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RESEARCH ARTICLE

Higher-Order Soliton Solutions for the Derivative Nonlinear Schrödinger Equation via Improved Riemann–Hilbert Method

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Abstract
In this paper we discuss an improved Riemann–Hilbert method, by which arbitrary higher-order soliton solutions for the derivative nonlinear Schrödinger equation can be directly obtained. The explicit determinant form of a higher-order soliton which corresponds to one pth order pole is given. Besides the interaction related to one simple pole and the other one double pole is considered.

Keywords Riemann–Hilbert problem · Inverse scattering transform · Higher-order soliton · The derivative nonlinear Schrödinger equation · Residue condition

1 Introduction

The derivative-type nonlinear Schrödinger equations have several applications in plasma physics and nonlinear fiber optics. In plasma physics, the equation (also called DNLS-I)

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0 \tag{1}$$

describes small-amplitude nonlinear Alfvén waves propagating parallel to the ambient magnetic field [9, 13]. In nonlinear optics, the modified NLS i.e. the equation (1) plus the nonlinear term $|q|^2 q$ describes the case of subpicosecond optical pulses [3, 6]. Recently Moses et al. has experimentally demonstrated that the equation (also called DNLS-II)

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$$\mathrm{i}q_t + q_{xx} + \mathrm{i}|q|^2 q_x = 0 \tag{2}$$

describes the propagation of the self-steepening optical pulses without self-phase modulation [14]. In the view of inverse scattering theory they are gauge equivalent [23] to the following equation [7] (also called DNLS-III)

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}q^2 q_x^* + \frac{1}{2}|q|^4 q = 0, \tag{3}$$

where the asterisk denotes the complex conjugate, so we take DNLS-III as an example to present our work. The above derivative-type equations are important integrable models. In addition, there are more general integrable generalizations, such as the high-order Kaup–Newell equation [18], the generalized mixed nonlinear Schrödinger equation [19], Kundu equation [11], Kundu–Eckhaus equation [5, 11]. Much research has been conducted for them, here we will not dwell on a detailed exposition of various results. The Eq. (3) are the compatible condition of the following linear differential equations

$$\psi_x = X\psi, \qquad \psi_t = T\psi, \tag{4}$$

where

$$X = -\mathrm{i}\lambda^2\sigma_3 + \lambda Q + \frac{\mathrm{i}}{2}|q|^2\sigma_3,$$
$$T = -2\mathrm{i}\lambda^4\sigma_3 + 2\lambda^3 Q + \mathrm{i}\lambda^2|q|^2\sigma_3 + \mathrm{i}\lambda\sigma_3 Q_x + \frac{\mathrm{i}}{4}|q|^4\sigma_3 - \frac{1}{2}(QQ_x - Q_x Q)$$

and the potential matrix

$$Q(x,t) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix},$$

σ_3 is one of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known that Zakharov first given higher-order solitons for the NLS equation corresponding to a double pole [24]. Subsequently higher-order solitons have also been studied for the modified KdV equation [22], the sine-Gordon equation [21] and so on. So far various methods have been developed to deal with higher-order solitons, for example the usual Riemann–Hilbert (RH) method [20], generalized Darboux transform [8, 12], $\bar{\partial}$-method [10], robust inverse scattering transform [4] et al.. In this paper, to avoid the difficulty of calculating residue conditions with multiple poles, different from the work [25, 26] we generalize Olmedilla's idea [16] to the framework for RH method and arbitrary higher-order soliton solutions for the derivative nonlinear Schrödinger equation can be directly obtained.

This paper is arranged as follows. In Sect. 2, we summary the inverse scattering method for DNLS-III. In Sect. 3, we derive the explicit determinant form of a higher-order soliton which corresponds to one pth order pole. In Sect. 4, the interaction related to one simple pole and the other one double pole is displayed.

2 Summary of the Inverse Scattering Method for the DNLS Equation

Firstly, we summarize the already well-known results [15] for the DNLS-III that will be used in our study. In this section we solve the initial value problem for the DNLS-III with the following zero boundary condition (ZBC) at $x \to \infty$:

$$\lim_{x\to\pm\infty} q(x,t) = 0, \tag{5}$$

meanwhile the complex function $q(x)$ satisfies

$$\int_{-\infty}^{\infty} |x^n||q(x)|dx < \infty.$$

2.1 The Direct Scattering Problem

2.1.1 Jost Solution and Analyticity

For the oriented curve $\Sigma = \mathbb{R} \bigcup \mathrm{i}\mathbb{R}$ (see Fig. 1) in complex λ-plane, we define $J_\pm$ as the Jost solutions of the Lax representation (4) which obey the boundary conditions

$$J_\pm(x,\lambda) \to e^{-\mathrm{i}\lambda^2 x\sigma_3}, \quad x \to \pm\infty. \tag{6}$$

Let

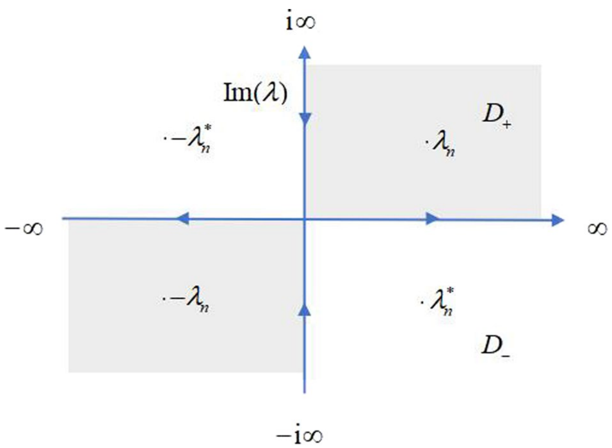

Fig. 1 The complex λ-plane, showing the regions $D_\pm$ where $\Re(\lambda)\Im(\lambda) > 0$ (grey) and $\Re(\lambda)\Im(\lambda) < 0$ (white), respectively

$$U_\pm(x,\lambda) = J_\pm(x,\lambda)e^{\mathrm{i}\lambda^2 x\sigma_3},$$

then

$$\begin{aligned} &U_{\pm x} = -\mathrm{i}\lambda^2[\sigma_3, U_\pm] + \lambda Q U_\pm + \frac{\mathrm{i}}{2}|q|^2\sigma_3 U_\pm, \\ &U_\pm(x,\lambda) \to \mathbb{I}, \quad x \to \pm\infty, \end{aligned} \tag{7}$$

which is equivalent to Volterra integral equation

$$U_\pm(x,\lambda) = \mathbb{I} + \int_{\pm\infty}^{x} e^{-\mathrm{i}\lambda^2\hat{\sigma}_3(x-y)}(\lambda Q(y) + \frac{\mathrm{i}}{2}|q|^2\sigma_3)U_\pm(y,\lambda)dy. \tag{8}$$

Denoting $D_\pm = \{\lambda \in \mathbb{C} | \pm \Re(\lambda)\Im(\lambda) > 0\}$, as shown in Fig. 1. By performing the Neumann series (cf. [1]) on the Volterra integral equations (8), we know that $U_{-1}(J_{-1})$ and $U_{+2}(J_{+2})$ can be analytically extended to D_+ and continuously extended to $D_+ \bigcup \Sigma$, while $U_{+1}(J_{+1})$ and $U_{-2}(J_{-2})$ can be analytically extended to D_- and continuously extended to $D_- \bigcup \Sigma$, where the subscripts 1 and 2 identify matrix columns, i.e., $U_\pm = (U_{\pm 1}, U_{\pm 2})$.

2.1.2 Scattering Matrix

Abel's theorem implies that for any solution $\psi(x,\lambda)$ of the Lax representation (4) one has $\partial_x(\det\psi) = 0$. Since $\lim_{x\to\pm\infty} J_\pm e^{\mathrm{i}\lambda^2\sigma_3 x} = I$ for $\lambda \in \Sigma$, we have

$$\det J_\pm(x,\lambda) = 1.$$

It follows that $\forall\lambda \in \Sigma$ both J_+ and J_- are two fundamental matrix solutions of the scattering problem (4). Define the scattering matrix $S(k)$

$$J_-(x,\lambda) = J_+(x,\lambda)S(\lambda), \tag{9}$$

where $S = \{s_{ij}\}$. Rewrite it by component

$$J_{-1}(x,\lambda) = s_{11}(\lambda)J_{+1}(x,\lambda) + s_{21}(\lambda)J_{+2}(x,\lambda), \tag{10}$$

$$J_{-2}(x,\lambda) = s_{12}(\lambda)J_{+1}(x,\lambda) + s_{22}(\lambda)J_{+2}(x,\lambda). \tag{11}$$

Furthermore we obtain

$$s_{11}(\lambda) = W(J_{-1}, J_{+2}), \quad s_{22}(\lambda) = W(J_{+1}, J_{-2}),$$

where $W(f,g)$ is the Wronskian of f and g and reflection coefficients

$$\rho(\lambda) = \frac{s_{21}}{s_{11}}, \quad \tilde{\rho}(\lambda) = \frac{s_{12}}{s_{22}}.$$

From the analytic property of Jost solutions, $s_{11}(\lambda)$ can be analytically extended to D_+ and continuously extended to $D_+ \bigcup \Sigma$, while $s_{22}(\lambda)$ can be analytically extended to D_- and continuously extended to $D_- \bigcup \Sigma$.

2.1.3 Symmetry Conditions and Discrete Spectrum

By virtue of the uniqueness of the Jost solutions, we have the following symmetry conditions

$$J_\pm(\lambda) = \sigma_3 J_\pm(-\lambda)\sigma_3, \quad J_\pm(\lambda) = \mathrm{i}\sigma_2 J^*_\mp(\lambda^*)(\mathrm{i}\sigma_2)^{-1}. \tag{12}$$

So

$$\begin{aligned} &s_{11}(\lambda) = s_{11}(-\lambda), \quad s_{12}(\lambda) = -s_{12}(-\lambda), \quad s_{21}(\lambda) = -s_{21}(-\lambda), \\ &s_{22}(\lambda) = s_{22}(-\lambda), \quad s_{11}(\lambda) = s^*_{22}(\lambda^*), \quad s_{12}(\lambda) = -s^*_{21}(\lambda^*) \end{aligned} \tag{13}$$

and

$$\rho(\lambda) = -\rho(-\lambda), \quad \tilde{\rho}(\lambda) = -\rho^*(\lambda^*). \tag{14}$$

If $s_{11}(\lambda_n) = 0, n = 1, \dots, N$, the eigenfunctions $J_{-1}(x, \lambda)$ and $J_{+2}(x, \lambda)$ at $\lambda = \lambda_n$ must be proportional, i.e.

$$J_{-1}(x, \lambda_n) = \gamma_n J_{+2}(x, \lambda_n), \tag{15}$$

where γ_n is a complex valued constant. Owing to the relations (13), we have

$$s_{11}(\lambda) = 0 \Longleftrightarrow s_{11}(-\lambda) = 0 \Longleftrightarrow s_{22}(-\lambda^*) = 0 \Longleftrightarrow s_{22}(\lambda^*) = 0,$$

then

$$\begin{aligned} J_{-1}(-\lambda_n) &= \hat{\gamma}_n J_{+2}(-\lambda_n), \\ J_{-2}(-\lambda^*_n) &= \check{\gamma}_n J_{+1}(-\lambda^*_n), \\ J_{-2}(\lambda^*_n) &= \tilde{\gamma}_n J_{+1}(\lambda^*_n), \end{aligned} \tag{16}$$

where $\hat{\gamma}_n = -\gamma_n, \check{\gamma}_n = \gamma^*_n, \tilde{\gamma}_n = -\gamma^*_n$. That is, the discrete spectrum is the set

$$Z = \{\lambda_n, -\lambda_n, \lambda^*_n, -\lambda^*_n\}.$$

This distribution is shown in Fig. 1.

2.1.4 Asymptotics as $\lambda \to \infty$

The Wentzel–Kramers–Brillouin (WKB) expansion can be used to derive the asymptotic of modified Jost solutions. In fact, we know that $U_\pm$ are analytic in $\mathbb{C}/\Sigma$, then we can write an asymptotic expansion for $U_\pm$ when $\lambda \to \infty$

$$U_\pm(x, \lambda) = U_{\pm,0} + \frac{U_{\pm,1}}{\lambda} + \frac{U_{\pm,2}}{\lambda^2} + O\Big(\frac{1}{\lambda^3}\Big). \tag{17}$$

Substituting the above expansion into the Eq. (7) and utilizing the expressions for s_{11} and s_{22}, we have

$$U_{\pm}(x,\lambda) \longrightarrow I, \quad |\lambda| \longrightarrow \infty$$

and

$$s_{11}(\lambda) \to 1, \quad s_{22}(\lambda) \to 1.$$

2.2 The Inverse Scattering Problem

2.2.1 Riemann–Hilbert Problem and Reconstruction Formula

In order to construct RH problem, introduce the sectionally meromorphic matrices

$$M_{+}(x,\lambda) = \left[\frac{U_{-1}}{s_{11}}, U_{+2}\right], \quad M_{-}(x,\lambda) = \left[U_{+1}, \frac{U_{-2}}{s_{22}}\right].$$

From the Eqs. (10) and (11) we obtain the jump condition

$$M_{+}(x,\lambda) = M_{-}(x,\lambda)(I + G(\lambda)),$$

where

$$G(\lambda) = \begin{pmatrix} -\rho(\lambda)\tilde{\rho}(\lambda) & \tilde{\rho}(\lambda)e^{-2\mathrm{i}\lambda^2 x} \\ \rho(\lambda)e^{2\mathrm{i}\lambda^2 x} & 0 \end{pmatrix}.$$

Recalling the asymptotic behavior of the scattering coefficients, it is easy to obtain that

$$M_{\pm}(x,\lambda) \to I, \quad |\lambda| \to \infty.$$

From the Eq. (7) we can reconstruct the potential $q(x,t)$ from the solution of the RH problem as follow

$$q(x,t) = 2\mathrm{i} \lim_{\lambda\to\infty} \lambda[M(\lambda;x,t)]_{12}. \tag{18}$$

2.2.2 Residue Conditions and Solution for RHP

To solve the RHP, introduce the Cauchy projectors $P^{\pm}$ over Σ:

$$P^{\pm}[f](\lambda) = \frac{1}{2\pi\mathrm{i}} \int_{\Sigma} \frac{f(\zeta)}{\zeta - (\lambda \pm \mathrm{i}0)} d\zeta.$$

If $f_{\pm}(\lambda)$ is analytic in the region $D_{\pm}$ and $f_{\pm}(\lambda) \to 0$ as $|\lambda| \to \infty$, then

$$P^{\pm}(f_{\mp})(\lambda) = 0, \quad P^{\pm}(f_{\pm})(\lambda) = \pm f_{\pm}(\lambda). \tag{19}$$

Furthermore, we can obtain the residue conditions from the equations (15) and (16)

$$\underset{\lambda=\lambda_n}{Res}\left(\frac{U_{-1}}{s_{11}}\right) = C_n e^{2i\lambda_n^2 x} U_{+2}(\lambda_n),$$
$$\underset{\lambda=-\lambda_n}{Res}\left(\frac{U_{-1}}{s_{11}}\right) = C_n e^{2i\lambda_n^2 x} U_{+2}(-\lambda_n),$$
$$\underset{\lambda=\lambda_n^*}{Res}\left(\frac{U_{-2}}{s_{22}}\right) = -C_n^* e^{-2i\lambda_n^{*2} x} U_{+1}(\lambda_n^*),$$
$$\underset{\lambda=-\lambda_n^*}{Res}\left(\frac{U_{-2}}{s_{22}}\right) = -C_n^* e^{-2i\lambda_n^{*2} x} U_{+1}(-\lambda_n^*)$$

where $C_n = \frac{\gamma_n}{s'_{11}(\lambda_n)}$.

Applying P^- to both sides of the expression (10) and P^+ to both sides of the expression (11), meanwhile, taking advantage of the formulae (19) and the above residue conditions, we have

$$\begin{aligned} U_{+1}(\lambda) =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{n=1}^{N} \frac{C_n e^{2i\lambda_n^2 x} U_{+2}(\lambda_n)}{\lambda - \lambda_n} \\ &+ \sum_{n=1}^{N} \frac{C_n e^{2i\lambda_n^2 x} U_{+2}(-\lambda_n)}{\lambda + \lambda_n} + \frac{1}{2\pi i} \int_{\Sigma} \frac{\rho e^{2i\zeta^2 x} U_{+2}}{\zeta - (\lambda - i0)} d\zeta, \\ U_{+2}(\lambda) =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{n=1}^{N} \frac{C_n^* e^{-2i\lambda_n^{*2} x} U_{+1}(\lambda_n^*)}{\lambda - \lambda_n^*} \\ &- \sum_{n=1}^{N} \frac{C_n^* e^{-2i\lambda_n^{*2} x} U_{+1}(-\lambda_n^*)}{\lambda + \lambda_n^*} + \frac{1}{2\pi i} \int_{\Sigma^-} \frac{\tilde{\rho} e^{-2i\zeta^2 x} U_{+1}}{\zeta - (\lambda + i0)} d\zeta. \end{aligned} \tag{20}$$

2.2.3 Time Evolution

The time evolution of the scattering data can be determined by the time part of the Lax representation (4). By the calculation (cf. [1] for details) we have

$$\lambda_n(t) = \lambda_n, \quad \rho(t) = \rho e^{4i\lambda_n^4 t}$$
$$\tilde{\rho}(t) = \tilde{\rho} e^{4i\lambda_n^4 t}, \quad C_n(t) = C_n e^{4i\lambda_n^4 t}.$$

2.3 The Soliton Solutions for the DNLS Equation

We now consider the potential $q(x, t)$ for which the reflection coefficient $\rho(\lambda)$ vanishes. As usual, in the case there is no jump from M_+ to M_- across the continuous spectrum, and the Eq. (20) reduce to an algebraic system. Next we take 1-soliton solution as an example.

Let $\rho(\lambda) = \tilde{\rho}(\lambda) = 0$ and $N = 1$. From the Eq. (20) and the formula (18), we can obtain 1-soliton solution

$$q = -4|C_1|e^{-\mathrm{i}\vartheta} Sech[2\eta(x + 4\xi t) + \mu + \mathrm{i}\nu]. \tag{21}$$

where

$$\lambda_1^2 = \xi + \mathrm{i}\eta, \quad \frac{|C_1|}{\eta}\lambda_1 = e^{-(\mu+\mathrm{i}\nu)},$$
$$\vartheta = \omega + 2\xi x + 4(\xi^2 - \eta^2)t, \quad \omega = arg(C_1).$$

Remark 1 If we consider N different zeros of $s_{11}(\lambda)$, By observation and calculation we can obtain the determinant form of N-soliton solution which is similar to the expression (29), this procedure will be elaborated in the next section.

3 Higher-Order Soliton Solutions for the DNLS Equation

In this section we generalize Olmedilla's idea to the framework for RH method. If the potential $q(x)$ decay rapidly enough at infinity, so that $\rho(\lambda)$ can be analytically continued above or below all poles $\{\pm\lambda_n\}_{n=1}^N$ and $\tilde{\rho}(\lambda)$ can be analytically continued below or above all poles $\{\pm\lambda_n^*\}_{n=1}^N$ (cf. [1, 2]). The Eq. (20) can be simplified by virtue of the residue theorem as follows:

$$\begin{aligned} U_{+1}(\lambda) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\rho(\zeta)e^{2\mathrm{i}\theta}U_{+2}(\zeta)}{\zeta - \lambda} d\zeta, \\ U_{+2}(\lambda) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi\mathrm{i}} \int_{\bar{\Gamma}} \frac{\tilde{\rho}(\zeta)e^{-2\mathrm{i}\theta}U_{+1}(\zeta)}{\zeta - \lambda} d\zeta, \end{aligned} \tag{22}$$

where $\theta = \zeta^2 x + 2\zeta^4 t$, Γ is the union of a contour from ∞ to $\mathrm{i}\infty$ that passes above all poles $\{\lambda_n\}_{n=1}^N$ and a contour from $-\infty$ to $-\mathrm{i}\infty$ that passes below all poles $\{-\lambda_n\}_{n=1}^N$, $\bar{\Gamma}$ is the union of a contour from $-\infty$ to $\mathrm{i}\infty$ that passes above all poles $\{-\lambda_n^*\}_{n=1}^N$ and a contour from ∞ to $-\mathrm{i}\infty$ that passes below all poles $\{\lambda_n^*\}_{n=1}^N$ in the Fig. 1.

Supposing that $\rho(\lambda)$ has a pth order pole, we consider the Laurent series expansion of $\rho(\lambda)$ around the point λ_n in the region D_+. From the symmetries (14), we have

$$\begin{aligned} \rho(\lambda) &= \rho_0(\lambda) + \sum_{k=1}^{p} \left[\frac{\rho_{-k}}{(\lambda - \lambda_n)^k} + (-1)^{p-1} \frac{\rho_{-k}}{(\lambda + \lambda_n)^k} \right], \\ \tilde{\rho}(\lambda) &= \tilde{\rho}_0(\lambda) - \sum_{k=1}^{p} \left[\frac{\rho_{-k}^*}{(\lambda - \lambda_n^*)^k} + (-1)^{p-1} \frac{\rho_{-k}^*}{(\lambda + \lambda_n^*)^k} \right], \end{aligned} \tag{23}$$

where $\rho_0(\lambda)$ and $\tilde{\rho}_0(\lambda)$ are analytic and satisfy the symmetries (14), $\rho_{-i}(i = 1, 2, \cdots, p)$ is a constant. Plugging the expressions (23) into (22), the integral equations (22) can be rewritten as

$$U_{+1}(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=1}^{p} \rho_{-k} \frac{\partial^{(k-1)}\left(\frac{e^{2i\theta}U_{+2}(\zeta)}{\zeta-\lambda}\right)}{\partial\zeta^{k-1}}|_{\zeta=\lambda_n}$$
$$- \sum_{k=1}^{p}(-1)^{k-1}\rho_{-k} \frac{\partial^{(k-1)}\left(\frac{e^{2i\theta}U_{+2}(\zeta)}{\zeta-\lambda}\right)}{\partial\zeta^{k-1}}|_{\zeta=-\lambda_n} + \frac{1}{2\pi i}\int_{\Sigma}\frac{\rho(\zeta)e^{2i\theta}U_{+2}(\zeta)}{\zeta-\lambda}d\zeta, \tag{24}$$

$$U_{+2}(\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=1}^{p} \rho^{*}_{-k} \frac{\partial^{(k-1)}\left(\frac{e^{-2i\theta}U_{+1}(\zeta)}{\zeta-\lambda}\right)}{\partial\zeta^{k-1}}|_{\zeta=\lambda^{*}_n}$$
$$+ \sum_{k=1}^{p}(-1)^{k-1}\rho^{*}_{-k} \frac{\partial^{(k-1)}\left(\frac{e^{-2i\theta}U_{+1}(\zeta)}{\zeta-\lambda}\right)}{\partial\zeta^{k-1}}|_{\zeta=-\lambda^{*}_n} + \frac{1}{2\pi i}\int_{\Sigma}\frac{\tilde{\rho}(\zeta)e^{-2i\theta}U_{+1}(\zeta)}{\zeta-\lambda}d\zeta. \tag{25}$$

For the case of a reflectionless potential, i.e. $\rho(\lambda) = \tilde{\rho}(\lambda) = 0$ when $\lambda \in \Sigma$, the integral equation (24) and (25) reduce to the system of linear equations. By calculation we have

$$\tilde{H}_p = \omega_p - \Omega_p H_p, \quad H_p = \Omega^{*}_p \tilde{H}_p, \tag{26}$$

where $\Omega = (F_{ij})_{p\times p}$ and F_{ij} is a 2×2 matrix. We denote

$$F^{11}_{ij} = \sum_{k=j}^{p} C^{j-1}_{k-1}\rho_{-k}\frac{\partial^{(k+i-j-1)}[(\zeta-\lambda)^{-1}e^{2i\theta}]}{\partial\lambda^{i-1}\partial\zeta^{k-j}}|_{\zeta=\lambda_n,\lambda=\lambda^{*}_n},$$
$$F^{12}_{ij} = \sum_{k=j}^{p} C^{j-1}_{k-1}(-1)^{k-1}\rho_{-k}\frac{\partial^{(k+i-j-1)}[(\zeta-\lambda)^{-1}e^{2i\theta}]}{\partial\lambda^{i-1}\partial\zeta^{k-j}}|_{\zeta=-\lambda_n,\lambda=\lambda^{*}_n},$$
$$F^{21}_{ij} = \sum_{k=j}^{p} C^{j-1}_{k-1}\rho_{-k}\frac{\partial^{(k+i-j-1)}[(\zeta-\lambda)^{-1}e^{2i\theta}]}{\partial\lambda^{i-1}\partial\zeta^{k-j}}|_{\zeta=\lambda_n,\lambda=-\lambda^{*}_n},$$
$$F^{22}_{ij} = \sum_{k=j}^{p} C^{j-1}_{k-1}(-1)^{k-1}\rho_{-k}\frac{\partial^{(k+i-j-1)}[(\zeta-\lambda)^{-1}e^{2i\theta}]}{\partial\lambda^{i-1}\partial\zeta^{k-j}}|_{\zeta=-\lambda_n,\lambda=-\lambda^{*}_n}$$

and

$$\tilde{H}_p = \begin{pmatrix} U_{+11}(\lambda^{*}_n) & U_{+11}(-\lambda^{*}_n) & \frac{\partial U_{+11}}{\partial\lambda}(\lambda^{*}_n) & \cdots & \frac{\partial^{(p-1)}U_{+11}}{\partial\lambda^{p-1}}(-\lambda^{*}_n) \end{pmatrix}^{T}_{1\times 2p},$$
$$H_p = \begin{pmatrix} U_{+21}(\lambda_n) & U_{+21}(-\lambda_n) & \frac{\partial U_{+21}}{\partial\lambda}(\lambda_n) & \cdots & \frac{\partial^{(p-1)}U_{+21}}{\partial\lambda^{p-1}}(-\lambda_n) \end{pmatrix}^{T}_{1\times 2p},$$
$$\omega_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}^{T}_{1\times 2p}.$$

From the formula (18), we have

$$q = -2i\Lambda_p\tilde{H}_p, \tag{27}$$

where

$$\Lambda_p = \begin{pmatrix} G_1 & \hat{G}_1 & G_2 & \hat{G}_2 & \cdots & G_p & \hat{G}_p \end{pmatrix}_{1\times 2p},$$
$$G_j = \sum_{k=j}^{p} C_{k-1}^{j-1} \rho_{-k}^{*} \frac{\partial^{(k-j)}(e^{-2i\theta})}{\partial \zeta^{k-j}}|_{\zeta=\lambda_n^*},$$
$$\hat{G}_j = \sum_{k=j}^{p} C_{k-1}^{j-1} (-1)^{k-1} \rho_{-k}^{*} \frac{\partial^{(k-j)}(e^{-2i\theta})}{\partial \zeta^{k-j}}|_{\zeta=-\lambda_n^*}.$$

Using the expression (26) and (27), we obtain

$$q = -2i\Lambda_p (I + \Omega_p \Omega_p^*)^{-1} \omega_p. \tag{28}$$

The expression (28) can written into determinant form

$$q = 2i \frac{\det \tilde{\Phi}_p}{\det \Phi_p}, \tag{29}$$

where

$$\Phi_p = I + \Omega_p \Omega_p^*, \quad \tilde{\Phi}_p = \begin{pmatrix} 0 & \Lambda_p \\ \omega_p & \Phi_p \end{pmatrix}.$$

Remark 2 If $\rho(\lambda)$ has r different poles, $\lambda_1, \lambda_2, \dots, \lambda_r$ in the region D_+, and their order are $p_1, p_2, \dots, p_r$ respectively. The process of dealing with the general case is similar to one pth order pole. To illustrate it we give an example of the interaction between one simple pole soliton and the other double pole soliton in the next section.

4 Example of the Solutions for DNLS Equation

4.1 The Double Soliton Solution

In this subsection we consider the soliton solution related to one double pole. Let

$$\begin{aligned} \rho(\lambda) &= \rho_0 + \frac{\rho_{-2}}{(\lambda-\lambda_1)^2} - \frac{\rho_{-2}}{(\lambda+\lambda_1)^2} + \frac{\rho_{-1}}{\lambda-\lambda_1} + \frac{\rho_{-1}}{\lambda+\lambda_1}, \\ \tilde{\rho}(\lambda) &= \tilde{\rho}_0 - \frac{\rho_{-2}^*}{(\lambda-\lambda_1^*)^2} + \frac{\rho_{-2}^*}{(\lambda+\lambda_1^*)^2} - \frac{\rho_{-1}^*}{\lambda-\lambda_1^*} - \frac{\rho_{-1}^*}{\lambda+\lambda_1^*}. \end{aligned} \tag{30}$$

Substituting the expressions (30) into the Eqs. (24) and (25), we obtain

$$H_2 = \omega_2 + \Omega_2 \tilde{H}_2, \quad \tilde{H}_2 = -\Omega_2^* H_2, \tag{31}$$

where

$$\omega_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T, \quad \Omega_2 = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$
$$H_2 = \begin{pmatrix} U_{+11}(\lambda_1^*) & U_{+11}(-\lambda_1^*) & \frac{\partial U_{+11}}{\partial \lambda}(\lambda_1^*) & \frac{\partial U_{+11}}{\partial \lambda}(-\lambda_1^*) \end{pmatrix}^T,$$
$$\tilde{H}_2 = \begin{pmatrix} U_{+21}(\lambda_1) & U_{+21}(-\lambda_1) & \frac{\partial U_{+21}}{\partial \lambda}(\lambda_1) & \frac{\partial U_{+21}}{\partial \lambda}(-\lambda_1) \end{pmatrix}^T,$$

and

$$F_{11}^{11} = \left[\frac{\rho_{-1} + \rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda_1^3 t)}{\lambda_1 - \lambda_1^*} - \frac{\rho_{-2}}{(\lambda_1 - \lambda_1^*)^2} \right] e^{2\mathrm{i}\theta}, \quad F_{11}^{22} = -F_{11}^{11},$$
$$F_{11}^{21} = \left[\frac{\rho_{-1} + \rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda_1^3 t)}{\lambda_1 + \lambda_1^*} - \frac{\rho_{-2}}{(\lambda_1 + \lambda_1^*)^2} \right] e^{2\mathrm{i}\theta}, \quad F_{11}^{12} = -F_{11}^{21},$$
$$F_{12}^{11} = \frac{\rho_{-2}}{\lambda_1 - \lambda_1^*} e^{2\mathrm{i}\theta}, \quad F_{12}^{21} = \frac{\rho_{-2}}{\lambda_1 + \lambda_1^*} e^{2\mathrm{i}\theta}, \quad F_{12}^{12} = F_{12}^{21}, \quad F_{12}^{22} = F_{12}^{11},$$
$$F_{21}^{11} = \left[\frac{\rho_{-1} + \rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda_1^3 t)}{(\lambda_1 - \lambda_1^*)^2} - \frac{2\rho_{-2}}{(\lambda_1 - \lambda_1^*)^3} \right] e^{2\mathrm{i}\theta}, \quad F_{21}^{22} = F_{21}^{11},$$
$$F_{21}^{21} = \left[\frac{\rho_{-1} + \rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda_1^3 t)}{(\lambda_1 + \lambda_1^*)^2} - \frac{2\rho_{-2}}{(\lambda_1 + \lambda_1^*)^3} \right] e^{2\mathrm{i}\theta}, \quad F_{21}^{12} = F_{21}^{21},$$
$$F_{22}^{11} = \frac{\rho_{-2}}{(\lambda_1 - \lambda_1^*)^2} e^{2\mathrm{i}\theta}, F_{22}^{21} = \frac{\rho_{-2}}{(\lambda_1 + \lambda_1^*)^2} e^{2\mathrm{i}\theta}, \quad F_{22}^{12} = -F_{22}^{21}, F_{22}^{22} = -F_{22}^{11}.$$

From the formula (29), we have

$$q = 2\mathrm{i} \frac{\det \tilde{\Phi}_2}{\det \Phi_2}, \tag{32}$$

where

$$\Phi_2 = I + \Omega_2 \Omega_2^*, \quad \tilde{\Phi}_2 = \begin{pmatrix} 0 & \Lambda_2 \\ \omega_2 & \Phi_2 \end{pmatrix}, \quad \chi = (4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda_1^3 t),$$
$$\Lambda_2 = \begin{pmatrix} \rho_{-1}^* - \rho_{-2}^* \chi & \rho_{-1}^* - \rho_{-2}^* \chi & \rho_{-2}^* & -\rho_{-2}^* \end{pmatrix}.$$

As shown in Fig. 2.

4.2 The Solution Related to One Simple Pole and the Other One Double Pole

In this subsection we consider the soliton solution related to one simplie pole and the other one double pole. Let

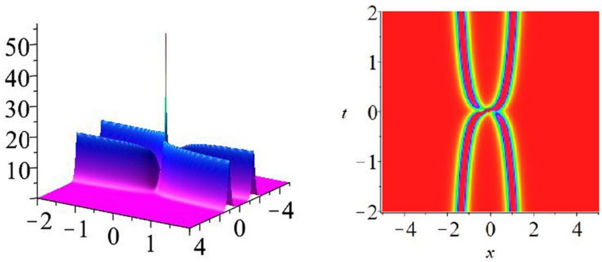

Fig. 2 $\rho_{-1} = \mathrm{i}, \rho_{-2} = 1, \lambda_1 = 1 + \mathrm{i}$, the left: the 3D image of $|q|^2$, the right: the density image of $|q|^2$

$$\begin{aligned}\rho(\lambda) &= \rho_0 + \frac{\rho_{-2}}{(\lambda-\lambda_1)^2} - \frac{\rho_{-2}}{(\lambda+\lambda_1)^2} + \frac{\rho_{-1}}{\lambda-\lambda_1} + \frac{\rho_{-1}}{\lambda+\lambda_1} + \frac{\varrho_{-1}}{\lambda-\lambda_2} + \frac{\varrho_{-1}}{\lambda+\lambda_2},\\ \tilde{\rho}(\lambda) &= \tilde{\rho}_0 - \frac{\rho^*_{-2}}{(\lambda-\lambda^*_1)^2} + \frac{\rho^*_{-2}}{(\lambda+\lambda^*_1)^2} - \frac{\rho^*_{-1}}{\lambda-\lambda^*_1} - \frac{\rho^*_{-1}}{\lambda+\lambda^*_1} - \frac{\varrho^*_{-1}}{\lambda-\lambda^*_2} - \frac{\varrho^*_{-1}}{\lambda+\lambda^*_2}.\end{aligned} \tag{33}$$

Plugging the expressions (33) into the Eqs. (24) and (25), we have

$$H_{1,2} = \omega_{1,2} + \Omega_{1,2}\tilde{H}_{1,2}, \quad \tilde{H}_{1,2} = -\Omega^*_{1,2}H_{1,2}, \tag{34}$$

where

$$\omega_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}^T, \quad \Omega_{1,2} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix},$$

$$\begin{aligned} H_{1,2} &= \begin{pmatrix} H^T_2 & U_{+11}(\lambda^*_2) & U_{+11}(-\lambda^*_2) \end{pmatrix}^T,\\ \tilde{H}_{1,2} &= \begin{pmatrix} \tilde{H}^T_2 & U_{+21}(\lambda_2) & U_{+21}(-\lambda_2) \end{pmatrix}^T \end{aligned}$$

and

$$\begin{aligned}
F^{11}_{13} &= \frac{\varrho_{-1}}{\lambda_2-\lambda^*_1}e^{2\mathrm{i}\theta}, \quad F^{21}_{13} = \frac{\varrho_{-1}}{\lambda_2+\lambda^*_1}e^{2\mathrm{i}\theta}, \quad F^{12}_{13} = -F^{21}_{13}, \quad F^{22}_{13} = -F^{11}_{13},\\
F^{11}_{23} &= \frac{\varrho_{-1}}{(\lambda_2-\lambda^*_1)^2}e^{2\mathrm{i}\theta}, \quad F^{21}_{23} = \frac{\varrho_{-1}}{(\lambda_2+\lambda^*_1)^2}e^{2\mathrm{i}\theta}, \quad F^{12}_{23} = F^{21}_{23}, \quad F^{22}_{23} = F^{11}_{23},\\
F^{11}_{31} &= \left[\frac{\rho_{-1}+\rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda^3_1 t)}{\lambda_1-\lambda^*_2} - \frac{\rho_{-2}}{(\lambda_1-\lambda^*_2)^2}\right]e^{2\mathrm{i}\theta}, \quad F^{22}_{31} = -F^{11}_{31},\\
F^{21}_{31} &= \left[\frac{\rho_{-1}+\rho_{-2}(4\mathrm{i}\lambda_1 x + 16\mathrm{i}\lambda^3_1 t)}{\lambda_1+\lambda^*_2} - \frac{\rho_{-2}}{(\lambda_1+\lambda^*_2)^2}\right]e^{2\mathrm{i}\theta}, \quad F^{12}_{31} = -F^{21}_{31},\\
F^{11}_{32} &= \frac{\rho_{-2}}{\lambda_1-\lambda^*_2}e^{2\mathrm{i}\theta}, \quad F^{21}_{32} = \frac{\rho_{-2}}{\lambda_1+\lambda^*_2}e^{2\mathrm{i}\theta}, \quad F^{12}_{32} = F^{21}_{22}, \quad F^{22}_{32} = F^{11}_{32},\\
F^{11}_{33} &= \frac{\varrho_{-1}}{\lambda_2-\lambda^*_2}e^{2\mathrm{i}\theta}, \quad F^{21}_{33} = \frac{\varrho_{-1}}{\lambda_2+\lambda^*_2}e^{2\mathrm{i}\theta}, \quad F^{12}_{33} = -F^{21}_{33}, \quad F^{22}_{33} = -F^{11}_{33}.
\end{aligned}$$

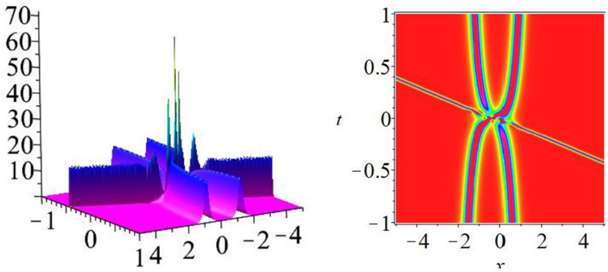

Fig. 3 $\rho_{-1} = \mathrm{i}, \rho_{-2} = 1, \varrho_{-1} = -1, \lambda_1 = 1 + \mathrm{i}, \lambda_2 = 2 + \mathrm{i}$, the left: the 3D image of $|q|^2$, the right: the density image of $|q|^2$

From the formula (29), we have

$$q = 2\mathrm{i}\frac{\det \tilde{\Phi}_{1,2}}{\det \Phi_{1,2}}, \tag{35}$$

where

$$\Phi_{1,2} = I + \Omega_{1,2}\Omega^*_{1,2}, \quad \tilde{\Phi}_{1,2} = \begin{pmatrix} 0 & \Lambda_{1,2} \\ \omega_{1,2} & \Phi_{1,2} \end{pmatrix},$$
$$\Lambda_{1,2} = \begin{pmatrix} \rho^*_{-1} - \rho^*_{-2}\chi & \rho^*_{-1} - \rho^*_{-2}\chi & \rho^*_{-2} & -\rho^*_{-2} & \varrho^*_{-1} & \varrho^*_{-1} \end{pmatrix}.$$

As shown in Fig. 3.

5 Conclusions and Discussions

We discussed the higher-order soliton solutions for DNLS-III equation by the improved Riemann–Hilbert method in detail. The main idea is to require the potentials $q(x)$ decay rapidly enough at infinity so that the reflection coefficient $\rho(\lambda)$ or $\tilde{\rho}(\lambda)$ can be analytically extended to the region $D_\pm$. For $\rho(\lambda)$ has a pth order pole, by virtue of the Laurent series expansion of $\rho(\lambda)$ we can obtain the explicit determinant form of higher-order soliton solutions. Moreover these results can be applied to the other derivative type NLS equations by gauge transform. In this paper the potentials $q(x)$ is considered under the ZBC, we know that under the nonzero boundary condition (NZBC) it is more complicated to solve double soliton solutions by the usual RH method in the literature [17], in the near future we will generalize this idea to the NZBC case.

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Data Availability Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Ethics Approval and Consent to Participate Not applicable and All authors consent to participate.

Consent for publication All authors consent for publication.

References

1. Ablowitz, M.J., Prinari, B., Trubatch, A.D.: Discrete and Continuous Nonlinear Schrödinger Systems. Cambridge University Press, Cambridge (2004)
2. Ablowitz, M., Kaup, D., Newell, A.C., Segur, H.: The inverse scattering transform-Fourier analysis for nonlinear problems. Stud. Appl. Math. **53**, 249–315 (1974)
3. Agrawal, G.P.: Nonlinear Fiber Optics. Academic Press, London (2007)
4. Bilman, D., Buckingham, R.: Large-order asymptotics for multiple-pole solitons of the focusing nonlinear Schrödinger equation. J. Nonlinear Sci. **29**, 2185–229 (2019)
5. Calogero, F., Eckhaus, W.: Nonlinear evolution equations, rescalings, model PDEs and their integrability. I. Inverse Probl. **3**, 229–262 (1987)
6. Doktorov, E.V.: The modified nonlinear Schrödinger equation: facts and artefacts. Eur. Phys. J. B **29**, 227–231 (2002)
7. Gerdjikov, V.S., Ivanov, M.I.: The quadratic bundle of general form and the nonlinear evolution equations. I. Expansions over the "squared" solutions are generalized Fourier transforms. Bulg. J. Phys. **10**, 13–26 (1983)
8. Guo, B., Ling, L., Liu, Q.P.: High-order solutions and generalized Darboux transformations of derivative nonlinear Schrödinger equations. Stud. Appl. Math. **130**, 317–344 (2013)
9. Kaup, D.J., Newell, A.C.: An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys. **19**, 798–801 (1978)
10. Kuang, Y., Mao, B., Wang, X.: Higher-order soliton solutions for the Sasa–Satsuma equation revisited via $\bar{\partial}$ method. J. Nonlinear Math. Phys. **30**, 1821–1833 (2023)
11. Kundu, A.: Landau–Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations. J. Math. Phys. **25**, 3433–3438 (1984)
12. Liu, S.Z., Zhang, Y.S., He, J.S.: Smooth positions of the second-type derivative nonlinear Schrödinger equation. Commun. Theor. Phys. **71**, 357–361 (2019)
13. Mjølhus, E.: On the modulational instability of hydromagnetic waves parallel to the magnetic field. J. Plasma Phys. **16**, 321–334 (1976)
14. Moses, J., Malomed, B.A., Wise, F.W.: Self-steepening of ultrashort optical pulses without self-phase-modulation. Phys. Rev. A **76**, 021802 (2007)
15. Nie, H., Zhu, J., Geng, X.: Trace formula and new form of N-soliton to the Gerdjikov-Ivanov equation. Anal. Math. Phys. **8**, 415–426 (2018)
16. Olmedilla, E.: Multiple pole solutions of the nonlinear Schrödinger equation. Phys. D **25**, 330–346 (1987)

17. Pichler, M., Biondini, G.: On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles. IMA J. Appl. Math. **82**, 131–151 (2017)
18. Qiao, Z.: A new completeiy integrable Liouville's system produced by the Kaup–Newell eigenvalue problem. J. Math. Phys. **34**, 3110–3120 (1993)
19. Qiao, Z.: A hierarchy of nonlinear evolution equations and finite-dimensional involutive systems. J. Math. Phys. **35**, 2971–2977 (1994)
20. Shchesnovich, V.S., Yang, J.: Higher-order solitons in the N-wave system. Stud. Appl. Math. **110**, 297–332 (2003)
21. Tsuru, H., Wadati, M.: The multiple pole solutions of the sine-Gordon equation. J. Phys. Soc. Jpn. **53**, 2908–2921 (1984)
22. Wadati, M., Ohkuma, K.: Multiple-pole solutions of the modified Korteweg–de Vries equation. J. Phys. Soc. Jpn. **51**, 2029–2035 (1982)
23. Wadati, M., Sogo, K.: Gauge transformations in soliton theory. J. Phys. Soc. Jpn. **52**, 394–398 (1983)
24. Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Sov. Phys. JETP **34**, 62–69 (1972)
25. Zhang, Y., Tao, X., Xu, S.: The bound-state soliton solutions of the complex modified KdV equation. Inverse Probl. **36**, 065003 (2020)
26. Zhang, Y., Wu, H., Qiu, D.: Revised Riemann–Hilbert problem for the derivative nonlinear Schrödinger equation: vanishing boundary condition. Theor. Math. Phys. **217**, 1595–1608 (2023)