The Evolution of Scalar Fields and Inflationary Cosmology

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Abstract

The thermal restoration of symmetry in Grand Unified Field Theories with Coleman-Weinberg symmetry breaking is considered. A concise rederivation of the high temperature approximation to the one loop effective potential is given and the critical temperature for the transition is then calculated exactly, within the one loop effective potential.

A number of new exact scalar field cosmologies are derived from effective potentials that are composed of two or more exponential terms. These solutions are considered from the standpoint of inflationary cosmology. In particular, solutions are presented that combine eras of exponential and power-law growth and which have a transition between inflationary and non-inflationary expansion. For some parameter choices, the density perturbation spectra produced by these models are significantly tilted with a minimal component of tensor perturbations.

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Contents

Chapter 1

Introduction

1.1 Scalar Fields: An Informal Overview

During the last three decades, scalar fields have assumed a central importance in particle physics and cosmology. In 1964, Higgs [86] proposed a model in which gauge particles acquired mass through an interaction with a massive scalar (or spinless) field with spontaneously broken symmetry. Previous attempts to construct field theories with massive gauge bosons had been plagued with incurable infinities. This method of mass generation was used by Weinberg [187] and Salam [160] to construct a model of the electroweak interactions which is both fully renormalisable (that is, all infinities one encounters during calculations can be consistently eliminated) and well tested experimentally.

In the 1970s it was realised that the standard model possessed a phase transition [94–97, 118] that could have a number of cosmological ramifications. It was Guth's invention of inflation in 1981 [79], though, that moved scalar fields to the centre of the cosmological stage. At this time the standard cosmological model was a homogeneous and isotropic (Robertson-Walker) universe full of either hot gas or dust. This model successfully explains most of the major observational features of the universe. However, it requires an unnatural fine tuning of its initial conditions and possesses other defects when examined in detail, which are summarised by Kolb and Turner [102] and Linde [122]. Guth's contribution was to realise that a scalar field with the right properties could bring about a period of exponentially rapid growth in the very early universe, leading to a massive amount of expansion. This phenomenon, which Guth christened inflation, provided a solution to most of the outstanding cosmological problems. Subsequently, inflationary cosmology has come to mean any model of the early universe that incorporates a sufficiently rapid period of expansion and many different schemes for bringing it about can be found in the literature. These generically employ the special properties of scalar fields, although exceptions to this rule do exist.

A second, and unexpected, achievement of inflationary cosmology was to provide a mechanism for the production of small density inhomogeneities that subsequently give rise to galaxies and clusters of galaxies. In the early 1970s, Harrison [83] and Zel'dovich [194] realised that small initial perturbations could seed the formation of galaxies. They independently described a scenario where the initial perturbations had a spectrum of Fourier components whose amplitude was independent of their wavelength, a situation that is now called a "scale free" spectrum. However, this begs the question of the origin of the initial inhomogeneities. In 1982 it was shown that small oscillations in the scalar field during inflation can provide density perturbations of the correct type and size, although this requires stringent constraints on the allowable parameter values [16, 80, 84, 173]. This discovery, together with the observation by the COBE satellite [169, 192] of tiny ripples in the microwave background that are a fossil of these perturbations, provides a crucial observational probe of the early universe and opens up the possibility of directly testing rival inflationary theories.

We now have a situation where the special properties of scalar fields are used to remove what might otherwise be insurmountable obstacles to the development of working models in particle physics and cosmology. The blot on this otherwise attractive landscape is that no-one has ever observed a fundamental particle without spin, and scalar fields have no concrete existence outside the imagination of theoretical physicists.

Over the next few years, accelerators now under construction should probe the energy ranges in which the Higgs must exist if the standard electroweak model is correct. More accurate observations of the Cosmic Microwave Background (especially those at a smaller angular scale than the COBE measurement) are currently underway and these have the potential to discriminate between different inflationary models.

1.2 User's Guide to the Thesis

This thesis looks at two different aspects of scalar fields. In Chapters 2 and 3, the critical temperature for the the phase transition in Coleman-Weinberg

GUTs is obtained exactly, within the one-loop effective potential. En route, I present a concise re-derivation of the finite temperature contribution to the effective potential. Chapters 4 to 9 deal with inflationary cosmological models sourced by a scalar field. Chapter 4 is an introduction to inflationary cosmology. Chapters 5 and 6 review exact solutions for inflationary models, and effective potentials composed of exponential terms. In Chapters 7 and 8 I present a range of new exact solutions, and in Chapter 9 I discuss the spectrum of primordial density perturbations that they produce.

This is the first doctoral thesis written by a student at the University of Canterbury that concentrates on the interface between particle physics and cosmology and, as such, breaks new ground for research within the department. During the time in which I was a graduate student I frequently found myself being asked questions by other students who were writing Masters' theses or Honours Projects, as well as the merely curious. However, when I began my work the same questions were usually resolved by a lengthy spell in the library.

I believe that the most likely readers of this thesis (other than its examiners) are future graduate students within the department. Consequently I have tried to ease their way by giving an accessible but relatively detailed discussion of the relevant aspects of quantum field theory and cosmology, as well as presenting some derivations that are often left implicit in standard treatments. Chapters 2 and 4 are reviews, although the derivation of the temperature dependent part of the effective potential is new. In Chapters 5 and 6 I introduce the formalism that is needed in the subsequent chapters. The original material in this thesis is concentrated in Chapters 3, 7, 8 and 9.

1.3 Units and Conventions

I have employed a spacetime metric with signature $(+,-,-,-)$ throughout this thesis. Unless otherwise noted, natural units are used where $\hbar = c = 1$. With this convention, Newton's constant, $G = 1/m_p^2$, where m_p is the Planck mass. In Chapters 5 through 8 (which focus on exact solutions to the Einstein field equations) I have set $m_p = \sqrt{8\pi}$. At other times, the Planck mass is taken to be $m_p = 1.22 \times 10^{19}\text{GeV}$.

Chapter 2

Development of Field Theory

In this chapter I present a brief description of perturbative quantum field theory and introduce the effective potential, $V_{\text{eff}}(\phi)$, of a scalar field ϕ. An introduction to modern quantum field theory is given in the books by (in increasing order of thoroughness) Ryder [159], Ramond [149] and Cheng and Li [39]. Brandenberger's review [28] gives an excellent discussion of quantum field theory in a cosmological setting and the use of the effective potential to analyse symmetry restoration.

2.1 The Effective Potential

We start with the usual functionals of quantum field theory. A full development begins with the Feynman Path Integral for a forced harmonic oscillator and generalises to a field, $\phi(x)$, that is defined at all spacetime points, x. The vacuum to vacuum amplitude is then defined to be

$$\langle \Omega \mid \Omega \rangle_J \equiv W[J] = N \int \mathcal{D}\phi e^{i\langle \mathcal{L} + J\phi \rangle} \tag{2.1}$$

when $\mathcal{L}$ is the lagrangian density, $J(x)$ is the source term and $\langle \mathcal{L} + J\phi \rangle$ denotes integration over all spacetime. For a scalar field the lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi). \tag{2.2}$$

The derivative term is the kinetic energy and the indices are raised and lowered with the usual metric of Minkowski spacetime. When field theories in an expanding universe are discussed later in this thesis, these derivatives will be defined in curved spacetime. The potential, $V(\phi)$, leads to the self interaction terms that characterise the theory. Renormalisation constraints require that the potential is at most a fourth order polynomial in ϕ.

The functional $W[J]$ can be expressed in terms of Green's functions and their associated Feynman diagrams. More usually, one systematically generates the allowable Feynman diagrams and then converts them into the Green's functions by using the Feynman rules. Both connected and disconnected diagrams are generated by $W[J]$, as illustrated in Figure 2.1.

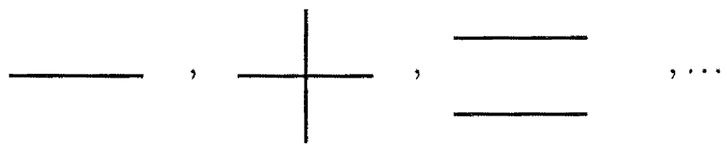


Figure 2.1 : The diagrams generated by $W[J]$, including both connected and disconnected diagrams. The disconnected diagrams are those that are composed of two or more separate pieces and correspond to physically distinct scattering events.

While $W[J]$ is the most obvious generalisation of the ordinary Feynman path integral, it is usual to define further functionals, each applying to a smaller subset of Feynman diagrams, but which become progressively more useful as calculational tools. The first reduction is in moving from $W[J]$ to $Z[J]$ where

$$W[J] \equiv e^{iZ[J]}. \tag{2.3}$$

Unlike $W[J]$, $Z[J]$ generates only connected diagrams.

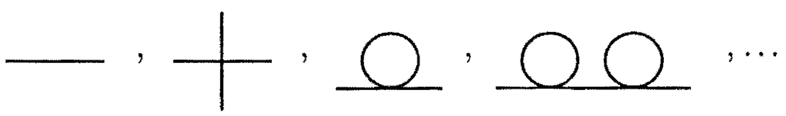


Figure 2.2 : The functional $Z[J]$ generates only connected diagrams, which are illustrated here. These are a subset of the diagrams generated by $W[J]$.

The next, and for our purposes crucial, step is the introduction of the effective action. The classical field, $\bar{\phi}$ is defined to be

$$\bar{\phi} = \frac{\delta Z[J]}{\delta J} \tag{2.4}$$

and the effective action is a functional of $\bar{\phi}$,

$$\Gamma[\bar{\phi}] = Z[J] - \langle J\bar{\phi}\rangle \tag{2.5}$$

and

$$J(x) = -\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)}. \tag{2.6}$$

The effective action is the generating functional of all the one particle irreducible (1PI) diagrams. These are the diagrams that cannot be broken into two separate pieces by severing a single internal line. In terms of the position space 1PI Green's functions, $\Gamma^{(n)}$, the effective action is

$$\Gamma[\bar{\phi}] = \sum_{n=2}^{\infty}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\bar{\phi}(x_1)\cdots\bar{\phi}(x_n)\Gamma^{(n)}(x_1,\cdots,x_n). \tag{2.7}$$

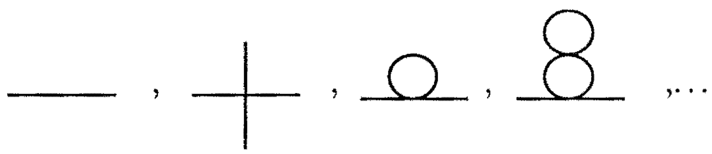


Figure 2.3 : The one particle irreducible diagrams that contribute to the effective action. None of these diagrams can be broken into two separate pieces by severing a single internal line.

Shifting to the momentum space 1PI Green's functions, $\tilde{\Gamma}^{(n)}$, and assuming that the field $\bar{\phi}$ is independent of position gives [28]

$$\Gamma[\bar{\phi}] = \sum_{n=2}^{\infty}\frac{1}{n!}\bar{\phi}^n\int d^4x_1\tilde{\Gamma}^{(n)}(0,\cdots,0) \tag{2.8}$$

where the Green's functions inside the integral are evaluated for vanishing external momentum. Finally, we can define the effective potential, $V_{\text{eff}}(\bar{\phi})$, in terms of the effective action:

$$\Gamma[\bar{\phi}] = -\int d^4x V_{\text{eff}}(\bar{\phi}). \tag{2.9}$$

For a spatially constant field configuration, then

$$V_{\text{eff}}(\bar{\phi}) = -\sum_{n=2}^{\infty}\frac{1}{n!}\bar{\phi}^n\tilde{\Gamma}^{(n)}(0,\cdots,0). \tag{2.10}$$

The effective potential can be viewed as the the generating functional for all of the 1PI diagrams, at vanishing external momentum. A detailed description of what has been hastily sketched here is given by Brandenberger [28]. The effective potential is crucial when studying symmetry breaking, as the value of $\bar{\phi}$ that minimises the effective potential will correspond to the lowest energy configuration of the field. The physical interpretation of the effective potential is discussed by Coleman [42] in a famous set of lectures. The effective potential is often given in terms of the loop expansion, which is determined by the number of distinct internal loops within the 1PI diagrams.

From this point on, I will simply denote the effective potential V and drop the bar on the classical field, $\bar{\phi}$. This follows the usual convention adopted in the literature, and involves no ambiguity since only the effective potential and classical field are referred to in the remainder of this thesis.

2.2 Spontaneous Symmetry Breaking

It has long been known that quantum field theories for massive, gauge particles cannot be renormalised. This problem is solved by allowing the gauge particles to interact with a scalar field that exhibits spontaneous symmetry breaking. Spontaneous symmetry breaking occurs in a scalar field theory when a global symmetry possessed by the lagrangian (for instance, rotational invariance) is not a symmetry of the vacuum. This happens when the minimum of the effective potential (which defines the vacuum) is displaced from the origin.

As an example, consider scalar electrodynamics (spinless, charged particles interacting through the exchange of "photons") which has the effective potential [168]

$$V(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{e^2}{2}A^\mu A_\mu \phi^2. \tag{2.11}$$

In this case, we find that $V(\phi)$ has a minimum at $\pm m/\sqrt{\lambda}$ and spontaneous symmetry breaking generates a non-zero vacuum expectation value $\langle \phi^2 \rangle = e^2 m^2/\lambda$. In quantum field theory each particle is associated with a propagator which, up to a multiplicative factor, is the inverse of the two-point momentum space 1PI Green's function. Particle states are identified with a propagating excitation of the field. However if the propagator has a pole at non-zero momentum, then excitations with a momentum (energy) less than the momentum at which the pole is located will not propagate. Physically, we therefore identify the position of the pole in the propagator with the particle's mass since it defines the minimum energy of a particle state. The terms that contribute the pole are those that are quadratic in the corresponding field. An $A^\mu A_\mu$ term not is renormalisable, and so cannot be included in the lagrangian, equation (2.11). However, the $A^\mu A_\mu \phi^2$ term is renormalisable, and once the scalar field acquires a non-zero expectation value through spontaneous symmetry breaking this term will contribute a pole to the propagator of the A-field. Thus, the photon (A field) acquires an effective mass-squared of $e^2 m^2/\lambda$. A similar mechanism works for other gauge particles and fermions.

2.3 The Coleman-Weinberg Mechanism

The symmetry breaking terms in the lagrangian, equation (2.11), are explicitly present in the tree level effective potential. However, a more subtle approach to symmetry breaking was introduced by Coleman and Weinberg in 1973 [43]. Since it is the effective potential that determines whether or not symmetry breaking will take place, an explicit "negative mass" term in the tree level potential is not a necessary condition for symmetry breaking. In particular, Coleman and Weinberg showed that one loop corrections to the effective potential can move the global minimum of the theory from the origin to a non-zero value of ϕ, which is the necessary and sufficient condition for symmetry breaking to occur. As a toy model consider a pure scalar field with the tree level potential

$$V(\phi) = \frac{\lambda}{4!}\phi^4. \tag{2.12}$$

This is a massless theory, since there is no quadratic term. Now consider the one loop corrections. All the higher order diagrams must be constructed out of the quartic vertex. The 1PI diagrams are illustrated in Figure 2.4.

The only diagrams that contribute to the effective potential at one loop level are the "ring diagrams". Any other diagrams are either reducible or contain more than one internal loop. If n is the number of vertices in the diagram, then for any n there is one and only one diagram that makes a contribution to the one loop potential. By examining the Feynman rules for a general diagram the infinite number of diagrams can be summed directly and the result integrated. This technique is not useful for higher order corrections to the effective potential, as the combinatorial factors involved are much more complex. However, higher order terms in the effective potential can be evaluated by constructing

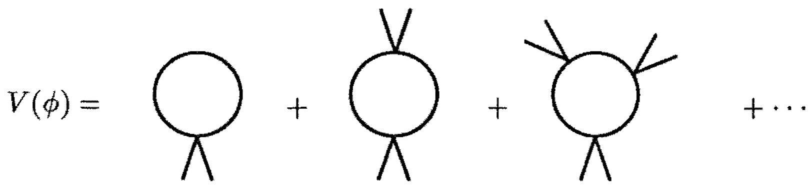


Figure 2.4 : The one loop, one particle irreducible diagrams obtained for the tree level effective potential.

a new theory in which the tadpole diagrams produce the loop expansion in the original theory [106].

Applying the Feynman rules to the n-vertex ring diagram gives the (Euclidean space) result,

$$\int \left[\frac{d^4k}{(2\pi)^4} \left(\frac{\lambda}{4!} \right)^n \times (4 \times 3)^n \times \left(\frac{\phi^2}{k^2} \right)^n \times \frac{1}{2n} \right]. \tag{2.13}$$

The terms in the integrand are, from left to right, the vertex factors, the topological weight of the diagram, the propagators (for the internal lines) and an overall symmetry factor. The integration is over the momentum of the internal loop, and all the external momenta have been set to zero. Only one momentum integration survives, since the diagrams are evaluated at vanishing external momentum and conservation of momentum at each vertex forces the momentum through each vertex to be identical. Hence, to one loop, the effective potential is

$$V(\phi) = \frac{\lambda}{4!}\phi^4 - \sum_{n=1}^{\infty} \int \frac{d^4k}{(2\pi)^4} \frac{1}{2n} \left(\frac{\lambda \phi^2}{2k^2} \right)^n . \tag{2.14}$$

The integral acquires the minus sign through being rotated to Euclidean space. Moving the summation term under the integral sign, we recognise the series form of $\log(1-x)$ which reduces the problem to a single integral. This integral can be performed, but it diverges for large k. We regularise with a momentum space cutoff at $k = \Lambda$,

$$V(\phi) = \frac{\lambda}{4!}\phi^4 - \frac{B}{2}\phi^2 - \frac{C}{4!}\phi^4 + \frac{\lambda \Lambda^2}{64\pi^2}\phi^2 + \frac{\lambda^2}{256\pi^2} \left[\log\left(\frac{\lambda\phi^2}{2\Lambda^2} \right) - \frac{1}{2} \right] \phi^4, \tag{2.15}$$

and dropping terms that vanish for large Λ. The B and C are the renormalisation counterterms. There are two renormalisation prescriptions that are encountered in the literature. The one I will use here was originally employed by Coleman and Weinberg [43], which is to fix

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=0} = 0 \quad , \quad \left.\frac{d^4V}{d\phi^4}\right|_{\phi=\mathcal{M}} = \lambda. \tag{2.16}$$

The coupling constant renormalisation must be performed for $\mathcal{M} \neq 0$, where $\mathcal{M}$ is an arbitrary mass scale, since the fourth derivative of the logarithmic term is singular at the origin. The other prescription sometimes encountered [119] is to perform the mass and coupling constant renormalisation at the same (non-zero) value of ϕ. After applying the renormalisation conditions the potential is independent of the cut off and,

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left[\log\left(\frac{\phi^2}{\mathcal{M}^2}\right) - \frac{25}{6}\right]. \tag{2.17}$$

This potential differs from the tree level potential since the origin, $\phi = 0$, is now a local *maximum*. Thus while the potential is not of the spontaneous symmetry breaking form at tree level, the first order corrections make symmetry breaking energetically favourable. This dynamical form of symmetry breaking is known as the Coleman-Weinberg mechanism.

The potential, equation (2.17), is a toy model since to produce mass in a realistic gauge theory we need to include gauge and fermion fields. Including boson and fermion terms coupled to the scalar Higgs adds diagrams of the form illustrated in Figure 2.5 to the one loop effective potential.

The logarithmic terms have a pre-factor that is proportional to the square of the coupling of the particle that forms the loop. Thus, if one particle has a larger coupling than the others, it will dominate the one loop corrections. For instance, scalar QED with $e^2 \gg \lambda$ has the one loop effective potential

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{3e^4}{64\pi^2}\phi^4\left[\log\left(\frac{\phi^2}{\sigma^2}\right) - \frac{25}{6}\right] \tag{2.18}$$

where we have *chosen* to perform the coupling constant renormalisation at $\mathcal{M} = \sigma$, the global minimum of the potential. By definition then,

$$V'(\sigma) = \left(\frac{\lambda}{6} - \frac{11e^4}{16\pi^2}\right)\sigma^3 = 0. \tag{2.19}$$

This redefines λ in terms of e, and physically it corresponds to the effective coupling being modified from the tree level coupling through quantum corrections to the potential. Since

$$\lambda = \frac{33}{8\pi^2}e^4 \tag{2.20}$$

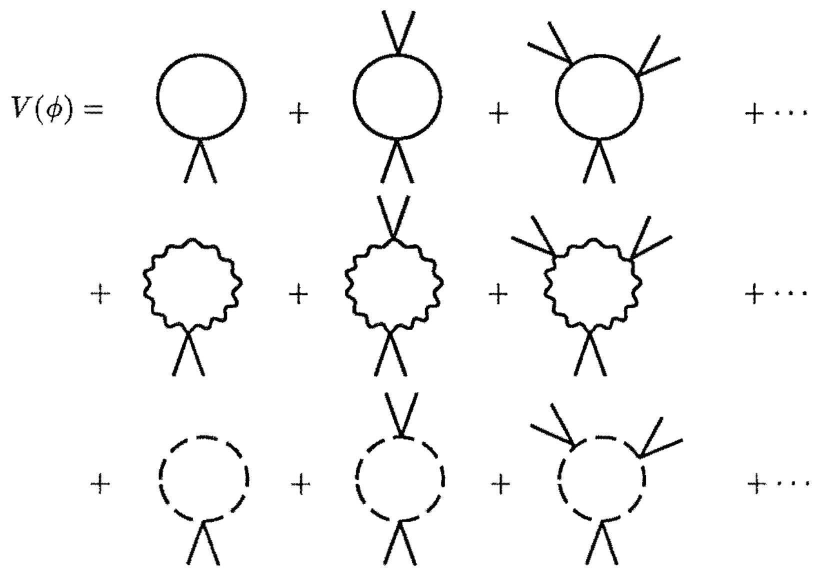


Figure 2.5 : The one loop, one particle irreducible diagrams that contribute to the effective potential in a general theory. The dashed lines are fermions and the wavy lines gauge bosons. The important feature is that the loops in each diagram are composed of identical particles, and this greatly facilitates calculations.

the scalar self coupling can be significantly increased over its tree level value. Inserting this value of λ into equation (2.18) gives

$$V(\phi) = \frac{3e^4}{64\pi^2}\phi^4 \left[\log\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right]. \tag{2.21}$$

This symmetry breaking mechanism can be applied to gauge theories. In particular, Grand Unified Field Theories (or GUTs), with Coleman-Weinberg symmetry breaking were the first apparently viable realisation of the inflationary paradigm [8, 120]. What ruined their chances as a candidate cosmological theory was the size of the scalar coupling, which must be extremely small to produce the correct primordial spectrum of density fluctuations. However, since the gauge coupling is large (but still less than unity, of course) and is constrained by the renormalisation group analysis of standard model couplings, the scalar self coupling is also large.

Considering only the gauge loops (as is usual), then the Coleman-Weinberg

GUT Higgs effective potential is [25]

$$V(\phi) = B\phi^4\left[\log\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right]. \tag{2.22}$$

The coefficient B depends on the number of heavy gauge bosons n and a group theoretical factor that is fixed by the particular symmetry breaking scheme being considered.

2.4 Temperature Dependent Terms

The last contribution to the effective potential I need to introduce is the finite temperature correction. The finite temperature terms usually remove the minima away from the origin at extremely high temperatures. Thus the form of the effective potential depends on the temperature. Spontaneous symmetry breaking is only energetically favourable below a critical temperature, and gauge theories will have a phase transition, or even a series of phase transitions. Thus any particles that derive their mass from their interaction with the spontaneously broken Higgs field will be massless before the phase transition takes place.

Finite temperature corrections are incorporated into quantum field theories by defining ensemble averages of the zero temperature operators [93]. At a finite temperature, T, the imaginary time Green's functions are periodic, with a period $\beta = 1/T$ (and Boltzmann's constant equal to unity). This modifies the Feynman rules used to derive the momentum space integrals, with the result that for a single scalar field,

$$V(\phi,T) = V(\phi,T=0) + \frac{T^4}{2\pi^2}\int_0^\infty dx\; x^2\log\left[1-\exp-\sqrt{x^2+m^2/T^2}\right] \tag{2.23}$$

where $m^2 = d^2V/d\phi^2$.

The temperature dependent part of the effective potential,

$$I(\tau) = \int_0^\infty dx\; x^2\log\left[1-\exp\left(-\sqrt{x^2+\tau^2}\right)\right], \tag{2.24}$$

must be evaluated. This calculation was first carried out by Dolan and Jackiw [58], and the result is an infinite series in τ. They perform the integral by "brute force", obtaining the terms of the series one by one. I have found a much simpler derivation of their (now standard) result, which leads directly to the full series expansion and involves much less labour than other approaches in the literature.

By expanding the logarithm in equation (2.24), integrating by parts and employing (3.479 1) of Gradshteyn and Ryzhik [77] we get

$$I(\tau) = -\sum_{m=1}^{\infty} \frac{\tau^2}{m^2} K_2(m\tau), \tag{2.25}$$

where K_ν is a modified Bessel function of the second kind of order ν [11,186]. Bollini and Giambiargi [26] use dimensional regularisation to derive the finite temperature effective potential in an arbitrary number of dimensions and Konoplich [103] obtains the same result with zeta function methods. The above series can be extracted immediately from their work but this derivation is more direct.

The series in equation (2.25) converges rapidly but we need to make its dependence on τ more transparent. Using a little-known result of Watson [185], $I(\tau)$ becomes

$$\begin{aligned} I(\tau) &= -\frac{\pi}{3}\sum_{m=1}^{\infty}\left[(\tau^2+4\pi^2m^2)^{\frac{3}{2}} - (2m\pi)^3 - 3m\pi\tau^2 - \frac{3\tau^4}{16m\pi}\right] \\ &\quad -\frac{\pi^4}{45} + \frac{\pi^2}{12}\tau^2 - \frac{\pi}{6}\tau^3 - \frac{\tau^4}{16}\left(\gamma - \frac{3}{4} + \log\frac{\tau}{4\pi}\right). \end{aligned} \tag{2.26}$$

When $\tau < 2\pi$, the binomial theorem can be used to re-express the infinite series in the above equation as a power series, giving

$$\begin{aligned} I(\tau) &= -\frac{\tau^4}{8}\sum_{m=1}^{\infty}(-1)^m \frac{(2m-1)!!}{2^m(m+2)!}\left(\frac{\tau}{2\pi}\right)^{2m}\zeta(2m+1) \\ &\quad -\frac{\pi^4}{45} + \frac{\pi^2}{12}\tau^2 - \frac{\pi}{6}\tau^3 - \frac{\tau^4}{16}\left(\gamma - \frac{3}{4} + \log\frac{\tau}{4\pi}\right). \end{aligned} \tag{2.27}$$

The high temperature approximation to the effective potential is obtained from equation (2.27) by taking the leading term(s) in τ. In particular, at very high temperatures only the term in τ^2 is retained. Inspection of the higher order terms shows that for $\tau \gtrsim 1$, some of the discarded terms are equal in magnitude to, or larger than, those that are kept. Therefore, high temperature approximations to $I(\tau)$ are unreliable unless $\tau \ll 1$.

The Bessel function identity used to obtain equation (2.26) is not well known. The only other application of these series that I am aware of is the calculation of some aerodynamic effects in wind-tunnels by Olver [145,153]. A striking feature of Watson's derivation is its similarity, both in overall approach and in the detailed working involved, to the dimensional regularisation of Feynman integrals [149,176]. He first derives a result for K_ν that becomes singular when ν takes integer or half integer values. These cases are then solved by adding a term that removes the singularity in the limit $\nu \to m/2$ for integer m. Interestingly enough, Bollini and Giambiargi [26] derive the finite temperature terms in spacetimes of non-integer dimensionality, and their result shows that the order of the Bessel function in the analogue of equation (2.25) is equal to half the spacetime dimension.

Finally, we can now write the finite temperature effective potential for a Coleman-Weinberg GUT. Including gauge loops only, the one loop finite temperature effective potential of the Higgs field, ϕ, is [25]

$$V(\phi,T) = B\phi^4 \left[\log\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] + \frac{DT^4}{2\pi^2} I(\tau), \tag{2.28}$$

where

$$\tau = \frac{M\phi}{T}, \qquad B = \frac{3nM^4}{64\pi^2}, \qquad D = 3n. \tag{2.29}$$

At zero temperature, the minimum of $V(\phi,T)$ is located at $\phi = \sigma$. The number of heavy gauge bosons is n and M is a coefficient determined by the gauge group and the coupling constant. In the case of $SU(5) \to SU(3) \times SU(2) \times U(1)$ [25],

$$M^2 = \frac{25}{8} g^2, \tag{2.30}$$

where $g^2/4\pi$ is the coupling constant.

Chapter 3

The Critical Temperature for Coleman-Weinberg GUTs

3.1 What is the Critical Temperature?

If a gauge theory has a phase transition at which spontaneous symmetry breaking becomes energetically favourable, then there is a critical temperature, T_c, associated with the transition [28]. Since the presence of symmetry breaking can be inferred from the shape of the effective potential, the mathematical problem reduces to finding how the location of the global minimum of $V(\phi, T)$ varies as a function of temperature. Two different definitions of the critical temperature are used in the literature. The first, T_c, is the temperature at which spontaneous symmetry breaking is energetically favourable, and this is probably more widely accepted. The second (used by Dolan and Jackiw [58]) is the temperature, T_c', at which the effective potential first acquires a minimum of any sort away from the origin. The difference between the two definitions is illustrated in Figure 3.1. The actual temperature at which the phase transition occurs may be much lower than the critical temperature, as there is often a small barrier at the origin which will lead to substantial supercooling.

A quick calculation of the critical temperature for the Coleman-Weinberg Higgs can be made by assuming that the high temperature approximation to the effective potential is valid. Taking only the first term in the high temperature expansion, equation (2.27), gives the effective potential,

$$V(\phi, T) = B\phi^4 \left[\log\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] + C\phi^2 T^2. \tag{3.1}$$

Assuming that the minimum is located at σ, the critical temperature is

$$T_c^2 = \frac{B\sigma^2}{2C}. \tag{3.2}$$

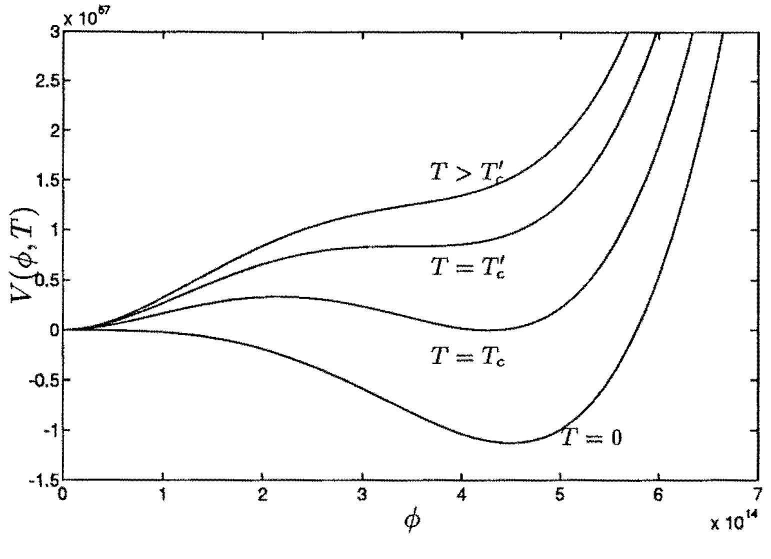


Figure 3.1 : The finite temperature effective potential for the Coleman Weinberg Higgs is plotted for a variety of temperatures. At the highest temperature, the only minimum in the potential is at the origin. When $T = T_c'$, the potential acquires a point of inflexion, and below this temperature symmetry breaking is possible. When $T = T_c$, the minimum at non-zero ϕ is degenerate with that at the origin, and at lower temperatures symmetry breaking is energetically favourable.

As Brandenberger [28] points out, this value is that which would have been obtained from dimensional analysis. For Coleman-Weinberg gauge theories, this approximation is not a good one and in the next section I calculate the critical temperature derived from the one-loop effective potential exactly, for an arbitrary GUT. This work was first reported in Easther and Moreau [66].

3.2 The Critical Temperature for Coleman-Weinberg GUTs

In order to calculate the critical temperature, the value of ϕ_c, the location of the minimum of $V(\phi, T)$ when $T = T_c$, must be known. By definition, at the critical temperature the minimum of $V(\phi, T)$ is degenerate with the minimum at the origin. This gives two equations that can be solved simultaneously for

ϕ_c and T_c,

$$\begin{aligned} V(0,T_c) &= V(\phi_c,T_c) && (3.3) \\ \frac{d}{d\phi}V(\phi,T)\bigg|_{\phi=\phi_c} &= 0. && (3.4) \end{aligned}$$

Substituting the generic form of the effective potential, equation (2.28), into equation (3.3) gives

$$\begin{aligned} \frac{B}{D}\frac{\phi^4}{T^4}\left[\log\left(\frac{\phi^2}{\sigma^2}\right)-\frac{1}{2}\right]+\frac{\tau^2}{24}-\frac{\tau^3}{12\pi}-\frac{\tau^4}{32\pi^2}\left(\gamma-\frac{3}{4}+\log\frac{\tau}{4\pi}\right) & \\ -\frac{\tau^4}{16\pi^2}\sum_{m=1}^{\infty}(-1)^m\frac{(2m-1)!!}{2^m(m+2)!}\left(\frac{\tau}{2\pi}\right)^{2m}\zeta(2m+1) &= 0, \quad (3.5) \end{aligned}$$

where we have anticipated that $\tau_c < 2\pi$ and used equation (2.27). From equation (3.4) we obtain (after multiplying by $\phi/4$)

$$\begin{aligned} \frac{B}{D}\frac{\phi^4}{T^4}\log\left(\frac{\phi^2}{\sigma^2}\right)+\frac{\tau^2}{48}-\frac{\tau^3}{16\pi}-\frac{\tau^4}{32\pi^2}\left(\gamma-\frac{1}{2}+\log\frac{\tau}{4\pi}\right) & \\ -\frac{\tau^4}{32\pi^2}\sum_{m=1}^{\infty}(-1)^m\frac{(2m-1)!!}{2^m(m+1)!}\left(\frac{\tau}{2\pi}\right)^{2m}\zeta(2m+1) &= 0. \quad (3.6) \end{aligned}$$

The logarithmic terms are eliminated by subtracting equation (3.6) from equation (3.5) and then using equation (2.29). This gives

$$\frac{\tau}{\pi}-6\sum_{m=1}^{\infty}(-1)^m\frac{m(2m-1)!!}{2^m(m+2)!}\left(\frac{\tau}{2\pi}\right)^{2m+2}\zeta(2m+1) = 1. \quad (3.7)$$

The solution of this equation is τ_c, the value of τ when the two minima are degenerate. Since it is a power series in τ, it can inverted by standard methods [135, §4.5], to get

$$\frac{\tau_c}{2\pi}=\frac{1}{2}+\sum_{n=4}^{\infty}b_n. \quad (3.8)$$

The terms b_n are given by

$$b_n=\frac{1}{n2^n}\sum_{d,e,f,\cdots}(-1)^{d+e+f+\cdots}\frac{n\cdots(n-1+d+e+f+\cdots)}{d!e!f!\cdots}(a_4)^d(a_6)^e(a_8)^f\cdots \quad (3.9)$$

with the sum running over all $d,e,f,\cdots$ such that $3d+5e+7f+\cdots=n-1$. The a_n are:

$$a_{2m+2}=(-1)^{m+1}\frac{3m(2m-1)!!}{2^m(m+2)!}\zeta(2m+1). \quad (3.10)$$

This is an exact result and will, in principle, yield τ_c to an arbitrary degree of precision. The first few terms are

$$\frac{\tau_c}{2\pi} \approx \frac{1}{2} - \frac{\zeta(3)}{64}\left[1 - \frac{\zeta(3)}{8} + \frac{15\zeta(5)}{256}\right] + \frac{3\zeta(5)}{1024} - \frac{9\zeta(7)}{16384} \qquad (3.11)$$

giving $\tau_c = 3.05$.

We are now in a position to calculate ϕ_c. From equations (2.25) and (3.4),

$$\begin{aligned} \phi_c^2 &= \sigma^2 \exp\left[-\frac{8}{\tau_c}\sum_{n=1}^{\infty}\frac{1}{n}K_1(n\tau_c)\right] \\ &= \frac{\sigma^2}{1.106}. \end{aligned} \qquad (3.12)$$

Finally, we obtain the critical temperature:

$$T_c = \frac{M\phi_c}{\tau_c}. \qquad (3.13)$$

Since $\tau_c > 1$, the high temperature approximation to the effective potential is not reliable when $T \approx T_c$ and $\phi \approx \phi_c$. Hence, calculations of T_c based on such an approximation cannot be trusted.

As an example, consider $SU(5)$ breaking to $SU(3) \times SU(2) \times U(1)$ with $g^2/4\pi = 1/40$ and $\sigma = 4.5 \times 10^{14} GeV$. In this case, $T_c = 1.39 \times 10^{14} GeV$. For comparison, the high temperature approximation, equation (3.2) gives $T_c \approx 6 \times 10^{13} GeV$, which differs by a factor of two from the value obtained here.

However, if the running gauge coupling is used [167], M has a logarithmic dependence on the temperature so we cannot assign it a constant value. For $SU(5) \to SU(3) \times SU(2) \times U(1)$ the one loop running coupling is

$$\frac{g^2}{4\pi} = \frac{3\pi}{10\log\frac{T^2}{\Lambda^2}}, \qquad (3.14)$$

where $\Lambda = 2.74 \times 10^6 GeV$. Let $\hat{T}_c$ be the critical temperature calculated using a running coupling. From equation (3.13) and the definition of M^2,

$$\hat{T}_c^2 \log\frac{\hat{T}_c^2}{\Lambda^2} = \frac{15\pi^2}{4}\frac{\phi_c^2}{\tau_c^2}. \qquad (3.15)$$

We obtain $\hat{T}_c$ in terms of T_c, the critical temperature calculated assuming a constant coupling. Define δ by

$$\hat{T}_c^2 = (1+\delta)T_c^2. \qquad (3.16)$$

If the value of $g^2/4\pi$ used to obtain T_c was reasonable, then δ may be estimated by substituting equation (3.16) into equation (3.15). Expanding the logarithm in δ, and retaining only the first power gives

$$\left[1+\log\frac{T_c^2}{\Lambda^2}\right]\delta = \frac{15\pi^2}{4}\left(\frac{\phi_c}{\tau_c T_c}\right)^2 - \log\frac{T_c^2}{\Lambda^2}, \tag{3.17}$$

which yields $\hat{T}_c = 1.43 \times 10^{14} GeV$.

3.3 Comments and Observations

At the critical temperature, T_c, we find that ϕ_c, the position of the minimum of the effective potential, is related to the zero temperature minimum, σ, by a multiplicative constant that is independent of the model's parameters. The critical temperature is expressed concisely in terms of τ_c, a dimensionless constant which does not depend on the particular parameter values of the Coleman-Weinberg model being considered.

I have calculated T_c for a Coleman-Weinberg model to one loop order without making use of a high temperature approximation to the effective potential. The contribution of higher loops to the value of T_c is expected to be of the order $e^2 T_c$ where e^2 is the coupling between gauge bosons and the Higgs [168]. In order for perturbation theory to be valid, this quantity must be much less than T_c, so the correction to T_c from higher loops will be small.

As Coleman and Weinberg [43] note, higher loop corrections can have the effect of converting the local maximum at the origin at $T = 0$ into a local minimum. However, since no graphs can affect the value of the temperature independent portion of $V(\phi, T)$ at $\phi = 0$, equation (3.3) is unaffected by adding higher order terms.

Using Dolan and Jackiw's definition of the critical temperature, T_c', $V(\phi, T_c')$ has a point of inflexion at the critical temperature and finding it requires us to solve for the values of ϕ and T for which the first and second derivatives of $V(\phi, T)$ are zero. These equations are similar to those solved in the previous section. I have carried out this calculation and it yields the result that the effective potential has a point of inflexion for $\tau = 2.055$. For $SU(5) \to SU(3) \times SU(2) \times U(1)$ this value corresponds to $T_c' = 1.67 \times 10^{14} GeV$.

Finally, when $T \approx T_c$ and $\phi \approx \phi_c$ we have shown that the high temperature approximation is unreliable for any Coleman-Weinberg model of the form given by equation (2.28). Thus, T_c cannot be calculated accurately with an approximation to the temperature dependent portion of the effective potential. However, we have obtained a concise expression for T_c in terms of τ_c and the parameters of the Coleman-Weinberg model which is exact to one loop order and allows the critical temperature to be obtained immediately for any specific model.

Chapter 4

Scalar Fields and Cosmology

In this chapter I review the "standard model" of cosmology which, in turn, requires a brief discussion of general relativity. The standard model successfully explains the observed expansion of the universe, the abundance of light elements (predicted via primordial nucleosynthesis) and the existence of the Cosmic Microwave Background, but it possesses several major deficiencies. In general, the shortcomings of the standard model are not that it is contradicted by observation, but rather that some of the observed features of the universe that would be predictions of a successful cosmology are incorporated into the construction of the standard model.

The most promising generalization of the standard model is inflationary cosmology. While inflation can be driven by several different mechanisms, I concentrate on models with a slowly rolling scalar field, since that is the particular subspecies of inflation that is being addressed in this thesis.

4.1 The Einstein Field Equations

The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{4.1}$$

where $g_{\mu\nu}$ is the spacetime metric, $R_{\mu\nu}$ the Ricci tensor and R its contraction, the curvature scalar. The source terms are all contained in the energy-momentum tensor, $T_{\mu\nu}$, which obeys the continuity constraint,

$$T^{\mu\nu}{}_{;\mu} \equiv 0. \tag{4.2}$$

In mks units, equation (4.1) has a factor of $1/c^4$ on the right hand side. Thorough treatments of general relativity are given in the textbooks by Misner, Thorne and Wheeler [131], Ohanian [142], Wald [181] and Weinberg [188].

For the purpose of this thesis, as in the large majority of modern cosmology, we do not employ the Einstein field equations in their most general form. Rather, we assume that the spatial portion of the metric is homogeneous and isotropic - or, equivalently, that it is invariant under spatial translations and rotations. Such a space is said to be maximally symmetric and imposing this restriction places a very strong constraint on the allowable range of models that may be constructed. This is reflected in the reduction of the ten partial differential equations in equation (4.1) to a single ordinary differential equation which describes the evolution of the maximally symmetric universe. The advantage, of course, is that the reduced equations are far more tractable.

A homogeneous and isotropic universe has no "privileged" location. The antecedents of such a cosmology can be traced to the atomist philosophers of classical Greece. They believed the universe to be infinite, in contrast to the bounded, geocentric cosmology of Aristotle which was the orthodox Western viewpoint during the Middle Ages. With the advent of the Renaissance, the earth was displaced from its special position at the centre of cosmos, most notably in the work of Giordano Bruno who posited an infinite universe with an infinite number of stars and planets. Harrison [82] traces the development of cosmological thought and concept of a spatially symmetric universe, in the context of a history of Olber's Paradox.[1]

The advent of General Relativity facilitated the introduction of an expanding universe which is spatially symmetric but whose properties evolve with time. Friedmann first described an expanding homogeneous and isotropic universe in 1922 [73, 74] before Hubble's discovery of galactic redshifts and the associated expansion of the universe. Lemaître independently proposed an expanding universe in 1927 [107][2], making a connection between his work and Hubble's observations. The line element and metric (equation (4.4)) for a spatially homogeneous and isotropic universe was first written down by Robertson [155–157] and Walker [182], and I will refer to a spatially homogeneous and isotropic spacetime as a Robertson-Walker universe in this thesis. For a careful account of the distinctions between these early papers see Felten and Isaacman [69]. The development of cosmology in the first half of the twentieth century is traced by North [141] while Overbye [146] gives an informal

[1] Olber's Paradox asks why, if the universe is infinite and uniformly populated with stars, there is not a star at every point in the night sky. Or, more simply, "Why is it dark at night?"

[2] This paper was translated and published in English in 1931 [108].

discussion of recent developments.

Initially the primary recommendation of the Robertson-Walker universe was its simplicity and symmetry. The seemingly reasonable assumptions of homogeneity and isotropy are dignified with the title, *The Cosmological Principle* [188], although applying a grandiose name does not manufacture observational support. Over the last decade, however, that observational support has been forthcoming. The COBE measurements of the Cosmic Microwave Background Anisotropy [169, 192] show that deviations from isotropy in the primordial universe were at the level of parts in a million. On the theoretical front, it is a *prediction* of inflation that the universe is locally Robertson-Walker, and spatially flat as well - neatly picking out the simplest possible cosmological model in three spatial dimensions.

The Robertson-Walker line element is

$$ds^2 = dt^2 - a(t)^2\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2\sin\theta^2 d\phi^2\right) \tag{4.3}$$

where k takes the values $\pm 1, 0$ for positive, negative and flat spatial curvature, respectively. The metric is diagonal, with the non-zero components

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a(t)^2\frac{1}{1-kr^2} & & \\ & & -a(t)^2 r^2 & \\ & & & -a(t)^2 r^2\sin\theta^2 \end{pmatrix}. \tag{4.4}$$

The expansion of the universe is described by the scale factor, $a(t)$. The spatial variables r, θ and ϕ are comoving coordinates - that is, they are "painted on" the expanding 3-surface. A physical length, l_{phys}, is related to a length in comoving coordinates, λ, by

$$l_{\text{phys}} = a(t)\lambda \tag{4.5}$$

which makes clear the reason for calling $a(t)$ the scale factor. This distinction will be important when we discuss the generation of perturbations in inflationary models. The mathematics is simplified by considering the comoving wavelength of a perturbation, but the physical size is what determines the observational consequences.

The matter content of any given model is incorporated through the energy-momentum tensor, $T_{\mu\nu}$. For a perfect fluid,

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}, \tag{4.6}$$

where ρ is the energy density and p the pressure.

From the Einstein field equations with the metric, equation (4.4), and the energy momentum tensor, equation (4.6), we obtain,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{4.7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{4.8}$$

The Hubble parameter, $H = \dot{a}/a$, and the dot denotes differentiation with respect to time. The constraint on $T_{\mu\nu}$, equation (4.2), applied to the energy momentum tensor for a perfect fluid, equation (4.6), gives

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho + p)\right]. \tag{4.9}$$

A particular model is specified by defining an equation of state - the relationship between ρ and p. In the next section I discuss the two phases of the standard cosmology and then apply the same formalism to produce the equations of motion for a scalar field and inflationary cosmology.

4.2 The Standard Cosmology

Most textbooks on general relativity include at least one chapter on cosmology. Weinberg's [188] account of the "standard model" of the Big Bang constitutes the canonical description of cosmology prior to the development of inflation.[3] Kolb and Turner [102] treat Big Bang cosmology as prelude to a description of inflation and other modern developments. Other useful texts are those of Börner [27], Collins, Martin and Squires [44] and Narlikar [140]. Raychaudhuri discusses different cosmological models [151] and Misner, Thorne and Wheeler [131] treat a variety of cosmological solutions to the Einstein field equations. Further details on the material in this section can be found in these books.

[3] Zel'dovich's text [195] is frequently accorded a similar status. However, since it is not held (to the best of my knowledge) in any New Zealand library it is difficult to consult.

We start by defining the critical density, ρ_c, and the parameter, Ω:

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{4.10}$$

$$\Omega = \frac{\rho}{\rho_c}. \tag{4.11}$$

Note that both Ω and ρ_c will vary as the universe evolves. By comparing these definitions with equation (4.7) we get the familiar results that when $\Omega < 1$, $k = -1$ and when $\Omega > 1$, $k = +1$. The flat case $\Omega = 1$ occurs for $k = 0$. In most of what follows, I assume that $k = 0$, which simplifies the algebra but does not obscure the details of the evolution of the primordial universe.

We begin by looking at Robertson-Walker cosmologies for universes that are composed of hot, relativistic gas and cold dust.

4.2.1 Hot Gas

For a relativistic gas, the equation of state is

$$p = \frac{1}{3}\rho. \tag{4.12}$$

Substituting this into equation (4.9) yields a differential relationship between ρ and a which, upon integration, connects the density to the expansion of the universe:

$$4\frac{\dot{a}}{a} = -\frac{\dot{\rho}}{\rho}$$

$$\Rightarrow \quad \rho \propto \frac{1}{a^4}. \tag{4.13}$$

The density and temperature of a hot gas are related by [136]

$$\rho = \frac{\pi^2}{30} N T^4. \tag{4.14}$$

The effective number of degrees of freedom, N, is

$$N = N_{\mathrm{boson}} + \frac{7}{8} N_{\mathrm{fermion}}, \tag{4.15}$$

where the factor of 7/8 arises due to the difference between Bose-Einstein and Fermi-Dirac statistics. In general, N varies as a function of temperature, since massive particles will cease to behave like a relativistic gas as the temperature falls. However, none of the results obtained in this thesis depend critically on the value of N, and we can safely treat it as a constant.

Comparing equations (4.13) and (4.14) gives the vital relationship

$$a \propto \frac{1}{T}. \tag{4.16}$$

This is solely a consequence of the conservation equation, and not any particular solution. Thus a small amount of radiation mixed into a universe full of cold dust (which is an accurate depiction of the Cosmic Microwave Background radiation in the present day universe) will continue to cool in inverse proportion to the expansion, even if the radiation energy density does not dominate the equation of motion.

When $k = 0$, we substitute for ρ in equation (4.7) and, after integrating, we find the familiar result,

$$a \propto t^{1/2}, \tag{4.17}$$

which describes the expansion of a spatially flat *radiation dominated* universe.

4.2.2 Cold Dust

In a universe full of cold dust there is no pressure, and the equation of state is simply

$$p = 0. \tag{4.18}$$

It is then trivial to find the connection between ρ and a. The conservation equation tells us that $a^3\rho$ is a constant, so

$$\rho \propto \frac{1}{a^3}. \tag{4.19}$$

Putting this result in equation (4.7), we then get

$$a \propto t^{2/3} \tag{4.20}$$

for a spatially flat *matter dominated* universe.

4.2.3 Horizons

When we seek a physical explanation for the observed isotropy and homogeneity of the universe, we must postulate a causal mechanism that will correlate regions of the universe which are separated by cosmological distances. For two points to be causally linked, a photon must be able to travel between them in a time less than the present age of the universe; points that are further apart cannot influence one another and are said to be *causally isolated*. If there are pairs of points that are causally isolated, then the cosmological model has a particle horizon [154], often referred to simply as a horizon. The horizon length, d_H, is thus the distance covered by a photon in the present age of the universe.

Mathematically, a photon travels on a null geodesic with $ds^2 = 0$. Assuming $d\theta = d\phi = 0$ (which is permissable in a homogeneous and isotropic universe) and using the metric, equation (4.4), gives

$$\begin{aligned} dt &= \frac{a(t)}{\sqrt{1-kr^2}}dr \\ \Rightarrow \int_0^t \frac{dt}{a(t)} &= \int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}} \end{aligned} \tag{4.21}$$

where t is the present age of the universe and r_H is the horizon distance in *comoving* co-ordinates. The physical distance to the horizon, $d_H = a(t)r_H$, is by definition:

$$\begin{aligned} d_H &= \int_0^{r_H} \sqrt{g_{rr}}dr \\ &= a(t)\int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}} \\ \Rightarrow d_H &= a(t)\int_0^t \frac{dt}{a(t)} \end{aligned} \tag{4.22}$$

where the last line follows from equation (4.21).

If $a(t) \propto t^p$ and $p \geq 1$ at small times, then d_H is infinite. However, for both the radiation and matter dominated cosmologies, $p < 1$ and the horizon length is finite. For instance, in the matter dominated model in flat ($k = 0$) space, $d_H = 3t$.[4] The horizon length defines the physical size of the universe because it is, quite literally, "as far as we can see". If the universe were not expanding, then $a(t)$ would be constant and $d_H = t$. The horizon length is a combination of the motion of the hypothetical photon and the subsequent expansion of the distance it has traversed due to the growth of the universe.

4.2.4 Hot Big Bang Cosmology

The standard model of cosmology begins with the universe full of immensely hot gas. As the universe expands, the gas cools, and at different temperatures

[4]This equation is not dimensionally incorrect. In natural units with $c = 1$, velocities are dimensionless and [length] $\equiv$ [time].

different physical processes take place - some of which leave imprints on the present universe, providing a sequence of windows into the first few moments of creation.

The dominant theme in the development of the universe is the formation of increasingly complex structure. At the highest temperatures, the spontaneously broken symmetry of the Higgs field is restored and all particles are massless, as described in Chapters 2 and 3. At a temperature of around 10^{14} GeV (10^{27}K) this symmetry can be broken, although substantial amounts of supercooling are possible. Below this transition, particles in the hidden sector of the GUT are massive, but electrons, muons and all other particles that acquire mass through their interaction with the electroweak Higgs remain massless until much lower temperatures are reached.

At a temperature of 10^3 GeV, the quark-hadron phase transition [152] takes place. At higher temperatures, ordinary nucleons are unstable, and the ability of quarks to form stable units is the first step in the formation of ever more complicated structures as the universe cools. At a few hundred GeV the electroweak phase transition occurs [56]. It has been carefully studied, but its precise explication depends both on sensitive calculations and obtaining accurate knowledge of the masses of the Higgs and the top quark.

Observationally, much effort has gone into measuring the relative abundances of light isotopes [183]. Nucleosynthesis proceeds rapidly at temperatures below 0.1 MeV, converting approximately 22% of the nucleons into helium-4 with traces of other isotopes. This provides a valuable window into the early universe since the pressures and temperatures in the Big Bang during primordial nucleosynthesis are markedly different from those found in the centres of stars. Consequently, the allowed reactions are different and yield isotopes that are not created by any other known astrophysical process. Observing the relative abundances of trace isotopes and then deconvolving the subsequent evolutionary effects from the primordial production is difficult and complex. However, these observations provide compelling evidence for the idea that the universe was once sufficiently hot and dense to support nuclear reactions of the kind envisaged for the Big Bang. Also of central importance is the constraint from nucleosynthesis calculations that baryonic matter can have a density that is no more than a few percent of the present critical density. Since observations of Ω tend to support the idea of a higher density [54, 158] the bulk of the matter in the universe consists of dark matter - invisible and non-baryonic.

This evolution occurs in the first three minutes after the Big Bang. When nucleosynthesis ceases, the universe is a plasma of electrons, protons and helium nuclei, awash in a bath of energetic photons. The density of radiation is much higher than that of matter, but this imbalance is slowly reversed as the universe cools, since the density of a photon gas drops faster during expansion than that of cold matter. Approximately 200,000 years after the Big Bang the matter and radiation densities are in equilibrium - and at later times the universe is matter dominated, expanding with $a \propto t^{2/3}$.

Photons cannot propagate through a hot plasma, so the primordial universe is opaque. Once the temperature falls to around $3,500$K, the electrons and protons combine to form hydrogen atoms. At this point the universe becomes transparent and the radiation decouples from the matter. The sea of photons that suffuses the present universe conforms to a blackbody distribution and has been cooled to its present value of 2.73K by the expansion of the universe. This is the Cosmic Microwave Background. The mean free path of a photon is much larger than the present horizon length so the forces that influenced it 15 billion years ago can be inferred from present day observations. Consequently, the Cosmic Microwave Background carries the delicate imprint of the density distribution of the universe when it was one millionth of its present age - giving what George Smoot calls the "baby photo of the universe".

4.3 Problems with the Standard Model

The standard model of the universe sketched in the last section is far from perfect. While it explains the overall form of the present universe, nagging questions linger. Here I will survey three of them in detail, although a much more extensive list can be found in the books of Kolb and Turner [102] and Linde [122].

4.3.1 The Horizon Problem

The concept of a horizon has already been introduced. The problem that horizons present for standard cosmology is this: the presently observable volume of the universe is composed of many identical subvolumes that evolved completely independently of each other. So - why do they all look the same? In particular, the Cosmic Microwave Background is isotropic to around 1 part in 10^5 - and the only way to bring this about in the conventional framework is to inject it through the initial conditions, which is not an explanation.

Consider the problem in more detail for a spatially flat Robertson-Walker model[5]. If a_o is the size of the scale factor today, and $a_{\rm dec}$ is the scale factor at decoupling, then the present temperature and the decoupling temperature are connected via equation (4.16),

$$\frac{a_o}{a_{\rm dec}} = \frac{T_{\rm dec}}{T_o} = \frac{3500}{2.73}. \tag{4.23}$$

The universe has been matter dominated throughout this interval, so equation (4.20) links the time at decoupling to the present time, since

$$\frac{a_o}{a_{\rm dec}} = \left(\frac{t_o}{t_{\rm dec}}\right)^{2/3}. \tag{4.24}$$

The present horizon volume is simply the volume of a sphere whose radius is equal to the present horizon length (dropping the H subscript for the remainder of this section),

$$V_o(t_o) = \frac{4}{3}\pi d_o^3. \tag{4.25}$$

At decoupling, this volume had a physical size

$$V_o(t_{\rm dec}) = V_o(t_o)\left(\frac{a_{\rm dec}}{a_o}\right)^3. \tag{4.26}$$

Since $d_H \propto t$, the horizon length at decoupling, $d_{\rm dec}$, is related to the present horizon length by

$$\begin{aligned} \frac{d_{\rm dec}}{d_o} &= \frac{t_{\rm dec}}{t_o} \\ \Rightarrow d_{\rm dec} &= d_o\left(\frac{a_{\rm dec}}{a_o}\right)^{3/2}. \end{aligned} \tag{4.27}$$

Thus the horizon volume *at decoupling*, $V_{\rm dec}(t_{\rm dec})$ is given by

$$V_{\rm dec}(t_{\rm dec}) = \frac{4}{3}\pi d_{\rm dec}^3 = \frac{4}{3}\pi d_o^3\left(\frac{a_{\rm dec}}{a_o}\right)^{9/2}. \tag{4.28}$$

The ratio of equations (4.28) and (4.26), evaluated at the same time, gives the number of independent horizon volumes at decoupling which now comprise the visible universe. Substituting gives

$$\frac{V_o(t_{\rm dec})}{V_{\rm dec}(t_{\rm dec})} = \left(\frac{a_o}{a_{\rm dec}}\right)^{3/2} = 4.6 \times 10^4. \tag{4.29}$$

[5]Restricting attention to the spatially flat case simplifies the algebra without obscuring the nature of the horizon problem.

That is, the visible universe is composed of around 46,000 regions that evolved completely independently from one another, and now are similar to within a few parts in a million.

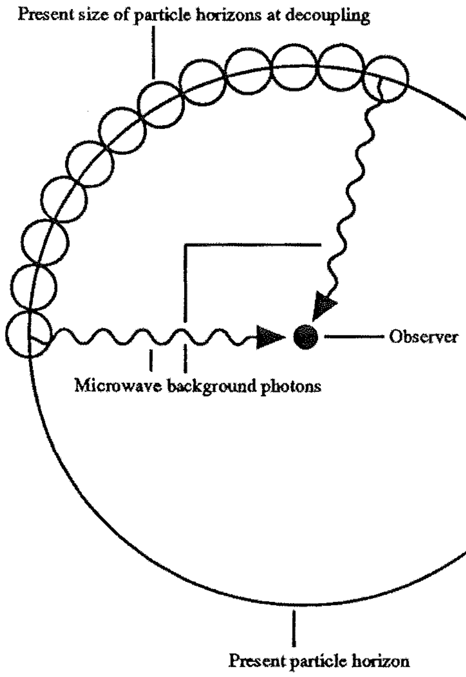


Figure 4.1 : The difference between the present particle horizon and the horizon size at decoupling is illustrated. Photons of the Cosmic Microwave Background coming from different directions in the sky show that the evolution within each causally isolated volume must have been the same.

4.3.2 The Flatness Problem

A similar problem that finds its roots in the initial conditions needed to obtain the present day universe is the flatness problem. As the universe expands, the quantity $|\Omega - 1|$ increases. Since $\Omega \approx 1$ today, then it must have been very close to unity immediately after the Big Bang.

As an illustrative example, consider a radiation dominated universe that cools from a GUT-scale temperature (10^{14}GeV or 10^{27}K) to the present microwave background temperature, 2.73K when $|\Omega - 1| = 0.5$. While this is an artificial example it accurately reflects the scale of the flatness problem in the real universe. By definition,

$$\Omega - 1 = \frac{\rho - \rho_c}{\rho_c}. \tag{4.30}$$

Using equations (4.7) and (4.10) we simplify the numerator, giving,

$$\Omega - 1 = \frac{k}{a^2 H^2}. \tag{4.31}$$

For radiation, $a \propto 1/T$ and $H \propto T^2$, so

$$\Omega - 1 \propto \frac{k}{T^2}. \tag{4.32}$$

Since T decreases as the universe expands, $\Omega = 1$ is an unstable fixed point and the universe will evolve away from it. The severity of the fine tuning problem for the example we are considering here is easily calculated:

$$\frac{\Omega_o - 1}{\Omega_{GUT} - 1} = \left(\frac{T}{T_{GUT}}\right)^2 = \left(\frac{2.73}{10^{27}}\right)^2 \approx 10^{-55}. \tag{4.33}$$

Thus, in order for the universe to be roughly as we see it today, the primordial universe must have been very close indeed to the critical density. Clearly any complete cosmological theory would explain this value, rather than require it as part of its initial conditions.

4.3.3 Structure Formation

The last difficulty faced by the standard model that I will discuss is the problem of structure formation. The Robertson-Walker metric is perfectly homogeneous and isotropic - but when viewed at "small" scales the physical universe is not. This is not an insurmountable difficulty, as the universe as a whole can still be modelled with the Robertson Walker metric, but this approach breaks down at smaller lengths.

The departures from homogeneity and isotropy can be described as small perturbations propagating in a Robertson-Walker background. This theory is well understood [188] and is summarised in Chapter 9. Small perturbations evolve according to a linear equation and, in a spatially flat Robertson-Walker model, can be expanded in a Fourier series. Harrison [83] and Zel'dovich [194] demonstrated independently that the presently observed structure of the universe can be produced if the perturbation spectrum is scale free, or the amplitudes of the Fourier modes are (initially) independent of their wavelength. At this point, the theory of structure formation is placed on the same footing as the Robertson-Walker model itself. The model produces something similar to the present universe, but requires special initial conditions - in this case the scale free perturbation spectrum.

4.4 Inflation for Fun and Profit

The solution to all the problems of standard cosmology can be found in inflation. Inflation can be brought about by a variety of physical mechanisms but broadly speaking, it is a period of pseudo exponential (or rapid power-law) growth in the primordial universe. Inflation typically takes place within 10^{-35} seconds of the Big Bang and it effectively prepares the universe in the very special set of initial conditions required for the standard cosmology to work. During inflation, the universe grows by a factor of at least e^{60} (or 60 e-foldings) and possibly much, much more. Without inflation the entire visible universe would be roughly the size of a large grapefruit: inflation is what puts the big into the Big Bang.

4.4.1 How Inflation Works

By definition, inflation is taking place whenever $\ddot{a} > 0$ [1]. Integrating, we can write this requirement as $a(t) \propto t^p$ with $p > 1$. In such a universe, $d_H \propto t$ and the comoving distance to the horizon, $d_H/a(t) \propto t^{1-p}$. If $p > 1$, the comoving distance to the horizon is decreasing, even though the physical volume contained within the horizon continues to expand. Hence in an inflating universe, points that were once visible are continually crossing the horizon and vanishing from view.

Consequently, a small volume of the pre-inflationary universe can be expanded until it is vastly greater than the current horizon size. After inflation ends, $p < 1$ and the comoving horizon length increases - however the points that are now crossing the horizon were correlated before inflation.

The flatness problem is also removed in the same way. For $a \propto t^p$, $H = p/t$. Substituting for ρ and a in equation (4.31) we have

$$\Omega - 1 = \frac{k}{p^2 t^{2(p-1)}} \tag{4.34}$$

and if $p > 1$, then Ω will move closer to unity as inflation continues.

The generation of density perturbations is less straightforward. Inflation takes points that are initially very close together and separates them by distances that are many times greater than the horizon scale. Thus microscopic, quantum, fluctuations are stretched until they attain macroscopic size. Furthermore, a computation of the size and shape of the fluctuation spectrum produced during inflation is roughly Harrison-Zel'dovich - although small departures from a truly scale free spectrum are common. Since these differences are model dependent, they will be valuable in differentiating between rival inflationary models.

4.4.2 The Field Equations

Most commonly, inflation is driven by a scalar field which is often referred to as the *inflaton*. The equations of motion are easy to write down. A minimally coupled scalar field has the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{4.35}$$

where $V(\phi)$ is the effective potential. The stress energy tensor for this field is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}. \tag{4.36}$$

Doughty [59] gives an introduction the stress energy tensor in flat spacetime. Since we are considering homogeneous and isotropic cosmologies, we can remove the spatial derivatives, leaving a diagonal tensor with the form of equation (4.6) from which we identify the density, ρ, and the pressure, p, for the field ϕ. However, since the pressure we have just defined can be negative (and, indeed, is negative during inflation), it does not correspond directly to a hydrodynamic pressure. Dropping the spatial derivatives we find

$$\rho = T_0{}^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4.37}$$

$$p = T_i{}^i = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{4.38}$$

where there is no implied summation on the $T_i{}^i$. Substituting for ρ and p in equations (4.7) and (4.8) gives

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \tag{4.39}$$

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left(V(\phi) - \dot{\phi}^2\right) \tag{4.40}$$

and equation (4.9) gives us the equation of motion for ϕ,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \tag{4.41}$$

These are the basic equations of inflationary cosmology.

Alternatively, the equation of motion can be found by varying the action,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\mathcal{L} \tag{4.42}$$

which, since we are working in curved spacetime, gives

$$\nabla_\lambda \frac{\partial \mathcal{L}}{\partial(\nabla_\lambda \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{4.43}$$

where ∇_λ is the covariant derivative. The second term on the left hand side contributes the $dV/d\phi$. The first term must be evaluated carefully, with the $3H\dot{\phi}$ term arising from the non-zero Christoffel symbols of the Robertson-Walker metric.

4.4.3 Inflation and the Slow Roll

Inflation is commonly analysed using the slow roll approximation. To make this approximation, we require that the energy density is dominated by the potential or that $\dot{\phi}^2$ is small and remains small for an appreciable time. The evolution is usually considered in a spatially flat universe, with $k = 0$. This is a legitimate assumption since an inflationary universe evolves towards $\Omega = 1$ and even if it is not a good assumption to begin with, it becomes a much better one as time passes. The more general problem of the dependence of inflationary models on their initial conditions and whether large initial inhomogeneities can prevent inflation from starting has been studied both numerically and analytically [76]. However, in the slow roll approximation it is assumed that

$$H^2 = \frac{8\pi G}{3}V \tag{4.44}$$

$$\dot{\phi} = -\frac{1}{3H}V'(\phi). \tag{4.45}$$

where the dash indications differentiation with respect to ϕ. There are three separate requirements that must be met for slow rolling to hold [111,174]. The first is that ρ be dominated by the potential energy, or

$$\frac{\dot{\phi}^2}{2} \ll V(\phi) \tag{4.46}$$

which is employed when writing down equation (4.44). Next, we require $\ddot{\phi}$, when calculated from equation (4.45), to be much smaller than the $3H\dot{\phi}$ term in equation (4.41), or

$$\frac{\ddot{\phi}}{3H\dot{\phi}} = \frac{1}{48\pi G}\left(\frac{V'}{V}\right)^2 - \frac{V''}{9H^2} \ll 1. \tag{4.47}$$

We can satisfy this requirement by demanding that both of the terms on the right hand side be very small, giving the second and third constraints,

$$|V''(\phi)| \ll 9H^2 \tag{4.48}$$

$$|V'/V| \ll \sqrt{48\pi G}. \tag{4.49}$$

In practice, a potential satisfying one of these criteria will usually satisfy both.

We can now write down approximate solutions to the Einstein field equations. Recalling the definition, $H = \dot{a}/a$, we have

$$a(t) = a_o \exp\left\{\int_0^t H(t)dt\right\}, \tag{4.50}$$

which, upon making the slow roll assumption, can be combined with equation (4.44) to give the quasi-exponential expansion characteristic of inflation.

4.5 A Survey of Inflationary Models

The concept of inflation was introduced by Guth in 1981 [79]. The idea that a period of rapid expansion in the primordial universe would solve the problems of the standard cosmology was not a new one. Olive [144] summarises "inflation before inflation" and with the benefit of hindsight he shows that inflation was anticipated by at least 15 years. In this section, I review fifteen different inflationary models. Broadly speaking, they fall into three classes. The first group of models have effective potentials motivated by different branches of field theory (and sometimes plucked out of thin air) serving as the source term in Einstein gravity. The second class involves different modifications to the gravitational sector of the lagrangian. The final subdivision contains broader groups of models that are not derived from any particular physical theory but are distinguished by their methodology.

Old Inflation

Old inflation is the name now given to Guth's original model. Instead of a slowly rolling field, it works by trapping ϕ in a local minimum of $V(\phi)$. If the barrier is high enough the field will remain trapped for a long time and sufficient inflation can take place. The field energy has two components - the magnitude of $V(\phi)$ at any given point and the spatial gradient. During inflation, the symmetry is unbroken. After the field tunnels through to the broken phase at a given spatial

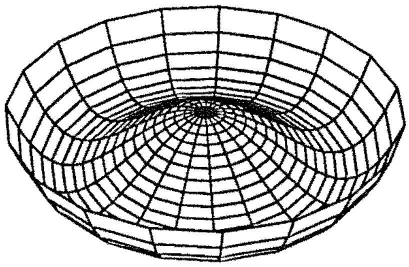


Figure 4.2 : When the field, ϕ, is at the peak of the "Mexican Hat potential" the configuration is rotationally symmetric. After symmetry breaking occurs and the field rolls down into the brim of the hat, the rotational symmetry is *spontaneously broken*.

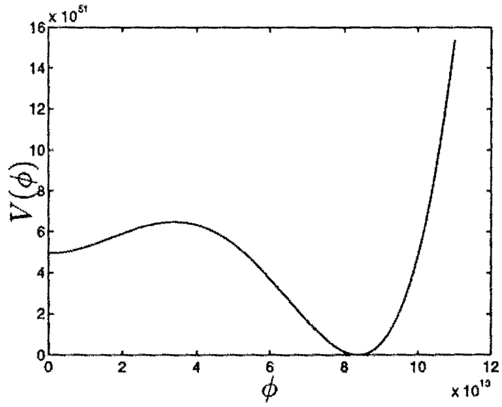


Figure 4.3 : This figure is a cross section of the effective potential that drives old inflation. Initially the field is at $\phi = 0$ and when it tunnels through the barrier it can evolve semi-classically to the global minimum.

point, a bubble of the broken phase forms in the unbroken background. Since the field, ϕ, is a discontinuous parameter the phase transition is first order.

It was recognised in Guth's original paper that old inflation is not a viable inflationary model, due to what is known as the "graceful exit" problem. In its more virulent form, which is the way it appears in old inflation, the phase transition never completes. The physical volume of space containing the field in the unbroken phase is continually increasing due to the expansion of the universe. If the probability of a phase transition at a given point is low, the total volume of space in the unbroken phase increases faster than it can be depleted by tunnelling. Thus the universe would consist of a set of disconnected bubbles (or domains) in the broken phase, in an unbroken sea. Even if the phase transition is completed, the universe can still contain domains where the field has broken in different directions. The domain walls are massive objects and produce a detectable anisotropy, which is not observed. So if inflation ends with a first order phase transition, it must proceed in such a way as to ensure that the domain walls do not produce a substantial anisotropy. Brandenberger [28] gives a thorough discussion of tunnelling in the expanding universe.

New Inflation

This was the first workable model. Originated independently by Linde [120] and Albrecht and Steinhardt [8] the potential is that of an $SU(5)$ GUT with Coleman-Weinberg symmetry breaking, as studied in Chapters 2 and 3.

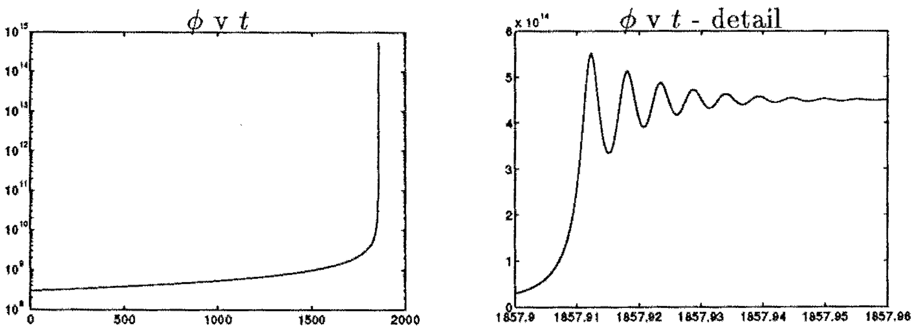


Figure 4.4 : The left hand plot shows the evolution of $\log(\phi)$ against time for new inflation when driven by a Coleman-Weinberg GUT. The field evolves slowly as long as it remains in the flat portion of the potential, but the final phase of the evolution to the minimum completes very rapidly. On the right hand plot, the oscillatory phase of the evolution of ϕ is plotted. During this phase, energy is efficiently transferred between the inflaton field and other species, reheating the universe. The plots have been produced by a numerical integration of the equations of motion, following Albrecht, Steinhardt, Turner and Wilczek [9].

In this case the domains are not a problem, as symmetry breaking occurs at the beginning of inflation and a single domain is inflated until the present visible universe would fit into it many times over. The second advance made by new inflation was that it could produce density perturbations of the required sort. However, this capability was also the undoing of new inflation, at least when based on a GUT Higgs. While the right spectrum of density perturbations is generated, their amplitude must be small, putting a strong constraint on the self coupling of ϕ and requiring it to be roughly 10^{-14}. Since radiative corrections to $V(\phi)$ ensure that the scalar self coupling is always roughly the same as the GUT coupling, around $1/40$, the original models of new inflation do not work. As the field becomes steeper, ϕ oscillates in the minimum of $V(\phi)$ and the universe reheats.

Chaotic Inflation

This is probably the simplest variant of inflation. Also introduced by Linde [121], it is based on a simple polynomial potential, for instance:

$$V(\phi) = \lambda\phi^4. \tag{4.51}$$

Linde's book [122] begins with a lengthy treatment of chaotic inflation, as an introduction to the inflationary paradigm. It is assumed that $V(\phi) \approx m_p^4$ at the Planck scale and $\lambda \leq 10^{-14}$, to ensure that the density perturbations of the correct amplitude are produced. The friction applied by the H term in equation (4.41) brings about slow rolling and exponential expansion. In both old and new inflation, ϕ had to have a special value for inflation to work; however, in chaotic inflation it merely needs to be sufficiently large.

Power-law Inflation

Power-law inflation was first introduced in 1984 by Abbott and Wise [1] and developed by Lucchin and Matarrese [124, 125] the following year. Usually an inflationary universe grows exponentially, but as its name suggests the hallmark of power-law inflation is $a(t) \propto t^p$ with $p > 1$. In its simplest form it is based on an exponential potential,

$$V(\phi) = \Lambda \exp(-\lambda\phi). \tag{4.52}$$

Exponential potentials arise in a number of settings and these will be explored in later chapters as a prelude to presenting exact solutions for potentials which are a combination of several exponential terms.

Natural Inflation

Natural inflation [5, 71] is based on the pseudo Nambu Goldstone boson, or axion, that was originally introduced to explain the strong CP problem of QCD. The potential is a periodic one,

$$V(\phi) = \Lambda^4(1 \pm \cos(\phi/f)) \tag{4.53}$$

where f and Λ are model parameters. Natural inflation produces a distinctive density perturbation spectrum with a significant departure from the Harrison-Zel'dovich scale free spectrum, and a minimal contribution from tensor modes. Of all the presently viable inflationary models it is the one most strongly motivated by particle physics.

Hybrid Inflation

This is a model based on two scalar fields, interacting through the Lagrangian,

$$V(\phi, \psi) = \frac{1}{4}\lambda(\psi^2 - M^2)^2 + \frac{1}{2}m^2\phi^2 + \lambda'\phi^2\psi^2, \tag{4.54}$$

and is again the work of Linde [116, 123]. It is also called Two Scale Inflation. Its attraction is the way in which inflation finishes, as it has a first order phase transition, rather than an oscillatory phase, but does not lead to bubble production.

Double Field Inflation

Double field inflation [4] is conceptually similar to hybrid inflation. Once again, it involves two real, interacting scalar fields. The difference is in the way that inflation ends. In double field inflation, bubble formation does occur but the role of the second field is to act as a catalyst and ensure that the bubbles percolate. Thus this model also avoids the problems that troubled old inflation.

Intermediate Inflation

Intermediate inflation [18, 22] uses a potential with little or no justification in particle physics, but has interesting properties and an exact solution. The potential is

$$V(\phi) = \frac{8A^2}{\beta + 4}\left[\frac{\phi}{\sqrt{2A\beta}}\right]^{-\beta}\left[6 - \frac{\beta^2}{\phi^2}\right] \tag{4.55}$$

where $\beta = 4(f^{-1} - 1)$, $0 < f < 1$. The scale factor is

$$a(t) = \exp\left(At^f\right), \tag{4.56}$$

giving an expansion that is *intermediate* between power-law and exponential inflation.

Soft Inflation

This is another two field model, and resembles extended inflation which is based on Brans-Dicke gravity [24]. In this case one of the two fields has an exponential potential, leading to a combined potential of the form

$$V(\phi, \psi) = e^{-\beta\phi} V(\psi) \tag{4.57}$$

where $V(\psi)$ can be one of a variety of different functions.

Electroweak Inflation

This version of inflation uses a modified version of the standard electroweak theory to produce inflation at electroweak energies, as opposed to GUT energies [100]. The scalar particle that drives inflation is a standard model singlet and the model is primarily of interest as a counterexample to the widespread assumption that inflation takes place at GUT energy scales. From an experimental viewpoint, any extension to the electroweak sector of particle physics is likely to be testable in terrestrial accelerator experiments.

Higher Order (R^2) Inflation

This is a generic class of models which work through modifications to the *gravitational* action. Standard, or Einstein, gravity arises though the variation of the very simple action,

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{4.58}$$

where R is the curvature scalar. Since other geometrical quantities - including higher powers of R - can be included in the gravitational lagrangian, Einstein gravity is not a unique theory, but it is the most simple. The addition of higher order terms can be motivated by various unification schemes, as discussed in Chapter 6.

The original variant of this model [172] predates Guth's paper and it must be stressed that a fundamental scalar field is not intrinsic to this species of inflation. A conformal transformation is frequently used to convert these models into a Einstein gravity coupled to a scalar field. This procedure was originally proposed by Higgs [85] and described independently by Whitt [189]. Teyssandier and Tourrenc [177] also give the transformation between R^2 gravity and an action linear in R, but do not perform the rescaling needed to recover the Einstein action. Barrow and Costakis [21] discuss the transformation between Einstein gravity and the more general class of theories where the lagrangian is any analytic function of R. Magnano, Ferraris and Francaviglia [127,128] and Jakubiec and Kijowski [89] consider theories whose lagrangians include terms in the Ricci and Riemann tensors. More recently Magnano and Sokołowski [129] have provided a thorough discussion of gravity theories whose lagrangian is a non-linear function of R, and the physical significance of these theories.

Carrying out the conformal transformation will also alter the particle lagrangian, by coupling it to the scalar field that enters the theory after the conformal transformation. The particle lagrangian thus takes on a non-standard form. However during inflation the energy-momentum tensor is dominated by the scalar field, and the other particles may be ignored until reheating takes place so that after the conformal transformation the usual machinery developed to understand inflationary theories with slowly rolling fields can then be applied.

Extended Inflation

Extended inflation [104] also relies on a modified gravity - this time a Brans-Dicke theory. While the simplest models required Brans Dicke theories that are incompatible with experimental bounds, later models attempted to work their way around this barrier. Extended inflation ends with a first order transition and bubble formation. The bubbles must not be big enough to leave a detectable imprint of the microwave background, and this requirement places additional constraints on the model [110,112]. Furthermore, the work of Sokołowski [170,171], Cho [40] and Magnano and Sokołowski [129] questions the physical significance of the Jordan-Brans-Dicke frame, creating further difficulties for extended inflation.

Kaluza Klein Inflation

Kaluza Klein theories [10, 12, 60, 105] were introduced 70 years ago as a way of unifying gravity and electromagnetism [92,98,99] by adding an extra *compactified* dimension - one that has the topology of a circle and a radius of curvature on the Planck scale. Adding further compactified dimensions has the effect of creating a space that is locally Minkowski space $\times$ some compact manifold. The gravitational metric in the higher dimensional space can be decomposed into the form of four dimensional Einstein gravity, one or more (possibily massless) scalar fields and one or more pure vector fields. The vector fields are interpreted as the gauge particles of non-gravitational interactions, whose gauge group is determined by the isometry group of the compact manifold.

In addition to the above "low energy" fields the four-dimensional theory generally also contains an infinite tower of charged massive states, known as *pyrgons* that are produced by a harmonic expansion about the vacua of the low energy fields. The pyrgons, which typically have Planck scale masses, can make a significant contribution to the present day density of the universe which in turn places constraints on the allowable range of Kaluza-Klein theories [101]. In practice, all but a finite number of states are discarded by truncating the particle spectrum in such a way as to remove the Planck-mass scale states. This procedure, known as the Kaluza-Klein ansatz must be performed carefully, so that the reduced theory is still consistent with the higher dimensional field equation [61, 62]. When reducing from a higher dimensional theory to four dimensional Einstein gravity, positivity of energy determines the choice of conformal rescaling, placing further constraints on the allowable range of low energy theories [40, 129, 170, 171].

Obviously, the existence of extra spatial dimensions would have cosmological implications and a number of authors have looked at higher dimensional cosmologies [41,72]. While Kaluza Klein models have not lived up to their initial promise as a way of achieving grand unification [191] some inflationary models have been constructed in higher dimensional spacetimes [2,3,13,88,143,165,166]. In particular, Pollock presents a model [148] where the expansion is faster than exponential.

Multiple Inflation

Multiple inflation is a generic title for models that combine a sequence of inflationary phases, often driven by a series of different mechanisms. In particular, Salopek, Bond and Bardeen [163] consider multiple inflation as a way of designing the spectrum of density perturbations.

Stochastic Inflation

Lastly, stochastic inflation is distinguished from other inflationary models by its emphasis on the random motion of the field, ϕ, instead of the semi-classical evolution through the effective potential. The evolution is analysed through the use of Langevin equations, rather than the semi-classical equation of motion, equation (4.41). Linde [122] gives a thorough discussion of the mechanism and consequences of stochastic inflation.

Once the field point moves into a region where the semiclassical evolution dominates the stochastic, it then proceeds according to the usual equations of motion. Before this time, the stochastic behaviour is just as likely to ϕ move away from the minimum of potential as it is to move towards it. In this case, new horizon sized volumes are being continuously created where ϕ has moved away from the minimum position that the semiclassical evolution tends towards. The physical picture is of a region dominated by a stochastic field, spawning horizon sized volumes, where ϕ behaves semi-classically and in which inflation eventually ceases leading to a "standard" Big Bang. In this picture, inflation need never finish (at least in some region of the universe), giving rise to the term *eternal inflation*. Furthermore, since inflation continues forever, it provides a potential solution to the initial singularity problem by removing the need for a beginning. The global topology of such a universe could be extremely complex. Recently, it has been used to analyse the *stationary universe* [117], which postulates a model where the proportion of the physical volume of the universe that has a given set of properties is time independent.

Chapter 5

Exact Solutions

In Chapter 4, I introduced inflationary cosmology and described the slow rolling approximation to the equations of motion. In this and the next three chapters, however, I focus on *exact* solutions to the equations of motion.

Inflationary cosmology is usually discussed in the context of the slow rolling approximation, and usually this is sufficient. So why bother with exact models at all? This question has a number of different answers, all of which will apply at some point in this work:

- The shortest answer is that an exact solution is interesting "because it's there". Any analytic solution is of intrinsic interest, and a search for it (in some eyes, at least) requires no further justification.

- Even though the exact solution to the equations of motion for a general potential is not known, specific exact solutions give insights into the wider problem.

- Exact models provide a useful check on any numerical or approximate calculations and so provide tests of the tools that are used to study "realistic models".

- Sometimes, the usual approximation schemes are not reliable and it is more convenient to work with exact models.

- It is not often appreciated that it is very easy to produce exact scalar field cosmologies. What is more difficult is finding *interesting* exact scalar field cosmologies. However, these do exist and, once discovered, they provide new options to the cosmological model builder.

- An exact solution can be used to approximate a wider class of models which resemble, but are not identical to, an exact model. This approach has to be applied with care, since an exact solution that is unstable against small perturbations is not representative of a larger region of the parameter space.[1]

When solving differential equations, a great deal of algebraic complexity can be removed by a sensible choice of parameters and variables. In particular, many exact solutions undergo considerable simplification if the field, ϕ, rather than the time, t, is used to parametrise the evolution. This formalism was first applied to inflationary cosmology by Muslimov [139] and Salopek and Bond [162]. It was extended by Lidsey [113] to Robertson Walker universes with non-zero spatial curvature. In this Chapter I review the exact solutions that are given in the literature using this formalism and we will see that it also provides a convenient way of organising these solutions. I will treat models where the potential consists only of exponential terms in Chapter 6. In Chapters 7 and 8, I present a large class of new solutions based on such potentials.

The method I use to find exact solutions is to specify a particular form of the density,[2] $\rho(\phi)$, and then run the field equations backwards to derive the potential that produced it. The choice of $\rho(\phi)$ is guided by our interest in potentials that have a "reasonable" form. However, as long as the integrals in equations (5.9) and (5.10) can be performed this method will work for an arbitrary $\rho(\phi)$. So, in principle at least, we could produce an infinite number of different exact solutions, each corresponding to a different potential.

Other authors have obtained exact solutions by constraining the field equations in some way and then solving the restricted problem. In particular, Lidsey [115] derives solutions by specifying $H(\phi)$. Ellis and Madsen [68] derive several exact solutions by fixing the functional form of the scale factor, $a(t)$, and then computing $V(\phi)$. While the calculation involved here is similar, they emphasise the desired expansion whereas my choice of $\rho(\phi)$ has been guided by a desire to produce potentials of a specific form. Cadoni [33] attempts to generate exact scalar cosmologies from D-dimensional solutions to the Einstein equations with a cosmological constant, but his approach is flawed [190].

[1]Most of the new solutions presented in this thesis are stable against small perturbations and exceptions to this are noted as they occur.

[2]In spatially flat models this is equivalent to specifying the Hubble parameter, $H(\phi)$.

It would have been possible for me to have written up this work by writing down the potential and then a solution for the motion, with little or no comment as to how the solution was obtained. However, I feel that is instructive to make explicit the methodology I have employed, as it will facilitate further work in this area.

A note on notation

In this, and the next three chapters, I employ natural units where $G = 1/m_p^2 = 1/8\pi$. This choice of units is convenient as it reduces the algebraic complexity of exact solutions.

5.1 Calculational Machinery For Exact Solutions

The equations of motion for the scalar field, ϕ, form an extremely nonlinear set of ordinary differential equations. We begin by casting them into a form that simplifies their analysis. There are several possible ways of doing this, but here the most productive approach is to use the field, ϕ, to parametrise the evolution, rather than the time, t. The development given here follows that of Lidsey [113].

With $G = 1/8\pi$, we have the following system of equations for a scalar field dominated cosmology:

$$H^2 = \frac{\rho}{3} - \frac{k}{a^2} \tag{5.1}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) \tag{5.2}$$

where ρ and p are given by equations (4.37) and (4.38). From the definition of H it follows that

$$\begin{aligned} \dot{H} &= \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 \\ &= \frac{\ddot{a}}{a} - H^2 \\ \Rightarrow \dot{H} &= -\frac{1}{2}\dot{\phi}^2 + \frac{k}{a^2}. \end{aligned} \tag{5.3}$$

Expanding equation (4.9) and eliminating the terms in $\dot{p}$ gives

$$\dot{\rho} = -3H(\rho + p) = -3H\dot{\phi}^2. \tag{5.4}$$

Dividing through by $\dot{\phi}$ we find

$$\frac{d\rho}{dt}\frac{dt}{d\phi} = \frac{d\rho}{d\phi} = -3H\dot{\phi}. \tag{5.5}$$

Hence, the time derivative of any quantity X can be written in terms of the field, ϕ, since

$$\dot{X} = \frac{dX}{d\phi}\frac{d\phi}{dt} = -\frac{\rho'}{3H}X', \tag{5.6}$$

where the dash denotes differentiation with respect to ϕ. Defining $\chi = a^2$, we have

$$\begin{aligned} H &= \frac{1}{2}\frac{\dot{\chi}}{\chi} \\ &= -\frac{\rho'}{6H}\frac{\chi'}{\chi} \\ \Rightarrow H^2 &= -\frac{\rho'}{6}\frac{\chi'}{\chi}. \end{aligned} \tag{5.7}$$

Equating equations (5.1) and (5.7) we get a first order, ordinary differential equation linking ρ and χ,

$$\rho'\chi' + 2\rho\chi = 6k, \tag{5.8}$$

which has the solution

$$a^2(\phi) = \exp\left[-2\int_{\phi_0}^{\phi}\frac{\rho}{\rho'}d\phi\right]\left\{a_0^2 + 6k\int_{\phi_0}^{\phi}\exp\left[2\int_{\phi_0}^{\phi}\frac{\rho}{\rho'}d\phi\right]\frac{d\phi}{\rho'}\right\}, \tag{5.9}$$

where $a_o = a(\phi_0)$. If we know ρ we can integrate equation (5.9) to find a. Similarly, we can integrate equation (5.5) to get the time

$$t - t_0 = -\int_{\phi_0}^{\phi}\frac{3H}{\rho'}d\phi \tag{5.10}$$

where H is given by equation (5.7). Lastly, the connection between ρ and $V(\phi)$ is found by noting that

$$V = \rho - \frac{1}{2}\dot{\phi}^2 = \rho - \frac{\rho'^2}{18H^2(\phi)} \tag{5.11}$$

and using equation (5.5).

In Chapter 4, I introduced the condition for inflation, $\ddot{a} > 0$. It is convenient to express this criterion in terms of $\epsilon = -\dot{H}/H^2$, so that inflation is taking place when $\epsilon < 1$. From equation (5.6) we have

$$\epsilon = \frac{\rho' H'}{3H^3}. \tag{5.12}$$

Similarly, $\Omega = \rho/\rho_c$ is given in terms of ϕ by

$$\Omega = \frac{\rho}{3H^2}. \tag{5.13}$$

If $k = 0$, then $\rho = 3H^2$ and these equations simplify accordingly, to give

$$V(\phi) = 3H^2 - 2H'^2, \tag{5.14}$$

$$\frac{a}{a_o} = \exp\left(-\frac{1}{2}\int_{\phi_0}^{\phi}\frac{H}{H'}d\phi\right), \tag{5.15}$$

$$t = -\frac{1}{2}\int_{\phi_0}^{\phi}\frac{1}{H'}d\phi, \tag{5.16}$$

$$\epsilon = 2\left(\frac{H'}{H}\right)^2. \tag{5.17}$$

In Robertson Walker universes with flat spatial hypersurfaces there is thus a simple connection between $V(\phi)$ and H, and it is possible to generate large numbers of exact solutions, each corresponding to a different potential.

5.2 Abel's Equation and Inflation

It is possible to transform equation (5.14) into Abel's differential equation. This is a promising simplification, as Abel's equation has been widely studied [138]. However, it is also frustrating as a general solution cannot be written down. Muslimov [139] analysed this equation in the context of inflation. Equations of the type of equation (5.14) were examined by Mitrinovich [132, 133], who also gives the transformation to Abel's equation. It is convenient to rescale ϕ, to $\psi = \sqrt{3/2}\phi$, and to write the equations as

$$\left(\frac{dH(\psi)}{d\psi}\right)^2 - H(\psi)^2 = -f^2(\psi) \tag{5.18}$$

where $f^2 = V(\psi)/3$ and we have assumed that the potential is always non-negative.

Now, make the substitution

$$H(\psi) = f(\psi)\cosh\left[\coth^{-1}(y(\psi))\right] = f(\psi)\frac{y(\psi)}{\sqrt{y^2(\psi)-1}}, \tag{5.19}$$

converting equation (5.18) into an Abel's equation for $y(\psi)$,

$$\frac{dy}{d\psi} = \frac{f'}{f}y^3 - y^2 - \frac{f'}{f}y + 1, \tag{5.20}$$

where the dash now denotes differentiation with respect to ψ.

5.2.1 Integrating Factor or Fiction

Muslimov claims that Abel's equation has an integrating factor [139]. The original paper has not been published in English, but his approach can be reconstructed. Consider Abel's equation in the form

$$\frac{dy(x)}{dx} = \sum_{j=0}^{3} f_j(x)y^j \tag{5.21}$$

and a hypothetical integrating factor,

$$\mu(x,y) = \left[\sum_{j=0}^{3} m_j(x)y^j\right]^{-1}. \tag{5.22}$$

We can write equation (5.21) in the form,

$$M(x,y)dx + N(x,y)dy = 0 \tag{5.23}$$

which after multiplying with an integrating factor, $\mu(x,y)$ becomes,

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0. \tag{5.24}$$

After multiplication by the integrating factor, equation (5.24) is exact if [196]

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}. \tag{5.25}$$

Identifying[3] M and N from equation (5.21), dividing through by μ^2 and setting the co-coefficients of each power of y to zero, we find the following system of equations:

$$\begin{aligned} m_0'(x) &= m_0 f_1 - m_1 f_0 && (5.26)\\ m_1'(x) &= 2m_0 f_2 - 2m_2 f_0 && (5.27)\\ m_2'(x) &= 3m_0 f_3 - 3m_3 f_0 + m_1 f_2 - m_2 f_1 && (5.28)\\ m_3'(x) &= 2m_1 f_3 - 2m_3 f_1 && (5.29) \end{aligned}$$

as well as the algebraic condition,

$$m_3 f_2 = m_2 f_3. \tag{5.30}$$

[3] Note that $N = -1$ when written in this form. There appears to be a sign error in Muslimov's paper, although it could be absorbed in a redefinition of the f_j.

However, for a general set of f_j this system of equations has no solution, since the four differential equations in four unknown functions are augmented by a further constraint, equation (5.30). While it is possible to simplify these equations in various ways, especially through a transformation that sets f_0 to zero, it does not appear to yield an integrating factor either for general Abel's equation, or the specific example we are concerned with here. With either a constant potential or an exponential potential the right hand side of equation (5.20) is a function of y only and the differential equation can be integrated directly, leading to a complete solution. While the transformation to Abel's equation appears to be a blind alley, it is useful to be able to express the field equations as a first order differential equation which contains no higher powers of its derivatives.

5.3 Summary of Existing Results

In this section I systematically review the exact solutions that are to be found in the literature. Various authors have had different motivations for studying exact models, which influences the approach they have taken in their work. In particular, it is frequently a tacit assumption that all exact solutions to the Einstein field equations sourced by a scalar field will be inflationary. However, not all scalar field cosmologies inflate and I will present examples where the motion is always inflationary, never inflationary or a mixture of inflationary and non-inflationary phases. The parameter ϵ, defined above will immediately show whether or not a given model is inflating. Ellis and Madsen [68] and Lidsey [115] give several examples that have not been reported in detail here. Barrow and Saich [23] consider exact models that contain both a scalar field and a perfect fluid.

The advantages of choosing to parametrise the motion in terms of ϕ are made apparent by this brief review as the algebraic complexity of the solutions is often reduced to a deceptively simple level by this substitution.

5.3.1 A Simple Case

As an introduction to the methodology I have used, consider a Hubble parameter that has the functional dependence on ϕ,

$$H = C\phi^2. \tag{5.31}$$

Inserting this form of H into equation (5.14) to equation (5.17) we find

$$\begin{aligned} V(\phi) &= -8C^2\phi^2 + 3C^2\phi^4, && (5.32) \\ a(\phi) &= a_o \exp\left[\frac{{\phi_0}^2 - \phi^2}{8}\right], && (5.33) \\ t(\phi) &= \frac{1}{4C}\log\left(\frac{\phi_0}{\phi}\right), && (5.34) \\ \epsilon(\phi) &= \frac{8}{\phi^2}. && (5.35) \end{aligned}$$

Inverting $t(\phi)$, we then get

$$\begin{aligned} \phi(t) &= \phi_0 e^{-4Ct}, && (5.36) \\ a(t) &= a_o \exp\left\{\frac{{\phi_0}^2}{8}\left[1 - \exp\left(-8Ct\right)\right]\right\}. && (5.37) \end{aligned}$$

This solution was first given by Madsen [126] and is illustrated in Figure 5.1.

Several features of this solution warrant comment. The first is that while $V(\phi)$ has the form of a symmetry breaking potential, it is not fully general as the coefficients of the ϕ^2 and ϕ^4 terms are not independent. Such constraints between coupling co-coefficients are extremely common in exact scalar models. The second limitation is that the initial conditions cannot all be specified freely. Physically, the field approaches the origin from positive infinity and is monotonically decreasing. While this restriction leads to little loss of generality for large values of ϕ, the behaviour of the field at late times is an asymptotic approach an unstable equilibrium point, so the solution is not stable against small perturbations at large times.

When $\phi^2 > 8$, $\epsilon < 1$ and the universe is inflating. At later times, the rate of growth slows and inflation comes to a halt. Since

$$\lim_{t\to\infty} a(t) = a_o \exp\left(\frac{{\phi_0}^2}{8}\right). \quad (5.38)$$

this solution has the unusual property that the scale factor, a, is bounded at all times.

5.3.2 Polynomial Potentials

Using this technique it is simple to produce solutions for a wider class of polynomial potentials [115], starting from

$$H(\phi) = A + B\phi + C\phi^2 \quad (5.39)$$

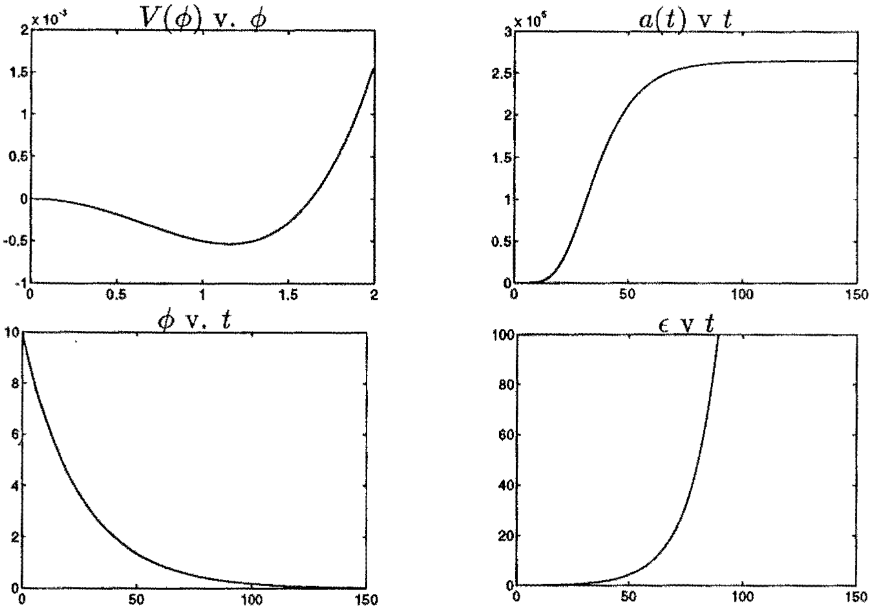


Figure 5.1 : The evolution of the solution to the potential, equation (5.32), is shown, when $C = 0.01$. The potential, $V(\phi)$ has the symmetry breaking form, and the field evolves smoothly towards the origin. The expansion is exponential for smaller values of ϕ, but once $\epsilon > 1$, inflation ceases. In this model, $a(t)$ does not expand indefinitely, but approaches a finite limit.

As long as $C \neq 0$ we can eliminate the term linear in ϕ through a redefinition of the field, $\psi = \phi + B/2C$, and $\alpha = A - B^2/4C$, which corresponds to

$$H(\psi) = \alpha + c\alpha\psi^2 \tag{5.40}$$

$$\begin{aligned}
V(\psi) &= 3\alpha^2 + (6\alpha^2 c - 8\alpha^2 c^2)\psi^2 + 3\alpha^2 c^2 \psi^4, && (5.41)\\
a(\psi) &= a_o \left[\frac{\psi_0}{\psi}\right]^{1/4c} \exp\left\{\frac{{\psi_0}^2 - \psi^2}{8}\right\}, && (5.42)\\
t(\psi) &= \frac{1}{4c}\ln\left(\frac{\psi_0}{\psi}\right), && (5.43)\\
\epsilon(\psi) &= 8\left[\frac{c\psi}{1 + c\psi^2}\right]^2. && (5.44)
\end{aligned}$$

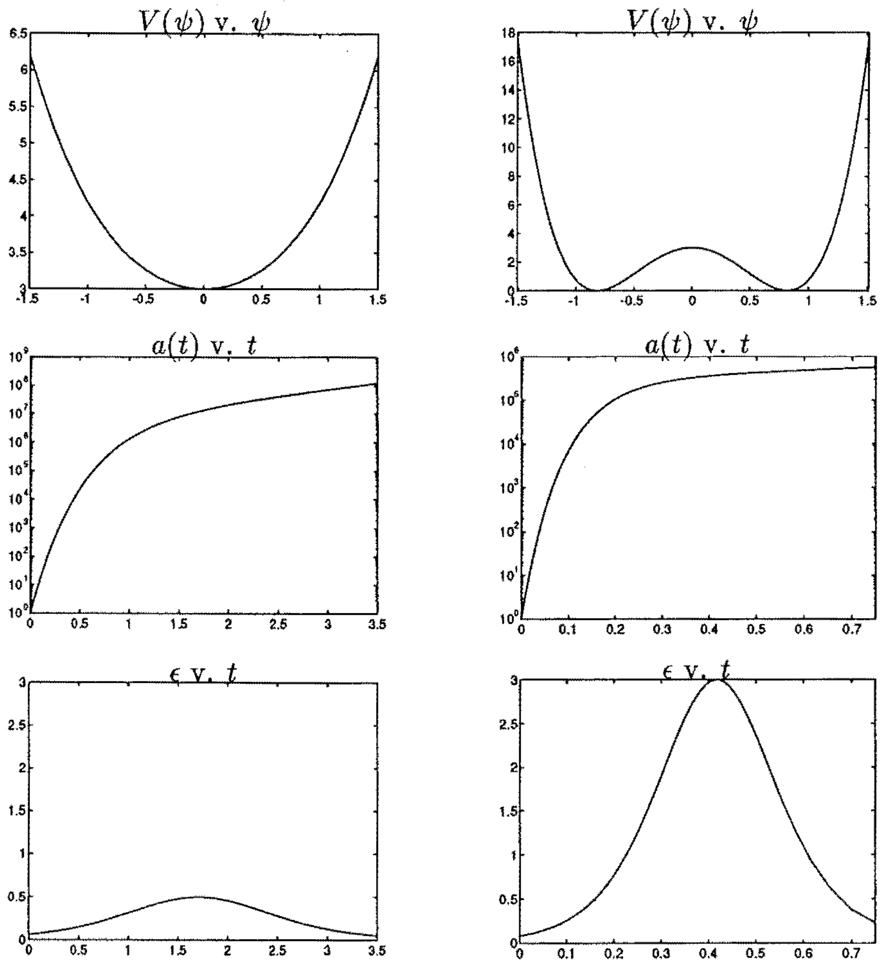


Figure 5.2 : Two solutions to the potential, equation (5.41) are shown, where $c = 0.25$ for the left hand column and $c = 1.5$ for the right hand column. The first case is always inflationary, since $\epsilon < 1$ at all times. Conversely, the right hand solution has two phases of pseudo-exponential growth, separated by a phase of "ordinary" expansion, when $\epsilon > 1$. The earlier phase is the inflationary expansion typical of a slowly rolling scalar field. The second occurs as the field slowly approaches the origin, and the non-zero value of the field at this point produces what is effectively a cosmological constant dominated universe with a de-Sitter expansion.

The properties of this solution depend strongly on the values of the constants α and c, as demonstrated in Figure 5.2. The model universe described by this solution is undergoing inflation when $\epsilon < 1$. A little calculus shows that the maximum value of ϵ is $2c$ and occurs when $\psi = 1/\sqrt{c}$. Since $\epsilon \approx 0$ when $\psi \rightarrow \infty$ and $\psi \approx 0$ the expansion for large values of c consists of two inflationary eras, separated by non-inflationary growth. This is less remarkable than it might appear - the first inflationary epoch is driven by the scalar field, but the second is simply the de Sitter expansion that is typical of a universe dominated by a cosmological constant.

When $c > 3/4$ the potential $V(\psi)$ has two degenerate global minima at $\psi \neq 0$, which is characteristic of a symmetry breaking potential. When $c > 3/2$ the minimum value of $V(\psi)$ becomes negative. In all cases, $\psi \rightarrow 0$ at large times. This exact solution, at large values of ϕ (or, equivalently, ψ) resembles models of chaotic inflation.

5.3.3 Intermediate Inflation

Unlike most species of inflation, intermediate inflation is not usually considered in the slow rolling mode [18,22]. While the potential, equation (4.55), has a complicated appearance, the solution is straightforward when parametrised by ϕ. Consider a model universe with the Hubble parameter,

$$H(\phi) = A\phi^{-m/2} \tag{5.45}$$

where $m > 0$, and from which we can obtain the solution,

$$\begin{aligned} V(\phi) &= 3A^2\phi^{-m} - \frac{A^2m^2}{2}\phi^{-(m+2)}, &(5.46)\\ a(\phi) &= a_o \exp\left(\frac{\phi^2}{2m}\right), &(5.47)\\ t(\phi) &= \frac{2}{Am(4+m)}\phi^{2+m/2}, &(5.48)\\ \epsilon(\phi) &= \frac{m^2}{2\phi^2}. &(5.49)\end{aligned}$$

The integration constants have been chosen to make $t = 0$ when $\phi = 0$. This is not a typical polynomial potential, as it contains negative powers of ϕ and is singular at the origin but tends towards zero at large values of ϕ. We can invert $t(\phi)$, giving

$$a(t) = a_o \exp\left(\gamma t^f\right) \tag{5.50}$$

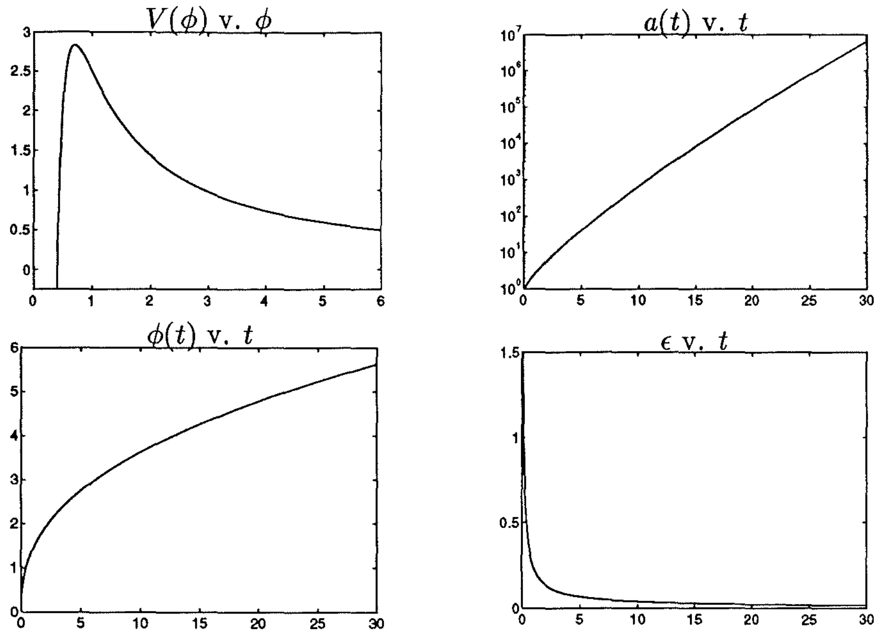


Figure 5.3 : The potential and corresponding solution is plotted for intermediate inflation with $m = 1$. The motion is initially non-inflationary, but as time continues $\epsilon \longrightarrow 0$ and inflation continues indefinitely.

where

$$f = \frac{4}{4+m} \qquad \text{and} \qquad \gamma = \left(\frac{Am(4+m)}{2}\right)^f .$$

At small values of ϕ this solution is not inflating, but when $\phi = \sqrt{2}m$, $\epsilon = 1$ and inflation commences, as depicted in Figure 5.3.

5.3.4 Trigonometric Potentials

Most of the usual elementary functions of mathematics appear in the effective potential of one or more inflationary models, and the trigonometric functions are no exception, with natural inflation being based on a simple periodic potential. It is straightforward to construct exact solutions from trigonometric functions, even if the results are a little odd. For instance, consider

$$H(\phi) = A\cos\left(\frac{\phi}{2f}\right) \tag{5.51}$$

with the corresponding exact solution

$$\begin{aligned}
V(\phi) &= A^2\left(\frac{3}{2}+\frac{1}{4f^2}\right)\left[\cos\left(\frac{\phi}{f}\right)+1\right]-\frac{A^2}{2f^2} && (5.52)\\
a(\phi) &= a_o\left(\frac{1-\cos\left(\frac{\phi}{f}\right)}{1-\cos\left(\frac{\phi_0}{f}\right)}\right)^{f^2} && (5.53)\\
t(\phi) &= \frac{2f^2}{A}\ln\left[\frac{\csc\frac{\phi}{2f}-\cot\frac{\phi}{2f}}{\csc\frac{\phi_0}{2f}-\cot\frac{\phi_0}{2f}}\right] && (5.54)\\
\epsilon(\phi) &= \frac{1}{2f^2}\tan^2\frac{\phi}{2f} && (5.55)
\end{aligned}$$

This potential is similar to that which drives natural inflation, equation (4.53) with a negative cosmological constant, $-A^2/2f^2$. When the model parameters (and the value of ϕ) are such that the cosmological constant term can be ignored, this model resembles natural inflation. However, the negative cosmological constant eventually causes the expansion of the universe to cease and recontract. The field, ϕ continues to increase with time, so the total energy density increases while the model contracts. Since we are considering a spatially flat model, H is strictly decreasing and once contraction starts it will not be reversed. A little analysis shows that $a \to 0$ as $t \to \infty$ so the future singularity is never actually attained. This solution is new, but Ellis and Madsen [68] have examined a similar potential in a Robertson-Walker universe with $k = 1$.

Chapter 6

Exponential Potentials

The majority of the new exact solutions presented in this thesis are based on potentials that are combinations of exponential terms. In this Chapter, I reformulate the equations of Section 5.2 to facilitate working with exponential terms and review the physical motivation for such potentials.

6.1 The Origin of Exponential Potentials

Inflationary cosmology has been described as "a paradigm in search of a model". Introducing a period of exponential (or rapid power-law) growth into the primordial universe eliminates most, if not all, of the problems that plague the standard model, as described in Chapter 4. What is still lacking, though, is a plausible physical mechanism to drive the inflationary growth. There is no shortage of ideas but none of these models has direct experimental support outside of inflationary cosmology - all rely on unobserved particles or extensions to Einstein gravity. While these models are not ruled out by experiment, they are not *required* by experiment either. In this climate, many researchers have simply written down a scalar field potential that produces interesting cosmological consequences and made the briefest of nods in the direction of physical motivation. My thesis is, of course, no exception to this rule.

Since I have focussed on exponential potentials I will briefly summarise the circumstances in which they do arise. However, while this summary provides the *motivation* for the work in this thesis, it does not provide a *derivation* for the potentials I consider.

A generic potential consisting of exponential terms has the form,

$$V(\phi) = \sum_{j=1}^{N} \Lambda_j \exp(-\lambda_j \phi). \tag{6.1}$$

In particular, we are interested in potentials where the λ_j are all integer multiples of the same constant, or

$$V(\phi) = \sum_{j=1}^{N} \Lambda_j \exp\left(-j\gamma\phi\right). \tag{6.2}$$

It is this form of the effective potential that is frequently attributed to superstring theory, where it appears as a result of a perturbative expansion. In addition, non-perturbative effects can result in potentials that are not expressible as a finite sum of exponentials, such as the double exponential in equation (6.13) below. The Λ_j are typically complicated functions of other fields or geometrical quantities, but they are frequently assumed to be fixed during inflation. While this assumption makes the analysis more tractable, it is not necessarily physically well motivated.

Generically, exponential potentials arise in modified theories of gravity and unification schemes such as superstring theory and supergravity. What all these theories have in common is their use of a conformal transformation to convert the gravitational lagrangian into the Einstein form and it is at this point the exponential potentials enter the picture [91]. Since conformal transformations are widely employed in models with complicated gravitational lagrangians or which are formulated in higher dimensional spaces, exponential potentials are reasonably widespread in high energy theory. On the other hand, many authors have been cavalier in claiming that any exponential potential is derived from one or more of these theories since there are often strong constraints on the allowable form of the potential, beyond simply consisting of exponential terms. Unless these constraints are satisfied in detail then it overstating the case to claim that a particular model is derived from supergravity or string theory. In the remainder of this section I review models that produce exponential potentials.

6.1.1 The Jordan-Brans-Dicke Metric

The simplest source of an exponential term is the Jordan-Brans-Dicke (JBD) scalar tensor theory of gravity when it is conformally transformed to Einstein gravity [30, 90]. The JBD action is

$$S_{\mathrm{JBD}} = \int d^4x\sqrt{g}\left[\phi R - \frac{\omega}{\phi}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - m^2\psi^2 - V(\psi)\right], \tag{6.3}$$

where ϕ is the Brans-Dicke scalar, or dilaton, and ψ is a scalar field that is representative of a larger matter lagrangian. The parameter ω is free. If φ

varies with time, then the gravitational coupling - which is the coefficient of R - is also time dependent but the couplings in the particle physics sector of the theory are time independent. We can redefine the metric through a conformal transformation,

$$\tilde{g}_{\mu\nu} = 2\kappa^2\phi g_{\mu\nu} \equiv \Omega^2 g_{\mu\nu}, \tag{6.4}$$

where $\kappa^2 = 8\pi G$.[1] Since the metric appears in several different guises in equation (6.3), the various terms within it transform differently. In particular R scales in proportion to the metric (through its definition), as do the kinetic terms in ψ and ϕ which have an implicit dependence on the metric through the summation over the dummy index. Taking all these factors into account gives

$$S_{\mathrm{JBD}} = \int d^4x\sqrt{\tilde{g}}\left[\frac{\tilde{R}}{2\kappa^2} - (\omega+\frac{3}{2})\frac{\partial_\mu\phi\partial^\mu\phi}{2\kappa^2\phi^2} - \frac{1}{2}\frac{\partial_\mu\psi\partial^\mu\psi}{\kappa^2\phi} - m^2\frac{\psi^2}{4\kappa^4\phi^2} - \frac{V(\psi)}{4\kappa^4\phi^2}\right]. \tag{6.5}$$

We now have a theory with a constant gravitational coupling and a more complicated particle spectrum with non-standard kinetic terms. However, this situation can be partially ameliorated by redefining ϕ,

$$\sigma = \frac{\sqrt{\omega+\frac{3}{2}}}{\kappa}\log 2\kappa^2\phi. \tag{6.6}$$

This is the point at which the exponential potentials enter, since powers of ϕ appear in the redefined lagrangian as exponentials of σ, so

$$\begin{aligned} S_{\mathrm{JBD}} &= \int d^4x\sqrt{\tilde{g}}\left[\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\exp\left(\frac{\kappa\sigma}{\sqrt{\omega+3/2}}\right)\partial_\mu\psi\partial^\mu\psi \right.\\ &\quad \left. - \exp\left(\frac{2\kappa\sigma}{\sqrt{\omega+3/2}}\right)m^2\psi^2 - \exp\left(\frac{2\kappa\sigma}{\sqrt{\omega+3/2}}\right)V(\psi)\right]. \end{aligned} \tag{6.7}$$

In more complicated theories the relationship between σ and ϕ will differ, but the presence of an exponential potential is a generic characteristic of theories that have been conformally transformed to resemble Einstein gravity. The equations (6.3) and (6.7) are equally "fundamental" - they represent a choice between Einstein gravity and time dependent couplings in the matter sector of the theory, or a time independent particle sector and a more complicated gravity sector. The complexity of the two actions is "conserved" but a choice has been made about where that complexity resides.

[1] In natural units, $\kappa = 1$ and drops out of these equations.

6.1.2 Higher Order Gravity, Extended Gravity and Supergravity

As described in Chapter 4, theories of gravity with an *extended* lagrangian can be conformally transformed to the Jordan-Brans-Dicke model. Wands [184] gives a careful account of the transformations involved as well as several explicit examples.

Similarly, supergravity theories can produce terms that are higher order in R, which can then be removed by a conformal transformation [189]. The action,

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{6M^2} + \mathcal{L}_{\text{particle}}\right) \tag{6.8}$$

has received considerable attention due to its relative simplicity [37]. The distinction between inflation based on higher order gravity and scalar field models is blurred by the conformal transformation. The lagrangian in the action, equation (6.8), becomes

$$\mathcal{L} = -\frac{1}{16\pi G}\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \tilde{\mathcal{L}}_{\text{particle}} \tag{6.9}$$

where

$$V(\phi) = \frac{3M^2}{32\pi G}\left[1 - \exp\left(-\sqrt{\frac{16\pi G}{3}}\phi\right)\right]^2 . \tag{6.10}$$

The theory is now Einstein gravity, with the addition of a new scalar degree of freedom that is formally equivalent to a scalar field.

Gravity theories that are polynomial in R will often, after a conformal transformation to the Einstein frame, give rise to a scalar potential that cannot be written in closed form. However, in regions of interest these potentials can be approximated by a sum of exponential terms [130], providing further motivation for studying exact solutions where the potential is of the form equation (6.1).

6.1.3 Superstrings

Superstrings [78] are frequently touted as candidates for a "theory of everything", although some of the excitement surrounding them in the early 1980s has faded. Superstring theories have potentials that contain exponential terms in the string dilaton [34,70,178] and employ conformal transformations to shift themselves into the Einstein frame. However, the form of the potentials that are obtained from string theory is subject to a number of constraints and once again it is important to stress the difference between motivation and derivation.

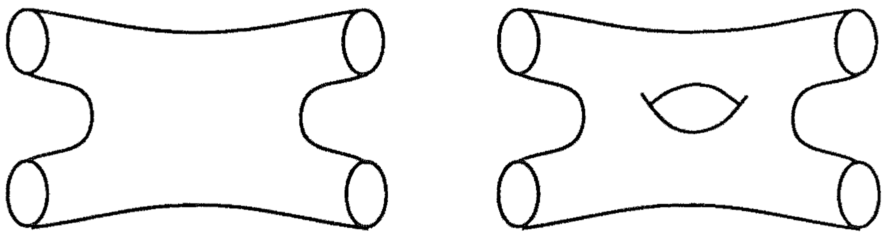


Figure 6.1 : Genus-0 (left) and genus-1 (right) string theory diagrams are illustrated. The genus of a manifold (such as the diagrams above) is the handles it possesses. This expansion in string theory strongly resembles the loop expansion in quantum field theory.

However, superstring theories can lead to potentials with the form of equation (6.2). In string theories, powers of $\exp(-\sqrt{2}\phi)$ appear as a loop counting parameter when the string effective action is expanded in terms of diagrams like those in Figure 6.1. Starting from the action [34,35,70,164],

$$\begin{aligned} S_{\text{string}} &= \int d^4x\sqrt{g}e^{\sqrt{2}\kappa\phi}\left\{\frac{R}{2\kappa^2}+\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi-m^2\psi^2\right. \\ &\qquad \left. -V^{(0)}-\sum_{G=1}\exp\left[\sqrt{2}\kappa\phi(G-1)\right]V^{(G)}\right\} \qquad (6.11) \end{aligned}$$

where ϕ is the string dilaton, G is the genus of terms in the string expansion, the V^G are constants and ψ is a (representative) massive scalar field. Campbell, Linde and Olive [36] transform this action to Einstein gravity with a potential that has the general form of equation (6.2), or

$$\begin{aligned} S_{\text{string}} &= \int d^4x\sqrt{\tilde{g}}\left\{\frac{\tilde{R}}{2\kappa^2}-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi\right. \\ &\qquad \left. -e^{-\sqrt{2}\kappa\phi}m^2\psi^2-\sum_{G=0}\exp\left[\sqrt{2}\kappa\phi(G+1)\right]V^{(G+1)}\right\}. \qquad (6.12) \end{aligned}$$

If the genus expansion is truncated after the first non-zero term, then this action has the general Jordan-Brans-Dicke form, but with no freedom in the choice of the ω parameter. The cosmological expansion for this action can be obtained but it is not inflationary.

After supersymmetry breaking, additional terms can appear in the effective potential. In particular, when supersymmetry breaking proceeds by gaugino condensation in the hidden sector of the theory, a "double exponential" potential,

$$V(\phi) = \exp\left(-ae^{-\sqrt{2}\kappa\phi}\right)\left[Ae^{\sqrt{2}\kappa\phi} + B + Ce^{-\sqrt{2}\kappa\phi}\right], \tag{6.13}$$

results after supersymmetry breaking [55,57]. Here a, A, B and C are constants determined by the theory. This potential is non-perturbative, so it will not reduce to a potential of the type equation (6.2) which is a finite sum of exponential terms.

Typically, the γ parameter in equation (6.2) is predicted[2] by string theory to take the value $\sqrt{2}$. In many cases, a smaller value of γ is required to accurately fit the available cosmological data, and this further reduces the latitude available when building inflationary models based on string theory. While the new exact solutions presented in this thesis have potentials of the type found in perturbative string theory, they are not obtained by solving the equations of motion derived from the string action. Rather, they are solutions to the equation of motion for a scalar field minimally coupled to Einstein gravity. Cosmological models based directly upon the superstring action have been examined by Casas, García-Bellido and Quirós [38,75], among others.

In 1993 Brustein and Steinhardt [31] argued that cosmological models based on the "standard formulation" of superstring theory face a number of difficulties if they are to be incorporated in a successful inflationary cosmology. In particular, they claim that the dilaton is likely to generally prevent inflation, or to be driven towards a minimum where the potential is either: (i) too steep, leading to an unacceptably large variation of the gravitational constant; or (ii) where the vacuum energy is negative, leading to an effective negative cosmological term which would be inconsistent with the observational evidence if sufficiently large. However, Brustein and Steinhardt hold the string moduli fields constant in their analysis and loosening this assumption may also weaken their conclusions. In particular, García-Bellido and Quirós [75] examine a model where the moduli do vary and arrive at a much more optimistic conclusion for string cosmology.

[2]The numerical value of γ is partly conventional, as changing the units ($m_p = 1$, for instance) or rescaling ϕ would both induce redefinitions of γ. However, the underlying physics is independent of the particular conventions that are chosen.

6.2 Exact Solutions with Exponential Potentials

Exact cosmological models based on exponential potentials are simplified by making the substitution,

$$x = \exp(-\xi\phi), \tag{6.14}$$

where ξ is a constant. In this section I present the revised equations of motion when the evolution is parametrised by x. The development is the same as that given in Section 4 and is spelled out in detail in my paper [64], so I just sketch it here.

The shift between x and t is obtained by rearranging equation (5.4),

$$\frac{dx}{dt} = -\frac{\xi^2 x^2}{3H}\frac{d\rho}{dx}. \tag{6.15}$$

Proceeding as we did in the previous chapter we find all the other model parameters in terms of $\rho(x)$:

$$\begin{aligned}
V(x) &= \rho + \frac{\chi\rho'}{3\chi'}. && (6.16)\\
a^2(x) &= \exp\left[-\frac{2}{\xi^2}\int_{x_0}^{x}\frac{\rho}{x^2\rho'}dx\right]\times \\
&\qquad \left\{a_0^2 + \frac{6k}{\xi^2}\int_{x_0}^{x}\exp\left[\frac{2}{\xi^2}\int_{x_0}^{x}\frac{\rho}{x^2\rho'}dx\right]\frac{dx}{x^2\rho'}\right\}, && (6.17)\\
t(x) &= -\frac{3}{\xi^2}\int_{x_0}^{x}\frac{H}{x^2\rho'}dx, && (6.18)\\
\epsilon(x) &= \xi^2 x^2\frac{\rho' H'}{3H^3}, && (6.19)\\
\Omega(x) &= \frac{\rho(x)}{3H^2(x)}, && (6.20)
\end{aligned}$$

where a_o and x_0 and the values of a and x when $t = 0$.

Setting $k = 0$, we find that for spatially flat models

$$\begin{aligned}
a(x) &= a_o \exp\left[-\frac{1}{2\xi^2}\int_{x_0}^{x}\frac{H}{x^2 H'}dx\right], && (6.21)\\
V(x) &= 3H^2 - 2\xi^2 x^2 H'^2, && (6.22)\\
t(x) &= -\frac{1}{2\xi^2}\int_{x_0}^{x}\frac{1}{x^2 H'}dx, && (6.23)\\
\epsilon(x) &= 2\left(\xi x\frac{H'}{H}\right)^2, && (6.24)
\end{aligned}$$

where we have expressed all quantities in terms of $H(x)$.

6.3 Existing Exact Solutions

Exact solutions for potentials containing a single exponential term with restrictions either on the initial conditions or the choice of model parameters have been discussed by many authors [17, 20, 32, 109, 125, 139, 150, 193]. The full solution was obtained by Salopek and Bond [162] and generalised by Lidsey to a two-field model [114]. The general solution for a single exponential potential in a Bianchi I universe is given by Aguirregabiria *et. al.* [6] and also generalised to inhomogeneous cases [7]. Solutions for potentials of the type $V(\phi) = A^2e^{2\lambda\phi} + B^2e^{-2\lambda\phi} - 2AB$ have been found by de Ritis *et. al.* [51–53].

6.3.1 Power-law Inflation

Power-law inflation [1, 17, 109, 125, 124, 150], where $a(t) \propto t^p$ and $p > 1$, is normally driven by a scalar field with $V(\phi) = \Lambda e^{-\lambda\phi}$.

In a spatially flat model, the form of $H(x)$ that leads to power-law inflation is extremely simple, namely,

$$H(x) = Ax. \tag{6.25}$$

Substituting equation (6.25) into equation (6.22) gives the potential,

$$V(x) = (3 - 2\xi^2)A^2x^2 \tag{6.26}$$

The parametric solution for $a(t)$ is

$$\begin{aligned} a(x) &= a_o \left(\frac{x_0}{x}\right)^{1/2\xi^2} && (6.27)\\ t(x) &= \frac{1}{2A\xi^2}\left(\frac{1}{x} - \frac{1}{x_0}\right). && (6.28) \end{aligned}$$

This solution is given by Muslimov [139] but we quote it for convenience as many of the models discussed later tend to this solution as the time, t, becomes large.

Inverting, to make t the independent variable,

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{1}{\xi}\log\left(1 + 2A\xi^2e^{-\xi\phi_0}t\right) && (6.29)\\ a(t) &= a_o\left(1 + 2A\xi^2e^{-\xi\phi_0}t\right)^{1/2\xi^2} && (6.30) \end{aligned}$$

For large t, $a(t) \propto t^p$ where $p = 1/2\xi^2$, which is power-law inflation if $\xi < \sqrt{1/2}$. While this is not the general solution, it is an attractor [81].

6.3.2 Özer and Taha's Result

Özer and Taha [147] obtain two, apparently distinct, solutions for the potential

$$V(\phi) = Ce^{-\phi} - De^{-2\phi}, \tag{6.31}$$

in a Robertson-Walker universe with positive spatial curvature. Their solutions constrain the initial conditions in terms of the coefficients C and D. Their first solution is

$$\begin{aligned} \phi(t) &= \phi_0 + \log\left[1 + \frac{t^2}{{a_o}^2}\right], & (6.32)\\ a(t) &= \sqrt{{a_o}^2 + t^2}, & (6.33) \end{aligned}$$

where the initial conditions are

$$\phi_0 = \log\frac{4D}{C}, \quad a_o = 4\frac{\sqrt{D}}{C}. \tag{6.34}$$

Their second solution, when expressed in terms of the time t, is

$$\begin{aligned} \phi(t) &= \phi_0 + \log\left[1 + \frac{t^2}{2{a_o}^2}\right], & (6.35)\\ a(t) &= a_o + \frac{t^2}{2a_o}, & (6.36) \end{aligned}$$

with initial conditions

$$\phi_0 = \log\frac{5D}{2C}, \quad a_o = \frac{5}{\sqrt{2}}\frac{\sqrt{D}}{C}. \tag{6.37}$$

While these solutions are distinct when parametrised by the time, the densities, $\rho(\phi)$, or equivalently, $\rho(x)$, for both solutions are identical up to terms that depend on the initial conditions, namely:

$$\rho(x) = \frac{6}{{a_o}^2 x_0}x - \frac{3}{{a_o}^2 {x_0}^2}x^2, \tag{6.38}$$

where the parameterisation $x = \exp(-\xi\phi)$, with $\xi = 1$ is being used. It is at this point we begin a search for new solutions in Robertson-Walker universes with non-zero spatial curvature.

Chapter 7

Exact Solutions for Spatially Curved Models

Most known exact inflationary solutions apply to spatially flat Robertson-Walker universes. However, in this chapter I present several new exact solutions for Robertson-Walker universes with $k = \pm 1$. I begin by extending Özer and Taha's result for positive spatial curvature and then give new solutions for the $k = -1$ case. These solutions are discussed in my paper, reference [64].

7.1 Robertson-Walker Models with Positive Spatial Curvature

Özer and Taha give two solutions, both of which have a density and potential of the type

$$\rho(x) = Ax - Bx^2, \tag{7.1}$$

$$V(x) = Cx - Dx^2, \tag{7.2}$$

where we have parametrised the exponential terms by x, equation (6.14), with $\xi = 1$. We search for potentials that have the form of equation (7.2) which give rise to a density of the form equation (7.1). In the process we will recover and extend the two solutions given by Özer and Taha.

We begin by substituting $\rho(x)$ into equations (6.17) and (6.16), giving

$$V(x) = \frac{5Ax_0(Aa_o{}^2x_0 - 6)x + 4(3A - 2ABa_o{}^2x_0{}^2 + 6Bx_0)x^2 + 2B(Ba_o{}^2x_0{}^2 - 3)x^3}{6x_0(Aa_o{}^2x_0 - 6) + 6(3 - Ba_o{}^2x_0{}^2)x}. \tag{7.3}$$

We can carry out this procedure for other choices of $\rho(x)$ and ξ. However, this case is special in that for other relatively simple forms of $\rho(x)$ in models with non-zero spatial curvature the corresponding potential is extremely complicated.

When $A = 6/a_o{}^2x_0$ or $B = 3/a_o{}^2x_0{}^2$ the potential, equation (7.3), simplifies to the form of equation (7.2). We consider these two special cases in turn.

7.1.1 Inflation with Non-Critical Density

First, set $A = 6/a_o{}^2 x_0$ in equation (7.1). Using equations (6.17) and (6.18) we find the following solution, parametrised by x:

$$\begin{aligned} V(x) &= \frac{2}{3}Ax - \frac{1}{3}Bx^2, & (7.4) \\ H^2(x) &= \frac{A}{6}x - Dx^2, & (7.5) \\ a(x) &= a_o\sqrt{\frac{x_0}{x}}, & (7.6) \\ t(x) &= a_o\left[\sqrt{\frac{x_0}{x} - Da_o{}^2x_0{}^2} - \sqrt{1 - Da_o{}^2x_0{}^2}\right], & (7.7) \end{aligned}$$

where we have identified $B = 3D$ from equations (7.2) and (7.4). Inverting gives $a(t)$ and $\phi(t)$,

$$\begin{aligned} \phi(t) &= \phi_0 + \log\left[Da_o{}^2x_0{}^2 + \left(\frac{t}{a_o} + \sqrt{1 - Da_o{}^2x_0{}^2}\right)^2\right], & (7.8) \\ a(t) &= a_o\sqrt{Da_o{}^2x_0{}^2 + \left(\frac{t}{a_o} + \sqrt{1 - Da_o{}^2x_0{}^2}\right)^2}, & (7.9) \end{aligned}$$

which we recognise as a generalisation of Solution I of Özer and Taha, equation (6.33). The distinction between this result and that of Özer and Taha is that only one of the parameters in this potential is fixed in terms of initial conditions.

The only constraint comes from the requirement that $H^2 \geq 0$ when $x = x_0$, so $D \leq 1/a_o{}^2x_0$. Setting $D = 1/a_o{}^2x_0$ recovers Özer and Taha's result. For this solution $\epsilon(x)$, equation (6.19), is

$$\epsilon(x) = \frac{1 - 2Da_o{}^2x_0x}{1 - Da_o{}^2x_0x} \qquad (7.10)$$

When $D > 0$, $a(t) \neq 0$ and $\epsilon < 1$ so this solution is always inflationary and non-singular, as Özer and Taha [147] point out for their special case. At large negative times, $a(t)$ is decreasing towards a finite minimum size, after which it will expand indefinitely.

While this is an inflationary solution, it is not spatially flat at late times, since

$$\Omega = \frac{2 - Da_o{}^2x_0x}{1 - Da_o{}^2x_0x} \qquad (7.11)$$

and $\Omega \to 2$ at small x (or large times). Further examples of inflationary models where $\Omega \neq 1$ can be found in Ellis *et. al.* [67] and Hübner and Ehlers [87].

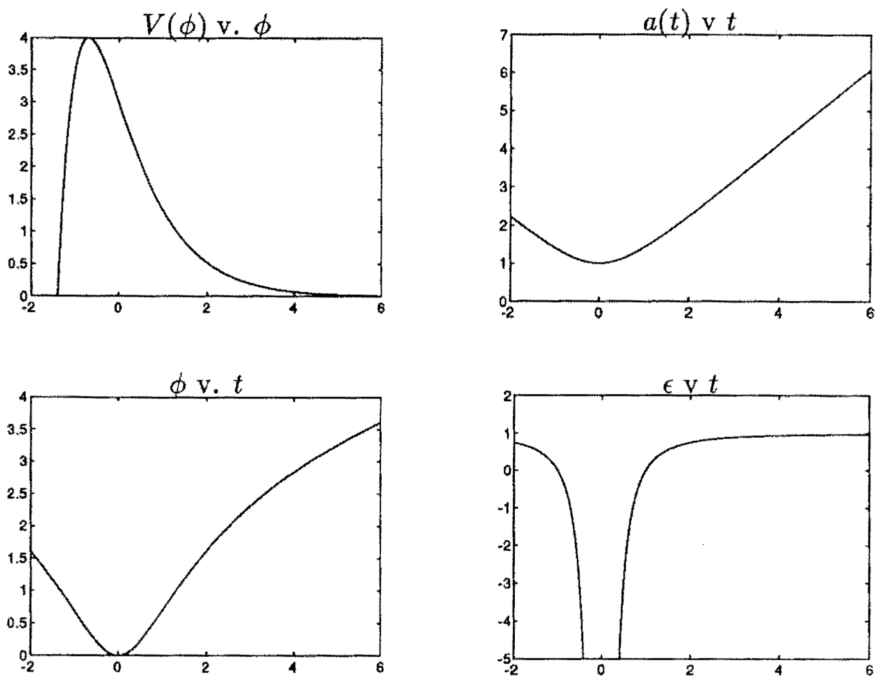


Figure 7.1 : The expansion for the solution equations (7.4) to (7.7) is shown. This model is non singular, as $a(t)$ is always non-zero. Physically, the universe is contracting at large, negative times and the field value is decreasing. Eventually the kinetic energy becomes zero and the field point is reflected off the potential, and the universe expands indefinitely. This model is always inflationary since $\epsilon < 1$ at all times.

The potential, equation (7.4), and a specific solution for a representative set of parameter values in plotted in Figure 7.1.

If $D < 0$ then $a(t) = 0$ at some finite, negative time and the resulting model universe does contain an initial singularity. In this case inflation never begins, since $\epsilon > 1$ at all times. At late times ϵ becomes arbitrarily close to unity, and the expansion approaches inflation "from beneath".

7.1.2 A Power-Law Solution

We obtain our second special case by putting $B = 3/a_o{}^2 x_0{}^2$, giving the solution

$$\begin{aligned} V(x) &= \frac{5}{6}Ax - \frac{2}{3}Bx^2, & (7.12) \\ H^2(x) &= \frac{2}{5}Cx - \frac{2}{3}Bx^2, & (7.13) \\ \phi(t) &= \phi_0 + \log\left[\frac{5}{Ca_o{}^2x_0} + \frac{Cx_0}{5}\left(\frac{t}{\sqrt{2}} + \frac{5}{C}\sqrt{\frac{C}{5x_0} - \frac{1}{a_o{}^2x_0{}^2}}\right)^2\right] & (7.14) \\ a(t) &= a_o\left[\frac{5}{Ca_o{}^2x_0} + \frac{Cx_0}{5}\left(\frac{t}{\sqrt{2}} + \frac{5}{C}\sqrt{\frac{C}{5x_0} - \frac{1}{a_o{}^2x_0{}^2}}\right)^2\right] & (7.15) \end{aligned}$$

where $C = 5A/6$.

To ensure that $H^2 \geq 0$, we need $C \geq 5/a_o{}^2x_0$. Setting $C = 5/a_o{}^2x_0$ recovers Solution II of Özer and Taha. This is also a non-singular model. The parameters ϵ and Ω are given by

$$\begin{aligned} \epsilon &= \frac{Ca_o{}^2x_0{}^2 - 10x}{2Ca_o{}^2x_0{}^2 - 10x} & (7.16) \\ \Omega &= \frac{2Ca_o{}^2x_0{}^2 - 5x}{2Ca_o{}^2x_0{}^2 - 10x} & (7.17) \end{aligned}$$

A representative solution is plotted in Figure 7.2. This solution is similar to the previous one as it is a non-singular inflationary model in a Robertson-Walker spacetime with positive curvature. However, the solution describes a universe where the density approaches the critical density, and $\Omega \to 1$. At large times, $a(t) \propto t^2$ and the solution resembles power-law inflation in flat spacetime. Since $\epsilon < 1$ at all times the solution is always inflationary.

7.1.3 A Single Exponential Potential

The solutions given thus far are obvious extensions of Özer and Taha's result. However, it is natural to ask if we can find exact solutions for models where the

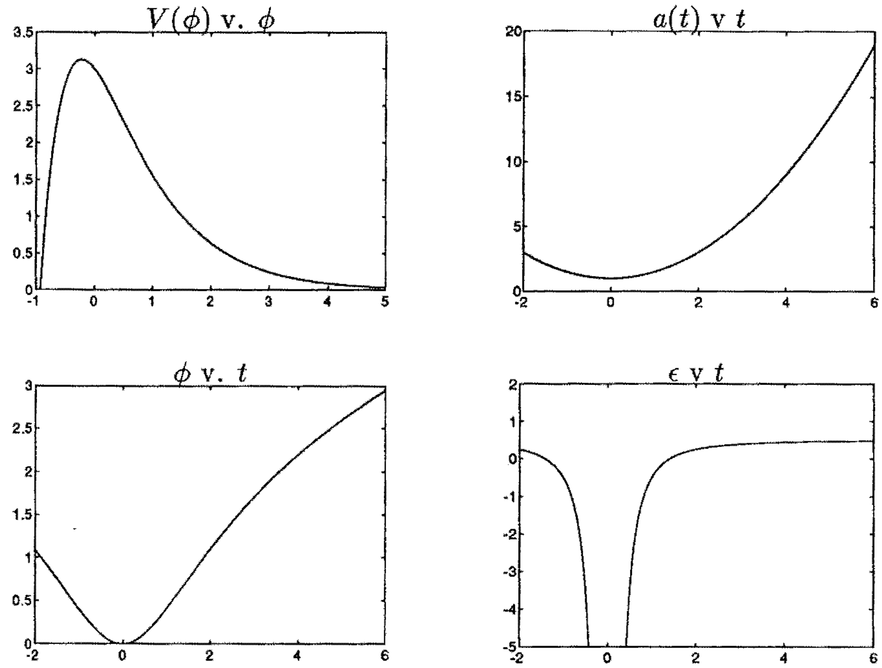


Figure 7.2 : The expansion for the solution equations (7.14) and (7.15) is shown. The features of this solution are similar to that equations (7.4) and (7.7) except that at large times, $\epsilon < 1$ and the universe approaches the critical density.

density takes the simple form,

$$\rho(x) = Ax \tag{7.18}$$

with an arbitrary value of ξ. Using the standard procedure we find that

$$V(x) = \frac{A{x_0}^{2/\xi^2}\left[x_0{a_o}^2A(8\xi^2 - \xi^4 - 12) + 36 - 6\xi^2\right]x^2 - 18x_0\xi^2x^{1+2/\xi^2}}{6{x_0}^{2/\xi^2}\left[{a_o}^2x_0A(\xi^2 - 2) + 6\right]x - 18\xi^2x_0x^{2/\xi^2}} \tag{7.19}$$

Choosing $A = 6/({a_o}^2x_0(2 - \xi^2))$ removes the dependence on x in the denominator and gives us the new solution,

$$V(x) \;=\; \frac{4}{{a_o}^2x_0(\xi^2 - 2)}x, \tag{7.20}$$

$$
\begin{aligned}
a(t) &= \frac{\xi}{\sqrt{2-\xi^2}}t + a_o, && (7.21)\\
\phi(t) &= \phi_0 + \frac{2}{\xi}\log\left[\frac{\xi}{a_o\sqrt{2-\xi^2}}t\right] && (7.22)\\
\Omega &= \frac{2}{\xi^2}, && (7.23)\\
\epsilon &= 1. && (7.24)
\end{aligned}
$$

A similar, restricted version of this solution (corresponding to $a_o = 0$) is given by Ellis and Madsen [68]. For these solutions to be real valued, $\xi < \sqrt{2}$. Similar solutions for a single exponential term and $\epsilon = 1$ can be found with $k = -1$ and for these, $\xi > \sqrt{2}$.

7.2 Robertson-Walker Models with Negative Spatial Curvature

The next possibility we consider is the existence of exact models in a Robertson-Walker universe with negative spatial curvature ($k = -1$). We look for potentials that give the density

$$\rho(x) = Ax + Bx^2, \quad \xi = 1. \tag{7.25}$$

From equations (6.17) and (6.16) and equation (7.25) we obtain

$$V(x) = \frac{5Ax_0(A{a_o}^2x_0+6)x + 4(3A+2AB{a_o}^2{x_0}^2+6Bx_0)x^2 + 2B(B{a_o}^2{x_0}^2-3)x^3}{6x_0(A{a_o}^2x_0+6)+6(3-B{a_o}^2{x_0}^2)x}. \tag{7.26}$$

Choosing either $A = -6/{a_o}^2x_0$ or $B = 3/{a_o}^2{x_0}^2$ simplifies the potential, equation (7.26), to the generic form given by equation (6.2). As is the case for Robertson-Walker universes with positive spatial curvature, other choices of $\rho(x)$ result in a very complicated potential.

7.2.1 Inflationary Solution with Negative Spatial Curvature

Putting $B = 3/{a_o}^2{x_0}^2$ gives the solution

$$
\begin{aligned}
V(x) &= \frac{5}{6}Ax + \frac{2}{3}Bx^2, && (7.27)\\
H^2(x) &= \frac{A}{3}x + \frac{2}{3}Bx^2, && (7.28)\\
\phi(t) &= \phi_0 + \log\left[\frac{Ax_0}{6}\left(\frac{t}{\sqrt{2}} + \frac{6}{A}\sqrt{\frac{1}{{a_o}^2{x_0}^2} + \frac{A}{6x_0}}\right)^2 - \frac{6}{A{a_o}^2x_0}\right], && (7.29)
\end{aligned}
$$

$$a(t) = a_o \left[\frac{Ax_0}{6} \left(\frac{t}{\sqrt{2}} + \frac{6}{A} \sqrt{\frac{1}{{a_o}^2 {x_0}^2} + \frac{A}{6x_0}} \right)^2 - \frac{6}{A{a_o}^2 x_0} \right]. \qquad (7.30)$$

This universe begins with an initial singularity ($a = 0$) and expands forever. A representative solution is displayed in Figure 7.3.

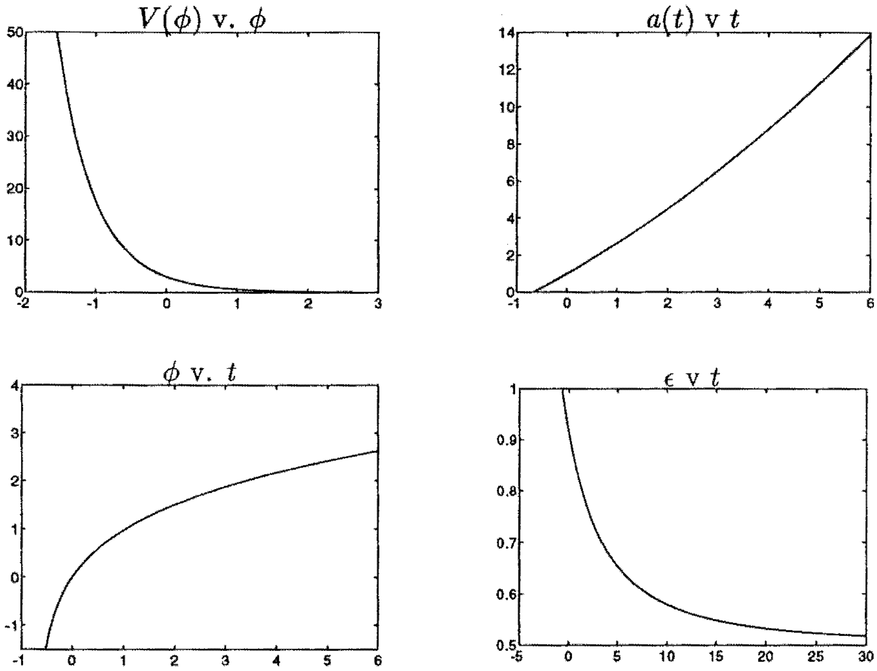


Figure 7.3 : The inflationary solution, equations (7.27) and (7.30), is plotted. In this case the model is singular, as $a(t)$ is zero at a finite time. Since $\epsilon < 1$ the model is always inflating and becomes spatially flat at large times.

For this solution, Ω and ϵ are given by

$$\epsilon = \frac{A{a_o}^2{x_0}^2 + 12x}{2A{a_o}^2{x_0}^2 + 12x}, \qquad (7.31)$$

$$\Omega = \frac{A{a_o}^2{x_0}^2 + 3x}{A{a_o}^2{x_0}^2 + 6x}. \qquad (7.32)$$

The limit of small x corresponds to large values of ϕ, and hence large times. Initially, $\Omega = 0.5$ and as $t \to \infty$, $\Omega \to 1$. Examining ϵ we see that the solution is always inflating, since $\epsilon < 1$ for all positive values of x. When the time is large, $a(t) \propto t^2$, and we again have power-law inflation in a spatially flat universe.

7.2.2 An Exact Closed Solution

Finally, when $A = -6/{a_o}^2 x_0$ the solution is markedly different from those we have examined so far. The scale factor $a(t)$ initially increases but the expansion comes to a halt at a finite time, after which the universe contracts back towards $a(t) = 0$. We have to patch two solutions together, one for the expanding phase, and one for the contracting phase, with the Hubble parameter, H, having the appropriate sign in equation (6.18). Defining $B = 3D$, we find

$$V(x) = \frac{2}{3}A + \frac{B}{3}x^2, \tag{7.33}$$

$$H^2(x) = \frac{A}{6}x + Dx^2, \tag{7.34}$$

$$a(x) = a_o\sqrt{\frac{x_0}{x}}, \tag{7.35}$$

$$t(x) = a_o\left[\sqrt{D{a_o}^2{x_0}^2 - 1} - \sqrt{D{a_o}^2{x_0}^2 - \frac{x_0}{x}}\right], \quad H > 0. \tag{7.36}$$

If $x < x_c$, where $x_c = 1/D{a_o}^2 x_0$, then $t(x)$ acquires a complex portion. However $H(x_c) = 0$ and $a(x_c)$ is the maximum value of $a(x)$. So when $t > t(x_c)$, $H < 0$ and the universe is contracting. In this case $V(x), H^2(x)$ and $a(x)$ are the same as those calculated above but

$$t(x) - t_1 = a_1\left[\sqrt{Da_1^2x_1^2 - \frac{x_1}{x}} - \sqrt{Da_1^2x_1^2 - 1}\right], \quad H < 0. \tag{7.37}$$

Choosing the initial conditions in equation (7.37) to be $a_1 = a(x_c)$, $t_1 = t(x_c)$ and $x_1 = x_c$ gives $t(x)$ during the contraction phase of the universe described by equations (7.33) to (7.35),

$$t(x) = a_o\left[\sqrt{D{a_o}^2{x_0}^2 - 1} + \sqrt{D{a_o}^2{x_0}^2 - \frac{x_0}{x}}\right], \quad H < 0. \tag{7.38}$$

These equations may now be inverted, giving

$$\phi(t) = \phi_0 + \log\left[D{a_o}^2{x_0}^2 - \left(\frac{t}{a_o} - \sqrt{D{a_o}^2{x_0}^2 - 1}\right)^2\right], \tag{7.39}$$

$$a(t) = a_o\sqrt{D{a_o}^2{x_0}^2 - \left(\frac{t}{a_o} - \sqrt{D{a_o}^2{x_0}^2 - 1}\right)^2}. \tag{7.40}$$

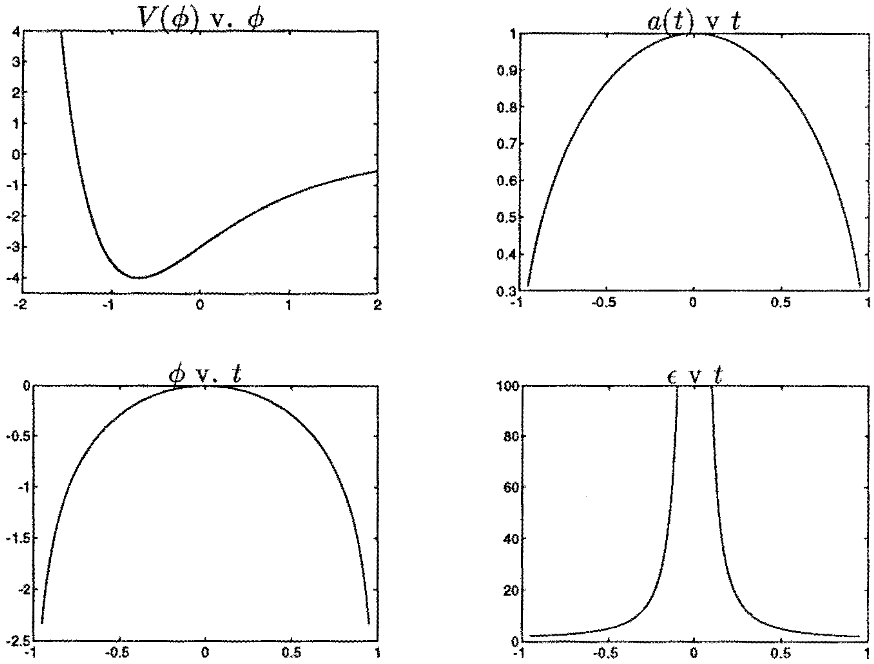


Figure 7.4 : This plot is of the solution equations (7.39) and (7.40). This solution does not inflate. Rather the universe expands to a maximum size and then contracts back to a singularity.

A representative solution is shown in Figure 7.4 and we can see that this is clearly not a viable inflationary model. Since the potential is not positive definite, the field can act as an effective negative cosmological constant and causes the universe to recontract. The field, ϕ, reaches a maximum value that coincides with the end of the expanding phase. At late times the energy density becomes indefinitely large, but once the collapse begins it will not be reversed. However, this solution is still an example of an exact scalar field cosmology in a curved spacetime and it serves as a strong reminder that not all exact solutions will be inflationary.

Chapter 8

Exact Solutions for Spatially Flat Models

In the previous chapter, we could only find a few exact solutions in Robertson-Walker universes with non-zero spatial curvature where the potential, $V(x)$, had a relatively simple form. For spatially flat models, however, life is considerably simpler. When $k = 0$ it is more convenient to specify $H(x)$ rather than $\rho(x)$. Any $H(x)$ that is a polynomial in x will, upon substitution into equation (6.22), give a potential that is automatically of the form equation (6.1) so we can find any number of exact spatially flat inflationary models.

We start by looking at new, exact solutions based on potentials that contain two or more exponential terms. These models are analagous to the new solutions I have found in curved Robertson-Walker spacetimes, but the simpler form of the equations when $k = 0$ allows us to examine more complicated models, which give rise to more interesting physics. The solutions reported in this chapter have been published in Easther [64].

8.1 Exponential Expansion from an Exponential Potential

We start with

$$H(x) = Ax - Bx^2, \quad B > 0 \tag{8.1}$$

which results in

$$V(x) = A^2(3 - 2\xi^2)x^2 + 2AB(4\xi^2 - 3)x^3 + B^2(3 - 8\xi^2)x^4. \tag{8.2}$$

From equations (6.21) to (6.23) it follows that

$$a(x) = a_o \left[\frac{{x_0}^2(A - 2Bx)}{x^2(A - 2Bx_0)}\right]^{1/4\xi^2} \tag{8.3}$$

$$t(x) = \frac{1}{A\xi^2}\left[\frac{1}{2}\left(\frac{1}{x} - \frac{1}{x_0}\right) + \frac{B}{A}\log\left(\frac{x_0(A - 2Bx)}{x(A - 2Bx_0)}\right)\right]. \tag{8.4}$$

This solution is plotted in Figure 8.1 for $\xi = 0.3$. It turns out that the properties of the solution depend much more strongly on ξ than upon the values of A and B. This is particularly true of the density perturbation spectrum it produces, which is discussed in Chapter 9.

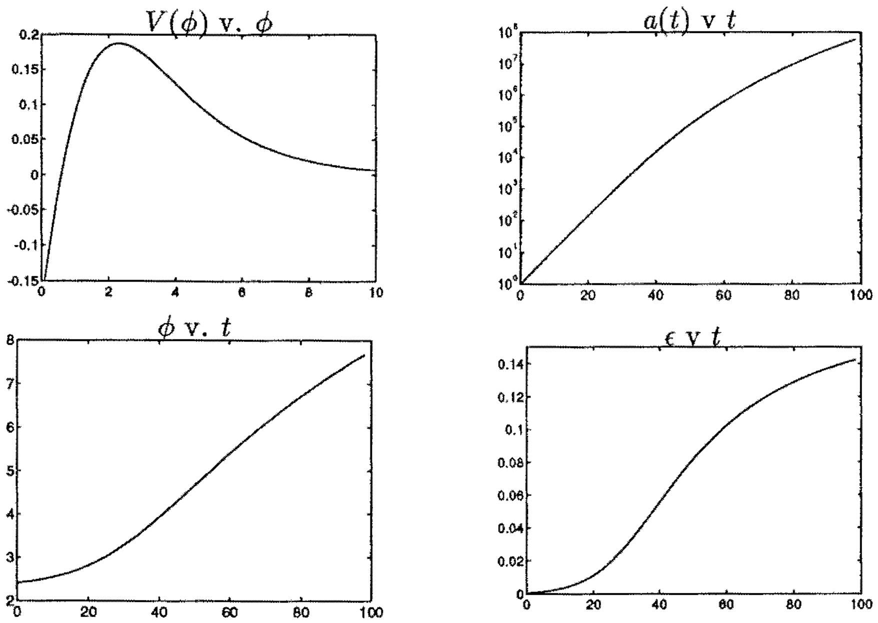


Figure 8.1 : The expansion for the potential equation (8.2) is illustrated, for $A = 1$, $B = 1$ and $\xi = 0.3$. The field rolls away from the maximum of the potential in the direction of increasing ϕ and the initially exponential expansion gives way to powerlaw inflation at later times. For $\xi = 0.3$, the potential is bounded below but the minimum is not shown on this plot.

This solution cannot be easily inverted, so we will work with the parametric form. We treat this case in some detail as much of the analysis here will apply with minor modifications to the other exact solutions discussed in this section.

The potential $V(x)$, equation (8.2), possesses turning points at

$$x_a = \frac{A}{2B}, \quad x_b = \frac{A}{B}\frac{3-2\xi^2}{3-8\xi^2}. \tag{8.5}$$

Since $x = \exp(-\xi\phi)$ is always positive, x_b only contributes an extremum to $V(\phi)$ if $x_b > 0$, so the number of turning points in the potential depends on ξ. For all values of ξ, x_a is a local maximum of $V(x)$. Thus the potential has several distinct forms, determined by the value of ξ. In particular, $V(\phi)$ is only bounded below when $\xi < \sqrt{3/8}$.

Like most exact scalar field cosmologies this solution applies to a restricted set of initial conditions, specifically $x \to x_a$ as $t \to -\infty$, so the field is always evolving away from the unstable equilibrium at x_a. Formally, $a(t) > 0$ for all negative times and this solution lacks an initial singularity, but small perturbations in ϕ or $\dot{\phi}$ when x is very close to x_a render the solution classically unstable in this region. However, the behaviour of the exact solution at later times is representative of a class of models that approximate these initial conditions and we discuss the solution in this context. If $x_0 > x_a$, then $t(x)$ increases with x. When $x \to \infty$ we find

$$t = \frac{1}{A\xi^2}\left[\log\left(\frac{2B}{2Bx_0 - A}\right) - \frac{1}{2x_0}\right], \quad a = 0. \tag{8.6}$$

and $a(x)$ becomes zero after a finite time. Thus this model universe collapses into a singularity for some choices of the initial conditions.

We now focus on the case where $x_0 < x_a$, and the expansion continues indefinitely. For this solution,

$$\epsilon(x) = 2\xi^2\left(\frac{A - 2Bx}{A - Bx}\right)^2. \tag{8.7}$$

If $\xi < \sqrt{1/2}$ then $\epsilon < 1$ for all $x < x_0$ and inflation continues forever. For larger values of ξ the inflationary phase will cease when $\epsilon = 1$, or

$$x = \frac{A}{B}\frac{2\xi - \sqrt{2}}{4\xi - \sqrt{2}}. \tag{8.8}$$

Furthermore, this solution has the capacity for both powerlaw and exponential inflation. During quasi-exponential expansion, H must be relatively constant on timescales of $1/H$, in which the universe expands by a single e-folding. This requirement is satisfied when $\epsilon \ll 1$. So if $x_0 \approx x_a$ (but not so close to the local maximum that the solution is unstable) then $a(t)$ is approximately exponential.

This solution thus exhibits the properties of the two major inflationary models at different stages in its evolution. If $\xi \lesssim 0.2$ and $x_0 \approx x_a$ then a large amount of exponential expansion is possible, with the number of e-foldings depending critically on ξ and the ratio x_0/x_a. Physically, since x_a is a local maximum, the potential is approximately flat when $x \approx x_a$. Thus both x and $V(x)$ are changing slowly, leading to an era of quasi-exponential expansion. As x moves further away from x_a the potential becomes steeper and the era of powerlaw expansion begins. Inspection of equation (8.4) shows that at late times (corresponding to small x) ϕ becomes arbitrarily large.

For most values of ξ, only two of the three coefficients in $V(x)$, equation (8.2), are independent. However, when $\xi = \sqrt{3/2}$, $\sqrt{3/4}$ or $\sqrt{3/8}$ one of the terms in $V(x)$ drops out, leaving only two nonzero terms which may be chosen arbitrarily. In particular, when $\xi = \sqrt{3/4}$, the potential is

$$V(\phi) = \frac{3}{2}A^2 \exp(-\sqrt{3}\phi) - 3B^2 \exp(-2\sqrt{3}\phi), \tag{8.9}$$

which are the first two terms in the expansion, equation (6.2), with $\gamma = \sqrt{3}$.

8.2 Modified Powerlaw Inflation

The next case we treat is superficially similar to the last, with

$$H(x) = Ax + Bx^2, \quad B > 0 \tag{8.10}$$

but the evolution we derive is markedly different. The corresponding solution is

$$\begin{aligned} V(x) &= A^2(3-2\xi^2)x^2 + 2AB(3-4\xi^2)x^3 + B^2(3-8\xi^2)x^4, & (8.11)\\ a(x) &= a_o\left[\frac{{x_0}^2(A+2Bx)}{x^2(A+2Bx_0)}\right]^{1/4\xi^2}, & (8.12)\\ t(x) &= \frac{1}{A\xi^2}\left[\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x_0}\right) - \frac{B}{A}\log\left(\frac{x_0(A+2Bx)}{x(A+2Bx_0)}\right)\right]. & (8.13) \end{aligned}$$

For this choice of $H(x)$, $V(x) \to \infty$ as $x \to \infty$ and $\xi \leq \sqrt{3/8}$. For other values of ξ, $V(x)$ is negative for large x. When $\sqrt{3/8} < \xi < \sqrt{3/2}$ the potential has a local maximum at

$$x_a = \frac{A}{B}\frac{2\xi^2-3}{3-8\xi^2}, \tag{8.14}$$

but for $\xi \geq \sqrt{3/2}$, $V(x) < 0$ for all allowable values of x. However, $a(t)$ is always increasing and there are no solutions for which the universe reaches a maximum size and then contracts back to a future singularity. This solution

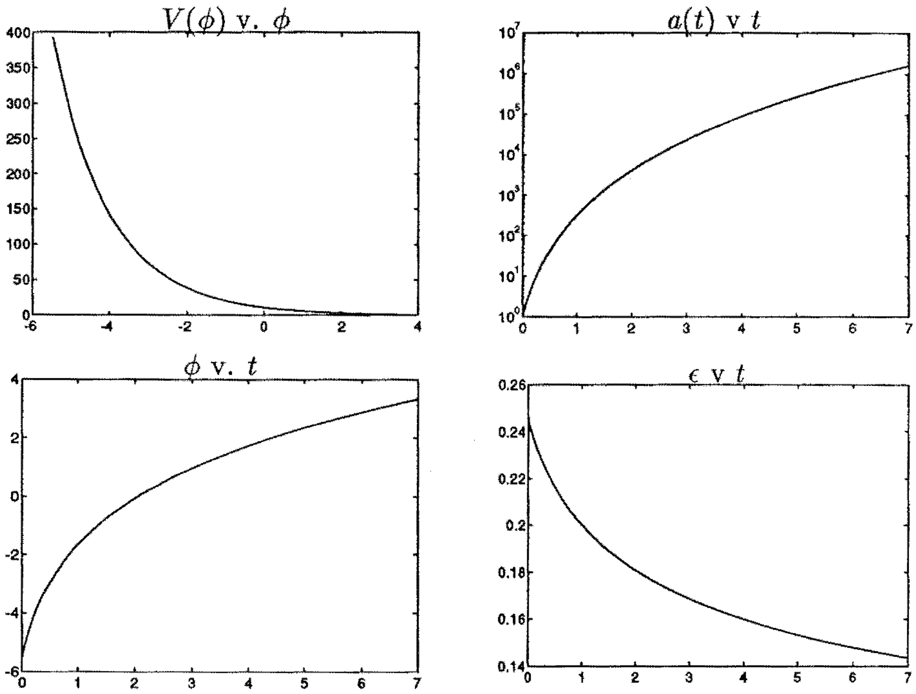


Figure 8.2 : The potential, equation (8.11), gives rise to motion that is always approximately powerlaw. The motion is plotted here for $\xi = 0.2$ and $A = B = 1$.

starts from a singularity though, since in the limit $x \to \infty$,

$$t = \frac{1}{A\xi^2}\left[\frac{B}{A}\log\left(\frac{A + 2Bx_0}{2Bx_0}\right) - \frac{1}{2x_0}\right], \quad a = 0. \tag{8.15}$$

In this instance, the logarithmic term in $a(x)$ never dominates and the expansion is always approximately powerlaw. A representative solution is plotted in Figure 8.2. Because this exact solution implicitly requires the value of $\dot{\phi}$ to be large when it is in the region containing the local maximum, the conditions that gave rise to a period of exponential expansion for the solution given by equations (8.3) and (8.4) are not satisfied in this case. Of course, the possibility of a period of exponential expansion in the general solution to the potential, equation (8.11), is not ruled out.

For this solution

$$\epsilon(x) = 2\xi^2 \left(\frac{A+2Bx}{A+Bx}\right)^2 . \tag{8.16}$$

If $\xi > \sqrt{1/2}$, $\epsilon > 1$ and there is no inflationary phase. Alternatively, if $\xi < \sqrt{1/8}$ the solution is inflationary at all times. For intermediate values of ξ, the inflationary era will be preceeded by a period of powerlaw expansion, $a \propto t^p$, but with $p < 1$. For all ξ, the field ϕ increases indefinitely in this solution.

Again, there are three special cases for which one of the terms in $V(x)$ is zero, and a potential with two terms and independent coefficients results.

8.3 Exact Solution for a Superstring Motivated Potential

Starting from $H = Ax - Bx^3$, we find the potential,

$$V(x) = A^2(3-2\xi^2)x^2 + 6AB(2\xi^2-1)x^4 + 3B^2(1-6\xi^2)x^6, \tag{8.17}$$

which has the form of equation (6.2) for all values of ξ. The corresponding exact solution is

$$\begin{aligned} a(x) &= a_o \left[\frac{{x_0}^3(A-3Bx^2)}{x^3(A-3B{x_0}^2)}\right]^{1/6\xi^2} && (8.18) \\ t(x) &= \frac{1}{2A\xi^2}\left\{\frac{1}{x} - \frac{1}{x_0} + \frac{C}{2}\log\left[\frac{(1-Cx)(1+Cx_0)}{(1-Cx_0)(1+Cx)}\right]\right\} && (8.19) \end{aligned}$$

where $C = \sqrt{3B/A}$.

For all ξ there is a local maximum at $x = 1/C$. For $x_0 > 1/C$, x will increase as the universe evolves, and it will eventually reach a maximum size and collapse towards a singularity. For $x_0 < 1/C$, the expansion will continue indefinitely. For this solution, ϵ has the form:

$$\epsilon = 2\xi^2\left(\frac{A-3Bx^2}{A-Bx^2}\right)^2 . \tag{8.20}$$

If $x_0 \approx 1/C$ then $a(t)$ will initially be approximately exponential, giving way to powerlaw behaviour at later times. Once again, inflation will cease and give way to ordinary expansion if $\xi > 1/\sqrt{2}$ and continue indefinitely for smaller values.

Again, there are three special values of ξ for which one of the terms drops out of the potential. In particular, when $\xi = 1/\sqrt{2}$, the potential is

$$V(\phi) = 2A^2\exp(-\sqrt{2}\phi) - 6B^2\exp(-3\sqrt{2}\phi). \tag{8.21}$$

Comparing with equation (6.2) we see that $\gamma = \sqrt{2}$, the value derived from superstring theory, although the term in $\exp(-2\sqrt{2}\phi)$ is missing.

Setting $H = Ax + Bx^3$ gives similar results to those found for $H = A + Bx^2$. The potential and the scale factor are found simply by changing the sign of B in equations (8.17) and (8.18). The time, $t(x)$ is

$$t(x) = \frac{1}{2A\xi^2}\left[\frac{1}{x} - \frac{1}{x_0} + C\left(\tan^{-1}(Cx) - \tan^{-1}(Cx_0)\right)\right]. \tag{8.22}$$

This model, which begins with a singularity, expands indefinitely and can have a mixture of non-inflationary and powerlaw expansion, depending on the value of ξ.

8.4 Other Spatially Flat Solutions

As I pointed out at the beginning of this chapter, an arbitrary number of spatially flat models with exact solutions can be generated. Since this is true, there is little virtue in merely cataloguing them. However, a recent paper by Barrow [19] examines potentials that are of the form

$$V(\phi) = V_0\phi^M \exp(A\phi^N). \tag{8.23}$$

I will briefly discuss the application of the methods used here to this case. Barrow is interested in models with

$$\phi = A(\log t - B)^n \tag{8.24}$$

and derives a variety of exact solutions after imposing this condition. However, his work does not exhaust the possibilities of these potentials. For instance, consider a Hubble parameter of the form

$$H(\phi) = A\phi^{M/2}\exp(\xi\phi). \tag{8.25}$$

Applying equations (5.14) and (5.15) we find the potential and scale factor

$$\begin{aligned} V(\phi) &= A^2\left[(3-2\xi^2)\phi^M - 2M\xi\phi^{M-1} - \frac{M^2}{2}\phi^{M-2}\right]\exp(2\xi\phi), \quad (8.26) \\ a(\phi) &= a_o\left(\frac{M+2\xi\phi}{M+2\xi\phi_0}\right)^{M/4\xi^2}\exp\left(\frac{\phi_0-\phi}{2\xi}\right). \quad (8.27) \end{aligned}$$

The time is given parametrically by equation (5.16). Leaving the integral unevaluated, we find

$$t(\phi) = -\frac{1}{A}\int_{\phi_0}^{\phi} \frac{y^{1-M/2}\exp\left(-\xi y\right)}{M+2\xi y}dy. \tag{8.28}$$

When evaluated with a specific value of M, $t(\phi)$ generally involves the exponential integral function, establishing that these solutions are different from Barrow's. Categorising these models by the connection between ϕ and t we see that we have produced a very wide range of exact models with a very small amount of effort.

8.5 Time Variation of Fundamental "Constants"

The new exact solutions presented in this thesis are all treated as scalar fields minimally coupled to Einstein gravity. However, if the models discussed here are taken to be the Einstein frame form of a string theory, Kaluza-Klein theory or Jordan-Brans-Dicke model extra physical considerations come into play. In particular, in these theories the inflaton is typically non-minimally coupled to the physical particles in the model. If the inflaton field continues to evolve at late times, then the fundamental gauge couplings will acquire a corresponding time dependence. Since there are strong experimental bounds on the variability of fundamental couplings (see Salam and Sezgin [161, pp 1252-1253] for a summary) these must be satisfied by any extension to Einstein gravity.

As a representative example, consider a model in which the inflaton, ϕ, is a fundamental dilaton. In this case the Einstein frame action, equation (6.12), has a non-minimal coupling between the ψ and ϕ fields. If ψ has a mass m, as in equation (6.11), then by equation (6.12) this non-minimal coupling will enter gravitational couplings as $Gme^{\kappa\phi/\sqrt{2}}$. The exponential can be absorbed into a redefined ϕ-dependent mass [36, 38], $\overline{m} = e^{\kappa\phi/\sqrt{2}}m$, or equivalently a ϕ-dependent gravitational constant, $\overline{G} = e^{\kappa\phi/\sqrt{2}}G$. Taking the latter alternative,

$$\left|\dot{\overline{G}}/\overline{G}\right| = \frac{\dot{\phi}}{\sqrt{2}} \tag{8.29}$$

where the dot denotes differentiation with respect to time. The best limit on the variation of the gravitational constant at the present epoch comes from the binary pulsar data [48].

The late time behaviour of some of the exact cosmologies discussed in this thesis is clearly at odds with the above bound. For example, the solution of

Özer and Taha equation (6.35) has $\dot{\phi}/\sqrt{2} \sim \sqrt{2}/t$ at late times. If the age of the universe is taken as $t \sim 1\text{–}2 \times 10^9$yr then this exceeds the binary pulsar bound by a couple of orders of magnitude.

For the solution to the potential, equation (8.2), the limit of large times corresponds to $x \to 0$. From the definition, $x = \exp(-\xi\phi)$, and equation (8.4) we find

$$\frac{d\phi}{dt} = 2\xi\left(Ax - Bx^2\right). \tag{8.30}$$

At small x, $t \sim 1/2\xi^2 Ax$, so the time dependence of the gravitational coupling is

$$\left|\dot{\overline{G}}/\overline{G}\right| = \frac{1}{\sqrt{2}\xi t}. \tag{8.31}$$

Like Özer and Taha's result, this exceeds the experimental bounds by two orders of magnitude since constraints from the allowable range of density perturbation spectra require that $\xi \lesssim 0.3$ (see Chapter 9). The other new solutions given in sections 8.2 and 8.3 have the same limit at late times, which follows from the potentials all having the same asymptotic form at small x.

The values of gauge coupling constants have an important effect in the nucleosynthesis era, which is probably the earliest epoch that one can discuss with real confidence; though much of theoretical cosmology today is concerned with earlier times. The exact details of how time-varying coupling "constants" affect nucleosynthesis are exceedingly involved, especially in models such as those arising from string theory, where many gauge couplings vary simultaneously. Producing a realistic exit to inflation while damping the variation of the inflaton sufficiently to satisfy nucleosynthesis bounds is a very challenging problem. These issues are discussed at length by Damour and Nordtvedt [49,50].

The bounds on the time dependence of coupling "constants" come from post inflationary physics (nucleosynthesis, solar system radar ranging and binary pular observations), however, and such 'late time' bounds do not exclude the models in this thesis as candidate descriptions of the inflationary era. If these models are to accurately depict the "real universe" after inflation then they must be modified to include fields that are not fundamental scalars (such as electrons and quarks) so that inflation ceases and the universe reheats, as well as damping the motion of ϕ at late times. Constraints on the inflationary epoch itself come from the primordial perturbation spectrum and the requirement that enough inflation occurs to solve the standard cosmological problems. These criteria are satisfied by some of the models developed here and are discussed at greater length in Chapter 9.

Chapter 9

Density Perturbations

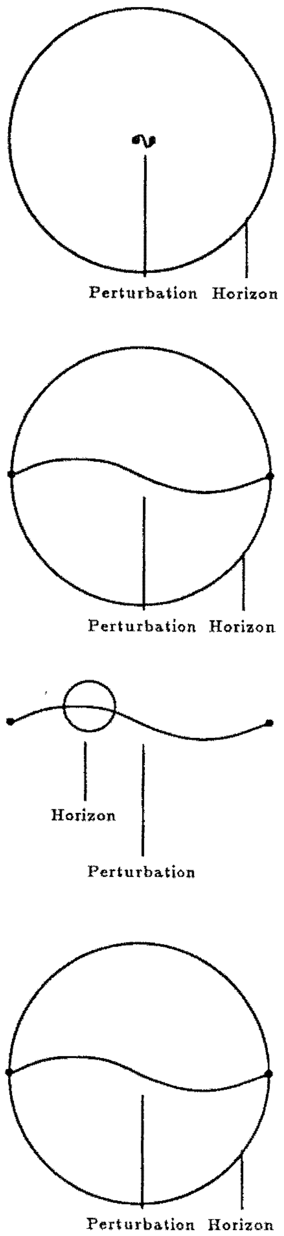
9.1 Preliminaries

In this chapter I derive and discuss the density perturbations predicted by the exact models introduced in Chapter 8. I begin by summarising the conventional approach to density perturbations and the connection between fluctuations in the inflaton field and presently observable density inhomogeneities. In the second section I introduce the mathematical machinery needed to calculate perturbation spectra in slow rolling inflation. I then apply this formalism to the models discussed in this thesis. I use a powerful approach, developed by Stewart and Lyth [175], which simplifies the calculation of the perturbations for any given model, although the overall problem remains complex, both from the computational and theoretical standpoints.

9.1.1 A Rough Outline

The inflaton field, ϕ, makes small quantum fluctuations about its semi-classical value. These fluctuations occur on a microscopic scale and in a universe undergoing "ordinary" or non-inflationary growth, they would never attain a macroscopic size. However, during inflation the comoving horizon length is decreasing and points initially close together will eventually be separated by distances many times greater than the horizon length, if inflation continues long enough. Thus local quantum fluctuations are stretched to the point where they seed structure formation in the visible universe. The process is illustrated schematically in Figure 9.1.

Inflation produces both scalar and tensor perturbations which are distinguished by the transformation properties of their respective contributions to the metric [14]. Density inhomogeneities are produced by the scalar perturba-



1. The perturbation is formed as a quantum mechanical fluctuation during inflation, and is initially much smaller than the horizon.

2. During inflation, the comoving horizon length decreases. Eventually, the perturbation length (which has a constant comoving size) is equal to the the horizon length. The perturbation then "leaves the horizon" and ceases to evolve, so its amplitude is fixed.

3. By the end of inflation the perturbation is many times larger than the horizon and has no influence on the evolution inside the horizon.

4. After inflation ends, the comoving size of the horizon increases. The horizon expands until it encompasses the perturbation, and it "re-enters the horizon" with the same amplitude it had when it left. The perturbation is now able to "seed" structure formation within the visible universe.

Figure 9.1 : A (cosmic) day in the life of a density perturbation

tions. By contrast, the temperature anisotropy in the microwave background observed by the COBE satellite have contributions from both scalar and tensor modes, particularly at larger angular scales.

9.1.2 The Mathematical Treatment of Perturbations

There are two parts to a mathematical treatment of fluctuations - their quantum mechanical production, and the subsequent evolution of the propagating fluctuation in an expanding Robertson-Walker universe. Since the fluctuations are necessarily small, we can view the problem as a perturbation of the original solution.

In my opinion, the most accessible discussions of the production of perturbations during inflation are those of Kolb and Turner [102], the recent review of Liddle and Lyth [111] and Turner's lectures [179] and paper [180]. An extremely thorough account is given by Mukhanov, Feldman and Brandenberger [137] and Brandenberger's lectures [29] are also useful.

Consider a Robertson-Walker universe (which includes the standard model of the Big Bang, and all the inflationary models discussed here) with a density, $\rho(t)$, pressure, $p(t)$ and Hubble parameter, $H(t)$. In the presence of a small perturbation these quantities depend on spatial position as well as time, so

$$\begin{aligned} \rho(x,t) &= \bar{\rho}(t) + \delta\rho(x,t) && (9.1)\\ p(x,t) &= \bar{p}(t) + \delta p(x,t) && (9.2)\\ H(x,t) &= \bar{H}(t) + \delta H(x,t) && (9.3) \end{aligned}$$

where the barred quantities are the average values. We restrict attention to perturbations in a flat Robertson-Walker universe, which is the case that is most applicable to inflation. When the perturbations are small, we can expand them in a Fourier series and analyse the components individually, so

$$\begin{aligned} \delta\rho(t,x) &= \sum_{\mathbf{k}} \delta\rho_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} && (9.4)\\ \Rightarrow \delta\rho(t,x) &= \frac{1}{(2\pi)^3}\int d^3k\,\delta\rho_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{x}} && (9.5) \end{aligned}$$

where k is the comoving wavenumber of the perturbation and volume factors have been suppressed. The corresponding wavelength, λ, is sometimes called a scale. Obviously, similar definitions can be made for δp, δH and any other perturbed quantity. As long the perturbation is small, we can ignore the back-reaction of the perturbation on the metric, and the Fourier components evolve independently.

It is useful to define the dimensionless quantity, δ, the density contrast:

$$\delta \equiv \frac{\delta\rho}{\rho} = \frac{1}{(2\pi)^3}\int d^3k\delta_{\mathbf{k}}e^{-i\mathbf{k}.\mathbf{x}}. \tag{9.6}$$

While the full details are given in the literature (in particular, this discussion follows that of Liddle and Lyth [111]) it is useful to summarise them here. When the pressure gradient is negligible compared to the density, the $\delta_{\mathbf{k}}$ obey the first order differential equation,

$$\frac{2H^{-1}}{5+3\omega}\frac{d}{dt}\left[\left(\frac{aH}{k}\right)^2\delta_{\mathbf{k}}\right] + \left(\frac{aH}{k}\right)^2\delta_{\mathbf{k}} = \frac{2+2\omega}{5+3\omega}\mathcal{R}_{\mathbf{k}}, \tag{9.7}$$

where $\omega = p/\rho$. The quantity $\mathcal{R}_{\mathbf{k}}$ on the right hand side is time independent, and is a measure of the curvature perturbation on each spatial hypersurface.

During an era when ω is constant, equation (9.7) has the solution,

$$\left(\frac{aH}{k}\right)^2\delta_{\mathbf{k}} = \frac{2+2\omega}{5+3\omega}\mathcal{R}_{\mathbf{k}}. \tag{9.8}$$

The full solution contains a second term that decays with time, but equation (9.8) adequately describes the perturbation for a constant ω and negligible pressure gradient. In particular, when the universe is matter dominated, $\omega = 0$ and

$$\left(\frac{aH}{k}\right)^2\delta_{\mathbf{k}} = \frac{2}{5}\mathcal{R}_{\mathbf{k}}. \tag{9.9}$$

The radiation dominated case has $\omega = 1/3$, so

$$\left(\frac{aH}{k}\right)^2\delta_{\mathbf{k}} = \frac{4}{9}\mathcal{R}_{\mathbf{k}}. \tag{9.10}$$

If ω is not constant, then the strict proportionality between $\mathcal{R}$ and δ is lost. However, we can express the "final" value of $\mathcal{R}$ in terms of the initial value, when the scale of comoving length k re-enters the horizon, in terms of a transfer function, $T(k)$, where

$$\mathcal{R}_{\mathbf{k}}(\text{final}) = T(k)\mathcal{R}_{\mathbf{k}}(\text{initial}) \tag{9.11}$$

or, equivalently,

$$\frac{1}{a}\left.\delta_{\mathbf{k}}\right|_{\text{final}} = AT(k)\left.\delta_{\mathbf{k}}\right|_{\text{initial}}. \tag{9.12}$$

The constant A is chosen to ensure that $T(k) = 1$ at large scales. The form of the transfer function is determined by the matter content of the universe. If, as seems likely, the universe is dominated by some type of dark matter then $T(k)$ will depend on the properties of the dark matter. Thus, studies of structure formation refer to CDM (Cold Dark Matter) or HDM (Hot Dark Matter). What we obtain from our calculations is the amplitude of a perturbation at horizon crossing. Since each scale crosses the horizon at a different time (with smaller scales re-entering the horizon at earlier times) we use the transfer function to describe the evolution at each scale after horizon crossing. The combination of the primordial spectrum produced by inflation and the transfer function gives predictions that can be compared with observations.

The last quantity we will need to introduce is the perturbation spectrum, P_δ. After matter domination,

$$P_\delta = \left(\frac{k}{aH}\right)^4 T(k)^2 \delta_H^2(k). \tag{9.13}$$

The primordial spectrum produced by inflation is δ_H^2, which may be expressed in terms of the curvature perturbation,

$$\delta_H^2(k) = \frac{4}{25} P_{\mathcal{R}}. \tag{9.14}$$

During inflation, δ_H^2 can be accurately represented as a power-law:

$$\delta_H^2 \propto k^{n-1}. \tag{9.15}$$

The quantity n is called the spectral index. The original model of Harrison and Zel'dovich has $n = 1$, so the magnitude of the initial density perturbation spectrum is "scale free" since it does not depend on k. Inflation approximates this requirement, but does not necessarily produce it exactly. Models where the index of the primordial perturbation spectrum differs significantly from 1 are referred to as being "tilted".

Inflation also produces gravitational waves, which give rise to tensor perturbations. A spectrum and its corresponding index can be defined for these perturbations, as well as isocurvature perturbations which relate to local changes in the equation of state.

9.2 The Inflationary Perturbation

Typically, the primordial spectrum of scalar (or density) fluctuations [111] produced during inflation is

$$P_{\mathcal{R}}^{1/2}(k) = \left.\frac{H^2}{2\pi\dot{\phi}}\right|_{aH=k} \tag{9.16}$$

while the tensor (or gravity wave) fluctuations have

$$P_{\psi}^{1/2}(k) = \left.\frac{H}{2\pi}\right|_{aH=k}. \tag{9.17}$$

When $k = aH$ the scale with comoving wavenumber k is equal in size to the horizon length of the inflationary universe. Since the perturbation does not evolve significantly after it leaves the horizon, evaluating at this point gives the primordial perturbation spectrum that seeds structure formation in the present universe. This is a first order result, and its utility is determined by how well the slow-rolling approximation is satisfied. The spectral indices, $n_{\mathcal{R}}$ and n_{ψ}, are defined to be

$$n_{\mathcal{R},\psi}(k) \equiv 1 + \frac{d\ln P_{\mathcal{R},\psi}}{dk}. \tag{9.18}$$

We will describe the perturbation spectrum in terms of the parameters[1], ϵ and η,

$$\epsilon = -\frac{\dot{H}}{H^2} \quad , \quad \eta = \frac{\ddot{\phi}}{H\dot{\phi}}. \tag{9.19}$$

During exponential inflation, $\epsilon \approx 0$ and η is usually assumed to be small as well. However the solutions we will be looking at here present a counterexample to this widespread belief. During exponential expansion, H is changing slowly and ϵ is necessarily small. However η is a measure of the ratio between $\dot{\phi}$ and $\ddot{\phi}$. Even if the velocity is small, so that the density is dominated by the potential, the acceleration can still be significant. This situation is realised by the models in the previous chapter where the field rolls away from a local maximum in the potential. The velocity is initially small, but as the field moves off the maximum into the steeper regions of the potential it rapidly picks up speed, and η is considerably larger than ϵ.

To first order, the spectral indices are

$$n_{\mathcal{R},1}(k) = 1 - 4\epsilon - 2\eta \tag{9.20}$$

$$n_{\psi,1}(k) = 1 - 2\epsilon \tag{9.21}$$

[1]This is the same ϵ that was introduced in Chapter 4.

Stewart and Lyth [175] give the spectral indices for the inflationary perturbations to second order in ϵ and η, and since these parameters are not always negligible for the exact solutions given in Chapter 8, we will need their results here. They are

$$\begin{aligned} n_{\mathcal{R},2}(k) &= 1-4\epsilon-2\eta-2(1+c)\epsilon^2+\frac{1}{2}(3-5c)\epsilon\eta \\ &\quad -\frac{1}{2}(3-c)\eta^2+\frac{1}{2}(3-c)\frac{\dddot{\phi}}{H\ddot{\phi}}\eta \qquad (9.22) \\ n_{\psi,2}(k) &= 1-2\epsilon-(3+c)\epsilon^2-(1+c)\epsilon\eta \qquad (9.23) \end{aligned}$$

where $c = 4(\ln(2)+\gamma)-5 \approx 0.08145$, and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The corresponding spectra are, to second order:

$$P_{\mathcal{R}}^{1/2} = \left[1+(2-\log 2-\gamma)(2\epsilon+\eta)-\epsilon\right] \left.\frac{H^2}{2\pi\dot{\phi}}\right|_{k=aH}, \qquad (9.24)$$

$$P_{\psi}^{1/2} = \left[1-(\log 2+\gamma-1)\epsilon\right] \left.\frac{H}{2\pi}\right|_{k=aH}. \qquad (9.25)$$

The perturbation spectrum for powerlaw inflation is known exactly [175]. This result is of interest to us, as the solutions discussed approach the powerlaw case at large times, and so the perturbation spectrum will similarly approach the form obtained for powerlaw inflation. I will use this fact to check the consistency of the approximations used here. Powerlaw inflation is obtained from a potential with a single exponential term,

$$\begin{aligned} V(x) &= \frac{m_p^4 A^2}{64\pi^2}x^2 \\ \Rightarrow V(\phi) &= \frac{m_p^4 A^2}{64\pi^2}\exp\left(-2\xi\frac{\sqrt{8\pi}}{m_p}\phi\right), \qquad (9.26) \end{aligned}$$

and the solution is given by equations (6.29) and (6.30). Taking $a(t) \propto t^p$, the perturbation spectrum and corresponding indices are

$$P_{\mathcal{R}}^{1/2}(k) \propto \left(\frac{1}{k}\right)^{1/p-1}, \qquad (9.27)$$

$$n_{\mathcal{R}} = \frac{p-3}{p-1} = \frac{1-6\xi^2}{1-2\xi^2}. \qquad (9.28)$$

The tensor perturbation spectrum is proportional to the scalar one, so $n_\psi = n_{\mathcal{R}}$. Applying the COBE constraints (which require $n \approx 1$) requires powerlaw inflation to have a large value of p, or conversely a small value of ξ. This in turn places places a physical constraint on inflationary models with exponential terms, since it rules out potentials of the form equation (6.26) unless $\xi \lesssim 0.1$ [110].

We will restrict attention to the potential, equation (8.2), whose solution is given by equations (8.3) and (8.4) as the working is similar for the other cases presented in the previous chapter. The functional form of ϵ and η can be extracted from the exact solution in terms of x,

$$\eta = 2\xi^2 \frac{4Bx - A}{A - Bx}, \tag{9.29}$$

$$\epsilon = 2\xi^2 \left(\frac{A - 2Bx}{A - Bx}\right)^2. \tag{9.30}$$

The parameters ϵ and η are plotted against the scale factor a in Figure 9.2 for representative parameter values. The initial numerical value of a is taken to be unity, but this is simply for convenience and has no physical consequences.

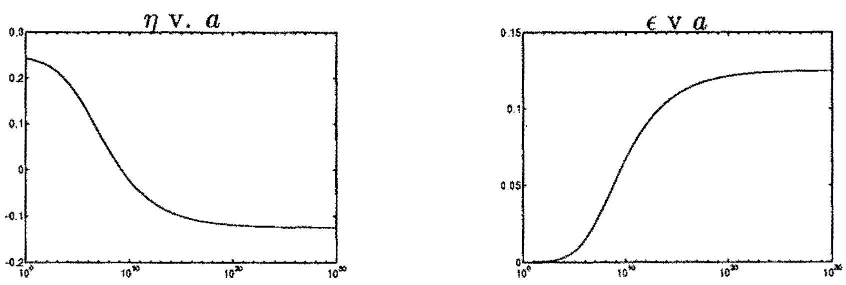


Figure 9.2 : The parameters ϵ and η for the solution to the potential, equation (8.2), are plotted for $A = B = 1$ and $\xi = 0.25$.

We can now calculate the spectral indices for the perturbations that these solutions produce. There are pronounced differences between the first and second order results, and we need to check whether the second order results are themselves unaffected by higher order corrections. The first and second order results for $n_{\mathcal{R}}$ are shown in Figure 9.3.

Since the solution asymptotically approaches that for a single powerlaw potential, the spectral indices are equal to the powerlaw case, given by equa-

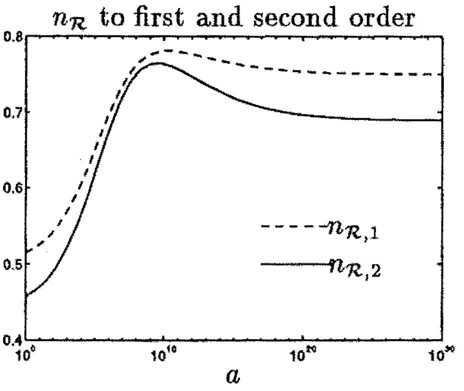


Figure 9.3 : The spectral index for the scalar perturbations arising from inflation driven by the potential, equation (8.2), is plotted for $\xi = 0.25$. The first and second order results have a similar overall shape, but the drop in $n_{\mathcal{R}}$ at large values of a is much less pronounced at first order.

tion (9.28), when $y = 1$. For $\xi \lesssim 0.3$, the second order results are within a few percent of their exact asymptotic values. At larger values of ξ even the second order result is unreliable - however this will not concern us here, as the values of $n_{\mathcal{R}}$ are too low for these models to be of practical interest. While the spectral index varies considerably over many e-foldings in a, all the perturbations that are accessible through astrophysical observations are formed over a range of a of a few thousand. On these scales, $n_{\mathcal{R}}$ is effectively constant.

9.3 Modelling the Real World

In this section I apply the available observational constraints to the exact models found in Chapter 8, and show that they can form viable models of the inflationary universe.

The most important constraint on inflationary models is the anisotropy in the microwave background, found by COBE [169,192]. In addition, we need to ensure that the energy density at the end of inflation is sufficiently large to reheat the universe to a high enough temperature for the baryon asymmetry to form.

What is lacking from the exact models found in the previous chapter is a natural end to inflation. While inflation does cease after a finite time in some of the scalar field cosmologies investigated there, the end of inflation is marked by the energy-momentum density becoming dominated by "ordinary" particles, the leptons and bosons that form the universe we currently observe. Since the exact models I have constructed contain no matter apart from the scalar field, the end of inflation, in this sense at least, is not accurately modelled. What we can do though, is to assume that the solutions given here apply only up to a time

t (or alternatively some value of the parameter, x). I will not propose a detailed mechanism for the end of inflation, other than to note that the approach taken here is that conventionally taken with "ordinary" powerlaw inflation.

9.3.1 The Data

From COBE and studies of galaxy distribution, we have the result that $n_{\mathcal{R}} \gtrsim 0.7$ [110]. If tensor perturbations make a significant contribution, the lower bound on the index is increased. On cosmological scales it is reasonable to assume that $\Omega = 1$. There is still no consensus of the present value of the Hubble parameter, although it is commonly parametrised[2] as

$$H_{\text{today}} = h100\text{kms}^{-1}\text{Mpc}^{-1}. \tag{9.31}$$

In natural units with $c = 1$, velocities are dimensionless and the numerical value of the Hubble parameter is reduced by 3×10^5, so

$$H_{\text{today}} = \frac{h}{3000}\text{Mpc}^{-1}. \tag{9.32}$$

The estimate of h that is most widely employed is $h = 0.5$ and that is the one I will use here. With $\Omega = 1$ and $h = 0.5$ the distance to the horizon is 12,000Mpc although neither the value of Ω nor h has a direct impact on the conclusions reached here.

If we assume a universe that is dominated by cold dark matter, then the transfer function has the analytic approximation [15]

$$T(k) = \frac{\log(1 + 2.34q)}{2.34q[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{1/4}} \tag{9.33}$$

where $q = k/(\Omega h^2) = 4k$. With pure cold dark matter and $n = 1$, the predicted density perturbation spectrum is larger than that which is observed. This difficulty can be resolved by tilting the perturbation spectrum (adopting a spectral index of less than unity) or mixing in some hot dark matter, which tends to wash out smaller scale perturbations. The composition of dark matter is beyond the scope of this thesis, and I will focus on models where the scalar perturbations have significant tilt. As summarised by Turner [179], requiring a tilted perturbation spectrum picks out natural inflation and powerlaw inflation produced by an exponential potential. These two models are distinguished

[2] A parsec (pc) is equal to 3.26 lightyears.

from one another by the tensor modes - natural inflation has only a small tensor contribution, but in powerlaw inflation it is significant. We will see that the solutions found in Chapter 8 that combine periods of exponential and powerlaw growth can also produce a significant degree of tilt, but the proportion of tensor modes varies over a wide range and depends on the parameter values.

9.3.2 The Matching Equation

In this section we will reinstate the factors of $8\pi G$ in the solutions, as it is convenient to work with temperatures and field quantities in GeV. It is straightforward to do this, and the results for the potential, equation (8.2), are

$$\begin{aligned}
V(x) &= \frac{m_p^4}{64\pi^2}\left[A^2(3-2\xi^2)x^2 + 2AB(4\xi^2-3)x^3 + B^2(3-8\xi^2)x^4\right] && (9.34)\\
a(x) &= a_o\left[\frac{{x_0}^2(A-2Bx)}{x^2(A-2Bx_0)}\right]^{1/4\xi^2}, && (9.35)\\
t(x) &= \frac{\sqrt{2\pi}}{m_p A\xi^2}\left[\frac{1}{x}-\frac{1}{x_0}+\frac{2B}{A}\log\left(\frac{x_0(A-2Bx)}{x(A-2Bx_0)}\right)\right], && (9.36)\\
H(x) &= \frac{m_p}{\sqrt{8\pi}}(Ax-Bx^2) && (9.37)
\end{aligned}$$

and

$$x = \exp\left(\frac{-\xi\sqrt{8\pi}}{m_p}\phi\right). \qquad (9.38)$$

Substituting $m_p = \sqrt{8\pi}$ will recover the earlier version of this solution.

The perturbations that we see on the sky today are generated roughly fifty e-foldings before the end of inflation. What we want to know is the "matching equation" that relates the size of perturbations at the present epoch to their size when they were produced during inflation [179]. A perturbation with physical wavelength λ_{phys} at the present epoch crosses the horizon at a time t_1, with physical wavelength λ_1^{-1} during the inflationary epoch. By definition,

$$\lambda_{\text{phys}} = \frac{a_{\text{today}}}{a_1}\lambda_1 = \frac{a_{\text{today}}}{a_1}\frac{1}{H_1} \qquad (9.39)$$

After the scale H_1 leaves the horizon, there are three distinct phases. Firstly there is the remainder of the inflationary growth, where $a(t)$ is determined by the particular inflationary model being considered. After inflation ceases the universe reheats. In some models reheating does not happen immediately, but the universe is dominated by coherent oscillations of the scalar field and

$a(t) \propto t^{2/3}$. The universe reheats to a temperature, $T_{\rm rh}$, and then cools to the present temperature of the microwave background, $2.73K$.

Define the end of inflation as occurring when $x = x_0$ and $a = a_o$. This parameter choice implies that the time is negative during inflation, but this is an artifact of the definition and has no physical significance. It is usual to normalise the scale factor so that $a_{\rm today} = 1$ and the comoving and physical sizes of features in the present universe coincide. This choice of a_o does not preclude us from making this normalisation. When $x = x_0$, $\rho = \mathcal{M}^4$. From equation (4.7) with $k = 0$ and $G = 1/m_p^2$ and equation (9.37) we find $\mathcal{M}$ in terms of x_0,

$$\mathcal{M} = \frac{3^{1/4} m_p}{\sqrt{8\pi}} \sqrt{A x_0 - B {x_0}^2}. \tag{9.40}$$

We now expand equation (9.39) to explicitly incorporate the three contributions to expansion since the end of inflation,

$$\lambda_{\rm phys} = \frac{a_{\rm today}}{a_{\rm rh}} \frac{a_{\rm rh}}{a_o} \frac{a_o}{a_1} \frac{1}{H_1}. \tag{9.41}$$

From equation (4.16) we have

$$\frac{a_{\rm today}}{a_{\rm rh}} = \frac{T_{\rm rh}}{T_{\rm today}} \tag{9.42}$$

where $T_{\rm today}$ is the present microwave background temperature. During the scalar field oscillations, $\rho \propto 1/a^3$, and to a sufficient degree of accuracy,

$$\frac{a_{\rm rh}}{a_o} = \left(\frac{\mathcal{M}}{T_{\rm rh}}\right)^{4/3}. \tag{9.43}$$

Finally, equations (9.35) and (9.37) give a_1 and H_1, so taking the logarithm of equation (9.41) gives

$$\begin{aligned}
\log \lambda &= \log \frac{T_{\rm rh}}{2.73\mathrm{K}} + \frac{4}{3} \log \frac{\mathcal{M}}{T_{\rm rh}} + \log \frac{a_o}{a(x_1)} - \log H(x_1) \\
\Rightarrow \log \lambda &= \log \frac{T_{\rm rh}}{10^{14}} + \log \frac{10^{14}\mathrm{GeV}}{2.73\mathrm{K}} + \\
&\quad + \frac{4}{3}\left[\log \frac{3^{1/4} m_p}{\sqrt{8\pi} 10^{14}} - \log \frac{T_{\rm rh}}{10^{14}} + 2 \log x_0 (A - B x_0)\right] \\
&\quad + \frac{1}{4\xi^2}\left[\log\left(\frac{A - 2Bx_0}{{x_0}^2}\right) + \log\left(\frac{x_1^2}{A - 2Bx_1}\right)\right] \\
&\quad - \log \frac{m_p}{\sqrt{8\pi}} - \log\left(x_1 (A - B x_1)\right)
\end{aligned} \tag{9.44}$$

While equation (9.44) is a complicated expression, the terms in it are mostly numerical constants or parameters that describe the exact solution. What it tells us is that perturbations giving rise to structure in the present universe at a characteristic length scale of λ were formed during inflation when $x = x_1$. Noting that $m_p = 1.22 \times 10^{19}$GeV and that 1GeV $= 1.1605 \times 10^{13}$K we obtain

$$\begin{aligned}\log\lambda &= 33.14 - \frac{1}{3}\log\frac{T_{\mathrm{rh}}}{10^{14}} + \frac{2}{3}\log\left(x_0(A - Bx_0)\right) + \frac{1}{4\xi^2}\log\left(\frac{A - 2Bx_0}{x_0{}^2}\right) \\ &+ \frac{1}{4\xi^2}\log\left(\frac{x_1^2}{A - 2Bx_1}\right) - \log\left(x_1(A - Bx_1)\right). \qquad (9.45)\end{aligned}$$

In the equation above, λ is specified in units of GeV^{-1}. The last step is to convert λ to megaparsecs. Since that 1Mpc $= 1.5637 \times 10^{38}$GeV^{-1} we finally get

$$\begin{aligned}\log\lambda &= -54.68 - \frac{1}{3}\log\frac{T_{\mathrm{rh}}}{10^{14}} + \frac{2}{3}\log\left(x_0(A - Bx_0)\right) + \frac{1}{4\xi^2}\log\left(\frac{A - 2Bx_0}{x_0{}^2}\right) \\ &+ \frac{1}{4\xi^2}\log\left(\frac{x_1^2}{A - 2Bx_1}\right) - \log\left(x_1(A - Bx_1)\right) \qquad (9.46)\end{aligned}$$

where λ is now given in megaparsecs.

9.3.3 Normalising to COBE

The detection of the anisotropy of the microwave background by the COBE satellite places a strong constraint on the amplitude of the primordial perturbation. Since the perturbation re-enters the horizon with the same amplitude it had when it left the horizon during inflation, normalising to COBE imposes into a constraint on the parameters of the particle physics model driving inflation.

Applying this constraint to the potential, equation (9.34), will eliminate one of A or B. Liddle and Lyth [111] express this constraint as

$$\left[\frac{2V^3}{\xi^2 x^2 V'^2}\right]^{1/4} = 6.2 \times 10^{16}\mathrm{GeV}. \qquad (9.47)$$

The working is simplified if we define

$$B = CA \quad , \quad x = \frac{1 - y}{2C}. \qquad (9.48)$$

With this redefinition, y increases with time to a maximum value of 1 and $y = 0$ corresponds the local maximum of the potential at $x = A/2B = 1/2C$.

Substituting equation (9.34) into equation (9.47) then gives C in terms of A, y and ξ,

$$C = 136\frac{A(1-y)(3+6y+3y^2-8\xi^2y^2)^{3/2}}{\xi y(3+3y+4\xi^2-8\xi^2 y)}. \tag{9.49}$$

We fix C at the value of y where the perturbations that have a characteristic scale equal to the present horizon length are produced.

When expressed in terms of y, ϵ and η have no dependence on C,

$$\epsilon = 8\xi^2\left(\frac{y}{1+y}\right)^2, \tag{9.50}$$

$$\eta = 4\xi^2\frac{2y-1}{1+y}, \tag{9.51}$$

which means that the spectral indices depend only on ξ. The coefficients of the exponential terms in the potential, A and B, determine the normalisation of the spectrum but not its shape.

If gravitational perturbations make a significant contribution at large scales the value of C will have to be reduced accordingly. The ratio, R, of scalar to tensor contributions to the primordial density fluctuation spectrum is proportional to the parameter ϵ. It is numerically equal to [110]

$$R \approx 12.4\epsilon \tag{9.52}$$

and is evaluated for the inflationary universe when perturbations equal to the present horizon scale are produced.

9.3.4 The Method

To build a realistic model, work through the following recipe:

1. Pick values of A and ξ.

2. Choose the value of y (which is equivalent to selecting ϕ) when the perturbations that now have a wavelength equal to the present horizon size are formed. Specifying y fixes C, via equation (9.49), and the spectral indices of the scalar and tensor perturbations through equations (9.22) and (9.23). Since the spectral indices depend on y, this step amounts to selecting the type of density perturbations the model will produce.

3. Now specify the temperature, $T_{\rm rh}$, to which the universe reheats. To ensure baryogenesis this should be at least equal to the electroweak scale

(10^2 - 10^3 GeV) and is often as much as 10^{12} or 10^{14} GeV. From equation (9.46) we can then solve (numerically) for x_0, the value of x at which inflation is assumed to cease.

4. The last step is to ensure that you have chosen values of A, y and T_{rh} are consistent with one another. The energy density at reheating is, from equation (4.14),

$$\rho_{\mathrm{rh}} = \frac{\pi^2}{30} N T_{\mathrm{rh}}^4 \tag{9.53}$$

where $10^2 \lesssim N \lesssim 10^3$. This must be less than the energy density at the end of inflation, which is obtained from equation (9.37) and equation (4.7) with $k = 0$, so

$$\frac{\pi^2}{30} N T_{\mathrm{rh}}^4 < \frac{3m_p^4}{64\pi^2}(Ax_0 - B{x_0}^2)^2. \tag{9.54}$$

If this condition is not satisfied then choose a lower value of T_{rh} or a smaller value of y and start again. Finally, when equation (9.49) is derived, it is assumed that the spectrum is dominated by the scalar perturbations. This is assumption not necessarily so, and if the tensor perturbations make a significant contribution then this normalisation will need to be adjusted [111].

9.3.5 Specific Models

First, consider a model with $\xi = 0.05$. In this case the inflation is always roughly exponential, the spectrum is very close to being exactly scale free and the gravitational perturbations make a negligible contribution. The indices over the full range of y are plotted in Figure 9.4. The parameter ϵ has a maximum value of $2\xi^2$ and so from equations (9.50) and (9.52) we see that the gravitational waves never contribute more than 5% of the total perturbation. This model is effectively scale free and therefore in good agreement with the COBE data.

When $\xi = 0.05$ the spectral indices are always effectively equal to unity. For the sake of a definite model, choose $A = 10^{-5}$, $y = 0.5$ and $T_{\mathrm{rh}} = 10^{14}$GeV. For these choices, the perturbations are normalised to the COBE data if $C \approx 0.1$ and inflation ends when $y = 0.6$. The fluctuations are almost exactly scale free. In this case the perturbations produced by the potential equation (9.34) are effectively the same as those found in standard powerlaw inflation, driven by equation (9.26). The proportion of gravitational waves is reduced by a factor

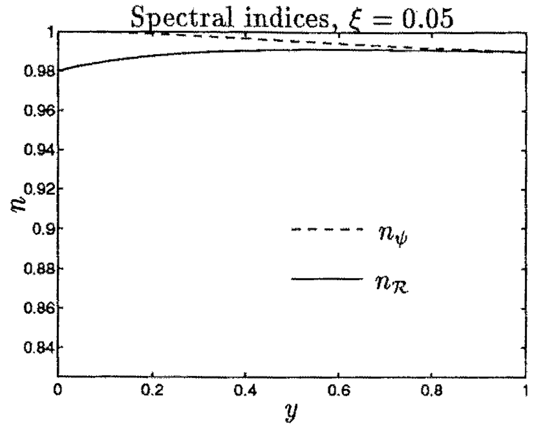


Figure 9.4 : The spectral indices $n_\mathcal{R}$ and n_ψ are plotted against y for $\xi = 0.05$. Since they are effectively unity at all times, this model will produce scale free perturbation spectra

of two, but since they only supply a small proportion of the total perturbation in both models the impact is small.

Larger values of ξ produce a much more interesting spectrum, as the spectral indices (particularly for the scalar perturbation) can take on a wide range of values. Since we are free to choose the point at which the perturbations that seed the structure in the present universe are produced, we have considerable latitude in determining the spectrum. In particular, a powerlaw model built on the potential, equation (9.26), with $\xi = 0.25$ will have a ratio of tensor to scalar perturbations of approximately 1.6. This, in turn, requires spectral index for the scalar perturbations to be closer to unity [110]. However, by arranging the model so that the presently observed perturbations are produced when the field point is close to the local maximum of the potential (corresponding to small y) the proportion of tensor perturbations is sharply reduced.

9.3.6 Summary

The magnitude of the primordial perturbations was fixed when we normalised to the COBE data. The test of inflationary models is in the spectral indices they predict, and the ratio of the scalar and tensor perturbations. Indeed, given accurate knowledge of this data (which is not yet available, but should be obtained in the future) it is possible to reconstruct the first few terms in the Taylor series expansion of the potential [45–47].

Since the potential equation (9.34) is constrained by the requirement that it

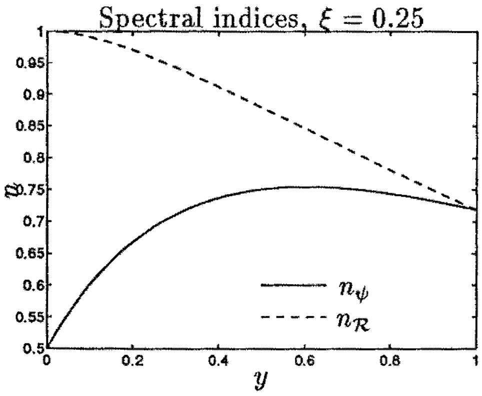


Figure 9.5 : The spectral indices $n_\mathcal{R}$ and n_ψ are plotted against y for $\xi = 0.25$. The perturbations that are cosmologically accessible are produced over a small range of y and so will have a constant spectral index.

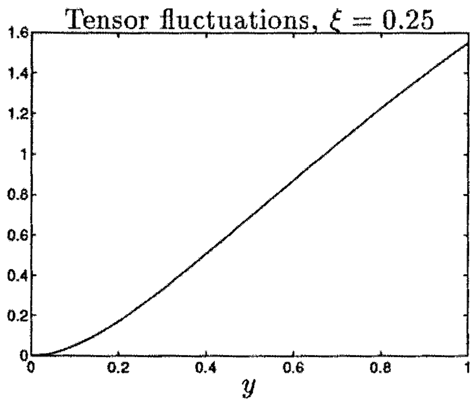


Figure 9.6 : The ratio of scalar to tensor perturbations for for $\xi = 0.25$ is plotted against y. At small values of y the contribution of tensor perturbation is greatly reduced.

can be solved exactly, any cosmological model based upon it is slightly artificial. Choosing a small value of ξ and with C given by equation (9.49) gives a scale free spectrum with a minimal contribution from tensor perturbations and fits the COBE data well. However, most inflationary models can be adjusted to give the same prediction so this cannot be construed as observational verification.

What is interesting, however, is that models of power-law inflation based on a single exponential potential, equation (9.26), need tight constraints on ξ if they are to work. Models based on the potential equation (9.34) work for a wider range of ξ even though the potential is constructed out of three terms, any one of which would not be able produce a realistic perturbation spectrum by itself. The reason is that the local maximum in equation (9.34) creates a relatively flat region in the potential, which reduces the deviations

Furthermore, the models resemble natural inflation in that for some parameter choices they can produce a strongly tilted spectrum of scalar perturbations with a minimal contribution from tensor modes.

The exact models introduced in Chapter 8 are not good candidates for a realistic model of inflation, since the coefficients of one of the exponential terms in equation (9.34) cannot be specified independently of the others (even before the COBE normalisation is applied). However, this constraint can be relaxed, and potentials that are of the general form equation (6.2) studied with the usual approximate and numerical techniques. As long as the coefficient of one of the exponential terms is negative (to produce the local maximum), then the corresponding inflationary model will reproduce the interesting features found here. Since these models both weaken the existing constraints on the allowable exponential terms, and because such potentials arise in a number of theoretical settings this work creates viable new options for the inflationary model builder.

Chapter 10

Conclusion

The work in this thesis is concerned with two distinct roles played by scalar fields in theoretical physics: symmetry breaking in gauge theories and inflationary cosmology. In Chapters 2 and 3, I presented a new and concise derivation of the standard finite temperature corrections to the effective potential. I then calculated the critical temperature for Grand Unified Theories with Coleman-Weinberg symmetry breaking. The calculation is exact to the first loop order, and applies to a general GUT.

In the second part of this thesis, I focussed on exact cosmological models where the source of the stress energy tensor is a fundamental scalar field. These models usually, but not always, led to an implementation of the inflationary paradigm. In Chapters 7 and 8 I presented new models based on scalar fields in which the potential is a sum of several exponential terms. These models exhibit a wide range of behaviour, including eras of both exponential and power-law growth. In Chapter 9 I examined the primordial density perturbation spectrum produced by the most interesting of these models. I show that it is possible to produce a sharply tilted spectrum of scalar perturbations, while holding the tensor contribution to a minimum.

The work done here is capable of extension in several directions. The technique I have used to produce exact scalar field cosmologies can be applied repeatedly to produce any number of new inflationary models. The modeller then has to discriminate between the large number of possible models, to focus on those with some interest beyond their mere existence. However, Barrow's recent paper (discussed at the end of Chapter 8) can be generalised significantly using this approach.

These models also point to a new method to producing significant tilt in the primordial perturbation spectrum. The proportion of tensor perturbations can be small, as in natural inflation, or large, as in conventional power-law inflation. The key ingredient in these models is a local maximum in the effective potential that consists of two or more exponential terms. In this thesis, only exact solutions have been considered but I intend to consider models of this type which do not admit exact solutions in the future.

Appendix A

Published papers

The new results in Chapters 2 and 3 are contained in the article:

Easther R J M and Moreau W R 1992 *Calculating the critical temperature in Coleman-Weinberg GUTs* J. Phys. G **18** 1869–1874. [66]

The derivation of the high temperature expansion of the effective potential (Chapter 2) also formed the subject of an unrefereed poster paper at the 1992 New Zealand Institute of Physics Conference:

Easther R J M 1992 *The High Temperature Approximation to the Finite Temperature Effective Potential* Poster paper at 1992 NZIP conference, preprint NZ-CAN-RE-92/2. [63]

The new, exact solutions presented in Chapters 7 and 8 form the basis of an article that will appear in Classical and Quantum Gravity

Easther R J M 1993 *Exact superstring motivated cosmological models* Class. Quantum Grav. **10** 2203-2215. [64]

These results were also presented in an abbreviated form in a poster paper given at the 37th Yamada Conference, *The Universe and Its Observational Quest* held in Tokyo, June 1993 and the proceedings are being published:

Easther R J M 1993 *Stringy Inflation: Exact cosmological solutions for scalar fields with string motivated potentials* in Sato K (ed.) *37th Yamada Conference: The Universe and Its Observational Quest* (Universal Academy Press, Tokyo) [65].

While I was a PhD student, I also collaborated on a paper for the American Journal of Physics that is currently being reviewed:

Moreau W, Easther R and Neutze R 1993 *The Relativistic (An)harmonic Oscillator* Am. J. Phys. (to appear) [134]

In this paper we present a relativistic generalisation of the standard one dimensional harmonic oscillator and solve the equations of motion for it. This work is not described in this thesis.

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