BERGMAN KERNEL, BALANCED METRICS AND BLACK HOLES

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ABSTRACT OF THE DISSERTATION

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In this thesis we explore the connections between the Kähler geometry and Landau levels on compact manifolds. We rederive the expansion of the Bergman kernel on Kähler manifolds developed by Tian, Yau, Zelditch, Lu and Catlin, using path integral and perturbation theory. The physics interpretation of this result is as an expansion of the projector of wavefunctions on the lowest Landau level, in the special case that the magnetic field is proportional to the Kähler form. This is a geometric expansion, somewhat similar to the DeWitt-Seeley-Gilkey short time expansion for the heat kernel, but in this case describing the long time limit, without depending on supersymmetry. We also generalize this expansion to supersymmetric quantum mechanics and more general magnetic fields, and explore its applications. These include the quantum Hall effect in curved space, the balanced metrics and Kähler gravity. In particular, we conjecture that for a probe in a BPS black hole in type II strings compactified on Calabi-Yau manifolds, the moduli space metric is the balanced metric.

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Dedication

Моим родителям

Table of Contents

Chapter 1

Introduction

The quantum mechanics of particle on a curved space is a prototypical example of the problem on the interface between geometry and physics. Since the pioneering work of DeWitt [1], where its path integral formulation was developed, many important results were obtained in this area. Among them the most well-known include the DeWitt-Seeley-Gilkey short time expansion of the heat kernel on a Riemannian manifold [2, 3, 4], the derivation of the Atiyah-Singer index theorem from supersymmetric quantum mechanics [5, 6], and relation to gravitational anomalies [7, 8].

In this thesis we explore the connection between the expansion of Bergman kernel on a Kähler manifold developed by Tian, Yau, Zelditch, Lu and Catlin [9, 10, 11, 12] and the quantum mechanics of a particle in a non-uniform magnetic field. This type of connection is not very well known to physicists, but certainly fits into the category above. We also discuss its possible applications.

Bergman kernel was introduced by S. Bergman in 1922 [13] in the context of conformal mapping problem of planar domains; similar kernel for functions on a disk was also considered by G. Szegő in 1921. It is a linear functional $\rho(x, y)$ on the space of holomorphic functions, which has reproducing property, i.e. it acts on the holomorphic function f by evaluating it at point, $\rho : f \to f(x)$. In higher dimensions and for compact manifolds there are no global holomorphic functions. However, on a Kähler manifold M the Bergman kernel can be introduced for sections s of holomorphic line bundle L (and its k'th tensor power L^k), which locally look like holomorphic polynomials of degree one (correspondingly, k). Following earlier work by Boutet de Monvel and Sjöstrand,

and by Fefferman [14], it was shown by Zelditch [10] and Catlin [11], that for large k the diagonal of the Bergman kernel $\rho(z, z)$ admits an expansion in $1/k$ with coefficients being *local* functions of Riemann tensor and its relatives.

Physically, this setup is very similar to the quantum mechanical problem of particle in magnetic field. Recall, that on the plane and in constant perpendicular uniform magnetic field the spectrum for this problem factorizes into infinite tower of Landau levels. Each level is highly degenerate, with total number of states controlled by the magnetic flux. The wavefunctions on the lowest Landau level (LLL) are holomorphic polynomials, up to an overall weight. The same problem can be considered on curved compact manifolds of any dimension. It turns out that the rich LLL structure is preserved if the magnetic field satisfies certain conditions of holomorphy and the manifold admits a Kähler metric. The LLL wavefunctions correspond to the sections s of line bundle, therefore the density matrix, projected on the lowest Landau level should be equal to the Bergman kernel.

To prove this analogy we consider the LLL density matrix, which can be defined in terms of the path integral. Rather than the standard short time expansion, the restriction to LLL corresponds to the zero temperature, or long time limit. We carefully define the path integral for a particle in magnetic field and compute the density matrix in the $T \to \infty$ limit, as a large magnetic field expansion. The result turns out to coincide with the Bergman kernel expansion, at least up to the second order in perturbation theory. Thus we provide the "physical proof" of the Tian-Yau-Zelditch expansion [15].

Let us now discuss the significance of this type of expansion from mathematical and physical perspective.

One of the most important applications regards the construction of Ricci-flat metrics on Kähler manifolds. In 10d superstring theory, the standard way to obtain a realistic 4d models of particle physics is to compactify extra dimensions on a 6d compact Kähler manifold [16]. Equations of motion require that the manifold should satisfy Calabi-Yau condition of zero first Chern class $c_1(M) = 0$, and admit a Ricci-flat metric. The exact Ricci-flat metric on Calabi-Yau manifold appears to be beyond the reach of the present day mathematics. However, it is of considerable interest for phenomenological applications of string theory to develop some approximation scheme to Ricci-flat and constant Ricci curvature metrics. This problem turns out to be related to the Tian-Yau-Zelditch expansion of the Bergman kernel in the following way.

It was proved by Yau in the case $c_1(M) = 0$ and by Aubin and Yau for $c_1(M) < 0$, that Ricci-flat metric exists in any Kähler class. The case $c_1(M) > 0$ turns out to be more complicated and the question of existence is related to a certain notion of stability, see e.g. [17] for a recent review. For $c_1(M) \geq 0$, it has been proposed by Yau [18] to approximate the metrics in a given Kähler class on M by the Bergman metrics. These metrics are constructed in the following way. A particular choice of the basis of sections s_α, $\alpha = 0, ..., N_k$ of the line bundle L^k defines an embedding of M into the complex projective space $\mathbb{CP}^{N_k}$ (Kodaira embedding), by sending a point $z \in M$ to the point $s_\alpha(z)$ in $\mathbb{CP}^{N_k}$. The Bergman metric is a pull-back to M under the Kodaira embedding of the Fubini-Study metric on $\mathbb{CP}^{N_k}$. Consider now a Kähler metric ω_g in the class $c_1(L)$ defined by the curvature of the line bundle L, and the basis of sections which is orthogonal under the standard $\mathcal{L}^2$ Hilbert space norm with respect to the Hermitian metric on L^k. Then the corresponding Bergman metric is in the same class as ω_g, and moreover, the two differ by a total derivative of the logarithm of the Bergman kernel. It follows then from the Tian-Yau-Zelditch theorem [9, 10], that an arbitrary metric in a given Kähler class can be approximated by the Bergman metric for $k \to \infty$. One can use this scheme to approximate Ricci-flat or constant Ricci curvature metric in a given Kähler class.

Of particular importance is the Bergman metric, for which the diagonal of the Bergman kernel is constant. This metric was introduced by Donaldson [19, 20] and is known as the balanced metric. It can be constructed numerically by solving a certain integral equation [21] and, for instance, can be used to find an approximation to the Ricci-flat metric on Calabi-Yau manifolds [22].

The physical meaning of the balanced metric is that it corresponds to a mixed state on LLL with constant density matrix on the diagonal. Such a mixed state satisfies what we call the "maximal entropy" property: the probability to find the system at a particular point on the manifold is independent of the point. This physical interpretation of the balanced metric has led us to consider its other possible applications, beyond the problem of the approximation of Ricci-flat metrics.

One rather straightforward application of balanced metrics concerns the Quantum Hall effect. It is well known, that Landau levels provide a theoretical framework for the QHE [23], explaining the formation of incompressible droplets of electrons as they fill Landau levels. The concept of QHE droplets has also been generalized to higher dimensions [24, 25], in particular to complex projective spaces [26, 27]. The defining property of the droplet is that it corresponds to a mixed state on LLL with constant density matrix. Therefore the balanced metric corresponds to QHE droplets on compact Kähler spaces, which have been considered in the literature.

Another physical system, where Landau levels on Kähler manifold play important role, is the Calabi-Yau black hole. For this black hole solution the famous Bekenstein-Hawking formula for the black hole entropy admits an explicit check by counting the number of the underlying microstates. This was done by Strominger and Vafa [28], who counted the microstates of a BPS bound state of Dirichlet branes with the same charge as the black hole. Their derivation is possible due to the observation of the attractor mechanism [29], which implies that entropies and numbers of microstates are independent of the moduli of the background, and flow to some particular values on the horizon. We propose, that not only the Kähler moduli of the black hole are fixed, but the whole metric is universal. Our observation is based on the near-horizon description of the black hole microstates as a quantum-mechanical system of $D0$ branes on the lowest Landau level on Calabi-Yau manifold [30]. Applying the maximal entropy principle to this system, we arrive at a conjecture that the effective metric seen by a probe on Calabi-Yau manifold is balanced metric [31].

We also consider another interesting application, which goes under the name "Kähler quantum gravity". According to the Tian-Yau-Zelditch theorem, for $k \to \infty$ the Bergman metrics become dense in the space of all Kähler metrics. Therefore the latter can be "replaced" by the space of Bergman metrics, which can be parameterized by large N positive hermitian matrices. It has been proposed by Zelditch, that random matrix measures on the Bergman metrics define a rather simple framework for construction of the theory of random metrics in higher dimensions. For instance, this would generalize the well-known two-dimensional Liouville gravity. Here we consider the simplest case of Gaussian measure and provide the solution of the theory in this case.

Let us also mention the applications of the Bergman kernel to the problem of geometric quantization. The basic idea, due to Berezin [32], is to consider M as a phase space, and try to quantize it. As a phase space, M must have a structure which can be used to define Poisson brackets; it is familiar [33] that this is a symplectic structure, i.e. a nondegenerate closed two-form ω. The space of holomorphic sections is the standard ingredient of geometric quantization [34]. It leads to a finite dimensional Hilbert space, whose dimension is roughly the phase space volume of M in units of $2\pi\hbar$. In this interpretation, the parameter k plays the role of $1/\hbar$, and thus the large k limit is semiclassical. The Bergman kernel in this framework is the "reproducing kernel" studied e.g. in [35] from the viewpoint of Landau levels. It can be used to define the symbol of an operator, the star product [36], and related constructions. We refer to [37, 38, 39] for the recent work on applications of the Bergman kernel to quantization of Kähler manifolds and related topics.

1.1 Outline of the thesis

We start in Chapter 2 with a review of the relevant mathematical concepts, with the emphasis on the connection to the quantum mechanics in a magnetic field. In §2.1 we recall the standard 2d Landau levels in uniform magnetic field, and explain basic ideas in this simple setup. In §2.2 we consider higher-dimensional generalization of this problem, and show that for compact Kähler manifolds and holomorphic magnetic field, i.e. holomorphic line bundle, the high degeneracy of the lowest Landau level is preserved. We also demonstrate, that the wavefunctions can be mathematically described as holomorphic sections of linear line bundle L^k, with the flux of magnetic field proportional to k. In §2.3 we review the construction of holomorphic sections and explain that they realize the Kodaira embedding of M into a projective space. This brings us to the subject of Bergman metrics, which are constructed by pulling back to M the Fubini-Study metric from the projective space. It follows from the Tian-Yau-Zelditch expansion of the density of states, that Bergman metrics approximate all Kähler metrics in a given Kähler class, as explained in §2.4. Donaldson's balanced metrics are introduced in §2.5. Since they play an important role in later discussion, we find it instructive to present their explicit construction on two dimensional sphere and torus. In §2.6 we explain how balanced metrics are related to QHE droplets. In §2.7 to consider the Gausian random matrix measure on the space of Bergman metrics and find all n-point correlators in this case.

In Chapter 3 we derive the Bergman kernel expansion from the path integral representation for the density matrix, which is the main result of the thesis. There is a huge body of work on the quantum mechanical path integral in curved space, mostly in relation to the short-time expansion and supersymmetry. Some of the challenging technical issues, concerning the proper definition of the measure, have been recently settled in Ref. [8]. For instance, there is a unique choice of hamiltonian, preserving Einstein invariance of the measure, which leads to the counterterm in lagrangian formulation, as explained in §3.3. We compute the propagator in §3.4 and then adopt normal Kähler coordinate frame to proceed to the standard perturbative expansion. We show that the infinite time limit is well defined, and all potential divergences due to contact terms, and linear and quadratic divergences in T, exactly cancel. We compute the first two terms in the large magnetic field $1/k$-expansion.

This result is generalized to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric quantum mechanics in §4.1 and §4.2. The perturbation theory expansion is T-independent due to supersymmetry, and the trace of the density matrix is fixed by the index formulas. In §4.3 we consider the most general choice of the $U(1)$ magnetic field, satisfying the condition of holomorphy, and show that the Bergman kernel expansion generalizes to this case as well. We calculate the first nontrivial term in the expansion.

In the concluding Chapter 5 we propose an application of balanced metrics to BPS Calabi-Yau black hole solution in type II string theory. In §5.1 we start by explaining the idea of the "effective metric", which is the metric seen by an observer in a certain quantum state. We then describe in §5.2 and §5.3 the black holes as system of probe branes, with $D4$-brane being a source of magnetic field on the Calabi-Yau, as seen by a $D2$ brane wrapped on the horizon. In §5.4 we propose that the $D2$ brane, which can be also described as a system of particles on the LLL, is in a state of maximal entropy, with the density matrix independent of a point on the manifold. This leads us to conjecture in §5.5 that the effective metric on the Calabi-Yau manifold, seen by $D2$ brane, is the balanced metric. We also discuss implications of this conjecture.

1.2 Summary of main results

Main results of the thesis include

- derivation of the Bergman kernel expansion from path inegral (Chapter 3),
- generalization of the Bergman kernel expansion to supersymmetric quantum mechanics, and to the most general $U(1)$ magnetic field (Chapter 4),
- proposal on the role of the balanced metric, as an effective metric for a $D2$ brane in Calabi-Yau black hole solution (Chapter 5),
- solution of the random Bergman metric theory with Gaussian measure (§2.7).

These results are partly based on two papers in collaboration with M. Douglas [15, 31] and on [40].

Chapter 2

Landau levels and holomorphic sections

2.1 Landau levels in two dimensions

The main ideas explored in this dissertation can be illustrated for the problem of particle on a plane subjected to an external magnetic field, considered by L. D. Landau in 1930 [41], and earlier by V. Fock in 1928 [42]. Here we recall the basic setup, contained e.g. in the Landau and Lifshitz textbook [43].

Consider the quantum-mechanical particle of mass m and charge e in magnetic field on a two dimensional plane (x, y). The Hamiltonian is

$$H = \frac{1}{2m}\left(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2$$

where $\vec{\nabla} = (\partial/\partial x, \partial/\partial y)$. We ignored the spin term, since it is inessential for our purposes. We will also work in units $m = e = c = 1$. It is instructive to rewrite it in complex coordinates on the plane $z = x + iy, \bar{z} = x - iy$

$$H = -2(\hbar\partial - iA)(\hbar\bar{\partial} - i\bar{A}) + \frac{1}{2}\hbar B, \tag{2.1}$$

where A and $\bar{A}$ are holomorphic and anti-holomorphic components of for the vector potential, and $B = 2i(\bar{\partial}A - \partial\bar{A})$ is the field strength. In the simplest case of everywhere constant and uniform the magnetic field one can easily solve the Shrödinger equation

$$H\psi_n = E_n\psi_n, \tag{2.2}$$

and determine the spectrum. It consists of an infinite tower of the "Landau levels" with the harmonic oscillator energies

$$E_n = \hbar B(n + \frac{1}{2}), \quad n = 0, 1, ...$$

and for each n there are many wavefunctions with the same energy. Note that the lowest Landau level (LLL) at the energy E_0 is special. In this case it is enough to solve a first order equation only

$$(\hbar\bar{\partial} - i\bar{A})\psi = 0, \tag{2.3}$$

then the full Shrödinger equation (2.2) holds automatically. In the symmetric gauge $A = \frac{1}{4i}B\bar{z}$, $\bar{A} = -\frac{1}{4i}Bz$ the wavefunction on the lowest level $E_0 = \hbar B/2$ are given by

$$\psi_\alpha(z) = z^\alpha \exp\left(-\frac{B}{4\hbar}|z|^2\right), \qquad \alpha = 0, 1, ..., N-1 \tag{2.4}$$

As we will see in a moment, the number of states N on the LLL is controlled by the total flux, and is infinite in the case at hand. Note also, that in this gauge the wavefunctions are chosen to be the eigenfunctions of the angular momentum operator, with the eigenvalue n, rather than of the p_x operator as in [43].

The main object of interest will be the mixed state comprised of the LLL states. It is described by the density matrix

$$\rho(z, z') = \sum_{\alpha=0}^{N-1} c_{\alpha\beta}\Psi_\alpha(z)\Psi^*_\beta(z'), \tag{2.5}$$

for some complex numbers coefficients $c_{\alpha\beta}$. Since the gap between E_0 and E_1 is proportional to the magnetic field strength, the reduction to lowest level is justified in the large field limit. We assume the normalization condition

$$\text{Tr } \rho = \int d^2z\sqrt{g}\,\rho(z, z) = N, \tag{2.6}$$

provided that the wavefunctions are properly normalized

$$\Psi_\alpha(z) = \sqrt{\alpha! 2^\alpha \left(\frac{\hbar}{B}\right)^{\alpha+1}}\, z^\alpha \exp\left(-\frac{B}{4\hbar}|z|^2\right)$$

with respect to the standard $\mathcal{L}^2$ inner product

$$\langle\Psi_\alpha, \Psi_\beta\rangle = \int d^2z\sqrt{g}\;\Psi^*_\alpha\Psi_\beta. \tag{2.7}$$

The question we would like to explore is *what are the conditions on the density matrix* (2.5) *such that the system is in the state of maximal entropy?*. To define "maximal entropy" we demand that each microstate in the mixed state (2.5) is realized with the same probability, or equivalently the matrix $c_{\alpha\beta}$ is identity

$$c_{\alpha\beta} = \delta_{\alpha\beta}. \tag{2.8}$$

Another definition of the maximally entropic state, is to require that the probability density $\frac{1}{N}\rho(z,z)\sqrt{g}d^2z$ to find the system at some point z is independent of z

$$\rho(z,z) = \text{const} = \frac{N}{\text{Vol}}, \tag{2.9}$$

i.e. the density matrix is constant on the diagonal. This can also be rephrased in the following way. Note, that each individual state is localized roughly along the annulus of radius $\sim n$ and area $\hbar/B$ (2.4). Then the maximally entropic state is a non-local state, that covers the space uniformly by a combination of individual localized states.

Generically, these two conditions do not have to be equivalent. In particular, it is not always possible to construct a mixed state, satisfying the second (strong) maximal entropy property. However, in the case under consideration the two definitions do coincide. Indeed, plugging (2.8) into the density matrix (2.5) we get for the diagonal

$$\rho(z,z) = \sum_{\alpha=0}^{N-1} \Psi_\alpha(z)\Psi_\alpha^*(z) = B/\hbar.$$

Therefore we recover (2.9) and vice versa. Note that $\text{Tr}\rho = \Phi/\hbar$, where $\Phi = B\cdot\text{Vol}$ is the total flux and Vol is the (regularized) volume of the space. This is exactly as expected from (2.6), confirming that for finite volume the number of states is proportional to flux. Technically, in this case both Vol and Φ are infinite, but we will avoid this problem in future by considering only compact manifolds.

The interplay between the two definitions of maximal entropy state, although somewhat trivial in this simple example, turns out to lead to interesting results for a more complicated setup of non-uniform magnetic field and non-flat backgrounds. This can be illustrated by considering non-constant magnetic field. In two dimensions it can be parameterized by a single real-valued function φ. In the gauge $A = i\partial\varphi$ the field strength is given by

$$B = -4\partial\bar{\partial}\varphi. \tag{2.10}$$

We still expect the rich LLL structure of the spectrum, which should arise as before by the reduction of the Shrödinger equation on the LLL to a first order equation. Naively it does not happen, since the last term in the hamiltonian (2.1) is non-constant $\frac{1}{2}\hbar B = -2\hbar\partial\bar{\partial}\varphi$. One way around this problem is to introduce the standard spin term $-\frac{1}{2}\hbar\sigma_3 B$, which cancels the previous term for spin-up configuration $\sigma_3|\uparrow\rangle = +|\uparrow\rangle$, see e.g. [44]. Another way, that we pursue here, is to consider the problem on a curved background, i.e. on a two-plane with a nontrivial metric $ds^2 = g_{z\bar{z}}|dz|^2$. Then the hamiltonian reads

$$H = -2g^{z\bar{z}}(\hbar\partial - iA)(\hbar\bar{\partial} - i\bar{A}) + \frac{1}{2}\hbar g^{z\bar{z}}B.$$

Now, if we adopt the condition

$$g^{z\bar{z}}B = \text{const}, \tag{2.11}$$

the Shrödinger equation reduces on the LLL to the first order PDE (2.3) and the zero-point energy is $E_0 = \frac{1}{2}\hbar g^{z\bar{z}}B$. The explicit solution for the LLL wavefunctions is

$$\psi_\alpha(z) = z^\alpha \exp\left(\frac{1}{\hbar}\varphi(z,\bar{z})\right), \quad \alpha = 0, 1, ..., N-1. \tag{2.12}$$

The degeneracy N of the LLL can now be determined, following the argument in Ref. [45]. The function φ tends to $-\frac{1}{2\pi}\Phi\log z$ as $z \to \infty$, where $\Phi = \int\sqrt{g}d^2\xi B(\xi)$ is the total flux, as follows from the solution of Poisson equation (2.10)

$$\varphi(z,\bar{z}) = -\frac{1}{2\pi}\int\sqrt{g}d^2\xi\log|z-\xi|B(\xi).$$

In order for the wavefunctions (2.12) to be renormalizable, the maximal power $N-1$ of the polynomial behavior at ∞ should be less than $\Phi/\Phi_0 - 1$, where $\Phi_0 = 2\pi\hbar$ is the quantum of flux. Therefore the total degeneracy N of the lowest level is equal to the integer number of flux units

$$N = [\Phi/\Phi_0].$$

Let us now consider the mixed state (2.5) in the case at hand. For any choice of the metric and the magnetic field one can build the maximally entropic state (2.8), provided the orthogonality condition (2.7) for wavefunctions is satisfied. However, the second definition (2.9) imposes a much stronger condition. It amounts to a nontrivial equation on the metric $g_{z\bar{z}}$ and the magnetic field φ

$$\rho[[g_{z\bar{z}}(z), \varphi(z)](z) = \text{const.} \tag{2.13}$$

One should also take into account the relation (2.11) between $g_{z\bar{z}}$ and φ. In general, (2.13) is a highly nonlinear constraint on the geometry of the manifold. The dependence of the density matrix on the metric and magnetic field comes from the exponential factor in the LLL wavefunction (2.12) as well as from the normalization condition (2.7), which in turn implicitly depends on the metric. The "maximally entropic" metric, satisfying Eq. (2.13), is an example of a *balanced metric*, which will study in more detail in §2.5.

It is interesting to determine the behavior of the density matrix as a function of metric. Due to general coordinate invariance, we may expect for it to be a function of the Ricci scalar, which is the only metric invariant in two dimensions. Indeed, for large magnetic fields, or equivalently large degeneracy of the LLL, one can find the following behavour for the large magnetic flux

$$\rho \sim N - 1 + \frac{1}{2}R + \mathcal{O}(1/N)$$

where R is the Ricci scalar, and the information on the magnetic field enters only through N. This expansion and its analog in higher dimensions correspond to the Bergman kernel expansion, which we derive in the next chapter. It follows from the previous equation, that the maximally entropic (in the strong sense) mixed state on LLL, prefers flatter backgrounds (near-constant R). This in turn implies, that the problem of finding Ricci-flat geometries can be translated into the problem of decribing lowest Landau level wavefunctions (2.3). One of the main goals of our work is to explore this interrelation.

2.2 Landau levels in higher dimensions

For potential geometric applications it is interesting to generalize the Landau problem to higher dimensional manifolds. Also one of the lessons of §2.1 is that due to large volume infinities, the lowest Landau level is most naturally defined for compact manifolds.

We consider a particle of unit mass and charge on a d-dimensional manifold M, in a general metric g_{ij} and magnetic field F_{ij}. It is described by a wave function $\psi(x;t)$ which satisfies the Schrödinger equation,

$$H\psi \equiv \frac{1}{2\sqrt{g}} D_i \ \sqrt{g} g^{ij} \ D_j \psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{2.14}$$

where $D_i = i\hbar\partial_i + A_i$ is the covariant derivative appropriate for a scalar wavefunction. As usual for topologically nontrivial manifolds, we treat this equation separately in each coordinate patch, and then sew the patches together by gauge and coordinate transformations. We consider the time-independent Schrödinger equation with fixed energy, and seek for the energy eigenstates

$$H\psi_i(x) = E_i \psi_i.$$

The case of two-dimensional Euclidean space $g_{ij} = \delta_{ij}$ with a constant magnetic field $F_{ij} = B\epsilon_{ij}$ was considered before. The main features of this case, that we would like to preserve in higher dimensions, include the spectral gap between the lowest and first excited level and the large number of localized states on the LLL. For flat backgrounds and uniform magnetic fields these properties can be easily generalized to $d = 2n$ dimensions. One can choose the coordinates such that the magnetic field lies in the 12, 34 planes and so forth, and $B_{12} > 0$, $B_{34} > 0$ etc. Then, considering the lowest Landau level we have

$$E_0 = \frac{\hbar}{2}(B_{12} + \ldots + B_{2n-1,2n}), \tag{2.15}$$

with states localized as before within each two-plane and, as usual, the gap between the zeroth and first excited levels is proportional to magnetic field.

For a nontrivial metric and magnetic field, while one might not at first expect a high degree of degeneracy, it still might be possible. When the magnetic field is much larger than the curvature of the metric, the intuition that wavefunctions localize should still be valid. Then, we might estimate the energy of a wavefunction in the lowest Landau level localized around a point x as Eq. (2.15), where the components B_{12}, B_{34} and so on are evaluated in a local orthonormal frame. If the energy E_0 in Eq. (2.15) is constant, then all states in the LLL will be degenerate, at least in the limit of large field stength.

The reason for large degeneracy of states was the factorization property of the Schrödinger equation (2.3). The same property occurs on complex manifolds, under certain conditions. Consider $2n$-dimensional manifold with local complex coordinates z^a, $\bar{z}^{\bar{a}}$ $(a, \bar{a} = 1, ..., n)$. The proper generalization of the splitting of the components of B Eq. (2.15) for nonconstant magnetic fieds, is that the magnetic should be a $(1,1)$-form, i.e. take nonzero values only for mixed components of the field strength $F_{a\bar{a}}$, with

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \tag{2.16}$$

this magnetic field corresponds to holomorphic line bundle, which we explain in detail in the next paragraph. The same condition should be imposed on the metric $g_{a\bar{a}}$ as well, with the corresponding metrics known as the *Kähler metrics*. Both Kähler metric and holomorphic field strength can be written as total derivatives $g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K,\ F_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} f$ for some local functions K and f. Then the identity

$$[D_i, D_j] = i\hbar F_{ij},$$

can be used to rewrite the Hamiltonian as

$$H = g^{a\bar{a}} D_a \bar{D}_{\bar{a}} + \frac{1}{2}\hbar g^{a\bar{a}} F_{a\bar{a}}. \tag{2.17}$$

Hence, if the following combination is constant

$$g^{a\bar{a}} F_{a\bar{a}} = \text{const}, \tag{2.18}$$

then any wave function satisfying

$$\bar{D}_{\bar{a}}\psi = 0 \tag{2.19}$$

will be degenerate and belong to the LLL. This argument also holds away from the strict large field limit.

The condition (2.18), with (2.11) being its special case, is known as Kähler-Yang-Mills equation, and is equivalent to Maxwell equation for $U(1)$ magnetic fields with $F^{0,2} = 0$. Indeed, for curved backgrounds the Maxwell equation is written as

$$D_a F^a{}_{\bar{b}} = 0, \quad D_{\bar{a}} F_a{}^{\bar{a}} = 0,$$

which is equivalent to

$$\partial(g^{a\bar{a}} F_{a\bar{a}}) = \bar{\partial}(g^{a\bar{a}} F_{a\bar{a}}) = 0,$$

and (2.18) follows. One can immediately write down a simple solution to (2.18)

$$F_{a\bar{a}} = k g_{a\bar{a}} \tag{2.20}$$

where the coefficient k is related to the total flux as $\Phi = \int F^n \sim k^n$. This is the simplest choice of the magnetic field that leads to a rich lowest Landau level on Kähler manifolds. However, there is also a more general magnetic field configurations, with similar property. For example in the case of a manifold with nontrivial cohomology class $b^{1,1}(M) > 1$, one can choose a more general holomorphic magnetic field

$$F_{a\bar{a}} = k g_{a\bar{a}} + u_{a\bar{a}}, \tag{2.21}$$

where $u_{a\bar{a}}$ is a $(1,1)$-form, not in the same cohomology class with $k g_{a\bar{a}}$. It turns out, that the choice of Eq. (2.20) leads to the Tian-Yau-Zelditch definition of the Bergman kernel, and Eq. (2.21) leads to its generalization considered in [48, 49, 50]. We consider the Bergman kernel for the first choice of the magnetic field in the next chapter, and for the second choice in §4.3. It also should be mentioned, that in the path integral formulation one can avoid the condition (2.18), and consider the field as in Eq. (2.21), not restricted by Maxwell equations.

2.3 Holomorphic sections and Kodaira embedding

The standard trick to simplify the equations Eq. (2.19), is to do a "gauge transformation" with a *complex* parameter $\theta(z,\bar{z})$. While at first this might seem to violate physical requirements such as unitarity of the Hamiltonian, in fact it is perfectly sensible as long as we generalize another ingredient in the standard definitions, namely the inner product on wave functions. Explicitly, we define the wave function in terms of another function $s(z)$, as

$$\psi(z) = e^{i\theta(z,\bar{z})}s(z), \qquad \bar{D}_{\bar{a}}\psi(x) = e^{i\theta(z,\bar{z})}\ (i\bar{\partial}_{\bar{a}} + \bar{A}_{\bar{a}} - \bar{\partial}_{\bar{a}}\theta)s(z). \tag{2.22}$$

This would be a standard $U(1)$ gauge transformation if $\theta(z,\bar{z})$ were real. By allowing complex $\theta(z,\bar{z})$, and assuming (2.16), we can find a transformation which trivializes all the antiholomorphic derivatives,

$$\bar{D}_{\bar{a}} \to \bar{\partial}_{\bar{a}}. \tag{2.23}$$

In this "gauge," wave functions in the LLL can be expressed locally in terms of holomorphic functions. The only price we pay is that the usual inner product (2.7) turns into an inner product which depends on an auxiliary real function,

$$h(z,\bar{z}) \equiv e^{-2\mathrm{Im}\theta(z,\bar{z})}, \tag{2.24}$$

as

$$(s,s') = \int_M \sqrt{g}\ h(z,\bar{z})\ \bar{s}(\bar{z})\ s'(z).$$

Taking into account the gauge transformations between coordinate patches, one would say that $s(z)$ is holomorphic sections of a holomorphic line bundle L evaluated in a specific frame, or L^k for the magnetic field proportional to k as in (2.20), while the quantity $h(z,\bar{z})$ defines a *hermitian metric* on the line bundle L. This setup appears in many mathematical applications, the most well known is probably the problem of "geometric quantization" [33]. Of special importance for us here is the relation to the problem of existence of Kähler-Einstein metrics, which by definition have Ricci tensor proportional to the metric itself, on Kähler manifolds. This relation is known as the Yau-Tian-Donaldson program [46, 9, 47], and is based on the concept of *Bergman metric*. Here and in the next paragraph we provide an introduction to this subject from the physics point of view, see also a nice review [17].

Following Tian [9], consider a compact Kähler manifold M of complex dimension n, and a line bundle L, called the "polarization" on M, with its k-th tensor power L^k. Consider some basis $s_0(z), ..., s_{N_k}(z)$ on the space $H^0(M, L^k)$ of all global holomorphic sections of L^k. Since s_α's are defined up to multiplication by a complex number, for k large enough this basis defines a *Kodaira embedding* of M into the projective space $\mathbb{CP}^{N_k}$ of sections, by sending a point z on M to a point $[s_0(z), ..., s_{N_k}(z)] \in \mathbb{CP}^{N_k}$.

For the hermitian metric h on L the Ricci curvature of the line bundle is a $(1,1)$-form

$$Ric(h) = -i\partial\bar{\partial}\log h$$

in the Chern class $c_1(L)$. Recall that the Kähler form ω_g is a positive $(1,1)$-form[1], related to the metric $g_{a\bar{a}}$ as $\omega_g = ig_{a\bar{a}}dz^a \wedge d\bar{z}^{\bar{a}}$ in local complex coordinates. The key idea is to consider L to be a positive line bundle, that is a holomorphic line bundle with a positive curvature $(1,1)$-form. Therefore we can consider Kähler metrics in the class $c_1(L)$. For instance, one can simply choose ω_g to be equal to the Ricci curvature of L, given by the expression above. The corresponding Kähler metric is said to be polarized with respect to L.

For the line bundle L^k the hermitian metric is just the k'th power h^k of the hermitian metric on L. Therefore the relation between the Ricci curvature of L^k and the polarized Kähler metric

$$kg_{a\bar{a}} = -\partial_a\bar{\partial}_{\bar{a}}\log h^k \tag{2.25}$$

is exactly as in Eq. (2.20), taking into account that $F_{a\bar{a}} = -iRic(h^k)$. Thus, in the physics language, the polarized Kähler metric is equivalent to the choice of magnetic

[1] A $(1,1)$-form μ is positive definite if the matrix of coefficients $-i\mu_{a\bar{a}}$ is positive definite everywhere on the manifold.

field (2.20), and the LLL wavefunctions correspond to the sections of L. To complete the analogy between the LLL and Kähler metrics, we still have to explain the role of the density matrix (2.5) and the maximal entropy condition (2.9).

2.4 Bergman metrics and Bergman kernel

The most obvious way to define a metric on a manifold is to consider its embedding into a larger space equipped with some metric, which then can be pulled back to the manifold. This induced metric will depend on details of embedding. For instance, one can always embed a real the manifold into $\mathbb{R}^n$ and pullback the usual euclidean metric. In the case of complex manifold the Kodaira embedding can be used for the same purpose. Since any basis of sections $[s_0(z), ..., s_{N_k}(z)]$ defines a holomorphic embedding of M into $\mathbb{CP}^{N_k}$, the standard Fubini-Study metric

$$g_{FS} = \frac{1}{k}\partial\bar{\partial}\log\sum_{\alpha=0}^{N_k}|s_\alpha|^2$$

on the projective space induces the *Bergman metric* $g_{FS}|_M$ on M

$$g_{FS}|_M = g + \frac{1}{k}\partial\bar{\partial}\log\left(h^k\sum_{\alpha=0}^{N_k}s_\alpha\bar{s}_\alpha\right), \tag{2.26}$$

where g is the polarized Kähler metric (2.25). This form of the Bergman metric implies that the corresponding Kähler form $\omega_{FS}|_M$ is in the same cohomology class as ω_g, since the expression inside the logarithm in the second term is a globally defined function. To prove the positivity of the Bergman metric one can compute the derivatives in (2.26)

$$g_{FS}|_M = \frac{1}{k}\left(\frac{(\partial s, \bar{\partial}\bar{s})}{(s,\bar{s})} - \frac{(\partial s, \bar{s})(s, \bar{\partial}\bar{s})}{(s,\bar{s})^2}\right)$$

and apply Cauchy-Bunyakovskii inequality for complete system.

Note, that the basis of sections used in the definition of the Bergman metric is not necessarily a normalized basis. In fact, a rotation of the basis s_α by any non-degenerate $(N_k+1)\times(N_k+1)$-matrix corresponds to a different embedding of M since $GL(N_k+1) = \mathrm{Aut}(\mathbb{CP}^{N_k})$ is the automorphism group of the projective space. Therefore the space $\mathcal{K}_k$ of Bergman metrics on the level k, in the Kähler class defined by ω_g, is the symmetric space

$$\mathcal{K}_k = GL(N_k+1)/U(N_k+1), \tag{2.27}$$

since (2.26) is obviously invariant under $U(N_k+1)$ rotation.

Consider next an *orthonormal* basis $s_0(z), \ldots, s_{N_k}(z)$ on the space $H^0(M, L^k)$ with respect to the standard L^2 norm

$$(s_\alpha, s_\beta) = \int_M \sqrt{g} h^k s_\alpha \bar{s}_\beta = \delta_{\alpha\beta}. \tag{2.28}$$

Note, that the integration measure here depends on the original metric g. In this case g and the corresponding Bergman metric $g_{FS}|_M$ differ by $\partial\bar{\partial}$ of the logarithm of the "density of states" function

$$\rho_k(z) = h^k \sum_{\alpha=0}^{N_k} s_\alpha(z)\bar{s}_\alpha(\bar{z}). \tag{2.29}$$

Obviously, $\rho_k(z)$ is independent of the choice of orthonormal basis, since all such bases are related by a unitary rotation. This function is the diagonal of the *Bergman kernel*. It is also a special case of a more general concept of reproducing kernel $K(x, y)$ on Hilbert spaces, which posses the property $(K(x, \cdot), K(\cdot, y)) = K(x, y)$.

It is interesting to look at the structure of the function (2.29) for large k. Zelditch [10] and Catlin [11] proved that it admits an asymptotic expansion in $1/k$ with coefficients, which can be expressed in terms of local invariants of the metric g, such as the Riemann tensor and its contractions. Several terms in this expansion were computed by Lu [12] with the following result up to the second order in $1/k$

$$\rho_k(z) = k^n + \frac{1}{2}k^{n-1}R + k^{n-2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \mathcal{O}(k^{n-3}). \tag{2.30}$$

The original computation is based on Tian's global peak section method [9], which is a technique to approximate sections of line bundle for large values of k. In Chapter 3 we reproduce this expansion from the path integral. Other methods to derive this result are the heat kernel approach of [49, 50] and the reproducing kernel approach of [51].

From this expansion it follows that any polarized Kähler metric g can be approximated by Bergman metrics as $k \to \infty$. Indeed, as follows from Eq. (2.30) and the definition (2.26), the difference between g and the corresponding Bergman metric

$$g - \frac{1}{k} g_{FS}|_M = \mathcal{O}(1/k^2)$$

vanishes in the large k limit. This result has also been obtained earlier by Tian [9] in C^2 topology and generalized by Ruan [52] to C^∞ topology. This statement together with the expansion Eq. (2.30) is usually refered to as the Tian-Yau-Zelditch theorem. Essentially it states that the space $\mathcal{K}$ of Kähler metrics, or equivalently, Kähler potentials in a cohomology class represented by ω_g

$$\mathcal{K} \cong \{\phi : \omega_g + i\partial\bar{\partial}\phi > 0\}$$

can be densely covered by the space of Bergman metrics

$$\mathcal{K} = \lim_{k\to\infty} \mathcal{K}_k,$$

where the potentials ϕ are of the form as in Eq. (2.26). One of the most utilized consequences of this statement is that one can "replace" $\mathcal{K}$ by $\mathcal{K}_k$, in order to proof existence ot the constant Ricci curvature Kähler metrics, as well as to construct their approximations.

2.5 Balanced metric

Based on the results of [9, 10, 11, 12] Donaldson proposed to consider the metrics with constant density function

$$\rho_k(z) = \text{const} = \frac{\dim H^0(M, L^k)}{\text{Vol}\, M}, \tag{2.31}$$

where the integration constant is fixed as before in (2.9) and we use the notation $\dim H^0(M, L^k) = N_k + 1$. The previous equation can be solved for h^k in terms of s_α's

$$h^k = \frac{\dim H^0(M, L^k)}{\text{Vol}\, M \cdot \sum_\alpha |s_\alpha|^2}. \tag{2.32}$$

as follows from (2.29). Plugging the solution back to the orthonormality condition Eq. (2.28) we get the integral equation for the sections

$$\frac{\dim H^0(M,L^k)}{\mathrm{Vol}\, M}\int_M \sqrt{g}\frac{s_\alpha \bar{s}_\beta}{\sum_\gamma s_\gamma \bar{s}_\gamma} = \delta_{\alpha\beta}. \tag{2.33}$$

Note, that here both the volume form and the basis of sections depend on h^k, which makes this condition highly nontrivial.

The meaning of the previous equation is that for a particular choice of basis the Kodaira embedding has diagonal 'moments-of-inertia' matrix $\langle\frac{s_\alpha \bar{s}_\beta}{|s|^2}\rangle$. Such an embedding $M \to \mathbb{CP}^{N_k}$ is called *balanced* [53] and the corresponding Bergman metric

$$g = -\partial\bar{\partial}\log h = \frac{1}{k}\partial\bar{\partial}\log\sum_\alpha |s_\alpha|^2$$

is known as the *balanced metric* (this concept probably appeared already in [54]). Using the expansion Eq. (2.30) and assuming existence of constant scalar curvature metric in the given Kähler class, Donaldson was able to prove [19, 20] that a unique balanced metric exists and, as $k \to \infty$, it tends to the metric of constant scalar curvature. The latter follows from the fact that R appears as a coefficient in the first nontrivial term in Eq. (2.30).

In general, the defining relation (2.33) is hard to solve. A more useful definition of the balanced metric was proposed in [21]. Consider two basic constructions. For a metric h^k on L^k, there is a Hermitian metric $\mathrm{Hilb}(h^k)$ on the vector space $H^0(M,L^k)$ defined as

$$||s||^2_{\mathrm{Hilb}(h^k)} = \frac{\dim H^0(M,L^k)}{\mathrm{Vol}\, M}\int_M \sqrt{g}|s|^2_{h^k}.$$

Also, given a hermitian metric G on the vector space $H^0(M,L^k)$ one can define a metric $FS(G)$ on L^k such that

$$\sum_\alpha |s_\alpha|^2_{FS(G)} = 1$$

for the orthonormal with respect to G basis of sections. The balanced condition is satisfied at a fixed point G^* of the T-map

$$T(G) = \mathrm{Hilb}(FS(G)),$$

or, equivalently, when

$$\mathrm{Hilb}(h^{k*}) = G^*, \quad FS(G^*) = h^{k*}. \tag{2.34}$$

Using the definitions above one can rewrite the T-map as an integral operator and the fixed point condition becomes an integral equation on $(N_k+1)\times(N_k+1)$ elements of the hermitian matrix $G_{\alpha\beta}$

$$T(G)_{\alpha\beta} = \frac{\dim H^0(M, L^k)}{\mathrm{Vol}\, M}\int_M \sqrt{g}\frac{s_\alpha \bar{s}_\beta}{(G^{-1})^{\gamma\delta}s_\gamma \bar{s}_\delta} = G_{\alpha\beta}.$$

Here again one has to keep in mind the dependence of the integration measure on $G_{\alpha\beta}$ due to Eq. (2.34). It was shown in [19, 21] that for any initial choice of the matrix G, the iterative procedure for T converges to the balanced embedding.

Physically, the balanced condition means that one can adjust *both* the metric and the magnetic field by choosing h^k and appropriate basis of sections, such that the density matrix is constant. Let us consider some simple examples, where the balanced metric can be found explicitly, namely a two dimensional sphere and a torus.

2.5.1 Balanced metric on $\mathbb{CP}^1$

Consider M to be a two-sphere $\mathbb{CP}^1$, after [21]. In homogeneous coordinates (z_0, z_1) the sections of L^k are all homogeneous polynomials of the form $z_0^i z_1^{k-i}$ of degree k. Introducing the projective coordinate $z = z_1/z_0$ we can write a basis of $H^0(M, L^k)$ as z^α, $\alpha = 0, ..., k$. The total number of sections is $N_k = k+1$. The balanced metric is equal to the standard round metric on sphere for all k

$$h^k = \frac{k+1}{\mathrm{Vol}\mathbb{CP}^1}\frac{1}{(1+|z|^2)^k},$$

$$g = -\partial\bar{\partial}\log\frac{1}{(1+|z|^2)} = \frac{1}{(1+|z|^2)^2},$$

where $\mathrm{Vol}\mathbb{CP}^1 = \pi$ in the round metric. Indeed, consider the basis of sections with the following normalizations

$$s_\alpha = \sqrt{C_k^\alpha}\, z^\alpha,$$

where C_k^α is a binomial coefficient. Using the identity

$$\int_{\mathbb{C}} \frac{dzd\bar{z}}{(1+|z|^2)^2} \frac{z^\alpha \bar{z}^\beta}{(1+|z|^2)^k} = \delta_{\alpha\beta} \frac{\pi}{k+1} [C_k^\alpha]^{-1}$$

one can see that the balanced condition (2.33) holds for the basis s_α. Therefore for all values of k the balanced metric is the same and it is equal to the metric of constant scalar curvature (round metric on sphere).

2.5.2 Balanced metric on T^2

The balanced metric on the abelian varieties in any dimension was constucted in [55]. Following this work, consider two dimensional torus T^2 with modular parameter τ, and a periodic complex coordinate $z \sim z+1 \sim z+\tau$. The only nontrivial global holomorphic section of L is the theta-function

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2 \tau + 2\pi i n z\right)$$

which has the following transformation properties under the action of modular group

$$\theta(z + s\tau + t, \tau) = \exp\left(-\pi i s^2 \tau - 2\pi i s z\right)\theta(z,\tau), \quad s,t \in \mathbb{Z}.$$

The standard flat metric on torus corresponds to the Kähler potential $h = \exp \frac{\pi}{2} \frac{(z-\bar{z})^2}{\mathrm{Im}\tau}$. This form of h can be uniquely fixed by the requirement that $h|\theta|^2$ is invariant under modular transformations.

Following [56], introduce the Heisenberg group Γ acting on the holomorphic function $f(z)$ by two operations

$$(S_b f)(z) = f(z+b), \qquad (T_a f)(z) = \exp\left(\pi i a^2 \tau + 2\pi i a z\right) f(z + a\tau).$$

for real numbers a, b. The theta functions with rational characteristics $a, b \in \mathbb{Z}[1/l]/\mathbb{Z}$ are defined as

$$\theta_{a,b}(z,\tau) = S_b T_a \theta = \sum_{n\in\mathbb{Z}} \exp\left(\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b)\right).$$

It is easy to see that $\theta_{a,b}(lz)$ form the basis of $H^0(T^2, L^{l^2-1})$. The corresponding Kodaira embedding is defined as

$$z \in T^2 \to (\theta_{0,0}(lz), \dots \theta_{1-1/l,1-1/l}(lz)) \in \mathbb{CP}^{l^2-1}.$$

Let us show that this map defines a balanced embedding. To see this, note that $\theta_{a,b}(z)$ is the basis of the space $\mathbb{V}_l$ of entire functions invariant under $l\Gamma$. There is a more convenient basis

$$s_c(z,\tau) = \sum_{n \in c + l\mathbb{Z}} \exp\left(\pi i n^2 \tau + 2\pi i n z\right), \quad c \in \mathbb{Z}[1/l]/l\mathbb{Z},$$

of eigenvectors of the shift operator for $a, b \in \mathbb{Z}[1/l]/\mathbb{Z}$

$$S_b s_c = \exp\left(abc\pi i\right) s_c, \quad T_a s_c = s_{c+a}.$$

Since the transformation between the bases is unitary

$$\theta_{a,b}(z,\tau) = \sum_{p \in \mathbb{Z}/l\mathbb{Z}} \exp\left(2\pi i b(p+a)\right) s_{p+a},$$

then it is enough to check the balanced condition for s_c. Introduce the Fubini-Study metric on $\mathbb{V}_l$ by

$$||f(lz)||_{FS} = \int_{T^2} \frac{|f(lz)|^2}{\sum_c |s_c(lz)|^2} \omega_{FS}, \tag{2.35}$$

where ω_{FS} is a pullback to the torus of the Fubini-Study metric on $\mathbb{CP}^{l^2-1}$ under the Kodaira embedding, defined above. It is not hard to see that S_b, T_a preserve this norm. Since the operators S_b are mutually commuting, their eigenvectors s_c form an orthonormal basis. Then the formula (2.35) is equivalent to the balanced condition (2.33). Therefore the balanced metric on T^2 is given by

$$g_{\text{bal}} = \frac{1}{l^2} \partial\bar{\partial} \log \sum_{a,b \in \mathbb{Z}[1/l]/\mathbb{Z}} |\theta_{a,b}(lz)|^2.$$

This metric is not flat, although it actually converges to the flat metric as $l \to \infty$.

It is interesting to compare the lowest Landau level density matrix in the balanced metric and in the flat metric. The former is obviously constant everywhere on the manifold. The latter has been considered in [57], where s_c's were chosen as an orthonormal basis of solutions of Shrödinger equation on the LLL, for constant magnetic field with flux $\sim N = l^2$. There it was found that the density matrix

$$\rho_N^{flat} = \sum_c h^{l^2} |s_c(lz)|^2$$

is non-constant, as one might expect from naive translation symmetry arguments. Instead it exhibits a pattern of evenly spaced bumps at the locations $\{n_1+n_2\tau\}/l^2$, $n_1, n_2 = 0, ..., l^2$ on the torus. Mathematically, this is a consequence of a theorem about the zeroes of theta functions, see e.g. §4 in [56]. Therefore for the flat metric and constant magnetic field the translational symmetry is broken down to $\mathbb{Z}_N \times \mathbb{Z}_N$. It is restored only if one introduces the balanced metric on the torus.

Other notable constructions of balanced metrics, considered in the literature, include numerical approximation of the Ricci flat Calabi-Yau metric on K3 surface [21] and on the quintic surface in $\mathbb{CP}^5$ [22].

2.6 Quantum Hall effect in higher dimensions

The standard example in which the projector on LLL appears in physics is the Quantum Hall effect, see e.g. the review [23]. In the standard setup of integer QHE one considers a system of non-interacting electrons on a two-dimensional plane subjected to a constant orthogonal magnetic field. At low temperatures and high values of the magnetic field only the lowest lying energy levels are important. Usually, incompressible droplets of electrons are considered in a confining potential V, with partly filled ground level and a number of filled states $K < N$. The edge dynamics of the droplets is of particular interest.

In recent years this problem has been much generalized. It has been considered for Riemann surfaces, see [58] and references therein; higher dimensional examples include the case of S^4 [24], $\mathbb{R}^4$ [25] and $\mathbb{CP}^n$ [26]; see also [27] for a review.

The case of $\mathbb{CP}^n$ is the first nontrivial case in which we can make contact with the balanced metrics. The choice made in [26] is the $U(1)$ background field proportional to the Ricci tensor

$$F_{a\bar{a}} \sim R_{a\bar{a}}, \tag{2.36}$$

which in turn is taken to be equal to the Fubini-Study metric on $\mathbb{CP}^n$. Therefore this equation is equivalent to Eq. (2.20). In the local projective coordinates $z_1, \ldots, z_n$, the LLL wave functions can be constructed explicitly

$$\psi_\alpha \sim \frac{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}}{(1+|z|^2)^{k/2}}, \quad \alpha = 1, \ldots, N \tag{2.37}$$

up to a normalization constant [27]. As in Eq. (2.22) they have the form of holomorphic function, weighted by the metric of the line bundle.

The dynamics of the droplet is characterized in the following way. One starts with diagonal density matrix ρ_0 with K states occupied, then the fluctuations, preserving the number of states correspond to the unitary transfomations $\rho_0 \to \rho = U\rho_0 U^\dagger$, and the equation of motion is the quantum Liouville equation

$$i\frac{\partial \rho}{\partial t} = [V, \rho].$$

The form of the droplet is determined by the form of the minima of the confining potential. In [26] the case of spherically symmetric potential $V = V(r = z\bar{z})$ was studied. In the limit of large number of states N (i.e. large magnetic field) and large number of fermions $K < N$ the density matrix has the form

$$\rho(r^2) = \Theta(r^2 - R_d^2),$$

where R_d is the radius of the droplet and Θ is the step function. In other words the density matrix is equal to constant in the region, occupied by the droplet. The presence of the boundary is due to partly filled LLL. For the completely filled lowest level the droplet is constant everywhere on the manifold. Therefore the droplet without a boundary is described by the mixed state, corresponding to the balanced metric.

One can also generalize the above construction to nonabelian background gauge fields. Since $\mathbb{CP}^n = SU(n+1)/U(n)$ and Lie algebra of $U(n) = U(1) \times SU(n)$, then in addition to $U(1)$ gauge field one can also turn on $SU(n)$ gauge field. In [26] the case of constant $SU(n)$ gauge field was considered, with the wavefunctions belonging to a certain $SU(n)$ representation.

The similar generalization of the Bergman kernel was considered in mathematical literature [48, 49]. One can consider a tensor product of L with a more general hermitian vector bundle $\mathcal{E}$, with the curvature $F^{\mathcal{E}}$. By analogy with Eq. (2.30), the Bergman kernel expansion exists

$$\rho(x) = k^n + k^{n-1}(R/2 \cdot \mathbf{1}_E + F^{\mathcal{E}}) + \dots.$$

The second term was computed in [49]. We derive this expansion for the abelian line bundle $\mathcal{E}$ in §4.3. It was shown in Ref. [48] that balanced metric exists in this case as well, and satisfies Hermitian-Einstein equation in large k limit.

2.7 Kähler quantum gravity

According to the Tian-Yau-Zelditch theorem, any Kähler metric can be approximated by a Bergman metric. For the metric in a Kähler class $c_1(L)$, the Bergman metric (or more precisely, its Kähler form) ω can be written in the following way

$$\omega(z) = \partial\bar{\partial} \log s_\alpha(z) H^{\alpha\beta} \bar{s}_\beta(z).$$

The positive $N \times N$ hermitian matrix H explicitly parameterizes the symmetric space $\mathcal{K}_k$ (2.27) of the Bergman metrics. The large N limit thus corresponds to the approximation of the full space $\mathcal{K}$ of Kähler metrics.

This setup turns out to provide a natural framework for the study of random measures on Kähler metrics, with the main goal being to give a definition to "Kähler quantum gravity", by analogy with the well known 2d Liouville quantum gravity. Here we consider the simplest example of Gaussian random matrix measure on the Bergman metrics.

A positive hermitian matrix H can always be put in the form $H = A^\dagger A$ for a general complex matrix A, defined up to a unitary rotation. We would like to compute the n-point correlation function in the large N limit, for the following Gaussian measure

$$\langle \omega(z_1) \dots \omega(z_n) \rangle = \int \omega(z_1) \dots \omega(z_n)\, e^{-Ntr A^\dagger A} DA,$$

where DA is the usual euclidean measure on complex matrices.

The following variation of the replica trick (see e.g. [59]) will be useful

$$\partial\bar{\partial} \log s_\alpha(z) H^{\alpha\beta} \bar{s}_\beta(z) = \lim_{m\to 0} \frac{1}{m} \partial\bar{\partial}(s_\alpha(z) H^{\alpha\beta} \bar{s}_\beta(z))^m.$$

The computation of the n-point function then reduces to the computation of ordinary matrix integrals of polynomial insertions of $A^\dagger A$. The propagator is given by

$$\langle (A^\dagger)^{\alpha\gamma} A^{\delta\beta} \rangle = \int (A^\dagger)^{\alpha\gamma} A^{\delta\beta}\, e^{-Ntr A^\dagger A} DA = \frac{1}{N} \delta^{\alpha\beta} \delta^{\gamma\delta}.$$

Using this we obtain

$$\begin{aligned}
&\langle (A^\dagger A)^{\alpha\beta} \rangle = \frac{1}{N} \delta^{\alpha\beta}, \\
&\langle (A^\dagger A)^{\alpha_1\beta_1} (A^\dagger A)^{\alpha_2\beta_2} \rangle = \delta^{\alpha_1\beta_1} \delta^{\alpha_2\beta_2} + \frac{1}{N} \delta^{\alpha_1\beta_2} \delta^{\alpha_2\beta_1}, \\
&\langle (A^\dagger A)^{\alpha_1\beta_1} \dots (A^\dagger A)^{\alpha_n\beta_n} \rangle = \delta^{\alpha_1\beta_1} \dots \delta^{\alpha_n\beta_n} + \qquad\qquad (2.38) \\
&+\frac{1}{N} (\delta^{\alpha_1\beta_2} \delta^{\alpha_2\beta_1} \delta^{\alpha_3\beta_3} \dots \delta^{\alpha_n\beta_n} + \text{all similar pairwise permutations}).
\end{aligned}$$

Thus one can immediately compute the one point function

$$\begin{aligned}
\langle \omega(z) \rangle &= \lim_{m\to 0} \frac{1}{m} \partial\bar{\partial}\ s_{\alpha_1}(z) \bar{s}_{\beta_1}(z) \dots s_{\alpha_m}(z) \bar{s}_{\beta_m}(z) \cdot \\
&\quad \cdot \langle (A^\dagger A)^{\alpha_1\beta_1} \dots (A^\dagger A)^{\alpha_m\beta_m} \rangle \qquad (2.39) \\
&= \lim_{m\to 0} \frac{1}{m} \partial\bar{\partial}\, [(s,\bar{s})^m + \mathcal{O}(1/N)] = \partial\bar{\partial} \log(s,\bar{s}) \equiv \omega_0, \qquad (2.40)
\end{aligned}$$

where we adopt the notation $(s,\bar{s}) = s_\alpha \delta^{\alpha\beta} \bar{s}_\beta$. Note, that this answer is actually exact to all orders in $1/N$. Here ω_0 is the "reference metric" in the Kähler class, for which the basis of sections is orthonormal. The final answer will of course depend on choice of the reference metric. This situation is completely analogous to the Liouville theory, which is also defined up to a choice of background.

The 2-point function is nontrivial only in the next-to-leading order, one would need the order-$1/N$ term in (2.38) to derive the following result (with a slight abuse of notation we call $s_1 = s(z_1)$, $s_2 = s(z_2)$, etc.):

$$\begin{aligned}
\langle \omega(z_1)\omega(z_2)\rangle &= \lim_{m_1,m_2\to 0} \frac{1}{m_1 m_2}\partial_1\bar\partial_1\partial_2\bar\partial_2\ s_{\alpha_1}(z_1)\bar s_{\beta_1}(z_1)\dots s_{\alpha_{m_1}}(z_1)\bar s_{\beta_{m_1}}(z_1)\cdot \\
&\quad \cdot s_{\gamma_1}(z_2)\bar s_{\delta_1}(z_2)\dots s_{\gamma_{m_2}}(z_2)\bar s_{\delta_{m_2}}(z_2)\cdot \\
&\quad \cdot\langle (A^\dagger A)^{\alpha_1\beta_1}\dots(A^\dagger A)^{\alpha_{m_1}\beta_{m_1}}(A^\dagger A)^{\gamma_1\delta_1}\dots(A^\dagger A)^{\gamma_{m_2}\delta_{m_2}}\rangle \\
&= \lim_{m_1,m_2\to 0} \frac{1}{m_1 m_2}\partial_1\bar\partial_1\partial_2\bar\partial_2 \left[(s_1,\bar s_1)^{m_1}(s_2,\bar s_2)^{m_2} + \right. \\
&\quad + \frac{1}{N}\frac{m_1(m_1-1)+m_2(m_2-1)}{2}(s_1,\bar s_1)^{m_1}(s_2,\bar s_2)^{m_2} + \\
&\quad \left. + \frac{m_1 m_2}{N}(s_1,\bar s_1)^{m_1-1}(s_2,\bar s_2)^{m_2-1}(s_1,\bar s_2)(s_2,\bar s_1) + \mathcal{O}(1/N^2)\right] \\
&= \langle\omega(z_1)\rangle\langle\omega(z_2)\rangle + \frac{1}{N}\partial_1\bar\partial_1\partial_2\bar\partial_2\ \frac{(s_1,\bar s_2)(s_2,\bar s_1)}{(s_1,\bar s_1)(s_2,\bar s_2)} + \mathcal{O}(1/N^2). \qquad (2.41)
\end{aligned}$$

Thus the connected 2-point function in the large-N limit is equal to

$$\begin{aligned}
\langle\omega(z_1)\omega(z_2)\rangle_c &= \langle(\omega(z_1)-\langle\omega(z_1)\rangle)(\omega(z_2)-\langle\omega(z_2)\rangle)\rangle \\
&= \frac{1}{N}\partial_1\bar\partial_1\partial_2\bar\partial_2\ \frac{(s_1,\bar s_2)(s_2,\bar s_1)}{(s_1,\bar s_1)(s_2,\bar s_2)}. \qquad (2.42)
\end{aligned}$$

Following the same technics one can derive the expression for the connected large-N n-point function

$$\langle\omega(z_1)\dots\omega(z_n)\rangle_c = \frac{1}{N^{n-1}}\partial_1\bar\partial_1\dots\partial_n\bar\partial_n \frac{\sum_{k=1}^{n-1}\prod_{i=1,\{i\sim i+n\}}^{n}(s_i,\bar s_{i+k})}{(s_1,\bar s_1)\cdots(s_n,\bar s_n)},$$

where in the numerator s_i is identified with s_{i+n}.

It is interesting, that the expression inside the derivatives in the two-point function (2.42) has an interesting geometric interpretation. It is equal to the exponent of the diastasic function $D(z_1,z_2)$, introduced by Calabi [60]

$$\frac{(s_1,\bar s_2)(s_2,\bar s_1)}{(s_1,\bar s_1)(s_2,\bar s_2)} = \exp(-D(z_1,z_2)) \qquad (2.43)$$

It can be shown that the diastatic function has the following short distance behaviour

$$D(z_1, z_2) \sim ||z_1 - z_2||^2_{\omega_0}$$

where the quantity on the rhs is the squared geodesic distance between the two points, computed in the reference metric. Hence, the Gaussian theory has a very peculiar "ultralocal" behaviour, with the propagator falling exponentially with the distance. It would be interesting to consider other random matrix measures as well, in particular, corresponding to the Liouville functional and higher Chen-Tian functionals [61].

Chapter 3

Path integral derivation of the Bergman kernel

3.1 LLL density matrix

In this chapter we consider the physical derivation of the asymptotic expansion of the Bergman kernel. Our basic result is to rederive the Tian-Yau-Zelditch expansion as the infinite time limit of the perturbative expansion for the quantum mechanical path integral [15].

Let us state the result for (nonsupersymmetric) quantum mechanics. We consider a compact Kähler manifold M, and a particle in magnetic field, with the field strength proportional to the Kähler form on the manifold, as in Eq. (2.20). More general magnetic field Eq. (2.21) will be considered in the next chapter. Now, given that there is a large degeneracy of ground states and thus a nontrivial LLL, it becomes interesting to study the projector on the LLL, or in other words the LLL density matrix

$$\rho \equiv \sum_{i;E_i=E_0} |i\rangle\langle i|.$$

If we shift the hamiltonian to set the ground state energy $E_0 = 0$, it can also be defined as the large time limit of propagation in Euclidean time. Regarded as a function of two variables, the projector ρ can be defined as a path integral by the standard Feynman-Kac formula.

It is not hard to see that the regime of large magnetic field is semiclassical, so that one can get an expansion of the LLL density matrix in the inverse magnetic field strength using standard perturbative methods. In the large k limit, the diagonal term satisfies

$$\rho(x,x) \sim k^n \left(1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \mathcal{O}((\hbar/k)^3)\right)$$

as an asymptotic expansion of the path integral. This formula coincides with the expansion of the Bergman kernel [12], thus providing the "physical proof" of the TYZ expansion.

In some ways this expansion is similar to the well known short time expansion of the heat kernel, except that its a long time expansion, because it projects on the ground states. Unlike other analogous results for ground states, it does not require supersymmetry, either for its definition or computation. Of course, similar results can be obtained for supersymmetric theories, our point is that that they do not depend on supersymmetry. Whether they depend ultimately on holomorphy is an interesting question, which we do not discuss here.

3.2 Path integral representation

The euclidean time path integral for a particle on a $2n$-dimensional Kähler manifold M with the magnetic field is given by

$$\rho(x_i, x_f) = \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \prod_{t_i<t<t_f} \det g_{a\bar{b}}(x(t))\, \mathcal{D}x^a \mathcal{D}\bar{x}^{\bar{b}}\ e^{-\frac{1}{\hbar}\int_{t_i}^{t_f} dt\,[g_{a\bar{b}}\dot{x}^a\dot{\bar{x}}^{\bar{b}} + A_a\dot{x}^a + \bar{A}_{\bar{a}}\dot{\bar{x}}^{\bar{a}}]} \tag{3.1}$$

Here we assume that $F_{ab} = F_{\bar{a}\bar{b}} = 0$ and work in the anti-holomorphic gauge $A_a = 0$ for the gauge connection[1]. We also set the non-zero components of the magnetic field strength to be aligned with the metric

$$F_{a\bar{b}} = \partial_a \bar{A}_{\bar{b}} = k g_{a\bar{b}} = k\partial_a\partial_{\bar{b}} K, \tag{3.2}$$

[1] Although the gauge, which trivializes anti-holomorphic derivatives is rather $\bar{A}_{\bar{a}} = 0$ (2.23), the difference between gauge choices is inessential, since the density matrix is a gauge invariant object. We find it convenient for technical reasons to work in the anti-holomorphic gauge.

as in Eq. (2.20), with $K = -\log h$ being the Kähler potential for the metric. Note that we chose the coupling to the vector potential such that the field strength, defined as in (3.2), is real, which corresponds to imaginary $\bar{A}_{\bar{b}}$.

Our goal is to compute the value of the density matrix (3.1) on the diagonal $x_i = x_f = x$ and project to the lowest Landau level, i.e. take a long time $T = t_f - t_i \to \infty$ limit. Note, that one cannot take this limit inside the path integral. That would simply suppress the kinetic term, and one would end up with a rather trivial answer. Thus, we must keep T finite in the process of calculation and take the $T \to \infty$ limit after computing the correlators. The long time limit is free of IR divergent terms, provided the path integral is properly regularized. It should also be mentioned, that by taking the long time limit we implicitly use the fact that there is a spectral gap between the lowest and first positive eigenvalues, which makes it a well-defined operation. The resulting expansion is a local asymptotic expansion in $\hbar/k$, whose coefficients at each order we would like to compute.

To keep track of the dependence on T we can introduce the rescaled time parameter

$$t = t_f + (t_f - t_i)\tau = t_f + T\tau$$

with $\tau \in [-1, 0]$. The classical solution for the trajectory with boundary conditions $x_i = x_f$ is just a constant. Introduce normal coordinates z^a, $\bar{z}^{\bar{a}}$ in the vicinity of the classical trajectory

$$x^a = x^a_f + z^a(\tau)$$

$$\bar{x}^{\bar{a}} = \bar{x}^{\bar{a}}_f + \bar{z}^{\bar{a}}(\tau)$$

The normalization factor $\mathcal{N}$ can be fixed by considering the standard normalization of the heat kernel in the case of non-coincident initial and final points, as e.g. in [62], and is equal to

$$\mathcal{N} = k^n,$$

where n is the complex dimension of the manifold.

3.3 Weyl-ordering counterterm

The Weyl-ordering counterterm in the path integral measure has been discovered by DeWitt [1]. In the hamiltonian framework the path integral corresponds to transition amplitude

$$K(x_i, x_f; T) = \langle x_f | e^{-(T/\hbar)\hat{H}} | x_i \rangle.$$

Since the kinetic term in $\hat{H}$ depends on the coordinate variable through the metric, there may be a discrepancy due to the operator ordering of momentum and coordinate variables, when constructing quantum hamiltonian. Different choices of ordering lead to different definition of path integral (this issue was studied in great detail in Ref. [63, 64, 8]). It can be shown that there is a unique choice the ordering of the hamiltonian which preserves general coordinate invariance

$$\begin{aligned} \hat{H} &= \frac{1}{2} G^{-1/4} (\hat{p}_i - iA_i) G^{ij} G^{1/2} (\hat{p}_j - iA_j) G^{-1/4} \\ &= \frac{1}{2} g^{-1/2} \hat{p}_a \, g^{a\bar{b}} \, g \, (\hat{\bar{p}}_{\bar{b}} - i\bar{A}_{\bar{b}}) g^{-1/2} + \frac{1}{2} g^{-1/2} (\hat{\bar{p}}_{\bar{b}} - i\bar{A}_{\bar{b}}) \, g^{a\bar{b}} \, g \, \hat{p}_a g^{-1/2}, \end{aligned} \tag{3.3}$$

where we specified our hamiltonian to the Kähler case. Here $G = \det g_{ij} = g^2 = (\det g_{a\bar{a}})^2$. To transform the hamiltonian framework to lagrangian we rewrite this expression in a Weyl-ordered form (see Eq. (3.24) in the Appendix) and then perform the Legendre transform with the generalized momenta

$$p_a = g_{a\bar{b}} \dot{\bar{z}}^{\bar{b}}, \qquad \bar{P}_{\bar{b}} = g_{a\bar{b}} \dot{z}^a + i\bar{A}_{\bar{b}}.$$

The following action, written in euclidean time, appears in the exponent of path integral

$$S = \int_{t_i}^{t_f} dt \left(g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} + \bar{A}_{\bar{b}} \dot{\bar{z}}^{\bar{b}} - \frac{\hbar^2}{4} R \right).$$

The Weyl-ordering corresponds to a "mid-point rule" prescription in the path integral, introduced in the next paragraph. The last term in this "quantum corrected" action is necessary, for instance, to obtain correct short time heat kernel expansion [8]. We will demonstrate, that leads to a correct long time as well.

3.4 Normal coordinates, free action and propagators

In the Kähler normal coordinate frame, defined in Appendix, all pure (anti-) holomorphic derivatives of the metric at a chosen point are set to zero. We will use the Kähler normal coordinate frame, centered at x, where the following expansions for the Kähler potential, metric and gauge connection hold up to the sixth order in derivatives

$$K(x^a+z^a,\bar{x}^{\bar{a}}+\bar{z}^{\bar{a}}) = g_{a\bar{b}}(x)z^a\bar{z}^{\bar{b}}+\frac{1}{4}K_{ab\bar{a}\bar{b}}(x)z^az^b\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}+\frac{1}{36}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^az^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}\bar{z}^{\bar{c}}+\dots,$$

$$\bar{A}_{\bar{b}}(x^a+z^a,\bar{x}^{\bar{a}}+\bar{z}^{\bar{a}}) = k\partial_{\bar{b}}K(x^a+z^a,\bar{x}^{\bar{a}}+\bar{z}^{\bar{a}})$$
$$= k\left(g_{a\bar{b}}(x)z^a+\frac{1}{2}K_{ab\bar{a}\bar{b}}(x)z^az^b\bar{z}^{\bar{a}}+\frac{1}{12}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^az^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}+\dots\right),$$
$$g_{a\bar{b}}(x^a+z^a,\bar{x}^{\bar{a}}+\bar{z}^{\bar{a}}) = g_{a\bar{b}}(x)+K_{ab\bar{a}\bar{b}}(x)z^b\bar{z}^{\bar{a}}+\frac{1}{4}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}+\dots$$

in self-explanatory notations. Note that we omitted terms which turn out not to be relevant up to the second order in $\hbar$. For example, the term with five derivatives of mixed type $K_{abc\bar{a}\bar{b}}(x)z^az^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}$ is non-zero in our coordinate frame, but it contributes to the density matrix only starting from $\hbar^3$, as one can check by power counting.

The determinant in the measure (3.1) can be raised to the exponent, using auxiliary anti-commuting ghost fields b^a and $c^{\bar{b}}$. The integral in these variables is the Berezin integral. The diagonal of the density matrix (3.1) can be now rewritten as

$$\rho(x) \quad = \quad \mathcal{N}\int_{z(-1)=0}^{z(0)=0}\mathcal{D}z^a(\tau)\mathcal{D}\bar{z}^{\bar{b}}(\tau)\mathcal{D}b^a(\tau)\mathcal{D}c^{\bar{b}}(\tau)\ e^{-\frac{1}{\hbar}S_0-\frac{1}{\hbar}S_{int}}$$

where we split the action into a free part

$$S_0 = \int_{-1}^{0}d\tau\ \left[\frac{1}{T}g_{a\bar{b}}(x)\dot{z}^a\dot{\bar{z}}^{\bar{b}}+kg_{a\bar{b}}(x)z^a\dot{\bar{z}}^{\bar{b}}+g_{a\bar{b}}(x)b^ac^{\bar{b}}\right], \tag{3.4}$$

and interaction part, with the terms up to the sixth order in derivatives of the Kähler potential

$$
\begin{aligned}
S_{int} \quad = \quad & \int_{-1}^{0} d\tau \left[\frac{1}{T} \left(K_{ab\bar{a}\bar{b}}(x) z^b \bar{z}^{\bar{a}} + \frac{1}{4} K_{abc\bar{a}\bar{b}\bar{c}}(x) z^b z^c \bar{z}^{\bar{a}} \bar{z}^{\bar{c}} \right) \dot{z}^a \dot{\bar{z}}^{\bar{b}} \right. \\
& + k \left(\frac{1}{2} K_{ab\bar{a}\bar{b}}(x) z^a z^b \bar{z}^{\bar{a}} + \frac{1}{12} K_{abc\bar{a}\bar{b}\bar{c}}(x) z^a z^b z^c \bar{z}^{\bar{a}} \bar{z}^{\bar{c}} \right) \dot{\bar{z}}^{\bar{b}} \\
& + \left(K_{ab\bar{a}\bar{b}}(x) z^b \bar{z}^{\bar{a}} + \frac{1}{4} K_{abc\bar{a}\bar{b}\bar{c}}(x) z^b z^c \bar{z}^{\bar{a}} \bar{z}^{\bar{c}} \right) b^a c^{\bar{b}} \\
& \left. - \frac{\hbar^2}{4} T \left(R(x) + \partial_c \bar{\partial}_{\bar{c}} R(x) z^c \bar{z}^{\bar{c}} \right) \right],
\end{aligned}
$$

here dots denote τ derivatives. The propagator for free theory (3.4)

$$
\langle \bar{z}^{\bar{b}}(\tau) z^a(\sigma) \rangle = \hbar g^{a\bar{b}} \Delta(\tau, \sigma), \tag{3.5}
$$

satisfies the following equation

$$
\left[-\frac{1}{T} \frac{d^2}{d\tau^2} + k \frac{d}{d\tau} \right] \Delta(\tau, \sigma) = \delta(\tau - \sigma),
$$

and the path integral boundary conditions translate into Dirichlet boundary conditions for Δ

$$
\Delta(-1, \sigma) = \Delta(0, \sigma) = \Delta(\tau, -1) = \Delta(\tau, 0) = 0
$$

The unique solution is

$$
\begin{aligned}
\Delta(\tau, \sigma) = \frac{1}{k(e^{kT} - 1)} \qquad & \Big\{ \theta(\tau - \sigma) e^{kT} (1 - e^{kT\tau})(1 - e^{-kT(\sigma+1)}) \\
& + \theta(\sigma - \tau)(1 - e^{-kT\sigma})(1 - e^{kT(\tau+1)}) \Big\},
\end{aligned} \tag{3.6}
$$

where the step-function is defined using the "mid-point rule"

$$
\theta(\tau - \sigma) = \begin{cases} 1, & \tau > \sigma \\ \frac{1}{2}, & \tau = \sigma \\ 0, & \tau \leq \sigma \end{cases} \tag{3.7}
$$

The non-zero value at coincident point depends on the choice of Weyl-ordering in the hamiltonian formulation and is crucial for obtaining correct results for the short-time heat kernel expansion from the path integral [8]. Ghost propagator can be regulated with the help of $\Delta(\tau, \sigma)$ in the following way

$$\langle b^a(\sigma) c^{\bar{b}}(\tau) \rangle = -\hbar g^{a\bar{b}} \delta(\tau - \sigma) = \hbar g^{a\bar{b}} \left(\frac{1}{T} {}^{\bullet\bullet}\Delta(\tau, \sigma) - k {}^{\bullet}\Delta(\tau, \sigma) \right)$$

where ${}^{\bullet}\Delta(\tau, \sigma) = d\Delta(\tau, \sigma)/d\tau$, etc.

3.5 Perturbation theory. First Order

Now we are ready to study the perturbation theory in $\hbar$ for the diagonal of the density matrix (3.4)

$$\rho(x) = \mathcal{N}(1 + \hbar\rho_1(x) + \hbar^2\rho_2(x) + \ldots).$$

From (3.5) the dimension of variable z is $\hbar^{1/2}$, therefore by power counting $\rho_n(x)$ should contain terms with $2n$ covariant derivatives of the metric. For example, at first order in $\hbar$ the only invariant of the metric is the Ricci scalar.

Expanding the interacting part of the exponent in the path integral (3.4) to the first order in $\hbar$, we have

$$\begin{aligned} \hbar\rho_1 &= -\frac{1}{\hbar} K_{ab\bar{a}\bar{b}} \int d\tau \, \left(\frac{1}{T} \langle z^b \bar{z}^{\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} \rangle|_\tau + \frac{k}{2} \langle z^a z^b \bar{z}^{\bar{a}} \dot{\bar{z}}^{\bar{b}} \rangle|_\tau + \langle z^b \bar{z}^{\bar{a}} b^a c^{\bar{b}} \rangle|_\tau \right) + \hbar \frac{T}{4} R \\ &= \hbar R \frac{1}{T} \int d\tau \, \left(\Delta({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta) + {}^{\bullet}\Delta\Delta^{\bullet} \right)|_\tau + \hbar \frac{T}{4} R = \hbar R I_1(T, k) + \frac{\hbar T}{4} R. \end{aligned} \tag{3.8}$$

and from here on the integration always runs from -1 to 0. Here we apply usual Wick rule to calculate the correlators. We also use the fact that $R_{a\bar{a}b\bar{b}}(x) = K_{ab\bar{a}\bar{b}}(x)$ in normal frame centered at x. To illustrate how the Wick rule works in this case, consider e.g. the first term here

$$\begin{aligned} \langle z^b \bar{z}^{\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} \rangle|_\tau &= \langle \bar{z}^{\bar{a}} z^b \rangle|_\tau \langle \dot{\bar{z}}^{\bar{b}} \dot{z}^a \rangle|_\tau + \langle \dot{\bar{z}}^{\bar{b}} z^b \rangle|_\tau \langle \bar{z}^{\bar{a}} \dot{z}^a \rangle|_\tau \\ &= g^{b\bar{a}} g^{a\bar{b}} \Delta(\tau, \tau) {}^{\bullet}\Delta^{\bullet}(\tau, \tau) + g^{a\bar{a}} g^{b\bar{b}} {}^{\bullet}\Delta(\tau, \tau) \Delta^{\bullet}(\tau, \tau), \end{aligned}$$

where we explicitly used (3.5). In general for a given polynomial in z and $\bar{z}$ one has to replace its expectation value by a sum of all possible pairwise "contractions" between

z's and $\bar{z}$'s. The same holds for the ghost fields b, c, the only subtlety one has take into account is that they anticommute.

This calculation elucidates the role of the ghosts. Their contribution cancels the contact terms, containing $\delta(0)$, which appear in second derivatives of the bosonic propagators at coincident points.

The values of the integrals used in the main text and their large time asymptotics are collected in Appendix. In $T \to \infty$ limit of the expression above becomes

$$\hbar\rho_1 = \left(-\frac{T}{4} + \frac{1}{2k}\right)\hbar R + \frac{\hbar T}{4} R = \frac{\hbar}{2k} R. \tag{3.9}$$

Note, that the Weyl-ordering counterterm (3.24) is necessary to cancel the large-T divergence. This calculation provides an independent check of the coefficient in front of this term[2], see also [8] for a related discussion.

3.6 Perturbation theory. Second Order

At the $\hbar^2$ order the following metric invariants can appear in the expansion: $\Delta R = g^{a\bar{a}}\partial_a\bar{\partial}_{\bar{a}}R$, $|\mathrm{Ric}|^2 = R_{a\bar{a}}R^{a\bar{a}}$, $|\mathrm{Riem}|^2 = R_{a\bar{a}b\bar{b}}R^{a\bar{a}b\bar{b}}$ and R^2. Therefore the second order correction splits into four components, corresponding to the listed invariants. The full second-order contribution reads

$$\begin{aligned}
\hbar^2\rho_2 = \quad & -\frac{1}{\hbar}K_{abc\bar{a}\bar{b}\bar{c}}\int d\tau \;\left(\frac{1}{4T}\langle z^b z^c \bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\dot{z}^a\dot{\bar{z}}^{\bar{b}}\rangle|_\tau\right.\\
&\left.+\frac{k}{12}\langle z^a z^b z^c \bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\dot{\bar{z}}^{\bar{b}}\rangle|_\tau + \frac{1}{4}\langle z^b z^c \bar{z}^{\bar{a}}\bar{z}^{\bar{c}} b^a c^{\bar{b}}\rangle|_\tau\right)\\
&+\frac{1}{2\hbar^2}K_{ab\bar{a}\bar{b}}K_{a'b'\bar{a}'\bar{b}'}\iint d\tau d\sigma\;\left(\frac{1}{T^2}\langle z^b\bar{z}^{\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{b}}|_\tau z^{b'}\bar{z}^{\bar{a}'}\dot{z}^{a'}\dot{\bar{z}}^{\bar{b}'}|_\sigma\rangle\right.\\
&+\frac{k}{T}\langle z^b\bar{z}^{\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{b}}|_\tau z^{a'}z^{b'}\bar{z}^{\bar{a}'}\dot{\bar{z}}^{\bar{b}'}|_\sigma\rangle + \frac{k^2}{4}\langle z^a z^b\bar{z}^{\bar{a}}\dot{\bar{z}}^{\bar{b}}|_\tau z^{a'}z^{b'}\bar{z}^{\bar{a}'}\dot{\bar{z}}^{\bar{b}'}|_\sigma\rangle\\
&+\frac{2}{T}\langle z^b\bar{z}^{\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{b}}|_\tau z^{b'}\bar{z}^{\bar{a}'}b^{a'}c^{\bar{b}'}|_\sigma\rangle + k\langle z^a z^b\bar{z}^{\bar{a}}\dot{\bar{z}}^{\bar{b}}|_\tau z^{b'}\bar{z}^{\bar{a}'}b^{a'}c^{\bar{b}'}|_\sigma\rangle\\
&\left.+\langle z^b\bar{z}^{\bar{a}}b^a c^{\bar{b}}|_\tau z^{b'}\bar{z}^{\bar{a}'}b^{a'}c^{\bar{b}'}|_\sigma\rangle\right) +
\end{aligned}$$

[2] Factor 1/4 here compared to 1/8 in [8] is due to our definition of scalar curvature (3.18) in Kähler case.

$$+\frac{\hbar T}{4}\partial_c\bar{\partial}_{\bar{c}}R\int d\tau\ \langle \bar{z}^{\bar{c}}z^c\rangle|_\tau + \frac{\hbar T}{4}R\cdot\hbar R I_1(T,k) + \frac{1}{2}\left(\frac{\hbar T}{4}R\right)^2 \qquad (3.10)$$

We start computation from the first line in this expression. Taking into account the identity (3.19), the first line reads

$$\begin{aligned}
&-\hbar^2(-\Delta R + 2|\mathrm{Ric}|^2 + |\mathrm{Riem}|^2)\int d\tau\ \frac{1}{T}({}^\bullet\Delta\Delta^\bullet\Delta + \Delta^2({}^\bullet\Delta^\bullet + {}^{\bullet\bullet}\Delta)/2)|_\tau \\
&= -\hbar^2(-\Delta R + 2|\mathrm{Ric}|^2 + |\mathrm{Riem}|^2)\, I_2(T,k) \\
&\approx -\hbar^2(-\Delta R + 2|\mathrm{Ric}|^2 + |\mathrm{Riem}|^2)\left(\frac{5}{6k^2} - \frac{T}{4k}\right),\ \text{ as } T\to\infty \qquad (3.11)
\end{aligned}$$

Consider now the second integral in (3.10). There are several nonequivalent ways to contract z variables, leading to different invariants. Contraction of each of the primed indices a', b', $\bar{a}'$, $\bar{b}'$ with a non-primed index, leads to the $|\mathrm{Riem}|^2$ structure. Such terms are given by the following expression

$$\begin{aligned}
&\frac{\hbar^2}{2}|\mathrm{Riem}|^2\int\!\!\int d\tau d\sigma\ \left(\frac{1}{T^2}(\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^\bullet\Delta^\bullet(\sigma,\tau){}^\bullet\Delta^\bullet(\tau,\sigma)\right. \\
&+\Delta(\sigma,\tau)\Delta^\bullet(\tau,\sigma){}^\bullet\Delta^\bullet(\sigma,\tau){}^\bullet\Delta(\tau,\sigma) + {}^\bullet\Delta(\sigma,\tau)\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\tau){}^\bullet\Delta^\bullet(\tau,\sigma) \\
&+{}^\bullet\Delta(\sigma,\tau)\Delta^\bullet(\tau,\sigma)\Delta^\bullet(\sigma,\tau){}^\bullet\Delta(\tau,\sigma)) + 2\frac{k}{T}(\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^\bullet\Delta^\bullet(\sigma,\tau){}^\bullet\Delta(\tau,\sigma) \\
&+{}^\bullet\Delta(\sigma,\tau)\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\tau){}^\bullet\Delta(\tau,\sigma)) + k^2\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^\bullet\Delta(\sigma,\tau){}^\bullet\Delta(\tau,\sigma) \\
&\left.-\Delta(\sigma,\tau)\Delta(\tau,\sigma)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma,\tau) - k{}^\bullet\Delta(\sigma,\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau,\sigma) - k{}^\bullet\Delta(\tau,\sigma)\right)\right) \\
&= \frac{\hbar^2}{2}|\mathrm{Riem}|^2\cdot I_4(T,k) \approx \frac{\hbar^2}{2}|\mathrm{Riem}|^2\left(\frac{7}{4k^2} - \frac{T}{2k}\right),\ \text{ as } T\to\infty \qquad (3.12)
\end{aligned}$$

If we contract only two of the prime indices with the two nonprime indices, we get the structure $|\mathrm{Ric}|^2$

$$\begin{aligned}
\frac{\hbar^2}{2}|\mathrm{Ric}|^2\int\!\!\int d\tau d\sigma \qquad &\left(\frac{1}{T^2}({}^\bullet\Delta(\tau){}^\bullet\Delta(\sigma){}^\bullet\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\tau) + {}^\bullet\Delta(\tau){}^\bullet\Delta^\bullet(\sigma)\Delta(\tau,\sigma)\Delta^\bullet(\sigma,\tau)\right. \\
&+{}^\bullet\Delta(\tau)\Delta(\sigma)\Delta^\bullet(\tau,\sigma){}^\bullet\Delta^\bullet(\sigma,\tau) + {}^\bullet\Delta(\tau)\Delta^\bullet(\sigma)\Delta(\tau,\sigma){}^\bullet\Delta^\bullet(\sigma,\tau) \\
&+\Delta^\bullet(\tau){}^\bullet\Delta(\sigma){}^\bullet\Delta^\bullet(\tau,\sigma)\Delta(\sigma,\tau) + \Delta^\bullet(\tau){}^\bullet\Delta^\bullet(\sigma){}^\bullet\Delta(\tau,\sigma)\Delta(\sigma,\tau) \\
&+\Delta^\bullet(\tau)\Delta(\sigma){}^\bullet\Delta^\bullet(\tau,\sigma){}^\bullet\Delta(\sigma,\tau) + \Delta^\bullet(\tau)\Delta^\bullet(\sigma){}^\bullet\Delta(\tau,\sigma){}^\bullet\Delta(\sigma,\tau) +
\end{aligned}$$

$$+{}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma)\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) + {}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau)$$

$$+{}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\sigma)\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau) + {}^{\bullet}\Delta^{\bullet}(\tau)\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)$$

$$+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) + \Delta(\tau){}^{\bullet}\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)$$

$$+\Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau) + \Delta(\tau)\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau))$$

$$+\frac{2k}{T}({}^{\bullet}\Delta(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) + {}^{\bullet}\Delta(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau)$$

$$+\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau) + \Delta^{\bullet}(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)$$

$$+{}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau) + {}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)$$

$$+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) + \Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau))$$

$$+k^2({}^{\bullet}\Delta(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau) + {}^{\bullet}\Delta(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)$$

$$+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau) + \Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau))$$

$$+\frac{2}{T}({}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\tau,\sigma)\Delta(\sigma,\tau) + {}^{\bullet}\Delta(\tau)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)$$

$$+\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau) + \Delta(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau))\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma) - k^{\bullet}\Delta(\sigma)\right)$$

$$+2k({}^{\bullet}\Delta(\tau)\Delta(\tau,\sigma)\Delta(\sigma,\tau) + \Delta(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau))\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma) - k^{\bullet}\Delta(\sigma)\right)$$

$$+\Delta(\tau,\sigma)\Delta(\sigma,\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau) - k^{\bullet}\Delta(\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma) - k^{\bullet}\Delta(\sigma)\right)$$

$$-\Delta(\sigma)\Delta(\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma,\tau) - k^{\bullet}\Delta(\sigma,\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau,\sigma) - k^{\bullet}\Delta(\tau,\sigma)\right)\bigg)$$

$$= \frac{\hbar^2}{2}|\mathrm{Ric}|^2 \cdot I_5(T,k) \approx \frac{\hbar^2}{2}|\mathrm{Ric}|^2\left(-\frac{T}{k} + \frac{3}{k^2}\right), \text{ as } T \to \infty \tag{3.13}$$

If we contract prime indices as well as nonprime indices only between each other, or in other words we contract separately z's and $\bar{z}$'s at point τ and z's and $\bar{z}$'s at σ, we get disconnected correlators only. The structure of such term is just $(\hbar R I_1)^2$. Adding up this term and last two terms from (3.10) we obtain the first order term (3.9) squared with the coefficient one-half

$$\frac{1}{2}(\hbar\rho_1)^2 = \frac{1}{2}\left(\hbar R I_1(T,k) + \frac{\hbar T}{4}R\right)^2 \approx \frac{\hbar^2}{8k^2}R^2, \text{ as } T \to \infty. \tag{3.14}$$

This term appears since we compute partition function, not the free energy, and therefore do not subtract integrals, corresponding to disconnected diagrams.

Finally the first term in the last line in (3.10) reads

$$\frac{\hbar^2 T}{4}\Delta R\int d\tau\ \Delta(\tau,\tau) = \frac{\hbar^2 T}{4}\Delta R I_3(T,k) \approx \hbar^2 \Delta R\left(-\frac{1}{2k^2}+\frac{T}{4k}\right),\ \text{ as } T\to\infty. \quad (3.15)$$

Let us now collect all the terms (3.11, 3.12, 3.13, 3.15) that contribute to ρ_2 and compute its $T\to\infty$ limit

$$\begin{aligned}\rho_2 \;&=\; (I_2(T,k)+TI_3(T,k)/4)\Delta R + (-2I_2(T,k)+I_5(T,k)/2)|\mathrm{Ric}|^2\\ &\quad +(-I_2(T,k)+I_4(T,k)/2)|\mathrm{Riem}|^2 + \frac{1}{2}(I_1(T,k)+T/4)^2R^2\\ &\approx \frac{1}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right),\ \text{ as } T\to\infty \end{aligned} \quad (3.16)$$

Now we are ready write down the full expansion of the density matrix (3.4) up to second order in $\hbar$

$$\rho = k^n\left(1+\frac{\hbar}{2k}R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \mathcal{O}((\hbar/k)^3)\right). \quad (3.17)$$

Note that this expansion is in perfect agreement with the expansion of Bergman kernel, obtained in [12].

3.7 Appendix

3.7.1 Curvatures

We follow conventions of [12]

$$\begin{aligned} R_{a\bar a b\bar b} &= \partial_b\bar\partial_{\bar b} g_{a\bar a} - g^{c\bar c}\partial_b g_{a\bar c}\bar\partial_{\bar b} g_{c\bar a},\\ R_{a\bar a} &= -g^{b\bar b}R_{a\bar a b\bar b},\\ R &= g^{a\bar a}R_{a\bar a},\\ \Delta R &= g^{a\bar a}\partial_a\bar\partial_{\bar a}R,\\ |\mathrm{Riem}|^2 &= R_{a\bar a b\bar b}R^{a\bar a b\bar b},\\ |\mathrm{Ric}|^2 &= R_{a\bar a}R^{a\bar a}. \end{aligned} \quad (3.18)$$

Let K be the Kähler potential for the metric

$$g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K.$$

In Kähler normal coordinate frame the Cristoffel symbols and all pure holomorphic derivatives of the metric vanish at the origin

$$K_{a\bar{a}b_1\ldots b_n}(x) = 0,$$

for any n. The following terms in the Taylor expansions of the Kähler potential, the metric, Rieman tensor and Ricci scalar are relevant for the present paper

$$K(x^a + z^a, \bar{x}^{\bar{a}} + \bar{z}^{\bar{a}}) = K(x) + K_{a\bar{b}}(x) z^a \bar{z}^{\bar{b}} + \frac{1}{4} K_{ab\bar{a}\bar{b}}(x) z^a z^b \bar{z}^{\bar{a}} \bar{z}^{\bar{b}} +$$

$$+ \frac{1}{36} K_{abc\bar{a}\bar{b}\bar{c}}(x) z^a z^b z^c \bar{z}^{\bar{a}} \bar{z}^{\bar{b}} \bar{z}^{\bar{c}} + \ldots,$$

$$g_{a\bar{b}}(x^a + z^a, \bar{x}^{\bar{a}} + \bar{z}^{\bar{a}}) = g_{a\bar{b}}(x) + K_{ab\bar{a}\bar{b}}(x) z^b \bar{z}^{\bar{a}} + \frac{1}{4} K_{abc\bar{a}\bar{b}\bar{c}}(x) z^b z^c \bar{z}^{\bar{a}} \bar{z}^{\bar{c}} + \ldots$$

$$R_{a\bar{a}b\bar{b}}(x^a + z^a, \bar{x}^{\bar{a}} + \bar{z}^{\bar{a}}) = K_{ab\bar{a}\bar{b}}(x) + K_{abc\bar{a}\bar{b}\bar{c}}(x) z^c \bar{z}^{\bar{c}} - g^{c\bar{c}}(x) K_{ab\bar{c}\bar{d}}(x) K_{cd\bar{a}\bar{b}}(x) z^d \bar{z}^{\bar{d}} + \ldots$$

$$R(x^a + z^a, \bar{x}^{\bar{a}} + \bar{z}^{\bar{a}}) = R(x) + (2 g^{a\bar{d}}(x) g^{d\bar{a}}(x) g^{b\bar{b}}(x) K_{ab\bar{a}\bar{b}}(x) K_{cd\bar{c}\bar{d}}(x) -$$

$$- g^{a\bar{a}}(x) g^{b\bar{b}}(x) K_{abc\bar{a}\bar{b}\bar{c}}(x) - g^{a\bar{a}}(x) g^{b\bar{b}}(x) g^{d\bar{d}}(x) K_{ab\bar{c}\bar{d}}(x) K_{cd\bar{a}\bar{b}}(x)) z^c \bar{z}^{\bar{c}} + \ldots$$

$$= R(x) + \partial_c \bar{\partial}_{\bar{c}} R(x) z^c \bar{z}^{\bar{c}}.$$

The following useful identity holds in the normal coordinate frame

$$g^{a\bar{a}} g^{b\bar{b}} K_{abc\bar{a}\bar{b}\bar{c}} = -\partial_c \bar{\partial}_{\bar{c}} R(x) + 2 R_{c\bar{d}} R^{\bar{d}}{}_{\bar{c}} + R_{c\bar{b}d\bar{d}} R^{\bar{b}d\bar{d}}{}_{\bar{c}}. \tag{3.19}$$

3.7.2 Hamiltonian

Here we rewrite the hamiltonian (3.3) in a Weyl-symmetric way. First we simplify the expression without the gauge potential

$$\begin{aligned} \hat{H} &= \frac{1}{2} g^{-1/2} \hat{p}_a \, g^{a\bar{b}} \, g \, \hat{\bar{p}}_{\bar{b}} g^{-1/2} + \frac{1}{2} g^{-1/2} \hat{\bar{p}}_{\bar{b}} \, g^{a\bar{b}} \, g \, \hat{p}_a g^{-1/2} = \frac{1}{2} (\hat{p}_a \, g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}} + \hat{\bar{p}}_{\bar{b}} \, g^{a\bar{b}} \hat{p}_a) \\ &+ \frac{\hbar^2}{4} \bar{\partial}_{\bar{b}} (g^{a\bar{b}} \partial_a \ln g) + \frac{\hbar^2}{4} \partial_a (g^{a\bar{b}} \bar{\partial}_{\bar{b}} \ln g) + \frac{\hbar^2}{4} g^{a\bar{b}} \bar{\partial}_{\bar{b}} \ln g \partial_a \ln g, \end{aligned} \tag{3.20}$$

where we use $\hat{p}_a = -i\hbar\partial_a$, $\hat{\bar{p}}_{\bar{a}} = -i\hbar\bar{\partial}_{\bar{a}}$. The Weyl-ordered form of the first term in the previous expression is

$$(\hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}})_W = \frac{1}{4}(\hat{p}_a \hat{\bar{p}}_{\bar{b}} g^{a\bar{b}} + \hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}} + \hat{\bar{p}}_{\bar{b}} g^{a\bar{b}} \hat{p}_a + g^{a\bar{b}} \hat{p}_a \hat{\bar{p}}_{\bar{b}})$$

Therefore

$$\begin{aligned} \frac{1}{2}(\hat{p}_a\, g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}} + \hat{\bar{p}}_{\bar{b}}\, g^{a\bar{b}} \hat{p}_a) &= (\hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}})_W + \frac{1}{8}([\hat{p}_a, [g^{a\bar{b}}, \hat{\bar{p}}_{\bar{b}}]] + [\hat{\bar{p}}_{\bar{b}}, [g^{a\bar{b}}, \hat{p}_a]]) \\ &= (\hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}})_W + \frac{\hbar^2}{4} R + \frac{\hbar^2}{4} g^{a\bar{b}} \Gamma^b_{ab} \Gamma^{\bar{c}}_{\bar{b}\bar{c}}. \end{aligned} \tag{3.21}$$

The last three terms in (3.20) can be written as

$$\begin{aligned} &\bar{\partial}_{\bar{b}}(g^{a\bar{b}} \partial_a \ln g) = \partial_a (g^{a\bar{b}} \bar{\partial}_{\bar{b}} \ln g) = -R - g^{a\bar{b}} \Gamma^b_{ab} \Gamma^{\bar{c}}_{\bar{b}\bar{c}}. \\ &g^{a\bar{b}} \bar{\partial}_{\bar{b}} \ln g \partial_a \ln g = g^{a\bar{b}} \Gamma^b_{ab} \Gamma^{\bar{c}}_{\bar{b}\bar{c}}. \end{aligned} \tag{3.22}$$

Using (3.21, 3.22) we get the expression for Weyl-ordered hamiltonian (3.20)

$$\hat{H} = (\hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}})_W - \frac{\hbar^2}{4} R. \tag{3.23}$$

Now it is straightforward to see that the similar expression holds in the presence of gauge connection (3.3). One just has to shift $\hat{\bar{p}}_{\bar{b}} \to \hat{\bar{p}}_{\bar{b}} - i\bar{A}_{\bar{b}}$ in the previous equation, and then order the part of $\hat{H}$ that involves $\bar{A}_{\bar{b}}$

$$\hat{H} = (\hat{p}_a g^{a\bar{b}} \hat{\bar{p}}_{\bar{b}})_W - \frac{\hbar}{2} g^{a\bar{b}} F_{a\bar{b}} - \frac{\hbar^2}{4} R. \tag{3.24}$$

For the field strength (3.2) the second term here is just a "zero-point energy" constant, which we subtract. However, it will play an important role in the next chapter, when we consider a more general magnetic field strength.

3.7.3 Integrals

Here we collect exact expressions for the integrals that appear in the main text. The following short hand notations are used

$$\begin{aligned} {}^{\bullet}\Delta(\tau,\sigma) &= d\Delta(\tau,\sigma)/d\tau, \quad \Delta^{\bullet}(\tau,\sigma) = d\Delta(\tau,\sigma)/d\sigma, \quad {}^{\bullet\bullet}\Delta(\tau,\sigma) = d^2\Delta(\tau,\sigma)/d\tau, \\ \Delta(\tau,\tau) &= \Delta(\tau), \quad {}^{\bullet}\Delta(\tau) = {}^{\bullet}\Delta(\tau,\sigma)|_{\sigma=\tau}, \end{aligned} \tag{3.25}$$

and so on.

$$
\begin{aligned}
I_1(T,k) &= \int d\tau\ \frac{1}{T}\left(\Delta({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta) + {}^{\bullet}\Delta\Delta^{\bullet}\right)|_\tau \\
&= -\frac{e^{kT}+1}{4k(e^{kT}-1)^2}\left(2+kT+e^{kT}(-2+kT)\right) \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
I_2(T,k) &= \int d\tau\ \frac{1}{T}({}^{\bullet}\Delta\Delta^{\bullet}\Delta + \Delta^2({}^{\bullet}\Delta^{\bullet} + {}^{\bullet\bullet}\Delta)/2)|_\tau = -\frac{1}{12k^2(e^{kT}-1)^3} \qquad (3.27) \\
&\left(10+3kT+3e^{kT}(6+7kT)+3e^{2kT}(-6+7kT)+e^{3kT}(-10+3kT)\right)
\end{aligned}
$$

$$
I_3(T,k) = \int d\tau\ \Delta(\tau,\tau) = \frac{1}{Tk^2(e^{kT}-1)}\left(2+kT+e^{kT}(-2+kT)\right) \qquad (3.28)
$$

$$
\begin{aligned}
I_4(T,k) &= \int\int d\tau d\sigma\ \Bigg(\frac{1}{T^2}(\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta^{\bullet}(\tau,\sigma) \\
&+\Delta(\sigma,\tau)\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta(\tau,\sigma) + {}^{\bullet}\Delta(\sigma,\tau)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta^{\bullet}(\tau,\sigma) \\
&+{}^{\bullet}\Delta(\sigma,\tau)\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta(\tau,\sigma)) + 2\frac{k}{T}(\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta(\tau,\sigma) \\
&+{}^{\bullet}\Delta(\sigma,\tau)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau){}^{\bullet}\Delta(\tau,\sigma)) + k^2\Delta(\sigma,\tau)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau){}^{\bullet}\Delta(\tau,\sigma) \\
&-\Delta(\sigma,\tau)\Delta(\tau,\sigma)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma,\tau) - k{}^{\bullet}\Delta(\sigma,\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau,\sigma) - k{}^{\bullet}\Delta(\tau,\sigma)\right)\Bigg) \\
&= \frac{1}{4k^2(e^{kT}-1)^4}\Big(7+2kT+12kTe^{kT}+2e^{2kT}(-7+2k^2T^2) \\
&-12kTe^{3kT}+e^{4kT}(7-2kT)\Big) \qquad (3.29)
\end{aligned}
$$

$$
\begin{aligned}
I_5(T,k) &= \int\int d\tau d\sigma\ \Bigg(\frac{1}{T^2}({}^{\bullet}\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) + {}^{\bullet}\Delta(\tau){}^{\bullet}\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) \\
&+{}^{\bullet}\Delta(\tau)\Delta(\sigma)\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau) + {}^{\bullet}\Delta(\tau)\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau) \\
&+\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) + \Delta^{\bullet}(\tau){}^{\bullet}\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau) \\
&+\Delta^{\bullet}(\tau)\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau) + \Delta^{\bullet}(\tau)\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau) \\
&+{}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma)\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) + {}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau)
\end{aligned}
$$

$$
\begin{aligned}
&+{}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\sigma)\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)+{}^{\bullet}\Delta^{\bullet}(\tau)\Delta^{\bullet}(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)\\
&+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)+\Delta(\tau){}^{\bullet}\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)\\
&+\Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta^{\bullet}(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau)+\Delta(\tau)\Delta^{\bullet}(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau))\\
&+\frac{2k}{T}({}^{\bullet}\Delta(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)+{}^{\bullet}\Delta(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau)\\
&+\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau)+\Delta^{\bullet}(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)\\
&+{}^{\bullet}\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau)+{}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)\\
&+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)+\Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta^{\bullet}(\sigma,\tau))\\
&+k^2({}^{\bullet}\Delta(\tau){}^{\bullet}\Delta(\sigma)\Delta(\tau,\sigma)\Delta(\sigma,\tau)+{}^{\bullet}\Delta(\tau)\Delta(\sigma)\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau)\\
&+\Delta(\tau){}^{\bullet}\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau)+\Delta(\tau)\Delta(\sigma){}^{\bullet}\Delta(\tau,\sigma){}^{\bullet}\Delta(\sigma,\tau))\\
&+\frac{2}{T}({}^{\bullet}\Delta^{\bullet}(\tau)\Delta(\tau,\sigma)\Delta(\sigma,\tau)+{}^{\bullet}\Delta(\tau)\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)\\
&+\Delta^{\bullet}(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau)+\Delta(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma)-k^{\bullet}\Delta(\sigma)\right)\\
&+2k({}^{\bullet}\Delta(\tau)\Delta(\tau,\sigma)\Delta(\sigma,\tau)+\Delta(\tau){}^{\bullet}\Delta(\tau,\sigma)\Delta(\sigma,\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma)-k^{\bullet}\Delta(\sigma)\right)\\
&+\Delta(\tau,\sigma)\Delta(\sigma,\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau)-k^{\bullet}\Delta(\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma)-k^{\bullet}\Delta(\sigma)\right)\\
&-\Delta(\sigma)\Delta(\tau)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\sigma,\tau)-k^{\bullet}\Delta(\sigma,\tau)\right)\left(\frac{1}{T}{}^{\bullet\bullet}\Delta(\tau,\sigma)-k^{\bullet}\Delta(\tau,\sigma)\right)\Bigg)=\\
=\;&\frac{1}{k^2(e^{kT}-1)^4}\Big(3+kT+e^{kT}(4+8kT+k^2T^2)+2e^{2kT}(-7+k^2T^2)+\\
&+e^{3kT}(4-8kT+k^2T^2)+e^{4kT}(3-kT)\Big) \qquad (3.30)
\end{aligned}
$$

Chapter 4

Generalizations of the Bergman kernel

4.1 $\mathcal{N} = 1$ supersymmetric quantum mechanics

4.1.1 Action, symmetries and propagators

One can obtain expansions similar to (3.17) in other quantum mechanical theories. Here we consider $(1, 1)$-supersymmetric particle on Kähler manifold with the magnetic field turned on. The action is

$$S = \int_{t_i}^{t_f} dt \left(g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} + \bar{\psi}^{\bar{a}} (g_{a\bar{a}} \dot{\psi}^a + \dot{x}^b \partial_b g_{a\bar{a}} \psi^a) + \bar{A}_{\bar{b}} \dot{\bar{x}}^{\bar{b}} + F_{a\bar{a}} \bar{\psi}^{\bar{a}} \psi^a \right) \tag{4.1}$$

This action is invariant under the following $\mathcal{N} = (1, 1)$ supersymmetry transformations

$$\begin{aligned} \delta x^a &= -\bar{\epsilon} \psi^a \\ \delta \bar{x}^{\bar{a}} &= -\epsilon \bar{\psi}^{\bar{a}} \\ \delta \psi^a &= \dot{x}^a \epsilon \\ \delta \bar{\psi}^{\bar{a}} &= \dot{\bar{x}}^{\bar{a}} \bar{\epsilon} \end{aligned} \tag{4.2}$$

if the metric is Kähler and if A_a, $\bar{A}_{\bar{a}}$ is a connection of holomorphic vector bundle

$$F_{ab} = F_{\bar{a}\bar{b}} = 0.$$

We repeat the previous setup with the field strength proportional to the metric, exactly as in Eq. (2.20). Consider now the path integral representation of this theory. If the boundary conditions for x and ψ fields are the same, no ghosts are needed in the action, because bosonic and fermionic determinants cancel in the measure. Moreover, the

Weyl-ordering counterterm cancels due to the contribution of fermionic terms. Bosonic propagator is the same a before, and fermionic propagator

$$\langle\bar{\psi}^{\bar{b}}(\tau)\psi^{a}(\sigma)\rangle = \hbar g^{a\bar{b}}\Gamma(\tau,\sigma)$$

satisfies

$$\left[\frac{d}{d\sigma} + Tk\right]\Gamma(\tau,\sigma) = -\delta(\tau-\sigma)$$

We would like to compute the "index density", i.e. the trace of the density matrix with alternating signs for fermionic states

$$\rho(x) = \lim_{T\to\infty} \mathrm{Tr}\ (-1)^F e^{-T\hat{H}},$$

without performing the x-integral. The right hand side here depends only on the bosonic "zero-mode" x, and all fermionic dependence is integrated out. Fermion number insertion $(-1)^F$ corresponds to periodic boundary conditions for fermions. In this case the propagator has the form

$$\Gamma(\tau,\sigma) = \frac{1}{1-e^{kT}}\left(e^{kT(\tau-\sigma)}\theta(\tau-\sigma) + e^{kT(\tau-\sigma+1)}\theta(\sigma-\tau)\right).$$

4.1.2 Perturbation theory

The calculation proceeds along the same lines as in nonsupersymmetric case. We use Kähler normal coordinates and expand the metric around the constant configuration x.

Free part of the action is given by

$$S_0 = \int_{-1}^{0} d\tau\ \left[\frac{1}{T}g_{a\bar{b}}(x)\dot{z}^a\dot{\bar{z}}^{\bar{b}} + kg_{a\bar{b}}(x)z^a\dot{\bar{z}}^{\bar{b}} + g_{a\bar{b}}(x)\bar{\psi}^{\bar{b}}\dot{\psi}^a + Tkg_{a\bar{b}}\bar{\psi}^{\bar{b}}\psi^a\right]. \tag{4.3}$$

The interaction part, up to the sixth order in derivatives of the Kähler potential, reads

$$\begin{aligned} S_{int} \;=\; & \int_{-1}^{0} d\tau\ \left[\frac{1}{T}\left(K_{ab\bar{a}\bar{b}}(x)z^b\bar{z}^{\bar{a}} + \frac{1}{4}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\right)\dot{z}^a\dot{\bar{z}}^{\bar{b}}\right. \\ & + k\left(\frac{1}{2}K_{ab\bar{a}\bar{b}}(x)z^az^b\bar{z}^{\bar{a}} + \frac{1}{12}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^az^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\right)\dot{\bar{z}}^{\bar{b}} \\ & + \left(K_{ab\bar{a}\bar{b}}(x)z^b\bar{z}^{\bar{a}} + \frac{1}{4}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^bz^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\right)\bar{\psi}^{\bar{b}}(Tk+\partial_\tau)\psi^a \\ & \left. + \left(K_{ab\bar{a}\bar{b}}(x)\bar{z}^{\bar{a}} + \frac{1}{2}K_{abc\bar{a}\bar{b}\bar{c}}(x)z^c\bar{z}^{\bar{a}}\bar{z}^{\bar{c}}\right)\dot{z}^b\bar{\psi}^{\bar{b}}\psi^a\right], \end{aligned} \tag{4.4}$$

At the first order in $\hbar$ we get

$$\rho_1(\mathcal{N}=1) = R\int d\tau \, \left(\frac{1}{T}({}^\bullet\Delta\Delta^\bullet + {}^\bullet\Delta^\bullet\Delta) + k^\bullet\Delta\Delta - \Delta\delta(0) + \Delta^\bullet\Gamma\right)|_\tau = 0,$$

so $\hbar^1$ term is exactly zero, even for finite T.

Computation at the second order in $\hbar$ proceeds in a similar fashion as in previous section, let us only mention one shortcut. Note, that each contraction of $\bar{\psi}(\sigma)$ and $(Tk + \partial_\tau)\psi(\tau)$ is proportional to delta-function $\delta(\tau,\sigma)$, exactly as contraction of ghosts b and c. Therefore the first three lines in the interaction lagrangian (4.4) generate the same terms as bosonic interaction lagrangian (3.5) and only the last line in (4.4) is a new one. With this observation the calculation simplifies significantly. We only give the final answer here

$$\begin{aligned} \rho_2(\mathcal{N}=1) &= -(I_2(T,k) + I_6(T,k))(-\Delta R + 2|\mathrm{Ric}|^2 + |\mathrm{Riem}|^2) \\ &\quad + (I_5(T,k)/2 + I_7(T,k) + I_8/2)|\mathrm{Ric}|^2 \\ &\quad + (I_4(T,k)/2 + I_6(T,k) - I_9(T,k))|\mathrm{Riem}|^2 \end{aligned} \tag{4.5}$$

and refer to Appendix for the values of the integrals here. The coefficients in front of $|\mathrm{Ric}|^2$ and $|\mathrm{Riem}|^2$ turn out to be T-independent, as a consequence of supersymmetry, and the answer for the density matrix up to the second order in $\hbar$ is

$$\rho(x)(\mathcal{N}=1) = k^n\left(1 + \frac{\hbar^2}{24k^2}\left(2\Delta R - |\mathrm{Ric}|^2 + |\mathrm{Riem}|^2\right) + \mathcal{O}(\hbar^3)\right)$$

This is consistent with the index theorem [5, 6]. According to the latter the x-integral of $\rho(x)(\mathcal{N}=1)$ is equal to the index of Dirac operator on the Kähler manifold M for which the exact answer is

$$\int_M dx\rho(x)(\mathcal{N}=1) = \mathrm{ind}D_A = \int_M \mathrm{ch}F \wedge \hat{A}(M).$$

If we plug $F = kg_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}$ and expand the A-roof genus $\hat{A}$ in powers of curvature tensors then the first two terms in this expression coincide with first two terms in the integrated density $\int \rho(x)(\mathcal{N}=1)$.

4.2 $N = 2$ supersymmetric quantum mechanics

4.2.1 Action, symmetries and propagators

The action is

$$\begin{aligned} S = \quad & \int_{t_i}^{t_f} dt \, \Big(g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} + \bar{\psi}_+^{\bar{a}} (g_{a\bar{a}} \dot{\psi}_+^a + \dot{x}^b \partial_b g_{a\bar{a}} \psi_+^a) \\ & + \bar{\psi}_-^{\bar{a}} (g_{a\bar{a}} \dot{\psi}_-^a + \dot{\bar{x}}^b \partial_b g_{a\bar{a}} \psi_-^a) + \bar{A}_{\bar{b}} \dot{\bar{x}}^{\bar{b}} + F_{a\bar{a}} (\bar{\psi}_+^{\bar{a}} \psi_+^a + \bar{\psi}_-^{\bar{a}} \psi_-^a) \Big) \, . \end{aligned} \tag{4.6}$$

The $\mathcal{N} = (2, 2)$ supersymmetry transformations

$$\begin{aligned} \delta x^a &= -\bar{\epsilon}_+ \psi_+^a - \bar{\epsilon}_- \psi_-^a \\ \delta \bar{x}^{\bar{a}} &= -\epsilon_+ \bar{\psi}_+^{\bar{a}} - \epsilon_- \bar{\psi}_-^{\bar{a}} \\ \delta \psi_+^a &= \dot{x}^a \epsilon_+ + \bar{\epsilon}_- \Gamma^a_{bc} \psi_-^b \psi_+^c \\ \delta \bar{\psi}_+^{\bar{a}} &= \dot{\bar{x}}^{\bar{a}} \bar{\epsilon}_+ + \epsilon_- \Gamma^{\bar{a}}_{\bar{b}\bar{c}} \bar{\psi}_-^{\bar{b}} \bar{\psi}_+^{\bar{c}} \\ \delta \psi_-^a &= \dot{x}^a \epsilon_- + \bar{\epsilon}_+ \Gamma^a_{bc} \psi_+^b \psi_-^c \\ \delta \bar{\psi}_-^{\bar{a}} &= \dot{\bar{x}}^{\bar{a}} \bar{\epsilon}_- + \epsilon_+ \Gamma^{\bar{a}}_{\bar{b}\bar{c}} \bar{\psi}_+^{\bar{b}} \bar{\psi}_-^{\bar{c}} \end{aligned} \tag{4.7}$$

leave the action invariant, if A is a holomorphic connection and also if the hermitian Yang-Mills equation is obeyed

$$g^{a\bar{b}} D_a F_{b\bar{b}} = 0,$$

which was not required in the previous case of one supersymmetry. We assume the field strength $F_{a\bar{b}} = k g_{a\bar{b}}$ (2.20), which satisfies this equation.

The object that we would like to compute is the Dolbeault index density, which corresponds to taking the supertrace over one species of fermions, and setting the zero modes of the second species of fermions to zero. To achieve this, we choose the following propagators for the fermions

$$\begin{aligned} \langle \bar{\psi}_+^{\bar{b}}(\tau) \psi_+^a(\sigma) \rangle &= \hbar g^{a\bar{b}} \Gamma_+(\tau, \sigma) \\ \langle \bar{\psi}_-^{\bar{b}}(\tau) \psi_-^a(\sigma) \rangle &= \hbar g^{a\bar{b}} \Gamma_-(\tau, \sigma), \end{aligned}$$

where Γ_+ satisfies periodic boundary conditions: $\Gamma_+(-1,\sigma) = \Gamma_+(0,\sigma)$, $\Gamma_+(\tau,-1) = \Gamma_+(\tau,0)$, and Γ_- satisfies Dirichlet b.c. $\Gamma_-(-1,\sigma) = \Gamma_-(\tau,0) = 0$. These propagators are given by

$$\begin{aligned}
\Gamma_+(\tau,\sigma) &= \frac{1}{1-e^{kT}}\left(e^{kT(\tau-\sigma)}\theta(\tau-\sigma) + e^{kT(\tau-\sigma+1)}\theta(\sigma-\tau)\right) \\
\Gamma_-(\tau,\sigma) &= e^{kT(\tau-\sigma)}\theta(\tau-\sigma),
\end{aligned}$$

One also has to add a pair of bosonic ghost fields $a^a, \bar{a}^{\bar{a}}$, coming from the path integral measure

$$S_{gh} = \int d\tau g_{a\bar{a}} a^a \bar{a}^{\bar{a}}.$$

4.2.2 Perturbation theory

The calculation proceeds along the same lines as in the previous two sections. Here we present the final answer for the index density

$$\begin{aligned}
\rho(x) &= k^n\left(1 + \hbar I_{13}(T,k)R + \hbar^2(I_{14}(T,k)\Delta R\right. \\
&\quad +(I_5(T,k)/2 + I_{11}(T,k) + I_{12}(T,k))|\mathrm{Ric}|^2 \\
&\quad \left. + (-I_2(T,k) + I_4(T,k)/2 + I_{10}(T,k))|\mathrm{Riem}|^2 + I_{13}^2(T,k)R^2/2) + \mathcal{O}(\hbar^3)\right) \\
&= k^n\left(1 + \frac{\hbar}{2k}R + \hbar^2(I_{14}(T,k)\Delta R - \frac{1}{6k^2}|\mathrm{Ric}|^2\right. \\
&\quad \left. + \frac{1}{24k^2}|\mathrm{Riem}|^2 + \frac{1}{8k^2}R^2) + \mathcal{O}(\hbar^3)\right) \\
&\approx k^n\left(1 + \frac{\hbar}{2k}R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right)\right. \\
&\quad \left. + \mathcal{O}((\hbar/k)^3)\right), \quad \text{as } T\to\infty. \qquad (4.8)
\end{aligned}$$

Notice, that the only term that depends on T here, is a total derivative. Therefore it is irrelevant for the index theorem [5, 6], that states

$$\int_M dx\rho(x)(\mathcal{N}=2) = \mathrm{ind}\ \bar{\partial}_A = \int_M \mathrm{ch}F \wedge \mathrm{Td}(M).$$

This index formula computes $\dim \sum_q (-1)^q H^{0,q}(M,L^k)$, which is equal to $\dim H^0(M,L^k)$ for large enough k. Therefore the number of holomorphic sections $\dim H^0(M,L^k)$ can be found for large k to be equal to

$$\dim\, H^0(M, L^k) = \int_M e^F \wedge \mathrm{Td}(M) = a_0 k^n + a_1 k^{n-1} + \ldots \tag{4.9}$$

and the coefficients a_i here are the integrals of the corresponding terms in the Tian-Yau-Zelditch expansion Eq. (2.30). This explains why $\mathcal{N} = 2$ and nonsupersymmetric Bergman kernel expansions coincide.

4.3 General magnetic field

The large field expansion of the path integral can be generalized [40] for the most general choice Eq. (2.21) of the $U(1)$ magnetic field

$$F_{a\bar{b}} = k g_{a\bar{b}} + u_{a\bar{b}}.$$

where $u_{a\bar{b}}$ is a $(1,1)$-form in $H^{1,1}(M)$, not necessarily in the same cohomology class as the Kähler form ω_g of the metric $g_{a\bar{b}}$. Mathematically, this setup corresponds to the tensor product of two line bundles $L^k \otimes \mathcal{E}$, with u being the Ricci curvature of $\mathcal{E}$. The Bergman kernel and balanced embeddings in this case and for more general vector bundles have been considered in [48].

Let $B_{\bar{b}}$ be the vector potential in anti-holomorphic gauge, with the field strength $F_{a\bar{b}} = \partial_a B_{\bar{b}}$. As before, the following expansion holds at the reference point in the Kähler normal coordinate frame

$$B_{\bar{b}} = F_{a\bar{b}} z^a + \frac{1}{2} F_{ab\bar{a}\bar{b}} z^a z^b \bar{z}^{\bar{a}} + \ldots$$

The quadratic part of free action (3.4) is now changed to

$$S_0 = \int_{-1}^{0} d\tau \left[\frac{1}{T} g_{a\bar{b}}(x) \dot{z}^a \dot{\bar{z}}^{\bar{b}} + F_{a\bar{b}}(x) z^a \dot{\bar{z}}^{\bar{b}} + g_{a\bar{b}}(x) b^a c^{\bar{b}} \right],$$

so that the equation for the propagator now contains two non-collinear matrices

$$\left[-\frac{1}{T} g_{a\bar{b}} \frac{d^2}{d\tau^2} + F_{a\bar{b}} \frac{d}{d\tau} \right] \Delta^{\bar{b}b}(\tau, \sigma) = \delta(\tau - \sigma) \delta_a{}^b. \tag{4.10}$$

The matrix-valued solution to this equation in the real case has been found in [65]. In the Kähler case the solution can be obtained by formally replacing $k \to F$ in (3.6)

$$\Delta^{\bar{b}b}(\tau,\sigma) = \left(\frac{1}{F(e^{FT}-1)}\left[\theta(\tau-\sigma)e^{FT}(1-e^{FT\tau})(1-e^{-FT(\sigma+1)})+\right.\right.$$
$$\left.\left.+\theta(\sigma-\tau)(1-e^{-FT\sigma})(1-e^{FT(\tau+1)})\right]\right)^{\bar{b}b}, \qquad (4.11)$$

where the expression on the rhs should be understood as a formal Taylor series in F and the product of two matrices is defined as a contraction by the metric, i.e. $(F^2)_{a\bar{a}} = F_{a\bar{b}}g^{b\bar{b}}F_{b\bar{a}}$ (we can always use the freedom of choosing $g_{a\bar{a}}(x)$ to be an identity matrix at the reference point).

The path integral (3.1) contains an overall normalization factor, equal to the determinant of the operator (4.10). Since we are interested in large k asymptotic expansion, the determinant can be computed by summing up the perturbation series for

$$e^{-\frac{1}{\hbar}\int_{-1}^{0} u_{a\bar{a}}z^a\dot{\bar{z}}^{\bar{a}}d\tau}$$

with the help of the old scalar-valued propagator (3.6). It is enough to compute connected diagrams only. Expanding the exponent into Taylor series and taking into account the following asymptotic of the integrals as $T \to \infty$

$$u_{a\bar{b}}\int_{-1}^{0} d\tau\langle z^a\dot{\bar{z}}^{\bar{b}}\rangle_c = \hbar\left(\frac{kT}{2}-1\right)\mathrm{Tr}\, u$$

and for $n > 1$

$$u_{a_1\bar{b}_1}\ldots u_{a_n\bar{b}_n}\int_{-1}^{0} d\tau_1\ldots\int_{-1}^{0} d\tau_n\langle z^{a_1}(\tau_1)\dot{\bar{z}}^{\bar{b}_1}(\tau_1)\ldots z^{a_n}(\tau_n)\dot{\bar{z}}^{\bar{b}_n}(\tau_n)\rangle_c = -\hbar^n(n-1)!\mathrm{Tr}\, u^n,$$

where $\mathrm{Tr}\, u^n = u_{a_1\bar{b}_1}g^{\bar{b}_1a_2}\ldots u_{a_n\bar{b}_n}g^{\bar{b}_na_1}$, we arrive at the following answer

$$\begin{aligned}\langle e^{-\frac{1}{\hbar}\int_{-1}^{0} u_{a\bar{a}}z^a\dot{\bar{z}}^{\bar{a}}d\tau}\rangle = \exp\langle e^{-\frac{1}{\hbar}\int_{-1}^{0} u_{a\bar{a}}z^a\dot{\bar{z}}^{\bar{a}}d\tau}\rangle_c &= \exp\left[\mathrm{Tr}\log(1+u) - \frac{kT}{2}\mathrm{Tr}\, u\right] = \\ &= \frac{\det F}{\det g}\exp\left(-\frac{kT}{2}\mathrm{Tr}\, u\right). \qquad (4.12)\end{aligned}$$

Note, that the expression in the exponent cancels with the magnetic field counterterm of (3.24), up to a “zero-point energy” constant, which can be omitted.

Now we would like to compute large k perturbative expansion to the first order in $\hbar$. The relevant interaction part of the action is

$$
\begin{aligned}
S_{int} \quad = \quad & \int_{-1}^{0} d\tau \left[\frac{1}{T} K_{ab\bar{a}\bar{b}}(x) z^b \bar{z}^{\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} + \frac{1}{2} F_{ab\bar{a}\bar{b}}(x) z^a z^b \bar{z}^{\bar{a}} \dot{\bar{z}}^{\bar{b}} + K_{ab\bar{a}\bar{b}}(x) z^b \bar{z}^{\bar{a}} b^a c^{\bar{b}} \right. \\
& \left. - \frac{\hbar}{2} T (g^{a\bar{a}}(x) F_{a\bar{a}b\bar{b}}(x) - F^{a\bar{a}}(x) K_{a\bar{a}b\bar{b}}(x)) z^b \bar{z}^{\bar{b}} - \frac{\hbar^2}{4} T R(x) \right],
\end{aligned}
$$

where the fourth term comes from the expansion of the second term in (3.24). In the first order in $\hbar$ we get the following correlator

$$
\begin{aligned}
\hbar\rho_1 \quad = \quad & -\frac{1}{\hbar} \int_{-1}^{0} d\tau \left[K_{ab\bar{a}\bar{b}} \left(\frac{1}{T} \langle z^b \bar{z}^{\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{b}} \rangle|_\tau + \langle z^b \bar{z}^{\bar{a}} b^a c^{\bar{b}} \rangle|_c \right) + \frac{1}{2} F_{a\bar{a}b\bar{b}} \langle z^a z^b \bar{z}^{\bar{a}} \dot{\bar{z}}^{\bar{b}} \rangle|_c - \right. \\
& \left. -\frac{\hbar}{2} T (g^{a\bar{a}} F_{a\bar{a}b\bar{b}} - F^{a\bar{a}} K_{a\bar{a}b\bar{b}}) \langle z^b \bar{z}^{\bar{b}} \rangle|_c \right] + \frac{\hbar}{4} T R. \qquad (4.13)
\end{aligned}
$$

The first two correlators in this expression can be rewritten as

$$
\begin{aligned}
& -\hbar K_{ab\bar{a}\bar{b}} \int_{-1}^{0} d\tau \frac{1}{T} ({}^\bullet\Delta^{a\bar{a}} \Delta^{\bullet b\bar{b}} + \Delta^{b\bar{a}} ({}^\bullet\Delta^{\bullet a\bar{b}} - T g^{a\bar{a}} \delta(0))) = -\frac{\hbar}{T} K_{ab\bar{a}\bar{b}} \int_{-1}^{0} d\tau \\
& \left[\left(-\frac{T}{2(-1+e^{FT})} \left(2e^{F(1+\tau)T} - (1+e^{FT}) \right) \right)^{a\bar{a}} \cdot \right. \\
& \cdot \left(\frac{T}{2(-1+e^{FT})} \left(2e^{-F\tau T} - (1+e^{FT}) \right) \right)^{b\bar{b}} + \qquad (4.14) \\
& \left. \left(\frac{1}{F(-1+e^{FT})} \left(1 - e^{-F\tau T} - e^{F(\tau+1)T} + e^{FT} \right) \right)^{a\bar{a}} \left(-\frac{FT^2}{2(-1+e^{FT})} (1+e^{FT}) \right)^{b\bar{b}} \right].
\end{aligned}
$$

The integral here takes values in a "tensor product" of two matrices with indices $a\bar{a}$ and $b\bar{b}$ respectively. It can be computed by standard methods in the infinite T limit, under the assumption that F is positively valued. This is a natural assumption from the point of view of the large k limit. The $T = \infty$ answer for the integral above is

$$
\hbar K_{ab\bar{a}\bar{b}} \left(g^{a\bar{a}} (F^{-1})^{b\bar{b}} - F^{a\bar{a}} (F^{-2})^{b\bar{b}} + \frac{T}{2} (F^{-1})^{a\bar{a}} F^{b\bar{b}} + \frac{T}{4} g^{a\bar{a}} g^{b\bar{b}} \right), \qquad (4.15)
$$

where F^{-1} is the inverse matrix to F, defined as $F_{a\bar{b}} g^{b\bar{b}} (F^{-1})_{b\bar{a}} = g_{a\bar{a}}$, or equivalently as a Taylor expansion around $g_{a\bar{a}}$

$$
(F^{-1})_{a\bar{a}} = \frac{1}{k} g_{a\bar{a}} - \frac{1}{k^2} g_{a\bar{b}} u^{b\bar{b}} g_{b\bar{a}} + \dots
$$

The term in front of $F_{a\bar{a}b\bar{b}}$ in Eq. (4.13) can be computed in the infinite time limit

$$\begin{aligned}
- \quad & \hbar F_{ab\bar{a}\bar{b}} \int_{-1}^{0} d\tau \Delta^{a\bar{a}\bullet}\Delta^{b\bar{b}} = \\
& = -\hbar F_{ab\bar{a}\bar{b}} \int_{-1}^{0} d\tau \frac{1}{F(-1+e^{FT})}(1+e^{FT}-e^{-F\tau T}-e^{F(\tau+1)T})^{a\bar{a}} \\
& \cdot \left(-\frac{T}{2(-1+e^{FT})}(2e^{FT(\tau+1)}-e^{FT}-1)\right)^{b\bar{b}} \approx \hbar F_{ab\bar{a}\bar{b}} \left((F^{-1})^{a\bar{a}}(F^{-1})^{b\bar{b}}\right. \\
& \left. +(F^{-2})^{a\bar{a}}g^{b\bar{b}} - \frac{T}{2}(F^{-1})^{a\bar{a}}g^{b\bar{b}} - [(F\otimes F+F^2\otimes g)^{-1}]^{a\bar{a}b\bar{b}}\right). \qquad (4.16)
\end{aligned}$$

where the last term is understood as an inverse of the tensor product of matrices.

The third term in Eq. (4.13) in the $T\to\infty$ limit is equal to

$$\begin{aligned}
& \hbar\frac{T}{2}(g^{a\bar{a}}F_{a\bar{a}b\bar{b}} - F^{a\bar{a}}K_{a\bar{a}b\bar{b}}) \int_{-1}^{0} d\tau \Delta^{b\bar{b}} \\
& \approx \hbar\frac{T}{2}(g^{a\bar{a}}F_{a\bar{a}b\bar{b}} - F^{a\bar{a}}K_{a\bar{a}b\bar{b}}) \left((F^{-1})^{b\bar{b}} - \frac{2}{T}(F^{-2})^{b\bar{b}}\right). \qquad (4.17)
\end{aligned}$$

Combining (4.15, 4.16) and (4.17) together with the contribution of the last term in (4.13) and the overall normalization (4.12) we get the following answer for the diagonal of the density matrix up to first order in $\hbar$

$$\begin{aligned}
\rho \quad = \quad & \frac{\det F}{\det g}\Big(1+\hbar F_{a\bar{a}b\bar{b}}((F^{-1})^{a\bar{a}}(F^{-1})^{b\bar{b}} - [(F\otimes F+F^2\otimes g)^{-1}]^{a\bar{a}b\bar{b}}) \\
& -\hbar(F^{-1})^{a\bar{a}}R_{a\bar{a}} + \mathcal{O}(\hbar^2)\Big).
\end{aligned}$$

As in the previous cases, all terms linear in T cancel[1]. One can immediately check that the standard TYZ-expansion is reproduced, by taking $F=\alpha g$, with ρ then reducing to the first two terms in Eq. (2.30).

We can extract the first term in $1/k$ expansion from the previous expression

$$\rho_k = k^n\left(1+\frac{\hbar}{2k}R+\frac{\hbar}{k}g^{a\bar{a}}u_{a\bar{a}}+\mathcal{O}(\frac{1}{k^2})\right). \qquad (4.18)$$

This expansion coincides with the expansion of the Bergman kernel for $L^k\otimes\mathcal{E}$, obtained by X. Wang [48], using the peak section method. As well as for the regular Bergman

[1]Note, that without the counterterms (3.24) the T-linear terms combine into the sum of $F^{-1}\cdot\Delta F + R/4$, where ΔF is the Hodge laplacian on $(1,1)$-forms.

kernel, the solution of the constant density matrix condition

$$\rho_k = \frac{\mathrm{Tr}\rho_k}{VolM}$$

defines a balanced hermitian metric $u_{a\bar{a}}(k)$ on the vector bundle $\mathcal{E}$. For $k \to \infty$ the metric $u(\infty)$ satisfies the hermitian Einstein equation

$$\frac{1}{2}R + g^{a\bar{a}}u_{a\bar{a}} = \mathrm{const} \tag{4.19}$$

as can be seen from the first term in the expansion.

4.4 Appendix

Here we collect the integrals, that appear in §4.1 and §4.2. In addition to previously introduced notations (3.25) we also use $\Gamma(\tau) = \Gamma(\tau,\tau)$

$$I_6(T,k) = \int d\tau\ \Gamma\Delta\Delta^{\bullet}|_{\tau} = \frac{1+e^{kT}}{4k^2(-1+e^{kT})^3}(3+kT+4kTe^{kT}+e^{2kT}(-3+kT)) \tag{4.20}$$

$$\begin{aligned} I_7(T,k) &= \iint d\tau d\sigma\ (\Gamma(\sigma)\,(({}^{\bullet}\Delta^{\bullet}(\tau)\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) + \Delta^{\bullet}(\tau){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) \\ &\quad + {}^{\bullet}\Delta(\tau)\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau) + \Delta(\tau){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau))/T \\ &\quad + k({}^{\bullet}\Delta(\tau)\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau) + \Delta(\tau){}^{\bullet}\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau)) \\ &\quad - \delta(0)\Delta^{\bullet}(\tau,\sigma)\Delta(\sigma,\tau)) + \delta(\tau,\sigma)\Delta^{\bullet}(\sigma)\Delta(\tau)\Gamma(\tau,\sigma)) \\ &= \frac{1+e^{kT}}{4k^2(-1+e^{kT})^4}(-5-2kT+e^{kT}(5-8kT-2k^2T^2) \\ &\quad + e^{2kT}(5+8kT-2k^2T^2)+e^{3kT}(-5+2kT)) \end{aligned} \tag{4.21}$$

$$\begin{aligned} I_8(T,k) &= \iint d\tau d\sigma\ (\Delta^{\bullet}(\tau,\sigma)\Delta^{\bullet}(\sigma,\tau)\Gamma(\tau)\Gamma(\sigma) - \Delta^{\bullet}(\tau)\Delta^{\bullet}(\sigma)\Gamma(\tau,\sigma)\Gamma(\sigma,\tau)) \\ &= \frac{1}{4k^2(-1+e^{kT})^3}(1+e^{kT}(5+4kT)+e^{2kT}(-5+4kT)-e^{3kT}) \end{aligned} \tag{4.22}$$

$$\begin{aligned}
I_9(T,k) &= \int\!\!\int d\tau d\sigma\ \Delta^\bullet(\tau,\sigma)\Delta^\bullet(\sigma,\tau)\Gamma(\tau,\sigma)\Gamma(\sigma,\tau) \\
&= \frac{e^{kT}}{k^2(-1+e^{kT})^4}(-(-1+e^{kT})^2+k^2T^2e^{kT}) \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
I_{10}(T,k) &= \int\!\!\int d\tau d\sigma\ \Big(-\frac{1}{2}\Delta^\bullet(\tau,\sigma)\Delta^\bullet(\sigma,\tau)(\Gamma_+(\tau,\sigma)\Gamma_+(\sigma,\tau)+\Gamma_-(\tau,\sigma)\Gamma_-(\sigma,\tau)) \\
&+ \frac{T^2}{2}\Gamma_+(\tau,\sigma)\Gamma_+(\sigma,\tau)\Gamma_-(\tau,\sigma)\Gamma_-(\sigma,\tau)\Big) \\
&= \frac{e^{kT}}{2(-1+e^{kT})^4k^2}((-1+e^{kT})^2-e^{kT}k^2T^2) \qquad (4.24)
\end{aligned}$$

$$\begin{aligned}
I_{11}(T,k) &= \int\!\!\int d\tau d\sigma\ \ (\Delta^\bullet(\tau,\sigma)\Delta^\bullet(\sigma,\tau)(\Gamma_+(\tau)+\Gamma_-(\tau))(\Gamma_+(\sigma)+\Gamma_-(\sigma)) \\
&- \Delta^\bullet(\tau)\Delta^\bullet(\sigma)(\Gamma_+(\tau,\sigma)\Gamma_+(\sigma,\tau)+\Gamma_-(\tau,\sigma)\Gamma_-(\sigma,\tau)) \\
&- T^2(\Gamma_+(\tau)\Gamma_+(\sigma)\Gamma_-(\tau,\sigma)\Gamma_-(\sigma,\tau)+\Gamma_-(\tau)\Gamma_-(\sigma)\Gamma_+(\tau,\sigma)\Gamma_+(\sigma,\tau)) \\
&+ \frac{2}{T}(\Gamma_+(\sigma)+\Gamma_-(\sigma))(\Delta^\bullet(\tau,\sigma)\Delta^\bullet(\sigma,\tau){}^\bullet\Delta(\tau) \\
&+ ({}^\bullet\Delta^\bullet(\tau)-T\delta(0))\Delta^\bullet(\tau,\sigma)\Delta(\sigma,\tau) \\
&+ ({}^\bullet\Delta^\bullet(\tau,\sigma)-T\delta(\tau-\sigma))\Delta^\bullet(\sigma,\tau)\Delta(\tau) \\
&+ ({}^\bullet\Delta^\bullet(\tau,\sigma)-T\delta(\tau-\sigma))\Delta(\sigma,\tau)\Delta^\bullet(\tau)) \\
&+ 2k(\Gamma_+(\sigma)+\Gamma_-(\sigma))(({}^\bullet\Delta^\bullet(\tau,\sigma)-T\delta(\tau-\sigma))\Delta(\sigma,\tau)\Delta(\tau) \\
&+ \Delta^\bullet(\tau,\sigma)\Delta(\sigma,\tau){}^\bullet\Delta(\tau)) \\
&+ 2T\Delta^\bullet(\tau)(\Gamma_+(\tau,\sigma)\Gamma_+(\sigma,\tau)\Gamma_-(\sigma)+\Gamma_-(\tau,\sigma)\Gamma_-(\sigma,\tau)\Gamma_+(\sigma))) \\
&= -\frac{1}{(-1+e^{kT})^4k^2}(3(-1+e^{kT})^2(1+e^{kT})+kT(1+e^{3kT}(-3+kT) \\
&\quad +e^{kT}(5+kT)+e^{2kT}(-3+2kT))) \qquad (4.25)
\end{aligned}$$

$$
\begin{aligned}
I_{12}(T,k) &= \int d\tau \ \ (\Delta(\Delta^{\bullet} + kT\Delta)(\Gamma_- + \Gamma_+) \\
&\quad -\frac{1}{T}(({}^{\bullet}\Delta - T\delta(0))\Delta^2 + 2\Delta\Delta^{\bullet\bullet}\Delta) - k^{\bullet}\Delta\Delta^2\Big)\,|_\tau \\
&= \frac{1}{6(-1+e^{kT})^3k^2}(1 + 9e^{kT}(2+kT) + 9e^{2kT}(-1+2kT) \\
&\quad +e^{3kT}(-10+3kT))
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
I_{13}(T,k) &= \int d\tau \ \ \Big(\frac{1}{T}(({}^{\bullet}\Delta^{\bullet} - T\delta(0))\Delta + {}^{\bullet}\Delta\Delta^{\bullet}) + k^{\bullet}\Delta\Delta - T\Gamma_+\Gamma_- \\
&\quad +\Delta^{\bullet}(\Gamma_+ + \Gamma_-))\,|_\tau = \frac{1}{2k}
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
I_{14}(T,k) &= \int d\tau \ \ \Big(\frac{1}{2T}(({}^{\bullet}\Delta^{\bullet} - T\delta(0))\Delta^2 + 2\Delta^{\bullet}\Delta\Delta^{\bullet}) \\
&\quad +\frac{k}{2}{}^{\bullet}\Delta\Delta^2 - T\Delta\Gamma_+\Gamma_- + \Delta\Delta^{\bullet}(\Gamma_+ + \Gamma_-)\Big)\,|_\tau \\
&= \frac{1}{6(-1+e^{kT})^3k^2}(1 - 6e^{kT} - 3e^{2kT}(-1+2kT) + 2e^{3kT})
\end{aligned} \tag{4.28}
$$

Chapter 5

Black holes and balanced metrics

5.1 Effective metric

In this chapter we propose a conjecture on the role of balanced metric, as an effective metric of the probe in the background of $N = 2$ supersymmetric black hole solution, and explore its consequences.

A famous problem in quantum gravity is to derive the Bekenstein-Hawking entropy of a black hole by counting its microstates. In string theory, this was first done by Strominger and Vafa [28]. They counted the microstates of a BPS bound state of Dirichlet branes with the same charge as the black hole, and then argued that the number of states was invariant under varying the string coupling, turning the bound state into a black hole.

This line of argument has been the basis for a great deal of work, generalizing the result to other systems and away from the semiclassical limit. One important element in such results is the claim that entropies and numbers of microstates are independent of the moduli of the background. An argument to this effect is provided by the attractor mechanism [29]. This was originally stated for BPS black holes in type II strings compactified on a Calabi-Yau manifold M, but the idea is probably more general (see [66] for a recent discussion). The attractor mechanism is based on the observation that the equations of motion for the moduli in a black hole background can be written in the form of gradient flow equations for the area of a surface of fixed radius as a function of the moduli. This flow approaches an attracting fixed point at the event horizon, with a definite value of the moduli and area. Thus, these values are insensitive to small variations of the initial conditions. By the Bekenstein-Hawking relation, this implies that the entropy is invariant under such variations.

It is plausible that other properties of the black hole microsystem share this type of universal behavior. For example, we might conjecture that not only the Kähler moduli of the Calabi-Yau metric near a black hole take universal values, but that the entire metric is universal, determined only by the charge and structure of the black hole and independent of the asymptotic moduli.

What would this mean? In classical supergravity, of course the metric is determined by the Einstein equation, reducing to the Ricci flatness condition for the source-free case. Thus the stronger conjecture is quite reasonable and indeed follows directly from the validity of supergravity. On the other hand, for a finite charge black hole preserving eight or fewer supercharges, one knows that these equations will get string theoretic (α' or g_s) corrections. Thus, while the stronger conjecture is still reasonable, it is not *a priori* clear either what the attractor CY metric should be, or what equations determine it.

Now, one reason the general question of finding exact metrics or even precisely defining corrected supergravity equations is hard, is that the metric and equations can be changed by field redefinitions, with no obvious preferred definition. For example, the metric g_{ij} could be redefined as $g_{ij} \to g_{ij} + \alpha R_{ij} + \beta (R^2)_{ij} + \ldots$. Unless we postulate an observable which singles out one definition, say measurements done by a point-like observer who moves on geodesics, there is no way to say which definition is right. This problem shows up in computing α' corrections in the sigma model as the familiar question of renormalization scheme dependence; in general there is no preferred scheme. We must first answer this question, to give meaning to the "CY attractor metric."

A nice way to answer this question is to introduce a probe brane, say a D0-brane, and study its world-volume theory. The kinetic term for its transverse coordinates is observable, and defines a unambiguous metric on the target space, including any α' corrections. While one can still make field redefinitions in the action, now these are just coordinate transformations. To make this argument straightforward, one requires that the mass (or tension) of the probe be larger than any other quantities under discussion, so that the action can be treated classically, and the metric read off from simple measurements.[1] For example, this is true for D0-branes in weakly coupled string theory, as their mass goes as $1/g_s$. One can then (in principle) define any term in the g_s expansion this way.

Both on general grounds [68] and in examples [69], the moduli space metric seen by a D-brane probe gets α' corrections, and for a finite size Calabi-Yau background it is not Ricci flat. The existing results are consistent with the first such correction arising from the standard $\alpha'^3 R^4$ correction to supergravity [70, 71], but pushing this to higher orders seems difficult.

Perhaps this problem becomes simpler in a black hole background. Rather than the D0, the probe brane we will use is a D2 or M2-brane wrapped on the black hole horizon. As discussed in [72, 73, 30, 74], such a brane, and D0-branes as well, in a near horizon BPS black hole background can preserve $SU(1,1|2)$ superconformal invariance. This is a symmetry of the $AdS_2 \times S^2$ near horizon geometry and thus this is as expected if multi-D0 quantum mechanics can be used as a dual gauge theory of the black hole. In these works, this quantum mechanics was argued to factorize into a space-time part, and an internal (Calabi-Yau) part; this second part describes motion of the probe in the Calabi-Yau and can be used to define a probe metric.

Given this system and its relation to the black hole, we will give a physical argument, based on the idea that a black hole must have "maximal entropy" no matter how this is defined, that suggests that the probe metric in such a black hole background is in fact

[1] This was the point of view taken in [67, 68]. Actually, one can in principle reconstruct a manifold with metric from quantum measurements (the spectrum and some position space observables), so one can work without this assumption.

the balanced metric, introduced in §2.5, for a particular level k, which depends on black hole charge. Thus the probe metric, satisfying the maximal entropy principle is not a Ricci flat metric. Nevertheless, one can define a certain large charge scaling limit where it approaches Ricci flat metric, with computable corrections in inverse powers in k.

Since our physical argument for the balanced metric does not assume the equations of motion, it illustrates a way to derive equations of motion from a maximum entropy principle. This idea was suggested some time ago by Jacobson [75, 76] and might have more general application.

5.2 BPS black holes and probes

Let us consider a BPS black hole solution in IIa theory compactified on a Calabi-Yau manifold M. Such a solution is characterized by discrete and continuous parameters. The discrete parameters are its electric and magnetic charges, which we take to be those of a system of D0, D2 and D4-branes. The continuous parameters are the values of the hypermultiplet moduli, namely the dilaton, complex structure moduli and their $N = 2$ supersymmetry partners. The vector multiplet (Kähler) moduli are determined by the attractor mechanism, as we review shortly.

By varying the dilaton to strong coupling, this theory is continuously connected to M theory compactified on $M \times S^1$. In this theory, the black hole can be thought of as a black string wrapped on S^1, and carrying S^1 momentum [77]. It will eventually turn out that our conjecture appears more natural in M theory, so let us start from that limit. To get a black string, we can wrap M5-branes on a four-cycle $[P] \in H_4(M, \mathbb{Z})$. By Poincaré duality $[P]$ can also be thought of as a class $p^A \omega_A$ in $H^2(M, \mathbb{Z})$, where we introduce a basis ω_A of $H^2(M, \mathbb{Z})$. In general, there are also electric charges q_A, corresponding to M2-branes wrapping dual two-cycles. We will set these to zero in the subsequent discussion.

According to the attractor mechanism, the Kähler class J_5 of the CY at the horizon of the black string is determined in terms of the charges p^A. Unlike in $d = 4$, in the black string solution, the volume V of the CY is a free parameter; thus we have (using 11d conventions of [78])

$$\frac{J_5}{V^{1/3}} = \frac{p^A \omega_A}{D^{1/3}},$$

where $D \equiv D_{ABC} p^A p^B p^C$ and D_{ABC} are the triple intersection numbers on the Calabi-Yau. We recall that $V = D_{ABC} J^A J^B J^C / 6$ where $J = J^A \omega_A$. It will also be useful to define

$$J_{CY} \equiv \frac{p^A \omega_A}{D^{1/3}} \tag{5.1}$$

which is independent of the overall scale of the charges. The corresponding supergravity solutions simplify in the near-horizon limit: the M theory solution approaches $AdS_3 \times S^2$ geometry

$$ds^2 = L^2 \left(\frac{-dt^2 + dx_4^2 + d\sigma^2}{\sigma^2} + d\Omega_{S^2}^2 \right). \tag{5.2}$$

with the following 4-form flux sourced by M5-branes

$$F_{(4)} \sim \frac{1}{L} \omega_{S^2} \wedge p^A \omega_A.$$

Now, by compactifying the black string on S^1 with the radius R_{10}, we obtain a 4d black hole with additional charge q_0, corresponding to momentum along the string. At this level, the discussion is simply mapped into IIa string theory, and the charges (q_0, q_A, p^A) correspond to D0, D2 and D4 brane charges. The radius L of S^2 and AdS_3 is related to the radius R_{10} of S^1 as $L \sim R_{10}\sqrt{D/q_0}$ and the volume of Calabi-Yau scales as $V \sim \alpha'^3 q_0 \sqrt{q_0/D}$. In $d = 4$, the overall scale of J and thus the volume V is also determined by the attractor mechanism, and Eq. (5.1) becomes

$$J_4 = \alpha' \sqrt{\frac{q_0}{D}} p^A \omega_A, \tag{5.3}$$

where as usual $\alpha' = l_p^3 / R_{10}$ and Q is the graviphoton charge (10d conventions correspond to those of [30]). The IIa supergravity solution in four dimensions approaches the $AdS_2 \times S^2$ near horizon geometry.

The metric on the CY M is determined by the Einstein equations,

$$R_{ij} - \frac{1}{2} R g_{ij} \sim (F^2)_{ij} + \delta + \mathcal{O}(l_p) \tag{5.4}$$

where $(F^2)_{ij}$ schematically denotes the contribution to stress-energy density due to the RR fields of the charged black hole, and δ denotes the contribution from brane sources [79]. The last term on the rhs represents string or M-theoretic corrections to this equation [71, 80].

In general, this metric depends on details of the state of the black hole. For example, if there are localized brane sources, the metric will depend on their location. However, we can simplify it by considering an appropriate state of the black hole. For general M, despite the non-zero rhs in Eq. (5.4), there exists a black hole state in which the sources are averaged in an analogous way, removing the localized terms, and leading to a Ricci flat metric on M in the supergravity limit.

At first it may seem that the field strength F would lead to a complicated position-dependent source; say concentrated on the cycles wrapped by the M5-branes. However, as observed in [74], the combination of the attractor mechanism and the equations of motion force the field strength to be proportional to the Kähler form on M,

$$F = dA = p^A \omega_A = QJ, \tag{5.5}$$

where Q is the graviphoton charge $Q \sim \sqrt{D/q_0}$. Note, that this condition appeared before in Eq. (2.20). This is true before adding α' or g_s corrections, and all the simple candidate corrections one can write down (such as powers of F and its derivatives, for example those which appear in the MMMS equation [81]), preserve Eq. (5.5). While there are still F^2 sources in Eq. (5.4), one can now check that these are canceled by terms coming from the space-time dependence of radius R_{10} of 11th dimension (in M theory) or the dilaton (in IIa). We omit the explicit check, instead pointing out that since the sources are constructed only from the metric tensor on M, they can at most add a cosmological constant term in Eq. (5.4), leading to a Kähler-Einstein metric. However, standard arguments imply that a solution exists only if this term is proportional to $c_1(M)$, which is zero for a Calabi-Yau manifold.

We have thus defined a preferred state of the black hole, for which the metric on M would be Ricci flat in the supergravity limit. To study corrections, we need to define this preferred state in a way which does not assume that we know the correct equations of motion or their solution in advance. Since the supergravity argument required us to average over all of the internal structure of the black hole, it is natural to define it as a mixed state of maximal entropy, as we will do below.

5.3 The probe theory

Having defined the state of the black hole under consideration, we now proceed to define the metric with l_p (11d Planck length) or α' corrections. As we discussed earlier, this can be made precise by introducing a probe, whose moduli space (space of zero energy configurations) includes M. This typically requires that the probe preserves some supersymmetry.

Now, the BPS black hole preserves an $N = 1$ supersymmetry, determined by the phase of its central charge Z, which is determined by the charges and attractor moduli. In asymptotically Minkowski space-time, introducing another BPS brane will typically break all of the supersymmetry. However, it was shown in [72] that in the near-horizon limit, a probe zerobrane can nevertheless preserve space-time supersymmetry, if it follows its "charged geodesic" (*i.e.* trajectory determined by the background metric and RR field). Even a collection of such branes with misaligned charges can preserve supersymmetry; consistent with this, the combined gravitational and RR potential energy of such a collection is additive.

Choosing a probe brane which preserves supersymmetry, one expects its configuration space to be some moduli space associated with the compactification space M. In the simplest example of a D0-brane, the moduli space is M itself. Another example for which the moduli space is M is a D2-brane wrapped around the S^2 horizon. Other choices, for example a Dp-brane wrapped on a p-cycle of M, would lead to different moduli spaces, related to the geometry of M.

While there will be a probe world-volume potential, this is determined by superconformal invariance [73] to be a function of the radius σ, but independent of the other coordinates. In particular, it is independent of position on M or other "internal" coordinates. Thus the world-volume theory includes a supersymmetric quantum mechanics on the moduli space M. Ground states of this quantum mechanics will correspond in the usual way to differential forms on M.

The argument we are about to make is clearest for the case of a probe D2 wrapping the S^2 horizon, so let us consider that. According to [73], the supersymmetry condition for such a brane forces it to the center of AdS_2 (in global coordinates), so there are no other moduli on which the probe metric on M can depend.

At leading order, the probe will see both the metric on M, and a magnetic field on M. The latter follows (in IIa language) from the D4 charge of the black hole: a probe D2 wrapping the horizon will see a background magnetic field on the CY M [30, 74],

$$F_{CY} = \int_{S^2} F_{(4)} = p^A \omega_A.$$

Mathematically, such a magnetic field defines a line bundle L over M; whose first Chern class is the D4 charge p^A. From Eq. (5.3), the Kähler class J is proportional to the first Chern class of L,

$$J = \frac{1}{Q} c_1(L).$$

5.4 Maximal entropy argument for the probe metric

Our argument will be based on two assumptions. First, the most symmetric state of a BPS black hole, and thus the state corresponding to the simplest metric on M, is a state of maximal entropy. Second, that there is a sense in which the black hole can be regarded as made up of constituents with "the same dynamics" as the probe. We will use this to argue that the probe should also be in a state of maximal entropy to get a simple result.

The first assumption is very natural and straightforward to explain. To define "maximal entropy," we look at the Hilbert space of BPS states of the black hole, call this $\mathcal{H}_{BH}$. By standard arguments going back to [28], these are BPS states of the quantum system describing the black hole, here a bound state of D0 and D4 branes. Let us denote an orthonormal basis of $\mathcal{H}_{BH}$ as $|h_\alpha\rangle$. Now, the states $|h_\alpha\rangle$ are pure states in the usual sense of quantum mechanics. The maximal entropy state of such a system is a mixed state, described by the density matrix

$$\rho_{BH} = \frac{1}{\dim \mathcal{H}_{BH}} \sum_\alpha |h_\alpha\rangle\langle h_\alpha|, \tag{5.6}$$

in which each pure state appears with equal probability. Thus, we have a clear definition of "maximal entropy" of the black hole.

The original description of the black hole Hilbert space $\mathcal{H}_{BH}$ [82] was in terms of a postulated bound state of D0-branes at each triple intersection of D4-brane on the Calabi-Yau. Denoting the number of triple intersections as k, one finds that the supergravity entropy formula can be matched if there is one D0 bound state for each value n of eleven-dimensional momentum, with 4 bosonic and 4 fermionic degrees of freedom, leading to a partition function

$$Z_{BH} = \prod_{i=1}^{k} \prod_{n\geq 1} \left(\frac{(1+q^n)}{(1-q^n)} \right)^4 \tag{5.7}$$

A later argument to the same effect [77] proceeds by lifting the black hole to M theory on $M \times S^1$, in which it becomes a wrapped M5-brane. First compactifying on M, a wrapped five-brane on a 4-cycle (or divisor) D becomes a black string. The string is then compactified on S^1 to obtain the black hole.

In this analysis, the string has world-sheet fields parameterizing the moduli space of degree N hypersurfaces P_N, which is precisely the projectivization of the space of sections of L. The resulting black string Hilbert space is that of a symmetrized orbifold $\mathrm{Sym}^M(\mathcal{M}(P_N))$ of the moduli space, constructed as

$$\prod_{i=1}^{k} \alpha_{-n_i}^{A_i}|0\rangle \tag{5.8}$$

with $\sum n_i = M$. Along with these moduli are additional fields (the dimensional reduction of the fivebrane two-form, and fermions), combining into $(0,4)$ supersymmetry multiplets with $4+4$ components. Finally, using the standard result for the density of states of a conformal field theory with central charge $c = 6k$, the entropy is

$$S = 2\pi\sqrt{\frac{c \cdot q_0}{6}}.$$

Another, more mathematical way to derive the multiplicity $4+4$ for each modulus, is to observe that the BPS states of $(0,4)$ supersymmetric quantum mechanics include all (p,q) forms taking values in the target space (here the moduli space of divisors),

$$\mathcal{H} = \oplus_{0\le p,q\le 3} H^p(M, \Omega^q \otimes L). \tag{5.9}$$

Since the divisor is ample, these vanish for $p > 0$, while the $q = 0,1,2,3$ terms have multiplicities $k, 3k, 3k, k$ (for large D). The even and odd q-forms then give rise to bosonic and fermionic moduli (respectively), whose quantization reproduces Eq. (5.7) or Eq. (5.8).

More recently, a related but not obviously identical description of the black hole Hilbert space has been developed, motivated by the idea that the black hole should be described by a superconformal matrix quantum mechanics of n D0-branes in the D4 background. [30, 74] In this picture, the basic object is a bound state of n D0-branes which can be thought of as a "fuzzy D2-brane," which arises from the matrix D0 theory by a Myers-type effect [83]. The general form of Eq. (5.7) then arises by summing over all partitions of the total D0 charge q_0.

Note that this second description is in terms of a supersymmetric quantum mechanics with target space the Calabi-Yau manifold M, very much like our probe theory. Indeed, the background RR field is postulated to appear as a non-trivial $U(1)$ magnetic field, of topological type exactly that of the bundle L.

A strategy to get the D0 matrix quantum mechanics on this background, pursued in [30], is to consider a D2-brane wrapped on the black hole horizon, an S^2. As is familiar (for example) in M(atrix) theory [84, 85], D0 matrix quantum mechanics contains bound configurations of N D0's which represent a wrapped or stretched D2. If we can reverse this identification, we can derive the matrix quantum mechanics from the D2 theory.

Of course, the D2 theory in this background is precisely the probe theory we discussed in section 5.3. Its full low-energy hamiltonian was found in [30]. It factorizes into an AdS_2 part and a CY part, with the latter being

$$H_{CY} = g^{a\bar{a}}(p_a - A_a)(\bar{p}_{\bar{a}} - A_{\bar{a}}). \tag{5.10}$$

Here the metric g is built from the Kähler form J and the gauge field has the field strength proportional to J as in Eq. (5.5). The general idea is then that, by promoting this quantum mechanics to matrix quantum mechanics, one would obtain a description of the black hole.

This brings us to our second assumption, that there is a sense in which the black hole can be regarded as made up of constituents with "the same dynamics" as the probe. If we grant the second description of the black hole, in terms of D0 matrix quantum mechanics, then clearly we can identify constituents with the same dynamics as our probe. As we mentioned, reproducing the black hole partition function Eq. (5.7) requires summing over configurations each labelled by a partition $\{n_i\}$ of the total D0 charge. Such a configuration is obtained by considering the matrix variables as a direct sum of blocks, each a matrix of dimension n_i. The dynamics of such a block is described by the $U(n_i)$ reduction of the matrix quantum mechanics, with interactions with the rest of the black hole produced by integrating out off-diagonal degrees of freedom. The supersymmetry of the combined system will cancel the relative potential between the blocks, and presumably makes the other induced interactions small.

Let us consider a sector with $n_1 = 1$, in other words containing single unbound D0. The dynamics of this D0 is approximately described by the $U(1)$ version of matrix quantum mechanics, in other words the theory discussed in [30]. The BPS Hilbert space of this theory is $\mathcal{H}$ defined in Eq. (5.9), and we see that this sits naturally in $\mathcal{H}_{BH}$.

Now, we will implement our second assumption, by deriving a natural maximal entropy state for the probe. By starting with the maximal entropy state Eq. (5.6) of the black hole in $\mathcal{H}_{BH}$ and tracing over all of the other degrees of freedom, we obtain a density matrix ρ over the Hilbert space $\mathcal{H}$. The result will be the standard quantum state of maximal entropy for this quantum mechanics, which assigns equal probability to each state in $\mathcal{H}$, given by the expression

$$\rho = \frac{1}{\dim \mathcal{H}} \sum_{\alpha} |h_\alpha\rangle\langle h_\alpha|, \tag{5.11}$$

Indeed, one might regard this choice of quantum state as the natural one whatever the probe is, without calling upon any relation to the black hole. However we spell out this step as it explains how we could, given a precise D0 quantum mechanics for the black hole, compute the probe state and observables.

Now, given ρ, we can ask, what is the probability to find the D2 probe at a given point $z \in M$. This will be

$$\rho(z, \bar{z}) = \langle z|\rho|z\rangle. \tag{5.12}$$

In general, the D2 will have "spin" degrees of freedom as well, corresponding to the degrees (p, q) of cohomology; let us fix these in the $p = q = 0$ sector.[2] By inserting explicit wave functions $\psi_\alpha(z, \bar{z})$, the density matrix can be written in position space as a kernel,

$$\rho(z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{1}{\dim H^0} \sum_{\alpha} \psi^*_\alpha(z_1, \bar{z}_1)\ \psi_\alpha(z_2, \bar{z}_2), \tag{5.13}$$

with Eq. (5.12) its values on the diagonal $z_1 = z_2 = z$.

[2]We comment on this point in section 5.5.

Note that, although the lowest Landau level wavefunctions satisfy the metric independent linear differential equation $\bar{D}h = 0$, their normalizations depend on the metric. Thus the kernel Eq. (5.13) depends on the specific choice of metric, not just the Kähler class.

Now, since the probe has maximal entropy, one would expect that this probability does not favor any particular point in moduli space, in other words

$$\rho(z, \bar{z}) = \text{constant}. \tag{5.14}$$

But this is not at all obvious from what we have said so far. We might regard it as a second, independent interpretation of the claim that the black hole has maximal entropy.

5.5 Balanced metric as maximally entropic metric

While from the point of view of an asymptotic observer, the first definition Eq. (5.11) of maximal entropy seems more natural, if we can only make measurements with the probe, the second definition seems more natural. Going further, to the extent that (following the arguments above) the probe can also be thought of as a constituent of the black hole, we might be able to reformulate black hole thermodynamics in terms of the second definition. In particular, the postulate that the black hole has maximal entropy, should imply that its constituents are equidistributed in moduli space. Otherwise, there would be a simple way for the system to increase its entropy

While not self-evident, it is an attractive hypothesis that the entropy should be maximal in both senses. As we have seen before, the two definitions of maximal entropy are not directly in conflict. Indeed, we could compute Eq. (5.12) from the definition Eq. (5.13), and check whether they agree. But since the actual wave functions and thus Eq. (5.13) depend on the details of the probe world-volume theory, in particular the metric, we need to know the probe metric to make this check.

Turning around this logic, we can regard the conjunction of Eq. (5.11) and Eq. (5.14) as a non-trivial condition on the probe metric. In fact, this is exactly the condition we encountered with in the §2.5: it implies that *the probe metric is the balanced metric.*

As we have demonstrated in Chapter 3, for the magnetic field proportional to the Kähler form (5.5, 2.20), the LLL density matrix Eq. (5.11) is equal to the Bergman kernel, at least for large k. Thus the constant probability distribution for the probe brane has a balanced metric as a solution, as previously for the Eq. (2.31).

Conversely, it is a theorem that (given suitable assumptions, which hold here), the balanced metric exists and is unique [19]; thus this is the only metric satisfying both Eq. (5.14) and Eq. (5.5). Thus, granting Eq. (5.5), our physical consistency condition between the two definitions of "maximal entropy," precisely picks out the balanced metric associated to the line bundle L whose first Chern class is the D4 charge.

Now, in the limit of a large charge black hole, in which the local curvatures and field strengths near the black hole become small, one would expect the probe metric to be approximately Ricci flat. To take the large charge limit, we scale up the D4 charge by a factor k, and take $A \to k\,A$. Mathematically, this corresponds to replacing the line bundle L by the line bundle L^k.

The claim is now that, in the large k limit, the balanced metric, defined by the maximal entropy property Eq. (5.14), should satisfy the supergravity equations of motion. By the discussion following Eq. (5.4), these equation imply that the metric on M will be Ricci flat. But *a priori*, the condition Eq. (5.14) has no evident connection with Ricci flatness or any other equation of motion. Thus this claim is in fact a nontrivial test of the conjecture, which it passes as we explain now.

Indeed, from the Tian-Yau-Zelditch asymptotic expansion (3.17) of the diagonal of density matrix Eq. (5.11) it follows that

$$\begin{aligned} P(z,\bar{z}) \quad = \quad & \frac{k^3}{\dim H^0}\left(1+\frac{1}{k}R+\right. \\ & \left.\frac{1}{k^2}\left(\frac{1}{3}\Delta R+\frac{1}{24}(|R_{a\bar{a}b\bar{b}}|^2-4|R_{a\bar{a}}|^2+3R^2)\right)+\ldots\right). \end{aligned} \tag{5.15}$$

Combining this expansion with Eq. (5.14) we obtain, that for sufficiently large k, the effective metric will have constant scalar curvature, up to corrections of order $1/k$. In the case at hand with $c_1(M) = 0$, this implies Ricci flatness, and the probe metric satisfies this test at leading order. Thus the basic consistency of our conjecture with supergravity is clear.

Let us also comment on the appropriate large k scaling limit in terms of black holes charges. Looking at Eq. (5.1) one sees that in M-theory settings the above scaling corresponds just to rescaling of the magnetic charges $p^A \to k\, p^A$. In the IIa set up, Eq. (5.3) and Eq. (5.5) tell us that in addition to rescaling the magnetic charges $p \to kp$, one also has to scale $q_0 \to kq_0$, so that the curvature of line bundle L^k scales as k times the metric. Such a scaling limit is described in [87] as the natural limit scaling up Kähler moduli.

Another variation of the conjecture is that, because the probe is a superparticle, we should also sum over spin states in the density matrix Eq. (5.13), and enforce Eq. (5.14) on the diagonal of this density matrix. In Chapter 4 we have shown how to define and compute the leading terms of this kernel using supersymmetric quantum mechanics. The resulting expansion has the leading nontrivial term, proportional to the Ricci scalar with a coefficient $|N - 1|$, where N is the number of supercharges. In the case at hand, the probe is an $N = 2$ supersymmetric quantum mechanics. Thus incorporating the spin states leads to the same basic result, that in the large volume limit the conjecture agrees with supergravity.

However, we have not been able to identify the subleading terms in Eq. (5.15), nor those in its supersymmetric analogs, with any known physical corrections. In particular, one might expect the famous coefficient $\zeta(3)$ of the R^4 correction to show up in this expansion, from both the IIA string and M theory points of view. On the other hand, it is clear from the nature of the expansion Eq. (5.15) that such transcendental coefficients will not appear.

It seems possible that the limits involved kill these particular terms, but one still needs to explain the other corrections. One might speculate that these are related to the much studied higher genus superpotential terms in RR backgrounds [88, 89], but we did not find evidence for this either. Or, it could be that the problem is essentially nonperturbative from the supergravity point of view, and that these terms are seeing another regime.

One might ask if the expression Eq. (5.5) for the magnetic field could also get stringy or M-theoretic corrections, which while preserving its cohomology class, nevertheless modify Eq. (5.15). This is possible, and as shown in §4.3, the expansion still exists, as well as the balanced metric. In that case, the infinite k scaling limit requires for the gauge connection to satisfy the hermitian Einstein equation (4.19).

Another possibility is that additional couplings on the probe world-volume are important. We assumed that the probe can be described purely in terms of a particle in a background metric and magnetic field, and then derived the balanced metric from the maximal entropy condition. Of course, there could be higher derivative terms, perhaps induced by interactions with the other constituents of the black hole. These might modify the wave functions so as to achieve Eq. (5.14) with a different metric.

Of course, the maximal entropy assumptions might not hold for these black holes. We nevertheless feel that our argument is making an important point. The assumptions do seem very natural in the context of this problem. Indeed, if we had a precise definition of the D0 matrix quantum mechanics suggested in [30, 74], we could in principle use it to compute the probe metric, and find out where the contradiction arises. Indeed, understanding this point might be a useful hint to how this (still mysterious) quantum mechanics works.

Furthermore our argument is very simple, indeed far simpler than the supergravity or topological string considerations one might compare it with. We believe it will find applications regardless of the fate of this conjecture.

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Chapter 6

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