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Complete integrability of geodesics in Sasaki-Einstein space $Y^{p,q}$ via action-angle variables

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Abstract. We describe the construction of the integrals of geodesic motions in the five-dimensional Sasaki-Einstein space $Y^{p,q}$. The complete integrability of geodesics makes possible the construction of the action-angle variables. We find that the Hamiltonian involves only four action variables which have the corresponding frequencies different from zero.

1. Introduction

Sasaki geometry, as an odd-dimensional counterpart of the Kähler geometry, has become of renewed interest in some modern developments in mathematics and physics [1, 2]. There has been particular interest in Sasaki-Einstein (SE) manifolds in connection with the AdS/CFT correspondence. The AdS/CFT correspondence conjectures that for a five-dimensional SE manifold Y_5, type IIB string theory on $AdS_5 \times Y_5$ is dual to a four-dimensional $\mathcal{N} = 1$ superconformal field theory. In dimension five, the SE $Y^{p,q}$ metrics have played a central role as they provide an infinite class of dualities.

The purpose of this paper is to describe the construction of the action-angle variables for the geodesics in SE space $Y^{p,q}$. The description of the integrability of geodesics in $Y^{p,q}$ space in terms of action-angle variables gives us a comprehensive geometric description of the dynamics. We find that one of the fundamental frequencies is zero giving way to chaotic behavior when the system is perturbed.

The organization of the paper is as follows. In the next Section we give a brief presentation of well-known results concerning Killing tensors and SE geometry. In Section 3 we describe the complete integrability of geodesic motions in $Y^{p,q}$ space. In Section 4 we construct the action-angle variables and the frequencies of the motions. The paper ends with conclusions in Section 5.

2. Preliminaries regarding Killing tensors and Sasaki-Einstein geometry

We consider a Riemannian manifold M with the metric g_M and let ∇ be its Levi-Civita connection.

A vector field field K_μ, associated with an isometry, satisfies the Killing equation

$$\nabla_{(\mu} K_{\nu)} = 0 \,, \tag{1}$$

where a round bracket denotes a symmetrization over the indices within.

A Stäckel-Killing (SK) tensor is a rank r symmetric tensor defined on M such that

$$\nabla_{(\mu} K_{\nu_1 \cdots \nu_r)} = 0 \,. \tag{2}$$

In the presence of a SK tensor the system of a free particle with the Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu \,, \tag{3}$$

admits the conserved quantity

$$K = K^{\mu_1 \cdots \mu_r} P_{\mu_1} \cdots P_{\mu_r} \,, \tag{4}$$

commuting with Hamiltonian (3) in the sense of Poisson brackets. Here P_μ are canonical momenta conjugate to the coordinates x^μ, $P_\mu = g_{\mu\nu} \dot{x}^\nu$ with overdot denoting proper time derivative.

Another important generalization of Killing vector fields is given by Killing-Yano (KY) tensors which are antisymmetric tensors and satisfy the equation

$$\nabla_{(\mu} \Psi_{\nu_1) \nu_2 \ldots \nu_r} = 0 \,. \tag{5}$$

There is a remarkable connection between these two generalizations of the Killing vectors. Namely, the partially contracted product of two KY tensors $\Psi^{i_1,\ldots,i_r}$ and $\Sigma^{i_1,\ldots,i_r}$ generates a SK tensor of rank 2:

$$K_{ij}^{(\Psi,\Sigma)} = \Psi_{ii_2\ldots i_r} \Sigma_j^{\ i_2\ldots i_r} + \Sigma_{ii_2\ldots i_r} \Psi_j^{\ i_2\ldots i_r} \,. \tag{6}$$

In order to find the conserved quantities it is necessary to know the Killing vectors and SK tensors. However the generalized Killing equations (2) and (5) are difficult to be solved. Sometimes it is possible to find the KY tensors using the geometrical properties of the manifold. Using (6) we are able to generate SK tensors and implicitly the conserved quantities (4). That is the case of the SE manifolds for which it is possible to construct their complete set of KY tensors.

A $(2n-1)$-dimensional manifold M is a *contact manifold* if there exists a 1-form η (called a contact 1-form) on M such that

$$\eta \wedge (d\eta)^{n-1} \neq 0 \,. \tag{7}$$

The *Reeb vector field* ξ dual to η satisfies:

$$\eta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner \, d\eta = 0 \,, \tag{8}$$

where $\lrcorner$ is the operator dual to the wedge product.

A contact Riemannian manifold (M, g_M) is *Sasakian* if its metric cone is Kähler [1]

$$C(M) \cong \mathbb{R}_+ \times M \quad , \quad g_{C(M)} = dr^2 + r^2 \, g_M \,. \tag{9}$$

Here $r \in (0, \infty)$ may be regarded as a coordinate on the positive real line $\mathbb{R}_+$.

If the Sasaki space is Einstein

$$\operatorname{Ric} g_M = 2(n-1) g_M \,, \tag{10}$$

then the metric cone is Ricci-flat ($\operatorname{Ric} g = 0$), i.e. Calabi-Yau manifold.

On a $(2n-1)$-dimensional SE manifold with the contact 1-form η there are the following KY tensors:

$$\Psi_k = \eta \wedge (d\eta)^k \quad , \quad k = 0, 1, \cdots, n-1 \,. \tag{11}$$

Besides these KY tensors, there are two additional KY tensors connected with the real and imaginary parts of the complex volume form of the Calabi-Yau metric cone $C(M)$ [3].

3. Complete integrability of geodesics in Sasaki-Einstein space $Y^{p,q}$

The metric of the Sasaki-Einstein space $Y^{p,q}$ is given by the line element [4]

$$ds^2 = \frac{1-y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta\, d\phi)^2$$
$$+ w(y)\left[d\alpha + \frac{a - 2y + y^2}{6(a - y^2)}[d\psi - \cos\theta\, d\phi]\right]^2, \tag{12}$$

where

$$w(y) = \frac{2(a - y^2)}{1 - y}, \tag{13}$$
$$q(y) = \frac{a - 3y^2 + 2y^3}{a - y^2}.$$

A detailed analysis of the SE metric $Y^{p,q}$ [5] showed that for

$$0 < a < 1, \tag{14}$$

one can take the range of the angular coordinates (θ, ϕ, ψ) to be $[0, 2\pi]$, while y lies between two zeros of $q(y)$, i.e. $y_1 \leq y \leq y_2$ with $q(y_i) = 0$. To be more specific, the roots y_i of the cubic equation

$$a - 3y^2 + 2y^3 = 0, \tag{15}$$

are real, one negative (y_1) and two positive, the smallest being y_2.

Finally, the period of α is chosen so as to describe a principal S^1 bundle over $B_4 = S^2 \times S^2$. For any p and q coprime, the space $Y^{p,q}$ is topologically $S^2 \times S^3$ and one may take [4, 5]

$$0 \leq \alpha \leq 2\pi\ell, \tag{16}$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}. \tag{17}$$

The contact 1-form η is

$$\eta = -2y d\alpha + \frac{1-y}{3}(d\psi - \cos\theta d\phi), \tag{18}$$

and the Reeb vector field is [4]

$$K_\eta = 3\frac{\partial}{\partial\psi} - \frac{1}{2}\frac{\partial}{\partial\alpha}. \tag{19}$$

In order to describe the conserved quantities for the geodesic motions we list the conjugate

momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$:

$$
\begin{aligned}
P_\theta &= \frac{1-y}{6}\dot\theta\,, \\
P_y &= \frac{1}{6p(y)}\dot y\,, \\
P_\alpha &= w(y)\left(\dot\alpha + f(y)\left(\dot\psi - \cos\theta\dot\phi\right)\right)\,, \\
P_\psi &= w(y)f(y)\dot\alpha + \left[\frac{q(y)}{9} + w(y)f^2(y)\right]\left(\dot\psi - \cos\theta\dot\phi\right)\,, \\
P_\phi &= \frac{1-y}{6}\sin^2\theta\dot\phi - \cos\theta P_\psi \\
&= \frac{1-y}{6}\sin^2\theta\dot\phi - \cos\theta w(y)f(y)\dot\alpha - \cos\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot\psi \\
&\quad + \cos^2\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot\phi\,,
\end{aligned}
\tag{20}
$$

where we introduced the functions

$$
f(y) = \frac{a - 2y + y^2}{6(a - y^2)}\,,
\tag{21}
$$

$$
p(y) = \frac{w(y)q(y)}{6} = \frac{a - 3y^2 + 2y^3}{3(1-y)}\,,
\tag{22}
$$

which simplify writing formulas.

Using the conjugate momenta (20), the Hamiltonian (3) governing the geodesic motions in $Y^{p,q}$ space is:

$$
\begin{aligned}
H = \frac{1}{2}\Bigg\{ &6p(y)P_y^2 + \frac{6}{1-y}\left(P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2\right) + \frac{1-y}{2(a-y^2)}P_\alpha^2 \\
&+ \frac{9(a-y^2)}{a - 3y^2 + 2y^3}\left(P_\psi - \frac{a - 2y + y^2}{6(a-y^2)}P_\alpha\right)^2\Bigg\}\,.
\end{aligned}
\tag{23}
$$

The integrals of motions associated with the isometry $SU(2) \times U(1) \times U(1)$ of the metric (23) are easily to find. First of all we observe that the angles (ϕ, ψ, α) are cyclic coordinates and consequently the corresponding momenta are conserved:

$$
\begin{aligned}
P_\phi &= c_\phi\,, \\
P_\psi &= c_\psi\,, \\
P_\alpha &= c_\alpha\,,
\end{aligned}
\tag{24}
$$

where $(c_\phi, c_\psi, c_\alpha)$ are some constants. P_ϕ corresponds to the third component of the $SU(2)$ angular momentum and P_ψ, P_α are associated with the $U(1)$ factors. Moreover the total $SU(2)$ angular momentum

$$
\vec{J}^{\,2} = P_\theta^2 + \frac{1}{\sin^2\theta}(P_\phi + \cos\theta P_\psi)^2 + P_\psi^2\,,
\tag{25}
$$

is also conserved.

Other conserved quantities can be constructed using the KY forms of the SE space $Y^{p,q}$ [6, 7, 8]. However the maximum number of conserved quantities which are functionally independent and in involution (their Poisson brackets are equal to zero) is *five* [6]. In what follows we choose these functionally independent integrals to be the the set $\mathbf{F} = (H, P_\phi, P_\psi, P_\alpha, \vec{J}^{\,2})$. The system is complete integrable and we can proceed to construct the action-angle variables.

4. Action-angle variables

The construction of the action-angle variables starts with fixing a level surface $\mathbf{F} = \mathbf{c}$ [9, 10]. We search a generating function S for the canonical transformation $(\mathbf{p}, \mathbf{q}) \to (\mathbf{J}, \mathbf{w})$ where $\mathbf{p}$ is represented by the momenta (20) and $\mathbf{q}$ is the set of coordinates $(\theta, \phi, y, \alpha, \psi)$.

The generating function S is

$$S(\mathbf{q}, \mathbf{c}) = \int_{\mathbf{F} = \mathbf{c}} \mathbf{p} \, d\mathbf{q} \,, \tag{26}$$

where $\mathbf{p} = \mathbf{p}(\mathbf{q}, \mathbf{c})$ by use of the equations of motion.

Since the Hamiltonian (23) has no explicit time dependence, we can write

$$S(\mathbf{q}, \mathbf{c}) = W(\mathbf{q}, \mathbf{c}) - Et \,, \tag{27}$$

where $W(\mathbf{q}, \mathbf{c})$ is the Hamilton characteristic function

$$W = \sum_i \int p_i dq_i \,, \tag{28}$$

and E is the energy $H = E$ corresponding to the chosen level surface $\mathbf{F} = \mathbf{c}$.

In the case of the geodesics in SE space $Y^{p,q}$ the variables in Hamilton-Jacobi equation are separable and consequently we seek a solution of the form

$$W(y, \theta, \phi, \psi, \alpha) = W_y(y) + W_\theta(\theta) + W_\phi(\phi) + W_\psi(\psi) + W_\alpha(\alpha) \,. \tag{29}$$

The action variables $\mathbf{J}$ are given by

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i; \mathbf{c})}{\partial q_i} dq_i \qquad \text{(no summation)} \,, \tag{30}$$

where $\oint$ means an integration over one cycle. The J_i's form n independent functions of the constants $\mathbf{c}$ and can be taken as a set of new constant momenta.

Finally, the angle variables can be found from the equation

$$w_i = \frac{\partial W}{\partial J_i} = \sum_{j=1}^n \frac{\partial W_j(q_j; J_1, \cdots, J_n)}{\partial J_i} \,, \tag{31}$$

having a linear evolution in time

$$w_i = \omega_i t + \beta_i \,, \tag{32}$$

with β_i other constants of integration and ω_i are frequencies associated with the periodic motion of q_i.

Applying the general prescriptions, we can manage to evaluate the action-angle variables for geodesics in $Y^{p,q}$. For the cyclic variables the task is easy as the Hamilton characteristic functions (28) related to cyclic variables are

$$\begin{aligned}
W_\phi &= P_\phi \phi = c_\phi \phi \,, \\
W_\psi &= P_\psi \psi = c_\psi \psi \,, \\
W_\alpha &= P_\alpha \alpha = c_\alpha \alpha \,,
\end{aligned} \tag{33}$$

with the constants c_ϕ, c_ψ, c_α introduced in (24). The corresponding action variables are

$$\begin{aligned}
J_\phi &= 2\pi c_\phi \,, \\
J_\psi &= 2\pi c_\psi \,, \\
J_\alpha &= 2\pi \ell c_\alpha \,.
\end{aligned} \tag{34}$$

For a fixed energy E and using (29), the Hamilton (23) can be written in the form:

$$
\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta}(c_\phi + \cos \theta_\psi)^2
$$
$$
= \frac{1-y}{3}E - p(y)(1-y)\left(\frac{\partial W_y}{\partial y}\right)^2 - \frac{(1-y)^2}{12(a-y^2)}c_\alpha^2 \tag{35}
$$
$$
- \frac{3(a-y^2)(1-y)}{2(a-3y^2+2y^3)}\left[c_\psi - \frac{a-2y+y^2}{6(a-y^2)}c_\alpha\right]^2 .
$$

We separated in the LHS the terms which depend only on θ, while the RHS depends only on y. Therefore we may set

$$
\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta}(c_\phi + \cos \theta c_\psi)^2 = c_\theta^2 , \tag{36}
$$

with c_θ another constant.

The action variable J_θ is given by the integral

$$
J_\theta = \oint d\theta \sqrt{c_\theta^2 - \frac{(c_\phi + c_\psi \cos \theta)^2}{\sin^2 \theta}} . \tag{37}
$$

Denoting by $\theta_\pm$ the roots of the function appearing in the square root, the complete cycle of θ involves going from θ_- to θ_+ and back to θ_-. The most efficient way to evaluate the integral (37) is to put $\cos \theta = t$, extend t to a complex variable z and interpret the integral as a closed contour integral in the complex z-plane [11]. The turning points of the t-motions are

$$
t_\pm = \frac{-c_\phi c_\psi \pm c_\theta \sqrt{c_\theta^2 + c_\psi^2 - c_\phi^2}}{c_\theta^2 + c_\psi^2} . \tag{38}
$$

They are real for

$$
c_\theta^2 + c_\psi^2 - c_\phi^2 \geq 0 , \tag{39}
$$

and situated in the interval $(-1, +1)$.

The contour of integration in the complex z-plane can be deformed to a large circular contour plus two contour integrals about the poles at $z = \pm 1$. The standard evaluation of the residues and taking into account the contribution of the large contour integral finally gives

$$
J_\theta = 2\pi \left[\sqrt{c_\theta^2 + c_\psi^2} - c_\phi\right] . \tag{40}
$$

Concerning the angular variable w_ϕ we have

$$
w_\phi = \frac{1}{2\pi}J_\phi + \frac{\partial W_\theta}{\partial J_\phi} . \tag{41}
$$

Putting $\cos \theta = t$, and using (34), (36), (40), the second term is

$$
\frac{\partial W_\theta}{\partial J_\phi} = -\frac{1}{2\pi}\int dt \frac{(J_\phi + J_\theta)t^2 + J_\psi t}{(1-t^2)\sqrt{-(J_\phi + J_\theta)^2 t^2 - 2J_\phi J_\psi t + (J_\theta^2 + 2J_\theta J_\phi - J_\psi^2)}}
$$
$$
= \frac{1}{2\pi}\int \frac{dt}{1-t^2}\frac{dt^2 + et}{\sqrt{a + bt + ct^2}} , \tag{42}
$$

where

$$
\begin{aligned}
a &= J_\theta^2 + 2 J_\theta J_\phi - J_\psi^2 \,, \\
b &= -2 J_\theta J_\psi \,, \\
c &= -(J_\theta + J_\phi)^2 \,, \\
d &= J_\theta + J_\phi \,, \\
e &= J_\psi \,.
\end{aligned}
\tag{43}
$$

We need the following integrals [12]

$$
\begin{aligned}
I_1(a,b,c;t) &= \int \frac{dt}{\sqrt{a + bt + ct^2}} = \frac{-1}{\sqrt{-c}} \arcsin\left(\frac{2ct + b}{\sqrt{-\Delta}} \right) \,, \\
I_2(a,b,c;t) &= \int \frac{dt}{t\sqrt{a + bt + ct^2}} = \frac{1}{\sqrt{-a}} \arctan\left(\frac{2a + bt}{2\sqrt{-a}\sqrt{a + bt + ct^2}} \right) \,,
\end{aligned}
\tag{44}
$$

evaluated for $c < 0$, $\Delta = 4ac - b^2 < 0$ and for $a < 0$ respectively. That is the case of the parameters a, b, c taking into account the constraint (39).

Using these integrals we finally get for the angular variable w_ϕ

$$
\begin{aligned}
w_\phi = &\frac{1}{2\pi} J_\phi - \frac{d}{2\pi} I_1(a, b, c; \cos\theta) \\
&- \frac{d + e}{4\pi} I_2(a + b + c, b + 2c, c; \cos\theta - 1) \\
&- \frac{e - d}{4\pi} I_2(a - b + c, b - 2c, c; \cos\theta + 1) \,.
\end{aligned}
\tag{45}
$$

For the action variable corresponding to y coordinate we have from (35)

$$
\begin{aligned}
\frac{\partial W_y}{\partial y} = &\left\{ \frac{1 - y}{a - 3y^2 + 2y^3} E - \frac{3}{a - 3y^2 + 2y^3} c_\theta^2 \right. \\
&- \frac{9(a - y^2)(1 - y)}{2(a - 3y^2 + 2y^3)^2} c_\psi^2 + \frac{3(a - 2y + y^2)(1 - y)}{2(a - 3y^2 + 2y^3)^2} c_\psi c_\alpha \\
&\left. - \frac{(1 - y)(2a + a^2 - 6ay - 2y^2 + 2ay^2 + 6y^3 - 3y^4)}{8(a - 3y^2 + 2y^3)^2(a - y^2)} c_\alpha^2 \right\}^{\frac{1}{2}} \,.
\end{aligned}
\tag{46}
$$

In contrast with the action variable J_θ (37), the evaluation of the action variable J_y involves an intricate integral as can be seen from (46). After all, the closed-form of J_y is not at all illuminating. For our purposes, it is enough to observe that J_y depends only on four constants of motion: $E, J_\theta, J_\alpha, J_\psi$. In consequence, the energy depends only on four action variables $J_y, J_\theta, J_\alpha, J_\psi$ representing a reduction of the number of action variables entering the expression of the energy of the system.

The explicit evaluation of the angular variables $w_\theta, w_\psi, w_\alpha, w_y$ is again intricate due to the absence of a simple closed-form for the action variable J_y. However, it is remarkable the fact that one of the fundamental frequencies (32)

$$
\omega_i = \frac{\partial H}{\partial J_i} \,,
\tag{47}
$$

is zero, namely

$$\omega_\phi = \frac{\partial H}{\partial J_\phi} = 0 \,, \tag{48}$$

since the action J_ϕ does not enter the expression of the energy.

Therefore the Hamiltonian (23) depends on that action variables for which the corresponding frequencies are different from zero. The presence of a vanishing frequency refers to the degeneracy of the system giving way to chaotic behavior when the system is perturbed [9] This fact was observed in several recent studies of non-integrability and chaotic behavior of some classical configurations of strings in the context of AdS/CFT correspondence (see e. g. [13, 14, 15]).

5. Conclusions

The existence of the conserved quantities are very important not only for the particle motions, but also leading to the separation of Hamilton-Jacobi and quantum Klein-Gordon, Dirac equations. The action-angle approach to the integrable geodesics represents a useful tool in the study of near-integrable systems (Kolmogorov-Arnold-Moser (KAM) theory) and for the quantization of integrable systems (Bohr-Sommerfeld rule).

It would be interesting to extend the action angle formulation to higher dimensional SE metrics relevant for the predictions of the AdS/CFT correspondence.

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