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ANALYTIC RENORMALIZATION AND RESIDUES OF FEYNMAN DIAGRAMS

Nikolay Nikolov, Dimitar Nedanovski

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Abstract

In the context of the Nikolov-Stora-Todorov renormalization prescription, we consider the notion of analytic residue of Feynman amplitude and propose a recursive procedure for analytic renormalization in position space.

Key words: renormalization, residues, Feynman amplitudes

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1. Introduction. The notion of analytic renormalization has been introduced by Speer [1,2]. It is based on the observation that multiplication of the propagator Δ_F, by Δ_F^{ε}, for some small enough noninteger complex exponent ε, gives a way for regularization of the Feynman amplitude. The analytic renormalization in [1,2] is carried out in momentum space (and uses a Schwinger-like parametric representation).

In the work [3] (see also [4] for subsequent developments) of Nikolov, Stora and Todorov was systematically developed a renormalization approach[1] (later we shortly refer to it as NST) in position space based on the techniques for continuation of homogeneous distributions. The analytic regularization naturally arises in the NST approach and is one of its pillars.

The current work is an addition to the NST approach, with regard to the construction of the renormalization procedure in therms of analytic regularization. In ([3], Sect. 5.2) the analytic renormalization for the 2-point amplitudes has been

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[1] for massless quantum field theories

carried out. Using the condition for causal[2] factorization (2.4), we propose (in Sect. 4) an algorithm for subtraction of the divergences from a general Feynman amplitude.

Basic notations: M is the Minkowski space, whose elements are denoted by $\mathrm{x} = (x^0, x^1, \dots, x^{D-1})$, y, ... The scalar product in M is $\mathrm{x} \cdot \mathrm{y} := -x^0 y^0 + x^1 y^1 + \cdots + x^{D-1} y^{D-1}$; $\mathrm{x}^2 = \mathrm{x} \cdot \mathrm{x}$. The complexification of the Minkowski space is denoted by $M_{\mathbb{C}} = M + iM$. Let $S \subset \mathbb{N}$ be a finite set. The Sth Cartesian power of M (respectively $M_{\mathbb{C}}$) is denoted by M^S (respectively $M_{\mathbb{C}}^S$). The set $\widehat{\Delta}_S := \big\{ (\mathrm{x}_j)_{j \in S} \in M^S : \mathrm{x}_j = \mathrm{x}_k \text{ for some } j, k \neq j \in S \big\}$ is the *large diagonal* in M^S, while $\Delta_S := \big\{ (\mathrm{x}_j)_{j \in S} \in M^S : \mathrm{x}_j = \mathrm{x}_k \ \forall\, j, k \in S \big\}$ is the *total diagonal* in M^S.

2. Axiomatic requirements on the renormalization maps in Minkowski space. For a finite set $S \subset \mathbb{N}$ of integer labels we introduce an algebra $\mathscr{O}_S$, which by definition is the linear span of all possible products of the form

$$G_S = \prod_{\substack{j,\,k\,\in\,S \\ j\,<\,k}} \frac{P_{jk}(\mathrm{x}_j - \mathrm{x}_k)}{\big((\mathrm{x}_j - \mathrm{x}_k)^2\big)^{N_{jk}}} = \prod_{\substack{j,\,k\,\in\,S \\ j\,<\,k}} G_{jk}\big(\mathrm{x}_j - \mathrm{x}_k\big), \tag{2.1}$$

where $P_{jk}(\mathrm{x})$ are homogeneous polynomials, $N_{jk} \in \mathbb{N}_0$, $G_{jk}(\mathrm{x}) \in \mathscr{O}_2 = \mathscr{O}_{\{1,2\}}$. Also we set $\mathscr{O}_n := \mathscr{O}_{\{1,\dots,n\}}$ so, $\mathscr{O}_S \cong \mathscr{O}_{|S|}$, where $|S|$ stands for the cardinality of S. The elements of $\mathscr{O}_S$ model the Feynman amplitudes of massless QFT and for the moment we consider them as meromorphic functions on $M_{\mathbb{C}}^S$.

In what follows we shall deal with *translation invariant* functions (distributions) on M^S (or, on $M_{\mathbb{C}}^S$), i. e. with functions (distributions) on M^S/M, where the quotient is taken with respect to the action of M on M^S by translations $(\mathrm{x}_j)_{j \in S} \mapsto (\mathrm{x}_j + \mathrm{x})_{j \in S}$.

We can construct distributions from the elements of $\mathscr{O}_S$ in various ways. One way is by taking boundary values with respect to tube domains $\mathcal{T}_{\vec{S}} \subset M_{\mathbb{C}}^S$. Every such tube $\mathcal{T}_{\vec{S}}$ is defined for an *ordered* finite set S. We shall write $\vec{S} = (S, \prec) = \big\langle j_1, \dots, j_n \big\rangle$, for he set $S = \{j_1, \dots, j_n\}$ equipped with a total order $j_1 \prec \cdots \prec j_n$ on it. For every such ordered set we have a standard backward tube domain associated to $\vec{S}$:

$$\mathcal{T}_{\vec{S}} := \Big\{ (\mathrm{x}_j)_{j\,\in\,S} \in M^S + iM^S : \mathrm{x}_{j_k} - \mathrm{x}_{j_{k+1}} \in M - iV_+ \text{ for } k = 1, \dots, n \Big\},$$

where V_+ is the open forward light-cone in M. We define a boundary value map with respect to the tube $\mathcal{T}_{\vec{S}}$:

$$\mathrm{b.v.}_{\vec{S}} : \mathscr{O}_S \to \mathscr{D}'\big(M^S/M\big) =: \mathscr{D}'_S \,.$$

[2]See [5] for a survey on the relativistic causality and renormalization in position space.

For G_S (2.1) we have:

$$\text{(2.2)}\quad \mathrm{b.v.}_{\vec{S}}\, G_S := \prod_{\substack{j,\,k\,\in\, S\\ j\,<\,k}} \frac{P_{jk}(\mathrm{x}_j - \mathrm{x}_k)}{\big((\mathrm{x}_j-\mathrm{x}_k)^2 \pm i0(x_j^0 - x_k^0)\big)^{N_{jk}}}\,, \quad (\pm) = \begin{cases} (+) \text{ if } j \prec k, \\ (-) \text{ if } k \prec j \end{cases}.$$

Let us note that the product in (2.1) is indexed with respect to the standard order $<$ coming from $\mathbb{N}$. But, for taking the boundary value $\mathrm{b.v.}_{\vec{S}}$ in (2.2) the order $\prec$, that is assumed in $\vec{S}$, plays an additional role and hence, the resulting boundary values $\mathrm{b.v.}_{\vec{S}}\, G_S$ will be in general different distributions on M^S/M when we change this order $\prec$ on $\vec{S}$. The maps $\mathrm{b.v.}_{\vec{S}}$ can be used in fact, to produce the Wightman functions of composite fields of any free Wightman fields.

Since if a boundary value of an analytic function vanishes on some open set, then the function is zero everywhere, we have that $\mathrm{b.v.}_{\vec{S}} : \mathscr{O}_S \to \mathscr{D}'_S$ *is an injection*, and also $\mathrm{b.v.}_{\vec{S}}$ maps commute with the action of the differential operators with polynomial coefficients. Another property of the boundary value maps is that they preserve the multiplication,

$$\mathrm{b.v.}_{\vec{S}}\big(G'G''\big) \,=\, \mathrm{b.v.}_{\vec{S}}\big(G'\big)\,\mathrm{b.v.}_{\vec{S}}\big(G''\big).$$

The renormalization gives rise to other linear maps (see [3], Sect. 5) of type $R : \mathscr{O}_S \to \mathscr{D}'_S$, which we shall axiomatically characterize here. Before that let us note that the linear spaces $\mathscr{O}_S$ and $\mathscr{D}'_S$ form *inductive systems* in $S \subset \mathbb{N}$ and hence, it is convenient to take the inductive limits:

$$\mathscr{O}_{\mathbb{N}} \,=\, \bigcup_{\substack{S\,\subset\,\mathbb{N}\\ \text{finite}}} \mathscr{O}_S \,=\, \bigcup_{n\,=\,1}^{\infty} \mathscr{O}_n\,, \qquad \mathscr{D}'_{\mathbb{N}} \,=\, \bigcup_{\substack{S\,\subset\,\mathbb{N}\\ \text{finite}}} \mathscr{D}'_S \,=\, \bigcup_{n\,=\,1}^{\infty} \mathscr{D}'_n\,.$$

Then, a renormalization map is a linear map:

$$\text{(2.3)}\qquad R : \mathscr{O}_{\mathbb{N}} \to \mathscr{D}'_{\mathbb{N}}\,, \quad \text{such that} \quad R\big(\mathscr{O}_S\big) \,\subseteq\, \mathscr{D}'_S$$

and satisfies the axiomatic conditions $(r1)$–$(r5)$ listed below.

$(r1)$ *Permutation symmetry.* First, we have a natural action $\sigma : \mathscr{O}_{\mathbb{N}} \to \mathscr{O}_{\mathbb{N}}$, $\sigma : \mathscr{D}'_{\mathbb{N}} \to \mathscr{D}'_{\mathbb{N}}$ of the permutations $\sigma \in \mathscr{S}(\mathbb{N})$, thus it is appropriate to request $\sigma \circ R = R \circ \sigma$.

$(r2)$ *Preservation of the filtrations.* The image $R(G_S)$ is an associate homogeneous distribution with a degree of associate homogeneity (as defined in [3], Sect. 3.1) less or equal to the degree of homogeneity of G_S.

$(r3)$ *Commutativity with multiplication by polynomials.* If p is a polynomial on M^S/M ($S \subset \mathbb{N}$), then $R(pG) = p\,R(G)$.

$(r4)$ *Causality.* For every disjoint union $S = S' \,\dot{\cup}\, S''$ we have

$$\text{(2.4)}\qquad R\big(G_S\big)\Big|_{\mathfrak{C}_{S';S''}} \,=\, R\big(G_{S'}\big)\,R\big(G_{S''}\big)\,\mathrm{b.v.}_{\overrightarrow{S'\,\dot{\cup}\,S''}}\big(G_{S',\,S''}\big)\Big|_{\mathfrak{C}_{S';S''}}$$

for every $G_S \in \mathscr{O}_S$ of the form (2.1). Our notations in Eq. (2.4) are the following: $\mathfrak{C}_{S';S''}$ is the open region:

$$\mathfrak{C}_{S';S''} := \Big\{ (\mathrm{x}_j)_{j\,\in S} \in M^S : \mathrm{x}_{j'} \gtrsim \mathrm{x}_{j''} \text{ for } j' \in S' \text{ and } j'' \in S'' \Big\}$$

(the relation $\mathrm{x}_{j'} \gtrsim \mathrm{x}_{j''}$ stands for $\mathrm{x}_{j''} \notin \mathrm{x}_{j'} - \overline{V}_+$); we consider some order on the sets S' and S'' and equip S with an order denoted by $\overrightarrow{S' \dot\cup S''}$ and induced by $S' \prec S''$; we also introduce the splitting

$$(2.5) \quad \begin{gathered} G_S = G_{S'}\, G_{S''}\, G_{S',\,S''} \,, \\ G_{S'} = \prod_{\substack{j,\,k\,\in S' \\ j\,\prec\,k}} G_{jk} \,, \quad G_{S''} = \prod_{\substack{j,\,k\,\in S'' \\ j\,\prec\,k}} G_{jk} \,, \quad G_{S',\,S''} = \prod_{\substack{j\,\in S' \\ j\,\in S''}} G_{jk} \,. \end{gathered}$$

The right hand side of Eq. (2.4) is well defined due to the following (see [3], Lemma 2.6):

Lemma 2.1. *The product* $R\big(G_{S'}\big)R\big(G_{S''}\big)$ *b.v.*$_{\overrightarrow{S' \dot\cup S''}}\big(G_{S',\,S''}\big)$ *of the three distributions* $R\big(G_{S'},\, R\big(G_{S''}\big)\big)$ *and b.v.*$_{\overrightarrow{S' \dot\cup S''}}\big(G_{S',\,S''}\big)$ *exists on* M^S/M*.*

($r5$) *Lorentz invariance.* The map $R : \mathscr{O}_{\mathbb{N}} \to \mathscr{D}'_{\mathbb{N}}$ intertwines the natural actions of the Lorentz group on $\mathscr{O}_{\mathbb{N}}$ and $\mathscr{D}'_{\mathbb{N}}$.

This completes our general axiomatic requirements on the renormalization map R in the Minkowski space.

3. Analytic regularization. The residue functional. For a finite index set $S \subset \mathbb{N}$ let us introduce a collection of complex parameters

$$(3.1) \qquad \boldsymbol{\varepsilon}_S = \big(\varepsilon_{j,k} \in \mathbb{C} : (j,k) \in S^{\times 2},\, j < k\big)\,.$$

(For a convenience $\boldsymbol{\varepsilon}_{k,j} := \boldsymbol{\varepsilon}_{j,k}$.) We say that the parameters $\boldsymbol{\varepsilon}_S$ are **generic** iff for every $S' \subseteq S$:

$$2\,|\boldsymbol{\varepsilon}_{S'}| \notin \mathbb{Z}\,, \quad \text{where} \quad |\boldsymbol{\varepsilon}_{S'}| := \sum_{j,\,k\,\in S';\; j\,<\,k} \varepsilon_{j,k}\,.$$

Now, let us set

$$(3.2) \qquad \varrho_S := \prod_{j,k\in S;j<k} \big((\mathrm{x}_j - \mathrm{x}_k)^2\big)^{\varepsilon_{j,k}}$$

Note that the function ϱ_S is not in $\mathscr{O}_S$. The product (3.2) can be understood as a well defined analytic function on every tube domain $\mathcal{T}_{\vec{S}}$, yet one has to be careful since ϱ_S depends on the tube $\mathcal{T}_{\vec{S}}$ (when we change the order in $\vec{S}$). Note that when we take a boundary value of the product (3.2) with respect to $\mathcal{T}_{\vec{S}}$ we obtain:

$$(3.3) \quad \text{b.v.}_{\vec{S}}\, \varrho_S = \prod_{\substack{j,k\in S \\ j<k}} \big((\mathrm{x}_j - \mathrm{x}_k)^2 \pm i0(x_j^0 - x_k^0)\big)^{\varepsilon_{j,k}} = \prod_{\substack{j,\,k\,\in S \\ j\,\prec\,k}} \big((\mathrm{x}_j - \mathrm{x}_k)^2 + i0(x_j^0 - x_k^0)\big)^{\varepsilon_{j,k}}\,,$$

where the ($\pm$) convention in (3.3) is the same as in Eq. (2.2) and the second equality is due to the formal symmetry of the product (3.2).

Theorem 3.1. *For every* $G_S \in \mathscr{O}_S$ *of the form* (2.1) *with* homogeneous *polynomials* P_{jk} *and for every* generic *set of parameters* $\boldsymbol{\varepsilon}_S$ *there exists a unique* homogeneous *distribution* $U_{\boldsymbol{\varepsilon}_S}(G) \in \mathscr{D}'_S$ *such that*

$$U_{\boldsymbol{\varepsilon}_S}(G_S)\mathfrak{C}_{S';S''} = U_{\boldsymbol{\varepsilon}_{S'}}(G_{S'})U_{\boldsymbol{\varepsilon}_{S''}}(G_{S''})\, b.v._{\overrightarrow{S'\cup S''}}(\varrho_{S',S''}G_{S',S''})\mathfrak{C}_{S';S''}, \tag{3.4}$$

where $\varrho_{S',S''} := \prod\limits_{j\in S';\ k\in S''} \big((\mathrm{x}_j - \mathrm{x}_k)^2\big)^{\varepsilon_{j,k}}$ *and we follow the same conventions as in (r4). The map* $U_{\boldsymbol{\varepsilon}_S}$ *extends to a linear map* $U_{\boldsymbol{\varepsilon}_S} : \mathscr{O}_S \to \mathscr{D}'_S$ *and* $U_{\boldsymbol{\varepsilon}_S}(G_S)$ *depend analytically on* $\boldsymbol{\varepsilon}_S$ *for generic values of the parameters* $\boldsymbol{\varepsilon}_S$*.*

Proof. The proof is by induction in $|S|$ and we briefly outline it here. Let $|S| = 2$ and take $S = \{1,2\}$, $G \in \mathscr{O}_2$. Then Eq. (3.4) reads

$$U_{\varepsilon_{1,2}}(G)\Big|_{\mathfrak{C}_{1;2}} = \mathrm{b.v.}_{\langle 1,2\rangle}\big(\varrho_{\{1,2\}}\,G\big)\Big|_{\mathfrak{C}_{1;2}},\quad U_{\varepsilon_{1,2}}(G)\Big|_{\mathfrak{C}_{2;1}} = \mathrm{b.v.}_{\langle 2,1\rangle}\big(\varrho_{\{1,2\}}\,G\big)\Big|_{\mathfrak{C}_{2;1}}. \tag{3.5}$$

One checks that the right hand sides of Eqs. (3.5) coincide on the intersection $\mathfrak{C}_{1;2} \cap \mathfrak{C}_{2;1}$. Hence, Eqs. (3.5) define a translation invariant distribution on $\mathfrak{C}_{1;2} \cup \mathfrak{C}_{2;1}$. But $\mathfrak{C}_{1;2} \cup \mathfrak{C}_{2;1} = M^{\times 2} \setminus \Delta_2$, where Δ_2 is the total diagonal $\{\mathrm{x}_1 - \mathrm{x}_2 = 0\}$. Taking into account the translation invariance we obtain by (3.5) a distribution $\overset{\bullet}{U}_{\varepsilon_{1,2}}(G)$ on $\big(M^{\times 2} \setminus \Delta_2\big)/M \cong M\setminus\{0\}$, which by construction is homogeneous of degree $\deg G + 2\boldsymbol{\varepsilon}_{1,2}$ if the degree of homogeneity of G is $\deg G$. Thus, according to the results of ([6], Sect. 3.2), $\overset{\bullet}{U}_{\varepsilon_{1,2}}(G)$ extends uniquely to a homogeneous distribution $U_{\varepsilon_{1,2}}(G)$ on M if $2\varepsilon_{1,2}$ is noninteger. This proves the basis of our induction.

Aside from some technicalities, the proof of the induction step is essentially the same as the proof of the base case. $\square$

By the construction in ([6], Sect. 3.2) of extension of homogeneous distributions of noninteger degree of homogeneity it follows that $U_{\boldsymbol{\varepsilon}_S}(G_S)$ will have a simple pole at $|\boldsymbol{\varepsilon}_S| = 0$. Thus, we can write

$$U_{\boldsymbol{\varepsilon}_S}(G_S) = \frac{1}{2|\boldsymbol{\varepsilon}_S|}\,\mathrm{Res}_S(G_S) + T_{\boldsymbol{\varepsilon}_S}(G_S), \tag{3.6}$$

where $\mathrm{Res}_S(G_S)$ is a distribution supported at the origin $0 \in M^S/M$ and $T_{\boldsymbol{\varepsilon}_S}(G_S)$ is analytic at $|\boldsymbol{\varepsilon}_S| = 0$ if $|\boldsymbol{\varepsilon}_{S'}|$ is noninteger for all $S' \subsetneq S$. Hence, we obtain a linear map

$$\mathrm{Res}_S : \mathscr{O}_S \to \mathscr{D}'_S\,, \tag{3.7}$$

which we call a **residue** of $G_S \in \mathscr{O}_S$. By the uniqueness of the extension of distributions of noninteger degree of homogeneity we obtain also that for every polynomial p_S on M^S/M:

$$U_{\boldsymbol{\varepsilon}_S}(p_S\,G_S) = p_S\,U_{\boldsymbol{\varepsilon}_S}(G_S)\,, \tag{3.8}$$

which further implies (see [3], Eq. 4.13) the following expansion of Res_S :

$$\mathrm{Res}_S\big(G_S\big) \;=\; \sum_{\mathbf{r}\,\in\,\mathbb{N}_0^{D(|S|-1)}} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!}\;\mathrm{res}_S\big(\mathbf{x}_S^{\mathbf{r}}\,G_S\big)\,\delta^{(\mathbf{r})}(\mathbf{x}_S)\,. \tag{3.9}$$

In (3.9) we use the following notations: $\mathbf{x}_S$ stands for some fixed set of linear coordinates in $M^S/M \cong M^{\times(|S|-1)}$; $\delta(\mathbf{x}_S)$ is the delta function with respect to the coordinates $\mathbf{x}_S$, and hence, its support is at $\mathbf{x}_S = 0$, which is the total diagonal Δ_S; $\mathbf{r}$ is a multiindex and we set $\mathbf{x}_S^{\mathbf{r}} := \prod\limits_{\xi\,=\,1}^{D(|S|-1)} (x^{\xi})^{r_{\xi}}$ and $\delta^{(\mathbf{r})}(\mathbf{x}_S) := \prod\limits_{\xi\,=\,1}^{D(|S|-1)} \big(\frac{\partial}{\partial x^{\xi}}\big)^{r_{\xi}}\delta(\mathbf{x}_S)$ if we enumerate the components of $\mathbf{x}_S$ and $\mathbf{r}$ with a single index $\xi = 1,\dots,D(|S|-1)$, i.e., $\mathbf{x}_S := (x^{\xi})$ and $\mathbf{r} = (r^{\xi})$. Thus, we have characterized the residue map (3.7) just by one linear functional

$$\mathrm{res}_S : \mathscr{O}_S \to \mathbb{C}\,.$$

Note that res_S is of degree $D(|S|-1)$, i. e., it vanishes if $\deg G + D(|S|-1) \neq 0$ since the delta function $\delta(\mathbf{x}_S)$ has a homogenous degree $-D(|S|-1)$.

4. Renormalization and pole subtractions. Our goal now is to extract a regular part in $U_{\boldsymbol{\varepsilon}_S}(G_S)$, which we shall denote by $R_{\boldsymbol{\varepsilon}_S}(G_S)$, and which is such that it is regular (analytic) at $\boldsymbol{\varepsilon}_S = 0$ and the map $R_S := R_{\boldsymbol{\varepsilon}_S}\Big|_{\boldsymbol{\varepsilon}_S\,=\,0}$ satisfies the axioms of Sect. 2. In Eq. (3.6) we have already seen a kind of a pole subtraction, but it only removes the singularity at $|\boldsymbol{\varepsilon}_S| = 0$, while there can be singularities at $|\boldsymbol{\varepsilon}_{S'}| = 0$ for $S' \subsetneqq S$. On the other hand, outside the total diagonal Δ_S the term with $\frac{1}{|\boldsymbol{\varepsilon}_S|}$ singularity in (3.6) vanishes and we can apply (3.4) on the open covering

$$\bigcup_{\substack{S\,=\,S'\,\dot\cup\,S''\text{ is a}\\ \text{proper partition}}} \mathfrak{C}_{S';S''} \;=\; M^S\backslash\Delta_S\,.$$

Iterating this procedure we can expect that there are singularities related to every partial diagonal in the large diagonal $\widehat{\Delta}_S$.

The decomposition of $U_{\boldsymbol{\varepsilon}_S}(G_S)$ into singular and regular parts generally should has the following form:

$$U_{\boldsymbol{\varepsilon}_S}\big(G_S\big) \;=\; \sum_{\substack{\mathfrak{P}\text{ is a}\\ S\text{–partition}}} \left(\prod_{S'\,\in\,\mathfrak{P}} \frac{1}{2|\boldsymbol{\varepsilon}_{S'}|}\,\mathrm{Res}_{S'}\big(G_{S'}\big)\right) R_{\boldsymbol{\varepsilon}_{\mathfrak{P}}}\Big(G_{S/\mathfrak{P}}\Big)\,. \tag{4.1}$$

However, (4.1) is not quite explicit and we have to make it more precise (cf. Eq. (4.4)). We start with explaining our notations. First of all, we assume in (4.1) that $G_S \in \mathscr{O}_S$ has the form (2.1). Then $G_{S'}$, for $S' \in \mathfrak{P}$ are defined according to the unique decomposition:

$$G_S = G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} G_{S'}, \tag{4.2}$$

where $G_{S'} \in \mathscr{O}_{S'}$ for $S' \in \mathfrak{P}$ and $G_{\mathfrak{P}} \in \mathscr{O}_{\mathfrak{P}}$:= the subalgebra of $\mathscr{O}_S$ generated by all $\mathscr{O}_{\{j,k\}}$ for $j,k \in S$, $j \nsim_{\mathfrak{P}} k$. We introduce a similar splitting for the set $\boldsymbol{\varepsilon}_S$:

$$\boldsymbol{\varepsilon}_S = \boldsymbol{\varepsilon}_{\mathfrak{P}} \cup \bigcup_{S' \in \mathfrak{P}} \boldsymbol{\varepsilon}_{S'}, \quad |\boldsymbol{\varepsilon}_S| = |\boldsymbol{\varepsilon}_{\mathfrak{P}}| + \sum_{S' \in \mathfrak{P}} |\boldsymbol{\varepsilon}_{S'}|.$$

Thus, we explained the meaning of $G_{S'}$ in Eq. (4.1). Concerning the meaning of "$G_{S/\mathfrak{P}}$", Eq. (4.1) is not precise. We have instead a well defined function $G_{\mathfrak{P}}$. Because of the presence of the product $\prod\limits_{S' \in \mathfrak{P}} \frac{1}{2|\boldsymbol{\varepsilon}_{S'}|} \operatorname{Res}_{S'}\big(G_{S'}\big)$ in (4.1), which is supported on $\Delta_{\mathfrak{P}}$, the function $G_{\mathfrak{P}}$ will be "restricted" to a function on $M^{S/\mathfrak{P}}$, where the quotient $S/\mathfrak{P}$ means $S/\sim_{\mathfrak{P}}$. We shall identify $S/\mathfrak{P}$ with the subset of S formed by the minimal elements of the sets $S' \in \mathfrak{P}$. Above, we put "restricted" in quotation marks since this restriction will also include derivatives because of the derivatives of the delta function. In more details, let us denote $\mathscr{D}'_{S,0} := \big\{u \in \mathscr{D}'\big(M^S/M\big) : \operatorname{supp} u \subseteq \{0\}\big\}$, i. e., the linear span of all delta function $\delta(\mathbf{x}_S)$ and its derivatives. Then, for an S-partition $\mathfrak{P}$ we set:

$$\mathscr{D}'_{\mathfrak{P},0} := \bigotimes_{S' \in \mathfrak{P}} \mathscr{D}'_{S',0}\,.$$

We have a unique linear map $n.f._{\mathfrak{P}}$ ("normal form"):

$$n.f._{\mathfrak{P}} : \mathscr{O}_{\mathfrak{P}} \otimes \mathscr{D}'_{\mathfrak{P},0} \to \mathscr{O}_{S/\mathfrak{P}} \otimes \mathscr{D}'_{\mathfrak{P},0}\,,$$

$$n.f._{\mathfrak{P}}\Bigg(G_{\mathfrak{P}} \otimes \prod_{S' \in \mathfrak{P}} \delta^{(\mathbf{r}_{S'})}\big(\mathrm{x}_{S'}\big)\Bigg) := \sum_{\mathbf{q}} \Big(G_{S/\mathfrak{P}}\Big)^{\mathbf{r}}_{\mathbf{q}} \otimes \prod_{S' \in \mathfrak{P}} \delta^{(\mathbf{q}_{S'})}\big(\mathrm{x}_{S'}\big)\,. \tag{4.3}$$

Let us give an example how (4.3) works: take $S = \{1,2,3,4\}$, $\mathfrak{P} = \big\{\{1,2\},\{3,4\}\big\}$, $G_S := A(12)B(13)C(24)D(34)$, where $A(jk) := A(\mathrm{x}_j - \mathrm{x}_k)$, ... Hence, $G_{\{1,2\}} = A(12)$, $G_{\{3,4\}} = D(34)$ and $G_{\mathfrak{P}} = B(13)C(24)$. Then if we denote $\delta(jk) := \delta(\mathrm{x}_j - \mathrm{x}_k)$, $\delta'(jk) := \partial_{\mathrm{x}_j}\delta(\mathrm{x}_j - \mathrm{x}_k)$, $C'(jk) := \partial_{\mathrm{x}_j}C(\mathrm{x}_j - \mathrm{x}_k)$, we have:

$$G_{\mathfrak{P}}\,\delta(12)\delta'(34) = B(13)C(13)\,\delta(12)\delta'(34) - B(13)C'(13)\,\delta(12)\delta(34)\,.$$

Now, after the preparation we made above, the precise form of Eq. (4.1) is:

$$U_{\boldsymbol{\varepsilon}_S}\big(G_S\big) = \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Bigg(\prod_{S' \in \mathfrak{P}} \frac{1}{2|\boldsymbol{\varepsilon}_{S'}|} \sum_{\mathbf{r}_{S'} \in \mathbb{N}_0^{|S'|-1}} \frac{(-1)^{|\mathbf{r}_{S'}|}}{\mathbf{r}_{S'}!} \operatorname{res}_{S'}\big(\mathrm{x}_{S'}^{\mathbf{r}_{S'}} G_{S'}\big)\Bigg)$$

$$\times \Bigg(\sum_q R_{\boldsymbol{\varepsilon}_{\mathfrak{P}}}\Big(\Big(G_{S/\mathfrak{P}}\Big)^{\mathbf{r}}_{\mathbf{q}}\Big) \prod_{S' \in \mathfrak{P}} \delta^{(\mathbf{q}_{S'})}\big(\mathrm{x}_{S'}\big)\Bigg)\,. \tag{4.4}$$

The terms in the sum in Eq. (4.4) that correspond to $\mathfrak{P} = \{S\}$ and $\mathfrak{P} = \big\{\{j\}_{j\,\in\,S}\big\}$ are: $\frac{1}{2|\boldsymbol{\varepsilon}_S|} \sum\limits_{\mathbf{r}\,\in\,\mathbb{N}_0^{D(|S|-1)}} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!}\,\mathrm{res}_S\big(\mathbf{x}_S^{\mathbf{r}}\,G_S\big)\,\delta^{(\mathbf{r})}(\mathbf{x}_S) = \frac{1}{2|\boldsymbol{\varepsilon}_S|}\,\mathrm{Res}_S\big(G_S\big)$ and $R_{\boldsymbol{\varepsilon}_S}\big(G_S\big)$, respectively. In particular, Eq. (4.4) can be used to define recursively $R_{\boldsymbol{\varepsilon}_S}$ as linear maps:

$$R_{\boldsymbol{\varepsilon}_S} : \mathscr{O}_S \to \mathscr{D}'_S \tag{4.5}$$

for every finite set S of indices. Note also that the part of the sum in Eq. (4.4) with $|\mathfrak{P}| > 1$ equals $T_{\boldsymbol{\varepsilon}_S}(G_S)$ in Eq. (3.6). We can summarize this section by the following

Theorem 4.1. *Equation* (4.4) *recursively defines linear maps* (4.5) *such that every distribution* $R_{\boldsymbol{\varepsilon}_S}(G_S)$ *is regular* (*analytic*) *at* $\boldsymbol{\varepsilon}_S = 0$ *and the map* $R_S := R_{\boldsymbol{\varepsilon}_S}\Big|_{\boldsymbol{\varepsilon}_S\,=\,0}$ *satisfies the axioms of Sect.* 2.

The proof of the theorem is by induction in $|S|$. Here we just outline the main steps: for $|S| = 2$ Eq. (4.4) reduces to Eq. (3.6) and $R_{\varepsilon_{1,2}} = T_{\varepsilon_{1,2}}$. For $|S| > 2$ we take the difference of both sides, restrict it on a configuration space $F_{\mathfrak{P}} := \big\{(\mathrm{x}_s)_{s\in S} \in M^S : \mathrm{x}_j \neq \mathrm{x}_k \text{ if } j \nsim_{\mathfrak{P}} k\big\}$ corresponding to partition $\mathfrak{P}$. Then, using the properties (see [3], Appx. B) of $F_{\mathfrak{P}}$ one shows under the inductive hypothesis that the difference is zero.

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Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
72 Tsarigradsko Shosse Blvd
1784 Sofia, Bulgaria
e-mail: `mitov@inrne.bas.bg`
`dnedanovski@inrne.bas.bg`