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# Canonical Quantization of Metric Tensor for General Relativity in Pseudo-Riemannian Geometry

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**Abstract**

By extending the four-dimensional semi-Riemann geometry to higher-dimensional Finsler/Hamilton geometry, the canonical quantization of the fundamental metric tensor of general relativity, i.e., an approach that tackles a geometric quantity, is derived. With this quantization, the smooth continuous Finsler structure is transformed into a quantized Hamilton structure through the kinematics of a free-falling quantum particle with a positive mass, along with the introduction of the relativistic generalized uncertainty principle (RGUP) that generalizes quantum mechanics by integrating gravity. This transformation ensures the preservation of the positive one-homogeneity of both Finsler and Hamilton structures, while the RGUP dictates modifications in the noncommutative relations due to integrating consequences of relativistic gravitational fields in quantum mechanics. The anisotropic conformal transformation of the resulting metric tensor and its inverse in higher-dimensional spaces has been determined, particularly highlighting their translations to the four-dimensional fundamental metric tensor and its inverse. It is essential to recognize the complexity involved in computing the fundamental inverse metric tensor during a conformal transformation, as it is influenced by variables like spatial coordinates and directional orientation, making it a challenging task, especially in tensorial terms. We conclude that the derivations in this study are not limited to the structure in tangent and cotangent bundles, which might include both spacetime and momentum space, but are also applicable to higher-dimensional contexts. The theoretical framework of quantization of general relativity based on quantizing its metric tensor is primarily grounded in the four-dimensional metric tensor and its inverse in pseudo-Riemannian geometry.

**Keywords:** fundamental metric tensor in pseudo-Riemann geometry; canonical quantization approach; gravitization of quantum mechanics; smooth continuous Finsler structure; discretized (quantized) Hamilton structure

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## 1. Introduction

The fundamental principles of general relativity (GR) and quantum mechanics (QM) are believed to be incompatible [1,2]. Despite the extensive investigations, a solution to this long-standing problem has yet to be found. A novel approach has been proposed, which involves a canonical quantization approach to quantize the metric tensor of GR in pseudo-Riemann geometry. The main assumption of this novel method is the requirement for a proper generalization of the current formulations of GR and QM. The integration of relativistic gravitational fields in the uncertainty principle [3–5], gravitational self-energy in the spontaneous collapse of superposition principle [6,7], and topology change or interaction in the principle of general covariance are essential ingredients for the generalization for QM [8]. The generalization for pseudo-Riemann geometry is discussed in the present paper; for more information, see Refs. [9–16].

Before the pseudo-Finsler geometry and its particular mathematical properties are addressed, it is worth recalling what was first considered for the possible generalization of pseudo-Riemann geometry [17]. In pseudo-Riemann geometry, the metric $g$ and the manifold $M$ with $\dim(M) = n$, that is $(M, g)$, there is a natural metric $\bar{g}$ on the tangent bundle $TM$, so $(TM, \bar{g})$ represents a generalization of the Riemann manifold. This manifold is known as the Sasaki metric [18–20]. Eduardo Caianiello introduced a similar procedure for quantization of GR [21,22]. The corresponding line element can be expressed in the local coordinates on $TM$, namely $(x, \dot{x})$, as

$$d\tilde{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{\mu\nu} D\dot{x}^\mu D\dot{x}^\nu, \tag{1}$$

where $g_{\mu\nu}$ is the fundamental metric tensor, the dot on top denotes the time derivative, and $D\dot{x}^\mu = d\dot{x}^\mu + \Gamma^\mu_{\nu\delta}\dot{x}^\nu dx^\delta$ is the covariant derivative. $\Gamma^\mu_{\nu\delta}$ is the affine connection. That is, the generalized metric tensor can be expressed as [23]

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu} + g_{\alpha\beta}\Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\gamma\nu}, \tag{2}$$

which leads to

$$\widetilde{g}_{(n+\mu)(n+\nu)} = g_{\mu\nu}, \tag{3}$$

$$g_{\alpha\beta}\Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\gamma\nu} = A_{\mu\nu}, \tag{4}$$

$$\widetilde{g}_{\mu(n+\nu)} = g_{\nu\sigma}\Gamma^\sigma_{\epsilon\mu}\dot{x}^\epsilon = B_{\mu\nu}. \tag{5}$$

The Latin letter indexes denote the space coordinates.

The additional component that, when is added to the conventional metric tensor, produces the generalized metric (2), is represented by $A_{\mu\nu}$. Now the Sasaki metric can be defined in line with Equation (2):

$$\widetilde{g}_{\mu\nu} = \begin{bmatrix} g_{\mu\nu} + A_{\mu\nu} & B_{\mu\nu} \\ B^T_{\mu\nu} & g_{\mu\nu} \end{bmatrix}. \tag{6}$$

By exploring the relationship between metric geometry, pseudo-Riemann geometry $(M, g)$, and the tangent bundle $TM$ geometry, which is supplied with the Sasaki metric $\widetilde{g}$, various types of rigidity have been suggested [24–26]; more details on the correct connections between $(M, g)$ and $(TM, \widetilde{g})$ can be found in the literature, such as Refs. [27,28]. The interconnection between difference space is contingent upon a specific parametrization. The extension to Finsler/Hamilton space, however, is there to introduce quantum mechanical components. Here, we convert the metric tensor on Finsler/Hamilton into a metric tensor on the pseudo-Riemann manifold.

As mentioned above, another type of generalization for the metric tensor which characterizes the Riemann geometry $(M, g)$, the canonical quantization approach proposed here, can be directly derived from the Finsler structure $F$, as discussed in Section 2. Consider an $n$-dimensional pseudo-Finslerian manifold $(M, F)$, where $F : TM \to \mathbb{R}$. Here, we define the coordinated $(x, \dot{x})$ as $(x^\mu, \dot{x}^\mu)$ the local coordinates on $TM$. Furthermore, $\pi : TM \setminus \{0\} \to M$ represents the natural projection. Similar to other pseudo-Finslerian quantities, the Finsler metric is a function on $TM$ [29] and can be related to essential quantities including the non-degenerate fundamental tensor [30]

$$g_{\mu\nu} = \frac{1}{2}\frac{\partial^2}{\partial\dot{x}^\mu\partial\dot{x}^\nu}F^2(x,\dot{x}), \tag{7}$$

and the Cartan tensor [31]

$$C_{\mu\nu\sigma} = \frac{1}{4}\frac{\partial^3}{\partial\dot{x}^\mu\partial\dot{x}^\nu\partial\dot{x}^\sigma}F^2(x,\dot{x}). \tag{8}$$

In Section 2, we introduce a discretized version (likely quantized) of Finsler structure, where the direction defined by $\dot{x}$ is replaced by that of the momentum space $p$. In this regard, a kinematic theory of a quantum particle that lives and moves in the discretized Finsler manifold to be utilized. Also, let us emphasize that the translation $\dot{x} \to m\dot{x} \to p$ is not exactly similar to the one proposed for Hamilton geometry [32]. The Finsler geometry characterized by $x$ and $\dot{x}$ is dual to the Hamilton geometry which is characterized by $x$ and $p$. Both types of geometries are mappings connecting the spaces of velocity and of momentum, respectively [33]. As soon as duality of both the velocity and the momentum spaces is nonlinear, the Legendre transformation does not linearly translate $\dot{x}$ in Finsler to $p$ in Hamilton geometry. The rigorous mathematical procedure we used to transform $F(x,\dot{x})$ to $F(x,m\dot{x}) \equiv H(x,p)$ is essentially based on the mathematical properties of the Finsler structure. We employ this homogeneous characteristic to introduce at least discreteness, if not quantization, into the tangent bundle.

In order to determine the mathematical equivalence between $F(x,p)$ and $H(x,p)$, let us analyze the procedure that strictly upholds the homogeneous property of $F(x,m\dot{x})$ and compare it with the Legendre transformation of $F(x,\dot{x})$ to $H(x,p)$. The Hamilton space is identified by the manifold $M$ and the Hamilton structure $H(x,p)$. This space is described by the cotangent space $T^*M \ni (x,p) \to H(x,p) \in \mathbb{R}$, where the Hamilton structure $H(x,p)$ is differentiable on the cotangent bundle $T^*M$ and continuous on the null section $\pi^* : T^*M \to M$. The function $TM \ni (x,\dot{x}) \to L(x,p) \in \mathbb{R}$ defines a Lagrangian space. It is quite straightforward that the Lagrangian space is characterized by the manifold $M$ and $L(x,\dot{x})$. The Legendre transformation leads to [34]

$$H(x,p) = p_a\dot{x}^a - L(x,\dot{x}), \tag{9}$$

where $\dot{x} = \{\dot{x}^a\}$ are solutions of $\dot{x}_b = \partial L(\hat{x},\hat{\dot{x}})/\partial\dot{x}^b$. In this regard, let us mention that the solution is obtained by identifying $L(x,\dot{x})$ with $F(x^a,m\dot{x}^b)$ so that

$$\dot{x}_b = \frac{\partial F(x^a,m\dot{x}^b)}{\partial\dot{x}^b} = \frac{m}{F(x^a,\dot{x}^b)}\left[\dot{x}^b + x^b\frac{\langle x^a,\dot{x}^b\rangle}{1-|x^a|^2}\right], \tag{10}$$

where indexes $a,b \in \{0,1,\cdots,7\}$. Also, it can be observed that Equation (9) is satisfied by this particular solution, wherein the indexes are lowered or raised with the corresponding metric tensor. One concludes that when $L(x,\dot{x})$ is equivalent to $F(x^a,m\dot{x}^b)$, then $F(x,p) \equiv H(x,p)$. This study focuses on a Finsler metric that can be derived from the Hessian of $F^2(x,m\dot{x})$. The derivation of the Hamilton metric to be addressed elsewhere.

The introduction of a discrete Finsler structure and its transformation into a Hamiltonian framework are examples of the proposed generalization of spacetime geometry. The four-dimensional limitation is a respectable approximation that appears to hold true at all levels. At relatively low (quantum) scale, this approximation has to be either decreased or removed. The discretized Finsler structure is transformed into a Hamiltonian framework in order to generalize the spacetime geometry and thereby attenuate the conventional approximation. We make a further step by introducing generalization to quantum mechanics by making further adjustments based on a relativistic generalized uncertainty principle (RGUP). Using RGUP, the quantum mechanical framework becomes compatible with finite gravitational fields. In this paper, we provide a first-principal derivation of the quantum mechanically corrected metric tensor. Various studies already cover physical grounding, predictive ability, and technical clarity; see, e.g., [9,10,12–16].

The paper is organized as follows. The discretized Finsler structure is discussed in Section 2. This Section also describes a discretized Hamilton geometry in which four-dimensional spacetime is coupled to four-dimensional momentum space. The anisotropic conformal transformation of arbitrary high-dimensional metric is introduced in Section 3. The utilization of the Klein metric is considered for pedagogical reasons, as it belongs to the simplest Finsler metrics. In Section 4, a detailed derivation of the inverse for this metric is outlined. The translation from the quantized metric and its inverse on a higher-dimensional Hamilton manifold to the fundamental metric tensors on the four-dimensional Riemann manifold are discussed in Sections 5 and 6, respectively. The conclusions are given in Section 7.

## 2. Discretized Finsler Structure

For a quantum particle with positive mass $m$, the Finsler structure can be reformulated as

$$F(x^\mu, m\dot{x}^\nu) \quad = \quad F(x^\mu, p^\nu), \qquad \forall m \in \mathbb{R}^+, \tag{11}$$

where $x^\mu$ and $p^\nu$ are quantum mechanical operators whose auxiliary four-vectors are $x^\mu \equiv \mathbf{x}_0^\mu = (x_0^0, x_0^i) = (ct, x_0^i)$ and $p^\mu \equiv \mathbf{p}_0^\mu = -i\hbar\partial/\partial\mathbf{x}_0^\mu = (p_0^0, p_0^i) = (E/c, p_0^i)$, where, $E$ represents energy, $t$ stands for time, $c$ denotes the speed of light, and $\hbar$ is the reduced Planck constant. For the sake of simplicity, we assume that the operators $x^\mu$ and $p^\mu$, as defined in QM without gravitational field, are considered to remain unchanged in generalized QM. QM without a gravitational field refers to the QM that relies on the Heisenberg uncertainty principle (HUP), which was developed in vanishing gravitational field [3,35–38]. For further details on recent studies in order to replace HUP by the generalized uncertainty principle (GUP), in which gravitational fields are included, see [3,37,38].

With respect to time operator $x_0^0$, we intentionally refer to Ref. [39] by Eric Galapon where the time operator is considered to behave like energy, is bounded and self-adjoint. To be more specific, we assume that the time operator is canonically conjugated to the Hamiltonian $H$. The Hamiltonian is considered to have discrete spectra. This choice is discussed shortly. This gives us reason to believe that this pair of operators follows the time–energy canonical commutation relation $[T, H] = i\hbar$. In addition, this enables us to represent the known Heisenberg equation of motion $dT/dt = [T, H]/i\hbar$ [39]. The latter indicates that time operators are always progressing in steps of parametric time $t$, which means that $T$ meets the covariance requirement $T(t) = T(0) + t$ for all times [39,40]. Given the semibounded Hamiltonian of quantum mechanics, we are aware of the historical rejection of the existence of any such kind of time operator [41,42]. As a non-self-adjoint, covariant, positive operator-valued measures (POVM) observable, time can now be reintroduced into quantum mechanics thanks to the recent critical finding that quantum observables need not be self-adjoint POVM [43]. In this regard, let us recall that Pauli's claim [44] has been refuted by several counterexamples, leading to a growing focus on the study of self-adjoint time operators. This highlights that the assumption of a self-adjoint operator canonically conjugate to a semibounded or discrete Hamiltonian does not contradict [45]. The statement that there are just non-self-adjoint time operators in single Hilbert quantum mechanics appears to be non-universal. The present study is an illustration of that type.

In the various attempts to overcome Paulis' highly-established no-go theorem, we consider bounded time operators, particularly for Hamiltonians that exhibit discrete spectra. The averaging involved in the proposed quantization, along with the lack of any other alternatives, suggests that Hamiltonians with discrete spectra are quite appropriate [9,10,12–16]. Other recent studies, such as Refs. [46,47], which have extended the bounded time operators to include the unbounded time operators and alternative formulations of Hilbert space, are discussed in another context, especially given the rigorous mathematical analysis presented in Ref. [48].

It is understandable that the Relativistic Generalized Uncertainty Principle (RGUP) [3] introduces modifications to the HUP where higher-order momentum terms are multiplied by the same order of the RGUP parameter. Therefore, the general Hamiltonian can be expanded in a Taylor series with respect to the RGUP parameter $\beta$. The corrections to the Hamiltonian $H_0 = p_0^2/(2m) + v(t)$, where $v(t)$ represents the correction part, are solely adjustments to the kinetic portion of the Hamiltonian, specifically the first term. This indicates that the corrections introduced are independent of the potential, represented by the second term. That is, we implement perturbations to $H$, which necessitates that the proposed RGUP corrections must be smaller than the uncertainties of the perturbations introduced to $H$. It is understandable that the present iteration of RGUP corresponds quite well to Hamiltonians with discrete spectra.

It was noted that GUP exhibits characteristics of being three-dimensional and non-relativistic, in addition to other shortcomings including breaking Lorentz covariance [5,49–52], violating linearity of the dispersion relations [53], and not coupling the uncertainty principle to the spacetime metric [54,55]. The relativistic generalization of the HUP was suggested to resolve the mentioned shortcomings that arise from HUP and GUP. The RGUP [3] is briefly reviewed in Appendix A. The RGUP impacts the operators and their auxiliary four-vectors so that $\mathbf{x}^\mu = \mathbf{x}_0^\mu$ and $\mathbf{p}^\mu = (1 + \beta p_0^\rho p_{0\rho})\mathbf{p}_0^\mu$, where $\rho$ is a dummy index and $\beta$'s upper bound can be empirically and/or observationally determined [56].

For the auxiliary four-vectors $x_0^\mu$, $\dot{x}_0^\mu$, and $p_0^\nu$, let us start with the conventional Finsler metric $F(x_0^\mu, \dot{x}_0^\nu)$, which appears to preserve the entire mathematical properties of the Finsler structure. To use the RGUP approach (Appendix A), consider the kinematics of the free-falling quantum particle, by which determining the uncertainty principle becomes possible through the utilization of energy and momentum. The positive mass preserves the homogeneous properties of $F$, resulting in $F(x, m\dot{x}) = mF(x, \dot{x})$. Then, RGUP can be directly applied on $F(x, m\dot{x})$. The RGUP also preserves the homogeneous properties of $F(x, p)$. In $F(x, p)$, the auxiliary four-moment becomes a subject of RGUP, $p_0 = \phi(p_0)p_0$, where $\phi(p_0) = 1 + \beta p_0^\rho p_{0\rho}$ and the generalization of $p_0$ is provided by the left-hand site. The function $\phi(p_0)$ assures that all quantities defining $\phi(p_0)$ obey the same dimension as the generalized $p_0$. The coefficients refer to the phase space coordinates $(x_0, p_0)$. Besides the positive homogeneity, i.e., $F$ is positively 1-homogeneous in $p_0$, the relativistic four-momentum so that $F(x, \phi(p_0)\, p) = \phi(p_0)\, F(x, p), \forall \phi(p_0) \in \mathbb{R}^+$; the resulting discretized $F(x, \phi(p_0)p)$ presumably preserves the following properties:

- non-degeneracy, i.e., the metric tensor is non-degenerate on $TM\backslash\{0\}$;
- smoothness, i.e., $F$ is smooth on $TM\backslash\{0\}$, on the complement of the zero section.

The function $\phi(p_0)$ has a few key roles. So far, we consider the extension of the HUP to RGUP, and then a successful incorporation gravitational effects into QM, avoiding the shortcomings that have been discussed earlier [3]. Other roles such as (i) generalizing the fundamental tensor and thereby quantizing GR [57], and (ii) retaining the special curvature of the Randers metric (Appendix B), are elaborated Section 3.

For the discretized Finsler–Hamilton structure $H(x_0, \phi(p_0)\, p_0)$, to be 1-homogeneous in $p_0$, the functions $\phi(p_0)$ must be 0-homogeneous in $p_0$. It is understandable that the

RGUP parameter $\beta$ is $-2$-homogeneous, because both $\beta$ and $p_0$ do not depend on $x_0$. $\beta = \beta_0 G/(c^3\hbar) = \beta_0(\ell_\mathtt{p}/\hbar)^2$ where $G$ is the gravitational constant and $\ell_\mathtt{p}$ is the Planck length [37,38]. As soon as the function $\phi(p_0)$ is positive and 0-homogeneous in $p_0$, $H(x_0, \phi(p_0)p_0) = \phi(p_0)\, H(x_0, p_0)$ is then 1-homogeneous in $p_0$.

In the discretized Hamilton space of the spacetime cotangent bundle, it is conjectured that the fiber manifold constitutes the 4-momentum forming momentum space, while the base manifold is represented by the spacetime [58]. This constructs the eight-dimensional geometry composed of four-dimensional spacetime and four-dimensional momentum space. By using a parameterization $\zeta$, a generic point can be characterized by the coordinates $x^a(\zeta) \equiv (x_0^\mu(\zeta), p_0^\nu(\zeta))$, which are phase space coordinates. Both position and momentum variables, $x_0$ and $p_0$, respectively, are appropriately parameterized. The application of the RGUP approach suggests that $p_0 \to (1 + \beta p_0^\rho p_{0\rho})p_0$ [37,38]. As so, then the quantized fundamental tensor can be derived as

$$\widetilde{g}_{\mu\nu} = \frac{1}{2}\frac{\partial^2}{\partial p_0^\mu \partial p_0^\nu} H^2(x_0^\mu, \phi(p_0) p_0^\nu). \tag{12}$$

The metric (12) is a Finsler-type metric because of the Hessian of the squared discretized structure. The coordinates in Finsler–Hamilton space are given as $x_0^a = (x_0^\mu, p_0^\nu)$.

By utilizing positive homogeneity, $F(x_0^\mu, \dot{x}_0^\nu)$ (Finsler) is transformed into $H(x_0^\mu, p_0^\nu)$ (Hamilton). This transformation allows us to suggest that the resulting expression of $H^2(x_0^\mu, \phi(p_0)\, p_0^\nu)$ may be equivalently expressed by the corresponding Finsler structure $F^2(x_0^\mu, \phi(p_0)\, p_0^\nu)$. For the sake of simplicity, let us suggest that the Klein metric represents $F(x_0^\mu, \phi(p_0)\, p_0^\nu)$ and thereby $H(x_0^\mu, \phi(p_0)\, p_0^\nu)$. The Klein metric on $\mathbb{B}^n(1)$ (Section 3), in which RGUP is integrated, can be given as

$$F^2(x_0^\mu, \phi(p_0)\, p_0^\nu) \quad = \quad \phi^2(p_0)\, \frac{|p_0^\nu|^2 - |x_0^\mu|^2 |p_0^\nu|^2 + \left\langle x_0^\mu, p_0^\nu \right\rangle^2}{(1 - |x_0^\mu|^2)^2}, \tag{13}$$

where $|\cdot|$ and $\langle\cdot\rangle$ are, respectively, the standard Euclidean norm and the inner product on $\mathbb{R}^n$. Accordingly, Equation (12) can be reexpressed as

$$\widetilde{g}_{\mu\nu}\big|_{\mathtt{Finsler}} = \frac{1}{2}\frac{\partial^2}{\partial p_0^\mu \partial p_0^\nu} F^2(x_0^\mu, \phi(p_0)\, p_0^\nu), \tag{14}$$

This represents an additional basis for asserting that the resulting metric is of the Finsler type.

It must be noted that the derivative of $\widetilde{g}_{\mu\nu}$ (14) is valid for arbitrary higher-dimensional structures. Section 3 provides a outline of all the relevant particularities. Henceforth, we denote an arbitrary high-dimensional structure using the structure $F$.

## 3. Anisotopic Conformal Transformation of Metric in Higher-Dimensional Geometry

Let us start with the components of the metric tensor associated with the arbitrary higher-dimensional structure $F$, i.e., that of $g_{\mu\nu}$ where

$$g_{\mu\nu} \quad = \quad \frac{\delta_{\mu\nu}}{1 - |x_0|^2} + \frac{x_0^\sigma x_0^\gamma \delta_{\nu\sigma}\delta_{\mu\gamma}}{(1 - |x_0|^2)^2}, \tag{15}$$

and $\delta_{\mu\nu}$ represents the Euclidean metric. Considering the anisotropic confromal transformation of $F$, one finds that

$$\bar{F} = \phi(p_0)F. \tag{16}$$

As discussed in Section 2, $\phi(p_0)$ is a 0-homogeneous function in $p_0$ and therefore can be expressed as

$$\phi(p_0) = 1 + \frac{\kappa}{(p_0^0)^2}F^2, \tag{17}$$

where $\kappa = \beta/(p_0^0)^2$.

To determine the components of the metric tensor associated with $\bar{F}$, namely $\bar{g}_{\mu\nu}$, we introduce the following lemma.

**Lemma 1.** *The first and second derivatives of the function $\phi(p_0)$, with respect to $p_0$, are given separately by [59,60]*

$$\begin{aligned} \phi_\mu &:= \dot{\partial}_\mu \phi(p_0) \\ &= \frac{2\kappa F}{(p_0^0)^3}\left(p_0^0 \ell_\mu - F\delta_{0\mu}\right), \end{aligned} \tag{18}$$

$$\begin{aligned} \phi_{\mu\nu} &:= \dot{\partial}_\nu \dot{\partial}_\mu \phi(p_0) \\ &= \frac{2\kappa}{(p_0^0)^2} g_{\mu\nu} - \frac{4\kappa F}{(p_0^0)^3}(\ell_\nu \delta_{0\mu} + \ell_\mu \delta_{0\nu}) + \frac{6\kappa F^2}{(p_0^0)^4}\delta_{0\nu}\delta_{0\mu}, \end{aligned} \tag{19}$$

*where $\dot{\partial}_\mu \equiv \partial/\partial \dot{x}^\mu$ and $\ell_\mu := \partial F/\partial p_0^\mu$.*

**Proof.** If $\phi_{\mu\nu} := \dot{\partial}_\nu \dot{\partial}_\mu \phi(p_0)$, Equation (19) reads

$$\begin{aligned} \phi_{\mu\nu} &= \left(\frac{2\kappa \ell_\nu}{(p_0^0)^3}\right)\left(p_0^0 \ell_\mu - F\delta_{0\mu}\right) - \frac{6\kappa F}{(p_0^0)^4}\delta_{0\nu}\left(p_0^0 \ell_\mu - F\delta_{0\mu}\right) \\ &+ \frac{2\kappa F}{(p_0^0)^3}\left(\delta_{0\nu}\ell_\mu + p_0^0 \ell_{\mu\nu} - \ell_\nu \delta_{0\mu}\right) \\ &= -\frac{4\kappa F}{(p_0^0)^3}(\ell_\nu \delta_{0\mu} + \ell_\mu \delta_{0\nu}) + \frac{2\kappa F}{(p_0^0)^2}\ell_\mu \ell_\nu + \frac{6\kappa F^2}{(p_0^0)^4}\delta_{0\nu}\delta_{0\mu} + \frac{2\kappa F}{(p_0^0)^2}\ell_{\mu\nu}, \end{aligned} \tag{20}$$

where

$$\ell_{\mu\nu} = \dot{\partial}_\mu \ell_\nu, \tag{21}$$

$$g_{\mu\nu} = F\ell_{\mu\nu} + \ell_\mu \ell_\nu. \tag{22}$$

□

**Proposition 1.** *Equation (14) expresses an anisotropic transformation of the metric tensor*

$$\begin{aligned} \widetilde{g}_{\mu\nu} = &\left(\phi^2(p_0) + \frac{2\kappa\phi(p_0)\,F^2}{(p_0^0)^2}\right) g_{\mu\nu} \\ &+ \frac{4\kappa F^2}{(p_0^0)^2}\left(2\phi(p_0) + \frac{\kappa F^2}{(p_0^0)^2}\right)\ell_\mu \ell_\nu - \frac{4\kappa F^3}{(p_0^0)^3}\left(2\phi(p_0) + \frac{\kappa F^2}{(p_0^0)^2}\right)(\ell_\mu \delta_{0\nu} + \ell_\nu \delta_{0\mu}) \\ &+ \frac{2\kappa F^4}{(p_0^0)^4}\left(3\phi(p_0) + \frac{2\kappa F^2}{(p_0^0)^2}\right)\delta_{0\mu}\delta_{0\nu}. \end{aligned} \tag{23}$$

**Proof.**

$$
\begin{aligned}
\tilde{g}_{\mu\nu} &= \phi^2(p_0)g_{\mu\nu} + F^2(\phi_\mu\phi_\nu + \phi(p_0)\,\phi_{\mu\nu}) + 2\phi(p_0)F(\ell_\mu\phi_\nu + \ell_\nu\phi_\mu) \\
&= \phi^2(p_0)\,g_{\mu\nu} \\
&\quad + F^2\left(\frac{2\kappa F}{(p_0^0)^3}\right)^2\left(p_0^0\ell_\mu - F\delta_{0\mu}\right)\left(p_0^0\ell_\nu - F\delta_{0\nu}\right) \\
&\quad + \phi(p_0)\,F^2\left(\frac{2\kappa}{(p_0^0)^2}\,g_{\mu\nu}|_{\texttt{Finsler}} - \frac{4\kappa F}{(p_0^0)^3}(\ell_\nu\delta_{0\mu} + \ell_\mu\delta_{0\nu}) + \frac{6\kappa F^2}{(p_0^0)^4}\delta_{0\nu}\delta_{0\mu}\right) \\
&\quad + 2\phi(p_0)\,F\left(\frac{2\kappa F}{(p_0^0)^3}\right)\left[\ell_\nu\left(p_0^0\ell_\mu - F\delta_{0\mu}\right) + \ell_\mu\left(p_0^0\ell_\nu - F\delta_{0\nu}\right)\right] \\
&= \left(\phi^2(p_0) + \frac{2\kappa\phi(p_0)\,F^2}{(p_0^0)^2}\right)g_{\mu\nu} \\
&\quad + \frac{4\kappa F^2}{(p_0^0)^2}\left(2\phi(p_0) + \frac{\kappa F^2}{(p_0^0)^2}\right)\ell_\mu\ell_\nu - \frac{4\kappa F^3}{(p_0^0)^3}\left(2\phi(p_0) + \frac{\kappa F^2}{(p_0^0)^2}\right)(\ell_\mu\delta_{0\nu} + \ell_\nu\delta_{0\mu}) \\
&\quad + \frac{2\kappa F^4}{(p_0^0)^4}\left(3\phi(p_0) + \frac{2\kappa F^2}{(p_0^0)^2}\right)\delta_{0\mu}\delta_{0\nu}. \qquad (24)
\end{aligned}
$$

□

Based on the function $\phi(p_0) = 1 + \frac{\kappa}{(p_0^0)^2}F^2$, Equation (23) can be rewritten as

$$
\begin{aligned}
\tilde{g}_{\mu\nu} &= \left(\phi^2(p_0) + 2\frac{\kappa\phi(p_0)}{(p_0^0)^2}F^2\right)g_{\mu\nu} \\
&\quad + \frac{4\kappa F^2}{(p_0^0)^2}\left\{\left[(p_0^0)^2(3\phi(p_0)-1)\ell_\mu\ell^\sigma\frac{\delta_{\mu\nu}}{\delta_{\mu\sigma}} - Fp_0^0(3\phi(p_0)-1)\ell^\sigma\left(\delta_{0\nu}\frac{\delta_{\mu\nu}}{\delta_{\mu\sigma}} + \delta_{0\mu}\frac{\delta_{\nu\mu}}{\delta_{\nu\sigma}}\right)\right.\right. \\
&\qquad \left.\left. + 2F^2(5\phi(p_0)-2)\delta_{0\mu}\delta_0^\sigma\frac{\delta_{\nu\mu}}{\delta_{\nu\sigma}}\right]\right\}g_{\mu\nu}, \qquad (25) \\
&= \left(\phi^2(p_0) + 2\frac{\kappa\phi(p_0)}{(p_0^0)^2}F^2\right)g_{\mu\nu} \\
&\quad + \frac{4\kappa F^2}{(p_0^0)^2}\left[(p_0^0)^2(3\phi(p_0)-1)\ell_\mu\ell^\sigma g_{\sigma\mu} - Fp_0^0(3\phi(p_0)-1)\ell^\sigma(\delta_{0\nu}g_{\sigma\mu} + \delta_{0\mu}g_{\sigma\nu})\right. \\
&\qquad \left. + 2F^2(5\phi(p_0)-2)\delta_{0\mu}\delta_0^\sigma g_{\sigma\nu}\right]. \qquad (26)
\end{aligned}
$$

This is the resulting generalized metric on the phase space manifold. According to the properties of $\phi(p_0)$, especially that $\phi(p_0)$ does not depend of $x_0$, the second term in Equation (26) is non-vanishing.

The anisotopic conformal transformation of the inverse metric reveals further properties of the metric, which are readily apparent upon derivation. Section 4 just below complete elaboration on the derivation of the inverse metric.

## 4. Anisotopic Conformal Transformation of Inverse Metric in Higher-Dimensional Geometry

Determining the transformation of the inverse metric under an anisotropic conformal transformation in higher dimensions, where the conformal factor is dependent on both position and direction, is quite challenging task [61]. Nevertheless, we have managed to derive this transformation in a tensorial form, incorporating the specific conformal factor.

The anisotropic conformal transformation of the inverse higher-dimensional metric tensor can be obtained by considering the following Lemma:

**Lemma 2.** *Let $(g_{\mu\nu})$ and $(m_{\mu\nu})$ be two $n \times n$ symmetric matrices and $C = (C_\mu)$ be an $n$-dimensional vector, which satisfy the relation [62,63]*

$$g_{\mu\nu} = m_{\mu\nu} + \lambda C_\mu C_\nu, \tag{27}$$

*where $\lambda$ is a constant. Then*

$$\det\left(g_{\mu\nu}\right) = \left(1 + \lambda C^2\right) \det\left(m_{\mu\nu}\right). \tag{28}$$

*Assuming that $(m_{\mu\nu})$ is non-degenerate with $\left(m_{\mu\nu}\right)^{-1} = (m^{\mu\nu})$ and $1 + \lambda C^2 \neq 0$, $(g_{\mu\nu})$ is invertible and $(g^{\mu\nu}) = \left(g_{\mu\nu}\right)^{-1}$ is given by*

$$g^{\mu\nu} = m^{\mu\nu} - \frac{\lambda}{1 + \lambda C^2} C^\mu C^\nu, \tag{29}$$

*where $C^\mu = m^{\mu\nu} C_\nu$ and $C = \sqrt{m^{\mu\nu} C_\mu C_\nu}$.*

**Proof.** Under proposition 11.2.1, Ref. [62] (p. 287) provides the proof of Lemma 2. The proof is also given with Lemma 1.1 in Ref. [63] (p. 3). □

Let $(M, F)$ be an $n$-dimensional manifold with the metric tensor $g_{\mu\nu}$. Consider the anisotropic transformation $\widetilde{F} = \phi(p_0) F$, where $\phi(p_0) = 1 + \kappa F^2/(p_0^0)^2$. Then, the metric $\widetilde{g}_{\mu\nu}$ corresponding to the higher-dimensional $F$ is given by

$$\begin{aligned} \widetilde{g}_{\mu\nu} \;=\;& \left(\phi^2(p_0) + \frac{2\,\phi(p_0)\,\kappa\,F^2}{(p_0^0)^2}\right) g_{\mu\nu} \\ & + \frac{4\,\kappa\,F^2}{(p_0^0)^2}\left(2\,\phi(p_0) + \frac{\kappa\,F^2}{(p_0^0)^2}\right) \ell_\mu\,\ell_\nu - \frac{4\,\kappa\,F^3}{(p_0^0)^3}\left(2\,\phi(p_0) + \frac{\kappa\,F^2}{(p_0^0)^2}\right) (\ell_\mu \delta_{0\mu} + \ell_\nu\,\delta_{0\nu}) \\ & + \frac{2\,\kappa\,F^4}{(p_0^0)^4}\left(3\,\phi(p_0) + \frac{2\,\kappa\,F^2}{(p_0^0)^2}\right) \delta_{0\mu}\,\delta_{0\nu}. \end{aligned} \tag{30}$$

Let us rewrite $\widetilde{g}_{\mu\nu}$ as

$$\begin{aligned} \widetilde{g}_{\mu\nu} \;=\;& \left(\phi^2(p_0) + \frac{2\,\phi(p_0)\,\kappa\,F^2}{(p_0^0)^2}\right) g_{\mu\nu} \\ & + \frac{4\,\kappa\,F^2}{(p_0^0)^2}\left(2\,\phi(p_0) + \frac{\kappa\,F^2}{(p_0^0)^2}\right)\left(\ell_\mu - \frac{F}{p_0^0}\,\delta_{0\mu}\right)\left(\ell_\nu - \frac{F}{p_0^0}\,\delta_{0\nu}\right) \\ & - \frac{2\,\kappa\,\phi(p_0)\,F^4}{(p_0^0)^4}\,\delta_{0\mu}\,\delta_{0\nu}. \end{aligned} \tag{31}$$

To find the inverse metric $\widetilde{g}^{\mu\nu}|_{\texttt{Finsler}}$ one can now apply Lemma 2. This allows to rewrite $\widetilde{g}_{\mu\nu}$ as

$$\widetilde{g}_{\mu\nu} = A\left(g_{\mu\nu} + \frac{1}{A}\,C_\mu\,C_\nu - \frac{2\,\kappa\,\phi(p_0)\,F^4}{(p_0^0)^4\,A}\,\delta_{0\mu}\,\delta_{0\nu}\right), \tag{32}$$

where

$$\begin{aligned} A &= \phi^2(p_0) + \frac{2\,\phi(p_0)\,\kappa F^2}{(p_0^0)^2}, \\ B &= 2\,\phi(p_0) + \frac{\kappa\,F^2}{(p_0^0)^2}, \\ C_\mu &= \sqrt{\frac{4\,\kappa\,B\,F^2}{(p_0^0)^2}}\left(\ell_\mu - \frac{F}{p_0^0}\,\delta_{0\mu}\right). \end{aligned} \tag{33}$$

Assuming that

$$m_{\mu\nu} = g_{\mu\nu} + \lambda\,C_\mu\,C_\nu \quad \text{with} \quad \lambda := 1/A, \tag{34}$$

$$m^{\mu\nu} = g^{\mu\nu} - \frac{\lambda}{1+\lambda\,C^2}\,C^\mu\,C^\nu, \tag{35}$$

where $C^\mu := g^{\mu\nu}\,C_\nu$. $C^2$ can be calculated as $C^2 = C^\mu\,C_\mu$,

$$\begin{aligned} C^2 &= \frac{4\,\kappa\,B\,F^2}{(p_0^0)^2}\left(\ell^\mu - \frac{F}{p_0^0}g^{0\mu}\right)\left(\ell_\mu - \frac{F}{p_0^0}\,\delta_{0\mu}\right) \\ &= \frac{4\,\kappa\,B\,F^2}{(p_0^0)^2}\left(1 - \frac{F}{p_0^0}\,\ell^0 - \frac{F}{p_0^0}\,\ell^0 + \frac{F^2}{(p_0^0)^2}\,g^{00}\right). \end{aligned} \tag{36}$$

Since $\ell^0 = p_0^0/F$, then $C^2$ reads

$$C^2 = \frac{4\,\kappa\,B\,F^2}{(p_0^0)^2}\left(\frac{F^2\,g^{00} - (p_0^0)^2}{(p_0^0)^2}\right). \tag{37}$$

From the Lemma 2, one gets

$$m^{\mu\nu} = g^{\mu\nu} - \frac{4\,\kappa\,B\,F^2\,(p_0^0)^4}{A\,(p_0^0)^4 + 4\,\kappa\,B\,F^2\left(F^2\,g^{00} - (p_0^0)^2\right)}\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right)\left(\ell^\nu - \frac{F}{p_0^0}\,g^{0\nu}\right). \tag{38}$$

Now $m^{\mu\nu}$ can be simplified as

$$m^{\mu\nu} = g^{\mu\nu} - \Theta\cdot\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right)\left(\ell^\nu - \frac{F}{p_0^0}\,g^{0\nu}\right), \tag{39}$$

where

$$\Theta := \frac{4\,\kappa\,B\,(p_0^0)^2\,F^2}{A\,(p_0^0)^4 + 4\,\kappa\,B\,F^2\left(F^2\,g^{00} - (p_0^0)^2\right)}. \tag{40}$$

When applying the Lemma 2, the derivation of the inverse metric $\widetilde{g}^{\mu\nu}$ leads to

$$\overline{m}_{\mu\nu} = m_{\mu\nu} + \overline{\lambda}\,\overline{C}_\mu\overline{C}_\nu, \tag{41}$$

where $\overline{\lambda} = -\frac{2\,\kappa\,\phi(p_0)\,F^4}{(p_0^0)^4\,A}$. For $\overline{C}_\mu = \delta_{0\mu}$, one derives

$$\overline{C}^\mu = m^{\mu\nu} = \left[g^{\mu\nu} - \Theta\cdot\left(\ell^\mu - \frac{F}{p_0^0}g^{0\mu}\right)\left(\ell^\nu - \frac{F}{p_0^0}\,g^{0\nu}\right)\right]\delta_{0\nu}, \tag{42}$$

which can be simplified as

$$\overline{C}^\mu = g^{0\mu} - \frac{\Theta\cdot\left((p_0^0)^2 - F^2\,g^{00}\right)}{p_0^0\,F}\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right). \tag{43}$$

Now, let us calculate $\overline{C}^2 = \overline{C}^\mu\,\overline{C}_\mu$:

$$\begin{aligned}\overline{C}^2 &= \left[g^{0\mu} - \frac{\Theta\cdot(p_0^2 - F^2\,g^{00})}{F\,p_0^0}\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right)\right]\delta_{0\mu} \\ &= g^{00} - \frac{\Theta\cdot\left((p_0^0)^2 - F^2\,g^{00}\right)^2}{F^2\,(p_0^0)^2}.\end{aligned} \tag{44}$$

Then, we construct $\widetilde{g}^{\mu\nu}$:

$$\begin{aligned}\widetilde{g}^{\mu\nu} &= \frac{1}{A}\Bigg\{g^{\mu\nu} - \Theta\cdot\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right)\left(\ell^\nu - \frac{F}{p_0^0}g^{0\nu}\right) \\ &\quad - \frac{\overline{\lambda}}{1+\overline{\lambda}\,\overline{c}^2}\left[g^{0\mu} - \frac{\Theta\cdot((p_0^0)^2 - F^2\,g^{00})}{p_0^0\,F}\left(\ell^\mu - \frac{F}{p_0^0}\,g^{0\mu}\right)\right] \\ &\qquad \left[g^{0\nu} - \frac{\Theta\cdot((p_0^0)^2 - F^2\,g^{00})}{p_0^0\,F}\left(\ell^\nu - \frac{F}{p_0^0}\,g^{0\nu}\right)\right]\Bigg\} \\ &= \frac{1}{A}\,g^{\mu\nu} - \frac{\Theta}{A}\left[1 + \frac{\overline{\lambda}\,\Theta\cdot((p_0^0)^2 - F^2\,g^{00})^2}{(p_0^0)^2\,F^2\,(1+\overline{\lambda}\,\overline{c}^2)}\right]\ell^\mu\,\ell^\nu \\ &\quad + \frac{\Theta}{A\,p_0^0\,F}\left[F^2 + \frac{\overline{\lambda}\,[(p_0^0)^2 - F^2\,g^{00}]}{1+\overline{\lambda}\,\overline{c}^2}\left(1 + \frac{\Theta\cdot[(p_0^0)^2 - F^2\,g^{00}]}{(p_0^0)^2}\right)\right]\left[\ell^\mu\,g^{0\nu} + \ell^\nu\,g^{0\mu}\right] \\ &\quad - \left\{\frac{\Theta\,F^2}{(p_0^0)^2} + \frac{\overline{\lambda}}{1+\overline{\lambda}\,\overline{c}^2}\left[1 + \frac{\Theta\cdot[(p_0^0)^2 - F^2\,g^{00}]}{(p_0^0)^2}\left(2 + \frac{\Theta\cdot[(p_0^0)^2 - F^2\,g^{00}]}{(p_0^0)^2}\right)\right]\right\}g^{0\mu}\,g^{0\nu} \\ &= \frac{1}{A}\,g^{\mu\nu} \\ &\quad + \left[\frac{4\,\kappa\,F^2\left(F^2\,\kappa + 2\,(p_0^0)^2\,\phi(p_0)\right)\left(2\,F^4\,g^{00}\,\kappa - 2\,F^2\,\kappa\,(p_0^0)^2 - (p_0^0)^4\,\phi(p_0)\right)}{(p_0^0)^2\,\phi^3(p_0)\left(6\,F^4\,g^{00}\,\kappa - 4\,F^2\,\kappa\,(p_0^0)^2 + (p_0^0)^4\,\phi(p_0)\right)\left(2\,F^2\,\kappa + (p_0^0)^2\,\phi(p_0)\right)}\right] \\ &\quad \times \ell^\mu\,\ell^\nu \\ &\quad + \left[\frac{4\,\kappa\,p_0^0\,F^3\left(F^2\,\kappa + 2\,(p_0^0)^2\,\phi(p_0)\right)}{\phi^2(p_0)\left(6\,F^4\,g^{00}\,\kappa - 4\,F^2\,\kappa\,(p_0^0)^2 + (p_0^0)^4\,\phi(p_0)\right)\left(2\,F^2\,\kappa + (p_0^0)^2\,\phi(p_0)\right)}\right] \\ &\quad \times\left[\ell^\mu\,g^{0\nu} + \ell^\nu\,g^{0\mu}\right] \\ &\quad + \left[\frac{-6\,F^4\,\kappa\,(p_0^0)^2}{\phi(p_0)\left(6\,F^4\,g^{00}\,\kappa - 4\,F^2\kappa\,(p_0^0)^2 + (p_0^0)^4\,\phi(p_0)\right)\left(2\,F^2\,\kappa + (p_0^0)^2\,\phi(p_0)\right)}\right] \\ &\quad \times g^{0\mu}\,g^{0\nu}.\end{aligned} \tag{45}$$

It is essential to resolve the distinct mathematical challenge that arises when trying to express the last three lines in terms of $g^{\mu\nu}$.

Both $\widetilde{g}_{\mu\nu}$ and its inverse tensor $\widetilde{g}^{\mu\nu}$ are metrics in arbitrary higher-dimensions. The transformation of these tensors into fundamental metric tensors is discussed in Section 5 just below.

## 5. Quantized Metric Tensor in Pseudo-Riemannian Geometry

When examining a relativistic particle with mass $m$ moving through a background spacetime, the following action functional represents the dynamics of the corresponding quantum field:

$$S[\dot{p}_0^\mu(\tau)] = -m\int_0^1 \sqrt{g_{\mu\nu}p_0^\mu p_0^\nu}d\tau, \tag{46}$$

where $\tau$ is a parametrization in a single coordinate for the worldline of a point particle on a one-dimensional manifold. The proper time $\tau$ along the world line is determined by the equation $d\tau^2 = g_{\mu\nu} p_0^\mu p_0^\nu = ds^2$. The effects of quantum gravity result in the identification of a zero-point length in spacetime [64,65]. Under the duality transformation $ds \rightarrow \ell_{\mathtt{P}}^2/ds$, the invariance of the path integral amplitude describes the way in which quantum gravity transforms the spacetime interval from $(x^{(i)} - y^{(f)})^2$ to $(x^{(i)} - y^{(f)})^2 + \ell_{\mathtt{P}}^2$, where $x^{(i)} = (x_0^{(i)\mu}, p_0^{(i)\nu})$ and $y^{(f)} = (x_0^{(f)\mu}, p_0^{(f)\nu})$ are initial and final locations [64]. The application of the proposed quantization approach to predict the zero-point length can be carried out elsewhere. Actually, GUP and the Duality Principle of the zero-point length of spacetime are what Equation (46) shows, along with the zero-point length of spacetime and duality.

Let us now focus on the line element and metric tensor on the higher-dimensional manifold. As elaborated in the above Sections, the higher-dimensional metric, such as a Finster-type metric, can be derived from the Hessian of the corresponding structure (14),

$$\widetilde{g}_{\mu\nu}\big|_{\mathtt{Finsler}} = \frac{1}{2}\frac{\partial^2}{\partial p_0^\mu \partial p_0^\nu}\left[\phi^2(p_0)\, F^2(x_0^\mu, p_0^\nu)\right], \tag{47}$$

where $F^2(x_0^\mu, p_0^\nu)$ can express any higher-dimensional metric, for instance, Klein metric [17]. The Euler theorem, as is briefly reviewed in Appendix C, describes a process for connecting the resulting metric to the structure $F^2(x_0^\mu, p_0^\nu)$, which—for instance—can be given by Equation (13). Equation (47) can be reexpressed as

$$\phi^2(p_0)\, F^2(x_0^\mu, p_0^\nu) \quad = \quad \widetilde{g}_{\mu\nu}\big|_{\mathtt{Finsler}}\, p_0^\mu p_0^\nu. \tag{48}$$

The Hessian matrix considers the derivative with regard to a particular variable as if the function is exclusively dependent on that variable. The proper mathematical derivative is defined by these two considerations. It is apparent that Equation (48) is related to the quadratic condition governing the line element as described in GR and therefore any attempts to correlate the corresponding line element with the one utilized in spacetime geometry neglect the significant advantages that phase space geometry provides.

Now, let us focus to the derivation associated with the proposed parametrization $\zeta$, which is characterized by a set of embedded four-coordinates. It is assumed that these coordinates effectively represent both geometries along with their corresponding metric tensor. The modified line element related to the higher-dimensional manifold

$$d\tilde{s}^2\Big|_{\mathtt{Finsler}} \quad = \quad \widetilde{g}_{\mu\nu}\big|_{\mathtt{Finsler}}\, dx^\mu(\zeta)\, dx^\nu(\zeta). \tag{49}$$

The line element obtained by the quantized fundamental metric tensor, $\widetilde{g}_{\alpha\beta}$, is

$$d\tilde{s}^2\Big|_{\mathtt{Riemann}} \quad = \quad \widetilde{g}_{\mu\nu}\big|_{\mathtt{Riemann}}\, d\zeta^\mu d\zeta^\nu. \tag{50}$$

The introducing of higher-dimensional coordinates $x^\mu = (x_0^\mu, \phi(p_0) p_0^\mu)$ extends the conventional spacetime framework incorporating the momentum space. On the other hand, the higher-dimensional coordinates permit the use of partial derivatives and quantum mechanical revisions such as RGUP. With the proposed parametrization, the modified line element associated with the $TM$ metric is formulated in the corresponding bases as $dx^\mu(\zeta)$ and $dx^\nu(\zeta)$ in Equation (49). The line element is accordingly modified by the quantized fundamental tensor $\widetilde{g}_{\mu\nu}$ and the bases $d\zeta^\mu$ and $d\zeta^\nu$ in Equation (50), so that

$$\widetilde{g}_{\mu\nu}\big|_{\mathtt{Riemann}} \quad = \quad \widetilde{g}_{\mu\nu}\big|_{\mathtt{Finsler}}\left[\frac{\partial x^\mu(\zeta)}{\partial \zeta^\mu}\,\frac{\partial x^\nu(\zeta)}{\partial \zeta^\nu}\right]. \tag{51}$$

At the proposed generic point, which is characterized by four-dimensional spacetime $x_0$ and four-dimensional momentum space $p_0$, at which the proposed parameterization is valid, we replace

$$\begin{aligned}\widetilde{g}_{\mu\nu}\big|_{\texttt{Riemann}} &= \widetilde{g}_{\mu\nu}\big|_{\texttt{Finsler}}\left[\left(\frac{\partial x_0^\mu}{\partial\zeta^\mu},\phi(p_0)\frac{\partial p_0^\mu}{\partial\zeta^\mu}\right)\left(\frac{\partial x_0^\nu}{\partial\zeta^\nu},\phi(p_0)\frac{\partial p_0^\nu}{\partial\zeta^\nu}\right)\right] \\ &= \widetilde{g}_{\mu\nu}\big|_{\texttt{Finsler}}\left[\frac{\partial x_0^\mu}{\partial\zeta^\mu}\frac{\partial x_0^\nu}{\partial\zeta^\nu}+\frac{\phi^2(p_0)}{\mathcal{F}^2}\frac{\partial p_0^\mu}{\partial\zeta^\mu}\frac{\partial p_0^\nu}{\partial\zeta^\nu}\right], \end{aligned} \tag{52}$$

where the normalization factor $\mathcal{F} = m\mathcal{A}$ represents the maximal proper force, where $\mathcal{A}$ stands for the maximum proper acceleration given by $\mathcal{A} = c^7/G\hbar$. In this context, $\mathcal{F}^2$ represents the normalization applied to the square of the resultant proper force. In the derivation of Equation (52), it was assumed that the products of mixed derivatives entirely vanish.

Several observations can be made regarding the line elements (49) and (50). First, the measurements associated with these line elements are no longer characterized by precision or continuity. Second, the proposed quantization introduces corresponding modifications to both line elements. Last, equating these two line elements allows for the assessment of their modifications, thereby promoting the relationship between manifolds of four dimensions and those of higher dimensions.

The quantum geometry predicts that the world lines correspond to physical particles with a maximum acceleration [21]. Caianiello proposed that the simplest theoretical framework with the maximal proper acceleration must satisfy physical invariance [66–68]. This is not the line element of classical four-dimensional spacetime, but rather an eight-dimensional phase space, for instance. The quantized four-dimensional first-order fundamental tensor can be derived from Equation (52).

**Lemma 3.** *By substituting Equation (26) into Equation (52), one obtains*

$$\begin{aligned}\widetilde{g}_{\mu\nu}\big|_{\texttt{Riemann}} &= \left[1+\left(\phi_4(p_0)+\phi_3(p_0)\frac{2\kappa}{(p_0^0)^2}F^2\right)\frac{\dot{p}_0^\mu\dot{p}_0^\nu}{\mathcal{F}^2}\right]g_{\mu\nu}\big|_{\texttt{Riemann}} \\ &+ \frac{\phi_2(p_0)}{\mathcal{F}^2}\frac{\partial p_0^\mu}{\partial\zeta^\mu}\frac{\partial p_0^\nu}{\partial\zeta^\nu}\,d_{\mu\nu}\big|_{\texttt{Finsler}}, \end{aligned} \tag{53}$$

*where* $\phi_2(p_0) = 1+2\beta p_0^\rho p_{0\rho}$, $\phi_3(p_0) = 1+3\beta p_0^\rho p_{0\rho}$, $\phi_4(p_0) = 1+4\beta p_0^\rho p_{0\rho}$, *and the symmetric tensor* $d_{\mu\nu}\big|_{\texttt{Finsler}}$ *is given as*

$$\begin{aligned} d_{\mu\nu}\big|_{\texttt{Finsler}} = \frac{4\kappa F^2}{(p_0^0)^2}\Big[&(p_0^0)^2(3\,\phi(p_0)-1)\,\ell_\mu\,\ell^\sigma\,g_{\sigma\mu} - F\,p_0^0\,(3\,\phi(p_0)-1)\,\ell^\sigma\left[\delta_{0\nu}\,g_{\sigma\mu}+\delta_{0\mu}\,g_{\sigma\nu}\right] \\ &+2\,F^2\,(5\,\phi(p_0)-2)\,\delta_{0\mu}\,\delta_0^\sigma\,g_{\sigma\nu}\Big]. \end{aligned} \tag{54}$$

**Proof.** Equation (26) represents the tensor $\widetilde{g}_{\mu\nu}\big|_{\texttt{Finsler}}$ along with its associated metrics, $g_{\mu\nu}\big|_{\texttt{Finsler}}$ and $d_{\mu\nu}\big|_{\texttt{Finsler}}$. The former metric is characterized by four dimensions, whereas the latter is defined in eight dimensions. By substituting Equation (26) into Equation (52), one gets

$$\begin{aligned}
\tilde{g}_{\mu\nu}\big|_{\texttt{Riemann}} &= \tilde{g}_{\mu\nu}\big|_{\texttt{Finsler}} \frac{\partial x_0^\mu}{\partial \zeta^\mu}\frac{\partial x_0^\nu}{\partial \zeta^\nu} + \frac{\phi^2(p_0)}{\mathcal{F}^2}\,\tilde{g}_{\mu\nu}\big|_{\texttt{Finsler}} \frac{\partial p_0^\mu}{\partial \zeta^\mu}\frac{\partial p_0^\nu}{\partial \zeta^\nu}, \\
&= \tilde{g}_{\mu\nu}\big|_{\texttt{Finsler}} \frac{\partial x_0^\mu}{\partial \zeta^\mu}\frac{\partial x_0^\nu}{\partial \zeta^\nu} \\
&+ \frac{\phi^2(p_0)}{\mathcal{F}^2}\left[\left(\phi^2(p_0) + 2\frac{\kappa\phi(p_0)}{(p_0^0)^2}F^2\right) g_{\mu\nu}\big|_{\texttt{Riemann}} + d_{\mu\nu}\big|_{\texttt{Finsler}}\right]\frac{\partial p_0^\mu}{\partial \zeta^\mu}\frac{\partial p_0^\nu}{\partial \zeta^\nu}.
\end{aligned} \tag{55}$$

The first term can be substituted by the line element in corresponding geometry:

$$g_{\mu\nu}\big|_{\texttt{Riemann}} = \tilde{g}_{\mu\nu}\big|_{\texttt{Finsler}} \frac{\partial x_0^\mu}{\partial \zeta^\mu}\frac{\partial x_0^\nu}{\partial \zeta^\nu}. \tag{56}$$

At relatively small $\beta$, the function $\phi(p_0)$ can be redefined

$$\begin{aligned}
\tilde{g}_{\mu\nu}\big|_{\texttt{Riemann}} &= \left[1 + \left(\phi_4(p_0) + \phi_3(p_0)\frac{2\kappa}{(p_0^0)^2}F^2\right)\frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathcal{F}^2}\right] g_{\mu\nu}\big|_{\texttt{Riemann}} \\
&+ \frac{\phi_2(p_0)}{\mathcal{F}^2}\frac{\partial p_0^\mu}{\partial \zeta^\mu}\frac{\partial p_0^\nu}{\partial \zeta^\nu}\, d_{\mu\nu}\big|_{\texttt{Finsler}},
\end{aligned} \tag{57}$$

where $\dot{p}_0^\mu = dp_0^\mu/\zeta^\mu$ and $\dot{p}_0^\nu = dp_0^\nu/\zeta^\nu$. The proper force squared, $\dot{p}_0^\mu \dot{p}_0^\nu$, is ad hoc normalized to the maximal proper force squared, $\mathcal{F}^2$, so that $0 \leq \dot{p}_0^\mu \dot{p}_0^\nu/\mathcal{F}^2 \leq 1$. The lower limit removes entirely the quantization contribution from the first line of Equation (57), while the upper limit maximizes it. Due to the complexity of the higher-dimensional metric $d_{\mu\nu}\big|_{\texttt{Finsler}}$, the second line of Equation (57) is not assessed; the assessment can be undertaken elsewhere. □

By examining only the first line of Equation (57), it becomes apparent that the quantized fundamental tensor can be expressed as a conformal transformation of its unquantized form. Given that all values within the square brackets are certainly positive, one concludes that $\tilde{g}_{\mu\nu}\big|_{\texttt{Riemann}} > g_{\mu\nu}\big|_{\texttt{Riemann}}$. The conformal coefficient, represented by the square brackets in Equation (57), indicates that the proposed quantization perspective entirely retains the unquantized fundamental tensor, thereby affirming the foundations of conventional GR. An essential aspect of the proposed quantization approach is its proposed quantization, which broadens the applicability of GR. The quantized GR, as it is predicted by its fundamental tensor, is no longer restricted to its traditional large-scale framework; it also finds relevance at lower (quantum) scales, where the second term in the conformal coefficient in the square brackets in Equation (57) becomes highly significant.

Section 6 just below outlines the derivatives of quantized fundamental inverse tensor in pseudo-Riemann geometry.

## 6. Quantized Inverse Tensor in Pseudo-Riemann Geometry

In Section 5, it was highlighted that the use of higher-dimensional coordinates $x_a = (x_{0\mu}, \phi(p_0)p_{0\nu})$ establishes a link between four-dimensional momentum space and the conventional four-dimensional spacetime:

$$\tilde{g}^{\mu\nu}\big|_{\texttt{Riemann}} = \tilde{g}^{\mu\nu}\big|_{\texttt{Finsler}} \frac{\partial x_\mu}{\partial \zeta_\mu}\frac{\partial x_\nu}{\partial \zeta_\nu}. \tag{58}$$

In phase space geometry, the modification proposed by the RGUP leads to

$$\tilde{g}^{\mu\nu}|_{\texttt{Riemann}} = \tilde{g}^{\mu\nu}|_{\texttt{Finsler}} \left[\frac{\partial x_{0\mu}}{\partial \zeta_\mu}\frac{\partial x_{0\nu}}{\partial \zeta_\nu} + \frac{\phi^2(p_0)}{\mathcal{F}^2}\frac{\partial p_{0\mu}}{\partial \zeta_\mu}\frac{\partial p_{0\nu}}{\partial \zeta_\nu}\right]. \tag{59}$$

**Lemma 4.** *By substituting Equation (45) into Equation (59), we obtain*

$$\tilde{g}_{\mu\nu}|_{\texttt{Riemann}} = \left[1 + \left(\frac{1}{1+\frac{\phi_1(p_0)}{\phi_2(p_0)}\frac{2\kappa}{(p_0^0)^2}F^2}\right)\frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathcal{F}^2}\right] g^{\mu\nu}|_{\texttt{Riemann}} + \frac{\phi_2(p_0)}{\mathcal{F}^2}\frac{\partial p_{0\mu}}{\partial \zeta_\mu}\frac{\partial p_{0\nu}}{\partial \zeta_\nu} d^{\mu\nu}|_{\texttt{Finsler}}, \tag{60}$$

*where* $\phi_1(p_0) = 1 + \beta p_0^\rho p_{0\rho}$ *and the symmetric tensor* $d^{\mu\nu}|_{\texttt{Finsler}}$ *is given as*

$$\begin{aligned} d^{\mu\nu}|_{\texttt{Finsler}} = & \left[\frac{4\kappa F^2\left(F^2\kappa + 2(p_0^0)^2\phi(p_0)\right)\left(2F^4 g^{00}\kappa - 2F^2\kappa(p_0^0)^2 - (p_0^0)^4\phi(p_0)\right)}{(p_0^0)^2\phi^3(p_0)\left(6F^4g^{00}\kappa - 4F^2\kappa(p_0^0)^2 + (p_0^0)^4\phi(p_0)\right)\left(2F^2\kappa + (p_0^0)^2\phi(p_0)\right)}\right] \\ & \times \ell^\mu \ell^\nu \\ + & \left[\frac{4\kappa p_0^0 F^3\left(F^2\kappa + 2(p_0^0)^2\phi(p_0)\right)}{\phi^2(p_0)\left(6F^4g^{00}\kappa - 4F^2\kappa(p_0^0)^2 + (p_0^0)^4\phi(p_0)\right)\left(2F^2\kappa + (p_0^0)^2\phi(p_0)\right)}\right] \\ & \times\left[\ell^\mu g^{0\nu} + \ell^\nu g^{0\mu}\right] \\ + & \left[\frac{-6F^4\kappa(p_0^0)^2}{\phi(p_0)\left(6F^4g^{00}\kappa - 4F^2\kappa(p_0^0)^2 + (p_0^0)^4\phi(p_0)\right)\left(2F^2\kappa + (p_0^0)^2\phi(p_0)\right)}\right] \\ & \times g^{0\mu}g^{0\nu}. \end{aligned} \tag{61}$$

**Proof.** The Equation (45) expresses the tensor $\tilde{g}^{\mu\nu}|_{\texttt{Finsler}}$ along with its associated metrics, $g^{\mu\nu}|_{\texttt{Finsler}}$ and $d^{\mu\nu}|_{\texttt{Finsler}}$. Substituting Equation (45) into Equation (59) results in

$$\begin{aligned} \tilde{g}^{\mu\nu}|_{\texttt{Riemann}} &= \tilde{g}^{\mu\nu}|_{\texttt{Finsler}}\frac{\partial x_{0\mu}}{\partial \zeta_\mu}\frac{\partial x_{0\nu}}{\partial \zeta_\nu} + \frac{\phi^2(p_0)}{\mathcal{F}^2}\tilde{g}^{\mu\nu}|_{\texttt{Finsler}}\frac{\partial p_{0\mu}}{\partial \zeta_\mu}\frac{\partial p_{0\nu}}{\partial \zeta_\nu}, \\ &= \tilde{g}^{\mu\nu}|_{\texttt{Finsler}}\frac{\partial x_{0\mu}}{\partial \zeta_\mu}\frac{\partial x_{0\nu}}{\partial \zeta_\nu} \\ &+ \frac{\phi^2(p_0)}{\mathcal{F}^2}\left[\left(\frac{1}{\phi^2(p_0) + 2\frac{\kappa\phi(p_0)}{(p_0^0)^2}F^2}\right)g^{\mu\nu}|_{\texttt{Riemann}} + d^{\mu\nu}|_{\texttt{Finsler}}\right]\frac{\partial p_{0\mu}}{\partial \zeta_\mu}\frac{\partial p_{0\nu}}{\partial \zeta_\nu}. \end{aligned} \tag{62}$$

The first term can be substituted by the line element in corresponding geometry:

$$g^{\mu\nu}|_{\texttt{Riemann}} = \tilde{g}^{\mu\nu}|_{\texttt{Finsler}}\frac{\partial x_{0\mu}}{\partial \zeta_\mu}\frac{\partial x_{0\nu}}{\partial \zeta_\nu} \tag{63}$$

At comparably small $\beta$, the function $\phi(p_0)$ can be approximated. Hence, the metric $g^{\mu\nu}|_{\texttt{Riemann}}$ can be redefined as

$$\tilde{g}^{\mu\nu}|_{\texttt{Riemann}} = \left[1 + \left(\frac{1}{1+\frac{\phi_1(p_0)}{\phi_2(p_0)}\frac{2\kappa}{(p_0^0)^2}F^2}\right)\frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathcal{F}^2}\right] g^{\mu\nu}|_{\texttt{Riemann}} + \frac{\phi_2(p_0)}{\mathcal{F}^2}\frac{\partial p_{0\mu}}{\partial \zeta_\mu}\frac{\partial p_{0\nu}}{\partial \zeta_\nu} d^{\mu\nu}|_{\texttt{Finsler}}. \tag{64}$$

The complexity of the higher-dimensional metric $d^{\mu\nu}|_{\mathtt{Finsler}}$ (61) results in the second line of Equation (64) not being evaluated. This evaluation can be performed elsewhere. □

## 7. Discussion and Conclusions

In this paper, we discussed the mathematical methodology for quantizing the fundamental metric tensor. This strategy is coherent from a mathematical point of view. The concept of geometric quantization presents a physical alternative to the numerous theories introduced in the last century that aimed to integrate quantum mechanics with GR and formulate a consistent theory of quantum gravity.

In this context, let us refer to Edward Witten's paper [69], which thoroughly discusses the challenges related to the quantization of gravity. In particular, the perspective on matters concerning differential geometry is limited to GR at the "trivial" (2+1) dimensions, which refers to flat spacetime. Witten's quantization process involves "constructing the classical phase space, defining Poisson brackets on this space, and then interpreting the functions on phase space as quantum mechanical operators" [69]. Furthermore [69], "the vierbein formalism of GR makes GR remarkably similar to a gauge theory." This specific formalism makes GR appear similar to a gauge theory. In this regard, it was concluded that achieving a "quantum gravity theory" is a significant challenge [69]. The vierbein formalism could be studied elsewhere. The approach proposed in the current paper is based on the quite straightforward consideration that the metric tensor defines the geometry. This is actually the main characteristic of conventional GR. The results obtained by us based on this are highly promising [9,10,12–16].

The foundation of GR is established on specific postulates and assumptions that are integral to the theory. When attempting to quantize GR, it becomes crucial to keep all postulates and assumptions preserved, as the unquantized version of GR has reliably passed all assessments conducted over the past century. Pseudo-Riemann geometry belongs to the assumptions and assumptions of the conventional GR. The quantization proposal made here focuses on the fundamental metric tensor, the quantity that encodes the entire geometric properties. The mathematical framework of GR certainly relies on this metric tensor. The construction of the affine connection is based on the fundamental tensor. Utilizing the affine connection allows for the derivation of the Riemann curvature tensor. Subsequently, the Riemann curvature tensor can be contracted to yield the Ricci curvature tensor, which further leads to the Ricci tensor. This sequence of derivations ultimately results in the formulation of the Einstein tensor. In the context of the Einstein field equations, Appendix E, the right-hand side of Equation (A16) is represented by the stress–energy tensor, which can be obtained from the Einstein–Hilbert action. It is worth noting that the Einstein–Hilbert action is also influenced by the fundamental tensor.

In this paper, we demonstrated the derivations of the quantized fundamental tensor, starting with the extension of the four-dimensional pseudo-Riemann geometry to the pseudo-Finsler geometry. We exclusively considered the homogeneity property of the pseudo-Finsler structure, presuming the kinematics of a free-falling quantum particle with a positive mass. The Finsler structure is subsequently translated into a Hamilton structure, where momentum replaces velocity. We also introduced a secondary physical component inspired by QM. It is crucial to generalize QM to accommodate the influences of relativistic gravitational fields. Thus, the momentum parameter is modified through the $\phi(p_0)$ function. Integration of momentum modification into the Hamilton structure allows for the straightforward deduction of the corresponding metric once the Hamilton structure is prepared including quantum mechanical ingredients.

The derivatives are valid in arbitrary higher dimensions. Particularly, the derivatives are not limited to the Hamiltonian space, which typically encompasses spacetime and momentum space. This principle is also applied for the inverse metric, as both can be represented as conformal transformations of their corresponding metrics. Given the context of GR within pseudo-Riemann geometry, the quantized metric in higher dimensions is shown needed to be translated to represent the fundamental tensor in four dimensions.

The main point of this paper was to present the "derivation of the quantized metric tensor". We have already considered the "geometric quantization" such as in our earlier studies [9,10,12–16]. Geometric quantization, as we impose it there, is a mathematical method for defining a quantum theory that corresponds to a specific classical theory, GR. The method we applied also relates to symplectic geometry, which is closely related to the Weyl quantization, one of the first attempts for natural quantization of GR. The Weyl's approach connects quantum mechanical observables, which are self-adjoint operators on a Hilbert space, with a real-valued function in classical phase space. We focused on position and momentum and their mapping to the Heisenberg group generators in particular. Groenewold's product of such a pair of observables serves as the foundation for our method [70,71]. In this regard, we emphasize that the approach we employed, being closely related to Weyl quantization, differs from Kirillov's "geometric quantization," known as Kirillov–Kostant–Souriau two-form which considers coadjoint orbit and calls for prequantization, polarization, and then metaplectic correction [72].

In summary, we have successfully derived the higher-dimensional quantized metric. Additionally, despite encountering significant mathematical challenges, we found a way to obtain the inverse metric. Furthermore, we have derived the fundamental tensor in four dimensions. Finally, it is now possible to construct Einstein field equations incorporating quantum revisions so that GR can be applied in both relatively large and relatively short quantum scales. The quantum mechanically revisited Einstein field equations are described in further detail in Appendix E.

As an outlook, we believe that a comprehensive study on how and why vierbein formalism appears in the proposed quantization to be conducted elsewhere.

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## Appendix A. Relativistic Generalized Uncertainty Principle (RGUP)

According to Refs. [4,14–16], the four-dimensional physical position and momentum coordinates of a test particle in curved spacetime are not canonically conjugate variables. To relax the energy regime of the possible implications of the GUP approach on relativistic quantum mechanics, quantum field theory, and quantum gravity, the physical position and momentum coordinates of a test particle are conjectured to be given in terms of their auxiliary four-vectors $x_0^\mu$ and $p_0^\nu$, i.e., $x^\mu\left(x_0^\mu, p_0^\mu\right)$ and $p^\mu\left(x_0^\mu, p_0^\mu\right)$. Under the general static metric $g^{\mu\nu}$, $\left[\hat{x}_0^\mu, \hat{p}_0^\nu\right] = i\hbar g^{\mu\nu}$ and $\hat{p}_0^\mu = -i\hbar\partial/\partial\hat{x}_{0\mu}$ [4,73,74]. The GUP corrections allow us to distinguish between the temporal and spatial components:

$$x^\mu = (x^0, x^i), \tag{A1}$$
$$p^\mu = (p^0, f(p^j)). \tag{A2}$$

With Snyder's algebra [75], we assume that [74]

$$x^\mu = (ct, x_0^i), \tag{A3}$$
$$p^\mu = \left(E/c, p_0^i\left(1 + \beta p_0^\rho p_{0\rho}\right)\right). \tag{A4}$$

The stress–energy tensor describing the test particle in curved spacetime is used to calculate $x_0^0 = ct$ and $p_0^0 = E/c$.

The relativistic generalized noncommutation relation [76], which is Lorentz covariant and preserves the linear law of addition of moments, is proposed as [4,74,77]

$$[\hat{x}^\mu,\, \hat{p}^\nu] = i\hbar\left[(1 + \beta_1 p^\rho p_\rho)g^{\mu\nu} + 2\beta_2 p^\mu p^\nu\right], \tag{A5}$$

where $\beta_1$ and $\beta_2$ are distinct RGUP parameters. Assuming Snyder's algebra [75] and an isotropic property of spacetime, we can revisit the physical space and momentum coordinates as

$$x^\mu = x_0^\mu, \tag{A6}$$
$$p^\mu = \left(1 + \beta p_0^\rho p_{0\rho}\right)p_0^\mu. \tag{A7}$$

The position and momentum noncommutation relation, Equation (A5), determines the uncertainties of their measurements based on the Robertson uncertainty relation [78] and the Schrödinger uncertainty relation [79]

$$\Delta x^\mu\, \Delta p^\nu \geq \frac{1}{2}|\langle[\hat{x}^\mu,\, \hat{p}^\nu]\rangle|. \tag{A8}$$

As a result, the uncertainties on the position and momentum measurements in curved spacetime could be estimated as

$$\Delta x^\mu\, \Delta p^\nu \gtrsim i\frac{\hbar}{2}\left[\langle g\rangle^{\mu\nu} + \beta_1\langle p^2\rangle - \beta_2(\Delta p^\nu)^2 + \beta_2\langle(p^\nu)^2\rangle\right]. \tag{A9}$$

Here, the Einstein summation is used to address uncertainty in the momentum operator $\Delta p$.

## Appendix B. Function $\phi(p_0)$ and the Curvature Properties of Randers Metric

Appendix A discussed how the function $\phi(p_0)$ affects the gravitation of the Heisenberg uncertainty principle. In this Appendix, we introduce how $\phi(p_0) = 1 + \beta p_0^\rho p_{0\rho}$ is capable of quantizing GR while retaining the special curvature properties of Randers' metric [80]

$$\phi(p_0) = 1 + \beta p_0^\rho p_{0\rho} = 1 + \beta F^2(x_0^\rho, p_0^\rho), \tag{A10}$$

where $\beta$ is the RGUP constant and $p_0$ is the auxiliary four-momentum with either a covariant or contravariant dummy index $\rho$.

For simplicity, we denote $F(x_0^\rho, p_0^\rho)$ as $F$. Multiplying both sides of Equation (A10) by $F$ and assuming $\bar{F} = \phi(p_0)\, F$ and $\bar{\beta} = \beta F^3$, one finds:

$$\bar{F} \quad = \quad (1+\beta F^2)F = F + \bar{\beta}. \tag{A11}$$

The Randers metric tensor can be obtained using Equation (A11). This is subject to the conditions on the right-hand side of Equation (A11). The first term, $F$, should be the Riemann metric: $F^2 = g_{\mu\nu} y^\mu y^\nu$, where $y \in T_p M$. At $p$, the second term, $\bar{\beta}$, must take the 1-form. $\bar{\beta} = b_i(x) y^i$, where $||\bar{\beta}||_p := \sup_{y \in T_p M}(\bar{\beta}(y)/F(y))$ is the expression to use. We discovered that $\bar{\beta}$ is 1-homogeneous in $\hat{p}_0$. To fulfill Randers' metric and thus the unification of gravity and electromagnetism, $\bar{\beta}$ has to be linear in $y^i$. With these two conditions, one draws the conclusion that the function $\phi(p_0)$ preserves the special curvature properties of Randers metric and plays the role of unifying gravity and electromagnetism [80], in addition to its crucial roles in gravitating QM and quantizing GR.

## Appendix C. Euler Theorem and Homogeneous Finsler–Hamilton Function

A function $F: V \to \mathbb{R}$ is positively homogeneous of degree $n \in \mathbb{R}$ and $F: TM\backslash\{0\}$ is a smooth function if $F(\lambda \mathbf{v}) = \lambda^n F(\mathbf{v})$ for all $\mathbf{v} = v^\mu \cdot e_\mu \in V$ and $\lambda \in \mathbb{R}^+$. The Finsler–Hamilton function $F$ is positively homogeneous of degree one, because $F(\lambda \mathbf{v}) = \lambda F(\mathbf{v})$ for all $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}^+$. Then, for 1-homogeneous $F$ in $p_0^\nu$, i.e., $n = 1$, and then, $\partial F / \partial p_0^\nu$ becomes homogeneous of degree zero in $p_0^\nu$ with

$$\frac{\partial F}{\partial p_0^\nu} \mathbf{v}\, v^\nu = F. \tag{A12}$$

To realize that Equation (A12) represents Euler theorem, let us assume a smooth curve $c: (a,b) \to M$ is stationary for $c \mapsto \int_a^b F \circ \hat{c}(t) dt$, where $\circ$ stands for group product and $t$ is a proper parametrization, if and only if at each value of $t$ exist local coordinates around $\hat{c}(t)$ such that

$$\frac{\partial F}{\partial x_0^\mu} \circ \hat{c} - \frac{d}{dt}\left(\frac{\partial F}{\partial p_0^\nu} \circ \hat{c}\right) = 0. \tag{A13}$$

For the $(x_0^\mu, p_0^\nu)$, $(\widetilde{x}_0^\mu, \widetilde{p}_0^\nu)$ local coordinates around $p_0^\nu \in T_x M$, where $\widetilde{x}_0^\mu = \widetilde{x}_0^\mu(x)$, $\widetilde{p}_0^\nu = \widetilde{p}_0^\nu(x,y)$, and $p_0^\nu$ are coordinates in the $\partial/\partial x_0^\mu$ basis, then $\partial/\partial x_0^\mu|_y$, $\partial/\partial p_0^\nu|_y$, $\partial/\partial \widetilde{x}_0^\mu|_y$, $\partial/\partial \widetilde{p}_0^\nu|_y \in T(TM\backslash\{0\})$ satisfy the transformation rules

$$\left.\frac{\partial}{\partial x_0^\mu}\right|_y \quad = \quad \frac{\partial \widetilde{x}_0^\gamma}{\partial x_0^\mu} \left.\frac{\partial}{\partial \widetilde{x}_0^\gamma}\right|_y + \frac{\partial^2 \widetilde{x}_0^\gamma}{\partial x_0^\mu \partial (x_0^\mu)^1} (p_0^\nu)^1 \left.\frac{\partial}{\partial \widetilde{p}_0^\gamma}\right|_y, \tag{A14}$$

$$\frac{\partial}{\partial p_0^\nu}|_y \quad = \quad \frac{\partial \widetilde{x}_0^\gamma}{\partial x_0^\mu} \left.\frac{\partial}{\partial \widetilde{p}_0^\gamma}\right|_y, \tag{A15}$$

which can be proved from $\widetilde{p}_0^\nu = \frac{\partial x_0^\nu}{\partial x_0^\gamma} p_0^\gamma$. Then, by implementing the transformation rules (A14) and (A15) to Equation (13), one arrives at Equation (48).

## Appendix D. Remarks on Proposed Quantization

The realization that the origin of quantization is intrinsically linked to the discretization of the energy spectrum implies that quantization must involve discretization. This realization motivated us to commence with the discretization of the Finsler (or Hamiltonian-type) structure. Thus, we have successfully quantized a free-falling particle, which originally necessitated a continuous energy spectrum.

As mentioned in Section 1, the primary objective is to reconcile the principles of QM and GR. The methodology outlined in this study involves several key steps. Initially, it is essential to incorporate gravitational effects into the fundamental framework of QM. Subsequently, the concept of quantum geometry must be introduced, which entails the addition of curvatures to replicate the proposed quantization of Riemann spacetime. To facilitate this, we have extended the four-dimensional Riemann manifold to a Finsler manifold. In order to revert the resulting quantized metric tensor to pseudo-Riemann geometry, we utilized the characteristic that the ratio of lengths between any two colinear vectors is independent of the underlying metric tensor. The fundamental aspects of the approach proposed are derived from this essential premise. It is crucial to note that our approach is predicated on the assumption that the measurements of both line elements are accurate and deterministic. Therefore, achieving a thorough quantization is reliant on the presence of a quantum line element and the consideration of uncertain measurements, as discussed in Appendix C.

To express the line element in a quantum terms, it is essential to integrate the probability distribution with the metric tensor and the 1-form $dx^\mu$. Alternatively, one might propose the introduction of noncommutation relations. It is worth noting that neither $g_{\mu\nu}$ nor $dx^\mu$, including their generalized forms, possess a straightforward noncommutation interpretation [81]. Conversely, the establishment of a noncommutative differential calculus [82,83] alongside a noncommutative metric tensor [84] facilitates the integration of these approaches into a coherent definition of a noncommutative measure for the line element [85]. Moreover, the revised approach to relativistic kinematics, characterized by pseudo-Finslerian geometry, provides insights into a potential vacuum state of quantum gravity at low energies [86]. The adoption of a Finslerian length element, along with RGUP discretization, leads to a quantization of spacetime. A thorough examination of this topic is warranted in future research.

Aside from any nonconservative manifestations of the limitations inherent in the line element measure, we have succeeded in deriving a quantized metric tensor, as shown in Equation (59). This derivation is characterized as a qualitative and approximate estimation. In this context, we nonconservatively underscore that the $g_{ab}$ term in Equation (59) is largely truncated. Therefore, some quantum mechanical aspects may not be sufficiently accounted for. This form of approximation is presently unavoidable, despite the inadequately justified physical and mathematical constraints it entails.

## Appendix E. Einstein Field Equations

The Einstein field equations are expressed as a second-order tensor. In traditional pseudo-Riemannian geometry, this is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R, \tag{A16}$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = g^{\mu\nu} R_{\mu\nu}$ the (Ricci) scalar curvature. All curvature quantities derive from the metric $g_{\mu\nu}$ via the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$. These quantities are all derived from the quantized Christoffel symbols $\widetilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\mu\nu}$. The latter enables the construction of the quantized Riemann curvature tensor

$$\widetilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu\, \widetilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu\, \widetilde{\Gamma}^\rho_{\mu\sigma} + \widetilde{\Gamma}^\rho_{\mu\lambda}\, \widetilde{\Gamma}^\lambda_{\nu\sigma} - \widetilde{\Gamma}^\rho_{\nu\lambda}\, \widetilde{\Gamma}^\lambda_{\mu\sigma}, \tag{A17}$$

which transforms as a $(1,3)$ tensor under diffeomorphisms.

First, by using the perturbative expansion and only keeping first-order corrections in $\delta\Gamma$, one obtains

$$\begin{aligned}\widetilde{R}^{\rho}_{\sigma\mu\nu} &= R^{\rho}_{\sigma\mu\nu} + \partial_\mu \delta\Gamma^{\rho}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\rho}_{\mu\sigma} + \delta\Gamma^{\rho}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\,\delta\Gamma^{\lambda}_{\nu\sigma} \\ &\quad -\delta\Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\nu\lambda}\,\delta\Gamma^{\lambda}_{\mu\sigma} + \mathcal{O}(\delta\Gamma^2). \end{aligned} \tag{A18}$$

Secondly, the quantized Ricci tensor is defined by contracting the first and third indices with the quantized inverse metric $\widetilde{g}^{\mu\nu}$.

$$\widetilde{R}_{\sigma\nu} = \widetilde{R}^{\lambda}{}_{\sigma\lambda\nu}, \tag{A19}$$

so that

$$\begin{aligned}\widetilde{R}_{\sigma\nu} &= R_{\sigma\nu} + \partial_\lambda \delta\Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \delta\Gamma^{\lambda}_{\lambda\sigma} + \delta\Gamma^{\lambda}_{\lambda\mu}\,\Gamma^{\mu}_{\nu\sigma} + \Gamma^{\lambda}_{\lambda\mu}\,\delta\Gamma^{\mu}_{\nu\sigma} \\ &\quad -\delta\Gamma^{\lambda}_{\nu\mu}\,\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\mu}\,\delta\Gamma^{\mu}_{\lambda\sigma} + \mathcal{O}(\delta\Gamma^2). \end{aligned} \tag{A20}$$

The quantized Ricci scalar is then derived through an additional contraction with $\widetilde{g}^{\sigma\nu}$:

$$\begin{aligned}\widetilde{R} &= \widetilde{g}^{\sigma\nu}\,\widetilde{R}_{\sigma\nu} \\ &= R + g^{\sigma\nu}\,\delta R_{\sigma\nu} - \Xi(p_0)\,\frac{\dot{p}_0^{\sigma}\dot{p}_0^{\nu}}{\mathscr{F}^2}\,R_{\sigma\nu} + \mathcal{O}(\delta^2), \end{aligned} \tag{A21}$$

where

$$\Xi(p_0) = \left(1 + 4\beta\, p_0^{\rho} p_{0\rho}\right) + \left(1 + 3\beta\, p_0^{\rho} p_{0\rho}\right) \frac{2\kappa}{(p_0^0)^2}\, K^2. \tag{A22}$$

with $\rho$ as a dummy index and $K$ representing the Klein metric which belongs to the simplest Finsler metrics. Finally, the quantized Einstein tensor is defined by

$$\widetilde{G}_{\sigma\nu} = \widetilde{R}_{\sigma\nu} - \tfrac{1}{2}\,\widetilde{g}_{\sigma\nu}\,\widetilde{R}, \tag{A23}$$

which, upon expanding to first order in $\delta g$ and $\delta R$, becomes

$$\begin{aligned}\widetilde{G}_{\sigma\nu} &= G_{\sigma\nu} + \delta R_{\sigma\nu} - \tfrac{1}{2}\,g_{\sigma\nu}\,\delta R - \tfrac{1}{2}\,\delta g_{\sigma\nu}\,R + \mathcal{O}(\delta^2), \\ &= G_{\sigma\nu} + \nabla_\lambda \delta\Gamma^{\lambda}_{\nu\sigma} - \nabla_\nu \delta\Gamma^{\lambda}_{\lambda\sigma} - \tfrac{1}{2}\,g_{\sigma\nu}\,g^{\alpha\beta}\,\delta R_{\alpha\beta} + \tfrac{1}{2}\,\Xi(p_0)\,\frac{\dot{p}_0^{\alpha}\dot{p}_0^{\beta}}{\mathscr{F}^2}\,g_{\sigma\nu}R_{\alpha\beta} \\ &\quad -\tfrac{1}{2}\,\Xi(p_0)\,\frac{\dot{p}_{0\sigma}\dot{p}_{0\nu}}{\mathscr{F}^2}\,R. \end{aligned} \tag{A24}$$

Regarding the physical implications, one observes that the momentum derivatives $\dot{p}_0^{\mu}$ and the maximal force scale $\mathscr{F}$ become dynamic during periods of high curvature, such as inflation, leading to a spacetime-dependent $\Xi(p_0)$. Then, in the semi-classical Einstein equations, $\delta G_{\sigma\nu}$ serves as a quantum mechanically corrected source term

$$\widetilde{G}_{\sigma\nu} = 8\pi G\,\langle T_{\sigma\nu}\rangle. \tag{A25}$$

The effective equation of state in the early universe appears to be modified by this finding, as well as the dispersion of gravitational waves and the expansion of perturbations. Thorough dimensional and tensorial bookkeeping, as demonstrated in the examples in this Appendix, allows for the consistent integration of these modifications into phenomenological and numerical studies of quantum gravitational effects in cosmology.

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