hermiticity of $\widehat{\Omega}$. And finally, trivial ghost–free BRST physical states

$$\Psi^0_{\mathrm{phys}} = \psi(q) \ , \quad \Upsilon^0_{\mathrm{phys}} = \chi(q) \ , \tag{6.1.13}$$

have ill–defined inner product $(\cdot\,,\cdot)^c_{\mathrm{BRST}}$.

So far, most of the BRST analysis done for the single–constrained system (Sect. 5.1.1) has been trivially extended for the case of two–constrained system. However, the nature of c in order to make the sesquilinear form (6.1.9) hermitian changes: If c is real, $c = \alpha$, hermiticity is implied[1]; if, however, c is purely imaginary, $c = i\alpha$, the sesquilinear form becomes skew–hermitian. In this system we opt for $c = -1$ and suppress the superscript c from (6.1.9). This choice will link the regularised inner product (4.2.21) to the structure of the group averaging formula when $i\widehat{\varrho}$ is fixed appropriately.

We choose the anti–hermitian gauge fixing fermion to be $i\widehat{\varrho} \equiv -\widehat{\lambda}^{\,a}\widehat{\bar{\rho}}_a$. It follows that $i[\widehat{\Omega}, \widehat{\varrho}] = i\lambda^1\widehat{p}_\theta + i\lambda^2\widehat{p}_\varphi + \rho^a\bar{\rho}_a$ when it acts on general BRST states (6.1.6). Then, after an elementary integration over the fermionic momenta. The Eq. (4.2.21) becomes

$$\big(\Psi^0_{\mathrm{phys}}, \Upsilon^0_{\mathrm{phys}}\big)^{\varrho}_{\mathrm{BRST}} = \int \mathrm{d}^2\lambda \, \mathrm{d}^3 q \ \psi^*(q) \big[\exp\big(i\lambda^1\widehat{p}_\theta + i\lambda^2\widehat{p}_\varphi\big)\chi\big](q) \ . \tag{6.1.14}$$

This formula resembles the structure of the averaging formula over the gauge group provided the operator $\exp(i\lambda^1\widehat{p}_\theta + i\lambda^2\widehat{p}_\varphi)$ and the λ–integrations can be appropriately defined. The task of the following section is to provide the required auxiliary structures in order to interpret (6.1.14) as the group averaging into the RAQ scheme.

6.1.2 Refined algebraic quantisation

Guided by Sect. 5.1.2 we wish to recognise the states in the expression (6.1.14) as genuine elements of some dense subspace of some auxiliary Hilbert space, and provide a precise

[1] This is a consequence of the identity

$$\int \mathrm{d}^2\rho \, \mathrm{d}^2\bar{\rho} \, \Psi^* \Upsilon = \left(\int \mathrm{d}^2\rho \, \mathrm{d}^2\bar{\rho} \, \Upsilon^* \Psi\right)^* ,$$

that holds for any two general BRST states of the form (6.1.6) (see also footnote 1 in page 101).

definition of the involved unitary operator. We choose $\mathcal{H}_{\text{aux}} := L^3(\mathbb{R}^3, \mathrm{d}^3 q)$ endowed with the positive definite auxiliary inner product

$$(\psi, \chi)_{\text{aux}} := \int_{\mathbb{R}^3} \mathrm{d}^3 q\ \psi^*(q)\chi(q)\ . \tag{6.1.15}$$

The basic quantum operators $\widehat{q}$ (which denotes $\widehat{\theta}$, $\widehat{\varphi}$ and $\widehat{x}$) and $\widehat{p}$ (which denotes $\widehat{p}_\theta$, $\widehat{p}_\varphi$ and $\widehat{p}_x$) act by multiplication and differentiation, respectively (*cf.* Eqs. (6.1.7a)). All of them, $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ obviously included, become essentially self–adjoint on the dense subspace $C_0^\infty(\mathbb{R}^3)$ of infinitely differentiable functions of compact support. We choose $\Phi = C_0^\infty(\mathbb{R}^3)$ as the dense subspace required by RAQ. Therefore

$$\widehat{U}(\lambda^a) := \exp\left(i\lambda^a \widehat{\gamma}_a\right) \tag{6.1.16}$$

becomes a unitary operator for each $\lambda^a \in \mathbb{R}$. The action of $\widehat{U}(\lambda^a)$ on $f \in \Phi$ is

$$\big(\widehat{U}(\lambda^a) f\big)(\theta, \varphi, x) := f(\theta + \lambda^1, \varphi + \lambda^2, x)\ , \tag{6.1.17}$$

so that the group of gauge transformations $\{\widehat{U}(\lambda^a) :\ \lambda^a \in \mathbb{R},\ a = 1, 2\}$ is isomorphic to $\mathbb{R}^2$. The group averaging formula, *cf.* Eq. (6.1.14), reads as

$$(f, g)_{\text{ga}} := \int \mathrm{d}^2\lambda\, \mathrm{d}^3 q\ f^*(q)\ \big[\widehat{U}(\lambda^a) g\big](q)\ . \tag{6.1.18}$$

Straightforward generalisations of the arguments given in Sect. 5.1.2 about (5.1.24) yield that the mapping

$$\begin{aligned} \eta(f)[g] := (f, g)_{\text{ga}} &= \int_{\mathbb{R}^2} \mathrm{d}^2\lambda \left(f, \widehat{U}(\lambda^a) g\right)_{\text{aux}} \\ &= \int_{\mathbb{R}^5} \mathrm{d}^2\lambda\, \mathrm{d}^3 q\ f^*(\theta, \varphi, x)\, g(\theta + \lambda^1, \varphi + \lambda^2, x) \end{aligned} \tag{6.1.19}$$

is a rigging map. Any $\eta(f)$ in the momentum space becomes

$$\eta(f) = (2\pi)^2 \int_{\mathbb{R}^3} \mathrm{d}^2 k\, \mathrm{d}x\ \delta(k_1)\delta(k_2)\, \widetilde{f}^*(k_1, k_2, x)\ , \tag{6.1.20}$$

so, the physical inner product defined by (4.1.9) takes the form

$$(\eta(g), \eta(f))_{\text{RAQ}} = \eta(f)[g] = (2\pi)^2 \int_{\mathbb{R}} \mathrm{d}x\ \widetilde{f}^*(0, 0, x)\, \widetilde{g}(0, 0, x)\ , \tag{6.1.21}$$

which is a non–trivial, hermitian, positive definite bilinear form. The Hilbert space $\mathcal{H}_{\text{phys}}$ is constructed by the Cauchy completion $\overline{\mathrm{Im}(\eta)}$. The averaging procedure projects out the gauge dependence of the wave functions, leaving us with the physical Hilbert space $\mathcal{H}_{\text{phys}} = L^2(\mathbb{R}, \mathrm{d}x)$.

6.2 Rescaled momentum constraints

We begin considering the following rescaling of the constraints (6.1.1):

$$\phi_1 := M(\theta, \varphi, x)\, p_\theta \approx 0 \;, \tag{6.2.1a}$$

$$\phi_2 := N(\theta, \varphi, x)\, p_\varphi \approx 0 \;, \tag{6.2.1b}$$

which is only a particular case of (4.1.23). In order not to change the constraint surface Γ defined by the original constraints (6.1.1), the scaling functions M and N are assumed to be real–valued, nonvanishing, smooth functions on the configuration space. These guarantees that the set of constraints (6.2.1) is regular and irreducible in the sense described in Sect. 2.2. The associated vector fields $X_1 := M(\theta, \varphi, x)\, \partial_\theta$ and $X_2 := N(\theta, \varphi, x)\, \partial_\varphi$ are then linearly independent at each point of the constraint surface. Again depending on the nature of the scaling functions, the vector fields X_a may or may not be complete on the phase space. Both scaling functions are assumed to be positive definite.

In contrast to the algebra (6.1.2), we have for the anholonomic basis (6.2.1) structure functions appearing on the RHS of their PBs

$$\{\phi_a, \phi_b\} = f_{ab}{}^{c}(q)\phi_c \;, \quad (a = 1, 2) \;, \tag{6.2.2}$$

given by

$$f_{11}{}^{a} = 0 = f_{22}{}^{a} \;, \tag{6.2.3a}$$

$$f_{12}{}^{1} = N(\partial_\varphi \ln M) \;, \tag{6.2.3b}$$

$$f_{12}{}^{2} = -M(\partial_\theta \ln N) \;. \tag{6.2.3c}$$

This kind of rescaling is a prototype of the way one can turn a closed gauge algebra, in this case an abelian one, into an open algebra by rescaling each constraint with some nonzero function. These class of rescalings are not harmful at the classical level. The Dirac observables of the theory are still functions on the reduced phase space $\Gamma_{\text{red}} = \{(x, p_x)\} \simeq \mathbb{R}^2$. In the following subsection we give a specific family of scaling functions which covers the spectrum of possibilities in the general rescaling of two momentum constraint with functions on the configuration space.

Rescaling constraints is not the only way in which one can produce open algebras from closed ones; for instance, from a set of reducible constraints whose PBs close with structure constants, one can extract a linearly independent subset of gauge generators whose PBs close with nonconstant structure functions [66].

6.2.1 Rescaling two momentum constraints: A family of scaling functions

We now make a particular choice of the scaling functions M and N. Let M and N be $f(x)e^{\kappa_1\varphi}$ and $g(x)e^{\kappa_2\theta}$, respectively, so that the relations (6.2.1) are reduced to

$$\phi_1 = f(x)e^{\kappa_1\varphi}p_\theta \approx 0 \ , \tag{6.2.4a}$$

$$\phi_2 = g(x)e^{\kappa_2\theta}p_\varphi \approx 0 \ , \tag{6.2.4b}$$

with κ_a real–valued parameters. The nonzero structure functions in the algebra of these specific constraints are

$${f_{12}}^1 = \kappa_1\, g(x)e^{\kappa_2\theta} \ , \tag{6.2.5a}$$

$${f_{12}}^2 = -\kappa_2\, f(x)e^{\kappa_1\varphi} \ , \tag{6.2.5b}$$

from which one can read the following interesting limits:

(1) Let $f(x) = f_0$, $g(x) = g_0$ for all $x \in \mathbb{R}$, with f_0 and g_0 real constants. Under these conditions we further analyse:

(1a) **[Structure functions on the gauge variables only]**. Case $\kappa_1 \neq 0$, $\kappa_2 \neq 0$. One obtains a set of first–class constraints where the structure functions of their algebra only depend on the gauge degrees of freedom θ and φ.

(1b) **[Constant structure functions]**. Case either κ_1 or κ_2 vanishes. A closed gauge algebra is obtained. For instance, if $\kappa_2 = 0$ and $\kappa_1 = \kappa \neq 0$, only one structure function is different from zero taking a constant value, namely

$${f_{12}}^1 = \kappa g_0 \ . \tag{6.2.6}$$

The gauge algebra $\{\phi_1, \phi_2\} = \kappa g_0 \phi_1$ turns out to be isomorphic to the Lie algebra of a triangular subgroup of $GL(2,\mathbb{R})$, a nonunimodular Lie group (see Sect. 6.3.2 below and Appendix E).

(1c) **[Constant scaling functions]**. Case $\kappa_1 = 0 = \kappa_2$. This corresponds to the trivial case of abelian constraints p_θ and p_φ rescaled by the constants f_0 and g_0, respectively. Then the structure functions vanish everywhere.

(2) Let $f(x)$ and $g(x)$ be general nonvanishing smooth functions. Under these conditions we further analyse:

(2a) **[Structure functions on the whole configuration space]**. Case $\kappa_1 \neq 0$, $\kappa_2 \neq 0$. This is the full case. The structure functions depend on all configuration variables, Eq. (6.2.5).

(2b) **[Gauge invariant structure functions]**. Case either κ_1 or κ_2 vanishes. In this case the structure functions are gauge invariant, that is, they only depend on the physical degree of freedom x. For example, if $\kappa_2 = 0$ and $\kappa_1 \equiv \kappa \neq 0$, the only nonzero ($x$–dependent) structure function is

$$f_{12}{}^{1} = \kappa g(x) \ . \tag{6.2.7}$$

The open algebra $\{\phi_1, \phi_2\} = \kappa g(x)\phi_1$ reduces to the case (1b) at each point x. Then the interpretation of the gauge group in this case is immediate: At a point x_0, the gauge group is the nonunimodular group of (1b) generated by the constraints ϕ_1 and ϕ_2 evaluated at x_0 (see Appendix E).

(2c) **[Gauge invariant scaling functions]**. Case $\kappa_1 = 0 = \kappa_2$. This corresponds to the case of rescaling the trivial abelian constraints $p_\theta \approx 0$ and $p_\varphi \approx 0$ with the nonvanishing functions $f(x)$ and $g(x)$ respectively. Such rescalings do not change the abelian nature of the original abelian constraints

$$\{\phi_1, \phi_2\} = 0 = \{\gamma_1, \gamma_2\} \ .$$

6.2.2 Canonical BRST analysis

In the present section we perform the canonical BRST analysis of the system of constraints (6.2.1). In the quantum mechanical analysis we leave the issues of operator domains for later.

The super phase space, where the constraints (6.2.1) are defined, has already been described in Sect. 6.1.1. The symplectic 2−form is given by (6.1.3). The nonminimal sector of constraints, $\pi_a \approx 0$, added to elevate the Lagrange multipliers as degrees of freedom, is not affected by the rescaling of constraints (6.2.1). According to the general discussion in Chap. 3, the BRST generator associated to the system of constraints (6.2.1) has the form

$$\Omega = \Omega^{\min} - i\rho^a \pi_a \tag{6.2.8}$$

with $\Omega^{\min}$ the BRST generator constructed from the minimal sector of constraints, $\phi_a \approx 0$; namely,

$$\Omega^{\min} = M(q)\, p_\theta\, C^1 + N(q)\, p_\varphi\, C^2 + \tfrac{1}{2} f_{ab}{}^{c}(q)\, C^a C^b \bar{\rho}_c \ . \tag{6.2.9}$$

The BRST generator in this case has to be of rank 1. Indeed, since the Latin indices take only two different values, 1 and 2, we have that the second (or higher) order structure functions $\overset{(n)}{U}_{[a_1 a_2 \ldots a_{n+1}]}{}^{[b_1 \ldots b_n]}(q)$, $n \geq 2$, vanish. Only the constraints ϕ_a on the original phase space determine the rank of the set of constraints in the BRST sense. The *ad* *hoc* introduced constraints $\pi_a \approx 0$ are abelian and have vanishing PB with the minimal sector of constraints.

If more than two constraints were present, the possibility of having higher order structure functions in the BRST generator is authentic; however, the following theorem precludes the associated BRST generator to have them, provided the constraints are all linear in momenta.

Theorem 6.2.1 *Let (q^i, p_i) be a representative point on a phase space on which a set of regular and irreducible constraints, linear in momenta, are defined. Then, the classical minimal BRST charge $\Omega^{\min}$ can be taken to be linear in the momenta $(p_i, \bar{\rho}_a)$.*

We place some comments here. First, a generalisation of this theorem, to cases where reducible linear–momentum constraints take place, can be found as Proposition 1 in Ref. [194], its proof can easily be specialised to prove the validity of the Theorem 6.2.1. Second, within Ω the momenta (π_a, ρ^a) of the nonminimal sector enter as in the general expression (3.4.21). Third, it is remarkable that even when the gauge algebra of constraints may contain structure functions of configuration variables, the cumbersome BRST generator (3.4.14) can be truncated to include the zero and first order structure functions only without losing any gauge information[2].

So, although we continue our analysis for the system of constraints (6.2.1), most of our results in this section can be generalised to more than two constraints linear in momenta.

The starting point in the canonical BRST–quantisation of the above systems, is to choose the elementary classical variables and then promote them into quantum operators that fulfil the basic (graded) commutation relations. These steps are achieved using the same choices as in Sect. 6.1.1. The sesquilinear form (6.1.9), with $c = -1$, will be the way we pair any two BRST general states (6.1.6). The basic hermitian operators act on BRST states as in (6.1.7). With these ingredients we have set the arena where we wish a nilpotent and hermitian quantum version of the BRST generator (6.2.9) acts on.

One can readily see that, in contrast to the one–rescaled–constraint system, various terms in $\widehat{\Omega}$ involve non–trivial ordering issues. The purely bosonic $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are specified by the symmetric operators

$$\widehat{\phi}_1 := -i\big(M(q)\partial_\theta + \tfrac{1}{2}(\partial_\theta M)\big) \ , \tag{6.2.10a}$$

$$\widehat{\phi}_2 := -i\big(N(q)\partial_\varphi + \tfrac{1}{2}(\partial_\varphi N)\big) \ . \tag{6.2.10b}$$

[2] An example with physical content where nonconstant structure functions appear in the algebra, and still its BRST generator is of rank 1 is that of general relativity in four dimensions when expressed in Ashtekar variables [195].

Those terms which contain structure functions possess ordering problems arising from the ghost and ghost–momenta graded commutation relations (6.1.8b). Their resolution can be written in two different, but equivalent, ways depending on the position of the fermionic momenta, these are

$$\widehat{\Omega}_R := \widehat{C}^a \left(\widehat{\phi}_a + \frac{i}{2} f_{ab}{}^b(q)\right) + \tfrac{1}{2} f_{ab}{}^c(q)\, \widehat{C}^a\, \widehat{C}^b\, \widehat{\bar{\rho}}_c - i\widehat{\pi}_a \widehat{\rho}^a \ , \tag{6.2.11a}$$

$$\widehat{\Omega}_L := \left(\widehat{\phi}_a - \frac{i}{2} f_{ab}{}^b(q)\right) \widehat{C}^a + \tfrac{1}{2} f_{ab}{}^c(q)\, \widehat{\bar{\rho}}_c\, \widehat{C}^a\, \widehat{C}^b - i\widehat{\rho}^a\, \widehat{\pi}_a \ . \tag{6.2.11b}$$

Both are hermitian in the sesquilinear form (6.1.9) and nilpotent. The BRST operator written as $\widehat{\Omega}_R$ (resp. $\widehat{\Omega}_L$) corresponds to the ordering where the fermionic momenta appear at the right (left) of the fermionic coordinates, that is, $\widehat{\Omega}_R$ (resp. $\widehat{\Omega}_L$) is in a $(C\bar{\rho}, \bar{C}\rho)$– $((\bar{\rho}C, \rho\bar{C})$–) ordering. The term $+\frac{i}{2} f_{ab}{}^b\, \widehat{C}^a$ (resp. $-\frac{i}{2} f_{ab}{}^b\ \widehat{C}^a$) in $\widehat{\Omega}_R$ $(\widehat{\Omega}_L)$ comes from reordering of ghost degrees of freedom. All these ambiguities are of order $\hbar$ (set equal to one here). In the limit $\hbar \to 0$, $\widehat{\phi}_1$ and $\widehat{\phi}_2$ go over into $M(q)p_\theta$ and $N(q)p_\varphi$, respectively, and the BRST operator written in either $\widehat{\Omega}_R$ or $\widehat{\Omega}_L$ form collapses into Ω, possessing hence the right classical limit. Note that in the case of unimodular behaviour in the gauge group ($f_{ab}{}^b(q) \equiv 0$), the order ambiguity introduced by the ghosts does not have any impact in the term linear in $\widehat{C}^a$ within the BRST operator.

We now define the following non–hermitian operators:

$$\widehat{\phi}'_a := \widehat{\phi}_a + \frac{i}{2} f_{ab}{}^b \ , \tag{6.2.12a}$$

$$\widehat{\phi}''_a := \widehat{\phi}_a - \frac{i}{2} f_{ab}{}^b \ , \tag{6.2.12b}$$

which, as can be seen by direct calculation, fulfil the algebra

$$[\widehat{\phi}'_a, \widehat{\phi}'_b] = i\, f_{ab}{}^c(q)\, \widehat{\phi}'_c \ , \tag{6.2.13a}$$

$$[\widehat{\phi}''_a, \widehat{\phi}''_b] = i\, \widehat{\phi}''_c\, f_{ab}{}^c(q) \ . \tag{6.2.13b}$$

These commutators guarantee $\widehat{\Omega}_R^2 = 0 = \widehat{\Omega}_L^2$. In the BRST formalism, $\widehat{\Omega}_R$ and $\widehat{\Omega}_L$ are anomaly free; whereas, in a Dirac–type quantisation only (6.2.12b) is considered to be anomaly free and therefore a good candidate to quantum mechanically represent the constraints in such formalism, especially if we decide not to use self–adjoint constraint operators. There is a Lie algebra morphism between ϕ_a and $\widehat{\phi}'_a$, *cf.* Eq. (6.2.2) and Eq. (6.2.13a). The absence of anomaly in the algebra of the constraints $\widehat{\phi}'_a$ follows from the ghost contribution to the naive $\widehat{\phi}_a$. On these grounds, Dirac condition on physical states reads

$$\widehat{\phi}'_a|\psi\rangle = 0 \quad \Leftrightarrow \quad \widehat{\phi}_a|\psi\rangle = -\frac{i}{2} f_{ab}{}^b|\psi\rangle \ . \tag{6.2.14}$$

In the canonical BRST quantum approach, physical states must be zero ghost numbered BRST invariant states. Trivial zero ghost numbered states are for example of the form (6.1.13). Considering $\widehat{\Omega}_R$ as the physical state selector, $\widehat{\Omega}_R\Psi = 0$, we have that all functions $\psi(q)$ that obey

$$\widehat{\phi}_a\,\psi(q) = -\frac{i}{2} f_{ab}{}^{b}\,\psi(q)\ , \tag{6.2.15}$$

are physical states, where their independence of the Lagrange multipliers comes from the nonminimal sector of the BRST operator. Thus, in the $(C\bar{\rho}, \bar{C}\rho)$−ordering Dirac states are recovered. However, these are not the only solutions to the physical conditions $\widehat{\Omega}_R\Psi = 0 = \widehat{G}\Psi$. Consistently employing the graded commutation relations between the basic operators, the BRST operator $\widehat{\Omega}_R$ can also be written as

$$\widehat{\Omega}_R = \widehat{\phi}'_a\,\widehat{C}^a - f_{12}{}^{1}(q)\,\widehat{C}^1\,\widehat{\bar{\rho}}_1\,\widehat{C}^2 - f_{12}{}^{2}(q)\,\widehat{C}^2\,\widehat{\bar{\rho}}_2\,\widehat{C}^1 - i\widehat{\pi}_a\widehat{\rho}^a\ . \tag{6.2.16}$$

Then it also accepts physical states (6.1.13). These states, are naturally solutions to $\widehat{\Omega}_L\Psi = 0$ and $\widehat{G}\Psi = 0$ simultaneously. In these states we focus our attention from now on.

The sesquilinear form (6.1.9) between physical states $\Psi^0_{\text{phys}} = \psi(q)$ needs to be regularised. The regularised inner product (4.2.21) is introduced with the anti–hermitian gauge fixing operator $i\widehat{\varrho} := -\widehat{\lambda}^a\,\widehat{\bar{\rho}}_a$ that mixes the minimal sector with the nonminimal one. By direct calculation it can be proved that on general BRST states (6.1.6) one has

$$i[\,\widehat{\Omega}_R, \widehat{\varrho}\,] = i\lambda^a\,\widehat{\phi}''_a - iu_a\widehat{C}^a + \rho^a\bar{\rho}_a = i[\,\widehat{\Omega}_L, \widehat{\varrho}\,]\ , \tag{6.2.17}$$

where $u_a := f_{ba}{}^{c}\,\lambda^b\bar{\rho}_c$.

For the regularised inner product (4.2.21) it is significant to provide an explicit formula for the exponential of the operator (6.2.17) as it acts on trivial BRST physical states. Afterwards, in RAQ we wish to use

$$\begin{aligned}\left(\Psi^0_{\text{phys}}, \Upsilon^0_{\text{phys}}\right)^{\varrho}_{\text{BRST}} &:= \left(\Psi^0_{\text{phys}}, e^{i[\widehat{\Omega},\widehat{\varrho}]}\,\Upsilon^0_{\text{phys}}\right)_{\text{BRST}} \\ &= -\int \mathrm{d}^2\lambda\,\mathrm{d}^3q\,\mathrm{d}^2\bar{\rho}\,\mathrm{d}^2\rho\,\psi^*(q)\left[\exp\left(i[\widehat{\Omega}, \widehat{\varrho}]\right)\chi\right](q)\end{aligned} \tag{6.2.18}$$

as an ansatz to construct a physical inner product; however, in order to interpret (6.2.18) as a group averaging formula, the fermionic variables need to be integrated out. For the cases where the structure functions are gauge invariant, the corresponding formula takes a simple form as we show in the next subsection.

Remarks

1. As it was mentioned, there is a canonical relation between the classical set of constraints $\{\gamma_a\}$ and its rescaled counterpart $\{\phi_a\}$ in the super phase space. One

can show that

$$\theta' := \theta \ , \quad \varphi' := \varphi \ , \quad x' := x \ , \quad \lambda'^a := \lambda^a \ , \tag{6.2.19a}$$

$$\rho'^a := \rho^a \ , \quad C'^1 := M(q)\, C^1 \ , \quad C'^2 = N(q) C^2 \ , \tag{6.2.19b}$$

$$p'_\theta := p_\theta - (\partial_\theta \ln M)\, C^1\, \bar\rho_1 - (\partial_\theta \ln N)\, C^2\, \bar\rho_2 \ , \tag{6.2.19c}$$

$$p'_\varphi := p_\varphi - (\partial_\varphi \ln M)\, C^1\, \bar\rho_1 - (\partial_\varphi \ln N)\, C^2\, \bar\rho_2 \ , \tag{6.2.19d}$$

$$p'_x := p_x - (\partial_x \ln M)\, C^1\, \bar\rho_1 - (\partial_x \ln N)\, C^2\, \bar\rho_2 \ , \tag{6.2.19e}$$

$$\pi'_a := \pi_a \ , \quad \bar C'_a := \bar C_a \ , \tag{6.2.19f}$$

$$\bar\rho'_1 := \frac{1}{M(q)}\, \bar\rho_1 \ , \quad \bar\rho'_2 := \frac{1}{N(q)}\, \bar\rho_2 \ , \tag{6.2.19g}$$

is a canonical transformation which takes the BRST generator $\Omega' = p'_\theta\, C'^1 + p'_\varphi C'^2 - i\rho'^a \pi'_a$, proper of the abelian system of constraints $\{p'_\theta, p'_\varphi\}$, into (6.2.8) with $\Omega^{\,\mathrm{min}}$ given by (6.2.9). ▲

6.2.3 Regularised inner product and gauge invariant structure functions

In this section we analyse the regularised BRST inner product for cases where the structure functions only depend on the true degree of freedom x. The cases of structure constants are trivially contained in this analysis.

Under the assumption that ${f_{ab}}^c$ only depends on x, from Appendix D in particular Eq. (D.12), we have that on trivial BRST physical states

$$\exp\big(i[\widehat\Omega, \widehat\varrho\,]\big)\psi = \exp\left(\bar\rho_a {\Big(\frac{e^{-\mathbf{u}} - \mathbb{1}}{\mathbf{u}}\Big)^a}_b\, \rho^b + i\lambda^a {\widehat\phi''}_a\right)\psi \ , \tag{6.2.20}$$

where the 2×2 matrix $\mathbf{u}$ explicitly reads

$$\mathbf{u} = \begin{pmatrix} {f_{21}}^1 \lambda^2 & {f_{12}}^1 \lambda^1 \\ {f_{21}}^2 \lambda^2 & {f_{12}}^2 \lambda^1 \end{pmatrix} \ . \tag{6.2.21}$$

Using (6.2.20) and solving the Gaussian integral in the Grassmann variables, the regularised BRST inner product (6.2.18) is reduced to

$$\big(\Psi^0_{\mathrm{phys}}, \Upsilon^0_{\mathrm{phys}}\big)^\varrho_{\mathrm{BRST}} = \int \mathrm{d}^2\lambda\, \mathrm{d}^3 q\, \det\left[\frac{\mathbb{1} - e^{-\mathbf{u}}}{\mathbf{u}}\right] \det{}^{1/2}\left[e^{\mathbf{u}}\right]\ \psi^*(q) \big[\exp\big(i\lambda^a \widehat\phi_a\big)\chi\big](q) \ . \tag{6.2.22}$$

The matrix $\mathbf{u}$ defined in (6.2.21) vanishes if the the gauge group shows unimodular behaviour, that is, ${f_{ab}}^b(q) = 0$ with $a, b = 1, 2$. In this limit, the determinants in the integrand become the identity. The structure of the regularised BRST inner product coincides with the group averaging formula of a unimodular group provided that the exponential of the constraint operators can be properly interpreted as unitary operator on some auxiliary Hilbert space and that, simultaneously, the integrals over the Lagrange multipliers converge in some sense.

The factor $\det^{1/2}[e^{\mathbf{u}}]$ in the measure of (6.2.22) follows from the use of the well–known matrix identity

$$\det(\exp(A)) = \exp(\mathrm{tr}\,(A)) \ , \tag{6.2.23}$$

applied to the inhomogeneous term in $\widehat{\phi}''_a$ (6.2.12b); this manipulation produces that only the symmetric operators $\widehat{\phi}_a$ (6.2.10) stands inside the exponential. In the case of structure constants in the gauge algebra the factor $\det^{1/2}[e^{\mathbf{u}}]$ coincides with the square root of the modular function $\Delta(g) = \det(\mathrm{Ad}_g)$. This factor turns the 'left–invariant measure', written in the adjoint representation of the group [189]

$$|j_l(\mathbf{u})| := \det\left[\frac{\mathbb{1} - e^{-\mathbf{u}}}{\mathbf{u}}\right] \ , \tag{6.2.24}$$

into the symmetric measure (*cf.* Eq. (4.1.16))

$$|j_0(\mathbf{u})| := \Delta^{1/2}(g)|\,j_l(\mathbf{u})| = \det\left[\frac{e^{\mathbf{u}/2} - e^{-\mathbf{u}/2}}{\mathbf{u}}\right] \ . \tag{6.2.25}$$

The change $\lambda^a \to -\lambda^a$ gives an alternative BRST regularised sesquilinear form; namely

$$\left(\Psi^0_{\mathrm{phys}}, \Upsilon^0_{\mathrm{phys}}\right)^{\varrho}_{\mathrm{BRST}} = \int \mathrm{d}^2\lambda\, \mathrm{d}^3 q\ \det\left[\frac{e^{\mathbf{u}} - \mathbb{1}}{\mathbf{u}}\right] \det^{-1/2}[e^{\mathbf{u}}]\ \psi^*(q)\big[\exp\big(-i\lambda^a\widehat{\phi}_a\big)\chi\big](q). \tag{6.2.26}$$

The 'right–invariant measure' emerges this time

$$|j_r(\mathbf{u})| := \det\left[\frac{e^{\mathbf{u}} - \mathbb{1}}{\mathbf{u}}\right] \ , \tag{6.2.27}$$

but it is compensated by the factor caused by ${f_{ab}}^b \neq 0$ which yields $\det^{-1/2}[e^{\mathbf{u}}]$, that is, $\Delta^{-1/2}(g)$, so that the symmetric measure is recovered (*cf.* Eq. (4.1.16))

$$|j_0(\mathbf{u})| := \Delta^{-1/2}(g)|\,j_r(\mathbf{u})| = \det\left[\frac{e^{\mathbf{u}/2} - e^{-\mathbf{u}/2}}{\mathbf{u}}\right]. \tag{6.2.28}$$

We end this section with some comments. First, formulae (6.2.22) and (6.2.26) are formally valid for nonconstant gauge invariant structure functions, which imply a gauge invariant measure. Second, in general, the inverse matrix of $\mathbf{u}$ may not exist, so we interpret $1/\mathbf{u}$ or $\mathbf{u}^{-1}$ as in Eq. (D.13). Third, the Eq. (6.2.22) shows that when the structure functions are constant, the regularised BRST inner product suitably reduces to the averaging over a Lie group in the measure adopted in Sect. 4.1.3 based on [96]. Fourth, to recover a full quantum theory, an averaging formula is not enough and it has to be supplemented with auxiliary additional structures: auxiliary Hilbert space, dense test space, dense definitions for the constraint operators, rigorous definition for the unitary action of the group, and a sense in which the averaging converges. These issues are generally delicate.

6.3 Refined algebraic quantisation: Artificial structure functions

In this section we sketch the RAQ of models with artificial structure functions introduced in Sect. 6.2.1. The auxiliary Hilbert space in the quantisation will be $\mathcal{H}_{\text{aux}} := L^2(\mathbb{R}^3, \mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}x)$ with the usual inner product

$$(\psi, \chi)_{\text{aux}} := \int_{\mathbb{R}^3} \mathrm{d}^3q\, \psi^*(q)\chi(q) \ , \tag{6.3.1}$$

where d^3q represents the Lebesgue measure $\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}x$ and q represents (θ, φ, x) together.

6.3.1 Gauge invariant scaling functions

Here, we will deal with scaling functions that only depend on the physical degree of freedom x as in the case of (2c) analysed in Sect. 6.2.1. The case of scaling constant functions, item (1c) in the same section, is trivially covered. In general, rescaling constraints by functions that only depend on true degrees of freedom (gauge invariant quantities) does not change the structure of an originally abelian algebra, and it turns closed gauge algebras into algebras with structure functions depending at most on the gauge invariant quantities.

Constraints in the case (2c) are

$$\phi_1 := f(x)p_\theta \approx 0 \ , \tag{6.3.2a}$$

$$\phi_2 := g(x)p_\varphi \approx 0 \ , \tag{6.3.2b}$$

where f and g are nonvanishing, smooth, and real–functions.

In the quantum theory, the basic quantum operators $\widehat{q}$ act by multiplication and the momentum operators $\widehat{p} := -i\partial_q$ act by derivation on elements in $\mathcal{H}_{\text{aux}}$. All of them become self–adjoint on the dense subspace $C_0^\infty(\mathbb{R}^3)$. The construction of the constraint operators presents no ordering difficulties,

$$\widehat{\phi}_1 := f(x)\widehat{p}_\theta \ , \tag{6.3.3a}$$

$$\widehat{\phi}_2 := g(x)\widehat{p}_\varphi \ , \tag{6.3.3b}$$

and become self–adjoint on the dense linear subspace $C_0^\infty(\mathbb{R}^3)$ of all compact supported smooth functions on $\mathbb{R}^3$. The unitary action of the group is then (*cf.* Eq. (5.6.9) and footnote 6 in page 114)

$$(\widehat{U}(\lambda^a)\psi)(\theta,\varphi,x) := \psi(\theta + f(x)\lambda^1, \varphi + g(x)\lambda^2, x) \ , \tag{6.3.4}$$

which shows that $\widehat{U}$ takes elements from $C_0^\infty(\mathbb{R}^3)$ onto $C_0^\infty(\mathbb{R}^3)$.

In this case, the group averaging formula reads

$$(\psi,\chi)_{\text{ga}} := \int \mathrm{d}^2\lambda \ \left(\psi, \widehat{U}(\lambda^a)\chi\right)_{\text{aux}} \ . \tag{6.3.5}$$

From (6.3.4), the multiplication law of the gauge group is $\widehat{U}(\lambda^a)\widehat{U}(\lambda'^a) = \widehat{U}(\lambda^a + \lambda'^a)$, therefore $\{\widehat{U}(\lambda^a) : \lambda^a \in \mathbb{R}\} \simeq \mathbb{R}^2$. We hence take the range of integration in this integral to be the whole $\mathbb{R}^2$. The regularised BRST inner product (6.2.18) indeed acquires the group averaging structure of (6.3.5) in the case where ${f_{ab}}^c = 0$, for all values of a, b and c.

Once again, it is convenient to map $\mathcal{H}_{\text{aux}}$ into $\widetilde{\mathcal{H}}_{\text{aux}} := L^2(\mathbb{R}^3, \mathrm{d}\Theta\mathrm{d}\Xi\mathrm{d}x)$ via the Hilbert space isomorphism:

$$\begin{aligned} \mathcal{H}_{\text{aux}} &\to \widetilde{\mathcal{H}}_{\text{aux}} \ , \\ \psi &\mapsto \widetilde{\psi} \ , \\ \widetilde{\psi}(\Theta,\Xi,x) &:= \sqrt{f(x)g(x)}\,\psi\big(\Theta f(x), \Xi\, g(x), x\big) \ . \end{aligned} \tag{6.3.6}$$

We now translate the group averaging formula (6.3.5) to $\widetilde{\mathcal{H}}_{\text{aux}}$. The Hilbert space $\widetilde{\mathcal{H}}_{\text{aux}}$ is endowed with the positive definite inner product

$$\left(\widetilde{\psi},\widetilde{\chi}\right)_{\widetilde{\text{aux}}} := \int_{\mathbb{R}^3} \mathrm{d}\Theta\mathrm{d}\Xi\mathrm{d}x \ \overline{\widetilde{\psi}(\Theta,\Xi,x)}\widetilde{\chi}(\Theta,\Xi,x) \ . \tag{6.3.7}$$

The group averaging sesquilinear form (6.3.5) becomes

$$\left(\widetilde{\psi},\widetilde{\chi}\right)_{\widetilde{\text{ga}}} = \int_{\mathbb{R}^2} \mathrm{d}^2\lambda \ \left(\widetilde{\psi}, \widetilde{U}(\lambda^a)\widetilde{\chi}\right)_{\widetilde{\text{aux}}} \ , \tag{6.3.8}$$

where the action of the gauge group in the new Hilbert space explicitly reads

$$\left(\widetilde{U}(\lambda^a)\widetilde{\chi}\right)(\Theta,\Xi,x) = \widetilde{\chi}(\Theta + \lambda^1, \Xi + \lambda^2, x) \ . \tag{6.3.9}$$

The system has then been mapped to that in which both functions f and g are the constant function 1. RAQ in $\widetilde{\mathcal{H}}_{\text{aux}}$ can hence be carried out as we did in Sect. 6.1.2. We choose smooth functions of compact support on $\mathbb{R}^3 = \{(\Theta,\Xi,x)\}$ as the linear dense subspace of test states required by the formalism; on this space, the integral (6.3.8) is well defined. At the end of the procedure, the averaging projects out all gauge dependence, (Θ,Ξ), from the wave functions and the physical Hilbert space becomes $L^2(\mathbb{R}, \mathrm{d}x)$. The generalisation to more constraints linear in momenta which originally are abelian is obvious.

6.3.2 Gauge invariant structure functions

In the previous subsection we analyse a system of constraints linear in momenta with scaling functions that only depend on the true degree of freedom x. This rescaling maintained the structure functions equal to zero. In order to obtain nonvanishing structure functions, the originally abelian constraints need to be modified by scaling functions that depend also on the gauge degrees of freedom. In Sect. 6.2.1, we modified the constraints γ_1 and γ_2 (6.1.1) by multiplying them with functions $M(\theta, \varphi, x) := f(x)e^{\kappa_1\varphi}$ and $N(\theta, \varphi, x) := g(x)e^{\kappa_2\theta}$, respectively. The case $\kappa_1 \equiv \kappa \neq 0$, $\kappa_2 = 0$ is considered in this section,

$$\phi_1 := f(x)e^{\kappa\varphi}p_\theta \approx 0 \ , \tag{6.3.10a}$$

$$\phi_2 := g(x)p_\varphi \approx 0 \ , \tag{6.3.10b}$$

whose gauge algebra presents structure functions only depending on the true degree of freedom x

$$\{\phi_1, \phi_2\} = f_{12}{}^1(x)\phi_1 = \kappa g(x)\phi_1 \ . \tag{6.3.11}$$

This system of constraints corresponds to the case (2b) of Sect. 6.2.1, but also contains (1b) as special case. Classically, the set of constraints (6.3.10) is equivalent to the original set (6.1.1), both generate the same constraint surface and have the same Dirac observables. The Hamiltonian vector fields of ϕ_1 and ϕ_2, restricted to the constraint surface $\Gamma = \{(\theta, \varphi, x, 0, 0, p_x)\} \simeq \mathbb{R}^4$, correspond to

$$X_1^+\big|_{(\theta,\varphi,x,p_x)} := f(x)e^{\kappa\varphi}\partial_\theta \ , \tag{6.3.12a}$$

$$X_2^+\big|_{(\theta,\varphi,x,p_x)} := g(x)\partial_\varphi \ . \tag{6.3.12b}$$

Each of these vector fields is complete on Γ and in particular on the configuration space $\mathbb{Q} = \{(\theta, \varphi, x)\} \simeq \mathbb{R}^3$. Given a starting point $q^0 \equiv (\theta_0, \varphi_0, x_0)$, the curve $\varsigma_1^{(q_0)}(t) = (\theta_0 + f(x_0)e^{\kappa\varphi_0}t, \varphi_0, x_0)$, which is well defined for all $t \in \mathbb{R}$, is an integral curve for X_1^+; similarly, with the same starting point the curve $\varsigma_2^{(q_0)}(t) = (\theta_0, \varphi_0 + g(x_0)t, x_0)$ is an integral curve for X_2^+.

In the quantum theory, the construction of the constraint operators is based on the prescription that $\widehat{q}$ acts by multiplication and $\widehat{p}$ by differentiation in the usual way. Building $\widehat{\phi}_a$ hence does not involve any serious ordering issues and read

$$\widehat{\phi}_1 := f(x)e^{\kappa\varphi}\widehat{p}_\theta \ , \tag{6.3.13a}$$

$$\widehat{\phi}_2 := g(x)\widehat{p}_\varphi \ . \tag{6.3.13b}$$

They obey the quantum algebra of constraints with the structure functions to the left of $\widehat{\phi}_1$,

$$[\,\widehat{\phi}_1, \widehat{\phi}_2\,] = i\kappa g(x)\widehat{\phi}_1 \ . \tag{6.3.14}$$

These constraint operators become symmetric with respect to the auxiliary inner product (6.3.1) on the dense subspace of smooth compactly supported functions on $\mathbb{R}^3$, $C_0^\infty(\mathbb{R}^3)$. Using the basic criterion for self–adjointness by Von Neumann, as it was employed in the previous chapter, it is not difficult to see that each pair of deficiency indices (n_+, n_-) associated to each constraint operator $\widehat{\phi}_a$ is $(0,0)$, the constraint operators are hence self–adjoint. The norm of the solutions to $\widehat{\phi}_a\psi = \pm i\psi$ for each a diverges either because of the behaviour at $\theta \to \infty$ or at $\theta \to -\infty$.

The algebra of the quantum constraints (6.3.13) exponentiates into a unitary representation $\widehat{U}$, of the group $B(2,\mathbb{R})$ at each point x, on $\mathcal{H}_{\text{aux}}$. This group is a subgroup of $GL(2,\mathbb{R})$ and consists of the upper triangular matrices (g_{ij}) such that $g_{11} > 1$ and $g_{22} = 1$. In the Appendix E we place the basic properties of this gauge group. In the decomposition (E.4), the group elements are represented by

$$\widehat{U}\Big[\exp\left(\beta\, T_1(x)\right)\Big] = \exp\left(i\beta\widehat{\phi}_1\right) \tag{6.3.15a}$$

$$\widehat{U}\left[\exp\left(\frac{\lambda^2}{2}\,T_2(x)\right)\right] = \exp\left(i\frac{\lambda^2}{2}\widehat{\phi}_2\right) \tag{6.3.15b}$$

where β is given by

$$\beta := \lambda^1\,\frac{\sinh\left(\frac{1}{2}\kappa g(x)\lambda^2\right)}{\frac{1}{2}\kappa g(x)\lambda^2} \ .$$

The structure of the quantum constraints (6.3.13) imply that each element (6.3.15) acts on $\mathcal{H}_{\text{aux}}$ as follows:

$$\left[\exp\left(i\beta\widehat{\phi}_1\right)\psi\right](\theta,\varphi,x) = \psi(\theta + \beta f(x)e^{\kappa\varphi}, \varphi, x) \ , \tag{6.3.16a}$$

$$\left[\exp\left(i\frac{\lambda^2}{2}\widehat{\phi}_2\right)\psi\right](\theta,\varphi,x) = \psi(\theta, \varphi + \lambda^2 g(x)/2, x) \ . \tag{6.3.16b}$$

Hence, in the decomposition (E.4), the action of a general element of $B(2,\mathbb{R})$ on a wave function produces a shift in the φ–direction by $\frac{1}{2}\lambda^2\,g(x)$, see Eq. (6.3.16b), which is followed by a shift in the θ–direction by $\beta f(x)e^{\kappa\varphi}$, see Eq. (6.3.16a), and ends with another shift in the φ–direction by $\frac{1}{2}\lambda^2\,g(x)$. The final result being

$$\left(\widehat{U}(g)\psi\right)(\theta,\varphi,x) = \psi\left(\theta + \beta f(x)e^{\kappa\varphi},\ \varphi + \lambda^2 g(x),\ x\right) \ . \tag{6.3.17}$$

We now wish to test the ‘group averaging’ ansatz (6.2.22), which allows structure functions depending on x, for the constraint algebra (6.3.14) and obtain a physical inner product. Directly from the symmetric measure (6.2.28), the definition of $\mathbf{u}$ (6.2.21), and the structure functions on the RHS of (6.3.11), a direct calculation shows that

$$|j_0(u)| = \frac{\sinh\left[\frac{1}{2}\lambda^2\kappa g(x)\right]}{\frac{1}{2}\lambda^2\kappa g(x)} \; . \tag{6.3.18}$$

This positive quantity coincides with the symmetric measure $\mathrm{d}_0 g$ independently obtained in Appendix E, Eq. (E.9).

Therefore the group averaging ansatz reads

$$\int \mathrm{d}^2\lambda\, \mathrm{d}^3 q\, \frac{\sinh\left[\frac{1}{2}\lambda^2\kappa g(x)\right]}{\frac{1}{2}\lambda^2\kappa g(x)}\, \psi^*(q)\big(\widehat{U}(g)\chi\big)(q) \; . \tag{6.3.19}$$

In order to make sense of this formula we now perform the following isomorphism:

$$\begin{aligned} \mathcal{H}_{\mathrm{aux}} &\to \widetilde{\mathcal{H}}_{\mathrm{aux}}, \\ \psi &\mapsto \widetilde{\psi}, \\ \widetilde{\psi}(\Theta, \Xi, x) &:= \sqrt{f(x)g(x)e^{\kappa\varphi}}\, \psi\big(\Theta f(x)e^{\kappa\varphi}, \Xi\, g(x), x\big) \; . \end{aligned} \tag{6.3.20}$$

The Hilbert space $\widetilde{\mathcal{H}}_{\mathrm{aux}}$ is endowed with the positive definite inner product (6.3.7). The sesquilinear form (6.3.19) is translated into

$$\int \mathrm{d}\lambda^1 \mathrm{d}\lambda^2\, \mathrm{d}\Theta \mathrm{d}\Xi \mathrm{d}x\, \frac{\sinh\left[\frac{1}{2}\lambda^2\kappa g(x)\right]}{\frac{1}{2}\lambda^2\kappa g(x)}\, \widetilde{\psi}^*(\Theta, \Xi, x)\, \widetilde{\chi}(\Theta + \beta, \Xi + \lambda^2, x) \; , \tag{6.3.21}$$

or written in a more familiar way

$$\int \mathrm{d}\beta \mathrm{d}\lambda^2\, \mathrm{d}\Theta \mathrm{d}\Xi \mathrm{d}x\, \widetilde{\psi}^*(\Theta, \Xi, x)\, \widetilde{\chi}(\Theta + \beta, \Xi + \lambda^2, x) \; , \tag{6.3.22}$$

where the definition of β, Eq. (E.5), was used. Allowing λ^a to take values over the whole real line, we have $\beta \in (-\infty, \infty)$ and the expression (6.3.22) is equivalent to (6.1.19). Therefore we have undone the rescaling at the quantum level. The system has been mapped to that in which the scaling functions are the identity.

RAQ in $\widetilde{\mathcal{H}}_{\mathrm{aux}}$ can now be carried out as in Sect. 6.1.2. Choosing smooth functions of compact support on $\mathbb{R}^3 = \{(\Theta, \Xi, x)\}$ as the test state space, on which (6.3.22) is well defined, one can verify that the averaging procedure projects out the gauge dependence on the wave functions leaving the physical Hilbert space $\mathcal{H}_{\mathrm{phys}} = L^2(\mathbb{R}, \mathrm{d}x)$.

6.3.3 Structure functions on full configuration space: Comments

Extension of the results presented in the previous section to the full case (2a), introduced in Sect. 6.2.1, does not seem immediate. Although the classical scaling functions in (6.2.4) will not introduce ordering issues in the construction of the quantum constraints,

$$\widehat{\phi}_1 := f(x) e^{\kappa_1 \varphi} \widehat{p}_\theta \ , \tag{6.3.23a}$$

$$\widehat{\phi}_2 := g(x) e^{\kappa_2 \theta} \widehat{p}_\varphi \ , \tag{6.3.23b}$$

and they can actually be achieved in an anomaly–free way

$$\big[\,\widehat{\phi}_1, \widehat{\phi}_2\,\big] = i f_{12}{}^{a}(q) \widehat{\phi}_a \ , \tag{6.3.24}$$

with $f_{ab}{}^{c}(q)$ given by (6.2.5), a group averaging ansatz must be provided.

The self–adjointness Von Neumann's criterion applied to each constraint operator $\widehat{\phi}_a$ gives the result that each pair of deficiency indices (n_+, n_-) is $(0,0)$. Therefore, the constraint operators exponentiate into a unitary operator. No necessity of an asymptotic analysis of the scaling functions is present, the current scaling functions do not impose any obstacle in the self–adjointness of constraints. However, a specific measure for the group averaging formula must be provided.

In the preceding section the corresponding measure was borrowed from the regularised BRST inner product once it was ghost–free. In the full case (2a), the corresponding BRST inner product unwieldy involves the ghost– and antighost–momenta, *cf.* Eq. (D.8), and it does not seem to be reduced to a Gaussian integral. The measure to use for the group averaging formula in this case is not clear. Nevertheless, we emphasise that while the search for rigging maps in this thesis used the hybrid BRST–group averaging as the starting point, the non–trivial part in showing that a rigging map is actually recovered was in the action of the quantum gauge transformations on the observables. In the previous chapter, particularly the Sect. 5.6.4, a direct analysis of these observables led us in fact to find more rigging maps than those suggested by the group averaging. Should notions of averaging be difficult to generalise to rescaling with more than one constraint, it may hence be well sufficient to focus directly on the action of the quantum gauge transformations on the observables.

CHAPTER 7

General Summary and Discussion

In the context of canonical BRST–quantisation and refined algebraic quantisation, in this dissertation we have dealt with one aspect in the quantisation of constrained systems: rescaling of first–class constraints by nonconstant functions on the configuration space. For completeness, in the first chapters we revisited various topics related to the subject. The standard material of general constrained systems, the BRST classical formalism, the refined algebraic quantisation scheme and the canonical BRST–quantisation were reviewed.

We have investigated the effects of rescaling constraints within the canonical BRST–quantisation and the refined algebraic quantisation procedure. Two different first–class constrained systems were considered: one with a single–constraint linear in one momentum, and the other, with two–constraints linear in momenta; both with a reduced phase space $\mathbb{R}^2$. Whereas rescaling first–class constraints does not affect the classical reduced phase space, and only may change the classification of the gauge algebra, at a quantum level its influence may be more radical. In the canonical BRST–quantisation, rescaling constraints may affect the final form of the Marnelius' regularised BRST inner product for physical states. In the refined algebraic quantisation approach, rescaling constraints alters the options one has to find a rigging map by which the constraints are implemented.

The quantum constraints were implemented by a BRST version of group averaging. In the study of rescaling a single constraint linear in one momentum, we found that the regularised BRST inner product can be written down in a ghost–free way. This allowed us to implement the regularised BRST inner product as a group averaging in the refined algebraic quantisation scheme. The asymptotic properties of the scaling function became essential in the construction of a self–adjoint constraint operator, depending on these properties, we found three qualitatively different cases. In case (i), the rescaled constraint operator is essentially self adjoint in the auxiliary Hilbert space, and the quantisation is equivalent to that with no scaling. In case (ii), the rescaled constraint operator can be symmetrically defined but has no self–adjoint extensions and no quantum theory is recovered. In case (iii), the rescaled constraint operator admits a family of self–adjoint extensions, and the choice of the extension to represent the constraint has significant effect on the quantum theory. In particular, the choice determines whether the quantum theory has superselection sectors.

Within case (iii), we analysed in full a subfamily of rescalings and self–adjoint extensions in which the superselection structures turned out to resemble closely that of Ashtekar–Horowitz–Boulware model [107]. There were however two significant differences, one conceptual and one technical. Conceptually, the superselection sectors in the Ashtekar–Horowitz–Boulware model are determined by the *classical* potential function in the constraint, while in the system studied in this dissertation the superselection sectors are determined by a quantisation ambiguity that has not counterpart in the classical system. Technically, in our system it is 'natural' to consider a wider family of self–adjoint extensions than the family of potential functions considered in [107], and we duly found a wider set of theories. In particular, while all the quantum theories in [107] have finite–dimensional Hilbert spaces, some of our quantum theories have separable Hilbert spaces, and some of them can even be realised as genuine Hilbert subspaces of the auxiliary Hilbert space.

Within those case–(iii) theories that we analysed in full, we found the quantum theory to be insensitive to the remaining freedom in the scaling function. We in particular discovered situations where the quantum gauge group is $\mathbb{R}$ for generic scaling functions but reduces to $U(1)$ in the special case of a constant scaling function: yet this difference between a compact and noncompact gauge group was irrelevant for the quantum theory. The quantum theory coincided with that which is obtained with the compact gauge group projection into the $U(1)$–invariant subspace of the auxiliary Hilbert space by taking an average over the $U(1)$ action. The formalism of refined algebraic quantisation is thus here able to handle seamlessly the transition between a compact and a noncompact gauge group.

In the generalisation of the aforementioned system to that in which two linear momentum constraints are present on the phase space $\mathbb{R}^6$, with reduced phase space $\mathbb{R}^2$, general rescaling does convey a change in the classification of the gauge algebra. Rescaling the constraints with non–gauge invariant scaling functions, turned the abelian gauge algebra into an open gauge algebra when the right choice of scaling functions is done. In contrast, when gauge invariant scaling functions were the only one used, the abelian property of the gauge group is maintained.

We provided a specific parameterised family of scaling functions such that, depending on the values taken by the parameters, the original two–constraints abelian gauge algebra either (1) is maintained, or (2) it corresponds to a gauge algebra with nonunimodular behaviour and nonconstant gauge invariant structure functions, or (3) it is a full open algebra, structure functions depending on all the configuration variables are present. One advantage of the models is that the chosen family of scaling functions permits the construction of self–adjoint constraint operators on the auxiliary Hilbert space. Issues on the asymptotic behaviour of the scaling functions to define self–adjointness of constraints at a quantum level were out of discussion in this family of toy models.

The canonical BRST–quantisation and the refined algebraic quantisation of case (1) corresponded to the quantisation of the rescaled constraints with gauge invariant scaling functions. The trivial case of multiplying the original constraints with nonvanishing constants was hence contained here. We found that the regularised BRST inner product indeed acquired the group averaging structure . We implemented it in the refined algebraic quantisation of the rescaled constrained system as a genuine rigging map. The structure of the scaling functions did not place any obstacle in the construction of self–adjoint extensions to represent the constraints, and the quantum theory turned out to be equivalent to that in which no scaling functions were present at all. The averaging projected out all gauge dependence of the quantum states, and the physical Hilbert space coincided with that expected on intuitive grounds.

In case (2), the regularised BRST inner product suggested a group averaging with a gauge invariant symmetric measure. This averaging was implemented in refined algebraic quantisation. The rescaled constraint operators were anomaly–free defined in the auxiliary Hilbert space as self–adjoint operators. After a specific choice in the factorisation of the gauge group elements, the rescaling was undone at the level of the group averaging and the refined algebraic quantisation of the system with the unscaled constraint was fully recovered. The refined algebraic quantisation of a system with structure functions, though artificially constructed and gauge invariant, was successfully written down. This became the first example of its kind known to the author were structure functions are fully treated in refined algebraic quantisation.

Within the specific family of rescaled constraints, in the case (3) we faced the most challenging difficulties. Although one can specify the constraints as self–adjoint operators on the auxiliary Hilbert space $L^2(\mathbb{R}^3, \mathrm{d}^3q)$ in anomaly–free way, a group averaging ansatz for the system must be provided. This seems to be rather intricate to obtain from the regularised BRST inner product as it takes an unwieldy ghost structure that does not seem to reflect a Gaussian integral in the fermionic momenta. A starting point for future investigations could be the determination of the convergence properties (if any) of the formal series (D.8). If this series converges to $\widehat{E}(t)$, it would give an explicit formula for $\widehat{E}(1)\chi = \exp\left(i[\widehat{\Omega}, \widehat{\varrho}]\right)\chi$ on trivial zero ghost numbered BRST physical states. Inserting this formula into the corresponding Marnelius' regularised inner product (6.2.18) would provide a candidate for a rigging map if the fermionic variables can be integrated out. The importance of providing some insight into the open algebra cases where the structure functions are *not* gauge invariant, even if artificially constructed, is that this property is shared by the gauge algebra of general relativity.

The classical formalism of constrained systems, in terms either of the original phase space or the ghost extended one, can be formally applied to field theories as it stands by: (i) regarding the index i which labels the discrete degrees of freedom as both a discrete index i, to numerate the fields, and a continuous index $\mathbf{x}$ for all points in space, that is, $i \to (i, \mathbf{x})$; (ii) regarding the index a_i or α_i that labels constraints as $a \to (a, \mathbf{x})$ or $\alpha_i \to (\alpha_i, \mathbf{x})$; (iii) interpreting the ordinary summations over the discrete indices as summations of the discrete indices *and* integrations over $\mathbf{x}$; finally, (iv) understanding the partial derivatives with respect to dynamical variables as functional derivatives with respect to the fields. The quantum formalism follows for field theories as it stands only at a heuristic level; for instance, in gauge field theories, subtleties arise when definitions of the Haar measure of the gauge group and auxiliary Hilbert space are meant to be given.

The problem of finding a physical inner product in the quantisation of Dirac constrained systems has never been solved in full generality. Although the study performed in this dissertation may be a welcome addition to our knowledge on the subject, it definitely makes evident that the formal answer to this issue given by the robust general BRST methods does not necessarily have mathematical meaning when more rigorous mathematical treatment is provided in examples that permit it.

APPENDIX A

Intertwining property of the rigging maps

In this Appendix we verify that the rigging maps (5.6.16) and (5.6.19) have the intertwining property (4.1.7d). This completes the proof of the Theorem 5.6.1 and the first item within the Theorem 5.6.2. We follow the method introduced in Appendix B of [107].

To begin, we only assume that R_n satisfies condition (i) of Sect. 5.6.3. The fork between the remaining conditions on α in Sect. 5.6.3 and Sect. 5.6.4 will take place after (A.3), when also condtion (iii) on the subset $\mathcal{P}$ is taken into account.

Let $\widehat{A} \in \mathcal{A}_{\text{phys}}^{(\star)}$, m and n be fixed integers, and $f, g \in \widetilde{\Phi}$ such that their only components in the decomposition (5.6.7) are respectively f_m and g_n. As $\widehat{U}_c(\lambda)$ is unitary and commutes with $\widehat{A}^\dagger$ on the relevant domain, we have $\left(\widehat{U}_c(-\lambda)f, \widehat{A}^\dagger g\right)_c = \left(f, \widehat{U}_c(\lambda)\widehat{A}^\dagger g\right)_c = \left(f, \widehat{A}^\dagger \widehat{U}_c(\lambda) g\right)_c = \left(\widehat{A}f, \widehat{U}_c(\lambda)g\right)_c$. Using (5.6.9) and (5.6.8), the leftmost and rightmost expressions yield

$$\int \mathrm{d}x\, e^{iR_m(x)\lambda}\, f_m^*(x)\, \big(\widehat{A}^\dagger g\big)_m(x) = \int \mathrm{d}x\, e^{iR_n(x)\lambda}\, \big(\widehat{A}f\big)_n^*(x)\, g_n(x)\ . \tag{A.1}$$

We denote the intervals in which R_q has no stationary points by I_{qr}, where the second index r enumerates the intervals with given q. We similarly denote the intervals in which R_q is constant by $\tilde{I}_{q\tilde{r}}$. We take these intervals to be open and inextendible, and we understand 'interval' to include half–infinite intervals and the full real line.

On the LHS (resp. RHS) of (A.1), we break the integral over $x \in \mathbb{R}$ into a sum of integrals over $\{I_{mr}\}$ and $\{\tilde{I}_{m\tilde{r}}\}$ ($\{I_{nr}\}$ and $\{\tilde{I}_{n\tilde{r}}\}$). By the assumptions about R_q, the sums contain at most finitely many terms.

Let R_{qr} be the restriction of R_q to I_{qr}, and let R_{qr}^{-1} be the inverse of R_{qr}. Changing the integration variable in each I_{mr} on the LHS to $s := R_{mr}(x)$ and in each I_{nr} on the RHS to $s := R_{nr}(x)$, we obtain

$$\sum_{\tilde r}\int_{\tilde I_{m\tilde r}} \mathrm{d}x\, e^{iR_m(x)\lambda}\, f_m^*(x)\,\big(\widehat{A}^\dagger g\big)_m(x) + \int \mathrm{d}s\, e^{i\lambda s}\sum_r\left[\frac{f_m^*\,\big(\widehat{A}^\dagger g\big)_m}{|R_m'|}\right]\big(R_{mr}^{-1}(s)\big)$$
$$= \sum_{\tilde r}\int_{\tilde I_{n\tilde r}} \mathrm{d}x\, e^{iR_n(x)\lambda}\,\big(\widehat{A}f\big)_n^*(x)\, g_n(x) + \int \mathrm{d}s\, e^{i\lambda s}\sum_r\left[\frac{\big(\widehat{A}f\big)_n^*\, g_n}{|R_n'|}\right]\big(R_{nr}^{-1}(s)\big)\ , \qquad \text{(A.2)}$$

where for given s the sum over r on the LHS (resp. RHS) is over those r for which s is in the image of R_{mr} (R_{nr}).

We now regard each side of (A.2) as a function of $\lambda \in \mathbb{R}$. On each side, the integral over s is the Fourier transform of an L^1 function and hence vanishes as $|\lambda| \to \infty$ by the Riemann–Lebesgue lemma, whereas the sum over $\tilde r$ is a finite linear combination of imaginary exponentials and does not vanish as $|\lambda| \to \infty$ unless identically zero. A peeling argument shows that (A.2) breaks into the pair

$$\sum_{\tilde r}\int_{\tilde I_{m\tilde r}} \mathrm{d}x\, e^{iR_m(x)\lambda}\, f_m^*(x)\,\big(\widehat{A}^\dagger g\big)_m(x) = \sum_{\tilde r}\int_{\tilde I_{n\tilde r}} \mathrm{d}x\, e^{iR_n(x)\lambda}\,\big(\widehat{A}f\big)_n^*(x)\, g_n(x)\ , \qquad \text{(A.3a)}$$

$$\int \mathrm{d}s\, e^{i\lambda s}\sum_r\left[\frac{f_m^*\,\big(\widehat{A}^\dagger g\big)_m}{|R_m'|}\right]\big(R_{mr}^{-1}(s)\big) = \int \mathrm{d}s\, e^{i\lambda s}\sum_r\left[\frac{\big(\widehat{A}f\big)_n^*\, g_n}{|R_n'|}\right]\big(R_{nr}^{-1}(s)\big)\ . \qquad \text{(A.3b)}$$

Suppose now that condition (ii) of Sect 5.6.3 holds. A peeling argument shows that the $\lambda-$independent component of (A.3a) reads

$$\eta_\infty(f)[\widehat{A}^\dagger g] = \eta_\infty(\widehat{A}f)[g]\ , \qquad \text{(A.4)}$$

where η_∞ is defined in (5.6.16). By linearity, (A.4) continues to hold for all f and g in $\widetilde{\Phi}$. η_∞ hence has the intertwining property (4.1.7d). This completes the proof of Theorem 5.6.1.

Suppose then that condition (ii$'$) of Sect. 5.6.4, holds. Hence only (A.3b) survives, there are not $\widetilde{I}$ intervals. Each side of relation (A.3b) is a Fourier transform of an L^1-function, hence we have the L^1 equality

$$\sum_r\left[\frac{f_m^*\,\big(\widehat{A}^\dagger g\big)_m}{|R_m'|}\right]\big(R_{mr}^{-1}(s)\big) = \sum_r\left[\frac{\big(\widehat{A}f\big)_n^*\, g_n}{|R_n'|}\right]\big(R_{nr}^{-1}(s)\big) \qquad \text{(A.5)}$$

which holds pointwise in s except at the stationary values of R_n and R_m. Note the similarity of this expression with Eq. (B.3) in Appendix B of Ref. [107]. Examination of the small s behaviour of the expression (A.5), by the technique of Appendix B in [107], shows that under the property (iii) of the index set $\mathcal{P}$

$$\eta_p(f)[\widehat{A}^\dagger g] = \eta_p(\widehat{A}f)[g]\ , \qquad \text{(A.6a)}$$
$$\eta_{p\epsilon}(f)[\widehat{A}^\dagger g] = \eta_{p\epsilon}(\widehat{A}f)[g]\ , \qquad \text{(A.6b)}$$

for all p and ϵ for which the maps η_p and $\eta_{p\epsilon}$ (5.6.19) are defined. By linearity, equations (A.6) continue to hold for all f and g in $\widetilde{\Phi}$. Each η_p and $\eta_{p\epsilon}$ hence has the intertwining property (4.1.7d). This completes the the proof of the first item within the Theorem 5.6.2.

APPENDIX B

Representation of $\mathcal{A}^{(\star)}_{\text{phys}}$

In this appendix we review the fact that $\mathcal{A}^{(\star)}_{\text{phys}}$ acts irreducibly on each of the Hilbert spaces $\mathcal{H}^{p}_{\text{phys}}$ and $\mathcal{H}^{p\epsilon}_{\text{phys}}$ of Sect. 5.6.4. Due to the similarity of the rigging maps η_p and $\eta_{p\epsilon}$ given in (5.6.19) to those reported in Ref. [107], we base this appendix on the Appendix C of that reference.

The cases of odd and even p are analysed separately.

Representation on $\mathcal{H}^{p}_{\text{phys}}$

Fix an odd $p \in \mathcal{P}$. In order not to clutter up the notation, the index p will be suppressed in most formulas.

We construct the following observables.

Using the notation introduced in the main text, let x_{pnj} and x_{pmk} be two zeroes. Define the following function from a neighbourhood U_{mj} of x_{mj} to a neighbourhood U_{nk} of x_{nk} by the formula

$$\begin{aligned} h_{nj;mk} : U_{mj} &\longrightarrow U_{nk} \\ : \ x \ &\longmapsto h_{mj;nk}(x) := \widetilde{R}^{-1}_{nk}\big(\widetilde{R}_{mj}(x)\big) \end{aligned} \tag{B.1}$$

where $\widetilde{R}_{mj} := R_m \restriction U_{mj}$ and $\widetilde{R}_{nk} := R_n \restriction U_{nk}$ are smooth functions that map their corresponding domains to a neighbourhood U_0 of $s = 0$. The inverse to each of these functions is well–defined and denoted by $\widetilde{R}^{-1}_{mj}$ and $\widetilde{R}^{-1}_{nk}$, respectively. The function $h_{mj;nk}$

is well–defined and smooth, and we can choose the domains to be pairwise disjoint and such that $h^{-1}_{mj;nk} := \widetilde{R}^{-1}_{mj} \circ \widetilde{R}_{nk} = h_{nk;mj}$.

For each x_{li}, we choose a real–valued smooth function ρ_{li} on $\mathbb{R}$ such that $\rho_{li}(x_{li}) \equiv 1$ and the support of ρ_{li} is contained in the domain of $h_{li;mj}$ for all x_{mj}.

We now define on $\widetilde{\Phi}$ the set of operators $\widehat{A}_{mj;nk}$ (an index p is suppressed here) such that if $f \in \widetilde{\Phi}$, $f(\omega, x) = \sum_l e^{i[l-\alpha(x)]\omega} f_l(x)$, then

$$\big(\widehat{A}_{mj;nk}\, f\big)(\omega, x) := e^{i[m-\alpha(x)]\omega} \big(\widehat{A}_{mj;nk}\, f\big)_m(x) \equiv e^{i[m-\alpha(x)]\omega} \rho_{mj}(x) f_n(h_{mj;nk}(x)) \tag{B.2}$$

where no sum is implicit. In words, $\widehat{A}_{mj;nk}$ takes $\sum_l e^{i[l-\alpha(x)]\omega} f_l(x)$ and discriminates from all the modes the nth, $e^{i[n-\alpha(x)]\omega} f_n(x)$, then modifies the coefficient by $\rho_{nk}(x) f_n(x)$ (which is nonzero only on U_{nk}), and finally maps the result to a vector whose dependence on ω is $e^{i[m-\alpha(x)]\omega}$ and its $x-$dependence is nonzero only near x_{mj}, $\rho_{mj}(x)\, f_n(h_{mj;nk}(x))$. Since the rightmost expression of (B.2) is nonzero only on U_{mj} no change in its value appears if we substitute $\alpha(x)$ by $\alpha(h_{nk,mj}(x))$ in the exponent.

$\widehat{A}_{mj;nk}$ has the following desirable properties: (a) It leaves invariant the test state space, *cf.* Eq. (B.2); (b) a direct calculation shows that $\widehat{A}_{mj;nk}$ commutes with $\widehat{U}_c(\lambda)$ on $\widetilde{\Phi}$; finally, (c) an explicit formula for $\widehat{A}^{\dagger}_{mj;nk}$ can be given

$$\widehat{A}^{\dagger}_{mj;nk}\, f = \widehat{A}_{nk;mj}\, f \ , \tag{B.3}$$

from which $\widehat{A}^{\dagger}_{mj;nk}$ is also defined on $\widetilde{\Phi}$, leaves the test state space invariant, and commutes with $\widehat{U}_c(\lambda)$.

With this preparation we prove the following proposition:

Proposition B.1 *Let $V \subset \mathcal{H}^p_{phys}$ be a linear subspace invariant under $\mathcal{A}^{(\star)}_{\text{phys}}$, $V \neq \{0\}$. Then $V = \mathcal{H}^p_{phys}$.*

Proof. Let $v \in V$, $v \neq 0$. Let $f \in \widetilde{\Phi}$ be such that $\eta_p(f) = v$. We have $f(\omega, x) = \sum_l e^{i[l-\alpha(x)]\omega} f_l(x)$. From (5.6.19a) it follows that there is at least one pair (n, k) such that $f_n(x_{nk}) \neq 0$. For each m and j such that x_{mj} exists, we define $w^{mj} := \widehat{A}_{mj;nk}\, f \in \widetilde{\Phi}$. It follows that $\eta_p(w^{mj}) \in V$, and from the construction of $\widehat{A}_{mj;nk}$ we see that for every $g \in \widetilde{\Phi}$,

$$\eta_p(w^{mj}) = \frac{f^*_n(x_{nk})}{|\alpha^{(p)}(x_{mj})\, N(x_{mj})|^{1/p}}\, g_m(x_{mj}) \ . \tag{B.4}$$

Comparison of (B.4) and (5.6.19a) shows that the set $\{\eta_p(w^{mj})\}$ spans $\mathcal{H}^p_{\text{phys}}$. ∎

Representation on $\mathcal{H}^{p\epsilon}_{\rm phys}$

Now fix an even $p \in \mathcal{P}$, and one of the values for ϵ so that solutions $x_{p\epsilon mj}$ exist.

No restriction of R_m to a neighbourhood U_{mj} of $x_{p\epsilon mj}$ has an inverse in this case. However we can give meaning to $h_{mj;nk}$ if we divide the U_{mj} (resp. U_{nk}) into two parts: U^l_{mj} (U^l_{nk}) and U^r_{mj} (U^l_{nk}), the former lying on the interval I_{ml} (I_{nl}) at the left of x_{mj} (x_{nj}) and the latter contained in the interval I_{mr} (I_{nr}) at the right of x_{mj} (x_{nk}). In this way, we break $h_{mj;nk}$ into its left and right parts,

$$\begin{aligned} h^{l,r}_{mj,nk} : U^{r,l}_{mj} &\longrightarrow U^{l,r}_{nk} \ , \\ : \ x \ &\longmapsto h^{l,r}_{mj,nk}(x) := (\widetilde{R}^{l,r}_{nk})^{-1}\big(\widetilde{R}^{l,r}_{mj}(x)\big) \ , \end{aligned} \tag{B.5}$$

where $\widetilde{R}^{l,r}_{mj} := R_m \restriction U^{l,r}_{mj}$ and $\widetilde{R}^{l,r}_{nk} := R_n \restriction U^{l,r}_{nk}$, each one being a well–defined smooth function with smooth inverse. If $\epsilon > 0$, both $\widetilde{R}^l_{mj}$ and $\widetilde{R}^l_{nk}$ are decreasing functions, whereas, both $\widetilde{R}^r_{mj}$ and $\widetilde{R}^r_{nk}$ are increasing functions. If $\epsilon < 0$, we have the opposite behaviour of $\widetilde{R}^{l,r}_{mj}$ and $\widetilde{R}^{l,r}_{nk}$.

After (B.5) all the arguments go through as for p odd. This completes the proof of item *2.* in the Theorem 5.6.2, and the proof of the theorem itself.

APPENDIX C

Lemmas on asymptotics

We use this appendix to record a pair of lemmas on asymptotics of integrals that occur in Sect. 5.6.4. With the notation $O(u)$ we mean that $u^{-1}O(u)$ remains bounded as $u \to 0$, $o(u)$ is such that $u^{-1}o(u) \to 0$ as $u \to 0$, and $o(1) \to 0$ as $u \to 0$ [196].

Lemma C.1 *Let* $f \in C_0^\infty(\mathbb{R})$, $L > 0$, $p \in \{1, 2, \ldots\}$ *and*

$$G_p(L) := \int_{-\infty}^{\infty} \mathrm{d}u\, f(u)\, \frac{\sin(Lu^p)}{u^p} \ . \tag{C.1}$$

As $L \to \infty$,

$$G_p(L) = \sum_{q=0}^{p-1} K_{p,q}\, f^{(q)}(0)\, L^{(p-1-q)/p} + O\big(L^{-1/p}\big) \ , \tag{C.2}$$

where

$$K_{p,q} = \frac{\sqrt{\pi}\, 2^{(q+1-p)/p}\Gamma\big(\frac{q+1}{2p}\big)}{p\, q!\, \Gamma\big(\frac{3p-q-1}{2p}\big)} \ . \tag{C.3}$$

Proof. (Sketch) We replace $f(u)$ in (C.1) by its Taylor series about the origin, including terms up to u^{p-1}, at the expense of an error that is $O\big(L^{-1/p}\big)$. The terms in the Taylor series give respectively the terms shown in (C.2) plus an error that is $O\big(L^{-1}\big)$. ∎

Let $f \in C_0^\infty(\mathbb{R})$ and $R \in C^\infty(\mathbb{R})$. Let R have at most finitely many zeroes and at most finitely many stationary points, and let all stationary points of R be of finite order. Denote the zeroes of R by x_{pj}, where $p \in \{1, 2, \ldots\}$ is the order of the lowest nonvanishing derivative of R at x_{pj} and j enumerates the zeroes with given p. For $L > 0$, let

$$I(L) := \int_{-\infty}^{\infty} \mathrm{d}x \; f(x) \, \frac{\sin\big[LR(x)\big]}{R(x)} \; . \tag{C.4}$$

Lemma C.2 *As* $L \to \infty$,

$$I(L) = \sum_{pj} I_{pj}(L) + o(1) \; , \tag{C.5}$$

where

$$I_{pj}(L) = K_{p,0} \left(\frac{p!}{\big|R^{(p)}(x_{pj})\big|} \right)^{1/p} f(x_{pj}) \, L^{(p-1)/p} \; + \; \sum_{q=1}^{p-1} A_{pjq} L^{(p-1-q)/p} \; + O\big(L^{-1/p}\big) \tag{C.6}$$

and the coefficients A_{pjq} *can be expressed in terms of derivatives of* f *and* R *at* x_{pj}.

Proof. (Sketch) Lemma C.1 and the techniques of Section II.3 in [196] show that the contribution from a sufficiently small neighbourhood of x_{pj} is I_{pj} (C.5). The techniques in Section II.3 in [196] further show that the contributions from outside these small neighbourhoods are $o(1)$. ∎

Note that $K_{1,0} = \pi$. This will be used to choose $\rho(L) = 2\pi$ in (5.6.21).

APPENDIX D

Calculation of $\exp\big(i[\,\widehat{\Omega},\widehat{\varrho}\,]\big)$

This Appendix is devoted to the analysis of $\exp\big(i[\,\widehat{\Omega},\widehat{\varrho}\,]\big)$ when acting on trivial BRST physical states. For the particular case of gauge invariant structure functions, a tractable expression is derived. The main tools used in this calculation are variations of those used in [89] within its Appendix A where a nonunimodular gauge algebra with structure constants is considered. In our case the calculation involves genuine structure functions on the configuration space.

From (6.2.17) we formally have

$$\exp\big(i[\,\widehat{\Omega},\widehat{\varrho}\,]\big)\Psi = \exp\Big(i\lambda^a\,\widehat{\phi}''_a - iu_a\widehat{C}^a + \rho^a\bar{\rho}_a\Big)\,\Psi\ . \tag{D.1}$$

The following definitions are found useful for our analysis:

$$\widehat{A} := \rho^a\,\bar{\rho}_a\,\mathbb{1},\quad \widehat{B} := -iu_a\,\widehat{C}^a \tag{D.2}$$

with u_a given in the main text. So that $i[\widehat{\Omega},\widehat{\varrho}\,] = i\lambda^a\,\widehat{\phi}''_a + \widehat{A} + \widehat{B}$. On general BRST wave functions the following commutators are shown to be satisfied:

$$[\,i\lambda^a\,\widehat{\phi}''_a, \widehat{A}+\widehat{B}\,] = \lambda^c(\partial_c\,u^a{}_b)\,\bar{\rho}_a\,\widehat{C}^b \tag{D.3}$$

$$[\,\widehat{A},\,\widehat{B}\,] = -\,\bar{\rho}_a\,u^a{}_b\,\rho^b \tag{D.4}$$

with $u^a{}_b = u^a{}_b(q,\lambda) := f_{cb}{}^a(q)\lambda^c$ and the notation ∂_a in this Appendix refers to $-iM(q)\partial_\theta$ and $-iN(q)\partial_\varphi$ when $a=1$ and $a=2$, respectively . Remember $q=(\theta,\varphi,x)$. Define the following operator

$$\widehat{E}(t) := \exp\big(it[\,\widehat{\Omega},\widehat{\varrho}\,]\big)\exp\big(-t\widehat{B}\big)\ . \tag{D.5}$$

Due to the property $\widehat{C}^a\chi = 0$ on trivial zero ghost number physical states χ, one has $\widehat{E}(1)\chi = \exp\left(i[\widehat{\Omega},\widehat{\varrho}]\right)\,\exp\left(-\,\widehat{B}\right)\chi = \exp\left(i[\widehat{\Omega},\widehat{\varrho}]\right)\chi$. Differentiating (D.5) with respect to t one obtains

$$\frac{\mathrm{d}\widehat{E}}{\mathrm{d}t} = \widehat{E}(t)\,\widehat{r}(t)\ , \tag{D.6}$$

with $\widehat{r}(t) \equiv \exp(t\widehat{B})\left(i\lambda^a\widehat{\phi}''_a + \widehat{A}\right)\exp(-t\widehat{B})$. Eq. (D.6) may be solved by iteration of the corresponding integral equation:

$$\widehat{E}(t) = \mathbb{1} + \int_0^t \mathrm{d}t_1\ \widehat{E}(t_1)\,\widehat{r}(t_1) \tag{D.7}$$

where the boundary condition $\lim_{t\to 0}\widehat{E}(t) = \mathbb{1}$ has been introduced, with the result that

$$\begin{aligned}\widehat{E}(t) = \mathbb{1} + \int_0^t \mathrm{d}t_1\ \widehat{r}(t_1) + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\ \widehat{r}(t_2)\widehat{r}(t_1) + \ldots \\ + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\ \widehat{r}(t_n)\widehat{r}(t_{n-1})\cdots\widehat{r}(t_1) + \ldots\ .\end{aligned} \tag{D.8}$$

In this expression the ordering of the operators is important and a 't–ordering' symbol may be used to rewrite this expression. Instead, using the Baker–Hausdorff lemma[1], a direct calculation shows that

$$\widehat{r}(t) = i\lambda^a\widehat{\phi}''_a - \bar{\rho}_a\left(e^{-t\mathbf{u}}\right)^a{}_b\,\rho^b - \lambda^c\bar{\rho}_a(\Delta_c(t))^a{}_b\,\widehat{C}^b\ , \tag{D.9}$$

with

$$(\Delta_c(t))^a{}_b \equiv \partial_c(t\mathbf{u})^a{}_b + \frac{1}{2!}[\partial_c(t\mathbf{u}),(t\mathbf{u})]^a{}_b + \frac{1}{3!}\big[[\partial_c(t\mathbf{u}),(t\mathbf{u})],(t\mathbf{u})\big]^a{}_b + \cdots\ , \tag{D.10}$$

where $\mathbf{u}$ is the matrix with coefficients $u^a{}_b(q,\lambda)$ and the square brackets refer to the usual commutator of matrices $[\mathbf{u},\mathbf{v}]^a{}_b := u^a{}_c\,v^c{}_b - v^a{}_c u^c{}_b$. From expression (D.9), one can see that for Eq. (D.8) the order of operators becomes irrelevant if the structure functions depend at most on the true degree of freedom, that is, $f_{ab}{}^c = f_{ab}{}^c(x)$ only. Indeed, in such cases $(\Delta_c(t))^a{}_b \equiv 0$ and it follows $[\,\widehat{r}(t),\widehat{r}(t')\,] = 0$. Each integrand on the RHS of (D.8) becomes symmetric in the parameters t_i, and one can convince oneself that the n–th integral satisfies the following:

$$n!\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\ \widehat{r}(t_n)\widehat{r}(t_{n-1})\cdots\widehat{r}(t_1) = \left[\int_0^t \mathrm{d}t'\ \widehat{r}(t')\right]^n\ .$$

Hence

$$\widehat{E}(t) = \exp\left(\int_0^t \mathrm{d}t'\,\widehat{r}(t')\right)\ ,\quad \left(f_{ab}{}^c = f_{ab}{}^c(x)\right)\ , \tag{D.11}$$

making use of (D.9) in our corresponding special case, a direct integration gives

$$\widehat{E}(1)\chi = \ \exp\left[i\lambda^a\widehat{\phi}''_a - \bar{\rho}_a\left(\frac{\mathbb{1}-e^{-\mathbf{u}}}{\mathbf{u}}\right)^a{}_b\rho_a\right]\chi\ , \tag{D.12}$$

[1] $e^{tX}Ye^{-tX} = Y + t[X,Y] + \frac{t^2}{2!}[X,[X,Y]] + \frac{t^3}{3!}[X,[X,[X,Y]]] + \ldots$

where the matrix involved in this equation is regarded as the expansion

$$\left(\frac{\mathbb{1}-e^{-\mathbf{u}}}{\mathbf{u}}\right)^{a}{}_{b} \equiv \delta^{a}{}_{b} - \frac{1}{2!}u^{a}{}_{b} + \frac{1}{3!}u^{a}{}_{c}u^{c}{}_{b} - \frac{1}{4!}u^{a}{}_{c}u^{c}{}_{d}u^{d}{}_{b} + \cdots . \tag{D.13}$$

APPENDIX E

The gauge group at x

In this Appendix we place the basic properties of the gauge group $B(2,\mathbb{R})$ and its Lie algebra at each point x.

The gauge group is the subgroup $B(2,\mathbb{R})$ of $GL(2,\mathbb{R})$ that is upper triangular with matrices such that $g_{11} > 0$ and $g_{22} = 1$. This is a two–dimensional, non–abelian group with x–dependent nonunimodular behaviour, that is, ${f_{ab}}^b(x) \neq 0$.

The Lie algebra $\mathfrak{b}(2,\mathbb{R})$, at each point x, is spanned by the following matrices with x dependent coefficients:

$$T_1(x) := \kappa g(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad T_2(x) := \kappa g(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{E.1}$$

which obey the algebra

$$\big[T_1(x), T_2(x)\big] = -\kappa g(x)\, T_1(x) \ . \tag{E.2}$$

The mapping $\phi_a \mapsto T_a$ becomes an anti–homomorphism of Lie algebras at each point x, *cf.* Eq. (6.3.11). Elements of $B(2,\mathbb{R})$ can be written as the exponential of $\lambda^a T_a(x)$, with $\lambda^a \in \mathbb{R}$,

$$\exp\left(\lambda^a T_a(x)\right) = \begin{pmatrix} e^{\lambda^2 \kappa g(x)} & \dfrac{\lambda^1}{\lambda^2}(e^{\lambda^2 \kappa g(x)} - 1) \\ 0 & 1 \end{pmatrix} =: g(\lambda^a) \ \in \ B(2,\mathbb{R}) \ , \tag{E.3}$$

from which $g^{-1}(\lambda^a) = g(-\lambda^a)$ and, in the limit $\lambda^a \to 0$, we have $g(0) = \mathbb{1}$.

An equivalent way to write elements of $B(2,\mathbb{R})$ is through the factorisation

$$g = \exp\left(\frac{\lambda^2}{2}T_2(x)\right)\exp\left(\beta T_1(x)\right)\exp\left(\frac{\lambda^2}{2}T_2(x)\right) \ , \tag{E.4}$$

with β uniquely defined by

$$\beta := \lambda^1 \, \frac{\sinh\left(\frac{1}{2}\kappa g(x)\lambda^2\right)}{\frac{1}{2}\kappa g(x)\lambda^2} \ . \tag{E.5}$$

A direct calculation shows that $g^{-1}\mathrm{d}g = W^a(x)T_a(x)$, from which one can read the left–invariant 1–forms

$$W^1(x) := \frac{1-e^{\lambda^2\kappa g(x)}}{\lambda^2\kappa g(x)}\,\mathrm{d}\lambda^1 + \frac{w(\lambda^a,x)}{\kappa g(x)}\,e^{-\lambda^2\kappa g(x)}\,\mathrm{d}\lambda^2 \ , \tag{E.6a}$$

$$W^2(x) := \ \mathrm{d}\lambda^2 \ , \tag{E.6b}$$

with $w(\lambda^a,x)$ a function whose specific dependence is irrelevant in the present analysis, henceforth we just denote it as w; a similar analysis shows that $\mathrm{d}gg^{-1} = \dot{W}^a(x)T_a(x)$, from which the following right–invariant 1–forms can be read:

$$\dot{W}^1(x) := \frac{e^{\lambda^2\kappa g(x)}-1}{\lambda^2\kappa g(x)}\,\mathrm{d}\lambda^1 + \left[\frac{\lambda^1}{\lambda^2}\left(1-e^{\lambda^2\kappa g(x)}\right) + \frac{w}{\kappa g(x)}\right]\mathrm{d}\lambda^2 \ , \tag{E.7a}$$

$$\dot{W}^2(x) := \ \mathrm{d}\lambda^2 \ . \tag{E.7b}$$

From the expressions (E.6) and (E.7) we respectively construct the left– and right–invariant measures forms

$$(\mathrm{d}_L g)(x) := W^1(x)\wedge W^2(x) = \frac{1-e^{-\lambda^2\kappa g(x)}}{\lambda^2\kappa g(x)}\,\mathrm{d}\lambda^1\mathrm{d}\lambda^2 \ , \tag{E.8a}$$

$$(\mathrm{d}_R g)(x) := \dot{W}^1(x)\wedge \dot{W}^2(x) = \frac{e^{\lambda^2\kappa g(x)}-1}{\lambda^2\kappa g(x)}\,\mathrm{d}\lambda^1\mathrm{d}\lambda^2 \ . \tag{E.8b}$$

The adjoint action of the group $B(2,\mathbb{R})$ on its Lie algebra $\mathfrak{b}(2,\mathbb{R})$ reads $(\mathrm{Ad}_g\,T_1)(x) = (gT_1g^{-1})(x) = e^{\lambda^2\kappa g(x)}\,T_1(x)$ and $(\mathrm{Ad}_g\,T_2)(x) = (gT_2g^{-1})(x) = \dfrac{\lambda^1}{\lambda^2}(1-e^{\lambda^2\kappa g(x)})\,T_1(x) + T_2(x)$. The modular function at each point x can then be obtained $\Delta(g) = \det(\mathrm{Ad}_g) = e^{\lambda^2\kappa g(x)}$. The symmetric measure, invariant under $g\mapsto g^{-1}$, is

$$(\mathrm{d}_0 g)(x) = ([\Delta(g)]^{1/2}\,\mathrm{d}_L g)(x) = ([\Delta(g)]^{-1/2}\,\mathrm{d}_R g)(x) = \frac{\sinh[\frac{1}{2}\lambda^2\kappa g(x)]}{\frac{1}{2}\lambda^2\kappa g(x)} \ . \tag{E.9}$$

References

[1] Dirac, P. A. M. (1964) *Lectures on Quantum Mechanics*. Belfer Graduate School of Sciences, Yeshiva University. (*1, 2, 4, 13, 16, 26, 39* and *71*).

[2] Noether, E. (1918) Invariante Variationsprobleme. *Nachr. v. d. Ges. d. Wiss. zu Göttingen*, pp. 235–257, also available at http://gdz.sub.uni-goettingen.de/. An English version by Tavel M. A. is available at http://cwp.library.ucla.edu/. (*1*).

[3] Nakanishi, N. and Ojima, I. (1990) *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*. Lecture Notes in Physics, Vol. 27, World Scientific. (*1*).

[4] Sundermeyer, K. (1982) *Constrained Dynamics*. Lecture Notes in Physics, Vol. 169, Springer–Verlag. (*1, 13, 26, 38, 41* and *42*).

[5] Rosenfeld, L. (1930) Zur Quantelung der Wellenfelder. *Ann. Phys.*, **5**, 113–152. (*2*).

[6] Bergmann, P. G. (1949) Non–linear field theories. *Phys. Rev.*, **75**, 680–685. (*2* and *26*).

[7] Anderson, J. L. and Bergmann, P. G. (1951) Constraints in covariant field theories. *Phys. Rev.*, **83**, 1018–1025. (*2, 26, 33* and *42*).

[8] Dirac, P. A. M. (1950) Generalized Hamiltonian dynamics. *Can. J. Math.*, **2**, 129–148. (*2*).

[9] Dirac, P. A. M. (1951) The Hamiltonian form of field dynamics. *Can. J. Math.*, **3**, 1–23. (*2*).

[10] Dirac, P. A. M. (1958) Generalized Hamiltonian dynamics. *Proc. Roy. Soc. Lond.*, **A246**, 326–332. (*2*).

[11] Dirac, P. A. M. (1959) Fixation of coordinates in the Hamiltonian theory of gravitation. *Phys. Rev.*, **114**, 924–930. (*2* and *43*).

[12] Salisbury, D. C. (2005) Peter Bergmann and the invention of constrained Hamiltonian dynamics. In *Proc. of the 7th International Conference on the History of General Relativity*, Tenerife, Spain, 10 – 14 March, also available at http://arxiv.org/abs/physics/0608067v1. (*2*).

[13] Salisbury, D. C. (2006) Rosenfeld, Bergmann, Dirac and the invention of constrained Hamiltonian dynamics. In *Proc. of the 11th Marcel Grossmann Meeting*, Berlin, Germany, 23 – 29 July, p. 2467–2469, Ed. by Kleinert H., Jantzen R. T., Ruffini R. World Scientific, also available at http://arxiv.org/abs/physics/0701299v1. (*2*).

[14] Salisbury, D. (2009) Leon Rosenfeld and the challenge of the vanishing momentum in quantum electrodynamics. *Stud. Hist. Philos. Mod. Phys.*, **40**, 363–373, also available at http://arxiv.org/abs/0904.3993v1. (*2*).

[15] Salisbury, D. C. (2010) León Rosenfeld's pioneering steps toward a quantum theory of gravity. In *Proc. of the 1st Mediterranean Conference on Classical and Quantum Gravity*, Crete, Grece, 14 – 18 Sep, 2009, Journal of Physics: Conference Series **222** (2010) 012052. IOP Publishing. (*2*).

[16] Feynman, R. P. and Hibbs, A. R. (1965) *Quantum Mechanics and Path Integrals*. McGraw–Hill. (*4*).

[17] Abers, E. S. and Lee, B. W. (1973) Gauge Theories. *Phys. Rept.*, **9**, 1–141. (*4*).

[18] Feynman, R. P. (1948) Space–time approach to nonrelativistic quantum mechanics. *Rev. Mod. Phys.*, **20**, 367–387. (*4*).

[19] Feynman, R. P. (1948) A relativistic cut–off for classical electrodynamics. *Phys. Rev.*, **74**, 939–946. (*4*).

[20] Feynman, R. P. (1948) Relativistic cutoff for quantum electrodynamics. *Phys. Rev.*, **74**, 1430–1438. (*4*).

[21] Feynman, R. P. (1949) The theory of positrons. *Phys. Rev.*, **76**, 749–759. (*4*).

[22] Feynman, R. P. (1949) Space–time approach to quantum electrodynamics. *Phys. Rev.*, **76**, 769–789. (*4*).

[23] Dyson, F. J. (1949) The Radiation theories of Tomonaga, Schwinger, and Feynman. *Phys. Rev.*, **75**, 486–502. (*4*).

[24] Dyson, F. J. (1949) The S matrix in quantum electrodynamics. *Phys. Rev.*, **75**, 1736–1755. (*4*).

[25] Feynman, R. P. (1963) Quantum theory of gravitation. *Acta Phys. Polon.*, **24**, 697–722. (*4* and *6*).

[26] Faddeev, L. D. and Popov, V. N. (1967) Feynman diagrams for the Yang–Mills field. *Phys. Lett.*, **B25**, 29–30. (*4, 43* and *55*).

[27] Popov, N. V. and Faddeev, L. D. (1967) Perturbation theory for gauge–invariant fields. In (2005) *50 Years of Yang–Mills Theory* with a preface by Gordon D. and Lee B. W., pp. 39–64. Ed. by 't Hooft G., World Scientific. Also quoted as Kiev Report No. ITP 67–36. (*4*).

[28] Faddeev, L. D. (1969) *Teoret. Mat. Fiz.*, **1**, 3–18, translation (1970) Feynman integral for singular Lagrangians. *Theor. Math. Phys.*, **1**, 1–13. (*5, 6, 37, 43* and *55*).

[29] Feynman, R. P. (1951) An operator calculus having applications in quantum electrodynamics. *Phys. Rev.*, **84**, 108–128. (*5*).

[30] Govaerts, J. and Scholtz, F. G. (2004) Revisiting the Fradkin–Vilkovisky theorem. *J. Phys. A*, **A37**, 7359–7380. (*5*).

[31] Grosse–Knetter, C. (1994) Equivalence of Hamiltonian and Lagrangian path integral quantization: Effective gauge theories. *Phys. Rev.*, **D49**, 1988–1995. (*5*).

[32] Grosse–Knetter, C. (1994) Effective Lagrangians with higher derivatives and equations of motion. *Phys. Rev.*, **D49**, 6709–6719. (*5*).

[33] Lopez–Lozano, L. and Toscano, J. (2002) Matthews' theorem in effective Yang–Mills theories. *Rev. Mex. Fis.*, **48**, 23–31. (*5*).

[34] Senjanovic, P. (1976) Path integral quantization of field theories with second class constraints. *Ann. Phys.*, **100**, 227–261, errata: (1991) **209**, 248. (*5* and *43*).

[35] Becchi, C., Rouet, A., and Stora, R. (1974) The abelian Higgs–Kibble model. Unitarity of the S operator. *Phys. Lett.*, **B52**, 344. (*5*, *6* and *63*).

[36] Becchi, C., Rouet, A., and Stora, R. (1975) Renormalization of the abelian Higgs–Kibble model. *Commun. Math. Phys.*, **42**, 127–162. (*5*, *6* and *63*).

[37] Tyutin, I. V. (1975) Gauge invariance in field theory and in statistical physics in the operator formulation. *Lebedev prepint FIAN*, No.39 (in Russian), unpublished. A translation at http://arXiv.org/abs/0812.0580v2. (*5*, *6* and *63*).

[38] Becchi, C., Rouet, A., and Stora, R. (1976) Renormalization of gauge theories. *Ann. Phys.*, **98**, 287–321. (*5*, *6* and *63*).

[39] Itzykson, B. and Zuber, J. B. (1980) *Quantum Field Theory*. International Series in Pure and Applied Physics, McGraw–Hill. (*6*).

[40] Ryder, L. H. (1985) *Quantum Field Theory*. Cambridge University Press, 1st edn. (*6*).

[41] Peskin, M. E. and Schroeder, D. V. (1995) *An Introduction to Quantum Field Theory*. Addison–Wesley Publishing Co. (*6*).

[42] Fisch, J. M. L. and Henneaux, M. (1989) Antibracket–Antifield formalism for constrained Hamiltonian systems. *Phys. Lett.*, **B226**, 80. (*6*).

[43] Dresse, A., Fisch, J. M. L., Gregoire, P., and Henneaux, M. (1991) Equivalence of the Hamiltonian and Lagrangian path integrals for gauge theories. *Nucl. Phys.*, **B354**, 191–217. (*6*).

[44] Nirov, K. and Razumov, A. (1992) Field antifield and BFV formalisms for quadratic systems with open gauge algebras. *Int. J. Mod. Phys.*, **A7**, 5719–5738. (*6*).

[45] Grigorian, G., Grigorian, R., and Tyutin, I. (1992) Equivalence of Lagrangian and Hamiltonian BRST quantizations: The general case. *Nucl. Phys.*, **B379**, 304–320. (*6*).

[46] Nirov, K. and Razumov, A. (1993) Equivalence between Lagrangian and Hamiltonian BRST formalisms. *J. Math. Phys.*, **34**, 3933–3953. (*6*).

[47] Fradkin, E. and Vilkovisky, G. (1975) Quantization of relativistic system with constraints. *Phys. Lett.*, **B55**, 224. (*6*).

[48] Fradkin, E. S. and Vilkovisky, G. A. (1977) Quantization of relativistic systems with constraints. Equivalence of canonical and covariant formalisms in quantum

theory of gravitational field. In *Proc. of the GR8 1977*, Waterloo, Ontario, Canada, 7 – 12 August, p. 401, also quoted as prepint (1977) CERN Report TH–2332. (*6*).

[49] Batalin, I. A. and Vilkovisky, G. A. (1977) Relativistic S matrix of dynamical systems with boson and fermion constraints. *Phys. Lett.*, **B69**, 309–312. (*6*).

[50] Fradkin, E. S. and Fradkina, T. E. (1978) Quantization of relativistic systems with boson and fermion first and second class constraints. *Phys. Lett.*, **B72**, 343. (*6*).

[51] Batalin, I. and Fradkin, E. (1983) A generalized canonical formalism and quantization of reducible gauge theories. *Phys. Lett.*, **B122**, 157–164. (*6* and *8*).

[52] Batalin, I. and Vilkovisky, G. (1981) Gauge algebra and quantization. *Phys. Lett.*, **B102**, 27–31. (*6*).

[53] Batalin, I. and Vilkovisky, G. (1983) Feynman rules for reducible gauge theories. *Phys. Lett.*, **B120**, 166–170. (*6*).

[54] Batalin, I. and Vilkovisky, G. (1983) Quantization of gauge theories with linearly dependent generators. *Phys. Rev.*, **D28**, 2567–2582, errata: (1984) **D30** 508. (*6*).

[55] Batalin, I. and Vilkovisky, G. (1985) Existence theorem for gauge algebra. *J. Math. Phys.*, **26**, 172–184. (*6*).

[56] Zinn–Justin, J. (1975) Renormalization of gauge theories. In *Trends in Elementary Particle Theory: Proc. of the 6th International Summer Institute 1974*, Bonn, Germany, pp. 2–39, Ed. by H. Rollnik and K. Dietz, Lecture Notes in Physics, Vol. 37, Springer–Verlag. (*6*).

[57] Kallosh, R. (1978) Modified Feynman rules in supergravity. *Nucl. Phys.*, **B141**, 141–152. (*6*).

[58] de Wit, B. and van Holten, J. (1978) Covariant quantization of gauge theories with open gauge algebra. *Phys. Lett.*, **B79**, 389. (*6*).

[59] Gomis, J., Paris, J., and Samuel, S. (1995) Antibracket, antifields and gauge theory quantization. *Phys. Rept.*, **259**, 1–145. (*6* and *24*).

[60] Henneaux, M. (1990) Lectures on the antifield–BRST formalism for gauge theories. *Nucl. Phys. Proc. Suppl.*, **18A**, 47–106. (*6*).

[61] Henneaux, M. (1985) Hamiltonian form of the path integral for theories with a gauge freedom. *Phys. Rept.*, **126**, 1–66. (*7, 8, 12, 13, 41, 46, 49, 50, 60, 65, 71* and *87*).

[62] Henneaux, M. (1985) Structure of constrained Hamiltonian systems and Becchi–Rouet–Stora symmetry. *Phys. Rev. Lett.*, **55**, 769–772. (*7*, *13*, *46* and *54*).

[63] Henneaux, M. and Teitelboim, C. (1988) BRST cohomology in classical mechanics. *Commun. Math. Phys.*, **115**, 213–230. (*7*, *13*, *45*, *58* and *65*).

[64] Govaerts, J. (1991) *Hamiltonian Quantisation and Constrained Dynamics*. Leuven University Press. (*7*, *12*, *13*, *26* and *87*).

[65] Henneaux, M. and Teitelboim, C. (1992) *Quantization of Gauge Systems*. Princeton University Press. (*7*, *9*, *12*, *13*, *21*, *23*, *26*, *33*, *41*, *42*, *45*, *60*, *61*, *71*, *87* and *91*).

[66] Kuchař, K. (1986) Hamiltonian dynamics of gauge systems. *Phys. Rev.*, **D34**, 3031–3043. (*8*, *10*, *11*, *71*, *75*, *85* and *128*).

[67] Kuchař, K. (1986) Covariant factor ordering of gauge systems. *Phys. Rev.*, **D34**, 3044–3057. (*8*, *11*, *71* and *75*).

[68] Hájíček, P. and Kuchař, K. (1990) Constraint quantization of parametrized relativistic gauge systems in curved space–times. *Phys. Rev.*, **D41**, 1091–1104. (*8*, *71* and *75*).

[69] McMullan, D. (1987) The use of ghost variables in the description of constrained systems. In *The physics of phase space: Nonlinear dynamics and chaos, geometric quantization, and Wigner function. Proc. of the 1st International Conference*, Maryland, USA, 20 – 23 May, 1986, Lecture Notes in Physics, Vol. 278, pp. 380–382, Springer–Verlag. (*8*, *13*, *65* and *87*).

[70] McMullan, D. and Paterson, J. (1988) Factor ordering and ghost variables. *Phys. Lett.*, **B202**, 358. (*8*, *65* and *87*).

[71] McMullan, D. and Paterson, J. (1989) Covariant factor ordering of gauge systems using ghost variables I: Constraint rescaling. *J. Math. Phys.*, **30**, 477. (*8*, *65* and *87*).

[72] McMullan, D. and Paterson, J. (1989) Covariant factor ordering of gauge systems using ghost variables II: States and observables. *J. Math. Phys.*, **30**, 487. (*8*, *65* and *87*).

[73] Kugo, T. and Ojima, I. (1979) Local covariant operator formalism of nonabelian gauge theories and quark confinement problem. *Prog. Theor. Phys. Suppl.*, **66**, 1. (*9*, *13*, *71* and *91*).

[74] Bognár, J. (1974) *Indefinite Inner Product Spaces*. Springer–Verlag. (*9*, *87*, *89*, *90* and *96*).

[75] Thomi, P. (1988) A remark on BRST quantization. *J. Math. Phys.*, **29**, 1014. (*9* and *100*).

[76] Slavnov, A. A. (1989) Physical unitarity in the BRST approach. *Phys. Lett.*, **B217**, 91–94. (*9*).

[77] Frolov, S. A. and Slavnov, A. A. (1989) Physical subspace norm in Hamiltonian BRST quantization. *Phys. Lett.*, **B218**, 461–464. (*9*).

[78] Hwang, S. and Marnelius, R. (1989) Principles of BRST quantization. *Nucl. Phys.*, **B315**, 638. (*9*, *13*, *71* and *87*).

[79] Hwang, S. and Marnelius, R. (1989) BRST symmetry and a general ghost decoupling theorem. *Nucl. Phys.*, **B320**, 476. (*9*, *13*, *71* and *87*).

[80] Razumov, A. V. and Rybkin, G. N. (1990) State space in BRST–quantization of gauge–invariant systems. *Nucl. Phys.*, **B332**, 209–223. (*9* and *96*).

[81] Rybkin, G. (1991) State space in BRST quantization and Kugo–Ojima quartets. *Int. J. Mod. Phys.*, **A6**, 1675–1692. (*9*).

[82] Marnelius, R. and Ögren, M. (1991) Symmetric inner products for physical states in BRST quantization. *Nucl. Phys.*, **B351**, 474–490. (*9*, *13*, *71*, *87*, *93*, *94* and *95*).

[83] Loll, R. (1992) Canonical and BRST–quantization of constrained systems. *Contemp. Math.*, **132**, 503–530. (*9*, *13*, *71* and *87*).

[84] Marnelius, R. (1993) General state spaces for BRST quantizations. *Nucl. Phys.*, **B391**, 621–650. (*9*, *13*, *71*, *87*, *93*, *95* and *103*).

[85] Marnelius, R. (1993) Simple BRST quantization of general gauge models. *Nucl. Phys.*, **B395**, 647–660. (*9*, *13*, *71*, *87*, *93*, *94* and *103*).

[86] Marnelius, R. (1994) Proper BRST quantization of relativistic particles. *Nucl. Phys.*, **B418**, 353–378. (*9*, *13*, *71* and *94*).

[87] Marnelius, R. (1994) Gauge fixing and abelianization in simple BRST quantization. *Nucl. Phys.*, **B412**, 817–834. (*9*, *13*, *71* and *94*).

[88] Nirov, K. and Razumov, A. (1994) Generalized Schrödinger representation in BRST quantization. *Nucl. Phys.*, **B429**, 389–406. (*9*).

[89] Marnelius, R. and Quaade, U. (1995) BRST quantization of gauge theories like $SL(2,\mathbb{R})$ on inner product spaces. *J. Math. Phys.*, **36**, 3289–3307. (*9, 13, 71, 74, 87, 93, 94* and *154*).

[90] Batalin, I. and Marnelius, R. (1995) Solving general gauge theories on inner product spaces. *Nucl. Phys.*, **B442**, 669–696. (*9, 13, 71, 87, 93* and *94*).

[91] Scholtz, F. G. and Shabanov, S. V. (1998) Gribov vs BRST. *Ann. Phys.*, **263**, 119–132. (*9* and *94*).

[92] Duchting, N., Shabanov, S. V., and Strobl, T. (1999) BRST inner product spaces and the Gribov obstruction. *Nucl. Phys.*, **B538**, 485–514. (*9* and *94*).

[93] Ashtekar, A., Lewandowski, J., Marolf, D., Mourao, J., and Thiemann, T. (1995) Quantization of diffeomorphism invariant theories of connections with local degrees of freedom. *J. Math. Phys.*, **36**, 6456–6493. (*10, 11, 13, 71, 74, 79* and *112*).

[94] Marolf, D. (1995) Refined algebraic quantization: Systems with a single constraint. In (1997) *Symplectic Singularities and Geometry of Gauge Fields*, Banach Center Publications, Polish Academy of Science, Institute of Mathematics Warsaw, pp. 331–344, also available at http://arXiv.org/abs/gr-qc/9508015v3. (*10, 13, 71* and *76*).

[95] Embacher, F. (1998) Hand–waving refined algebraic quantization. *Hadronic J.*, **21**, 337–350, also available at http://arxiv.org/abs/gr-qc/9708016v2. (*10, 13* and *71*).

[96] Giulini, D. and Marolf, D. (1999) A uniqueness theorem for constraint quantization. *Class. Quant. Grav.*, **16**, 2489–2505. (*10, 11, 13, 71, 74, 76, 82, 83, 84* and *135*).

[97] Giulini, D. and Marolf, D. (1999) On the generality of refined algebraic quantization. *Class. Quant. Grav.*, **16**, 2479–2488. (*10, 13, 71, 74* and *78*).

[98] Marolf, D. (2000) Group averaging and refined algebraic quantization: Where are we now?. In *Proc. of the 9^{th} Marcel Grossmann Meeting*, Rome, Italy, 2 – 8 July, p. 1348, Ed. by Gurzadyan V. G., Jantzen R. T., Ruffini R. World Scientific, also available at http://arxiv.org/abs/gr-qc/0011112v1. (*10, 11, 13, 71, 72, 74* and *86*).

[99] Higuchi, A. (1991) Quantum linearization instabilities of de Sitter space–time. 1. *Class. Quant. Grav.*, **8**, 1961–1981. (*10, 72* and *80*).

[100] Higuchi, A. (1991) Quantum linearization instabilities of de Sitter space–time. 2. *Class. Quant. Grav.*, **8**, 1983–2004. (*10, 72* and *80*).

[101] Marolf, D. (1994) The Spectral analysis inner product for quantum gravity. In *Proc. of the 7^{th} Marcel Grossmann Meeting*, California, USA, 24 − 30 July, Ed. by Keiser M and Ruffini R. World Scientific, also available at http://arXiv.org/abs/gr-qc/9409036. (*10*).

[102] Marolf, D. (1995) Observables and a Hilbert space for Bianchi IX. *Class. Quant. Grav.*, **12**, 1441–1454. (*10*).

[103] Gomberoff, A. and Marolf, D. (1999) On group averaging for $SO(n,1)$. *Int. J. Mod. Phys.*, **D8**, 519–535. (*10, 74* and *79*).

[104] Louko, J. and Rovelli, C. (2000) Refined algebraic quantization in the oscillator representation of $SL(2,\mathbb{R})$. *J. Math. Phys.*, **41**, 132–155. (*10, 74* and *83*).

[105] Louko, J. and Molgado, A. (2004) Group averaging in the (p,q) oscillator representation of $SL(2,\mathbb{R})$. *J. Math. Phys.*, **45**, 1919–1943. (*10, 74* and *83*).

[106] Louko, J. and Molgado, A. (2005) Refined algebraic quantisation with the triangular subgroup of $SL(2,\mathbb{R})$. *Int. J. Mod. Phys.*, **D14**, 1131. (*10, 74, 83* and *84*).

[107] Louko, J. and Molgado, A. (2005) Superselection sectors in the Ashtekar–Horowitz–Boulware model. *Class. Quant. Grav.*, **22**, 4007–4020. (*10, 74, 79, 119, 143, 146, 147* and *149*).

[108] Louko, J. (2006) Group averaging, positive definiteness and superselection sectors. *J. Phys. Conf. Ser.*, **33**, 142–149. (*10, 72* and *74*).

[109] Marolf, D. and Morrison, I. A. (2009) Group averaging for de Sitter free fields. *Class. Quant. Grav.*, **26**, 235003. (*10* and *74*).

[110] Ashtekar, A. (1991) *Lectures on Non–perturbative Canonical Gravity.* Lecture notes prepared in collaboration with R. S. Tate. World Scientific Publishing. (*10* and *72*).

[111] Tate, R. (1992), *An Algebraic Approach to the Quantization of Constrained Systems: Finite Dimensional Examples.* Ph.D. Thesis. Syracuse University. Also available at http://arxiv.org/abs/gr-qc/9304043v2. (*10* and *74*).

[112] Ashtekar, A. and Tate, R. S. (1994) An Algebraic extension of Dirac quantization: Examples. *J. Math. Phys.*, **35**, 6434–6470. (*10, 72, 73* and *74*).

[113] Groenewold, H. J. (1946) On the principles of elementary quantum mechanics. *Physica*, **12**, 405–460. (*10* and *70*).

[114] Ashtekar, A. (1980) On the relation between quantum and classical observables. *Comm. Math. Phys.*, **71**, 59–64. (*10* and *72*).

[115] Gotay, M. J. (1995) On a full quantization of the torus. In *Quantization, Coherent States and Complex Structures*, pp. 55–62, Ed. by Antonie J. P. et. al., Plenum. (*10* and *70*).

[116] Gotay, M. J. (1998) Obstructions to quantization. In (1998) *The Juan Simo Memorial Volume*. Ed. by J. Marsden and S. Wiggins. Springer. The paper is substantially revised and updated version of the paper (1996) Gotay, M. J., Grundling H. B. and Tuynman G. M. *J. Nonlinear Sci.* **6**, 469–498. (*10* and *70*).

[117] Gotay, M. J. (1999) On the Groenewold–Van Hove problem for $\mathbb{R}^{2n}$. *J. Math. Phys.*, **40**, 2107–2116. (*10* and *70*).

[118] Giulini, D. (2003) That strange procedure called quantization. In *Proc. of the 271st WE–Heraeus Seminar. Quantum Gravity: From Theory to Experimental Search*, Bad Honnef, Germany, February 26 – March 1, 2002, Lecture Notes in Physics, Vol. 631, pp. 17–40, Springer–Verlag. (*10* and *70*).

[119] Ashtekar, A. and Horowitz, G. T. (1982) On the canonical approach to quantum gravity. *Phys. Rev.*, **D26**, 3342–3353. (*11* and *79*).

[120] Gotay, M. (1986) Constraints, reduction, and quantization. *J. Math. Phys.*, **27**, 2051–2066. (*11*).

[121] Romano, J. D. and Tate, R. S. (1989) Dirac versus reduced space quantization of simple constrained systems. *Class. Quant. Grav.*, **6**, 1487. (*11* and *71*).

[122] Duval, C., Ellhadad, J., Gotay, M. J., and Tuynman, G. M. (1989) Nonunimodularity and the quantization of the pseudo–rigid body. In *Proc. of the CMR Workshop on Hamiltonian Systems, Transformation Groups and Spectral Theorem*, Montréal, Canada, October, Les Publications du CMR, 1990. (*11*).

[123] Tuynman, G. M. (1990) Reduction, quantization, and unimodular groups. *J. Math. Phys.*, **31**, 83. (*11*).

[124] Shvedov, O. Y. (2002) On correspondence of BRST–BFV, Dirac and refined algebraic quantizations of constrained systems. *Ann. Phys.*, **302**, 2–21. (*11*, *71*, *87*, *92*, *93*, *94*, *97* and *103*).

[125] Woodhouse, N. M. J. *Geometric Quantization*. Oxford Mathematical Monographs, Oxford Science Publications, 2nd edn. (*12*).

[126] Klauder, J. R. (1999) Universal procedure for enforcing quantum constraints. *Nucl. Phys.*, **B547**, 397–412. (*12*).

[127] Govaerts, J. and Klauder, J. R. (1999) Solving gauge invariant systems without gauge fixing: The physical projector in $(0+1)-$dimensional theories. *Ann. Phys.*, **274**, 251–288. (*12*).

[128] Klauder, J. R. (2000) Quantization of constrained systems. In (2001) *Methods of Quantization*, Lecture Notes in Physics, Vol. 572, pp. 143–182. Ed. by Latal H. and Schweiger W., Springer. Also available at http://arXiv.org/abs/hep-th/0003297v1. (*12*).

[129] Kempf, A. and Klauder, J. R. (2001) On the implementation of constraints through projection operators. *J. Phys. A*, **A34**, 1019–1036. (*12*).

[130] Thiemann, T. (2006) The Phoenix project: Master constraint program for loop quantum gravity. *Class. Quant. Grav.*, **23**, 2211–2248. (*12*).

[131] Dittrich, B. and Thiemann, T. (2006) Testing the master constraint programme for loop quantum gravity. I. General framework. *Class. Quant. Grav.*, **23**, 1025–1066, Ibid.: Testing the master constraint programme for loop quantum gravity. II. Finite dimensional systems, 1067–1088. Testing the master constraint programme for loop quantum gravity. III. $SL(2,\mathbb{R})$ models, 1089–1120. Testing the master constraint programme for loop quantum gravity. IV. Free field theories, 1121–1142. Testing the master constraint programme for loop quantum gravity. V. Interacting field theories, 1143–1162. (*12*).

[132] Halliwell, J. J. (1988) Derivation of the Wheeler–De Witt equation from a path integral for minisuperspace models. *Phys. Rev.*, **D38**, 2468. (*12*).

[133] Halliwell, J. J. and Hartle, J. B. (1991) Wave functions constructed from an invariant sum over histories satisfy constraints. *Phys. Rev.*, **D43**, 1170–1194. (*12*).

[134] Ferraro, R., Henneaux, M., and Puchin, M. (1994) Path integral and solutions of the constraint equations: The case of reducible gauge theories. *Phys. Lett.*, **B333**, 380–385. (*12*).

[135] Gogilidze, S. A., Khvedelidze, A. M., and Pervushin, V. N. (1996) On abelianization of first class constraints. *J. Math. Phys.*, **37**, 1760–1771. (*12, 61* and *85*).

[136] Han, M. and Thiemann, T. (2010) On the relation between operator constraint–, master constraint–, reduced phase space–, and path integral quantisation. *Class. Quant. Grav.*, **27**, 225019. (*12*).

[137] Han, M. and Thiemann, T. (2010) On the relation between rigging inner product and master constraint direct integral decomposition. *J. Math. Phys.*, **51**, 092501. (*12* and *13*).

[138] Lanczos, C. (1949) *The Variational Principles of Mechanics*. Univeristy of Toronto Press. (*13* and *20*).

[139] Ter Haar, D. (1961) *Elements of Hamiltonian Mechanics*. North–Holland Publishing Co. (*13* and *17*).

[140] Abraham, R. and Marsden, J. E. (1978) *Foundations of Mechanics*. The Benjamin/Cummings Publishing Co., 2nd edn. (*13* and *17*).

[141] Goldstein, H. (1980) *Classical Mechanics*. Addison–Wesley Publishing Company, 2nd edn. (*13*, *17* and *112*).

[142] Landau, L. D. and Lifschitz, E. M. (1981) *Course in Theoretical Physics: Mechanics*, vol. 1. Butterworth–Heinenann, 3rd edn. (*13* and *17*).

[143] Arnold, V. I. (1989) *Mathematical Methods of Classical Mechanics* . Springer–Verlag, 2nd edn. (*13*, *17* and *22*).

[144] Doughty, N. A. (1990) *Lagrangian Interaction: An Introduction to Relativistic Symmetry in Electrodynamics and Gravitation*. Addisson–Wesley Publishing Co. (*13* and *17*).

[145] José, J. V. and Saletan, E. J. (1998) *Classical Dynamics: A Contemporary Approach*. Cambridge University Press. (*13*, *17*, *19*, *21* and *23*).

[146] Gelfand, I. M. and Fomin, S. V. (2000) *Calculus of Variations*. Dover Publications, Inc., Dover edn. (*13* and *18*).

[147] Greiner, W. (2003) *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics*. Springer. (*13* and *17*).

[148] Thornton, S. T. and Marion, J. B. (2004) *Classical Dynamics of Particles and Systems*. Thomson Brooks/Cole, 5th edn. (*13* and *17*).

[149] Sundarshan, E. C. G. and Mukunda, N. (1974) *Classical Dynamics: A Modern Perspective*. John Wiley & Sons. (*13*, *26*, *28* and *29*).

[150] Hanson, A., Regge, T., and Teitelboim, C. (1976) *Constrained Hamiltonian Systems*. Academia Nazionale dei Lincei. (*13* and *26*).

[151] Gitman, D. M. and Tyutin, I. V. (1990) *Quantization of Fields with Constraints*. Springer–Verlag. (*13*, *16*, *26* and *36*).

[152] Dresse, A., Fisch, J., Henneaux, M., and Schomblond, C. (1988) Consistent elimination of redundant second class constaints. *Phys. Lett.*, **B210**, 141–146. (*13*, *41* and *43*).

[153] Wipf, A. W. (1994) Hamilton's formalism for systems with constraints. *Proc. of the 117th WE Heraeus Seminar. Canonical Gravity: From Classical to Quantum*, Bad Honnef, Germany, 13 – 17 September 1993, Lecture Notes in Physics, Vol. 343, pp. 22–58, Springer–Verlag. (*13* and *26*).

[154] Miskovic, O. and Zanelli, J. (2003) Dynamical structure of irregular constrained systems. *J. Math. Phys.*, **44**, 3876–3887. (*13* and *37*).

[155] Thiemann, T. (2007) *Modern Canonical Quantum Gravity*. Cambridge University Press. (*13* and *71*).

[156] Kaminski, W., Lewandowski, J., and Pawlowski, T. (2009) Quantum constraints, Dirac observables and evolution: Group averaging versus Schrodinger picture in LQC. *Class. Quant. Grav.*, **26**, 245016. (*13*).

[157] Louko, J. and Martinez–Pascual, E. (2011) Constraint rescaling in refined algebraic quantisation: Momentum constraint. *J. Math. Phys.*, **52**, 123504. (*14*).

[158] Gitman, D. M. and Kupriyanov, V. G. (2006) Action principle for so–called non–Lagrangian systems. *PoS*, **IC2006**, 016. (*16*).

[159] Gitman, D. M. and Kupriyanov, V. G. (2007) On the action principle for a system of differential equations. *J. Phys.*, **A40**, 10071–10081. (*16*).

[160] Courant, R. and Hilbert, D. (1989) *Methods of Mathematical Physics*, vol. 1–2. John Wiley & Sons. (*18*).

[161] Spivak, M. (1965) *Calculus on Manifolds*. Addison–Wesley Publishing Co. (*19*).

[162] Gavrilov, S. P. and Gitman, D. M. (1993) Quantization of systems with time dependent constraints. Example of relativistic particle in plane wave. *Class. Quant. Grav.*, **10**, 57–67. (*26*).

[163] Henneaux, M., Teitelboim, C., and Zanelli, J. (1990) Gauge invariance and degree of freedom count. *Nucl. Phys.*, **B332**, 169–188. (*32*, *42* and *47*).

[164] Castellani, L. (1982) Symmetries in constrained Hamiltonian systems. *Ann. Phys.*, **143**, 357. (*42*).

[165] Baulieu, L. and Henneaux, M. (1986) Hamiltonian analysis of gauge theories with interacting $p-$forms. *Nucl. Phys.*, **B277**, 268. (*43*).

[166] Hori, T. and Kamimura, K. (1985) Canonical formulation of superstring. *Prog. Theor. Phys.*, **73**, 476. (*43*).

[167] Gogilidze, S. A., Khvedelidze, A. M., and Pervushin, V. N. (1996) On admissible gauges for constrained systems. *Phys. Rev.*, **D53**, 2160–2172. (*43*).

[168] Faddeev, L. D. and Slavnov, A. A. (1991) *Gauge Fields: Introduction to Quantum Theory*. Addisson–Wesley Publishing Co. (*43*).

[169] McMullan, D. (1987) Yang–Mills theory and the Batalin–Fradkin–Vilkovisky formalism. *J. Math. Phys.*, **28**, 428–437. (*45*).

[170] Browning, A. D. and McMullan, D. (1987) The Batalin, Fradkin, and Vilkovisky formalism for higher–order theories. *J. Math. Phys.*, **28**, 438–444. (*45*).

[171] Fisch, J. M. L., Henneaux, M., Stasheff, J., and Teitelboim, C. (1989) Existence, uniqueness and cohomology of the classical BRST charge with ghosts of ghosts. *Commun. Math. Phys.*, **120**, 379–407. (*45*).

[172] Figueroa–O'Farrill, J. M. (1989), *BRST Cohomology and its Applications to Two Dimensional Conformal Field Theory.* Unpublished Ph.D. Thesis. Stony Brook University. (*45*).

[173] Batalin, I. A. and Vilkovisky, G. A. (1984) Closure of the gauge algebra, generalized Lie equations and Feynman rules. *Nucl. Phys.*, **B234**, 106–124. (*61*).

[174] Dirac, P. A. M. (1958) *The Principle of Quantum Mechanics*. The International Series of Monographs in Physics. Oxford University Press, 4th edn. (*70* and *86*).

[175] Geroch, R. (1985) *Mathematical Physics*. The University of Chicago Press. (*72*).

[176] Louko, J. (1993) Holomorphic quantum mechanics with a quadratic Hamiltonian constraint. *Phys. Rev.*, **D48**, 2708–2727. (*74*).

[177] Reed, M. and Simon, M. (1980) *Methods of Modern Mathematical Physics: Functional Analysis*, vol. 1. Academic Press Inc., 2nd edn. (*75* and *109*).

[178] Hájíček, P. (1994) Quantization of systems with constraints. In *Proc. of the 117th WE Heraeus Seminar. Canonical Gravity: From Classical to Quantum*, Bad Honnef, Germany, 13 – 17 September 1993, Lecture Notes in Physics, Vol. 343, pp. 113–149, Springer–Verlag. (*75*).

[179] Bohm, A. and Gadella, M. (1969) *Dirac Kets, Gamow Vectors and Gel'fand Triplets*. Lectures Notes in Physics Vol. 348, Ed. by Bohm and Dollard, Springer–Verlag. (*75*).

[180] Gel'fand, I. M. and Vilenkin, N. Y. (1964) *Generalized Functions: Applications of Harmonic Analysis*, vol. 4. Academic Press, translated version of Obobshchennye Funktsh, Nekotorye Primeneniya Garmonicheskogo Analiza, Vypusk 4, Moscow, 1961. (*75*).

[181] de la Madrid, R. (2005) The role of the rigged Hilbert space in quantum mechanics. *Eur. J. Phys.*, **26**, 287. (*75*).

[182] Boulware, D. G. (1983) Comment on 'On the canonical approach to quantum gravity'. *Phys. Rev.*, **D28**, 414–416. (*79*).

[183] Isham, C. J. (1999) *Modern Differential Geometry for Physicists*. Lecture Notes in Physics, Vol. 61, World Scientific, 2 edn. (*81*).

[184] Hall, B. C. (2003) *Lie Groups, Lie Algebras, and Representations: An elementary introduction*. Graduate Texts in Mathematics, vol. 222. Springer–Verlag. (*81* and *84*).

[185] Lee, J. M. (2003) *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics, vol. 218. Springer–Verlag. (*81* and *106*).

[186] Fecko, M. (2006) *Differential Geometry and Lie Groups for Physicists*. Cambridge University Press. (*81*).

[187] Duval, C., Elhadad, J., Gotay, M. J., Sniatycki, J., and Tuynman, G. M. (1991) Quantization and bosonic BRST theory. *Ann. Phys.*, **206**, 1–26. (*84*).

[188] Tuynman, G. M. (1990) Reduction, quantization, and nonunimodular groups. *J. Math. Phys.*, **31**, 83–91. (*84*).

[189] Rossmann, W. (2002) *Lie Groups: An Introduction Through Linear Groups*. Oxford Graduate Texts in Mathematics, vol. 5, Oxford Univ. Press. (*84* and *135*).

[190] Carslaw, H. S. (1921) *Introduction to the Theory of Fourier's Series and Integrals*. McMillan and Co., 2nd edn. (*104*).

[191] Reed, M. and Simon, M. (1980) *Methods of Modern Mathematical Physics: Fourier Analysis, Self–adjointness*, vol. 2. Academic Press Inc., 2nd edn. (*105*, *109* and *114*).

[192] Bonneau, G., Faraut, J., and Valent, G. (2001) Self–adjoint extensions of operators and the teaching of quantum mechanics. *Am. J. Phys.*, **69**, 322. (*109*).

[193] Blank, J., Exner, P., and Havlíček, M. (2008) *Theoretical and Mathematical Physics: Hilbert Space Operators in Quantum Physics*. Springer. (*109* and *114*).

[194] Ferraro, R., Henneaux, M., and Puchin, M. (1993) On the quantization of reducible gauge systems. *J. Math. Phys.*, **34**, 2757–2778. (*131*).

[195] Ashtekar, A., Mazur, P., and Torre, C. (1987) BRST structure of general relativity in terms of new variables. *Phys. Rev.*, **D36**, 2955–2962. (*131*).

[196] Wong, R. (2001) *Asymptotic Approximations of Integrals*. SIAM, Philadelphia. (*152* and *153*).

Glossary of Symbols

S	Action functional	*16*
L	Lagrangian	*17*
L_i	Euler derivative of L with respect to q^i	*17*
$\mathbb{Q}$	Configuration manifold	*18*
H	Canonical Hamiltonian function	*19*
$\mathbf{T}^*\mathbb{Q}$	Phase space manifold	*20*
$\{f, g\}$	Poisson bracket between f and g	*21*
$\delta G/\delta y^\alpha$	Variational derivative of G with respect to y^α	*23*
$\delta S/\delta y^\alpha(t)$	Functional derivative of S	*24*
$H^{(1)}$	Primary Hamiltonian function	*34*
$\approx$	Weakly symbol	*34*
ϕ_μ	Generic symbol for a Hamiltonian constraint	*37*
Γ	Surface defined by all the constraints present in the theory	*37*
χ_α	Second–class constraints on the original phase space	*40*
γ_a	First–class constraints on the original phase space	*41*
$\{F, G\}^*$	Dirac bracket between F and G	*42*
H_E	Extended Hamiltonian	*46*
λ^a	Lagrange multiplier	*48*

π_a	Conjugate momentum to the Lagrange multiplier λ^a	*48*
$\mathbf{T}^*_\lambda\mathbb{Q}$	Lagrange–multiplier enlarged phase space	*48*
$\overset{(k)}{U}$	$k-th$ order structure function	*54*
η^α	Generic symbol for a classical ghost variable	*55*
$\mathcal{P}_\alpha$	Generic symbol for a classical ghost–momenta variable	*55*
$\epsilon(\cdot)$ or $\epsilon.$	Grassman parity function	*55*
$\dfrac{\partial^\ell f}{\partial\theta}\left(\dfrac{\partial^r f}{\partial\theta}\right)$	Left (Right) partial derivative of f with respect to θ	*56*
$\mathrm{gh}(\cdot)$	Ghost number function	*57*
Ω	BRST classical generator	*58*
$\bar{C}_a, \rho^a$	Antighost, antighost–momentum	*60*
$C^a, \bar{\rho}_a$	Ghost, ghost–momentum	*61*
$\overset{(k)}{A}$	$k-th$ order coefficient in the BRST extension of the Dirac observable A_0	*66*
$\mathcal{H}_{\mathrm{aux}}$	Auxiliary Hilbert space	*71*
η	Rigging map	*71*
$\mathcal{H}_{\mathrm{phys}}$	Physical Hilbert space	*72*
$\mathcal{A}_{\mathrm{aux}}$	Free associate algebra of kinematical operators	*73*
$\mathcal{A}^{(\star)}_{\mathrm{aux}}$	$\star-$ free associate algebra of kinematical operators	*73*
$(\cdot,\cdot)_{\mathrm{aux}}$	Inner product defined in $\mathcal{H}_{\mathrm{aux}}$	*74*
$\widehat{\gamma}_a$	Quantum constraint operator	*75*
$\widehat{U}(g)$	Unitary gauge operator	*75*
$\Phi', \widetilde{\Phi}'$	Algebraic dual to Φ, $\widetilde{\Phi}$	*75*
$\Phi, \widetilde{\Phi}$	Test state space in $\mathcal{H}_{\mathrm{aux}}$	*75*
$\mathcal{A}^{(\star)}_{\mathrm{phys}}$	$\star-$ algebra of (strong) physical observables	*76*
$(\cdot,\cdot)_{\mathrm{RAQ}}$	Physical inner product defined in RAQ through a rigging map	*78*
$\mathrm{d}_L g$	Left–invariant measure on a Lie group G	*81*
$\mathrm{d}_R g$	Right–invariant measure on a Lie group G	*82*
$\Delta(g)$	Modular function of a Lie group G	*82*
$\mathrm{d}_0 g$	Symmetric measure on a Lie group G	*82*

$\mathrm{d}g$	Haar measure on a locally compact Lie group	*84*
$\mathcal{B}_{\mathrm{aux}}$	BRST free associative algebra of kinematical operators	*88*
$\mathcal{B}_{\mathrm{aux}}^{(\star)}$	$\star$–BRST free associative algebra of kinematical operators	*88*
$\widehat{G}$	Ghost number operator	*88*
$\widehat{\Omega}$	BRST operator	*89*
$\mathcal{V}_{\mathrm{BRST}}$	BRST state space	*89*
$(\cdot\,,\cdot)_{\mathrm{BRST}}$	Indefinite BRST inner product on $\mathcal{V}_{\mathrm{BRST}}$	*89*
$\mathrm{Im}(\cdot)$	Image of a mapping	*90*
$(\cdot\,,\cdot)_{\mathrm{BM}}$	Batalin–Marnelius–Shvedov's physical inner product	*94*
$(\cdot\,,\cdot)^{\varrho}_{\mathrm{BRST}}$	Regularised inner product for BRST–invariant states	*95*