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A class of entanglement witnesses and a realignment-like criterion

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We present a multi-parameter family of positive maps between spaces of arbitrary but finite dimensions. This framework facilitates the construction of Entanglement Witnesses (EWs) specifically designed for systems living in $d_1 \times d_2$ dimensions. A sufficient condition for positivity is presented. Interestingly, it is shown that all EWs constructed this way are equivalent to a single realignment-like criterion which for d_1 and d_2 are different is in general stronger than the original realignment criterion. We illustrate effectiveness of this criterion considering examples of Positive Partial Transpose entangled states in 4×4 and 3×4 dimensions.

Quantum entanglement is one of the crucial resource for modern quantum technologies like quantum cryptography, quantum communication, and quantum computation[1,2]. One of the main problem of the entanglement theory is to devise appropriate criteria which enable one to distinguish separable and entangled states[2,3]. The identification of entangled states, in particular, presents a formidable challenge, underscored by its classification as an NP-Hard problem in mathematical terms[4]. Hence, the pursuit of viable techniques for detecting entangled states takes on paramount importance. Entangled states are referred to as states that are not separable, and separable states refer to states that can be represented as follows[5]:

$$\rho = \sum p_i \rho_i^{(1)} \otimes \rho_i^{(2)},\qquad(1)$$

where $0 \leq p_i \leq 1$ and $\sum p_i = 1$. It is not easy at all to determine whether or not a given state can be expressed in the aforementioned form. As a result, there is a pressing need for more effective methods for the detection of entanglement. Over time, diverse techniques have emerged for this purpose. For low-dimensional bipartite systems, i.e. qubit-qubit and qubit-qutrit, the separability poblem is completely solved by the well known Peres-Horodecki criterium which states that a state is separable if and only if it is positive under partial transpose (PPT), that is, $\rho^\Gamma = (I \otimes T)\rho \geq 0$[6,7]. However, for higher dimensional systems the separability problem is notoriously difficult. In higher dimensions one should search for other criteria of entanglement, many of which have been reviewed in[3]. The most important ones are called Entanglement Witnesses (EW), (cf.[8] for a review). An EW refers to a Hermitian operator W, which has negative eigenvalues but has the following property:

$$\text{Tr}(W\sigma) \geq 0, \quad \forall \sigma \in S,$$

where S is the set of all separable states. Therefore, a negative expectation value for W on a state ρ, then means that ρ is entangled. Alternatively, one can think of W as a block-positive operator, i.e.

$$\langle \psi, \phi | W | \psi, \phi \rangle \geq 0, \ \forall \ \psi \in H_1 \ \forall \ \phi \in H_2,$$

which has negative eigenvalue when acting on the whole Hilbert space $H_1 \otimes H_2$. Hence, any entangled state ρ of a bipartite system can be detected a suitable entanglement witness such that $Tr(W\rho) < 0$[2,3,8,9].

The well-known criterion based on positive maps[2,7] states that if ρ is separable, then $(I \otimes \Phi)\rho \geq 0$ for all positive maps Φ. Actually, it is well known that any entanglement witness W can be written in the form

$$W = (I \otimes \Phi)|\phi^+\rangle\langle\phi^+|,$$

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where Φ is a positive (but not completely positive) map, and $|\phi^+\rangle = \frac{1}{\sqrt{d}}\sum_{\mu=0}^{d-1}|\mu\mu\rangle$, is a maximally entangled state[7]. In other words, W is the Choi-matrix of the map Φ, which due to its lack of complete positivity can have negative eigenvalue.

Apart from positive maps or entanglement witnesses criteria there exist other well known effective methods for entanglement detection. One example is a realignment (or computable crossnorm (CCNR)) criterion[10-12]. This criterion proved to be very effective in the detection of PPT entangled states[2] (cf. the recent review[13]).

In this paper we construct a large class of positive maps which satisfy simple condition which does guarantee the positivity of the map. Equivalently, we provide a large class of entanglement witnesses in $d_1 \otimes d_2$ quantum systems. The structure of this paper is as follows: In Sect. 2 we present the main results (Theorem 1). Interestingly, several well known maps/witnesses belong to this class. We illustrate this result by several examples. Moreover, we relate our result to so called mirrored witnesses[14,15]. Section 3 shows that our class of witnesses is essentially equivalent to a single realignment-like criterion (Theorem 3). The power of this criterion is then illustrated by two examples of $4 \otimes 4$ and $3 \otimes 4$ systems. Final conclusions are collected in Sect. 4.

Results
Notations and conventions

Let H_d be the Hilbert space of dimension d and $L(H_d)$ the space of linear operators acting on H_d. The convex set of positive operators is denoted by $L(H_d)^+$. A vector space of linear operators $L(H_d)$ is equipped with the Hilbert-Schmidt inner product $\langle A, B \rangle := \mathrm{Tr}(AB^\dagger)$. In what follows we consider maps $\Phi : L^+(H_{d_1}) \longrightarrow L^+(H_{d_2})$.

Consider two orthonormal bases of Hermitian operators in $L(H_{d_1})$:

$$\left\{ \Gamma_0 = I_{d_1}/\sqrt{d_1},\ \Gamma_i;\ i = 1, \ldots, d_1^2 - 1 \right\}, \quad \mathrm{Tr}(\Gamma_\alpha \Gamma_{\alpha'}) = \delta_{\alpha\alpha'},$$

and in $L(H_{d_2})$:

$$\left\{ \Omega_0 = I_{d_2}/\sqrt{d_2},\ \Omega_k;\ k = 1, \ldots, d_2^2 - 1 \right\}, \quad \mathrm{Tr}(\Omega_\beta \Omega_{\beta'}) = \delta_{\beta\beta'},$$

where Γ_i $(i > 0)$ and Ω_k $(k > 0)$ are traceless operators.

A state of a quantum system is denoted by $\rho \in D(H_d)$, where $D(H_d) \subset L(H_d)^+$ is the convex subset of positive operators with unit trace. Any Hermitian operator $X \in L^+(H_{d_1})$ can be expanded as

$$X = \sum_{\alpha=0}^{d_1^2-1} x_\alpha \Gamma_\alpha = \frac{x_0}{\sqrt{d_1}} I_{d_1} + \mathrm{x} \cdot \boldsymbol{\Gamma}, \tag{2}$$

where $x_0 \in \mathbb{R}$ and $\mathrm{x} = (x_1, \ldots, x_{d^2-1})$ is a generalized Bloch vector[16-18]. When X is a rank-1 operator, it necessarily holds that $\mathrm{Tr}\,X^2 = (\mathrm{Tr}\,X)^2$ which leads to

$$\|\mathrm{x}\|^2 = x_0^2(d_1 - 1). \tag{3}$$

where $\|\mathrm{x}\|^2 = \sum_{i=1}^{d_1^2-1} x_i^2$. Similarly, any Hermitian operator Y in $L(H_{d_2})$ can be represented as

$$Y = \sum_{\beta=0}^{d_2^2-1} y_\beta \Omega_\beta = \frac{y_0}{\sqrt{d_2}} I_{d_2} + \mathrm{y} \cdot \boldsymbol{\Omega}. \tag{4}$$

Consider now a linear map $\Phi : L(H_{d_1}) \to L(H_{d_2})$ defined by

$$\Phi(\Gamma_\alpha) = \sum_{\beta=0}^{d_2^2-1} R_{\beta\alpha}\Omega_\beta, \tag{5}$$

or equivalently

$$\Phi(X) = \sum_{\alpha=0}^{d_1^2-1}\sum_{\beta=0}^{d_2^2-1} R_{\beta\alpha}\,\mathrm{Tr}(\Gamma_\alpha X)\Omega_\beta, \tag{6}$$

where the matrix R is parameterized as follows

$$R = \begin{pmatrix} R_{00} & \mathrm{s} \\ \mathrm{t} & \Lambda \end{pmatrix} \tag{7}$$

with $t \in R^{d_2^2-1}$, $s \in R^{d_1^2-1}$, and $\Lambda \in M_{d_2^2-1 \times d_1^2-1}$ is a real matrix. This map induces the following affine transformation on the level of generalized Bloch vectors

$$\begin{pmatrix} y_0 \\ y \end{pmatrix} = \begin{pmatrix} R_{00} & s \\ t & \Lambda \end{pmatrix} \begin{pmatrix} x_0 \\ x \end{pmatrix}. \tag{8}$$

Equivalently one has the following relation

$$\Phi(X) = \Phi(x_0 \Gamma_0 + x \cdot \mathbf{\Gamma}) = (R_{00} x_0 + s \cdot x) \Omega_0 + (x_0 t + \Lambda x) \cdot \mathbf{\Omega}. \tag{9}$$

Note that in particular

1. Φ is trace-preserving iff $s = 0$ and $R_{00} = \sqrt{\frac{d_1}{d_2}}$,

2. Φ is unital iff $t = 0$ and $R_{00} = \sqrt{\frac{d_2}{d_1}}$. Finally, let us recall the following result (known as a Mehta's lemma, cf. Refs.[19,20]).

Proposition 1 *Let A be a Hermitian $d \times d$ matrix. If*

$$\operatorname{Tr} A^2 \leq \frac{(\operatorname{Tr} A)^2}{d-1}, \tag{10}$$

then A is positive matrix.

A class of positive maps

Given a quantum map Φ characterized by the affine transformation on the generalized Bloch vector $x \longrightarrow \Lambda x + x_0 t$, our goal is to find the condition on this affine map which guarantees positivity of Φ.

Theorem 1 *A map Φ, implementing the affine transformation (8), is positive if the following condition holds*

$$\sqrt{d_2 - 1}\|t\| + \sqrt{d_1 - 1}\|s\| + \sqrt{(d_1 - 1)(d_2 - 1)}\|\Lambda\|_\infty \leq R_{00}, \tag{11}$$

where $\|\Lambda\|_\infty$ is the operator norm of Λ and is equal to λ_{max} which is the largest singular value of Λ.

Proof Due to linearity of the map, it suffices to prove the statement only for rank-1 positive operators *X*. For such *X* (see (3)), we have

$$\operatorname{Tr} \Phi(X) = y_0 \sqrt{d_2} = (R_{00} x_0 + s \cdot x) \sqrt{d_2}$$

and

$$\operatorname{Tr} \Phi(X)^2 = \sum_{\beta=0}^{d_2^2-1} y_\beta^2 = (R_{00} x_0 + s \cdot x)^2 + \|x_0 t + \Lambda x\|^2.$$

Inserting these two expressions in Mehta's lemma we find that the sufficient condition for positivity of $\Phi(X)$ reads

$$(R_{00} x_0 + s \cdot x)^2 + \|x_0 t + \Lambda x\|^2 \leq \frac{(R_{00} x_0 + s \cdot x)^2 d_2}{d_2 - 1}. \tag{12}$$

or, equivalently,

$$\|x_0 t + \Lambda x\| \leq \frac{|R_{00} x_0 + s \cdot x|}{\sqrt{d_2 - 1}}. \tag{13}$$

The above condition still depends upon the input (x_0, x). Moreover, it is highly asymmetric in (t, s). Now, using (13) we derive a condition which does not depend on (x_0, x) and uses only the parameter R_{00}, vectors t, s, and the matrix Λ.

Let us observe that

$$\|x_0 \mathrm{t} + \Lambda \mathrm{x}\| \leq \|x_0 \mathrm{t}\| + \|\Lambda \mathrm{x}\| \leq |x_0| \|\mathrm{t}\| + \|\Lambda\|_\infty \|\mathrm{x}\| = |x_0| \left(\|\mathrm{t}\| + \|\Lambda\|_\infty \sqrt{d_1 - 1} \right), \tag{14}$$

where in the last equality we have used the relation $\|\mathrm{x}\| = |x_0| \sqrt{d_1 - 1}$. Moreover,

$$R_{00} |x_0| - \|\mathrm{s}\| \|\mathrm{x}\| \leq |R_{00} x_0 + \mathrm{s} \cdot \mathrm{x}|, \tag{15}$$

and hence using again $\|\mathrm{x}\| = |x_0| \sqrt{d_1 - 1}$, one finds

$$|x_0| \left(R_{00} - \|\mathrm{s}\| \sqrt{d_1 - 1} \right) \leq |R_{00} x_0 + \mathrm{s} \cdot \mathrm{x}|. \tag{16}$$

This way we found an upper bound

$$\|x_0 \mathrm{t} + \Lambda \mathrm{x}\| \leq |x_0| \left(\|\mathrm{t}\| + \|\Lambda\|_\infty \sqrt{d_1 - 1} \right), \tag{17}$$

for the lhs of (13), and a lower bound

$$|x_0| \left(R_{00} - \|\mathrm{s}\| \sqrt{d_1 - 1} \right) \leq |R_{00} x_0 + \mathrm{s} \cdot \mathrm{x}|, \tag{18}$$

for the rhs of (13). It is, therefore, clear that if

$$\|\mathrm{t}\| + \|\Lambda\|_\infty \sqrt{d_1 - 1} \leq \frac{R_{00} - \|\mathrm{s}\| \sqrt{d_1 - 1}}{\sqrt{d_2 - 1}}, \tag{19}$$

holds, then (13) is trivially satisfied and hence the map Φ is positive. After rearranging the terms one obtains

$$\sqrt{d_2 - 1} \|\mathrm{t}\| + \sqrt{d_1 - 1} \|\mathrm{s}\| + \sqrt{(d_1 - 1)(d_2 - 1)} \|\Lambda\|_\infty \leq R_{00} \tag{20}$$

which ends the proof. $\square$

Corollary 1 *When $d_1 = d_2 = 2$, the above condition simplifies to*

$$\|\mathrm{t}\| + \|\mathrm{s}\| + \|\Lambda\|_\infty \leq R_{00}. \tag{21}$$

For unital trace-preserving maps in equal dimensions $d_1 = d_2 = d$, where $\mathrm{t} = \mathrm{s} = 0$, the above condition simplifies to

$$\|\Lambda\|_\infty \leq \frac{1}{d - 1}. \tag{22}$$

In particular for $d_1 = d_2 = 2$, formula (22) reduces to $\|\Lambda\|_\infty \leq 1$ and defines a necessary and sufficient condition for positivity for trace-preserving unital maps.

It should be stressed that condition (11) is sufficient but not necessary for a map Φ to be positive.

Examples of entanglement witnesses and positive maps
Let us review some well known examples EWs and positive maps in connection to a sufficient condition (11).

Example 1 Transposition map T is unital and trace-preserving and hence $\mathrm{s} = \mathrm{t} = 0$, and $R_{00} = 1$. Taking $\{\Gamma_\alpha\}$ as generalized Gell-Mann matrices one finds

$$T(\Gamma_\alpha) = \pm \Gamma_\alpha,$$

depending on whether Γ_α is symmetric or anti-symmetric. Hence $\|\Lambda\|_\infty = 1$ and the condition (11) is satisfied only for $d_1 = d_2 = 2$.

Example 2 Reduction map $\Phi : L(H_d) \to L(H_d)$ is defined via[21]

$$\Phi(X) = I_d \operatorname{Tr} X - X. \tag{23}$$

One finds $s = t = 0$ and $R_{00} = d - 1$. Now,

$$\Phi(\Gamma_k) = -\Gamma_k, \quad k = 1, \dots, d^2 - 1,$$

and hence $\|\Lambda\|_\infty = 1$. Clearly condition (11) is satisfied since

$$\|\Lambda\|_\infty = \frac{R_{00}}{d - 1}, \tag{24}$$

for all dimensions d. Note, that reduction map can be immediately generalized as follows[22]

$$\Phi(X) = I_d \operatorname{Tr} X - \sum_{k,\ell=1}^{d^2-1} \Lambda_{\ell k} \operatorname{Tr}(\Gamma_k X)\Gamma_l, \tag{25}$$

with arbitrary $\Lambda_{k\ell}$ satisfying $\|\Lambda\|_\infty \leq 1$. In particular for $\Lambda_{k\ell}$ being an orthogonal matrix.

Example 3 Consider a map $\Phi : L(H_{d_1}) \to L(H_{d_2})$ defined via

$$\Phi(X) = I_{d_2} \operatorname{Tr} X - \sum_{\alpha=0}^{d_1^2-1} \sum_{\beta=0}^{d_2^2-1} O_{\beta\alpha} \operatorname{Tr}(\Gamma_\alpha X)\Omega_\beta, \tag{26}$$

where O is an $d_2^2 \times d_1^2$ isometry, i.e. if $d_1 \geq d_2$ one has $OO^{\mathrm{T}} = I_{d_2^2}$. It provides a generalization of (25). In[24] it is shown that (26) defines a positive map. For Φ defined in (26) one has

$$R_{00} = \sqrt{d_1 d_2} - O_{00} \quad s_k = O_{0k}, \quad t_\ell = O_{\ell 0} \quad \Lambda_{k\ell} = -O_{k\ell}.$$

Example 4 Any 2-qubit entanglement witness W can be represented as follows

$$W = \sum_{\mu,\nu=0}^{3} T_{\mu\nu} \sigma_\mu \otimes \sigma_\nu, \tag{27}$$

with $\sigma_\mu \in \{I, \sigma_x, \sigma_y, \sigma_z\}$, and 16 real parameters $T_{\mu\nu}$. Authors of Ref.[25] analyzed a particular scenario when one has an access to the limited resources and consider EW with diagonal correlation tensor $T_{kl} = c_k \delta_{kl}$ ($k, l = 1, 2, 3$) only, that is,

$$W = \alpha I \otimes I + \sum_{k=1}^{3} \left(t_k I \otimes \sigma_k + s_k \sigma_k \otimes I \right) + \sum_{k=1}^{3} \lambda_k \sigma_k \otimes \sigma_k, \tag{28}$$

with real parameters $\alpha, s_k, t_k, \lambda_k$. This reduces the number of independent parameters from 6 elements $T_{kl} = T_{lk}$ to 3 real parameters λ_k. The above form is justified by the fact that the mean values of single qubit operators $\sigma_k \otimes \mathbb{I}$ and $\mathbb{I} \otimes \sigma_k$ for $i = 1, 2, 3$ can be derived by simply ignoring the statistics on one side. There are six 1-parameter families of rank-1 projectors $|\varphi\rangle\langle\varphi|$ of the form (28):

$$|\varphi_1\rangle = a |\phi^+\rangle + b |\phi^-\rangle, \quad |\varphi_2\rangle = a |\psi^+\rangle + b |\psi^-\rangle, \quad |\varphi_3\rangle = a |\phi^+\rangle + b |\psi^+\rangle,$$
$$|\varphi_4\rangle = a |\phi^-\rangle + b |\psi^-\rangle, \quad |\varphi_5\rangle = a |\phi^+\rangle + ib |\psi^-\rangle, \quad |\varphi_6\rangle = a |\phi^-\rangle + ib |\psi^+\rangle,$$

where $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are the Bell states, and $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. Hence, there are six 1-parameter families of optimal EWs of the form $W_k := |\varphi_k\rangle\langle\varphi_k|^\Gamma$. For example one finds for W_1: $R_{00} = a^2 + b^2 = 1$, $s = t = (0, 0, 2ab)$, and $\lambda_1 = -\lambda_2 = a^2 - b^2$, $\lambda_3 = a^2 + b^2 = 1$. Hence, inequality (11) reduces to $4|ab| + 1 \leq 1$ which implies that either $(a = 1, b = 0)$ or $(a = 0, b = 1)$. It shows that only four optimal EWs $|\phi^\pm\rangle\langle\phi^\pm|^\Gamma$ and $|\psi^\pm\rangle\langle\psi^\pm|^\Gamma$ satisfy inequality (11).

A sufficient condition for complete positivity

As a byproduct of our analysis let us formulate the sufficient condition for complete positivity.

Proposition 2 *A positive map* (5) *is a completely positive if the following condition holds*:

$$\|s\|^2 + \|t\|^2 + \mathrm{Tr}(\Lambda\Lambda^T) \leq \frac{R_{00}^2}{d_1 d_2 - 1}. \tag{29}$$

Proof According to a celebrated theorem of Choi[26], a map is completely positive if its Choi matrix

$$C_\Phi = d_1 (I_{d_1} \otimes \Phi)(|\phi^+\rangle\langle\phi^+|) = \sum_{m,n=1}^{d_1} |m\rangle\langle n| \otimes \Phi(|m\rangle\langle n|), \tag{30}$$

is positive. One finds

$$C_\Phi = \sum_{\alpha,\beta} R_{\beta\alpha}\Gamma_\alpha^T \otimes \Omega_\beta, \tag{31}$$

and hence

$$C_\Phi = \frac{R_{00}}{\sqrt{d_1 d_2}} I_{d_1} \otimes I_{d_2} + \frac{1}{\sqrt{d_2}} s \cdot \Gamma^T \otimes I_{d_2} + \frac{1}{\sqrt{d_1}} I_{d_1} \otimes t \cdot \Omega + \sum_{k,i} \Lambda_{ki}\Gamma_i^T \otimes \Omega_k, \tag{32}$$

from which we find

$$\mathrm{Tr}(C_\Phi) = R_{00}\sqrt{d_1 d_2}, \qquad \mathrm{Tr}(C_\Phi^2) = R_{00}^2 + \|s\|^2 + \|t\|^2 + \mathrm{Tr}(\Lambda^T\Lambda). \tag{33}$$

Mehta's Lemma (10) implies (29) which completes the proof. □

Example 5 As an example consider the trace-preserving qubit maps Φ with diagonal elements $\Lambda_{ij} = \delta_{ij}\lambda_i$. Then the condition for positivity of the map is given from (11) to be

$$\sqrt{t_1^2 + t_2^2 + t_3^2} + \lambda_{max} \leq 1, \tag{34}$$

and the condition of complete positivity is given from (29) to be

$$\|t\|^2 + \sum_i \lambda_i^2 \leq \frac{1}{3}. \tag{35}$$

We shoud emphasize that since the Mehta's condition is a sufficient conditions, the above inequalities determine only a subset of positive and completely positive maps. For example, if we restrict ourselves to unital maps, then the condition (29) is given by

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq \frac{1}{3},$$

which is a sphere of radius $r = \frac{1}{\sqrt{3}}$ and is a subset of the Fujiwara-Algoet tetrahedron[23]

$$|\lambda_1 + \lambda_2| \leq 1 + \lambda_3 \qquad |\lambda_1 - \lambda_2| \leq 1 - \lambda_3.$$

A comparison of the volumes of these two sets shows that 30 percents of the completely positive unital maps for qubits are detected by this simple use of Mehta's criteria, figure (1).

Mirorred witnesses

The concept of a mirrored entanglement witness was introduced in Ref.[14]: given an entanglement witness W one defines a family

$$W_\mu := \mu I_{d_1} \otimes I_{d_2} - W. \tag{36}$$

It is proved[14] that there exists a minimal μ_0 such that for all $\mu \geq \mu_0$, W_μ is non-negative on separable states. If $W_M := W_{\mu_0}$ is an EW one calls (W, W_M) a mirrored pair of EWs[14].
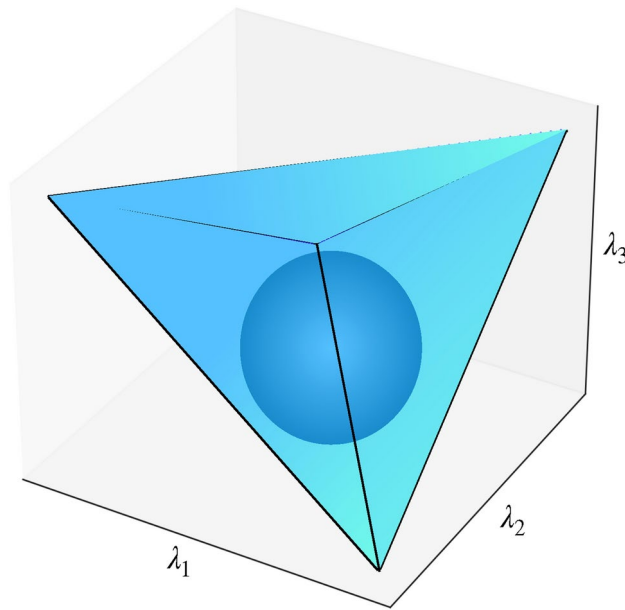
Figure 1. Example of qubit maps, with eigenvalues of affine transformation given by λ_1, λ_2 and λ_3. Restricting to unital maps, the tethdrahedron shows the complete set of unital maps as derived from the Fujiwara-Algoet condition, the sphere shows those maps which are derived from Mehta's condition. Its volume is approximately 30 percent of the volume of the tethrahedron.

Proposition 3 *If W satisfies* (11), *then for* $\mu \geq \frac{2R_{00}}{\sqrt{d_1 d_2}}$, W_μ *is non-negative on separable states.*

Proof indeed, if W is characterized by $(R_{00}, \mathrm{s}, \mathrm{t}, \Lambda)$ an operator W_μ is characterized by $(\mu\sqrt{d_1 d_2} - R_{00}, -\mathrm{s}, -\mathrm{t}, -\Lambda)$. Hence, if W satisfies (11) and $\mu\sqrt{d_1 d_2} - R_{00} \geq R_{00}$, then W_μ satisfies (11) as well. A condition $\mu\sqrt{d_1 d_2} - R_{00} \geq R_{00}$ provides a lower bound $\mu \geq \frac{2R_{00}}{\sqrt{d_1 d_2}}$ $\square$

Example 6 Consider an EW corresponding to the celebrated Choi map[14,15]

$$W = \sum_{i=0}^{2} \left(2|i,i\rangle\langle i,i| + |i,i-1\rangle\langle i,i-1| - 3|\phi^+\rangle\langle\phi^+| \right), \tag{37}$$

where $|\phi^+\rangle = (|0,0\rangle + |1,1\rangle + |2,2\rangle)/\sqrt{3}$ is a Bell state. One easily finds $R_{00} = 2$, $\mathrm{s} = \mathrm{t} = 0$, and $\|\Lambda\|_\infty = 1$, and hence

$$\|\Lambda\|_\infty = \frac{R_{00}}{2}, \tag{38}$$

that is, the inequality (11) is saturated. One has $\frac{2R_{00}}{\sqrt{d_1 d_2}} = \frac{4}{3}$ and reproduces $\mu_0 = \frac{4}{3}$ found in Ref.[15].

Entanglement Witnesses and realignment-like criterion

Consider a bi-partite state living in $H_{d1} \otimes H_{d_2}$ and represented by

$$\rho = \sum_{\alpha,\beta} C_{\beta\alpha} \Gamma_\alpha^T \otimes \Omega_\beta. \tag{39}$$

Due to the celebrated realignment criterion[10–12] one has

Proposition 4 *If ρ is separable, then*

$$\|C\|_1 \leq 1. \tag{40}$$

Actually, the above criterion was recently generalized as follows[27]: let us introduce two square diagonal matrices:

$$D_1(x_1) = \text{diag}\{x_1, 1, \dots, 1\}, \quad D_2(x_2) = \text{diag}\{x_2, 1, \dots, 1\},$$

where $D_1(x_1)$ is $d_1^2 \times d_1^2$ and $D_2(x_2)$ is $d_2^2 \times d_2^2$, and the real parameters $x_1, x_2 \geq 0$. One proves[27]

Theorem 2 *If ρ is separable, then*

$$\|D_1(x_1)CD_2(x_2)\|_1 \leq \mathcal{N}_1(x_1)\mathcal{N}_2(x_2), \tag{41}$$

where

$$\mathcal{N}_1(x_1) = \sqrt{\frac{d_1 - 1 + x_1^2}{d_1}} \quad \mathcal{N}_2(x_2) = \sqrt{\frac{d_2 - 1 + x_2^2}{d_2}}, \tag{42}$$

for arbitrary $x_1, x_2 \geq 0$.

Note that for $(x_1, x_2) = (1, 1)$ the inequality (41) reproduces CCNR criterion. Moreover, for $(x_1, x_2) = (0, 0)$ it reproduces separability criterion derived by de Vicente[28]. If $d_1 = d_2$, then CCNR criterion is stronger than the de Vicente criterion[28]. However, for bipartite states ρ with maximally mixed marginals $\rho_1 = I_{d_1}/d_1$ and $\rho_2 = I_{d_2}/d_2$, the de Vicente criterion is stronger than CCNR if $d_1 \neq d_2$, and they are equivalent if $d_1 = d_2$[28]. Here we prove the following result:

Theorem 3 *The criterion* (41) *corresponding to* $(x_1, x_2) = (0, 0)$ *is equivalent to the entire class of entanglement witnesses characterized by* (11).

Proof consider an EW

$$W = \sum_{\alpha, \beta} R_{\beta\alpha} \Gamma_\alpha^T \otimes \Omega_\beta = R_{00}\Gamma_0 \otimes \Omega_0 + \mathbf{s} \cdot \mathbf{\Gamma} \otimes \Omega_0 + \Gamma_0 \otimes \mathbf{t} \cdot \mathbf{\Omega} + \sum_{k,i} \Lambda_{ki} \Gamma_i^T \otimes \Omega_k, \tag{43}$$

and a bipartite state $\rho \in D(H_{d_1} \otimes H_{d_2})$ be a bi-partite density matrix given by

$$\rho = \sum_{\alpha, \beta} C_{\beta\alpha} \Gamma_\alpha^T \otimes \Omega_\beta = \frac{1}{\sqrt{d_1 d_2}}\Gamma_0 \otimes \Omega_0 + \mathbf{r_1} \cdot \mathbf{\Gamma} \otimes \Omega_0 + \Gamma_0 \otimes \mathbf{r_2} \cdot \mathbf{\Omega} + \sum_{k,i} Q_{ki} \Gamma_i^T \otimes \Omega_k. \tag{44}$$

One finds

$$\text{Tr}(W\rho) = \frac{R_{00}}{\sqrt{d_1 d_2}} + \mathbf{r_1} \cdot \mathbf{s} + \mathbf{r_2} \cdot \mathbf{t} + \text{Tr}(Q\Lambda^T). \tag{45}$$

In the above expression, the triple $(\mathbf{r_1}, \mathbf{r_2}, Q)$ represent the data of the density matrix and the triple $(\mathbf{s}, \mathbf{t}, \Lambda)$ represent the data of the entanglement witness W. Given ρ let us minimize $\text{Tr}(W\rho)$ over all EWs from the class (11). one obviously has

$$\mathbf{s} = -x\mathbf{r_1}, \qquad \mathbf{t} = -y\mathbf{r_2}, \tag{46}$$

where x and y are positive constants. To choose Λ, we note the singular value decomposition of $Q = Oq O'$, where O and O' are orthogonal matrices and $\mathbf{q}_{ij} = q_i\delta_{ij}$, where q_i are singular values of Q. Let $\Lambda = O'^T\boldsymbol{\lambda}O^T$, where $\boldsymbol{\lambda}_{ij} = -\lambda_{max}\delta_{ij}$.

One has

$$\text{Tr}(Q\Lambda^T) = -\lambda_{max}\sum_i q_i = -\lambda_{max}\|Q\|_1. \tag{47}$$

Putting everything together, we find

$$\text{Tr}(\rho W) = \frac{R_{00}}{\sqrt{d_1 d_2}} - x\|\mathbf{r_1}\|^2 - y\|\mathbf{r_2}\|^2 - \lambda_{max}\|Q\|_1, \tag{48}$$

where $\|Q\|_1$ is the trace norm of Q. Now we have to minimize this with respect to the data of the entanglement witness (x, y, λ_{max}) while respecting the condition (11), which now takes the form

$$y\sqrt{d_2 - 1}\|\mathbf{r_2}\| + x\sqrt{d_1 - 1}\|\mathbf{r_1}\| + \lambda_{max}\sqrt{(d_1 - 1)(d_2 - 1)} \leq R_{00}. \tag{49}$$

The above condition defines a simplex in the Cartesian space (x, y, λ_{max}) with vertices $v_0 = (0, \ 0, \ 0)$, and

$$v_1 = \left(\frac{R_{00}}{\|\mathbf{r_1}\|\sqrt{d_1 - 1}}, \ 0, \ 0\right), \quad v_2 = \left(0, \ \frac{R_{00}}{\|\mathbf{r_2}\|\sqrt{d_2 - 1}}, \ 0\right), \quad v_3 = \left(0, \ 0, \ \frac{R_{00}}{\sqrt{(d_1 - 1)(d_2 - 1)}}\right). \tag{50}$$

Since $\mathrm{Tr}(W\rho)$ is a linear function of (x, y, λ_{max}), it achieves its minimum on one of the above vertices. One easily finds the value of $\mathrm{Tr}(W_i\rho)$ corresponding to the vertex v_i

$$\mathrm{Tr}(W_i\rho) = \frac{R_{00}}{\sqrt{d_1 d_2}} F_i, \quad i = 0, \ 1, \ 2, \ 3, \tag{51}$$

where $F_0 = 1$, and

$$F_1 = 1 - \|\mathbf{r_1}\|\sqrt{\frac{d_1 d_2}{d_1 - 1}}, \quad F_2 = 1 - \|\mathbf{r_2}\|\sqrt{\frac{d_1 d_2}{d_2 - 1}}, \quad F_3 = 1 - \|Q\|_1\sqrt{\frac{d_1 d_2}{(d_1 - 1)(d_2 - 1)}}. \tag{52}$$

Positivity of the reduced density matrices $\rho_1 \in D(H_{d_1})$ and $\rho_2 \in D(H_{d_2})$ imply that F_1 and F_2 are always non-negative. The condition $F_3 \geq 0$ is equivalent to

$$\|Q\|_1 \leq \sqrt{\frac{(d_1 - 1)(d_2 - 1)}{d_1 d_2}}, \tag{53}$$

which reproduces (41) for $(x_1, x_2) = (0, 0)$. $\square$

Example 7 In Ref.[29] the following family of PPT entangled states in $C^4 \otimes C^4$ is introduced

$$\rho = \frac{1}{N}\begin{pmatrix} 1 & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & 1 \\ . & \frac{1}{a} & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\ . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & a & . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & a & . & . & . & . & . & 1 & . & . & . & . & . \\ 1 & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & 1 \\ . & . & . & 1 & . & . & \frac{1}{a} & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . \\ . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\ 1 & . & . & . & . & 1 & . & . & . & a & . & . & 1 & . & . & 1 \\ . & . & . & . & 1 & . & . & . & . & . & \frac{1}{a} & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & 1 & . & \frac{1}{a} & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . & . & . & a & . & . \\ 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & 1 \end{pmatrix}, \tag{54}$$

where dots represent zeros, and $N = 4a + \frac{4}{a} + 8$ stands for the normalization factor, and it is shown that their entanglement can be detected by an entanglement witness, called W_{ext}, which shows that these states are entangled in the interval $0 < a < 1$. We now show that the interval of entanglement of these states is in fact much larger and covers the whole positive line $0 < a < \infty$, except the point $a = 1$. To show this, the Q matrix corresponding to (54) is calculated, which turns out to be
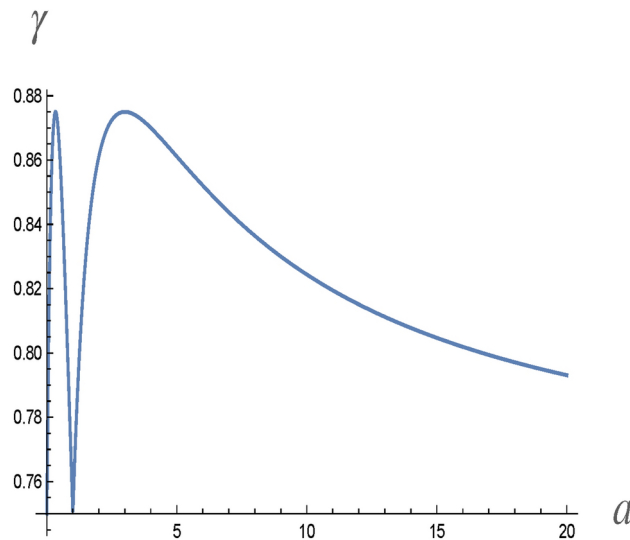
$$Q = \frac{1}{(a + 1)^2}\left(A \oplus A' \oplus B\right),$$

where

$$A = \begin{pmatrix} \frac{a}{4} & . & . & . & . & \frac{a}{4} \\ . & \frac{a}{2} & . & . & . & . \\ . & . & \frac{4}{4} & \frac{a}{4} & . & . \\ . & . & \frac{a}{4} & \frac{4}{4} & . & . \\ . & . & . & . & \frac{a}{2} & . \\ \frac{a}{4} & . & . & . & . & \frac{a}{4} \end{pmatrix}, \quad A' = \begin{pmatrix} -\frac{a}{4} & . & . & . & . & \frac{a}{4} \\ . & . & . & . & . & . \\ . & . & -\frac{a}{4} & -\frac{a}{4} & . & . \\ . & . & -\frac{a}{4} & -\frac{a}{4} & . & . \\ . & . & . & . & . & . \\ \frac{a}{4} & . & . & . & . & -\frac{a}{4} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{(a-1)^2}{8} & -\frac{a^2+2a-3}{8\sqrt{3}} & -\frac{(a-1)a}{2\sqrt{6}} \\ \frac{3a^2-2a-1}{8\sqrt{3}} & -\frac{(a-1)^2}{24} & -\frac{a^2+a-2}{6\sqrt{2}} \\ \frac{a-1}{2\sqrt{6}} & \frac{2a^2-a-1}{6\sqrt{2}} & -\frac{(a-1)^2}{12} \end{pmatrix},$$

with singular values given as

$$\left\{\frac{|a-1|}{4|a+1|} \ (\times 2), \ \frac{|a|}{2(a+1)^2} \ (\times 6), \ \frac{(a-1)^2}{4(a+1)^2} \right\}.$$

γ

Figure 2. A state (54) is entangled for $0 < a < \infty$; $a \neq 1$.

This leads to

$$\|Q\|_1 = \frac{1}{4}\frac{(a-1)^2}{(a+1)^2} + 3\frac{|a|}{(a+1)^2} + \frac{1}{2}\left|\frac{a-1}{a+1}\right|.$$

Figure 2 shows that the value of $\gamma = \|Q\|_1$ is always larger than $\sqrt{\frac{(d_1-1)(d_2-1)}{d_1 d_2}} = \frac{3}{4}$ except for $a = 1$.

Note that by $a < \infty$, we are emphasizing that a should be a finite value. In fact, in the limit $a \to \infty$, the state approaches to

$$\rho_\infty = \frac{1}{4}\Big(|03\rangle\langle03| + |10\rangle\langle10| + |11\rangle\langle11| + |22\rangle\langle22|\Big),$$

which is obviously separable. Incidentally, this is the limiting case where γ approaches the value $\frac{3}{4}$.

Example 8 In Ref.[9] Terhal introduced a family of UPB for $3 \otimes n$ systems. Here we take $n = 4$ and show that our criterion (53) can detect the corresponding PPT entangled state in $3 \otimes 4$. The UPB states for the 3×4 system are:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes (|1\rangle - |3\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}|1\rangle \otimes (|2\rangle - |3\rangle), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle \otimes (|0\rangle - |3\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle,$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \otimes |1\rangle, \quad |\psi_5\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |0\rangle) \otimes |2\rangle, \quad |\psi_6\rangle = \frac{1}{\sqrt{12}}\sum_{i=0}^{2}\sum_{j=0}^{3}|i\rangle \otimes |j\rangle.$$

The PPT entangled state $\rho = \frac{1}{6}(I - \sum_{i=0}^{6}|\psi_i\rangle\langle\psi_i|)$ turns out to be

$$\rho = \frac{1}{60}\left(\begin{array}{ccc|ccc|ccc|ccc}
5 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 5 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 \\
\hline
-1 & 5 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
5 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & 5 & -1 & -1 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & 5 \\
\hline
-1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & 5 & -1 & -1 \\
-1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & 5
\end{array}\right). \tag{55}$$

The corresponding Q matrix is given by

$$Q = \frac{1}{30} \left(\begin{array}{c|c|c} -\mathbb{J}_{3\times 6} & 0_{3\times 6} & A \\ \hline 0_{3\times 6} & 0_{3\times 6} & 0_{3\times 3} \\ \hline B & 0_{2\times 6} & 0_{2\times 3} \end{array} \right),$$

where

$$A = \left(\begin{array}{ccc} 3 & \sqrt{3} & \sqrt{\frac{3}{2}} \\ 0 & -2\sqrt{3} & \sqrt{\frac{3}{2}} \\ -3 & -\sqrt{3} & \sqrt{\frac{3}{2}} \end{array} \right), \quad B = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -2\sqrt{3} & \sqrt{3} & \sqrt{3} \end{array} \right),$$

and the 3×6 matrix $\mathbb{J}_{ij} = 1$. The singular values of Q are found to be

$$\left\{ \frac{1}{2\sqrt{10}}, \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{2}} \right\},$$

and we find

$$\|Q\|_1 = \frac{1}{2\sqrt{10}} + \frac{4}{5\sqrt{2}} = \frac{8 + \sqrt{5}}{10\sqrt{2}},$$

which is greater than $\sqrt{\frac{(d_1-1)(d_2-1)}{d_1 d_2}} = \frac{1}{\sqrt{2}}$. This shows that the PPT state (55) is entangled.

Conclusions

Mehta's lemma provides a simple sufficient condition for a Hermitian matrix to be positive. We have used this lemma to construct a large family of positive maps and characterized them in terms of their corresponding affine transformations, i.e. transformation of the so-called Bloch vector $\mathrm{r} \longrightarrow \Lambda \mathrm{r} + x_0 \mathrm{t}$. Using the correspondence between entanglement witnesses and positive (but not completely positive) maps, a multi-parameter family of entanglement witnesses for bipartite systems with different dimensions was constructed. This family is uniquely characterized by $\{R_{00}, \mathrm{s}, \mathrm{t}, \Lambda\}$ satisfying the very condition (13). Actually, several well-known positive maps belong to this family. Prominent examples are the reduction map[21], its generalizations[22], and the Choi map for $d_1 = d_2 = 3$. This shows that both decomposable and non-decomposable positive maps can be constructed this way. Interestingly, the transposition map belongs to this family but only if $d_1 = d_2 = 2$. Let us recall, that due to the seminal Peres-Horodecki results[6,7] transposition map detects all entangled states whenever $d_1 d_2 \leq 6$. Hence the proposed construction is important for systems that allow for PPT entangled states. Both transposition, reduction, any many other well-known positive maps are both trace-preserving and unital. It should be stresses that presented method enables one to construct positive maps which are neither unital nor trace-preserving by including two nontrivial vectors s and t.

Interestingly, it is shown that this class of witnesses is equivalent to a single realignment-like criterion already analyzed by de Vicente[28]. It simply means that any entangled state which is detected by de Vicente criterion is also detected by some map, or equivalently an entanglement witness, from the above family. It should be stressed that contrary to the realignment-like criterion the method based on a particular entanglement witness does not require full state tomography and hence usually is more experimental friendly. The power of this criterion is illustrated by two examples of PPT entangled states in $4 \otimes 4$ and $3 \otimes 4$ systems.

It would be interesting to extend proposed construction for multipartite scenario. The multipartite generalization of realignment-like criteria was already analyzed by several authors[27,30–32]. Now, on the level of maps instead of positive maps $\Phi : L(H_{d_1}) \to L(H_{d_2})$ one considers linear maps

$$\Phi : L(H_{d_1} \otimes \ldots \otimes H_{d_{n-1}}) \to L(H_{d_n}), \tag{56}$$

which are positive but only on product positive operators from $L^+(H_{d_1}) \otimes \ldots \otimes L^+(H_{d_{n-1}})$. Construction of such maps is much more demanding than construction of positive maps $\Phi : L(H_{d_1}) \to L(H_{d_2})$. Using similar techniques based on generalized Bloch vectors we plan to provide a similar construction of linear maps satisfying (56). In particular this method enables one to construct maps which are neither unital nor trace-preserving.

Another interesting problem is related to optimality. Optimal entanglement witnesses are of particular importance in entanglement theory[2,8,33]. It would be interesting to decide which sets of parameters satisfying

$$\sqrt{d_2 - 1}\|\mathrm{t}\| + \sqrt{d_1 - 1}\|\mathrm{s}\| + \sqrt{(d_1 - 1)(d_2 - 1)}\|\Lambda\|_\infty = R_{00}, \tag{57}$$

that is, saturating inequality (11), give rise to optimal entanglement witnesses. We postpone these questions for the future work.

Data availability
The datasets involved in this study can be obtained from the corresponding author upon request.

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Declarations

Competing interests
The authors declare no competing interests.

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